



On the regularity of the pressure field of relaxed solutions to Euler equations with variable density



Juliana Conceição Precioso

Departamento de Matemática, Universidade Estadual Paulista, 15054-000, S.J.R. Preto, SP, Brazil

ARTICLE INFO

Article history:

Received 17 May 2010

Available online 29 June 2013

Submitted by W. Layton

Keywords:

Euler equations

Incompressible flows

Relaxed solutions

ABSTRACT

In this paper we improve the regularity in time of the gradient of the pressure field in the solution of relaxed version of variational formulation proposed by V. I. Arnold and by Y. Brenier, for the incompressible Euler equations with variable density. We obtain that the pressure field is not only a measure, but a function in $L^2_{loc}((0, T); BV_{loc}(D))$ as an extension of the work of Ambrosio and Figalli (2008) in [1] to the variable density case.

© 2013 Elsevier Inc. All rights reserved.

1. Introduction

The motion of an incompressible inviscid fluid inside a N -dimensional domain D is described by the Euler equations:

$$\begin{cases} \rho [\partial_t u + (u \cdot \nabla)u] + \nabla p = 0, & \text{in } [0, T] \times D, \\ \partial_t \rho + u \cdot \nabla \rho = 0, & \text{in } [0, T] \times D, \\ \nabla \cdot u = 0, & \text{in } [0, T] \times D, \\ u(t, \cdot) \cdot \hat{n}|_{\partial D} = 0 & \text{on } [0, T] \times \partial D, \end{cases} \quad (1)$$

where, $\rho = \rho(t, x)$ denotes the density of the fluid, $u = u(t, x)$ is the velocity field and $p = p(t, x)$ is the scalar pressure. As observed by V. I. Arnold, the incompressible inviscid flows can be interpreted as geodesics on the group of volume-preserving maps endowed with the left-invariant Riemannian structure inherited from the Lie algebra of divergence-free vector fields with the L^2 metric. In the sequel we explain Arnold's interpretation. Let $D \subset \mathbb{R}^N$ be a compact domain and

$$G(D) = \{h \in C^1(D; D) \mid \det D_x h = 1; h \text{ diffeomorphism}\},$$

the set of volume-preserving C^1 -diffeomorphisms of D , endowed with the left invariant Riemannian structure as in [2]. Let $u = u(x, t)$ a solution of the incompressible Euler equations in D . If $(t, x) \mapsto g_u(t, x)$ denotes the flow induced by u , then each trajectory $t \mapsto g_u(t, x)$ is a geodesic on $G(D)$.

Let $g = g(t)$ be a path on $G(D)$. The action is defined by

$$A = A(g) \equiv \frac{1}{2} \int_0^T \|\partial_t g\|_{L^2}^2 dt.$$

The existence of minimal geodesics in $G(D)$ is equivalent to the existence of minimizers of the action. For a discuss about existence of minimal geodesics, see [5,8] and the references therein.

In this work we focus on the case $D = [0, 1]^N$. In [4], Yann Brenier introduced a weak formulation to the problem of existence of minimal geodesics for the case of homogeneous (constant density) fluids. He also showed the existence

E-mail address: precioso@ibilce.unesp.br.

of such a solution. This formulation consists of re-interpreting the geodesic as a measure c on $[0, T] \times D \times D$, with $(x, a) \in D \times D$ representing position x and Lagrangian marker a , while relaxing the minimization problem. For a discussion of the connection between this problem and optimal mass transportation, see [9]. The solution to the relaxed problem is a pair of measures (c, m) , where m is a vector-valued measure representing a weak version of the Euler flow u . The Euler–Lagrange equations associated to the relaxed problem are called *relaxed Euler equations*. Recently, we extended Brenier’s results to the case of incompressible fluids with variable density, deriving a nonhomogeneous version of the relaxed Euler equations. In [1], Ambrosio and Figalli obtained an improved regularity result for the pressure associated with Brenier’s optimal solutions. It is natural to ask whether the improved regularity of the pressure, obtained by Ambrosio and Figalli, in [1], extends to the nonhomogeneous case. The present work is devoted to answering this question.

In [6], we proved that the distributions $\partial_{x_i} p$ are locally finite measures in $[0, T] \times D$, however this information is not sufficiently to guarantee that the pressure p is in fact a function (because to a lack of time regularity). We show that a sufficiently condition, to have $p \in L^2_{loc}((0, T); L^{d/d-1}_{loc}(D))$, is that $\partial_{x_i} p \in L^2_{loc}((0, T); \mathcal{M}_{loc}(D))$, $i = 1, \dots, d$.

2. Lagrangian flow as minimizer of action

Let D be the unit cube $[0, 1]^N$ of \mathbb{R}^N and $Q = [0, T] \times D$, $T > 0$. The incompressible Euler equations (1) can be re-written in Lagrangian form as

$$\begin{cases} \rho_0(x) \partial_t^2 g(t, x) + \nabla p(t, g(t, x)) = 0, \\ \rho(t, g(t, x)) = \rho(t = 0, x) = \rho_0(x), \\ \det D_x g(t, x) = 1. \end{cases} \quad (2)$$

In [2], Arnold considered the homogeneous case $\rho_0 \equiv 1$ and observed that, if $g = g(t, x)$ is the Lagrangian flow associated with a solution of the (homogeneous) Euler equations, then g is a geodesic in the group $G(D)$.

In this case, a minimal geodesic g connecting the identity and $h \in G(D)$ also minimizes the action:

$$A(g) = \frac{1}{2} \int_0^T \int_D \left| \frac{\partial g}{\partial t}(t, x) \right|^2 dx dt,$$

among all smooth paths $t \in [0, T] \mapsto g(t, \cdot) \in G(D)$ such that $g(0) = id$ and $g(T) = h$.

As observed in [3], there is also a least action principle for solutions of (2), where the action is now given by

$$A_{\rho_0} = A_{\rho_0}(g) \equiv \frac{1}{2} \int_0^T \int_D \rho_0(x) |\partial_t g(t, x)|^2 dx dt. \quad (3)$$

If g is a smooth solution of (2) then g is a minimizer of the action A_{ρ_0} , see [3].

Given a smooth path $g = g(t)$ on $G(D)$ set $\rho = \rho(t, y) \equiv \rho_0(t, (g(t, \cdot)^{-1}(y)))$ and define the vector field $u = u(t, x) \equiv \partial_t g(t, (g(t, \cdot)^{-1}(x)))$, so that $\partial_t g(t, y) = u(t, g(t, y))$. Then, switching back to Eulerian coordinates, the problem of minimizing the action A_{ρ_0} can be reformulated in terms of the functional

$$K[u] = \frac{1}{2} \int_Q |u(t, y)|^2 \rho(t, y) dy dt, \quad (4)$$

where $u \in V$ is such that $g_u(T, \cdot) = h(\cdot)$ and where $\rho(t, g_u(t, x)) = \rho_0(x)$.

3. The relaxed problem

In this section we will consider a relaxed problem where the nonlinear constraint $g \in G(D)$, is embedded into a larger, linear, admissible set.

We will assume throughout the remainder of this paper that

$$\rho_0 \in C^0(D), \quad \rho_0 > 0. \quad (5)$$

Set $Q' = Q \times D_a$, where $Q = [0, T] \times D_x$. For each $\mu \in \mathcal{BM}(Q')$, we define a measure μ_{ρ_0} through the pairing

$$\langle \mu_{\rho_0}, f \rangle \equiv \langle \mu, \rho_0 f \rangle, \quad (6)$$

for any test function $f \in C^0(Q')$.

For each $u \in V$ let $g = g(t, a) = g_u(t, a)$ be the associated flow. Define $(c, m) \in \mathcal{BM}(Q') \times (\mathcal{BM}(Q'))^N$ by

$$(c(t, x, a), m(t, x, a)) \equiv (\delta(x - g(t, a)), u(t, x) \delta(x - g(t, a))). \quad (7)$$

Thus, for any $f \in C^0(Q')$, we have

$$\begin{aligned}\langle c(t, x, a), f(t, x, a) \rangle &= \int_{Q'} f(t, x, a) dc(t, x, a) = \int_Q f(t, g(t, a), a) dadt, \\ \langle m(t, x, a), f(t, x, a) \rangle &= \int_{Q'} f(t, x, a) dm(t, x, a) = \int_Q f(t, g(t, a), a) u(t, g(t, a)) dadt \\ &= \int_Q f(t, g(t, a), a) \partial_t g(t, a) dadt.\end{aligned}$$

Consider the pair (c_{ρ_0}, m_{ρ_0}) associated with (c, m) through (6).

For each such (c_{ρ_0}, m_{ρ_0}) constructed above we have

$$\partial_t c_{\rho_0} + \nabla_x \cdot m_{\rho_0} = 0, \quad (8)$$

and

$$\int_D \frac{c_{\rho_0}(t, x, a)}{\rho_0(a)} da = 1. \quad (9)$$

Moreover, for any $f \in C^0(Q')$ such that $\partial_t f$ and $\nabla_x f$ are continuous in Q' , we have

$$\langle c_{\rho_0}, \partial_t f \rangle + \langle m_{\rho_0}, \nabla_x f \rangle = \int_D \rho_0(a) [f(T, h(a), a) - f(0, a, a)] da, \quad (10)$$

which is a weak formulation of (8).

Also, if $\bar{f} \in C^0(Q)$ then we have

$$\left\langle \frac{c_{\rho_0}}{\rho_0}, \bar{f} \right\rangle = \int_Q \bar{f}(t, g(t, a)) dt da = \int_Q \bar{f}(t, x) dt dx. \quad (11)$$

Analogously to what was done in [4], we rewrite the functional $K = K[u] : V \rightarrow \mathbb{R}$, (see (4)), in terms of the measures (c_{ρ_0}, m_{ρ_0}) as follows:

$$K[u] = K[c_{\rho_0}, m_{\rho_0}] = \sup_X \{ \langle c_{\rho_0}, F \rangle + \langle m_{\rho_0}, \Phi \rangle \}, \quad (12)$$

where

$$X = \left\{ (F, \Phi) \in C^0(Q') \times (C^0(Q'))^N \mid F(t, x, a) + \frac{1}{2} |\Phi(t, x, a)|^2 \leq 0 \right\}.$$

We have seen that, given $u \in V$, there is a pair (c_{ρ_0}, m_{ρ_0}) of measures such that (10) and (11) hold and (12) follows. In view of these observations we introduce the relaxed problem:

Find a pair $(\bar{c}_{\rho_0}, \bar{m}_{\rho_0}) \in \mathcal{A}$ such that

$$K[\bar{c}_{\rho_0}, \bar{m}_{\rho_0}] = \min_{(c_{\rho_0}, m_{\rho_0}) \in \mathcal{A}} K[c_{\rho_0}, m_{\rho_0}], \quad (13)$$

where the admissible set \mathcal{A} is given by

$$\mathcal{A} = \left\{ (c_{\rho_0}, m_{\rho_0}) \in \mathcal{BM}(Q') \times (\mathcal{BM}(Q'))^N \mid K[c_{\rho_0}, m_{\rho_0}] < +\infty \text{ and } (c_{\rho_0}, m_{\rho_0}) \text{ satisfy (10) and (11) for all test functions } f \in C^0(Q') \text{ with } \partial_t f \text{ and } \nabla_x f \in C^0(Q') \right\}.$$

We refer to [6] for general results on the existence of minimizing pairs $(\bar{c}_{\rho_0}, \bar{m}_{\rho_0})$ and described here two properties of minimizing pairs $(\bar{c}_{\rho_0}, \bar{m}_{\rho_0})$ that will be used in this work:

A. The kinetic energy,

$$\int_{D \times D} \frac{1}{2} |v(t, x, a)|^2 \bar{c}_{\rho_0}(t, dx, da) \quad (14)$$

is time-independent and bounded.

B. There exists a distribution $p = p(t, x)$, $(t, x) \in Q^\circ$, satisfying

$$\nabla p(t, x) = -\partial_t \int_D v(t, x, a) \bar{c}_{\rho_0}(t, x, da) - \nabla_x \cdot \int_D (v \otimes v)(t, x, a) \bar{c}_{\rho_0}(t, x, da), \quad (15)$$

in the sense of distributions.

For details of the proof of these results, see [6].

4. A difference quotients estimate

For the proof of our result, we will need to recall an approximation of the pressure field obtained in [6] as a generalization of that obtained in [4].

Let us consider the Banach space $E = C(Q') \times (C(Q'))^N$ and define two convex functions

$$\alpha_{\rho_0} = \alpha_{\rho_0}(F, \Phi) \equiv \alpha(\rho_0 F, \Phi) \quad \text{and} \quad \beta_{\rho_0} = \beta_{\rho_0}(F, \Phi) \equiv \beta(F, \Phi),$$

for $(F, \Phi) \in E$, where α and β are the functions defined by Brenier in [4], namely,

$$\alpha(\rho_0 F, \Phi) = \begin{cases} 0, & \text{if } \rho_0 F + \frac{1}{2} |\Phi|^2 \leq 0 \\ +\infty, & \text{otherwise} \end{cases} \quad \text{and} \quad \beta(F, \Phi) = \begin{cases} \langle \bar{c}, F \rangle + \langle \bar{m}, \phi \rangle, & \text{if (16) follows} \\ +\infty & \text{otherwise} \end{cases}$$

where

$$\begin{aligned} \exists p \in C^0(Q) \text{ and } \phi \in C^0(Q') \text{ with } \partial_t \phi \text{ and } \nabla_x \phi \in C^0(Q') \\ \text{such that } F(t, x, a) + \partial_t \phi(t, x, a) + p(t, x) = 0 \text{ and } \Phi(t, x, a) + \nabla_x \phi(t, x, a) = 0, \end{aligned} \quad (16)$$

and $(\bar{c}_{\rho_0}, \bar{m}_{\rho_0})$ is the fixed minimizing pair. Following the steps of the proof of Brenier, we used the Fenchel–Rockafeller duality Theorem to show, in [7], that

$$\sup_{(F, \Phi) \in E} \{-\alpha_{\rho_0}(-F, -\Phi) - \beta_{\rho_0}(F, \Phi)\} = \inf_{(c, m) \in E'} \{\alpha_{\rho_0}^*(c, m) + \beta_{\rho_0}^*(c, m)\},$$

where $\alpha_{\rho_0}^*$ and $\beta_{\rho_0}^*$ denote the Legendre–Fenchel transforms of α_{ρ_0} and β_{ρ_0} respectively.

Note that

$$\alpha_{\rho_0}^*(c, m) = \frac{1}{2} \langle c, \rho_0 |v|^2 \rangle = \frac{1}{2} \int_{Q'} |v|^2 dc_{\rho_0}$$

and

$$\beta_{\rho_0}^*(c, m) = \begin{cases} 0, & \text{if } \langle \bar{c} - c, \partial_t \phi + p \rangle + \langle \bar{m} - m, \nabla_x \phi \rangle = 0, \quad \forall p, \phi \\ +\infty & \text{otherwise.} \end{cases}$$

It is easy check that $\beta_{\rho_0}^*(c, m) = 0$ if, and only if, the admissible conditions (10) and (11) are satisfied. Then, the minimum of the action coincides with the dual problem

$$\sup_{(F, \Phi) \in E} \{-\alpha_{\rho_0}(-F, -\Phi) - \beta_{\rho_0}(F, \Phi)\},$$

which can be written as

$$\sup_{p, \phi} \langle c, \rho_0 \partial_t \phi + p \rangle + \langle m, \rho_0 \nabla_x \phi \rangle \quad \text{with } \rho_0 \partial_t \phi + \frac{1}{2} \rho_0 |\nabla_x \phi|^2 + p \leq 0.$$

Then, by duality we can conclude that for any $\varepsilon > 0$, there exist $p_\varepsilon(t, x)$ and $\phi_\varepsilon(t, x, a)$ satisfying

$$\rho_0 \partial_t \phi_\varepsilon + \frac{1}{2} \rho_0 |\nabla_x \phi_\varepsilon|^2 + p_\varepsilon \leq 0 \quad \text{and} \quad \frac{1}{2} \langle c_{\rho_0}, |v|^2 \rangle \leq \langle c, \rho_0 \partial_t \phi_\varepsilon + p_\varepsilon \rangle + \langle v c, \rho_0 \nabla_x \phi_\varepsilon \rangle + \varepsilon^2.$$

As shown in [4] and generalized in [6], from this one deduces the estimate

$$\frac{1}{2} \int_{Q'} |v - \nabla_x \phi_\varepsilon|^2 dc_{\rho_0} \leq \varepsilon^2.$$

We remark that, by adding to the ϕ_ε a suitable function of time, one can always assume $\int_D p_\varepsilon(t, x) dx = 0$, for all $t \in [0, T]$. As shown in [4] and in [6] for the case of variable density, the family p_ε is compact in the sense of distributions and, therefore, there exists a cluster point p . Moreover, since that for any optimal solution (c_{ρ_0}, m_{ρ_0}) , any limit point p of the family p_ε satisfies (15) in the sense of distributions, then ∇p is uniquely determined, and this enforces the convergence of the whole family (∇p_ε) to ∇p in the sense of distributions.

Now, we presented a regularity result on $\nabla_x \phi_\varepsilon$.

Proposition 4.1. Let $0 < \tau < T$ e $Q'_\tau = [\tau, T - \tau] \times D \times D$. If $\omega : D \rightarrow \mathbb{R}^2$ is a smooth divergence-free vector tangent to ∂D , and $g_\omega(\cdot, x) : \mathbb{R} \rightarrow D$ is the integral curve of ω starting at x . Then, for any optimal solution (c_{ρ_0}, m_{ρ_0})

$$K[c_{\rho_0}, m_{\rho_0}] = \frac{T}{2} \int_Q |v(t, x, a)|^2 c_{\rho_0}(t, x, a) dx da,$$

$$\int_{Q'_\tau} |\nabla_x \phi_\epsilon(t, x, a) - v(t, x, a)|^2 dc_{\rho_0}(t, x, a) \leq C\epsilon^2, \quad (17)$$

$$\int_{Q'_\tau} |\nabla_x \phi_\epsilon(t, x, a)|^2 dc_{\rho_0}(t, x, a) \leq C, \quad (18)$$

$$\int_{Q'_\tau} |\nabla_x \phi_\epsilon(t + \eta, g_\omega(\delta, x), a) - \nabla_x \phi_\epsilon(t, x, a)|^2 dc_{\rho_0}(t, x, a) \leq (\epsilon^2 + \eta^2 + \delta^2) C, \quad (19)$$

for all η, δ and $\epsilon > 0$ sufficiently small, where C depending only on D, T, τ and ω .

This result is a non-trivial adaptation of the analogous result in the homogeneous case, discussed in [4]. For more details, see [6].

5. Regularity of pressure field

In [6], we proved that the distributions $\partial_{x_i} p$ are locally finite measures in $(0, T) \times D$, but due to a lack of time regularity this is not sufficient to imply that p is a function. In this section, as made in [1], we refine a little bit this results. To prove that the pressure field is a function in $L^2_{loc}((0, T); BV_{loc}(D))$ we will begin with the following result, which is crucial for our purposes. The proof follows the Ambrosio and Figalli's arguments (homogeneous case) and will be pointed only those parts where fundamental changes were needed.

Theorem 5.1. Let $\tau \in (0, T)$ and let $w : \bar{D} \rightarrow \mathbb{R}^n$ be a smooth divergence-free vector field tangent to ∂D . Then, there exists a constant $C = C(w, \tau, T, \rho_0)$ such that

$$|\langle \nabla p \cdot w, \xi f \rangle| \leq C \|f\|_\infty \|\xi\|_{L^2(0, T)}, \quad \forall \xi \in C_0^\infty((\tau, T - \tau); [0, +\infty)), f \in C_0^\infty((0, T) \times D).$$

Proof. For the proof, we consider $\xi \in C_0^\infty(\tau, T - \tau)$ nonnegative, $\eta \in (0, \frac{\tau}{2})$ and $\epsilon > 0$ and define I by

$$\begin{aligned} I &= \int_0^T \int_D \xi(t) \left| \int_0^1 [p_\epsilon(t + \eta\theta, g_\omega(\delta, x)) - p_\epsilon(t + \eta\theta, x)] d\theta \right| dx dt \\ &= \int_{Q'} \xi(t) \left| \int_0^1 [p_\epsilon(t + \eta\theta, g_\omega(\delta, x)) - p_\epsilon(t + \eta\theta, x)] d\theta \right| dc(t, x, a) \\ &\leq I_1 + I_2 + I_3, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_{Q'} \xi(t) \left| \int_0^1 \left[\left(\rho_0 \partial_t \varphi_\epsilon + \rho_0 \frac{1}{2} |\nabla_x \varphi_\epsilon|^2 + p_\epsilon \right) (t + \eta\theta, g_\omega(\delta, x), a) \right. \right. \\ &\quad \left. \left. - \left(\rho_0 \partial_t \varphi_\epsilon + \frac{1}{2} \rho_0 |\nabla_x \varphi_\epsilon|^2 + p_\epsilon \right) (t + \eta\theta, x, a) \right] d\theta \right| dc, \\ I_2 &= \int_{Q'} \xi(t) \left| \int_0^1 [\rho_0(a) \partial_t \varphi_\epsilon(t + \eta\theta, g_\omega(\delta, x), a) - \rho_0(a) \partial_t \varphi_\epsilon(t + \eta\theta, x, a)] d\theta \right| dc \quad \text{and} \\ I_3 &= \int_{Q'} \xi(t) \left| \int_0^1 \left[\rho_0(a) \frac{1}{2} |\nabla_x \varphi_\epsilon|^2(t + \eta\theta, g_\omega(\delta, x), a) - \rho_0(a) \frac{1}{2} |\nabla_x \varphi_\epsilon|^2(t + \eta\theta, x, a) \right] d\theta \right| dc, \end{aligned}$$

where in the last step we use that $\int_{Q'} [\rho_0 \partial_t \varphi_\epsilon + \rho_0 \frac{1}{2} |\nabla_x \varphi_\epsilon|^2 + p_\epsilon] dc \leq \epsilon^2$.

Using the estimatives obtained in the Proposition 4.1 and observing that the initial density $\rho_0 = \rho_0(a) > 0$ belongs to $C^0(D)$, after the adaptation in the definition of I to the non-homogeneous case, the estimate of the term I follows Ambrosio and Figalli's. Then, we can bound above I as follows:

$$\begin{aligned} &C(\epsilon + \eta + \delta) (\xi^2(t) |\nabla_x \phi_\epsilon|^2 dc_{\rho_0} + \|\xi\|_{L^\infty} (\epsilon^2 + \eta^2 + \delta^2))^{\frac{1}{2}} + 2 \int_{Q'} \int_0^1 [\xi(t - \theta\eta) - \xi(t)] \partial_t \phi_\epsilon d\theta dc_{\rho_0} \\ &+ C \frac{\delta}{\eta} (\epsilon + \eta + \delta) \|\xi\|_{L^2} + \|\xi''\|_{L^\infty} \eta^2 C (\|\nabla_x \phi_\epsilon\|_{L^2} + C(\epsilon + \eta)) + 2 \|\xi\|_{L^\infty} \epsilon^2 + C \|\xi\|_{L^\infty} \epsilon. \end{aligned}$$

Now, recalling the definition of I , we integrate $p_\varepsilon \xi$ against a function $f \in C_0^\infty((0, T) \times D)$ and pass to the limit as $\varepsilon \rightarrow 0$, with $\eta = \delta$ frozen, to obtain

$$\frac{1}{\delta} \left| \int_0^1 \langle q, \xi(t) [f(t - \delta\theta, g_\omega(-\delta, x)) - f(t - \delta\theta, x)] \rangle d\theta \right| \leq C \|f\|_{L^\infty} (\|\xi\|_{L^2} + \delta \|\xi''\|_{L^\infty} + \delta \|\xi\|_{L^\infty}),$$

for any limit point q of p_ε in the sense of distributions. So, letting $\delta \rightarrow 0$, we obtain $|\langle \nabla q \cdot w, \xi f \rangle| \leq C \|f\|_\infty \|\xi\|_{L^2}$. But $\nabla p_\varepsilon \rightarrow \nabla p$ implies that $\nabla p = \nabla q$ and concludes the proof. \square

Now, we presented the analogous of the [Corollary 3.3, [1]] for the variable density case:

Corollary 5.2. *For $n \geq 2$ and for all smooth subdomains $D' \subset\subset D$ there exists $q \in L_{loc}^2((0, T); BV_{loc}(D')) \subset L_{loc}^2((0, T); L_{loc}^{\frac{n}{n-1}}(D'))$, with $\nabla q = \nabla p$ in the sense of distributions in $(0, T) \times D$.*

In view of the Theorem 5.1, the proof of this result is a straightforward adaptation of the proof of [Corollary 3.3, [1]].

We add a comment. This result will be of great importance to improve the consistence results obtained in [6]. More precisely, was showed in [6] that, if the solution of the relaxed system has a particular structure, then this solution gives rise to a (classical) weak solution of the Euler equations (this result is new even to the homogeneous case).

It is a natural question to ask whether the hypothesis on the regularity of the path $t \rightarrow g(t; x)$, which we assumed to be C^∞ in the consistency result, can be weakened. The reason for requiring that g be C^∞ is in order to properly define the restriction of the distribution gradient of the pressure to a suitable surface. Then, once higher regularity of pressure field has been established in this paper, it may be possible try to weaken considerably the regularity requirement on g .

Acknowledgment

The author gratefully acknowledges FAPESP Thematic Project #2007/51490-7.

References

- [1] L. Ambrosio, A. Figalli, On the regularity of the pressure field of Brenier's weak solutions to incompressible Euler equations, *Calc. Var. Partial Differential Equations* 31 (2008) 497–509.
- [2] V.I. Arnold, Sur la Géométrie Différentielle des Groupes de Lie de dimension infinie et ses applications à L'Hidrodynamique, *Ann. Inst. Fourier (Grenoble)* 16 (1966) 319–361.
- [3] Y. Brenier, The Least action principle and the related concept of generalized flows for incompressible perfect fluids, *J. Amer. Math. Soc.* 2 (2) (1989) 225–255.
- [4] Y. Brenier, Minimal geodesics on groups of volume-preserving maps and generalized solutions of the Euler equations, *Comm. Pure Appl. Math.* 52 (1999) 411–452.
- [5] D.G. Ebin, J. Marsden, Groups of diffeomorphisms and the motion of an incompressible fluid, *Ann. of Math.* 92 (1970) 102–163.
- [6] M.C. Lopes Filho, H.J. Nussenzweig Lopes, J.C. Precioso, Least action principle and the incompressible Euler equations with variable density, *Trans. Amer. Math. Soc.* 363 (2011) 2641–2661.
- [7] J.C. Precioso, Equações relaxadas para hidrodinâmica ideal, não homogênea, Tese de doutorado, IMECC-UNICAMP, 2005, http://www.mat.ibilce.unesp.br/personal/juliana/Equacoes_relaxadas.pdf.
- [8] A.I. Shnirelman, The geometry of the groups of diffeomorphisms and the dynamics of an ideal incompressible fluid, *Mat. Sb. (N.S.)* 128 (170) (1985) 82–109, 144.
- [9] C. Villani, Topics in Optimal Transportation, in: Graduate Studies in Math., vol. 58, Amer. Math. Soc., Providence, 2003.