

On the Hamilton–Jacobi method in classical and quantum nonconservative systems

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In this work we show how to complete some Hamilton–Jacobi solutions of linear, nonconservative classical oscillatory systems which appeared in the literature, and we extend these complete solutions to the quantum mechanical case. In addition, we obtain the solution of the quantum Hamilton–Jacobi equation for an electric charge in an oscillating pulsing magnetic field. We also argue that for the case where a charged particle is under the action of an oscillating magnetic field, one can apply nuclear magnetic resonance techniques in order to find experimental results regarding this problem. We obtain all results analytically, showing that the quantum Hamilton–Jacobi formalism is a powerful tool to describe quantum mechanics.
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1. Introduction

In 1924 the physicist Max Born put forward for the first time the name “quantum mechanics” in the literature [1]. In that work, quantum mechanics denoted a theoretical framework of atomic and electronic motion, which was understood in the same level of generality and consistency as the classical mechanics laws. Approximately one year after that work, in 1925, the historic paper presented by Heisenberg and entitled “Quantum-theoretical reinterpretation of kinematic and mechanical relations” [2] showed a new quantum-theoretical quantity which contains information about the measurable line spectrum of an atom. Motivated by Heisenberg’s work, Born, Jordan, and Heisenberg published the articles “On quantum mechanics” [3] and “On quantum mechanics II” [4], which were the first comprehensive explanations of quantum mechanics. It is worth mentioning that those works used a matrix framework.

On the other hand, Dirac independently formulated a consistent algebraic framework for quantum mechanics [5], where the equations were obtained with no use of matrix theory.

However, it was only in 1926 that the Schrödinger formalism (SF) appeared in the literature. Since then, day after day, numerous problems linked to quantum mechanics have been analyzed rigorously in the literature [6–10]. Formal developments have arisen, in particular to deepen comprehension of quantum fields. Quantum canonical transformations have attracted interest since the incipient development of the theory about a century ago.

Although the SF is the prevailing framework, alternative formalisms emerged. For instance, the path integral formulation plays a prominent role in quantum field theory [12].

The basic postulates of a third version for the study of quantum mechanics have also been proposed, namely a quantum version of the Hamilton–Jacobi formalism [13], where a better understanding of the quantum Hamilton–Jacobi theory and its consequences was presented. Moreover, in that work the authors showed applications of the quantum Hamilton–Jacobi formalism (QHJF) for the calculation of the propagators of the harmonic oscillator potential and of the same potential with time-dependent parameters. Here, it is important to highlight that Leacock and Padgett (LP) [14] and, independently, Gozzi [15] are a few names who have worked this formalism out. For instance, LP developed the QHJF for the case of conservative systems, where the main feature of their theory is the definition of the quantum action variable which permits the determination of the bound-state energy levels without solving the dynamic equation [14]. On the other hand, Castro and Dutra (CD) have obtained the QHJF through basic postulates similar to the case of the Heisenberg picture [13]. An important feature in CD’s work is the straightforward equivalence of the QHJF with both the Feynman and Schrödinger formalisms.

Currently we can find in several areas of physics a considerable amount of work dedicated to studying the QHJF. Among the different research areas, we can find an interesting connection of quantum Hamilton–Jacobi theory with supersymmetric quantum mechanics (SUSYQM) [16,17]. In this case, the quantum momenta of supersymmetric partner potentials are connected via linear fractional transformations. Moreover, in the SUSYQM context, a connection between fractional and ordinary SUSYQM has been shown by Dauod and Kibler [18]. Another line of investigation comes from one-dimensional scattering problems in the framework of the QHJF [19]. In addition, Roncadelli and Schulman solved the quantum Hamilton–Jacobi equation, by a prescription based on the propagator of the Schrödinger equation [20]. It provided the use of quantum Hamilton–Jacobi theory, developing an unexpected relation between operator ordering and the density of paths around a semiclassical trajectory. Related to it, black hole tunneling procedures have been placed as prominent methods to calculate the temperature of black holes using the Hamilton–Jacobi technique in the Wentzel, Kramers, and Brillouin (WKB) approximation [21–23]. Various types of black holes have been studied in the context of tunneling of fermions and bosons as well [21–24]. Tunneling procedures are used to investigate black hole radiation, by taking into account classically forbidden paths that particles go through, from the inside to the outside of black holes. Moreover, quantum WKB approaches were employed to calculate corrections to the Bekenstein–Hawking entropy for the Schwarzschild black hole [25].

As can be seen in [26], the problem of the electron quantum dynamics in the hydrogen atom has been modeled exactly by QHJF, where the quantization of energy, angular momentum, and the action variable originate from the electron complex motion. In addition, the shell structure observed in the hydrogen atom arises from the structure of the complex quantum potential, from which the quantum forces acting upon the electron can be uniquely determined.

Moreover, much has been learned regarding the QHJF in recent years, when several developments have been accomplished in the literature. These include the definability of time parameterization of trajectories [27], corrections for any soliton equation for which action–angle variables are known [28], lattice theories [29], gauge invariance in loop quantum cosmology [30], treatment of the relativistic double ring-shaped Kratzer potential [31], shape-invariant potentials in higher dimensions [32], application to the photodissociation dynamics of NOCl [33], and Dirac–Klein–Gordon systems [34].

Furthermore, Vujanovic and Strauss [35] developed a series of calculations using the classical Hamilton–Jacobi method to study linear nonconservative systems. In order to obtain solutions for

the cases studied, the authors used an expression for the classical action that contains only the quadratic term, which reads:

$$S_{\text{VS}}(x, t) = \frac{\alpha(t)}{2} x^2. \quad (1)$$

Despite the fact that this term does not alter the classical solution, we shall show here that it does not hold for the quantum mechanical case. In fact, when quantum systems are approached, we shall study Hamilton's principal function S given by a polynomial of x , which is written in the form

$$S(x, t) = \frac{\alpha(t)x^2}{2} + \xi(t)x + \zeta(t). \quad (2)$$

In fact, the linear term is necessary for the development of the quantum propagator. Hence, this term cannot be neglected when quantum solutions are regarded. In addition, in order to deal with a more interesting application from the point of view of QHJF, we will study the problem of an electric charge in an oscillating pulsed magnetic field [38,39].

This paper is organized as follows. In the next section, we present a complete review of the QHJF and its basic postulates. In Sect. 3, we show an illustration of the QHJF for the standard case of the harmonic oscillator. In Sect. 4, we apply the ideas to analyze the driven oscillator case. Section 5 is devoted to the resonance example. In Sect. 6, we show an application of the Hamilton–Jacobi formalism to the problem related to the quantum dynamics of an electric charge in an oscillating pulsing magnetic field. We end with some general remarks and conclusions in Sect. 7.

2. A brief review of the Hamilton–Jacobi formalism

In this section we will present a review of the QHJF and its basic postulates. We present a prescription for obtaining the quantum Hamilton–Jacobi equation from the classical one. At this point, it is important to remark that this approach is analogous to the Heisenberg prescription, which makes a link between the Poisson brackets and quantum commutation relations. Here, we follow the work presented by CD [13], and revisit the QHJF as well.

Let us start by remembering that the Hamilton principal function, or action, S_{cl} , is a generating function of the canonical transformation $(\vec{r}, \vec{p}) \mapsto (\vec{r}', \vec{p}')$, which generates new time-dependent variables \vec{r}' and \vec{p}' with a null Hamiltonian. In this case, the classical Hamilton–Jacobi equation reads

$$H(\vec{r}, \vec{\nabla} S_{\text{cl}}, t) + \frac{\partial S_{\text{cl}}}{\partial t} = 0, \quad (3)$$

where $\vec{\nabla} S_{\text{cl}} = \vec{p}$. It is worth pointing out that the above classical Hamilton–Jacobi equation provides a successful form for establishing the equations of motion of a mechanical system.

Following the approach given in [13], where the authors used classical mechanics as a short wavelength limit of wave mechanics, and by taking into consideration the similarity with the electromagnetic quantities and their limits to geometrical optics, it was postulated that the quantum wave amplitude has the form

$$\Psi = 2^{-1/2} \exp(iS/\hbar), \quad (4)$$

where S is the quantum Hamilton principal function, or complex action, \hbar represents the Planck constant, and $2^{-1/2}$ is a factor introduced for convenience. In order to accomplish the transition from the classical Hamilton–Jacobi equation to the quantum case, one defines the momentum in the operatorial form, given by

$$\vec{p}_{\text{op}} = \vec{\nabla}S - i\hbar\vec{\nabla}. \quad (5)$$

Hence the classical momentum is obtained in the limit $\hbar \rightarrow 0$, where the commutation relations are established. Thus, when the Hamiltonian has the standard form

$$H = \frac{\vec{p}^2}{2m} + V, \quad (6)$$

one can find, using (3) and (5), the following quantum Hamilton–Jacobi equation (QHJE):

$$\frac{1}{2m}(\vec{\nabla}S)^2 + \frac{\partial S}{\partial t} + V = \frac{i\hbar}{2m}\nabla^2 S, \quad (7)$$

where the Hamiltonian operator coming from (5) was applied in the identity, so that

$$\frac{1}{2m} \left[\vec{\nabla}S \cdot \left(\frac{\hbar}{i} \right) \vec{\nabla} - \hbar^2 \nabla^2 \right] + 1 = 0. \quad (8)$$

In the next sections we will show how linear, strictly nonconservative, oscillatory systems with one degree of freedom may be analyzed within the quantum Hamilton–Jacobi framework. The motivation for this study is that linear dissipative systems, possessing even one degree of freedom, have not been analyzed in the context of the quantum Hamilton–Jacobi method, despite their practical, theoretical, and pedagogical interest. Moreover, using a complete solution of the Hamilton–Jacobi equation, we find in a more pedagogical form the propagators. In this case, the method does not use path integral or eigenfunction techniques. Furthermore, our approach avoids the difficulties found in the usual methods such as finding a recurrence formula to calculate the infinite integrals or manipulating the special functions that in general appear in the eigenvalue problem.

3. Harmonic oscillator

A particularly important physical system is the harmonic oscillator. There exist a large number of important physical applications for it, such as the vibrations of the atoms of a molecule about their equilibrium position, or even an electromagnetic field, for instance. In fact, whenever the behavior of a physical system in the neighborhood of a stable equilibrium position is studied, one obtains equations which, in the limit of small oscillations, are those of a harmonic oscillator.

Let us start our study with a straightforward example of the harmonic oscillator. The associated quantum Hamilton–Jacobi equation is provided by [13]:

$$\frac{\partial S}{\partial t} + \frac{1}{2} \left(\frac{\partial S}{\partial x} \right)^2 + \frac{\omega^2 x^2}{2} = \frac{i\hbar}{2} \frac{\partial^2 S}{\partial x^2}. \quad (9)$$

The substitution of (2) into the QHJE (9) generates a polynomial equation leading to a system of first-order coupled differential equations for the arbitrary coefficients introduced in (2). The polynomial

equations can be split into the following set of first-order nonlinear differential equations:

$$\begin{aligned}\dot{\alpha}(t) + \alpha^2(t) + \omega^2 &= 0, \\ \dot{\xi}(t) + \alpha(t)\xi(t) &= 0, \\ \dot{\zeta}(t) + \frac{\xi^2(t)}{2} - \frac{i\hbar}{2}\alpha(t) &= 0,\end{aligned}\tag{10}$$

yielding the general solutions

$$\begin{aligned}\alpha(t) &= -\omega \tan(\omega t + c_1), \\ \xi(t) &= c_2 \sec(\omega t + c_1), \\ \zeta(t) &= -\frac{c_2^2}{2\omega} \tan(\omega t + c_1) + \frac{i\hbar}{2} \ln[\cos(\omega t + c_1)] + c_3,\end{aligned}\tag{11}$$

where c_1 , c_2 , and c_3 are arbitrary integration constants. Hence, a complete solution of (1) is given by

$$S(x, t) = -\frac{\omega}{2} \tan(\omega t + c_1) x^2 + c_2 \sec(\omega t + c_1) x - \frac{c_2^2}{2\omega} \tan(\omega t + c_1) + \frac{i\hbar}{2} \ln[\cos(\omega t + c_1)] + c_3.\tag{12}$$

It is worth emphasizing that in the limit $\hbar \rightarrow 0$, the classical Hamilton principal function is reobtained. The general solution for the classical case of the Hamilton–Jacobi equation can be obtained from the constraint $\frac{\partial S}{\partial c_1} = B$, where B is a constant. Furthermore, it is straightforward to verify that the classical solution is given by

$$x_{\pm}(t) = \frac{c_2 \sin(\omega_0 t + c_1)}{\omega_0} \pm \left[-\frac{c_2^2}{\omega_0^2} - \frac{2B}{\omega_0} \right]^{1/2} \cos(\omega_0 t + c_1).\tag{13}$$

By analyzing the classical case for Eq. (12), the solution can also be immediately determined by $\frac{\partial S}{\partial c_2} = B$. Thus, in this case the classical solution contains two integration constants, as should be expected, since the equation of motion is a second-order one.

Moreover, by using Eq. (12), the solution for the problem consists in obtaining the quantum propagator, by imposing the following boundary condition: [13]

$$S(x, 0) = \hbar k x.\tag{14}$$

Therefore, $c_1 = 0$, $c_2 = \hbar k$, and $c_3 = 0$.

The concept of propagators is of great importance in quantum physics, and in Feynman's formulation, particularly. All the time evolution of a given system may be obtained through the propagators [13]. They are used mostly to calculate the probability amplitude for particle interactions using Feynman diagrams.

The propagator can be obtained by considering a physical wave packet

$$\Psi(x, t) = \frac{1}{\sqrt{2\pi}} \int dk \Phi(k) \exp \left[\frac{i}{\hbar} S_k(x, t) \right],\tag{15}$$

where $S_k(x, t)$ denotes the quantum Hamilton principal function Eq. (12) if the boundary condition

$$S_k(x, 0) \equiv S(x, 0) = \hbar k x\tag{16}$$

is taken into account. Inserting the Fourier transform $\Phi(k) = \frac{1}{\sqrt{2\pi}} \int dx \Psi(x, 0) \exp(-ikx)$ into Eq. (15) yields

$$\Psi(x, t) = \int dk K(x, t; \tilde{x}, 0) \Psi(\tilde{x}, 0), \quad (17)$$

where the propagator reads

$$K(x, t; \tilde{x}, 0) = \frac{1}{2\pi} \int dk \exp \left\{ \frac{i}{\hbar} [S(x, t) - \hbar k \tilde{x}] \right\}. \quad (18)$$

We observe that the constant c_2 is related to the term that generates the quantum propagator. It is important to remark that this constant appears in the linear term of Eq. (2). Hence, we conclude that the linear term must also compose the principal Hamilton function, in order to construct the quantum propagator.

By substituting the solution (12) and imposing the initial conditions (16) on the expression of the propagator and integrating in k , one gets

$$K(x, t; \tilde{x}, 0) = \left(\frac{\omega}{2\pi i \hbar \sin(\omega t)} \right)^{1/2} \exp \left\{ \frac{i\omega}{2\hbar \sin(\omega t)} [(x^2 + \tilde{x}^2) \cos(\omega t) - 2x\tilde{x}] \right\}. \quad (19)$$

The quantum propagator can be alternatively constructed [13], by imposing that

$$K(x, t; \tilde{x}, 0) = \exp \left[\frac{i}{\hbar} S(x, t; \tilde{x}, 0) \right], \quad (20)$$

where S represents the quantum solution of the Hamilton–Jacobi equation.

The propagator must satisfy the condition

$$\lim_{t \rightarrow 0^+} K(x, t; \tilde{x}, 0) = \delta(x - \tilde{x}), \quad (21)$$

where $\delta(x - \tilde{x})$ represents the Dirac delta function. For our purposes it is useful to employ the following representation:

$$\delta(x - \tilde{x}) = \lim_{t \rightarrow 0^+} (\pi \lambda t)^{-\frac{1}{2}} \exp \left[-\frac{(x - \tilde{x})^2}{\lambda t} \right]. \quad (22)$$

By using Eqs. (20)–(22), we determine

$$c_1 = \frac{\pi}{2}, \quad c_2 = \omega \tilde{x}, \quad c_3 = -i \frac{\hbar}{2} \ln \left(\frac{i\omega}{2\pi \hbar} \right). \quad (23)$$

By substituting Eq. (23) in Eq. (20), the propagator is reduced to the form presented in (19).

We emphasize that the linear term in S is quite necessary. We are now going to implement this approach in similar cases which, to our knowledge, have not been taken into account in the literature, at least from the point of view of the quantum Hamilton–Jacobi formalism.

4. Driven oscillator

Driven harmonic oscillators are damped oscillators further affected by an externally applied force. The potential of a driven harmonic oscillator can describe many phenomena in physics, such as superconducting quantum-interference devices [36] and magnetohydrodynamics [37].

Its classical equation of motion reads

$$\ddot{x} + \omega x^2 = h \cos(\Omega t), \quad (24)$$

and the corresponding Lagrangian can be written as

$$L = \frac{1}{2} \left[(\dot{x} - \dot{f}(t))^2 - \omega^2 (x - f(t))^2 \right], \quad (25)$$

where $f(t) = \left(\frac{h \cos(\Omega t)}{\omega^2 - \Omega^2} \right)$. The following Hamiltonian is then derived:

$$H = \frac{p^2}{2} + \dot{f}(t)p + \frac{\omega^2}{2} [x - f(t)]^2. \quad (26)$$

Hence, the Hamilton–Jacobi equation assumes the form

$$\frac{\partial S}{\partial t} + \frac{1}{2} \left(\frac{\partial S}{\partial x} \right)^2 - \left[\frac{h \Omega \sin(\Omega t)}{\omega^2 - \Omega^2} \right] \frac{\partial S}{\partial x} + \frac{1}{2} \omega^2 \left[x - \left(\frac{h \cos(\Omega t)}{\omega^2 - \Omega^2} \right) \right]^2 = \frac{i \hbar}{2} \frac{\partial^2 S}{\partial x^2}, \quad (27)$$

and the principal Hamilton function is represented by

$$S(x, t) = \frac{1}{2} \alpha(t) [x - f(t)]^2 + \xi(t) [x - f(t)] + \zeta(t). \quad (28)$$

By substituting Eq. (28) into Eq. (27), the quantum Hamilton principal function reads

$$\begin{aligned} S(x, t) = & -\frac{1}{2} \omega \tan(\omega t + c_1) \left[x - \left(\frac{h \cos(\Omega t)}{\omega^2 - \Omega^2} \right) \right]^2 + c_2 \sec(\omega t + c_1) \left[x - \left(\frac{h \cos(\Omega t)}{\omega^2 - \Omega^2} \right) \right] \\ & - \frac{c_2^2}{2\omega} \tan(\omega t + c_1) + \frac{i \hbar}{2} \ln [\cos(\omega t + c_1)] + c_3. \end{aligned} \quad (29)$$

The limit $\hbar \rightarrow 0$ leads to the classical case, and the solution is obtained by imposing $\frac{\partial S}{\partial c_1} = B$, implying that

$$x_{\pm}(t) = \frac{c_2 \sin(\omega t + c_1)}{\omega} \pm \left[-\frac{c_2^2}{\omega^2} - \frac{2B}{\omega} \right]^{1/2} \cos(\omega t + c_1) + \left(\frac{h \cos(\Omega t)}{\omega^2 - \Omega^2} \right). \quad (30)$$

Our result can lead to the one in [35], with some mathematical manipulations. The above solution can also be obtained by imposing $\frac{\partial S}{\partial c_2} = B$.

For the quantum case, once again the condition

$$S(x, 0) = \hbar k x \quad (31)$$

is imposed, which implies that

$$c_1 = 0, \quad c_2 = \hbar k, \quad c_3 = \frac{\hbar k h}{\omega^2 - \Omega^2}. \quad (32)$$

Remembering that $f(t) = \left(\frac{h \cos(\Omega t)}{\omega^2 - \Omega^2} \right)$, and imposing the described conditions in (31) and (32), the propagator reads

$$\begin{aligned} K(x, t, \tilde{x}, 0) = & \left(\frac{\omega}{2\pi i \hbar \sin(\omega t)} \right)^{1/2} \exp \left\{ \frac{i\omega}{2\hbar \sin(\omega t)} \left[\left(\left(x - \frac{h \cos(\Omega t)}{\omega^2 - \Omega^2} \right)^2 \right. \right. \right. \\ & \left. \left. \left. + \left(\tilde{x} - \frac{h \cos(\Omega t)}{\omega^2 - \Omega^2} \right)^2 \right) \cos(\omega t) - 2 \left(x - \frac{h \cos(\Omega t)}{\omega^2 - \Omega^2} \right) \left(\tilde{x} - \frac{h \cos(\Omega t)}{\omega^2 - \Omega^2} \right) \right] \right\}. \end{aligned} \quad (33)$$

From the initial condition of the second method, the propagator can be obtained if we choose

$$c_1 = \frac{\pi}{2}, \quad c_2 = \omega(\tilde{x} - f(t)), \quad c_3 = -i\frac{\hbar}{2} \ln\left(\frac{i\omega}{2\pi\hbar}\right), \quad (34)$$

which leads to the result in (33).

5. Resonances

Resonance occurs when a given system is driven to oscillate by another vibrating system with greater amplitude at a specific preferential frequency. This occurs with all types of waves, such as mechanical, electromagnetic, and quantum wave functions.

Let us consider the following equation:

$$\ddot{x} + \omega^2 x = h \cos(\omega t). \quad (35)$$

The Lagrangian reads

$$L = \frac{1}{2} \left[\dot{x} - \frac{1}{2} h t \cos(\omega t) - \frac{h}{2\omega} \sin(\omega t) \right]^2 - \frac{1}{2} \omega^2 \left[x - \frac{h t}{2\omega} \sin(\omega t) \right], \quad (36)$$

whereas the Hamiltonian is given by

$$H = \frac{1}{2} p^2 + \left[\frac{1}{2} h t \cos(\omega t) + \frac{h t}{2\omega} \sin(\omega t) \right] p + \frac{1}{2} \omega^2 \left[x - \frac{h t}{2\omega} \sin(\omega t) \right]^2. \quad (37)$$

Hence the corresponding quantum Hamilton–Jacobi equation becomes

$$\frac{\partial S}{\partial t} + \frac{1}{2} \left(\frac{\partial S}{\partial x} \right)^2 + f(t) \left(\frac{\partial S}{\partial x} \right) + \frac{1}{2} \omega^2 [x - f(t)]^2 = \frac{i\hbar}{2} \left(\frac{\partial^2 S}{\partial x^2} \right). \quad (38)$$

Considering Eq. (28), where $f(t) = \left(\frac{h t}{2\omega} \sin(\omega t) \right)$, and applying it in Eq. (38), it follows that

$$\begin{aligned} S(x, t) = & -\frac{1}{2} \omega \tan(\omega t + c_1) \left[x - \left(\frac{h t}{2\omega} \sin(\omega t) \right) \right]^2 + c_2 \sec(\omega t + c_1) \left[x \right. \\ & \left. - \left(\frac{h t}{2\omega} \sin(\omega t) \right) \right] - \frac{c_2^2}{2\omega} \tan(\omega t + c_1) + \frac{i\hbar}{2} \ln[\cos(\omega t + c_1)] + c_3. \end{aligned} \quad (39)$$

In the classical case we have the solutions

$$x_{\pm}(t) = \frac{c_2 \sin(\omega t + c_1)}{\omega} \pm \left[-\frac{c_2^2}{\omega^2} - \frac{2B}{\omega} \right]^{1/2} \cos(\omega t + c_1) + \left(\frac{h t}{2\omega} \sin(\omega t) \right). \quad (40)$$

By imposing the condition

$$S(x, 0) = \hbar k x - \hbar k f(t), \quad (41)$$

the quantum propagator for the resonance reads

$$\begin{aligned} K(x, t; \tilde{x}, 0) = & \left(\frac{\omega}{2\pi i \hbar \sin(\omega t)} \right)^{1/2} \exp \left\{ \frac{i\omega}{2\hbar \sin(\omega t)} \left[\left(\left(x - \frac{h t}{2\omega} \sin(\omega t) \right)^2 + \right. \right. \right. \\ & \left. \left. \left. + \left(\tilde{x} - \frac{h t}{2\omega} \sin(\omega t) \right)^2 \right) \cos(\omega t) - 2 \left(x - \frac{h t}{2\omega} \sin(\omega t) \right) \left(\tilde{x} - \frac{h t}{2\omega} \sin(\omega t) \right) \right] \right\}. \end{aligned} \quad (42)$$

On the other hand, if we try to construct the propagator from the initial conditions procedure, we find

$$c_1 = \frac{\pi}{2}, \quad c_2 = \omega [\tilde{x} - f(t)], \quad c_3 = -i\frac{\hbar}{2} \ln \left(\frac{i\omega}{2\pi\hbar} \right). \quad (43)$$

With these values, the propagator (20) leads to the form given by Eq. (42).

6. Electric charge in an oscillating pulsed magnetic field

In this section, we show an application of the Hamilton–Jacobi formalism to a problem related to the quantum dynamics of an electric charge in an oscillating pulsed magnetic field [38]. It becomes important then to analyze, through a parallel formalism, the validity of the solutions presented, since the systems can describe experimental measurements in nuclear magnetic resonance techniques [39].

We consider an electric charge e in an oscillating pulsed magnetic field given by

$$\vec{B}(t) = B_1 \cos(\omega t) \hat{i} + B_2 \sin(\omega t) \hat{j} + B_0 \hat{k}. \quad (44)$$

The Lagrangian for a charge in an electromagnetic field reads

$$L = \frac{m\vec{v}^2}{2} + \frac{e}{c} \vec{A} \cdot \vec{v} - e\phi(\vec{r}), \quad (45)$$

where $\vec{A} = -\frac{1}{2}(\vec{r} \times \vec{B})$ and $\phi(\vec{r})$ denotes the scalar potential. The Hamiltonian is usually written as

$$H = \frac{1}{2m} \left(\vec{p} - \frac{e}{c} \vec{A} \right)^2 + e\phi(\vec{r}), \quad (46)$$

or explicitly, as

$$\begin{aligned} H = & \frac{p_x^2 + p_y^2 + p_z^2}{2m} + \frac{m\gamma^2 B_0^2}{8c^2} (x^2 + y^2) + \frac{m\gamma^2 B_1^2}{8c^2} [z^2 + (x \sin(\omega t) - y \cos(\omega t))^2] \\ & - \frac{m\gamma^2 B_0 B_1}{4c^2} z(y \sin(\omega t) - x \cos(\omega t)) + \frac{\gamma B_1}{2c} \cos(\omega t) L_x - \frac{\gamma B_1}{2c} \sin(\omega t) L_y + \frac{\gamma B_0}{2c} L_z + e\phi(\vec{r}), \end{aligned} \quad (47)$$

where $\gamma \equiv \frac{e}{m}$. Substituting $\vec{p} = \vec{\nabla}S + \frac{\hbar}{i}$ and $H = -\frac{\partial S}{\partial t}$ yields

$$\begin{aligned} & \frac{1}{2m} (\vec{\nabla}S)^2 + \frac{m\gamma^2 B_0^2}{8c^2} (x^2 + y^2) + \frac{m\gamma^2 B_1^2}{8c^2} [z^2 + (x \sin(\omega t) - y \cos(\omega t))^2] \\ & - \frac{m\gamma^2 B_0 B_1}{4c^2} z(y \sin(\omega t) - x \cos(\omega t)) + \frac{\gamma B_1}{2c} \cos(\omega t) L_x - \frac{\gamma B_1}{2c} \sin(\omega t) L_y \\ & + \frac{\gamma B_0}{2c} L_z + e\phi(\vec{r}) + \frac{\partial S}{\partial t} = \frac{i\hbar}{2m} \nabla^2 S, \end{aligned} \quad (48)$$

where the Hamilton principal function reads

$$\begin{aligned} S(x, y, z, t) = & \frac{1}{2} [\alpha_1(t)x^2 + \alpha_2(t)y^2 + \alpha_3(t)z^2] + \xi_1(t)x + \xi_2(t)y + \xi_3(t)z \\ & + \zeta_1(t)xy + \zeta_2(t)xz + \zeta_3(t)yz + \lambda_1(t) + \lambda_2(t) + \lambda_3(t). \end{aligned} \quad (49)$$

It is worth realizing that in the limit when \hbar goes to zero we obtain the respective classical Hamilton–Jacobi equation and solution. In the particular case where $B_1 = 0$, $\phi(\vec{r}) = 0$, and $\alpha_3(t) = \zeta_1(t) = \zeta_2(t) = \zeta_3(t) = 0$, we find Eq. (49) with the Hamilton–Jacobi equation. Thus, the substitution of (49) into Eq. (48) generates a polynomial equation leading to a set of first-order ordinary differential equations.

Therefore, after resolving the corresponding set of nonlinear differential equations, the quantum Hamilton principal function reads

$$S(x, y, z, t) = -\frac{m\omega \tan(\omega t + c_1)}{2}(x^2 + y^2) + \left(\frac{\sigma}{m} - c_2 \tan(\omega t + c_1)\right)x + \left(\frac{\sigma}{m} \tan(\omega t + c_1) + c_2\right)y + i\hbar \ln[\cos(\omega t + c_1)] - \frac{1}{2m} \tan(\omega t + c_1) \left[\frac{c_2^2}{\omega} + \frac{1}{\omega} \left(\frac{\sigma}{\omega}\right)^2 \right] - \frac{c_3^2 t}{2m} + c_3 z + c_4 + c_5 + c_6. \quad (50)$$

The solution consists in obtaining the quantum propagator if we impose the following boundary condition [13]:

$$S(x, y, z, 0) = \hbar k_x x + \hbar k_y y + \hbar k_z z. \quad (51)$$

Hence we obtain

$$c_1 = 0, \quad c_2 = \hbar k_y, \quad c_3 = \hbar k_z, \quad \sigma = \omega \hbar k_x, \quad c_4 + c_5 + c_6 = 0. \quad (52)$$

Now, using

$$K(x, y, z, t; \tilde{x}, \tilde{y}, \tilde{z}, 0) = (2\pi)^{-3} \int d^3k \exp \left\{ \frac{i}{\hbar} [S_k(x, y, z, t) - S(\tilde{x}, \tilde{y}, \tilde{z}, 0)] \right\}, \quad (53)$$

substituting the solution which is in accordance with the initial conditions imposed into the expression of the propagator and integrating in k , we arrive at

$$K(x, y, z, t; \tilde{x}, \tilde{y}, \tilde{z}, 0) = \left(\frac{m\omega}{2\pi i \hbar \sin(\omega t)} \right) \left(\frac{m}{2\pi i \hbar t} \right)^{1/2} \exp \left\{ \frac{im}{2\hbar} \left[\omega \cot(\omega t) \left((x - \tilde{x})^2 + (y - \tilde{y})^2 \right) + 2\omega (x\tilde{y} - \tilde{x}y) + \frac{(z - \tilde{z})^2}{t} \right] \right\}. \quad (54)$$

This leads to a two-dimensional oscillator in the xy plane and a free particle in the direction $0z$.

On the other hand, the problem of an electric charge in an oscillating pulsed magnetic field can be approached through SF. In fact, the Schrödinger equation reads

$$i\hbar \frac{\partial \psi}{\partial t} = -\vec{\mu} \cdot \vec{B}(t) \psi, \quad (55)$$

where μ is a magnetic moment and is represented, according to Ref. [39], by $\vec{\mu} = \gamma \vec{L}$, where \vec{L} represents the angular momentum. Now, we perform a rotation in the reference system where the

z axis is stationary, namely

$$\begin{aligned}x &= \bar{x} \cos(\delta t) - \bar{y} \sin(\delta t), \\y &= \bar{x} \sin(\delta t) + \bar{y} \cos(\delta t), \\z &= \bar{z}.\end{aligned}\tag{56}$$

Hence, the Schrödinger equation reads

$$i\hbar \frac{\partial \psi}{\partial \tau} = -\gamma \left[\left(B_0 + \frac{\omega}{\gamma} \right) L_{\bar{z}} + B_1 L_{\bar{x}} \right] \psi.\tag{57}$$

For an effective static field,

$$B_{\text{ef}} = \left(B_0 + \frac{\omega}{\gamma} \right) \hat{k} + B_1 \hat{i}.\tag{58}$$

Therefore, the possibility suggested by the authors of [38] is not valid for the studied system, although it is correct for a differential equation of first order.

Rewriting the expression of the magnetic field (44) only with the part oscillating in the x direction,

$$B(t) = B_1 \cos(\omega t) \hat{i},\tag{59}$$

implies that

$$\vec{A} = \frac{B_1 \cos(\omega t)}{2} \hat{k} - \frac{B_1 \cos(\omega t)}{2} \hat{j}.\tag{60}$$

Now, applying this result to Eq. (46), the Hamiltonian reads

$$H = \frac{1}{2m} \left(\vec{\nabla} p \right)^2 + \frac{m\gamma^2}{8c^2} (y^2 + z^2) B_1^2 \cos^2(\omega t) - \frac{\gamma B_1 \cos(\omega t)}{2c} L_x + e\phi.\tag{61}$$

Using the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = H \psi,\tag{62}$$

and the Hamiltonian given by (61), yields

$$\begin{aligned}-\frac{\hbar^2}{2m} \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \right) + \frac{m\gamma^2}{8c^2} (y^2 + z^2) B_1^2 \cos^2(\omega t) \psi \\ - \frac{\gamma B_1 \cos(\omega t)}{2c} L_x \psi + e\phi \psi = i\hbar \frac{\partial \psi}{\partial t}.\end{aligned}\tag{63}$$

Now we make a rotation in the coordinate system around the x axis ($x = \bar{x}$) to cancel the angular momentum operator L_x . Equation (63) then reads

$$\begin{aligned}-\frac{\hbar^2}{2m} \left(\frac{\partial^2 \psi}{\partial \bar{x}^2} + \frac{\partial^2 \psi}{\partial \bar{y}^2} + \frac{\partial^2 \psi}{\partial \bar{z}^2} \right) + \frac{m\gamma^2}{8c^2} (\bar{y}^2 + \bar{z}^2) B_1^2 \cos^2(\omega t) \psi \\ - \left[\frac{\gamma B_1 \cos(\omega t)}{2c} + \dot{\alpha} \right] L_{\bar{x}} \psi + e\phi \psi = i\hbar \frac{\partial \psi}{\partial \tau}.\end{aligned}\tag{64}$$

We choose the arbitrary angle α conveniently to guarantee that the coefficient of the term $L_{\bar{x}}$ vanishes identically, implying that

$$\dot{\alpha} = -\frac{\gamma B_1 \cos(\omega t)}{2c}. \quad (65)$$

Substituting this value into Eq. (64), we can rewrite it as

$$-\frac{\hbar^2}{2m} \left(\frac{\partial^2 \psi}{\partial \bar{x}^2} + \frac{\partial^2 \psi}{\partial \bar{y}^2} + \frac{\partial^2 \psi}{\partial \bar{z}^2} \right) + \frac{\dot{\alpha} m}{2} (\bar{y}^2 + \bar{z}^2) \psi + e\phi \psi = i\hbar \frac{\partial \psi}{\partial \tau}. \quad (66)$$

Now, taking the separation of variables

$$\psi(\bar{x}, \bar{y}, \bar{z}, \tau) = \varphi_1(\bar{x}, \tau) \varphi_2(\bar{y}, \tau) \varphi_3(\bar{z}, \tau) \quad (67)$$

yields

$$i\hbar \left(\frac{\dot{\varphi}_1}{\varphi_1} + \frac{\dot{\varphi}_2}{\varphi_2} + \frac{\dot{\varphi}_3}{\varphi_3} \right) = -\frac{\hbar^2}{2m} \left(\frac{1}{\varphi_1} \frac{\partial^2 \varphi_1}{\partial \bar{x}^2} + \frac{1}{\varphi_2} \frac{\partial^2 \varphi_2}{\partial \bar{y}^2} + \frac{1}{\varphi_3} \frac{\partial^2 \varphi_3}{\partial \bar{z}^2} \right) + \frac{\dot{\alpha}^2 m}{2} (\bar{y}^2 + \bar{z}^2) + e\phi. \quad (68)$$

Making $\phi(\bar{x}, \bar{y}, \bar{z}) = \phi_1(\bar{x}) + \phi(\bar{y}) + \phi(\bar{z}) = 0$, and organizing the terms, it follows that

$$\left[\frac{\hbar^2}{2m} \left(\frac{\partial^2 \varphi_1}{\partial \bar{x}^2} \right) + i\hbar \dot{\varphi}_1 \right] \frac{1}{\varphi_1} = Q_1, \quad (69)$$

$$\left[\frac{\hbar^2}{2m} \left(\frac{\partial^2 \varphi_2}{\partial \bar{y}^2} \right) + i\hbar \dot{\varphi}_2 \right] \frac{1}{\varphi_2} - \frac{\dot{\alpha}^2 m}{2} \bar{y}^2 = Q_2, \quad (70)$$

$$\left[\frac{\hbar^2}{2m} \left(\frac{\partial^2 \varphi_3}{\partial \bar{z}^2} \right) + i\hbar \dot{\varphi}_3 \right] \frac{1}{\varphi_3} - \frac{\dot{\alpha}^2 m}{2} \bar{z}^2 = Q_3. \quad (71)$$

By writing

$$\varphi_3(\bar{z}, \tau) = \chi_3(\bar{z}, \tau) \exp\left(-\frac{iQ_3\tau}{\hbar}\right), \quad (72)$$

Eq. (71) reads

$$-\frac{\hbar^2}{2m} \left(\frac{\partial^2 \chi_3}{\partial \bar{z}^2} \right) + \frac{\dot{\alpha}^2 m \bar{z}^2}{2} \chi_3 = i\hbar \frac{\partial \chi_3}{\partial \tau}. \quad (73)$$

Moreover, by performing the transformation

$$\chi_3(\bar{z}, \tau) \mapsto \chi_3(\tilde{z}, T), \bar{z} = s(T)\tilde{z}, \quad (74)$$

one obtains

$$-\frac{\hbar^2}{2m} \left(\frac{\partial^2 \chi_3}{\partial \tilde{z}^2} \right) + \frac{m\dot{\alpha}^2 s^2 \tilde{z}^2}{2} \chi_3 = i\hbar \mu \left(\frac{\partial \chi_3}{\partial T} - \frac{\dot{s}}{s} \tilde{z} \frac{\partial \chi_3}{\partial \tilde{z}} \right), \quad (75)$$

where $\mu = \frac{dT}{d\tau}$.

We redefine

$$\chi_3(\tilde{z}, T) = \sigma_3(\tilde{z}, T) \exp[i f(\tilde{z}, T)], \quad (76)$$

which, once substituted in the previous equation, yields

$$\left\{ i\mu \frac{\partial}{\partial T} + \frac{1}{2ms^2} \frac{\partial^2}{\partial \tilde{z}^2} - \frac{m\dot{\alpha}^2 s^2 \tilde{z}^2}{2} + \frac{1}{2ms^2} \left[i \frac{\partial^2 f}{\partial \tilde{z}^2} - \left(\frac{\partial f}{\partial \tilde{z}} \right)^2 \right] \right. \\ \left. + \mu \hbar \frac{\dot{s}}{s} \tilde{z} \frac{\partial f}{\partial \tilde{z}} - \mu \hbar \dot{f} \right\} \sigma + \left\{ \frac{\hbar}{2ms^2} \left(2i \frac{\partial f}{\partial \tilde{z}} \right) - i\mu \frac{\dot{s}}{s} \tilde{z} \right\} \frac{\partial \sigma}{\partial \tilde{z}} = 0. \quad (77)$$

In addition, making $\frac{\partial f}{\partial \tilde{z}} = m\mu s \dot{s} \tilde{z}$ implies that

$$f(\tilde{z}, T) = \frac{m\mu s \dot{s} \tilde{z}^2}{2} + f_T(T), \quad (78)$$

where $f_T(T)$ is an arbitrary function of the rescaled time T . Substituting this function into Eq. (77), one can rewrite it as

$$\left\{ i\mu \frac{\partial}{\partial T} + \frac{1}{2ms^2} \frac{\partial^2}{\partial \tilde{z}^2} - \frac{\mu}{2} \left[\frac{m\dot{\alpha}^2 s^2}{\mu} - \mu m \dot{s}^2 + \frac{d}{dT} (m\mu s \dot{s}) \right] \tilde{z}^2 + \frac{i}{2} \mu \frac{\dot{s}}{s} - \mu \dot{f}_T \right\} \sigma = 0. \quad (79)$$

Now, we choose the arbitrary function $f_T(T)$ to guarantee that the two last terms on the left-hand side above are eliminated, by setting

$$\frac{df}{dT} = \frac{i\dot{s}}{2s}. \quad (80)$$

Integrating this equation yields

$$f_T(T) = i \ln s^{1/2}. \quad (81)$$

On the other hand, defining

$$\Omega^2 \equiv \frac{\mu}{ms^2} \frac{d}{dT} (m\mu s \dot{s}) - \mu^2 \left(\frac{\dot{s}}{s} \right)^2, \quad (82)$$

and substituting Eqs. (81) and (82) into Eq. (79), a compact form is achieved:

$$\mu \left\{ i \frac{\partial}{\partial T} + \frac{1}{2\mu ms^2} \frac{\partial^2}{\partial \tilde{z}^2} - \frac{ms^2}{2\mu} [\dot{\alpha}^2 + \Omega^2] \tilde{z}^2 \right\} \sigma(\tilde{z}, T) = 0. \quad (83)$$

Now, making the identification

$$m_0 \equiv ms^2 \mu = \text{const}, \quad \frac{ms^2}{\mu} (\dot{\alpha}^2 + \Omega^2) \equiv m_0 \omega_0^2, \quad (84)$$

with $m = m_0$, and substituting these values into (83), we obtain

$$\mu \left(i \frac{\partial}{\partial T} + \frac{1}{2m_0} \frac{\partial^2}{\partial \tilde{z}^2} - \frac{1}{2} m_0 \omega_0^2 \tilde{z}^2 \right) \sigma(\tilde{z}, T) = 0. \quad (85)$$

For this, we make the transformation $s = v^{-1}$ in (85) so that it can be rewritten as

$$\ddot{v} + \omega_0^2 v = \frac{\dot{\alpha}^2}{v^3}, \quad (86)$$

and consequently we get

$$\ddot{v} + \xi^2 v = 0, \quad (87)$$

where $\xi^2 = \left(\omega_0^2 - \frac{\dot{\alpha}^2}{\mu^2}\right)$. In this form the problem has been transformed into a classical harmonic oscillator with time-dependent frequency. We can particularize this problem by requiring that $\xi = \text{const}$, thus obtaining the solution

$$v = A \cos(\eta T + \delta), \quad (88)$$

so that

$$\mu = \dot{\alpha}(\omega_0^2 - \eta^2)^{-1/2}, \quad (89)$$

and

$$s = A^{-1} \sec(\eta T + \delta). \quad (90)$$

Therefore, it is easy to check that the conditions (84) are true and, therefore, the problem is reduced when a particular case is required.

7. Conclusion

We have studied classical and quantum solutions for harmonic oscillator-like systems, further encompassing the driven case and with resonances as well, by using the Hamilton–Jacobi method. For the quantum case, the propagator allows the study of the time evolution of the system, if we take into account the Hamilton principal function with a linear term. This term is shown to be essential to obtain the respective quantum propagators of the systems studied. Therefore, it can be verified that the Hamilton–Jacobi quantum formalism is an alternative version for the quantum mechanical formulation, obtaining the classical limit when $\hbar \rightarrow 0$.

After that, we computed, through this approach, the propagator for an electric charge in a oscillating magnetic field. Since we observed that the Schrödinger approach to this problem in the literature presents a technical flaw, we computed its solutions also through the SF.

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