Acta Scientiarum

http://www.uem.br/acta
ISSN printed: 1806-2563
ISSN on-line: 1807-8664
Doi: 10.4025/actascitechnol.v35i2.13547

Stochastic models with heteroskedasticity: a Bayesian approach for Ibovespa returns

Sandra Cristina de Oliveira[1*] and Marinho Gomes de Andrade[2]

*[1]Campus Experimental de Tupã, Universidade Estadual Paulista, Av. Domingos da Costa Lopes, 780, 17602-660, Tupã, São Paulo, Brazil. [2]Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos, São Paulo, Brazil. *Author for correspondence. E-mail: sandra@tupa.unesp.br*

ABSTRACT. Current research compares the Bayesian estimates obtained for the parameters of processes of ARCH family with normal and Student's *t* distributions for the conditional distribution of the return series. A non-informative prior distribution was adopted and a reparameterization of models under analysis was taken into account to map parameters' space into real space. The procedure adopts a normal prior distribution for the transformed parameters. The posterior summaries were obtained by Monte Carlo Markov Chain (MCMC) simulation methods. The methodology was evaluated by a series of Bovespa Index returns and the predictive ordinate criterion was employed to select the best adjustment model to the data. Results show that, as a rule, the proposed Bayesian approach provides satisfactory estimates and that the GARCH process with Student's *t* distribution adjusted better to the data.

Keywords: ARCH family, Bayesian analysis, MCMC methods, financial returns.

Modelos estocásticos com heterocedasticidade: uma abordagem Bayesiana para os retornos do Ibovespa

RESUMO. O objetivo deste trabalho foi comparar as estimativas Bayesianas obtidas para os parâmetros de processos da família de modelos auto-regressivos com heterocedasticidade condicional – ARCH considerando distribuição normal e *t* de *student* para a distribuição condicional da série de retornos. Adotou-se a distribuição a priori não-informativa e considerou-se uma reparametrização dos modelos estudados para mapear o espaço dos parâmetros no espaço real. Este procedimento permite adotar distribuição a priori normal para os parâmetros transformados. Os sumários a posteriori foram obtidos por meio dos métodos de simulação de Monte Carlo em Cadeias de Markov (MCMC). A metodologia foi avaliada considerando uma série de retornos do Ibovespa e, para selecionar o modelo de melhor ajuste aos dados, foi utilizado o critério da densidade preditiva ordenada. Os resultados mostram que, de um modo geral, a abordagem Bayesiana proposta fornece estimativas satisfatórias e que o processo GARCH com distribuição *t* de *student* apresenta melhor ajuste aos dados.

Palavras-chave: família ARCH, análise Bayesiana, métodos MCMC, retornos financeiros.

Introduction

A large variety of models exists to estimate the volatility of financial assets return series. The most common in the literature are the Autoregressive Conditional Heteroskedasticity (ARCH) model, suggested by Engle (1982), and its extension, the Generalized ARCH (GARCH) models, proposed by Bollerslev (1986). The models characterize a non-linear dependence among returns due to the serial dependency of conditional variance.

Since volatility at a specific time depends on the past values of the series, the determination of maximum likelihood estimators (MLE) of parameters of ARCH family models require the maximization of a non-linear function, and thus, estimates could only be obtained numerically. Engle (1982) suggested Newton's method as an iterative method to calculate maximum likelihood estimates. Such procedure relaxes the imposed restrictions to parameters (they must be positive and their sum must be less than one) which warrant stationary covariance. On the other hand, certain difficulties are involved in the determination of asymptotic characteristics of MLE with restrictions. It may lead to local maxima, since characteristics such as asymptotic normality do not allow restrictions. Further, procedures for the identification, adjustment and diagnosis of models and prediction of econometric series values require the characteristics of asymptotic theory. Since models are distant from linearity, the estimators'

asymptotic characteristics may only be verified for very long series and, in general, are more appropriate with symmetrical distribution for errors and with normal distribution for data. Bollerslev (2008) provided an extensive review of the models' characteristics.

Under a Bayesian approach, Geweke (1989) provided one of the first investigations for ARCH family models in which a special reparameterization case employed non-informative prior distributions. Estimates of parameters were obtained from Monte Carlo simulation algorithms. Nakatsuma (2000) used normal prior distributions for the parameters of ARMA-GARCH models and the Metropolis-Hastings algorithm to determine posterior summaries. Further, Polasek (2001) suggested a hierarchic structure for PAR-ARCH from a Bayesian approach by using Monte Carlo Markov Chain simulation methods. Within the context of unobserved component models, Giakoumatos et al. (2005) suggested a Bayesian approach for ARCH models with auxiliary variables (PITT; WALKER, 2005).

Whereas Ausín and Galeano (2007) recently suggested a Bayesian approach for GARCH models with errors generated by Gaussian mixtures, Barreto et al. (2008) compared Bayesian and Maximum Likelihood methods by simulated series, following ARCH processes, with different orders and under conditions of finite and infinite variance. Moreover, Andrade and Oliveira (2011) presented a Bayesian approach for ARCH models with normal prior distributions for their respective parameters and compared credibility intervals with bootstrap intervals by employing index return series of the Brazilian financial market.

The Stock Exchange Index of São Paulo (Ibovespa) is the most important index for the average performance of the Brazilian market shares rates. Its relevance is due to the fact that Ibovespa portrays the behavior of the main stocks and shares negotiated at Bovespa and also its history. In fact, the Bovespa index maintains the integrity of a historical series and did not undergo methodological modifications since its establishment in 1968[1]. Ibovespa's basic aim is to be an average indicator of the main transacted shares and a profile of cash negotiations in the Bovespa exchange. Stocks with the highest participation (in terms of volume) on the exchange are selected to compose the index. Ibovespa's behavior has been widely investigated in the literature owing to its economical relevance and several authors have used Ibovespa return series for modeling and for comparing different models. Morettin (2008), Andrade and Oliveira (2011), and Oliveira and Andrade (2012) are among the many authors who investigated the above-mentioned index and who employed different time series models from a classical or Bayesian approach.

Since the context and the relevance of the ARCH family models in the solution of problems in the economical and financial areas due to their applicability and interpretation (the relations between returns and volatility) have been provided, current investigation compares the Bayesian estimates obtained for the parameters of AR(p)-ARCH(q), ARCH(q) and GARCH(q,r) models, taking into account normal and Student's t distributions for the conditional distribution of the Ibovespa's financial returns series. Non-informative prior distributions, foregrounded on Geweke (1989), were suggested and a reparameterization of the models studied was taken into account for each case to map the parameter's space on real space. The procedure adopts normal prior distributions for the transformed parameters. Posterior summaries were obtained by MCMC simulation methods.

Stochastic models with heteroskedasticity for time series

AR-ARCH models

The regression model proposed by Engle (1982) with its mean non-zero and expressed as a linear combination of exogenous variables, exhibits the following structure:

$$y_t = x_t\beta + z_t \quad (1)$$

$$y_t \mid \Omega_{t-1} \sim P\left(x_t\beta, h_t\right) \quad (2)$$

$$h_t = \alpha_0 + \sum_{j=1}^{q} \alpha_j z_{t-j}^2 \quad (3)$$

$$z_t = y_t - x_t\beta \quad (4)$$

in which:

y_t represents a returns series;

$P(.)$is a parametric distribution, usually Normal or Student's t;

x_t is a vector of exogenous variables which may include values of y_t outdated by time; β is the vector of unknown parameters;

Ω_{t-1} is the set of information available up to time t - 1.

[1]BM&FBOVESPA was established in May 2008 and combined the Bolsa de Mercadorias & Futuros (BM&F) and Bolsa de Valores de São Paulo (Bovespa). It became the biggest Stock Exchange institution in Latin America, the second in the Americas and the third in the world (PORTAL DO INVESTIDOR, 2011).

Only AR(*p*)-ARCH(*q*) models have been investigated in current research, in which $x_t = (y_{t-1}, y_{t-2},..., y_{t-p})'$. Therefore,

$$y_t = \sum_{i=1}^{p} \beta_i y_{t-i} + z_t \quad (5)$$

Let z_t be the process that satisfies model $z_t = h_t^{1/2}\varepsilon_t$ in which $\{\varepsilon_t, t \geq 0\}$ is a sequence of independent randomized variables, identically distributed with mean zero and variance 1, regardless of x_t. In practice, it is commonly supposed that $\varepsilon_t \sim N(0,1)$ or $\varepsilon_t \sim t_v$ (Student's *t* distribution with *v* degrees of freedom). The model defined in (1)-(4) may be interpreted by disturbances in linear regression which follow an autoregressive conditional heteroskedasticity of the order *q*.

So that model (1)-(4) may be plausible ($h_t > 0$ during *t*), there must be $\alpha_0 > 0$ and $\alpha_j \geq 0, j = 1,...,q$. Further, process y_t has finite variance and, therefore, stationary covariance if, and only if, all the roots of the polynomials $1 - \sum_{i=1}^{p} \beta_i l^i$ and $1 - \sum_{j=1}^{q} \alpha_j l^j$ are outside the unit radius circle (ENGLE, 1982: Theorem 2). When these conditions are satisfied, unconditional variance y_t may be given by $\frac{\alpha_0}{\left(1 - \sum_{j=1}^{q} \alpha_j\right)} \Bigg/ \left(1 - \sum_{i=1}^{p} \beta_i \rho_i\right)$, where $\rho_i = \frac{\gamma_i}{\gamma_0}$, with $\gamma_0 = V(y_t) = E(y_t^2)$ and $\gamma_i = E(y_t y_{t-i})$, $i = 1,...,p$. Therefore, $\sum_{j=1}^{q} \alpha_j < 1$ and $\sum_{i=1}^{p} |\beta_i| < 1$ is the sufficient condition so that the process has a stationary covariance.

Let $Y = \{y_t, t = 1,2,...,T\}$ be a trajectory of the process y_t and that x_t involves only past "*p*" values of y_t. If normality is held for ε_t, the likelihood function of y_t, $t = p+q+1,...,T$, conditioned to $p+q$ first observations (presumed to be known) is defined as:

$$L(Y \mid \alpha, \beta) = (2\pi)^{-\frac{(T-p-q)}{2}} \prod_{t=p+q+1}^{T} \left(\frac{1}{h_t}\right)^{1/2} \exp\left\{-\frac{(y_t - x_t\beta)^2}{2h_t}\right\} \quad (6)$$

where:

$\alpha = (\alpha_0, \alpha_1,..., \alpha_q)'$;

$\beta = (\beta_1,..., \beta_p)'$.

Supposing that ε_t has standard Student's *t* distribution such that $t_v \sim$ Student's *t* with *v* degrees of freedom, or rather, $\varepsilon_t = \frac{t_v}{\sqrt{v/(v-2)}}$, the likelihood function of y_t, $t = q+1,...,T$, conditioned to $p+q$ first observations is given by (MORETTIN, 2008):

$$L(Y \mid \alpha, \beta) = \prod_{t=p+q+1}^{T} \frac{\Gamma((v+1)/2)}{\Gamma(v/2)\sqrt{(v-2)\pi}} \left(\frac{1}{h_t}\right)^{1/2} \left(1 + \frac{(y_t - x_t\beta)^2}{h_t(v-2)}\right)^{\frac{-(v+1)}{2}} \quad (7)$$

with $\alpha = (\alpha_0, \alpha_1,..., \alpha_q)'$, $\beta = (\beta_1,..., \beta_p)'$ and for some *v*.

ARCH models

If the regression model proposed by Engle (1982), defined by the expressions (1)-(4), considers $\beta = 0$, then the new structure may be summarized as

$$z_t \mid \Omega_{t-1} \sim P(0, h_t) \quad (8)$$

$$h_t = \alpha_0 + \sum_{j=1}^{q} \alpha_j z_{t-j}^2 \quad (9)$$

where:

z_t represents return series;

$P(\cdot)$ is the parametric distribution;

Ω_{t-1} is the set of information available up to time *t* - 1.

The interpretation for the model defined in (8)-(9) is that *returns* in linear regression follow an autoregressive conditional heteroskedasticity of the order *q*.

Similarly, so that model (8)-(9) is plausible ($h_t > 0$ during *t*), there must be $\alpha_0 > 0$ and $\alpha_j \geq 0$ for $j = 1,...,q$. Further, process z_t has finite variance and therefore stationary covariance if, and only if, all the roots of the polynomial $1 - \sum_{j=1}^{q} \alpha_j l^j$ lie outside the unit radius circle. It may be thus shown that the unconditional variance of z_t is given by $\alpha_0 \Big/ \left(1 - \sum_{j=1}^{q} \alpha_j\right)$, whose condition for stationary covariance process is $\sum_{j=1}^{q} \alpha_j < 1$.

Let $Z = \{z_t, t = 1,2,...,T\}$ be a trajectory of the process z_t. If normality holds for ε_t, the likelihood

function of z_t, $t = q+1,...,T$, conditioned to q first observations (presumed to be known) is given by

$$L(Z|\alpha) = (2\pi)^{-\frac{(T-q)}{2}} \prod_{t=q+1}^{T} \left(\frac{1}{h_t}\right)^{1/2} \exp\left\{-\frac{z_t^2}{2h_t}\right\} \quad (10)$$

with $\alpha = (\alpha_0, \alpha_1, ..., \alpha_q)'$.

Presuming that ε_t has a standard Student's t distribution, the likelihood function of z_t, $t = q+1,...,T$, conditioned to q first observations, is given by (MORETTIN, 2008):

$$L(Z|\alpha) = \prod_{t=q+1}^{T} \frac{\Gamma((v+1)/2)}{\Gamma(v/2)\sqrt{(v-2)\pi}} \left(\frac{1}{h_t}\right)^{1/2} \left(1 + \frac{z_t^2}{h_t(v-2)}\right)^{\frac{-(v+1)}{2}} \quad (11)$$

with $\alpha = (\alpha_0, \alpha_1, ..., \alpha_q)'$ and for some v.

GARCH models

The regression model proposed by Engle (1982) and generalized by Bollerslev (1986), with means zero and expressed as a linear combination of exogenous variables, have a structure summarized as

$$z_t | \Omega_{t-1} \sim P(0, h_t) \quad (12)$$

$$h_t = \alpha_0 + \sum_{j=1}^{q} \alpha_j z_{t-j}^2 + \sum_{i=1}^{r} \lambda_i h_{t-i} \quad (13)$$

where:

z_t is a return series;

$P(\cdot)$ is a parametric distribution;

Ω_{t-1} is the set of information available up to time $t-1$.

When $r = 0$, the process is reduced to an ARCH(q).

Let z_t be a process that satisfies the model $z_t = h_t^{1/2}\varepsilon_t$ such that $\{\varepsilon_t, t \geq 0\}$ is a sequence of independent randomized variables and identically distributed with mean zero and variance 1. An interpretation for the model defined in (13)-(14) is that disturbances in linear regression follow a generalized ARCH process respectively of the orders q and r.

So that model (12)-(13) be plausible ($h_t > 0$ during t), there must be $q > 0$, $r \geq 0$, $\alpha_0 > 0$, $\alpha_j \geq 0$, $j = 1,2,...q$, $\lambda_i \geq 0$, $i = 1,2,...r$. Thus, it may be demonstrated that unconditional variance of z_t is given by $\alpha_0 \Big/ \left(1 - \sum_{j=1}^{q} \alpha_j - \sum_{i=1}^{r} \lambda_j\right)$, whose condition for stationary covariance is $\sum_{j=1}^{q} \alpha_j - \sum_{i=1}^{r} \lambda_j < 1$.

Let $Z = \{z_t, t = 1,2,...,T\}$ be a trajectory of process z_t and $\theta = (\alpha_0, \alpha_1, ..., \alpha_q, \lambda_1, \lambda_2, ..., \lambda_r)'$. Presuming normality holds for ε_t, the likelihood function of z_t, $t = q+r+1,...,T$, conditioned to $q+r$ first observations (presumed to be known), is given by:

$$L(Z|\theta) = (2\pi)^{-\frac{(T-q-r)}{2}} \prod_{t=q+r+1}^{T} \left(\frac{1}{h_t}\right)^{1/2} \exp\left\{-\frac{z_t^2}{2h_t}\right\} \quad (14)$$

Presuming that ε_t has a standard Student's t distribution, or rather, $\varepsilon_t = \frac{t_v}{\sqrt{v/(v-2)}}$ such that $t_v \sim$ Student's t with v degrees of freedom, the likelihood function of z_t, $t = q+r+1,...,T$, conditioned to $q+r$ first observations, is given by (MORETTIN, 2008):

$$L(Z|\theta) = \prod_{t=q+r+1}^{T} \frac{\Gamma((v+1)/2)}{\Gamma(v/2)\sqrt{(v-2)\pi}} \left(\frac{1}{h_t}\right)^{1/2} \left(1 + \frac{z_t^2}{h_t(v-2)}\right)^{\frac{-(v+1)}{2}} \quad (15)$$

with $\theta = (\alpha_0, \alpha_1, ..., \alpha_q, \lambda_1, \lambda_2, ..., \lambda_r)'$ and for some v.

The likelihood functions described in (6), (7), (10), (11), (14) and (15) may be maximized with regard to the respective unknown parameters.

In Bayesian context, the employment of the conditioned likelihood function instead of the exact likelihood function may be undertaken without any great precision loss in the estimates. This is due to the fact that one of the main advantages of Bayesian inference lies in the possibility of adjusting models, even in small samples.

Bayesian approach

Taking into consideration returns trajectory $Y = \{y_t, t = 1,2,...,T\}$, the Bayesian approach for the inference of parameters of AR(p)-ARCH(q), ARCH(q) and GARCH(q,r) processes starts from the joint likelihood functions defined by the trajectory with a prior density for parameters (that reflects previous knowledge on the distribution of these parameters) by Bayes's rule:

$$\pi(\bullet | Y) \propto L(Y | \bullet)\pi(\bullet) \quad (16)$$

The expression $\pi(\bullet | Y)$ is called posterior distribution of parameter(s) of interest and explains how these randomized variables are distributed after data have been complied with.

In current research, a non-informative prior distribution is proposed for the parameters of AR(p)-ARCH(q) and ARCH(q) processes based on the suggestion by Geweke (1989), defined as:

$$\pi(\varphi) \propto \begin{cases} \left(\frac{1-\sum_{j=1}^{q}\alpha_j}{\alpha_0}\right) & ,\alpha_0>0,\alpha_j\geq 0,j=1,...,q \\ 0 & ,\text{otherwise} \end{cases} \quad (17)$$

Conditioned to $\alpha_j = 0, j = 1,...,q$, it is Jeffreys' non-variant prior distribution for the normal linear regression model. Reparameterization is also considered and consists of

$$\phi_j = \log\left(\frac{\alpha_j - a_j}{b_j - \alpha_j}\right) \quad j = 0,1,2,...,q \quad (18)$$

Since the necessary condition of stationary covariance warrants variation intervals for the parameters of ARCH family processes, there are intervals $\lfloor a_j, b_j \rfloor$, $j = 0,1,...,q$ with $a_j > 0$ and $b_j < 1$, such that $a_j \leq \alpha_j \leq b_j$ which may be also defined as $a_0 \leq \alpha_0 \leq b_0$, with $a_0 > 0$ and $b_0 \leq E\left(z_t^2\right)$. Above analysis leads towards a choice of transformation (18), which maps the intervals $(-\infty,+\infty)$ within the dominion (a_j, b_j) and vice-versa. It also decreases the rejection rate of the simulation algorithm MCMC and accelerates its convergence process. Values for a_j and b_j may be chosen based on some prior information, for instance, previous studies on the series under analysis.

When reparameterization, defined in (18), is employed,

$$\pi(\varphi) \propto \begin{cases} \left(\frac{1}{k(\varphi)}\right)^{\frac{1}{2}} & ,\ -\infty < \phi_j < \infty,\ j = 0,1,...,q \\ 0 & ,\text{otherwise} \end{cases} \quad (19)$$

where:

$\varphi = (\phi_0, \phi_1, ..., \phi_q)'$;

$k(\varphi) = \alpha_0 \Big/ \left(1 - \sum_{j=1}^{q}\alpha_j\right)$;

$\alpha_j = \frac{b_j e^{\phi_j} + a_j}{1 + e^{\phi_j}}, j = 0,1,...,q$.

Similarly, prior distribution for parameters of GARCH (q,r) processes based on the proposal of Geweke (1989) and on the reparameterization $\phi_j = \log\left(\frac{\theta_j - a_j}{b_j - \theta_j}\right)$, when $\theta_j = \alpha_j$ or $\lambda_j, j = 0,1,...,q+r$, is given by:

$$\pi(\varphi) \propto \begin{cases} \left(\frac{1}{k(\varphi)}\right)^{\frac{1}{2}} & ,\ -\infty < \phi_j < \infty,\ j = 0,1,...,q+r \\ 0 & ,\text{otherwise} \end{cases} \quad (20)$$

where:

$\varphi = (\phi_0, \phi_1, ..., \phi_{q+r})'$;

$k(\varphi) = \alpha_0 \Big/ \left(1 - \sum_{j=1}^{q}\alpha_j + \sum_{i=1}^{r}\lambda_j\right)$.

It should be emphasized that in current research a_j and b_j were defined by analyses undertaken by Andrade and Oliveira (2011), in which Ibovespa returns series were widely debated and discussed.

Therefore, volatility h_t as a function of model parameters, duly transformed into ϕ_j, causes posterior joint distributions for φ, described in Table 1.

Posterior densities have forms that are only similar to those of density functions of known probability. Consequently, the analytic calculation of the parameters' quantities of interest, such as means, mode, medians, standard deviation and others becomes impossible (KOTZ et al., 2000). The issue may be solved by MCMC simulation methods, specifically the Metropolis-Hastings algorithm. Representative samples of posterior distributions of Table 1 may be produced and recover the parameters estimated by inverse transformation.

Metropolis-Hastings Algorithm

The Metropolis-Hastings (M-H) algorithm is an iterative scheme to produce indirectly a density sample when direct production from this density is unknown. Algorithm comprises a choice of nucleus, or rather, a transition density q, and generate from this nucleus by using an acceptance criterion p of the generated value to warrant that the sample obtained is representative of the generated sample (GAMERMAN; LOPES, 2006).

M-H algorithm in current research follows specifically the steps below:

Step 1: Attribute arbitrary initial rates $\varphi^{(0)} = \left\{\phi_j^{(0)}\right\}$ and start the iteration counter in $l = 1$.

Table 1. Posterior joint distributions.

AR(*p*)-ARCH(*q*) Model	
Normal distribution for ε_t (M1)	$\pi(\varphi \mid Y) \propto \left(\frac{1}{k(\varphi)}\right)^{1/2} \times \prod_{t=p+q+1}^{T} \left(\frac{1}{h_t(\varphi)}\right)^{1/2} \exp\left\{-\frac{(y_t - x_t\beta)^2}{2h_t(\varphi)}\right\}$
Student's t distribution for ε_t (M2)	$\pi(\varphi \mid Y) \propto \left(\frac{1}{k(\varphi)}\right)^{1/2} \times \prod_{t=p+q+1}^{T} \frac{\Gamma((v+1)/2)}{\Gamma(v/2)\sqrt{(v-2)\pi}} \left(\frac{1}{h_t(\varphi)}\right)^{1/2} \left(1+\frac{(y_t - x_t\beta)^2}{h_t(\varphi)(v-2)}\right)^{\frac{-(v+1)}{2}}$
ARCH(*q*) model	
Normal distribution for ε_t (M3)	$\pi(\varphi \mid Z) \propto \left(\frac{1}{k(\varphi)}\right)^{1/2} \times \prod_{t=q+1}^{T} \left(\frac{1}{h_t(\varphi)}\right)^{1/2} \exp\left\{-\frac{z_t^2}{2h_t(\varphi)}\right\}$
Student's t distribution for ε_t (M4)	$\pi(\varphi \mid Z) \propto \left(\frac{1}{k(\varphi)}\right)^{1/2} \times \prod_{t=q+1}^{T} \frac{\Gamma((v+1)/2)}{\Gamma(v/2)\sqrt{(v-2)\pi}} \left(\frac{1}{h_t(\varphi)}\right)^{1/2} \left(1+\frac{z_t^2}{h_t(\varphi)(v-2)}\right)^{\frac{-(v+1)}{2}}$
GARCH(*q*,*r*) model	
Normal distribution for ε_t (M5)	$\pi(\varphi \mid Z) \propto \left(\frac{1}{k(\varphi)}\right)^{1/2} \times (2\pi)^{-\frac{(T-q-r)}{2}} \prod_{t=q+r+1}^{T} \left(\frac{1}{h_t(\varphi)}\right)^{1/2} \exp\left\{-\frac{z_t^2}{2h_t(\varphi)}\right\}$
Student's t distribution for ε_t (M6)	$\pi(\varphi \mid Z) \propto \left(\frac{1}{k(\varphi)}\right)^{1/2} \times \prod_{t=q+r+1}^{T} \frac{\Gamma((v+1)/2)}{\Gamma(v/2)\sqrt{(v-2)\pi}} \left(\frac{1}{h_t(\varphi)}\right)^{1/2} \left(1+\frac{z_t^2}{h_t(\varphi)(v-2)}\right)^{\frac{-(v+1)}{2}}$

Step 2: Generate a new rate $\phi_j^{'}$ from the transition nucleus or density $q\left(\varphi^{(l-1)}, \varphi^c\right)$. In this case, the non-informative prior distributions lead towards posterior ones which do not present known expression to generate candidate parameters. Gaussian nucleus $N(\varphi)$ is thus produced so that posterior densities may be written as $\pi^*(\varphi \mid Y) = \frac{\pi(\varphi \mid Y)N(\varphi)}{N(\varphi)}$, $N(\varphi) \neq 0$, consequently, $\varphi^c = \left(\phi_0^{(l-1)}, \phi_1^{(l-1)}, ..., \phi_{j-1}^{(l-1)}, \phi_j^{'}, \phi_{j+1}^{(l-1)}, ..., \phi_J^{(i-1)}\right)$.

Step 3: Calculate the probability of acceptance of the new value generated as $\phi_j^{'}$:

$$p\left(\phi_j^{(l-1)}, \phi_j^{'}\right) = \begin{cases} \min\left\{1, \frac{\pi^*\left(\varphi^c \mid Y\right)}{\pi^*\left(\varphi^{(l-1)} \mid Y\right)}\right\} & \text{, if } \pi^*\left(\varphi^{(l-1)} \mid Y\right) \neq 0 \\ 1 & \text{, otherwise} \end{cases}$$

Step 4: Generate a uniform randomized variable $u \sim U(0,1)$ and make:

$$\phi_j^{(l)} = \begin{cases} \phi_j^{'} & \text{, if } u \leq p\left(\phi_j^{(i-1)}, \phi_j^{'}\right) \\ \phi_j^{(i-1)} & \text{, otherwise} \end{cases}$$

Step 5: Increase *l* and repeat Steps 2 to 4 till convergence is warranted. Finally, recover the estimated parameters by inverse transformation.

Presupposing a quadratic loss function, parameters' Bayesian estimates should be expressed as an evaluation of hope of a function of interest $g(\bullet)$ with regard to posterior distribution $\pi(\bullet \mid Y)$. Every time there is $E[g(\bullet)]$, an approximation for the integrity of the desired function may be obtained by Monte Carlo simulation.

Several criteria exist to select models within a Bayesian context. The predictive ordinate criterion (POC) based on density was used in current research, which was built from distribution y_{T+m}, conditioned to data Y and to the parameters of each model analyzed. POC comprises the choice of model *l* which presents the *highest* rate of $\hat{c}(l)$ (a quantity obtained by the predictive density of Monte Carlo estimate). Gamerman and Lopes (2006) give greater details on the subject.

Evaluation of parameters' estimate methods

Below are given the results obtained by implementing the modeling proposed for the parameter inference of the processes AR(*p*)-ARCH(*q*), ARCH(*q*) and GARCH(*q*,*r*), taking into account normal and Student's *t* distributions for ε_t.

The historical series under analysis gives information on the final indexes of the Bovespa (Ibovespa) registered between January 2, 1996 and February 1, 1999, a total of 651 information items. Choice of the period is due to its importance for Brazilian and world economy. In fact, events which generated behavioral changes with irregular frequencies occurred and produced economical and financial impacts during the period. Since current

research estimates the parameters for the evaluation of methods of time series modeling (and not the prediction of time series), the period may not necessarily be a recent occurrence.

Let p_t be the final result of the Bovespa index (Ibovespa) on a trading day. Since $P_t = \ln p_t$, then returns are given by $y_t = \ln(p_t/p_{t-1}) = P_t - P_{t-1}$. Figure 1 illustrates Bovespa indexes and their respective returns.

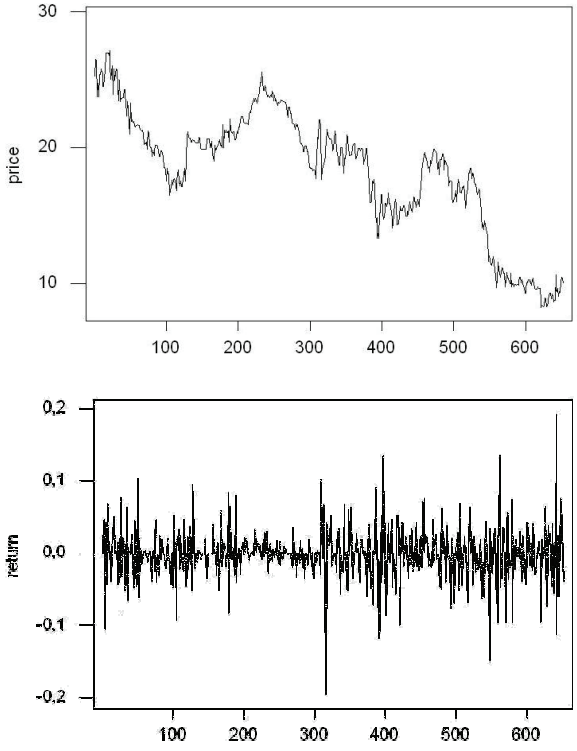

Figure 1. Price and return of the Ibovespa series.

The graphs in Figure 2 show the behavior of returns series of Ibovespa, y_t^2. In fact, the series is correlated and such behavior is typically associated with that of the models of the ARCH family.

A chain of 50,000 iterations was simulated within the implementation of the Metropolis-Hastings algorithm. Moreover, 50% of values were discarded to decrease the effect of initial conditions. Values, spaced in fives, totaling a sample of 5,000 observations, were established. Algorithm convergence was verified by Geweke criterion at 5% significance, under the null hypothesis Ho (GEWEKE, 1992). Parameter convergence was established for values obtained by Geweke's diagnosis between -1.96 and 1.96.

Adjusted models to the Ibovespa series, according to POC, were respectively ARCH(3), AR(3)-ARCH(6) and GARCH(1,1).

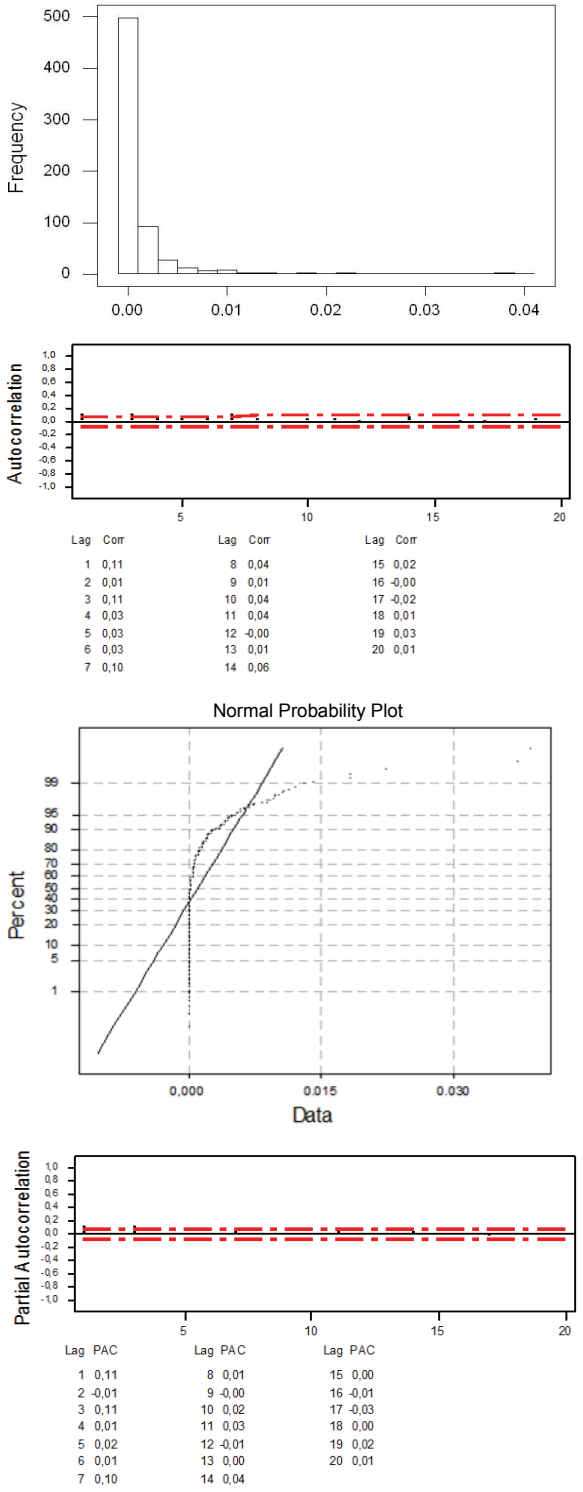

Figure 2. Histogram, normal probability plot, autocorrelation and partial autocorrelation of y_t^2 - Ibovespa series.

Tables 2, 3 and 4 show Bayesian estimates for the adjusted models, where M1 and M2: ARCH processes; M3 and M4: AR-ARCH processes; M5 and M6: GARCH processes, with normal and Student's *t* distributions for ε_t, respectively.

Table 2. Bayesian estimates of ARCH models – M1 and M2.

	M1			M2		
Parameter	Mean	Standard deviation	Credibility interval (95%)	Mean	Standard deviation	Credibility interval (95%)
α_0	0.00061	0.00005	0.00052;0.00070	0.00070	0.00003	0.00060;0.00071
α_1	0.16350	0.01113	0.14187;0.18514	0.21672	0.00591	0.20524;0.22858
α_2	0.06065	0.01139	0.03885;0.08257	0.05228	0.00205	0.04818;0.05624
α_3	0.33162	0.02134	0.28888;0.37152	0.34961	0.00926	0.33156;0.36665

Table 3. Bayesian estimates of AR-ARCH models – M3 and M4.

	M3			M4		
Parameter	Mean	Standard deviation	Credibility interval (95%)	Mean	Standard deviation	Credibility interval (95%)
α_0	0.00048	0.00003	0.00042;0.00056	0.00050	0.00003	0.00044;0.00056
α_1	0.21579	0.04142	0.13543;0.29508	0.18757	0.00567	0.17646;0.19868
α_2	0.08565	0.02706	0.03649;0.14026	0.06340	0.00331	0.05691;0.06988
α_3	0.41482	0.04587	0.32397;0.49956	0.38771	0.00900	0.37007;0.40535
β_1	-0.09571	0.02879	-0.15259;-0.03821	-0.10020	0.00577	-0.11151;-0.08889
β_2	-0.01679	0.02853	-0.07421;0.03983	-0.02365	0.00542	-0.03427;-0.01303
β_3	-0.13483	0.02876	-0.19079;-0.07799	-0.12111	0.00531	-0.13152;-0.11070
β_4	-0.16456	0.02804	-0.21835;-0.10972	-0.15437	0.00522	-0.16460;-0.14414
β_5	-0.14604	0.02958	-0.20397;-0.08949	-0.13210	0.00689	-0.14560;-0.11860
β_6	-0.08765	0.02993	-0.14678;-0.02814	-0.07353	0.00695	-0.08715;-0.05991

Table 4. Bayesian estimates of GARCH models – M5 and M6.

	M5			M6		
Parameter	Mean	Standard deviation	Credibility interval (95%)	Mean	Standard deviation	Credibility interval (95%)
α_0	0.00007	1.5e-009	0.000069;0.000071	0.00007	1.4e-009	0.000069;0.000071
α_1	0.15010	6.7e-004	0.14870;0.15140	0.15035	6.5e-004	0.14920;0.15100
β_1	0.75050	1.3e-003	0.74790;0.75290	0.75027	1.2e-003	0.74850;0.75250

According to values obtained by POC by the adjustments of the proposed models, the model that best adjusts itself to the Ibovespa series was the GARCH(1,1) process with Student's t distribution for ε_t, or rather, model M6, as Table 5 shows.

Table 5. Criterion of model selection.

Model	POC ($\hat{c}(l)$)
M1	1.163e+018
M2	1.190e+018
M3	1.155e+019
M4	1.264e+019
M5	1.173e+019
M6	1.859e+019

Figure 3 shows estimated volatility from estimates obtained within the Bayesian approach with non-informative prior distribution by the best adjustment model, or rather, GARCH(1,1) with Student's t distribution for ε_t. Several events (Russia's default in August 1998; Asian crisis in October 1998; devaluation of the Brazilian real in January 1999), which generated important behavioral changes, occurred during the period under analysis, and caused changes in the behavior of volatility which were entirely identified by the adjusted model. The above changes impacted the prices of stocks and their relationship with the market.

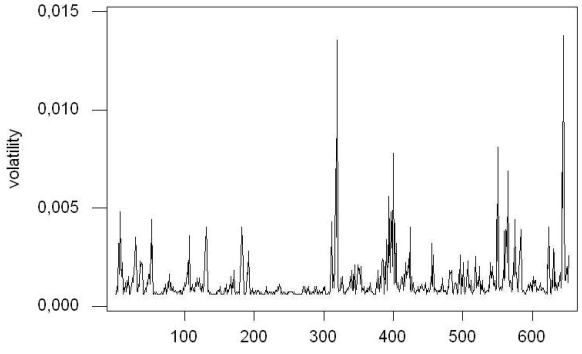

Figure 3. Estimated volatility – Ibovespa series: Model M6.

Conclusion

Although the above is a highly simplified representation of the data-generating process of conditional returns data, normal distribution is widely employed in the estimation of volatility models. However, in certain situations, it is more appropriate to presume that ε_t has Student's t distribution, as may be seen in current case.

Results show that, as a rule, the Bayesian approach provides satisfactory estimates and is entirely viable in returns modeling. In fact, it makes feasible the incorporation of experts' experience in finance which is a highly relevant issue within the analysis of economical and financial series.

The proposal of non-informative prior distributions, coupled to a reparameterization of the models under analysis, provides a faster convergence of the inference process of parameters of ARCH family models by MCMC methods.

Acknowledgements

The authors thank to the referees for all the suggestions and, CAPES and FUNDUNESP (Process 331/2010-DFP) for financial support.

References

ANDRADE, M. G.; OLIVEIRA, S. C. A comparative study of Bayesian and Maximum Likelihood approaches for ARCH models with evidence from Brazilian financial series. **New Mathematics and Natural Computation**, v. 7, n. 2, p. 347-361, 2011.

AUSÍN, M. C.; GALEANO, P. Bayesian estimation of the Gaussian mixture GARCH model, **Computational Statistics and Data Analysis**, v. 51, n. 5, p. 2636-2652, 2007.

BARRETO, G. A.; OLIVEIRA, S. C.; ANDRADE, M. G. Estimação de parâmetros de modelos ARCH(*p*): abordagem Bayes-MCMC versus máxima verossimilhança. **Revista Brasileira de Estatística**, v. 69, n. 230, p. 7-24, 2008.

BOLLERSLEV, T. Generalized autoregressive conditional heteroskedasticity. **Journal of Econometrics**, v. 31, n. 3, p. 307-327, 1986.

BOLLERSLEV, T. **Glossary to ARCH (GARCH)**. Research paper, school of economics and manangement. Aarthus: University of Aarthus, 2008. Available from: <http://ssrn.com/abstract=1263250>. Access on: Jan. 11, 2011.

ENGLE, R. Autoregressive conditional heteroscedasticity with estimates of the variance of UK inflation. **Econometrica**, v. 50, n. 4, p. 987-1007, 1982.

GAMERMAN, D.; LOPES, H. F. **Markov chain Monte Carlo**: stochastic simulation for Bayesian inference. London: Chapman and Hall, 2006.

GEWEKE, J. Exact predictive densities for linear models with ARCH disturbances. **Journal of Econometrics**, v. 40, n. 1, p. 63-86, 1989.

GEWEKE, J. Evaluating the accuracy of sampling-based approaches to calculating posterior moments (with discussion). **Bayesian Statistics**, v. 4, n. 1, p. 169-193, 1992.

GIAKOUMATOS, S.; DELLAPORTAS, P.; POLITIS, D. Bayesian analysis of the unobserved ARCH model using auxiliary variable sampling. **Statistics and Computing**, v. 15, n. 2, p. 103-111, 2005.

KOTZ, S.; BALAKRISHNAN, N.; JOHNSON, N. L. **Continuous multivariate distributions, models and applications**. Wiley Series in Probability and Statistics: Applied Probability and Statistics. New York: Wiley-Interscience, 2000. v. 1.

MORETTIN, P. A. **Econometria financeira**: um curso em séries temporais financeiras. São Paulo: Edgard Blücher Ltda., 2008.

NAKATSUMA, T. Bayesian analysis of ARMA-GARCH Models: a Markov chain sampling approach. **Journal of Econometrics**, v. 95, n. 1, p. 57-69, 2000.

OLIVEIRA, S. C.; ANDRADE, M. G. Comparison between the complete Bayesian method and empirical Bayesian method for ARCH models using Brazilian financial time series. **Pesquisa Operacional**, v. 32, n. 2, p. 293-313, 2012.

PITT, M.; WALKER, S. Constructing stationary time series models using auxiliary variables with applications. **Journal of the American Statistical Association**, v. 100, n. 470, p. 554-564, 2005.

POLASEK, W. **MCMC methods for periodic AR-ARCH models**. Research review. Basel: University of Basel, 2001.

PORTAL DO INVESTIDOR. Available from: <http://www.portaldoinvestidor.gov.br>. Access on: Aug. 20, 2011.

Received on May 26, 2011.
Accepted on September 12, 2011.