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Immersions in the metastable dimension range via the normal bordism approach

C. Biasi [a,*], D.L. Gonçalves [b,1], A.K.M. Libardi [c]

[a] *Departamento de Matemática, ICMC-USP, Caixa Postal 668, 13560-970, São Carlos, SP, Brazil*
[b] *IME-USP, Caixa Postal 66281, Agência Cidade de S. Paulo, 05315-970, São Paulo, SP, Brazil*
[c] *Departamento de Matemática, IGCE-UNESP, Bela Vista, 13506-700, Rio Claro, SP, Brazil*

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Abstract

Let $f : M \to N$ be a continuous map between two closed n-manifolds such that $f_* : H_*(M, \mathbb{Z}_2) \to H_*(N, \mathbb{Z}_2)$ is an isomorphism. Suppose that M immerses in $\mathbb{R}^{n+k}$ for $5 \leqslant n < 2k$. Then N also immerses in $\mathbb{R}^{n+k}$. We use techniques of normal bordism theory to prove this result and we show that for a large family of spaces we can replace the homology condition by the corresponding one in homotopy. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper we are concerned with the following immersion problem:

Let M and N be closed smooth connected n-dimensional manifolds and let $f : M \to N$ be a continuous map. Suppose that M immerses in $\mathbb{R}^{n+k}$, for some k, with $5 \leqslant n < 2k$. Under which conditions on f does N immerse in $\mathbb{R}^{n+k}$?

For the special case where f is a homotopy equivalence between M and N, if M immerses in $\mathbb{R}^{n+k}$ for some $k \geqslant [n/2] + 2$, Glover and Mislin in [5] proved that N also immerses in $\mathbb{R}^{n+k}$. They also proved that in the case where M and N are connected simple[2] orientable and closed smooth manifolds and the 2-localizations M_2 and N_2 are

* Corresponding author.
E-mail addresses: biasi@icmc.sc.usp.br (C. Biasi), dlgoncal@ime.usp.br (D.L. Gonçalves), alicekml@ms.rc.unesp.br (A.K.M. Libardi).

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2 A manifold X is simple if $\pi_1(X)$ acts trivially on $\pi_*(X)$.

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homotopy equivalent, if M immerses in $\mathbb{R}^{n+k}$ for some $k \geqslant [n/2]+1$, then N immerses in $\mathbb{R}^{n+2[k/2]+1}$.

Later, Glover and Homer in [3,4] proved the same result under weaker hypotheses on M and N, for example, nilpotent manifolds. In fact, in [3] they considered the more general problem where M immerses in another manifold.

In our work, the manifolds M and N are not required to be nilpotent. Concerning the dimension of the space that M and N immerse into, our results are improvement of the results in [5] by 1, and are the same as the ones from [4]. Nevertheless we do not require $H^k(M,\mathbb{Q})=0$, as well k be an odd number, as it is required in [4]. We use a normal bordism approach to investigate this problem. We prove the following main results:

Theorem A. *Let M and N be closed smooth connected n-manifolds and let $f: M \to N$ be a continuous map such that*

$$f_*: H_i(M,\mathbb{Z}_2) \to H_i(N,\mathbb{Z}_2)$$

is an isomorphism for $i \geqslant 0$.

Then if M immerses in $\mathbb{R}^{n+k}$ for $5 \leqslant n < 2k$, so does N.

Theorem B. *Let $f: \widetilde{M} \to M$ be a finite regular covering of a closed connected smooth n-manifold M.*

Suppose $\pi = \pi_1(M)/f_\#\pi_1(\widetilde{M})$ is of odd order and operates trivially on $H_*(\widetilde{M},\mathbb{Z}_2)$. Then if $\widetilde{M}$ immerses in $\mathbb{R}^{n+k}$ for $5 \leqslant n < 2k$, so does M.

In the next theorem, for $f: M \to N$, $\pi_i(N,M)$ means $\pi_i(Z_f, M)$, where Z_f is the mapping cylinder of f.

Theorem C. *Let M and N be closed smooth connected n-manifolds, $f: M \to N$ a continuous map and $H = f_\#\pi_1(N)$. Suppose the inclusion $i: H \to \pi_1(N)$ induces isomorphisms in homology with $\mathbb{Z}_2$ coefficients, the index $[\pi_1(N), H]$ is finite and odd, and $\pi_i(N,M)$ is an odd torsion group for $1 < i \leqslant n$. Then if M immerses in $\mathbb{R}^{n+k}$ for $5 \leqslant n < 2k$, so does N.*

The work is divided in four sections. In Section 2 we review the normal bordism theory and in Section 3 we prove Theorems A and B. In Section 4 we show that, for a family of spaces, larger than the family of nilpotent spaces, we can replace the homology hypotheses given in Theorem A by the corresponding one in homotopy. Finally, in Section 5 we analyze algebraic conditions in terms of the fundamental group of the spaces, in order to have an isomorphism of homology with $\mathbb{Z}_2$ coefficients and then we prove Theorem C. We finish the section by giving an example where we have an isomorphism in homotopy modulo finite odd groups, but not an isomorphism of homology with $\mathbb{Z}_2$ coefficients.

2. Normal bordism

Given a topological space X and a virtual bundle ϕ, $\Omega_i(X,\phi)$ denotes the ith normal bordism group of X with coefficient ϕ. We adopt the Salomonsen convention.

Let M be a closed smooth n-manifold and X a connected smooth $(n+k)$-manifold.

Let us consider $h : M \to X$ a map with $\dim \nu = k+\ell$, for ℓ large. Here ν denotes the stable normal bundle of h.

Then, by Hirsch [9], h is homotopic to an immersion if and only if there is a monomorphism from $M\times\mathbb{R}^\ell$ to ν or equivalently geom $\dim(\nu)\leqslant k$.

In [13,14], Koschorke defines an invariant $\omega_k(\nu)\in\Omega_{n-k-1}(M\times P^\infty,\phi)$ which is an obstruction to the existence of a monomorphism from $M\times\mathbb{R}^\ell$ into ν.

We recall that $\phi=\lambda\otimes(\nu-\varepsilon^\ell)-TM$ is a virtual vector bundle over $M\times P^\infty$, where λ denotes the canonical line bundle over the real projective space P^∞. For the definition and more details about normal bordism see [12] or [17].

So geom $\dim(\nu)\leqslant k$ if and only if $\omega_k(\nu)=0$, provided $n<2k$. See [13,14].

Salomonsen defines a fibre bundle $\pi_M:\widetilde{V}_k(\Psi)\to M$ such that the existence of a cross section $s:M\to\widetilde{V}_k(\Psi)$ implies that geom $\dim(\Psi)\leqslant k$. Here Ψ is a virtual bundle over M. In order to study whether cross sections exist we consider $(\pi_M)_*:\Omega_n(\widetilde{V}_k(\Psi),TM^0)\to\Omega_n(M,TM^0)$, where TM^0 is the virtual vector bundle $TM-\varepsilon^n$. We note that if geom $\dim(\Psi)\leqslant k$ then $(\pi_M)_*$ is onto. We can also consider the following exact sequence, for $5\leqslant n<2k$ [17].

$$\cdots\to\Omega_n\big(\widetilde{V}_k(\Psi),TM^0\big)\xrightarrow{(\pi_M)_*}\Omega_n\big(M,TM^0\big)\xrightarrow{\gamma_M}\Omega_{n-k-1}\big(M\times P^\infty,\phi\big)\to\cdots. \quad\text{(I)}$$

The mapping γ_M is defined by the construction of the sequence and

$$\phi=-(n-k-1)\lambda-\lambda\otimes\Psi+TM^0.$$

Let $[M]=[M,1_M,t_M]\in\Omega_n(M,TM^0)$ be the fundamental class of M where

$$t_M:TM\oplus\varepsilon^n\cong\varepsilon^n\oplus TM$$

is the isomorphism which interchange factors. If we consider $\Psi=h^\star TX-\varepsilon^k\oplus TM$, then we have that $\gamma_M[M]=\omega_k(\nu)$.

For the next lemma we recall that $F(q)$ is the monoid of degree 1 or -1 pointed maps of S^q and $F=\bigcup F(q)$. For a connected finite CW complex X, let $\alpha\in[X,BF]$ be a stable bundle over X and we define α_p as the composition of α and the canonical map $BF\to(BF)_p$, where $(BF)_p$ is the p-localization of BF (p prime or 0). The map α represents a fibre bundle $\pi:E\to M$ with fibre S^r, r large, and α_p represents the fibre bundle $(\pi)_p:(E)_p\to(M)_p$ with fibre $(S^r)_p$. The Thom complexes $T(\alpha)$ and $T(\alpha_p)$ are respectively the mapping cones of π and π_p.

Let $\mathcal{C}$ denote the class of all torsion groups where the torsion is odd.

Lemma 2.1. *Let $f:M\to N$ be a mapping between closed connected smooth n-manifolds such that*

$$f_*:H_i(M,\mathbb{Z}_2)\to H_i(N,\mathbb{Z}_2)$$

is an isomorphism for $i \geqslant 0$.

Then we have:

(1) $f_* : \Omega_n(M, f^*TN^0) \to \Omega_n(N, TN^0)$ *is a* $\mathcal{C}$*-isomorphism.*

(2) $f^*(\beta_2) = \alpha_2$, *where* $\alpha = \nu_M$ *and* $\beta = \nu_N$ *are the stable normal bundles over* M *and* N *respectively.*

Proof. The first part of this lemma is a special case of Lemma 2 in [17].

For the second part, let us define $\theta \in [M, BF]$ by $\theta_p = \alpha_p$, $p \neq 2$, and $\theta_2 = f^*(\beta_2)$. We observe that for $p \neq 2$, $T(\theta_p)$ is S-reducible, because $\alpha_p = (\nu_M)_p$.

We now use the same techniques of [5] in the proof of their Proposition 4.1, but applied to the Thom complexes. We point out that since f_* is an isomorphism for $i \geqslant 0$, we have homotopy equivalences between the Thom complexes $(T(f^*\beta))_2$ and $(T(\beta))_2$. But $(T(f^*\beta))_2 = (T(f^*\beta_2))_2 = (T(\theta_2))_2 = (T(\theta))_2$ and since $T(\beta)$ is S- reducible, it follows that $T(\theta)$ is S-reducible at 2. Therefore $T(\theta)$ is S-reducible and by Proposition 5.6 [18], we have that $\theta = \alpha$. □

3. Proofs of Theorems A and B

Proof of Theorem A. Let us consider the following commutative diagram

$$\begin{tikzcd}
{} \arrow[d] & & {} \arrow[d] \\
\Omega_n(\widetilde{V}_k(\Psi_M), f^*TN^0) \arrow[rr, "G_*"] \arrow[d, "(\pi'_M)_*"] & & \Omega_n(\widetilde{V}_k(\Psi_N), TN^0) \arrow[d, "(\pi_N)_*"] \\
\Omega_n(M, f^*TN^0) \arrow[rr, "f_*"] \arrow[d, "\gamma_M"] & & \Omega_n(N, TN^0) \arrow[d, "\gamma_N"] \\
\Omega_{n-k-1}(M \times P^\infty, \phi_M) \arrow[rr, "F_*"] \arrow[d] & & \Omega_n(N \times P^\infty, \phi_N) \arrow[d] \\
{} & & {}
\end{tikzcd}$$

where the vertical exact sequences are obtained from (I) and G_* and F_* are given in [17], $\Psi_M = \varepsilon^{n+k} - TM \oplus \varepsilon^k$ and $\Psi_N = \varepsilon^{n+k} - TN \oplus \varepsilon^k$. We observe that $(\pi'_M)_*$ is the induced map of π_M in normal bordism groups with virtual bundle f^*TN^0.

Since f_* is a $\mathcal{C}$-isomorphism by Lemma 2.1, there exists an odd number ℓ_1 such that

$$\ell_1 \cdot [N] = f_*(x), \quad \text{for some } x \in \Omega_n\bigl(M, f^*TN^0\bigr).$$

If we prove that $(\pi'_M)_*$ is a $\mathcal{C}$-epimorphism, then there exists an odd number ℓ_2 such that $\ell_2 \cdot x = (\pi'_M)_*(y)$ where $y \in \Omega_n(\widetilde{V}_k(\Psi_M), TM^0)$.

Using the commutative diagram we have that $(\pi_N)_* G_*(y) = f_*(\pi'_M)_*(y) = f_*(\ell_2 \cdot x) = \ell_2 \cdot \ell_1[N]$ and so $\ell_2 \cdot \ell_1 \cdot \gamma_N[N] = 0$.

Since the elements of the image of γ_N have order a power of 2, by Theorem 10.2 in [17], and $\ell_2 \cdot \ell_1$ is an odd number, $\gamma_N[N] = 0$ and N immerses in $\mathbb{R}^{n+k}$.

Thus, all we need to prove is that if M immerses in $\mathbb{R}^{n+k}$, then $(\pi'_M)_*$ is a $\mathcal{C}$-epimorphism and this is what we will do.

We recall that $\alpha = \nu_M$ and $\beta = \nu_N$, as in the Lemma 2.1.

We use the following commutative diagram, equivalent to the top commutative diagram above,

$$\begin{tikzcd}
\Pi^S_{n+p}(T(f^*\widehat{\beta})) \arrow[r, "G_*"] \arrow[d, "(\pi'_M)_*"] & \Pi^S_{n+p}(T(\widehat{\beta})) \arrow[d, "(\pi_N)_*"] \\
\Pi^S_{n+p}(T(f^*\beta)) \arrow[r, "f_*"] & \Pi^S_{n+p}(T(\beta))
\end{tikzcd}$$

where $\widehat{\beta}$ denotes the pull back of β by π_N.

We observe that

$$\big(T(f^*\beta)\big)_2 = \big(T(f^*\beta_2)\big)_2 = \big(T(\alpha_2)\big)_2 = \big(T(\alpha)\big)_2,$$

where the second equality is given by Lemma 2.1. We also have that $(T(f^*\widehat{\beta}))_2 = (T(\widehat{\alpha}))_2$, where $\widehat{\alpha}$ denotes the pullback of α by π_M.

Now, if M immerses in $\mathbb{R}^{n+k}$ then

$$(\pi_M)_* : \Omega_n\big(\widetilde{V}_k(\Psi_M), TM^0\big) \to \Omega_n\big(M, TM^0\big)$$

is an epimorphism and then

$$\Pi^S_{n+p}\big((T(\widehat{\alpha}))_2\big) \to \Pi^S_{n+p}\big((T(\alpha))_2\big)$$

is a $\mathcal{C}$-epimorphism or equivalently $\Pi^S_{n+p}((Tf^*\widehat{\beta})_2) \to \Pi^S_{n+p}((Tf^*\beta)_2)$ is a $\mathcal{C}$-epimorphism and the result follows. □

Proof of Theorem B. If π has odd order and since the coefficient group of the homology group is $\mathbb{Z}_2$, the divisibility condition of Theorem 10.8.8 in [11] is satisfied.

Therefore $H_*(\widetilde{M}, \mathbb{Z}_2) \approx H_*(M, \mathbb{Z}_2)$ and the result follows by Theorem A. □

4. Applications

Throughout this paragraph we will consider M, N connected closed and smooth n-manifolds and $f : M \to N$ a continuous map.

We consider the following cases:

(1) f is a homotopy equivalence.

(2) M and N are nilpotent orientable and f is such that the induced $f_2 : M_2 \to N_2$, from the 2-localizations M_2 of M into N_2 of N, is a homotopy equivalence.

(3) M and N are $\mathcal{C}$-nilpotent manifolds and f is such that

$$f_{\#1} : \pi_1(M) \to \pi_1(N) \text{ is a } \mathcal{C}\text{-epimorphism}$$

and

$$f_{\#i} : \pi_i(M) \to \pi_i(N) \text{ is a } \mathcal{C}\text{-isomorphism}$$

for $i < n$ and a $\mathcal{C}$-epimorphism for $i = n$. For details about nilpotent spaces see [10] and for $\mathcal{C}$-nilpotent spaces see [7].

(4) $f : M \to N$ is such that

$$f_{\#1} : \pi_1(M) \to \pi_1(N) \text{ is an epimorphism}$$

and

$$f_{\#i} : \pi_i(M) \to \pi_i(N) \text{ is a } \mathcal{C}\text{-isomorphism for } i < n$$
$$\text{and a } \mathcal{C}\text{-epimorphism for } i = n.$$

(5) f is an orientation true map (i.e., the loops α and $f_{\#1}(\alpha)$ have the same sign for all $\alpha \in \pi_1(M)$) such that

$$f_* : H_i(M, \widetilde{\mathbf{Z}}) \to H_i(N, \widetilde{\mathbf{Z}})$$

is an isomorphism for $i \geqslant 0$, where $\widetilde{\mathbf{Z}}$ denotes the twisted integer coefficients over M respectively N, associated to $w_1(M)$ respectively $w_1(N)$.

In all the cases above we have that if M immerses in $\mathbb{R}^{n+k}$ for $5 \leqslant n < 2k$, so does N. Under the hypotheses of cases (1) or (2), the result follows immediately from Theorem A.

For the third and the fourth cases, we use the Proposition 3.5 in [7] and Theorem 1.1 in [2] respectively in order to show that $f_* : H_i(M, \mathbb{Z}_2) \to H_i(N, \mathbb{Z}_2)$ is an isomorphism for $i \geqslant 0$. So, by Theorem A, the result follows.

The last case follows from Theorem 3.5 in [1].

Theorem B has an interesting consequence:

Corollary 4.1. *If $H_i(M, \mathbb{Z}_2) = 0$ or $\mathbb{Z}_2$, for all i, and if G is an odd order group which acts freely on M, then if M immerses in $\mathbb{R}^{n+k}$ for $5 \leqslant n < 2k$, so does M/G.*

5. The mod 2 homology isomorphism condition

In this section we will assume that $f_{i\#} \cdot \pi_i(M) \to \pi_i(N)$ satisfies the hypotheses: $f_{i\#}$ is a $\mathcal{C}$-isomorphism for $1 < i < n$, $\mathcal{C}$-epimorphism for $i = n$, $\mathcal{C}$-injective for $i = 1$ and the index $[\pi_1(N), f_{1\#}(\pi_1(M))]$ is finite and odd. We will study the question of how to decide when a map $f : M \to N$ induces an isomorphism $f_{i*} : H_i(M, \mathbb{Z}_2) \to H_i(N, \mathbb{Z}_2)$ for $0 \leqslant i \leqslant n$.

We will also provide an example where this does not happen. As before, $\mathcal{C}$ stands for the class of all torsion groups where the torsion is odd.

Proposition 5.1. *Given $f: M \to N$ as above, there is a finite cover $p: \overline{N} \to N$ and a lifting*

$$\begin{tikzcd} & \overline{N} \arrow[d] \\ M \arrow[ur, "\overline{f}"] \arrow[r, "f"'] & N \end{tikzcd}$$

such that $\overline{f}_{*i}: H_i(M, \mathbb{Z}_2) \to H_i(\overline{N}, \mathbb{Z}_2)$ is an isomorphism for $i \geqslant 0$.

Proof. Let $\overline{N}$ be the cover which corresponds to the subgroup $f_{1\#}(\pi_1(M))$ and $\overline{f}$ a lift of f. By Theorem 1.1 of [2] the result follows. □

Corollary 5.2. *Suppose that $f_\#(\pi_1(M))$ is a normal subgroup of $\pi_1(N)$. If $H_i(M, \mathbb{Z}_2) = 0$ or $\mathbb{Z}_2$, for all i, then if M immerses in $\mathbb{R}^{n+k}$ for $5 \leqslant n < 2k$, so does N.*

Proof. The normal subgroup condition implies that the covering is regular [11]. The result follows now from Proposition 5.1 and Theorem B. □

For the next proposition let us consider the commutative diagram

$$\begin{tikzcd} \widetilde{\overline{N}} \arrow[r, "\widetilde{p}"] \arrow[d] & \widetilde{N} \arrow[d] \\ \overline{N} \arrow[r, "p"] \arrow[d] & N \arrow[d] \\ K(\pi_1(\overline{N}), 1) \arrow[r, "p'"] & K(\pi_1(N), 1) \end{tikzcd}$$

where $\widetilde{\overline{N}}$, $\widetilde{N}$ are the universal covers of $\overline{N}$, N respectively, and $K(\pi, 1)$ are Eilenberg–MacLane spaces.

Proposition 5.3. *The induced homomorphisms $p_{i*}: H_i(\overline{N}, \mathbb{Z}_2) \to H_i(N, \mathbb{Z}_2)$ are isomorphisms for all $i \geqslant 0$ if $p'_{i*}: H_i(\pi_1(\overline{N}), \mathbb{Z}_2) \to H_i(\pi_1(N), \mathbb{Z}_2)$ is an isomorphism for all $i \geqslant 0$, where H_* means group homology.*

Proof. Since $\widetilde{\overline{N}}$ and $\widetilde{N}$ are simply connected and $\widetilde{p}_\#: \pi_i(\widetilde{\overline{N}}) \to \pi_i(\widetilde{N})$ is a $\mathcal{C}$-isomorphism for $i \geqslant 0$, we have that $\widetilde{p}_{i*}: H_i(\widetilde{\overline{N}}, \mathbb{Z}_2) \to H_i(\widetilde{N}, \mathbb{Z}_2)$ is an isomorphism. Now we consider the Serre spectral sequence of the fibrations and the induced map. The result follows from the spectral mapping theorem. □

Remark. The Corollary 5.2 generalizes the Proposition 10.8.8 of [11].

From now on let us denote $H = \pi_1(\overline{N})$, $G = \pi_1(N)$ and $j : H \mapsto G$ the inclusion. We have that the index $[G, H]$ is odd.

Proposition 5.4. *The homomorphism $j_{i*} : H_i(H, \mathbb{Z}_2) \to H_i(G, \mathbb{Z}_2)$ is an epimorphism for $i \geqslant 0$.*

Proof. It is equivalent to show that the induced map in cohomology is injective. For this we consider the composite of the corestriction with the restriction (see Chapter II, Sections 2.3 and 2.4 in [19]) $\tau : H^i(G, \mathbb{Z}_2) \to H^i(H, \mathbb{Z}_2) \to H^i(G, \mathbb{Z}_2)$, which is multiplication by $\ell = [G, H]$. Since ℓ is odd this is an isomorphism. So it follows that the first map is injective and hence the result. □

Corollary 5.5. *If G is finite of odd order, then $H_i(G, \mathbb{Z}_2) = 0$, $i > 0$.*

Proof. We take H the trivial group. By Proposition 5.3 $H_i(G, \mathbb{Z}_2) = 0$, $i > 0$. □

Proof of Theorem C. The result follows directly from Propositions 5.1, 5.2 and Theorem A. □

In general one can not expect to have $j_{i*} : H_i(H, \mathbb{Z}_2) \to H_i(G, \mathbb{Z}_2)$ an isomorphism for all $i > 0$.

Example 5.6 (*Algebraic*). Let us consider the following extension

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \to G \to \mathbb{Z}_3 \to 0$$

where the action $\omega : \mathbb{Z}_3 \to \mathrm{Aut}(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z})$ is given by

$$\omega(1) = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the extension corresponds to a nontrivial element of $H^2(\mathbb{Z}_3, \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}) \approx \mathbb{Z}_3$. By using Chapter IV, Section 7 [15], it is a standard calculation that $H^2(\mathbb{Z}_3, \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}) \simeq \mathbb{Z}_3$. So let $H = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.

We will use the Lyndon–Hochschild–Serre spectral sequence, in order to compute $H_*(G, \mathbb{Z}_2)$. The E^2 term is $E^2_{p,q} = H_p(\mathbb{Z}_3, H_q(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}; \mathbb{Z}_2))$.

So

$$E^2_{p,q} = \begin{cases} 0 & q \geqslant 4; \\ H_p(\mathbb{Z}_3, \mathbb{Z}_2) & q = 0, 3; \\ H_p(\mathbb{Z}_3, \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2) & q = 1; \\ H_p(\mathbb{Z}_3, \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2) & q = 2. \end{cases}$$

Where the local coefficient is given by

$$\omega_1(1) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \omega_2(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

for the cases $q = 1$, $q = 2$, respectively.

Again, using [15] Chapter IV, Section 7, we get

$$H_p(\mathbb{Z}_3, \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & p = 0, \\ 0 & p \neq 0, \end{cases}$$

where the action ω is either ω_1 or ω_2. So we get

$$H_*(G, \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 \quad * = 0, 3; \\ H_1(H\ , \mathbb{Z}_2, \oplus\mathbb{Z}_2 \oplus \mathbb{Z}_2) \approx \mathbb{Z}_2; \\ H_2(H\ , \mathbb{Z}_2, \oplus\mathbb{Z}_2 \oplus \mathbb{Z}_2) \approx \mathbb{Z}_2. \end{cases}$$

Remarks.

- It is not difficult to see that the group G is not nilpotent but it is certainly infra-abelian.
- Let $\omega : Q \to SL(n, \mathbb{Z})$ be an action. It is natural to look for weaker hypothesis than the one that ω is trivial which still implies that $H_i(\underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{n}, \mathbb{Z}_2) \to H_i(G, \mathbb{Z}_2)$ is an isomorphism for all i, where G is any extension of $\underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{n}$ by $Q \in \mathcal{C}$ with action ω. The condition that the reduced action $\omega_2 : Q \to SL(n, \mathbb{Z}_2)$ is nilpotent, implies such isomorphism. Unfortunately this condition is equivalent to the condition that ω is trivial. For, either by Proposition 5 in [6] or the argument given in the proof of Proposition 1, we have that ω_2 nilpotent implies necessarily that ω_2 is trivial, since every element of Q is torsion of odd order. Using now Theorem IX.7 in [16], we conclude that the triviality of ω follows from the one of ω_2.

Example 5.7 (*Geometric*). Finally we will show how to realize geometrically the Example 5.6. Let us consider any action $\omega : Q \to Aut(\underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{n})$. Consider the action of Q on $\mathbb{R}^n$, $Q \times \mathbb{R}^n \to \mathbb{R}^n$ given by the linear transformation which have matrices $\omega(Q)$. This action certainly induces an action on the torus $T^n = \underbrace{S^1 \times \cdots \times S^1}_{n}$, which we also denote by ω. Now let M be a manifold where Q acts freely. So there is a free action $Q \times T^n \times M \to T^n \times M$ given by $(q, x, y) \to (q \cdot x, q \cdot y)$. Therefore, we have the map $T^n \times M \to (T^n \times M)/Q$, where $(T^n \times M)/Q$ is the orbit space, which is a finite regular cover of a compact manifold.

So we get a short exact sequence

$$1 \to \pi_1(T^n \times M) \to \pi_1\left(\frac{T^n \times M}{Q}\right) \to Q \to 1$$

which is

$$1 \to \mathbb{Z}^n \oplus \pi_1(M) \to G \to Q \to 1$$

where $G = \pi_1((T^n \times M)/Q)$. Now we apply the above procedure to Example 5.6. Let ω be the action given by the upper left corner 2×2 submatrix of the 3×3 matrix given in Example 5.6. Let M be the circle S^1 where Z_3 acts freely by rotating 120 degrees. So we get a finite cover $T^3 \to N$ where the three manifold N has homology given by $H_i(N, \mathbb{Z}_2) \approx \mathbb{Z}_2$ for $3 \geqslant i \geqslant 0$. Certainly $H_i(T^3, \mathbb{Z}_2) \to H_i(N, \mathbb{Z}_2)$ is not an isomorphism for all i and the result follows. Observe that $H^i(N, \mathbb{Z}_2) \approx \mathbb{Z}_2$ for $3 \geqslant i \geqslant 0$ which is the same as the $H^*(RP^3, \mathbb{Z}_2)$, where RP^3 is the 3-projective space. It is not hard to show that the cohomology ring structures of these two spaces are different.

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