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Blocking Radial Diffusion in a Double-Waved Hamiltonian Model

Caroline G. L. Martins^{1,4}, F. A. Marcus², I. L. Caldas³ and R. Egydio de Carvalho¹

¹ UNESP-Univ Estadual Paulista; Instituto de Geociências e Ciências Exatas; Departamento de Estatística, Matemática Aplicada e Computação, Av. 24A, 1515, 13506-900 Rio Claro, SP, Brazil.

² Universidade de São Paulo; Departamento de Engenharia Naval e Oceânica 05508-970 São Paulo, SP, Brazil.

³ Universidade de São Paulo; Instituto de Física 05315-970 São Paulo, SP, Brazil.

⁴ Instituto Tecnológico da Aeronáutica; Departamento de Física 12228-900 São José dos Campos, SP, Brazil.

E-mails: carolinegameiro@gmail.com, albertus@if.usp.br, ibere@if.usp.br, regydio@rc.unesp.br

Abstract A non-twist Hamiltonian system perturbed by two waves with particular wave numbers can present Robust Tori, barriers created by the vanishing of the perturbing Hamiltonian at some defined positions. When Robust Tori exist, any trajectory in phase space passing close to them is blocked by emergent invariant curves that prevent the chaotic transport. We analyze the breaking up of the RT as well the transport dependence on the wave numbers and on the wave amplitudes. Moreover, we report the chaotic web formation in the phase space and how this pattern influences the transport.

1. Introduction

A Hamiltonian model with two waves was introduced by W. Horton [1, 4] to describe drift waves, originated by particles drift proportional to $\vec{E} \wedge \vec{B}$ in nonuniform plasmas, propagating in a magnetic toroidal and an electric radial fields. The model has been applied in many works as to investigate the influence of the equilibrium electric and magnetic fields on the radial transport as to analyze experimental results [2, 3].

Robust Tori [5-7] are dynamical barriers present in Hamiltonian models that survive intact under the action of generic perturbations [8]. Our proposal is to show that the two-wave Hamiltonian model contains RT, for certain parameters set, and even if other waves are included in the system they do not break up themselves. This is an important fact since the creation of barriers in Hamiltonian systems has been a remarkable subject in physics, mainly in Hamiltonians used for plasma studies [9, 10].

We are going to consider the behavior of an integrable Hamiltonian H_0 under the effect of a small amplitude perturbation described as,

$$H(x, y, t) = H_0(x) + \varepsilon H_1(x, y, t) \quad (1)$$

where ε is the control perturbation parameter. We also consider the transport along the “axis x ”, called radial transport [6, 7], due to the waves, in poloidal sections of the tokamak. The Hamiltonian describing that transport is [1],

$$H(x, y, t) = H_0(x) + \sum_n A_n \sin(k_{xn}x) \cos(k_{yn}y - \omega_n t) \quad (2)$$

where $H_0(x)$ is an unperturbed integrable Hamiltonian perturbed by a collection of waves. The coordinates x and y are the canonical momentum and its conjugate coordinate respectively. In this work we consider only two waves.

The canonical transformation mediated by the generating function, $F(x, y', t) = x(y' - u_1 t)$ from where $y = \frac{\partial F(x, y', t)}{\partial x} = y' - u_1 t$, $x' = \frac{\partial F(x, y', t)}{\partial y'} = x$, $\frac{\partial F(x, y', t)}{\partial t} = -x u_1$ and

$H(x, y, t) = H(x', y', t) + \frac{\partial F(x, y', t)}{\partial t}$, leads the Hamiltonian from Eq. (2) in the following equation,

$$H(x, y, t) = H_0(x) - u_1 x + A_1 \sin(k_{x1}x) \cos(k_{y1}y) + A_2 \sin(k_{x2}x) \cos(k_{y2}(y - ut)) \quad (3)$$

where $u = (\omega_2 / k_{y2}) - (\omega_1 / k_{y1})$ is the difference of the phase velocities between the two waves, while $u_1 = (\omega_1 / k_{y1})$ is the phase velocity of the first wave, (A_1, A_2) and (k_{x1}, k_{y1}) are the amplitudes and the wave numbers of the first and second waves respectively.

An important characteristic of the unperturbed Hamiltonian system is its rotation number,

$$\Omega(x) = \frac{dH_0(x)}{dx} \quad (4)$$

because depending if $\Omega(x)$ is monotonic or not the system is twist or non-twist [11, 12, 13] respectively. Many models are described by non-twist Hamiltonians, for instance orbits in particle accelerators [14], plasma wave heating [15] and fluid dynamics [16]. The unperturbed Hamiltonian $H_0(x)$ that we will consider here presents a non-twist profile and is giving by,

$$H_0(x) = Ax^3 + Bx^2 + Cx - u_1 x \quad (5)$$

where A, B and C are constants numbers.

2. Hamiltonian approach for one wave

At first, we consider the Hamiltonian of Eq. (3) with only one wave, choosing $A_2=0$, so that,

$$H(x, y, t) = Ax^3 + Bx^2 + Cx - u_1 x + A_1 \sin(k_{x1}x) \cos(k_{y1}y) \quad (6)$$

We note that the perturbed Hamiltonian vanishes when $\cos(k_{y1}y) = 0$ or $\sin(k_{x1}x) = 0$. However, looking at the equations of motion,

$$\begin{aligned}\dot{x} &= A_1 k_{y1} \sin(k_{x1} x) \sin(k_{y1} y) \\ \dot{y} &= [3Ax^2 + 2Bx + C - u_1] + A_1 k_{x1} \cos(k_{x1} x) \cos(k_{y1} y)\end{aligned}\quad (7)$$

we can see that for $\cos(k_{y1} y) = 0$ the motion still exists in x and y coordinates. So, the RT will appear only in the cases where the perturbation and also the motion in at least one coordinate vanish [6]. Our interest is about the radial transport, then we will analyze the case when $\sin(k_{x1} x) = 0$. For this condition, the perturbation and the motion in the coordinate x vanish (see Eqs. (6) and (7)). However, the motion still exists along the coordinate y . There will be straight lines with $x = \text{constant}$, corresponding to the RT, at $x = \frac{n_1 \pi}{k_{x1}}$, for integers n_1 .

Figure 1 shows the Poincaré section for the one-wave Hamiltonian, with islands that look like cells flatted by straight lines. These lines, in black, are the RT previously mentioned. The Hamiltonian of Eq. (7) presents only one wave and the system is integrable, i.e. there is not any chaotic orbit.

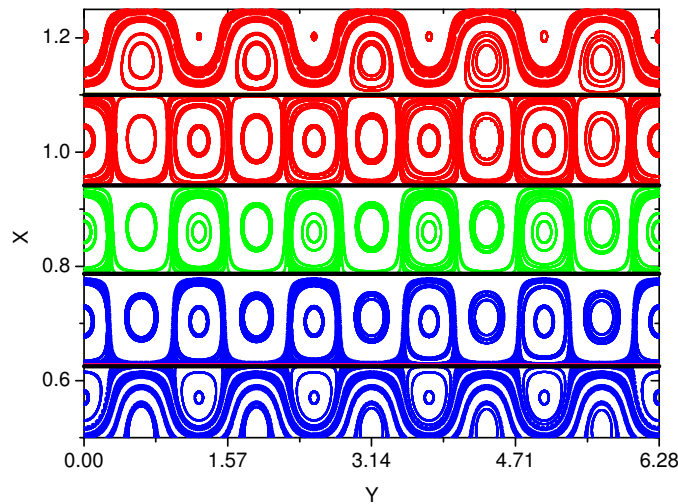


Figure 1 – Poincaré section for the Hamiltonian of Eq. (7) with one wave, $A_2=0$, and $A = -2$; $B = 5.25$ and $(C-u_1) = -4.5$

3. Two-waves Hamiltonian approach

The addition of another wave in the Hamiltonian of Eq. (7) will break the integrability of the system and we get this through the equation,

$$H(x, y, t) = Ax^3 + Bx^2 + Cx - u_1 x + [A_1 \sin(k_{x1} x) \cos(k_{y1} y) + A_2 \sin(k_{x2} x) \cos(k_{y2} (y - ut))] \quad (8)$$

The equations of motion are given by,

$$\begin{aligned}\dot{x} &= A_1 k_{y1} \sin(k_{x1} x) \sin(k_{y1} y) + A_2 k_{y2} \sin(k_{x2} x) \sin(k_{y2} (y - ut)) \\ \dot{y} &= [3Ax^2 + 2Bx + C - u_1] + A_1 k_{x1} \cos(k_{x1} x) \cos(k_{y1} y) + A_2 k_{x2} \cos(k_{x2} x) \cos(k_{y2} (y - ut))\end{aligned}\quad (9)$$

We can see that the perturbation in square brackets in Eq. (8) vanishes for $\cos(k_{y1}y) = \cos(k_{y2}(y - ut)) = 0$, but looking at the equations of motion above, Eq. (9), we note that the motion does not vanish with only this condition analogously as mentioned in section 2.

On the other hand, for $\sin(k_{x1}x) = \sin(k_{x2}x) = 0$, there will be lines with $x = \text{constant}$ at $x = \frac{n_1\pi}{k_{x1}} = \frac{n_2\pi}{k_{x2}}$ for integers n_1 and n_2 . Consequently, the relation between the wave numbers should

satisfy $k_{x2} = m k_{x1}$. If m is an integer, all RT will remain intact, otherwise only some percentage will survive. If the wave numbers obey the condition above, all RT will remain intact under the addition of the second wave blocking the radial transport.

Fig. 2 shows two different situations for the two-wave Hamiltonian model of Eq. (8). Fig. 2(a) shows the Poincaré section for the case already described in the literature [1, 2, 3, 4]. Adding the second wave, the integrability of the system is broken and there are chaotic orbits spread along the phase space, i.e. the particles move along the radial coordinate x forming a chaotic web. The colours blue, green and red represent trajectories of initial conditions given on different regions in the phase space and they show the mixing radial transport done by the particles. There are no barriers in the phase space along the *axis* y . Fig. 2(b) shows the Poincaré section for the particular case $k_{x2} = m k_{x1}$, and the RT (lines in black) remain intact under the addition of the second wave. We see that there is no mixture of the blue, green and red, showing that the RT are blocking the radial diffusion.

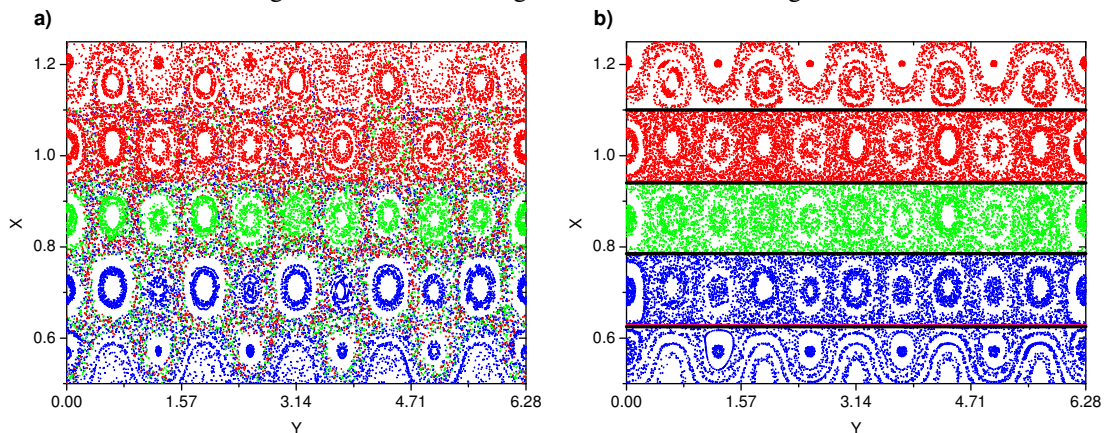


Figure 2 - Poincaré sections for the Hamiltonian with two waves of Eq. (8); **a)** for $k_{x2} \neq m k_{x1}$ without RT. **b)** for $k_{x2} = m k_{x1}$ with RT in black.

4. Diffusion coefficient and robust tori

The onset of chaos in the phase space takes place when the second wave is added and consequently the transport can be evaluated by calculating the diffusion coefficient from the orbits. The calculation of the local diffusion coefficient, for finite time, is done considering the non-dimensional equation [2]:

$$D = \frac{1}{2tN} \sum_{i=1}^N [x_i(t) - x_i(0)]^2 \quad (10)$$

where $N=1000$ is the total number of initial conditions, which are uniformly distributed through the grid $x:[0.50; 1.25]$ and $y:[0; 2\pi]$, and $t=150$ is, in fact, the number of iterations for each initial condition.

In Fig. 3(a) we show the dependence of the radial diffusion coefficient on the wave number k_{x2} . We used in the numerical simulations the following parameters $k_{x1}=20$; $k_{y1}=3$. As we can see, the lowest values of the diffusion occur for $k_{x2}=m.k_{x1}=20$ for $m=1$ and for $k_{x2}=m.k_{x1}=40$ for $m=2$. The radial diffusion coefficient tends to zero for all m integer, but it is not zero for small non-integer m .

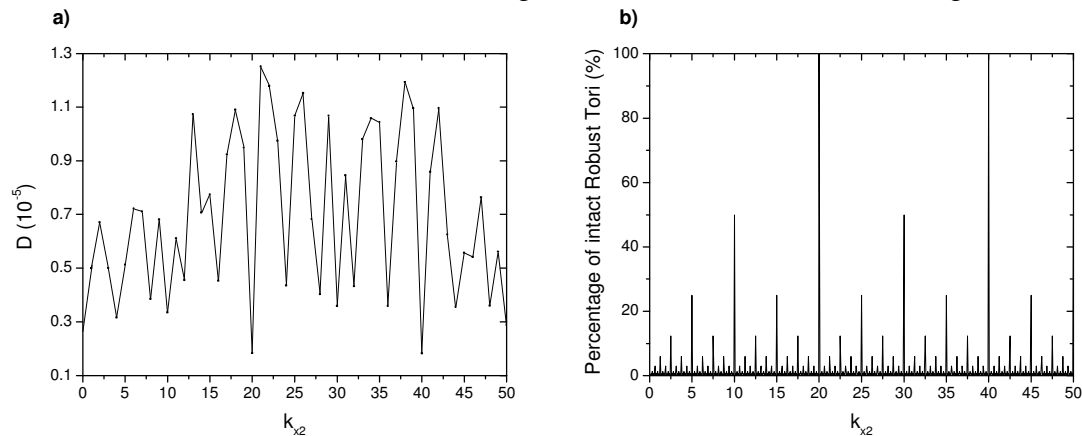


Figure 3 – Comparison between **a)** Diffusion coefficient and **b)** Percentage of intact robust tori.

The behavior of the radial diffusion coefficient is explained from Fig. 3(b), which shows the percentage of intact remaining RT after the addition of the second wave, considering an initial amount of 40 RT in the phase space. For the wave numbers $k_{x2} = 20$ and $k_{x2} = 40$ all RT still exist because the wave numbers satisfy the particular solution we present here, $k_{x2} = m.k_{x1}$ with m an integer. However, we can observe that a portion of RT also exists when m is a non-integer.

The existence of RT affects directly the particles diffusion in the radial direction. For the particular solution introduced here, the radial diffusion coefficient is zero because all RT are preserved. However, for some intermediate values of k_{x2} some RT still remain intact, decreasing the radial transport.

5. Chaotic web and the breaking up of the robust tori

In order to analyze the breaking up of the RT and the formation of a chaotic web in the phase space we performed a colour map to indicate how many iterations are necessary for the initial conditions to reach the reference line $x=0.9408$ for different values of (A_2/A_1) . A grid of initial conditions is used with y in the range $[0, 2\pi]$ and x in the range $[0.6283, 0.9408]$.

Figure 4 shows the colour maps for different values of (A_2/A_1) . We emphasize that the particular solution for the RT preservation was not used in this case. We used $k_{x2} \neq m.k_{x1}$ to show how the RT are broken by the time we increase the rate for the amplitudes of the two waves.

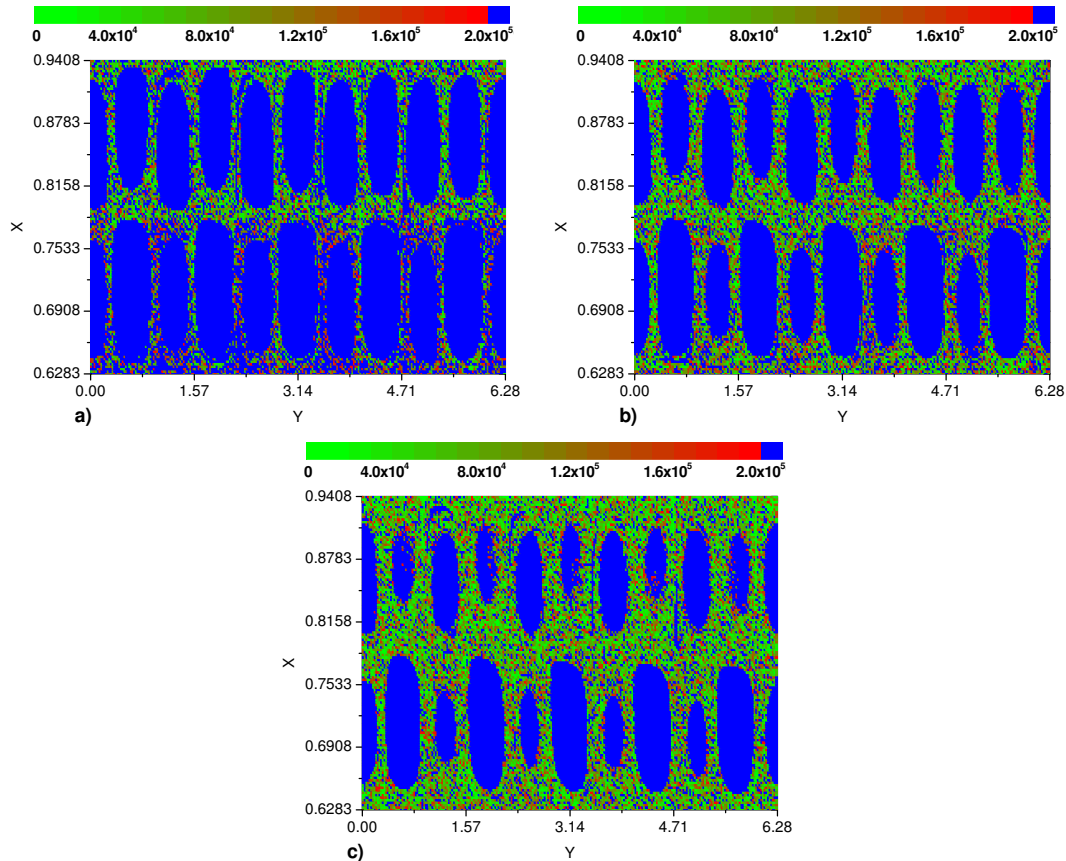


Figure 4 – Colour map showing how many iterations are necessary for the initial conditions to reach the reference line $x=0.9408$ for **a)** $(A_2/A_1)=0.05$ **b)** $(A_2/A_1)=0.1$ **c)** $(A_2/A_1)=0.15$

A trajectory with a chosen initial condition is considered trapped if the number of iteration to reach the reference line is higher than 6×10^5 and the colour that corresponds to this situation is blue. As can be seen in Fig. 4(a), with $(A_2/A_1)=0.05$, all RT are destroyed and a chaotic web appears in the phase space. The chaotic web is created around the hyperbolic fixed points, doing a channel for the radial transport. As we increase the amplitudes rate, the chaotic web also increases, as can be seen in Fig. 4(b) so that a large number of trajectories achieve the reference line with only a few iterations. The enlargement of the chaotic web is clearer when we increase the rate of the amplitudes to $(A_2/A_1)=0.15$, as is showed in Fig. 4(c), and the area covered by the colour blue, which corresponds to the trapped trajectories, decreases.

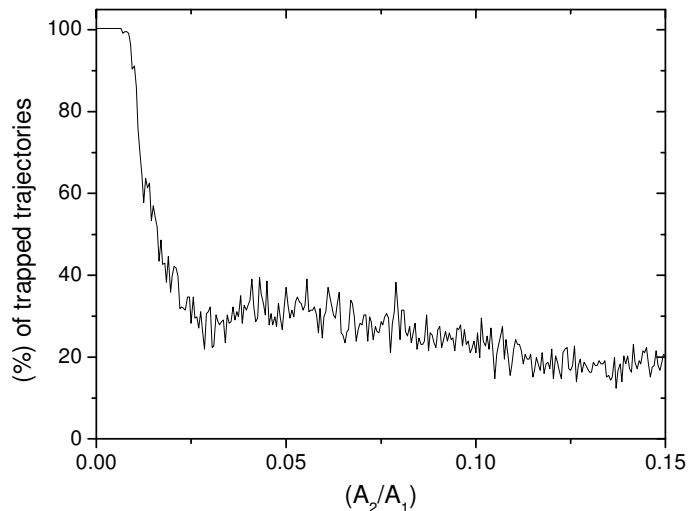


Figure 5 – Percentage of the trapped trajectories in the RT located at $x=0.62831853$ as a function of the rate of the amplitudes (A_2/A_1) of the first and the second wave.

In order to determine how many trajectories (generated by chosen initial conditions) have been trapped in the line with $x=0.62831853$ formed by one robust torus, we count how many dots in blue exist in this line. We depict in Fig. 5 the behavior of the breaking up of the robust torus. How was expected for the integrable case (with only one wave), $(A_2/A_1)=0$, all trajectories are trapped, showing the presence of the robust torus. After the breaking up of the robust torus, the number of trapped trajectories in $x=0.62831853$ decreases, with an oscillatory behavior.

6. Conclusions

The onset of chaos in the phase space takes place when the second wave is added and consequently the destruction of the manifolds of the hyperbolic fixed points creates a chaotic web in phase space, forming a channel for the radial transport. The increase of the wave amplitudes is the responsible for the enlargement of the chaotic web, improving the radial transport.

Previous studies [9, 17] have shown the importance of decreasing the radial transport induced by drift waves to improve the plasma confinement in tokamaks. It is also reported that similar Hamiltonians to the one presented in this paper have been used to study transport but only few works were dedicated to control chaos in these systems [2, 10].

Robust Tori are dynamical barriers that block the chaotic radial transport and a particular solution that preserves all RT in phase space is presented. We point out that the particular two wave solution presented in this paper can be extended for many waves and the non perturbed Hamiltonian H_0 does not influence the formation of RT along the x -direction. The condition to introduce RT can be rewritten as $k_{x(n+1)} = m_n \cdot k_{xn}$ with n an integer $\in [1, (N-1)]$, where $(N-1)$ is the number of waves

considered. The coefficients m_n have to obey the condition, $\prod_{n=1}^{N-1} m_n = \text{integer number}$. The multiplication of the coefficients m_n has to be an integer because all radial wave numbers k_{xn} have to be a multiple of the first radial wave number k_{x1} , to keep intact all RT in phase space.

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