RESSALVA

Atendendo solicitação da autora,
o texto completo desta tese será
disponibilizado somente a partir
de 27/03/2022

UNIVERSIDADE ESTADUAL PAULISTA
"JÚLIO DE MESQUITA FILHO"
Campus de São José do Rio Preto

Fabiola Roxana Villanueva

Contributions in Interval Optimization and Interval Optimal Control

PhD Thesis

Post-Graduation in Mathematics

São José do Rio Preto

2020

Fabiola Roxana Villanueva

Contributions in Interval Optimization and Interval Optimal Control

Tese apresentada como parte dos requisitos para obtenção do título de Doutor em Matemática, junto ao Programa de Pós-Graduação em Matemática, do Instituto de Biociências, Letras e Ciências Exatas da Universidade Estadual Paulista "Júlio de Mesquita Filho", Câmpus de São José do Rio Preto.

Financiadora: CAPES

Prof. Dr. Valeriano Antunes de Oliveira

São José do Rio Preto

2020

V718c

Villanueva, Fabiola Roxana
Contributions in interval optimization and interval optimal control / Fabiola Roxana Villanueva. -- São José do Rio Preto, 2020
189 f. : il.

Tese (doutorado) - Universidade Estadual Paulista (Unesp), Instituto de Biociências Letras e Ciências Exatas, São José do Rio Preto
Orientador: Valeriano Antunes de Oliveira

1. Matemática. 2. Controle ótimo intervalar. 3. Otimização intervalar. 4. Análise de intervalos (Matemática). I. Título.

Sistema de geração automática de fichas catalográficas da Unesp. Biblioteca do Instituto de Biociências Letras e Ciências Exatas, São José do Rio Preto. Dados fornecidos pelo autor(a).

Essa ficha não pode ser modificada.

Fabiola Roxana Villanueva

Contributions in Interval Optimization and Interval Optimal Control

Tese apresentada como parte dos requisitos para obtenção do título de Doutor em Matemática, junto ao Programa de Pós-Graduação em Matemática, do Instituto de Biociências, Letras e Ciências Exatas da Universidade Estadual Paulista "Júlio de Mesquita Filho", Câmpus de São José do Rio Preto.

Financiadora: CAPES

Comissão Examinadora

Prof. Dr. Valeriano Antunes de Oliveira
UNESP - São José do Rio Preto
Orientador

Prof. Dr. Geraldo Nunes Silva
UNESP - São José do Rio Preto

Prof^{a}. Dr^{a}. Lucelina Batista dos Santos
UFPR - Curitiba

Prof. Dr. Yurilev Chalco Cano
UTA - Chile

Prof^{a}. Dr^{a}. Ulcilea Alves Severino Leal
UFTM - Minas Gerais

São José do Rio Preto
27 de março de 2020

Dedicated to
my mother, Flor.

ACKNOWLEDGMENTS

I am grateful to God and my angels: Alberto (†), Carlos (†), and Gloria (†) for being present all the time, remember me that I am not alone, and that I am strong. I thank my loved mother for her support, for encourage me every day, and tell me that I can. Without each one of them I would not be who I am and I would not be where I am.

I want to thank for thinking, worrying, and making me feel always loved to my uncles: Alejandra, Consuelo, Eraclia, Julieta, Limbert, Lore, and Pablo; my cousins: Jimena, Sandra, and Wilbert; and my dear nephews: Adhemar, Estrella, and Iker.

Thanks to my teachers Dr. Efraín Cruz Mullisaca and Dr. Yurilev Chalco Cano for having believed in me from the beginning.

I sincerely thank my advisor Dr. Valeriano Antunes de Oliveira for his corrections, suggestions, infinite patience, and dedication since I arrived, from 2013 until now. You were the best!

Thanks to my PhD Commission: Dr. Geraldo Nunes Silva, Dra. Lucelina Batista dos Santos, and Dra. Ulcilea Alves Severino Leal. I have an enormous respect for each one of you!

I also thank Dr. Weldon A. Lodwick for always being willing to help, and for always encouraging me to study.

I am happy for the friends I made, they were not only in bad but also in good moments, they become the family when your family is far from you.

I am really thankful to: Adriana, Aline, Ana Cristina, Anayle, Bianca, Brenda, Bruna, Dani, Fernanda, Geovana, Gisele, Gislaine, Ju, Lais, Lelia, Lilian, Patricia Cano, Patricia

Scolari, Paula, Simone, Thais, Thayra, and Vanessa. I love to share all our "festinhas" with food, production, and a lot of glitter, and I will always remember the nerves and emotion on stage. It really was a pleasure to dance with you.

Thanks to Andrea, Carlos, Dalva, Flávia, and João, who always cared about my well−being, with you it was like be at home.

Thanks to my "amigos da praça e dos rolés épicos": Elaine, Felipe, Julia, Kassia, Le, Leticinha, Naty, Reuma, Stéfano, Vava, and Welker. You are special and I will never forget you.

Thanks to my "amigos extrangeiros", who gave their sincere friendship and shared their culture with me: Ana Maria, Angélica Otero, Angélica Pérez, Arturo, Bia, Bruno, Chelo, Costa Rica, Daimon, Débora, Diana, Erick, Jacqui, Janelys, Manu, Marco, Mariana, Mélanie, Paola, Pedro, Sérgio, Tuany, and many more. Some are still here near me, others returned to their respective countries, but each one of you will always be in my heart.

Thanks to my "amigos da salinha": Daniel, Daniella, Gislaine, Gustavo, Ismael, Jairo, Jennifer, Karina, Luana, Marcel, Rodrigo Barbosa, Rodrigo Ishizaka, Willian, and Yen. With friendship, conversations, and laughters, you made hard days better ones.

I want to give a special thank to Dr. Tiago Mendonça da Costa, sometimes he discouraged me, but he encouraged me too, now I know, it was to bring out the best in me, thanks!

Until to finish, thanks to Ado, Andi, Carito, Charlie, Clau, Cyn, David (†), Deni, Franz, Jhuli, Jos, Juanjo, Liz, Magui, Marilu, Nestor, Omar, Oscar, Rodri, Ron, and Rosa, the friends who always were present throughout the distance and the years.

Thanks to the IBILCE employees who, directly or indirectly, contributed along these years, specially to the DMAP employees: Getúlio, Pinceli, and Tiago.

This Thesis was financed in part by the "Coordenação de Aperfeiçoamento de Pessoal de Nível Superior − Brasil (CAPES) − Finance Code 001."

“Let nothing disturb thee;
Let nothing dismay thee:
All things pass;
God never changes.
Patience attains
All that it strives for.
He who has God
Finds he lacks nothing:
God alone suffices.”
Saint Teresa of Ávila (2002, p. 288), [62].

RESUMO

Neste trabalho, primeiramente, serão apresentados problemas de otimização nos quais a função objetivo é de múltiplas variáveis e de valor intervalar e as restrições de desigualdade são dadas por funcionais clássicos, isto é, de valor real. Serão dadas as condições de otimalidade usando a $E-$diferenciabilidade e, depois, a $gH-$diferenciabilidade total das funções com valor intervalar de várias variáveis. As condições necessárias de otimalidade usando a $gH-$diferenciabilidade total são do tipo KKT e as suficientes são do tipo de convexidade generalizada. Em seguida, serão estabelecidos problemas de controle ótimo nos quais a função objetivo também é com valor intervalar de múltiplas variáveis e as restrições estão na forma de desigualdades e igualdades clássicas. Serão fornecidas as condições de otimalidade usando o conceito de Lipschitz para funções intervalares de várias variáveis e, logo, a $gH-$diferenciabilidade total das funções com valor intervalar de várias variáveis. As condições necessárias de otimalidade, usando a $gH-$diferenciabilidade total, estão na forma do célebre Princípio do Máximo de Pontryagin, mas desta vez na versão intervalar.

Palavras Chave: Problemas de otimização intervalar, condições de tipo Karush$-$Kuhn$-$Tucker, condições suficientes, problemas de controle ótimo intervalar, Principio do Máximo de Pontryagin local intervalar.

ABSTRACT

In this work, firstly, it will be presented optimization problems in which the objective function is interval$-$valued of multiple variables and the inequality constraints are given by classical functionals, that is, real$-$valued ones. It will be given the optimality conditions using the $E-$differentiability and then the total $gH-$differentiability of interval$-$valued functions of several variables. The necessary optimality conditions using the total $gH-$differentiability are of $KKT-$type and the sufficient ones are of generalized convexity type. Next, it will be established optimal control problems in which the objective function is also interval$-$valued of multiple variables and the constraints are in the form of classical inequalities and equalities. It will be furnished the optimality conditions using the Lipschitz concept for interval$-$valued functions of several variables and then the total $gH-$differentiability of interval$-$valued functions of several variables. The necessary optimality conditions using the total $gH-$differentiability is in the form of the celebrated local Pontryagin Maximum Principle, but this time in the intervalar version.

Keywords: Interval optimization problems, Karush$-$Kuhn$-$Tucker$-$type conditions, sufficient conditions, interval optimal control problems, interval local Pontryagin Maximum Principle.

LIST OF FIGURES

LIST OF SYMBOLS

Description

$\mathbb{R}$	Euclidean vectorial space;
$\mathbb{R}^n$	$n-$dimensional Euclidean vectorial space;
e_j	$j-$th canonical direction in $\mathbb{R}^n$;
$\mathbb{R}^n_+$	nonnegative orthant of $\mathbb{R}^n$;
$\mathring{\mathbb{R}}^n_+$	positive orthant of $\mathbb{R}^n$;
$\mathcal{K}_C(\mathbb{R}), \mathcal{K}_C$	interval space, set of all bounded closed intervals in $\mathbb{R}$;
$(\mathcal{K}_C)^n$	$n-$cartesian interval space, $n \in \mathbb{N}$;
$[\underline{a}, \overline{a}]$	element of $\mathcal{K}_C$ and such that $\underline{a} \leq \overline{a}$;
$\underline{a}$	real left extremum of the interval $[\underline{a}, \overline{a}]$;
$\overline{a}$	real right extremum of the interval $[\underline{a}, \overline{a}]$;
$[a, a]$	degenerate interval;
$\ll_{LU}$	order relation LU, where $\ll_{LU} \in \{\leqq_{LU}, \leq_{LU}, \leq_{\underline{LU}}, \leq_{\overline{LU}}, <_{LU}\}$;
$M(x^*)$	set of Fritz$-$John nonzero multipliers associated to x^*;
f	real$-$valued function of one variable or several variables and sometimes, if there is no confusion, it also means a vector$-$valued function of one variable or several variables;
$\begin{bmatrix} f_1 & f_2 & \dots & f_q \end{bmatrix}^\top$	vector function $f : \mathbb{R}^p \to \mathbb{R}^q$;
F	interval$-$valued function of one variable or several variables;
f'	classic derivative of a real$-$valued function of one variable;
f'_-, f'_+	classical left and right lateral derivative, respectively;
$\frac{\partial f}{\partial x_i}$	classical $i-$th partial derivative;
$\frac{\partial f}{\partial x_i^-}, \frac{\partial f}{\partial x_i^+}$	$i-$th left$-$partial and $i-$th right$-$partial derivative, respectively;

Description

$\frac{\partial f}{\partial x_i^\sharp}$	$i-$th partial derivative sharp, that means the classical $i-$th partial derivative, or the $i-$th left$-$partial derivative, or the $i-$th right$-$partial derivative;
df	classical diferencial function;
∇f	classical gradient;
$(\nabla f)_-, (\nabla f)_+$	left gradient and right gradient, respectively;
$\nabla_\sharp f$	gradient sharp, that means the classical gradient, or the left gradient, or the right gradient;
Jf	classical Jacobian;
Df	generalized Jacobian;
$D_x f$	partial generalized Jacobian with respect to x;
F'_E	extremal derivative, $E-$derivative, of an interval$-$valued function of one variable or several variables;
F'_{gH}	generalized Hukuhara derivative, $gH-$derivative, of an interval$-$valued function of one variable or several variables;
$\frac{\partial_{gH} F}{\partial x_i}$	$i-$th partial interval$-$valued $gH-$derivative of F;
$\frac{\partial_{gH} F}{\partial x_i^-}, \frac{\partial_{gH} F}{\partial x_i^+}$	$i-$th left$-$partial and $i-$th right$-$partial interval$-$valued $gH-$derivative of F, respectively;
$D_{gH} F$	interval$-$valued $gH-$differential function (or interval$-$valued total $gH-$derivative function) of F;
$\nabla_{gH} F$	$gH-$gradient of an interval$-$valued function of several variables;
$f'(x; d)$	directional derivative of f at x in the direction d;
$F'_{gH}(x; d)$	directional $gH-$derivative of F at x in direction d;
$\langle a, b \rangle$	classical inner product between $a, b \in \mathbb{R}^n$;
$A + B$	addition between $A, B \in \mathcal{K}_C$;
$\lambda \cdot A$	product between $\lambda \in \mathbb{R}$ and $A \in \mathcal{K}_C$;
$A - B$	difference between $A, B \in \mathcal{K}_C$;
$A \otimes B$	product between $A, B \in \mathcal{K}_C$;
$A \ominus_{gH} B$	$gH-$difference between $A, B \in \mathcal{K}_C$;
$A \oplus B$	addition between $A, B \in \mathcal{K}_C^n$;
$\lambda \odot A$	product between $\lambda \in \mathbb{R}$ and $A \in \mathcal{K}_C^n$;
S°	polar cone of S;
S^a	set of accumulation points of S;
$\top$	transpose of a matrix;

Description

$\mathcal{L}$	Lebesgue subsets of a given interval $[a,b]$;
$\mathcal{B}^m$	Borel sets of $\mathbb{R}^m$;
$\mathcal{L}\times\mathcal{B}^m$	product $\sigma-$algebra;
$\mathrm{Gr}\,U$	graph of U;
$W^{1,1}([0,T];\mathbb{R}^n), W^{1,1}$	space of all absolutely continuous functions $f:[0,T]\to\mathbb{R}^n$;
$\mathcal{M}([0,T];\mathbb{R}^m)$	space of all Lebesgue$-$measurable functions $f:[0,T]\to\mathbb{R}^m$;
$L_\infty([0,T];\mathbb{R}^m), L_\infty$	space of essentially bounded functions $f:[0,T]\to\mathbb{R}^m$;
$L_p([a,b];\mathbb{R}^n), L_p$	space of $n-$tuples of measurable functions $f(t)=(f_1(t),\ldots,f_n(t))$ defined on $[a,b]$ with summable $p-$th power, where $1\le p<+\infty$;
$C([0,T];\mathbb{R}^n), C$	space of continuous functions $f:[0,T]\to\mathbb{R}^n$;
$\lvert\cdot\rvert$	norm in $\mathbb{R}$;
$\lVert\cdot\rVert_{\mathbb{R}^n}$	norm in $\mathbb{R}^n$;
$\lVert\cdot\rVert_{W^{1,1}}$	norm in $W^{1,1}$;
$\lVert\cdot\rVert_{L_\infty}$	norm in L_∞;
ess sup	essential supremum;
$\lVert\cdot\rVert_C$	norm in C;
$\lVert\cdot\rVert_{C\times L_\infty}$	norm in $C\times L_\infty$;
$\lVert\cdot\rVert_{\mathcal{K}_C}$	quasi$-$norm in $\mathcal{K}_C$;
N_S^P	proximal normal cone to S;
N_S	limiting normal cone to S;
$\overline{T}_S$	Clarke tangent cone to S;
∂f	limiting Mordukhovich's subdifferential of f;
$\partial_x f$	partial limiting Mordukhovich's subdifferential of f with respect to x;
$\mathrm{dom}\,f$	domain of f;
$\mathrm{epi}\,f$	epigraph of f;
$\mathrm{co}\,(S)$	convex hull of S;
a.e.	almost everywhere. A property holds almost everywhere if the set for which the property does not hold has measure zero (in the sense of Lebesgue);
w.r.t.	with respect to;
$\mathrm{int}\,(S)$	interior of the set S;
$\mathrm{int}\,(A)$	interior of the $n-$tuple of intervals A;
$\mathrm{cl}\,(S)$	closure of the set S;

Description

0	element 0 of $\mathbb{R}$;
$\boldsymbol{0}$	element $(0,\ldots,0)$ of $\mathbb{R}^n$;
$[0,0]$	zero interval of $\mathcal{K}_C$;
$\mathbb{B}_{\mathbb{R}^n}$	open unit ball of center $\boldsymbol{0}$ and radius 1 in $\mathbb{R}^n$;
$\overline{\mathbb{B}_{\mathbb{R}^n}}$	closed unit ball of center $\boldsymbol{0}$ and radius 1 in $\mathbb{R}^n$;
$\mathbb{B}_{\mathbb{R}^n}(x,\epsilon)$	open ball of center x and radius ϵ in $\mathbb{R}^n$;
$\overline{\mathbb{B}_{\mathbb{R}^n}}(x,\epsilon)$	closed ball of center x and radius ϵ in $\mathbb{R}^n$;
$\mathbb{B}_{C\times L_\infty}((x,u),\epsilon)$	open ball of center (x,u) and radius ϵ in $C\times L_\infty$;
$\Delta_\beta\, f$	difference operator of f;
$\mathcal{F}_\beta$	$\beta-$admissible directions set;
d_S	distance function associated with the set S;
d_H	Pompeiu$-$Hausdorff metric;
$\mathrm{len}(A)$	length of the interval $A=[\underline{a},\overline{a}]$;
a^C	center of the interval $A=[\underline{a},\overline{a}]$;
a^R	radius of the interval $A=[\underline{a},\overline{a}]$;
$(a^C;a^R)$	center$-$radius representation of the interval $A=[\underline{a},\overline{a}]$;
$(0;0)$	center$-$radius representation of the interval $[0,0]$;
$[\underline{a}\bigvee\overline{a}]$	means that $\underline{a}$ and $\overline{a}$ are the extremum of the interval, but $\underline{a}\leq\overline{a}$ is not necessarily satisfied;
E	topological linear space;
E'	topological dual space;
$\mathcal{K}$	cone;
$\mathcal{K}^*$	dual cone;
S^*	annihilator of $S\subset E'$.

CONTENTS

CHAPTER 1

INTRODUCTION

The theory of interval optimization problems and interval optimal control problems, in which the objective functions are interval−valued and possibly of multiple variables, plays an increasingly important role since such problems deal with numerically inexact data and uncertain environments, they allow developments of physical theories as well as descriptions of real−life situations, and they have a large number of applications.

For instance, consider a portfolio management problem and assume that there are 2 investment, see Osuna-Gómez et al., [65].

Denote the return of the $j-th$ investment as R_j, $j \in \{1,2\}$, and the proportion of the total investment funds to this investment as x_j, and let us say that

$$x_1 + x_2 = 1 \text{ million dollars.}$$

Since R_j's vary due to uncertainties, these are assumed to be the random variables which can be represented by the average vector

$$\gamma = (\gamma_1, \gamma_2),$$

where γ_j is the average rate for R_j, and the covariance matrix is denoted by

$$\sigma_{jk},$$

where σ_{jk} is the covariance between R_j and R_k with $\sigma_{jk} = \sigma_{kj}$.

The average return associated with the portfolio $x = (x_1, x_2)$ is given by

$$\gamma_1 x_1 + \gamma_2 x_2, \qquad \text{which it is desirable to be as much as possible!}$$

and the risk of investment can be formulated by

$$f(x) = \frac{1}{2}\sum_{j=1}^{2}\sum_{k=1}^{2}\sigma_{jk}x_j x_k, \qquad \text{which it is desirable to be as little as possible!}$$

From a practical point, it is difficult to specify the coefficients, γ_j and σ_{jk}. But, there are cases where one can estimate the upper and lower bound of the coefficients.

Then, the *nonlinear multiobjective interval−valued optimization* is:

$$(IOP)\begin{cases} \text{minimize} & \left(-([\underline{\gamma}_1, \overline{\gamma}_1]x_1 + [\underline{\gamma}_2, \overline{\gamma}_2]x_2), \sum_{j=1}^{2}\sum_{k=1}^{2}[\underline{\sigma}_{jk}, \overline{\sigma}_{jk}]x_j x_k\right) \\ \text{subject to} & x_1 + x_2 = 1 \text{ million dollars}, \\ & x_1, x_2 \geq 0, \end{cases}$$

$$\Leftrightarrow \begin{cases} \text{minimize} & F(x) = (F_1(x), F_2(x)) \\ \text{subject to} & -x \leq 0 \\ & x - 1 \leq 0, \\ & x \in \mathbb{R}, \end{cases}$$

where F_1 and F_2 are interval−valued functions (wich will be explained later) defined by $F_1(x) = [\underline{f}_1(x), \overline{f}_1(x)]$, $F_2(x) = [\underline{f}_2(x), \overline{f}_2(x)]$, and

$$x = x_1, \qquad x_2 = 1 - x,$$

$$\underline{f}_1(x) = (\overline{\gamma}_2 - \overline{\gamma}_1)x - \overline{\gamma}_2, \qquad \overline{f}_1(x) = (\underline{\gamma}_2 - \underline{\gamma}_1)x - \underline{\gamma}_2,$$

$$\underline{f}_2(x) = (\underline{\sigma}_{11} - 2\underline{\sigma}_{12} + \underline{\sigma}_{22})x^2 + 2(\underline{\sigma}_{12} - \underline{\sigma}_{22})x - \underline{\sigma}_{22},$$

and

$$\overline{f}_2(x) = (\overline{\sigma}_{11} - 2\overline{\sigma}_{12} + \overline{\sigma}_{22})x^2 + 2(\overline{\sigma}_{12} - \overline{\sigma}_{22})x - \overline{\sigma}_{22}.$$

Now, consider the example of the temperature control of an oven where the conditions for control existence and uniqueness of solutions are satisfied (namely, the compactness of the admissible control set, the convexity and continuity of the cost functional regarding the control), see Campos, [12], and Campos et al., [13].

A certain material goes through a sequence of five ovens. Denote $x_0 = 40°C$ the initial temperature of the material, x_k, $k \in \{1, \ldots, 5\}$, the interval state that represents the material temperature at the exit of the oven k, and u_{k-1}, $k \in \{1, \ldots, 5\}$, the interval control that represents the oven temperature at the stage k. The interval difference equation is given by

$$x_{k+1} = [0.2, 0.8] \otimes x_k + 0.5 \cdot u_k, \qquad k \in \{0, \ldots, 4\}.$$

For the analyzed interval model, the loss of the material temperature at the exit of the oven from one stage to another is uncertain and is represented by the interval $[0.2, 0.8]$ instead of a real and fixed number. This interval shows a great uncertainty regarding the loss of the material temperature and may come from the lack of knowledge of the modeler.

The objective of the problem is to get x_5 as close as possible to, let us say, the temperature $T = 150°C$, spending the least amount of energy, and where the interval cost functional has the form

$$\text{minimize} \quad C(x, u) = 500 \cdot (x_5 - [150, 150])^2 + \sum_{k=0}^{4} u_k^2.$$

To get x_5 to the temperature $T = 150°C$ means that the distance between them is "small" according to the definition of the distance between two intervals given in Chalco-Cano et al., [15], for example. The value 500 that appears in the objective functional represents a penalty for the model. In addition, a lower cost for the objective functional is expected if the loss of the material temperature is small.

Then, the *interval optimal control problem* is given by

$$(IOCP) \begin{cases} \text{minimize} \quad C(x, u) = 500 \cdot (x_5 - [150, 150])^2 + \displaystyle\sum_{k=0}^{4} u_k^2 \\ \text{subject to } x_{k+1} = [0.2, 0.8] \otimes x_k + 0.5 \cdot u_k, \qquad k \in \{0, \ldots, 4\}, \end{cases}$$

where x_k is the state variable with initial condition equals to $T = 40°C$, u_k is the control variable and $C(x, u)$ is the interval cost functional.

In the last decades, the interval optimization and the interval optimal control theory have been investigated by many authors, see, e. g., Ahmad et al., [1]; Chalco-Cano et al., [16]; Chalco-Cano et al., [17]; Chanas and Kuchta, [21]; Costa et al., [23]; Inuiguchi and Kume, [38]; Ishibuchi and Tanaka, [39]; Jiang et al., [41]; Montemanni and Gambardella, [56]; Osuna-Gómez et al., [64]; Osuna-Gómez et al., [65]; Pal et al., [66], [67]; Singh et al., [77]; Stefanini and Arana-Jiménez, [81]; Wu, [85], [87]; Zhang et al., [88]; and the references therein.

Necessary optimality conditions for classical optimal control problems are present in the well known Pontryagin Maximum Principle, which is a generalization of the classic Euler−Lagrange and Weierstrass conditions. This principle gives first order conditions which brings the characterization of optimal (extremal) control processes candidates, see e. g., Athans and Falb, [3]; Bellaassali and Jourani, [7]; Bonnel and Kaya, [10]; Clarke, [22]; Dubovitskii and Milyutin [27], [28]; Gamkrelidze, [35]; Girsanov, [36]; Kien et al., [44]; Mordukhovich, [58], [59]; Pinch, [68]; Pontryagin et al., [69], [70]; Vinter, [83]; Zhu, [89]; among others. Now, regarding necessary optimality conditions for interval optimal control problems that are in the Pontryagin Maximum Principle form, are seldom at hand in the literature. Nonetheless, it can be found necessary optimality conditions for fuzzy optimal control problems, see e. g., Fard et al., [29]; Fard and Salehi, [30]; Fard and Zadeh, [31]; Farhadinia, [32], [33]; Najariyan and Farahi, [60], [61]; Leal, [46], [47]; Soolaki et al., [78], [79]; and its references.

On the one hand, Farhadinia, [32], studied necessary optimality conditions for fuzzy variational problems using the fuzzy differentiability concept of Buckley and Feuring, see [11], but this work was generalized by Fard et al., [29]; Fard and Salehi, [30]; Fard and Zadeh, [31]. In Fard et al., [29], it was presented the fuzzy Euler−Lagrange condition for fuzzy constrained and unconstrained variational problems under the generalized Hukuhara differentiability of several variables. Fard and Salehi, [30], investigated fuzzy fractional Euler−Lagrange equations for fuzzy fractional variational problems using their generalized fuzzy fractional Caputo−type derivatives. In Fard and Zadeh, [31], it was used the $\alpha-$differentiability concept, obtained and extended the fuzzy Euler−Lagrange condition. Farhadinia, [33], derived the Pontryagin Minimum Principle for fuzzy optimality control problems based on the concepts of differentiability and integrability of a fuzzy mapping parameterized by the left and right−hand functions of its $\alpha-$level sets. In Najariyan and Farahi, [60], it was used a procedure to solve linear time invariant dynamical systems with fuzzy parameters, and they applied it to solve a fuzzy optimal controlled system by using the $\alpha-$cuts and the Pontryagin Maximum Principle of Pinch, [68]. Najariyan and Farahi, [61], studied and solved fuzzy linear time varying control systems and fuzzy optimal control problems, with fuzzy boundary conditions governed by fuzzy differential equations, using the Heaviside functions concept, the generalized Hukuhara derivative, and the Pontryagin Maximum Principle of Pinch, [68]. In Leal, [46], it was considered interval optimal control problems and given optimality conditions using the extremal differentiability, the $gH-$differentiability for several variables, the $\lambda-$single−level differentiability for several variables, and assumptions of convexity. Soolaki et al., [78], studied fuzzy problems with boundary conditions and fuzzy problems with holonomic constraints, and obtained Euler−Lagrange equations and transversality conditions. In Soolaki et al., [79], using the Pontryagin Maximum Principle of Pinch, [68], a form of the Hamiltonian, and the generalized Hukuhara differentiability, it was obtained fuzzy solutions by solving a system of differential equations.

On the other hand, Ahmad et al., [1]; Chalco-Cano et al., [16]; Chalco-Cano et al., [17]; Osuna-Gómez et al., [64]; Singh et al., [77]; Stefanini and Arana-Jiménez, [81]; Wu, [85], [87]; Zhang et al., [88], studied optimality conditions for interval optimization problems in several variables. For instance, in Wu, [85], [87], it is presented sufficient optimality conditions using the Hukuhara differentiability and convexity assumptions. In Zhang et al., [88], it is also presented sufficient optimality conditions using the same concept of differentiability, but under invexity assumptions.

Since the Hukuhara differentiability is restrictive (see Chalco-Cano et al., [20], and Diamond, [26]), the authors Ahmad et al., [1]; Chalco-Cano et al., [16]; Chalco-Cano et al., [17]; Osuna-Gómez et al., [64]; Singh et al., [77], have been obtained optimality conditions using the concept of continuously generalized Hukuhara differentiability for several variables (see Chalco-Cano et al., [17]) or $gH-$differentiability for several variables (see Chalco-Cano et al., [16]; Osuna-Gómez et al., [64]). In Chalco-Cano et al., [17]; Singh et al., [77], sufficient conditions are obtained through convexity concepts. In Ahmad et al., [1], sufficient conditions are given, but with invexity and pre−invexity concepts. In Chalco-Cano et al., [16], sufficient conditions are obtained using concepts of pseudoinvexity. In Osuna-Gómez et al., [64], necessary and sufficient conditions are given, but for the unconstrained case, where the sufficient conditions are obtained using concepts of pseudoinvexity different from Chalco-Cano et al., [16].

Despite the use of the continuously generalized Hukuhara differentiability has the purpose to deal with interval optimization problems using a less restrictive differentiability concept, this concept is also restrictive.

In Stefanini and Arana-Jiménez, [81], it is presented a new concept of differentiability: the total generalized Hukuhara differentiability for interval−valued functions with several variables, which extends the generalized Hukuhara differentiability given in Stefanini and Bede, [82], and is more convenient than the concept of continuously $gH-$differentiability of Ahmad et al., [1]; Chalco-Cano et al., [16]; Chalco-Cano et al., [17]; Osuna-Gómez et al., [64]. Stefanini and Arana-Jiménez, [81], gave optimality conditions for interval optimization problems, where they used the linear independence constraint qualification for the necessary conditions and the convexity concept for the sufficient ones.

This Thesis is focused mainly in the use of:

1. the total and directional generalized Hukuhara differentiability, a more suitable concept, given by Stefanini and Arana-Jiménez, [81];

2. the positive linear independence constraint qualification or Mangasarian−Fromovitz constraint qualification (introduced by Mangasarian and Fromovitz, [54], and used by many authors such as Davis, [24]; Qi and Wei, [71]; Robinson, [74]; Robinson and Meyer, [75]; among others) and the positive linear independence regularity condition (an adaptation of Maciel et al., [52]). The Mangasarian−Fromovitz constraint qualification is more general than the linear independence constraint qualification used in Stefanini and Arana-Jiménez, [81], but the positive linear independence regularity condition is not comparable with the linear independence constraint qualification, since the positive linear independence regularity condition is not a constraint qualification, the objective functions are present in the positive linear independence regularity condition;

3. generalized convexity;

4. Dubovitskii−Milyutin approach, see Girsanov, [36].

The main purposes of this thesis are present optimality conditions for interval optimization problems and interval optimal control problems, both with a single interval−valued objective function of several variables.

This Thesis is organized in the following sequence.

In order to reach our important results, it will be required some notations and mathematical background that will be used throughout the Thesis: the optimality for multi−objective programming problems: the Karush−Kuhn−Tucker conditions; the optimality for multi−objective control problems: the Maximum Principle and the MP−pseudoinvexity; the interval space $\mathcal{K}_C$; the limit, the continuity, the differentiability, and the integrability of interval−valued functions; classical integral equations; interval optimization problems; topological properties, cones, and dual cones. These are in Chapter 2.

In the third chapter, it will be considered the interval optimization problem with inequality constraints. It will be shown that to determining the LU−solutions of an interval optimization problem is equivalent to determining the solutions of a classical bi$-$objective optimization problem. Moreover, using the concept of $E-$differentiability, it will be given necessary conditions of Fritz John and Karush$-$Kuhn$-$Tucker$-$types.

The fourth chapter is dedicated to present an interval optimal control problem. In this chapter, it will be used the Lipschitz concept of an interval$-$valued function of several variables. It will be presented the weak (strong) optimal $LU-$processes, the optimal $LU-$processes, and their local $W^{1,1}$ and L_∞ definitions, respectively. Next, the Hamilton function, and the (locally) Lipschitz continuous concept of interval$-$valued functions it will be given. It will be shown that to determining the optimal $LU-$processes of an interval optimal control problem is equivalent to determining the solutions of a classical bi$-$objective control problem. Therefore, the necessary and sufficient optimality conditions for the weak optimal $LU-$processes are easily presented.

In the fifth chapter, it will be used the total and directional generalized Hukuhara differentiability. The necessary optimality conditions are Fritz John$-$type, $KKT-$type and strict $KKT-$type, and the sufficient optimality conditions are of generalized convexity type. To obtain the necessary optimality conditions it will be used the positive linear independence constraint qualification and the positive linear independence regularity condition. Considering the sufficient ones, it will be given definitions of weak (strong) $KKT-$interval and $KKT-$interval pseudoinvexities, and with these it will be shown that $KKT-$solutions are weak (strong) $LU-$solutions and $LU-$solutions, respectively. Also, it will be given definitions of strict $KKT-$interval and partial$-\underline{LU}(\overline{LU})$ strict $KKT-$interval pseudoinvexities, and with these it will be shown that strict $KKT-$solutions are $LU-$solutions and $\underline{LU}(\overline{LU})-$solutions, respectively. Furthermore, it will be given the converses of each one of these results. The $\underline{LU}$ and $\overline{LU}$ partial order relation given here, are adaptations of Arana-Jiménez and Neto, [2], but are new in interval optimization. Also, the constraint qualification that is used in this chapter is more general than the constraint qualification used in Stefanini and Arana-Jiménez, [81], so, the $KKKT-$types optimality conditions are original in the intervalar Theory. As far as it is known, the pseudoinvexities defined here, are similar to the pseudoinvexities of Chalco-Cano et al., [16], but in Chalco-Cano et al., [16], it is used different assumptions such as the differentiability. Finally, the pseudoinvexities of this chapter, lead to get sufficient optimality conditions under more geral assumptions than those used in Stefanini and Arana-Jiménez, [81], since the authors Stefanini and Arana-Jiménez of [81], used just convexity.

The sixth chapter, using the total and directional generalized Hukuhara differentiability with several variables, is devoted to show an interesting original result for interval optimization problems with abstract constraints. It will be defined optimal $LU-$solution for the problem in question. Next, it will be considered some basic assumptions, definitions, and properties of them. It will be described important cones, and it will be presented some properties. Using well known results of the Theory of extremum problems, based on the Dubovitskii$-$Milyutin formalism (see Girsanov, [36]), it will be achieved a desired result.

In the seventh chapter, it will be obtained necessary optimality conditions for interval optimal control problems in the Lagrange form using again the total and directional generalized Hukuhara differentiability, for problems with several variables. First, it will be given the optimal $LU-$process concepts for the considered problems: with right endpoint fixed and then with free right endpoint. Next, it will be given some basic assumptions, definitions, and properties of them. It will be characterized important cones and it will be given some properties. Using again results of the Theory of extremum problems, it will be obtained an interval local Pontryagin Maximum Principle. All this study is based on the Dubovitskii$-$Milyutin formalism given in Girsanov, [36]. The originality resides in the fact that in the Dubovitskii$-$Milyutin formalism, the objective functional must take values in a topological linear space, which is not the case of the interval control problem. Therefore, it is necessary to define directions of decrease for interval$-$valued functions and characterize geometrically the optimal solutions, for then apply the Lemma of Dubovitskii$-$Milyutin.

Finally, the eighth chapter is committed to give general conclusions of this Thesis and some suggestions of future research.

CHAPTER 8

CONCLUSIONS AND FUTURE PERSPECTIVES

In this Thesis, applying classical results, necessary conditions of Fritz John and Karush$-$Kuhn$-$Tucker$-$types for interval optimization problems, using the $E-$differentiability, were obtained.

Also, applying classical results, we gave not only necessary but also sufficient optimality conditions for interval optimal control problems. To this, we showed that a solution of an interval optimal control problem is a solution of a classical bi$-$objective control problem and conversely.

We presented a new approach to interval optimization using the total and directional generalized Hukuhara differentiability, for problems with several variables. With the Gordan Transposition Theorem, we obtained a Fritz John$-$type condition and with a constraint qualification or a regularity condition, we obtained interval Karush$-$Kuhn$-$Tucker$-$type conditions. Also, we defined, for interval optimization problems, the following concepts: $KKT-$solutions, strict $KKT-$solutions, weak (strong) $KKT-$interval, $KKT-$interval, strict $KKT-$interval, and partial$-\underline{LU}(\overline{LU})$ strict $KKT-$interval pseudoinvexities. Moreover, we proved that a $KKT-$solution for a weak (strong) $KKT-$interval pseudoinvex problem and for a $KKT-$interval pseudoinvex problem is a weak (strong) $LU-$solution and $LU-$solution, respectively. Also, we proved that a strict $KKT-$solution for a strict $KKT-$interval pseudoinvex problem and for a partial$-\underline{LU}(\overline{LU})$ strict $KKT-$interval pseudoinvex problem is a $LU-$solution and $\underline{LU}(\overline{LU})-$solution, respectively. As far as it knows, strict $KKT-$type optimality conditions are news in the interval optimization area. Furthermore, it is important to mention that these concepts of generalized invexity were proved to be the most general as possible.

Using the total and directional generalized Hukuhara differentiability and known results from the general extremum problem theory, we obtained necessary optimality condition for abstract interval optimization problems, which is a novelty in the interval optimization.

We gave necessary optimality conditions for interval optimal control problems in the Lagrange forms using again the total and directional generalized Hukuhara differentiability. With general extremum problem theory, we obtained an interval local Pontryagin Maximum Principle. In the literature, this work is outstanding since there is nothing like this in the interval optimal control area.

In future researches, we would like:

1. to provide applications of the proposed results;

2. to obtain an interval Pontryagin Maximum Principle;

3. to consider a more general problem of interval optimal control, in which there are also constraints on the phase coordinates;

4. to extend the optimality conditions obtained for interval optimization problems and for interval optimal control problems to the case of Fuzzy optimization problems and to the case of Fuzzy optimal control problems, respectively;

5. to give sufficient conditions for interval control problems, in which beyond of the objective functional be interval−valued, the variables also be intervals, not only for interval control problems but also for interval optimization.

BIBLIOGRAPHY

[1] Ahmad, I., Singh, D., and Dar, B. A. Optimality conditions for invex interval valued nonlinear programing problems involving generalized H-derivative. *Filomat 30*, 8 (2016), 2121–2138.

[2] Arana-Jiménez, M., and Salles Neto, L. L. Sufficient condition for partial efficiency in a bicriteria nonlinear cutting stock problem. *RAIRO Operation Research 51*, 3 (2017), 709–717.

[3] Athans, M., and Falb, P. L. *Optimal control: An introduction to the theory and its applications.* McGraw-Hill, Inc., New York, 1966.

[4] Aubin, J.-P., and Cellina, A. *Differential inclusions*, vol. 264 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, 1984. Set-valued maps and viability theory.

[5] Aubin, J. P., and Frankowska, H. *Set-Valued Analysis*, vol. 2 of *Systems & Control: Foundations & Applications.* Birkhäuser, Boston, Mass, USA, 1990.

[6] Aumann, R. J. Integrals of set-valued functions. *Journal of Mathematical Analysis and Applications 12*, 1 (1965), 1–12.

[7] Bellaassali, S., and Jourani, A. Necessary optimality conditions in multiobjective dynamic optimization. *SIAM Journal on Control and Optimization 42*, 6 (2004), 2043–2061.

[8] Bigi, G. *Optimality and Lagrangian regularity in vector optimization.* Ph. D. Thesis, Dipartimento di Matematica, Università di Pisa, Italy, 1999.

[9] Bigi, G., and M., P. Regularity conditions in vector optimization. *Optimization Theory and Applications 102* (1999), 83–96.

[10] Bonnel, H., and Kaya, C. Y. Optimization over the efficient set of multi-objective convex optimal control problems. *Optimization Theory and Applications 147*, 1 (2010), 93–112.

[11] Buckley, J. J., and Feuring, T. Introduction to fuzzy partial differential equations. *Fuzzy Sets and Systems 105*, 2 (1999), 241–248.

[12] Campos, J. R. *Problemas de Controle Ótimo Intervalar e Intervalar Fuzzy.* Ph. D. Thesis, Universidade Estadual Paulista, Brazil, 2018.

[13] Campos, J. R., Assunção, E., Silva, G. N., Lodwick, W. A., Teixeira, M. C. M., and Maqui-Huamán, G. G. Fuzzy interval optimal control problem. *Fuzzy Sets and Systems 385* (2020), 169–181.

[14] Chalco-Cano, Y., Flores-Franulič, A., and Román-Flores, H. Ostrowski Type inequalities for interval-valued functions using generalized Hukuhara derivative. *Computational and Applied Mathematics 31* (2012), 457–472.

[15] Chalco-Cano, Y., Lodwick, W. A., and Bede, B. Single-level constraint interval arithmetic. *Fuzzy Sets and Systems 257* (2014), 146–168.

[16] Chalco-Cano, Y., Lodwick, W. A., Osuna-Gómez, R., and Rufián-Lizana, A. The Karush-Kuhn-Tucker optimality conditions for fuzzy optimization problems. *Fuzzy Optimization and Decision Making 15*, 1 (2016), 57–73.

[17] Chalco-Cano, Y., Lodwick, W. A., and Rufián-Lizana, A. Optimality conditions of type KKT for optimization problem with interval-valued objective function via generalized derivative. *Fuzzy Optimization and Decision Making 12*, 3 (2013), 305–322.

[18] Chalco-Cano, Y., Maqui-Huamán, G. G., Silva, G. N., and Jiménez-Gamero, M. D. Algebra of generalized Hukuhara differentiable interval-valued functions: Review and new properties. *Fuzzy Sets and Systems 375* (2019), 53–69.

[19] Chalco-Cano, Y., and Román-Flores, H. On new solutions of fuzzy differential equations. *Chaos Solitons Fractals 38* (2008), 112–119.

[20] Chalco-Cano, Y., Román-Flores, H., and Jiménez-Gamero, M. D. Generalized derivative and π-derivative for set-valued functions. *Information Sciences 181*, 11 (2011), 2177–2188.

[21] Chanas, S., and Kuchta, D. Multiobjective programming in optimization of interval objective functions - A generalized approach. *European Journal of Operational Research 94*, 3 (1996), 594–598.

[22] Clarke, F. H. *Optimization and nonsmooth analysis*, vol. 5 of *Classics in Applied Mathematics.* Society for Industrial and Applied Mathematics (SIAM, Philadelphia), 1990.

[23] Costa, T. M., Chalco-Cano, Y., Lodwick, W. A., and Silva, G. N. Generalized interval vector spaces and interval optimization. *Information Sciences 311* (2015), 74–85.

[24] Davis, C. Theory of positive linear dependence. *American Journal of Mathematics 76* (1954), 733–746.

[25] de Oliveira, V. A., and Silva, G. N. On sufficient optimality conditions for multiobjective control problems. *Global Optimization 64*, 4 (2016), 721–744.

[26] Diamond, P., and Kloeden, P. E. *Metric spaces of fuzzy sets: Theory and applications.* World Scientific, 1994.

[27] Dubovitskii, A. Y., and Milyutin, A. A. *The extremum problem in the presence of constraints*, vol. 4 of *Dokl. Akad. Nauk SSSR 149.* 1963.

[28] Dubovitskii, A. Y., and Milyutin, A. A. *The extremum problem in the presence of constraints*, vol. 3 of *Zh. Vychisl. Mat. i Mat. Fiz. 5.* 1965.

[29] Fard, O. S., Borzabadi, A. H., and Heidari, M. On fuzzy Euler-Lagrange equations. *Annals of Fuzzy Mathematics and Informatics 7*, 3 (2014), 447–461.

[30] Fard, O. S., and Salehi, M. A survey on fuzzy fractional variational problems. *Computational and Applied Mathematics 271* (2014), 71–82.

[31] Fard, O. S., and Zadeh, M. S. Note on "Necessary optimality conditions for fuzzy variational problems". *Advanced Research in Dynamical and Control Systems 4*, 3 (2012), 1–9.

[32] Farhadinia, B. Necessary optimality conditions for fuzzy variational problems. *Information Science 181*, 7 (2011), 1348–1357.

[33] Farhadinia, B. Pontryagin's Minimum Principle for fuzzy optimal control problems. *Iranian Journal of Fuzzy Systems 11*, 2 (2014), 27–43.

[34] Fréchet, M. *Sur les ensembles de fonctions et les opérations linéaires.* Les Comptes rendus de l'Académie des sciences, France, 1907.

[35] Gamkrelidze, R. V. *Principles of optimal control theory (Transl. from the Russian by K. Malowski. Transl. ed. by and with a foreword by L. D. Berkovitz)*, vol. 7 of *In Mathematical Concepts and Methods in Science and Engineering.* Plenum Press, New York, London, 1978.

[36] Girsanov, I. V. *Lecture notes in economics and mathematical systems*, vol. 67 of *Operations Research, Computer Science, Social Science.* Springer-Verlag, Berlin, Heidelberg, New York, 1972.

[37] Goetschel, R., and Voxman, W. Elementary fuzzy calculus. *Fuzzy Sets and Systems 18*, 1 (1986), 31–43.

[38] Inuiguchi, M., and Kume, Y. Goal programming problems with interval coefficients and target intervals. *European Journal of Operational Research 52*, 3 (1991), 345–360.

[39] Ishibuchi, H., and Tanaka, H. Multiobjective programming in optimization of the interval objective function. *European Journal of Operational Research 48*, 2 (1990), 219–225.

[40] Jensen, J. L. W. V. Sur les fonctions convexes et les inégalités entre les valeurs moyennes. *Acta Mathematica 30*, 1 (1905), 175–193.

[41] Jiang, C., Han, X., Liu, G. R., and Liu, G. P. A nonlinear interval number programming method for uncertain optimization problems. *European Journal of Operational Research 188*, 1 (2008), 1–13.

[42] Kaleva, O. Fuzzy numbers fuzzy differential equations. *Fuzzy Sets and Systems 24*, 3 (1987), 301–317.

[43] Kantorovich, L. V., and Akilov, G. P. *Functional analysis in normed spaces.* Nauka, Moscow, Fizmatgiz, 1959.

[44] Kien, B. T., Wong, N. C., and Yao, J. C. Necessary conditions for multi-objective optimal control problems with free end-time. *SIAM Journal on Control and Optimization 47*, 5 (2008), 2251–2274.

[45] Kulisch, U. W., and Miranker, W. L. *Computer arithmetic in theory and practice.* Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1981. Computer Science and Applied Mathematics.

[46] Leal, U. A. S. *Incerteza Intervalar em Otimização e Controle*. Ph. D. Thesis, Universidade Estadual Paulista, Brazil, 2015.

[47] Leal, U. A. S., Lodwick, W. A., Silva, G. N., and Maqui-Huamán, G. G. Interval optimal control for uncertain problems. *Fuzzy Sets and Systems* (2019), 1–13.

[48] Leal, U. A. S., Silva, G. N., and Lodwick, W. A. Multi-objective optimization in optimal control problem with interval-valued objetive function. *XXXV CNMAC 3* (2015).

[49] Lima, E. L. *Curso de Análise - Volume 2*. Rio de Janeiro, IMPA, 2014.

[50] Luenberger, D. G. *Linear and nonlinear programming*, vol. 2. Addison-Wesley Publishing Company, Nova York, 1986.

[51] Luhandjula, M. K., and Rangoaga, M. J. An approach for solving a fuzzy multi-objective programming problem. *European Journal of Operational Research 232* (2014), 249–255.

[52] Maciel, M. C., Santos, S. A., and Sottosanto, G. N. Regularity conditions in differentiable vector optimization revisited. *Optimization Theory and Applications 142*, 2 (2009), 385–398.

[53] Mangasarian, O. L. *Nonlinear programming*, vol. 10 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1994. Corrected reprint of the 1969 original.

[54] Mangasarian, O. L., and Fromovitz, S. The Fritz John necessary optimality conditions in the presence of equality and inequality constraints. *Mathematical Analalysis and Applications 17* (1967), 37–47.

[55] Miettinen, K. M. *Nonlinear multiobjective optimization*, vol. 12 of *International Series in Operations Research & Management Science*. Kluwer Academic Publishers, Boston, MA, 1999.

[56] Montemanni, R., and Gambardella, L. M. A branch and bound algorithm for the robust spanning tree problem with interval data. *European Journal of Operational Research 161*, 3 (2005), 771–779.

[57] Moore, R. E., Kearfott, R. B., and Cloud, M. J. *Introduction to interval analysis*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2009.

[58] Mordukhovich, B. S. *Variational analysis and generalized differentiation I. Basic theory, Grundlehren der Mathematischen Wissennschaften [Fundamental principles of mathematical science]*, vol. 330. Springer, Berlin, 2006.

[59] Mordukhovich, B. S. *Variational analysis and generalized differentiation II. Applications, Grundlehren der Mathematischen Wissennschaften [Fundamental principles of mathematical science]*, vol. 331. Springer, Berlin, 2006.

[60] Najariyan, M., and Farahi, M. H. A new approach for the optimal fuzzy linear time invariant controlled system with fuzzy coefficients. *Iran Journal of Fuzzy Systems 10*, 3 (2014), 21–35.

[61] Najariyan, M., and Farahi, M. H. A new approach for solving a class of fuzzy optimal control systems under generalized Hukuhara differentiability. *Journal of the Franklin Institute* (2015).

[62] of Ávila, T. *The complete works of Saint Teresa of Ávila (Translated and edited by E. Allison Peers. From the critical edition of P. Silverio de Santa Teresa, C. D.), Poem IX*, vol. 3. Burns & Oates. A Continuum imprint, London, New York, 2002.

[63] Osuna-Gómez, R., Chalco-Cano, Y., Hernández-Jiménez, B., and Costa, T. M. gH-differentiability of interval-valued functions of several variables. Applications to interval optimization theory. Submitted for publication, 2019.

[64] Osuna-Gómez, R., Chalco-Cano, Y., Hernández-Jiménez, B., and Ruiz-Garzón, G. Optimality conditions for generalized differentiable interval-valued functions. *Information Sciences 321* (2015), 136–146.

[65] Osuna-Gómez, R., Hernández-Jiménez, B., Chalco-Cano, Y., and Ruiz-Garzón, G. New efficiency conditions for multiobjective interval-valued programming problems. *Information Sciences 420* (2017), 235–248.

[66] Pal, B. B., Kumar, M., and Sen, S. *A priority based interval-valued goal programming approach for land utilization planning in agricultural system: A case study.* 2nd International Conference on Computing, Communication and Networking Technologies, ICCCNT 2010, 2010.

[67] Pal, B. B., Kumar, M., and Sen, S. A priority-based goal programming method for solving academic personnel planning problems with interval-valued resource goals in university management system. *International Jornal of Applied Management Science 4*, 3 (2012), 284–312.

[68] Pinch, E. R. *Optimal control and the calculus of variations.* Oxford Science Publications, Oxford University Press, New York, 1993.

[69] Pontryagin, L. S., Boltyanskii, V. G., Gamkrelidze, R. V., and Mishchenko, E. F. *The mathematical theory of optimal processes (Translated by D. E. Brown. A Pergamon Press Book).* The Macmillan Co., New York, 1964.

[70] Pontryagin, L. S., Boltyanskii, V. G., Gamkrelidze, R. V., and Mishchenko, E. F. *The mathematical theory of optimal processes Ed. and with a preface by R. V. Gamkrelidze (Transl. from the Russian by K. N. Trirogoff. Transl. ed. by L. W. Neustadt. With a preface by L. W. Neutad and K.N. Trirogoff. Reprint of the 1962 English translation. Classics of Soviet Mathematics).* Science Publichers, New York, 1986.

[71] Qi, L., and Wei, Z. On the constant positive linear dependence condition and its application to sqp methods. *SIAM Journal on Optimization 10* (2000), 963–981.

[72] Riesz, F. *Sur une espèce de géométrie analytique des systèmes de fonctions sommables.* Comptes rendus de l'Académie des sciences, France, 1907.

[73] Riesz, F. *Sur les opérations fonctionnelles linéaires.* Comptes rendus de l'Académie des sciences, France, 1909.

[74] Robinson, S. First order conditions for general nonlinear optimization. *SIAM Journal on Applied Mathematics 30* (1976), 597–607.

[75] Robinson, S., and Meyer, R. Lower semicontinuity of multivalued linearization mapping. *SIAM Journal on Control 11* (1973), 525–533.

[76] Sawaragi, Y., Nakayama, H., and Tanino, T. *Theory of multiobjective optimization*, vol. 176 of *Mathematics in Science and Engineering.* Elsevier Science, 1985.

[77] Singh, D., Dar, B., and Kim, D. S. KKT optimality conditions in interval valued multiobjective programming with generalized differentiable functions. *European Journal of Operational Research 254*, 1 (2016), 29–39.

[78] Soolaki, J., Fard, O. S., and Borzabadi, A. H. Generalized Euler-Lagrange equations for fuzzy fractional variational calculus. *Mathematical Communications 21* (2016), 199–218.

[79] Soolaki, J., Fard, O. S., and Borzabadi, A. H. A necessary condition of Pontryagin Type for fuzzy control systems. *Computational and Applied Mathematics* (2016).

[80] Stefanini, L. A generalization of Hukuhara difference and division for interval and fuzzy arithmetic. *Fuzzy Sets and Systems 161*, 11 (2010), 1564–1584.

[81] Stefanini, L., and Arana-Jiménez, M. Karush-Kuhn-Tucker conditions for interval and fuzzy optimization in several variables under total and directional generalized differentiability. *Fuzzy Sets and Systems 362* (2019), 1–34.

[82] Stefanini, L., and Bede, B. Generalized Hukuhara differentiability of interval-valued functions and interval differential equations. *Nonlinear Analysis-Theory, Methods and Applications 71*, 3 (2009), 1311–1328.

[83] Vinter, R. *Optimal control.* Birkhaüser, Boston, Massachusetts, 2000.

[84] Wu, H.-C. *Evaluate fuzzy optimization problems based on biobjective programming problems*, vol. 47. Computer and Mathematics with Applications, 2004.

[85] Wu, H.-C. The Karush-Kuhn-Tucker optimality conditions for the optimization problem with fuzzy-valued objective function. *Mathematical Methods of Operations Research 66*, 2 (2007), 203–224.

[86] Wu, H.-C. The Karush-Kuhn-Tucker optimality conditions in an optimization problem with interval-valued objective function. *European Journal of Operational Research 176* (2007), 46–59.

[87] Wu, H.-C. The Karush-Kuhn-Tucker optimality conditions in multiobjective programming problems with interval-valued objective functions. *European Journal of Operational Research 196*, 1 (2009), 49–60.

[88] Zhang, J., Liu, S., Li, L., and Feng, Q. The KKT optimality conditions in a class of generalized convex optimization problems with an interval-valued objective function. *Optimization Letters 8*, 2 (2014), 607–631.

[89] Zhu, Q. J. Hamiltonian necessary conditions for a multiobjective optimal control problem with endpoint constraints. *SIAM Journal on Control and Optimization 39*, 1 (2000), 97–112.