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J. Borrego-Morell & A. Sri Ranga

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RESEARCH ARTICLE

Orthogonal polynomials on the unit circle satisfying a second-order differential equation with varying polynomial coefficients

J. Borrego-Morell[a] and A. Sri Ranga[b]

[a]Departamento de Matemática, Universidade Federal do Rio de Janeiro, Polo de Xerém, Rio de janeiro, Brazil; [b]Departamento de Matemática Aplicada, Universidade Estadual Paulista, Campus São José do Rio Preto, São Paulo, Brazil

ABSTRACT

Consider the linear second-order differential equation

$$A_n(z)y'' + B_n(z)y' + C_n y = 0, \tag{1.1}$$

where $A_n(z) = a_{2,n}z^2 + a_{1,n}z + a_{0,n}$ with $a_{2,n} \neq 0$, $a_{1,n}^2 - 4a_{2,n}a_{0,n} \neq 0$, $\forall n \in \mathbb{N}$ or $a_{2,n} = 0$, $a_{1,n} \neq 0$, $\forall n \in \mathbb{N}$, $B_n(z) = b_{1,n} + b_{0,n}z$ are polynomials with complex coefficients and $C_n \in \mathbb{C}$. Under some assumptions over a certain class of lowering and raising operators, we show that for a sequence of polynomials $(\phi_n)_{n=0}^{\infty}$ orthogonal on the unit circle to satisfy the differential equation (1.1), the polynomial ϕ_n must be of a specific form involving and extension of the Gauss and confluent hypergeometric series.

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1. Introduction

The Bochner classification theorem [1] (also given by Routh in 1885, [2]) characterizes, under a complex linear change of the variable z, the sequences $(y_n)_{n=0}^{\infty}$ of orthogonal polynomials with respect to a positive Borel measure having finite moments of all orders that simultaneously solve a second-order differential of the form

$$A(z)y'' + B(z)y' + C_n y = 0,$$

where A,B are polynomials of degree 2 and 1, respectively, $C_n \in \mathbb{C}$. Such sequences of polynomials turn out to be the classical families of orthogonal polynomials Laguerre, Jacobi and Hermite.

Askey [3] introduced the two-parameter system $\{R_n, S_n\}_{n\geq 0}$ of polynomials given by

$$\begin{aligned} R_n(z;\alpha,\beta) &= {}_2F_1(-n, \alpha+\beta+1; \beta-\alpha+1-n; z), \\ S_n(z;\alpha,\beta) &= R_n(z;\alpha,-\beta), \end{aligned} \tag{1.2}$$

and pointed out that this system is biorthogonal with respect to the complex-valued weight of beta type $\omega(\theta) = (1-e^{i\theta})^{\alpha+\beta}(1-e^{-i\theta})^{\alpha-\beta} = (2-2\cos\theta)^{\alpha}(-e^{i\theta})^{\beta}$,

CONTACT J. Borrego-Morell jborrego@xerem.ufrj.br, jbmorell@gmail.com

$\theta \in [-\pi, \pi], \Re(\alpha) > -\frac{1}{2}$. That is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} R_n(e^{i\theta};\alpha,\beta)S_m(e^{-i\theta};\alpha,\beta)\omega(\theta)\,d\theta$$
$$= \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+\beta+1)\Gamma(\alpha-\beta+1)}\frac{n!}{(2\alpha+1)_n}\delta_{n,m},$$

where Γ denotes the Euler Gamma function. The bi-orthogonality was stated in [3] in a slightly different form and a formal proof was given later in [4, p.16–17]. Other proofs of the bi-orthogonality have been given by several authors, please see [5] for some historical considerations.

The author in [6] using a different approach also proved that when $\alpha \in \mathbb{R}, \alpha > -\frac{1}{2}$ and $i\beta \in \mathbb{R}$, the sequence $(R_n(z;\alpha,\beta))_{n=0}^{\infty}$ is orthogonal with respect to the weight ω, which is now positive and can be given by $\omega(\theta) = 2^{2\alpha}\,e^{(\pi-\theta)\Im(\beta)}\sin^{2\alpha}(\theta/2)$.

We also mention another known system of orthogonal polynomials on the unit circle of hypergeometric type which arise in a class of random unitary matrix ensembles, cf. [7], the circular Jacobi polynomials, defined as $C_n(z;a) = {}_2F_1(-n, a+1; -a+1-n; z)$. These polynomials are a particular case of the R_n, by taking $\alpha = 2a$ and $\beta = 0$, we obtain C_n.

From known results on hypergeometric functions, the element R_n in the orthogonal system $(R_n(z;\alpha,\beta))_{n=0}^{\infty}$ satisfies the differential equation

$$z(1-z)y'' + (\beta-\alpha+1-n-(-n+2+\alpha+\beta)z)y' + n(\alpha+\beta+1)y = 0.$$

Hence, it is natural to question if there exists other classes of orthogonal polynomials on the unit circle satisfying a linear second-order differential equation similar to the Jacobi, Hermite and Laguerre systems of orthogonal polynomials. The above differential equation satisfied by the sequence $(R_n)_{n=0}^{\infty}$ suggest that we should consider a differential equation with varying coefficients in the index n and the associated sequence of orthogonal polynomials as solution. In the present manuscript, under some assumptions on a certain class of lowering and raising operators, we give a necessary condition for the existence of a sequence of orthogonal polynomials solving the differential equations.

We state the results and notation in the subsection below and in Section 2 we prove the results.

1.1. Statement of the results

Let μ be a probability measure supported on an infinite subset of the unit circle. We say that $(\phi_n)_{n=0}^{\infty}$ is the sequence of orthonormal polynomials with respect to μ if

$$\int_{|z|=1} \phi_m(z)\overline{\phi_n(z)}\,d\mu(z) = \delta_{m,n},$$

where $\phi_n(z) = \kappa_n z^n + l_n z^{n-1} +$ lower order terms and $\kappa_n > 0$. Let $\Phi_n(z) = \phi_n(z)/\kappa_n$ be the monic polynomials.

A general background to orthogonal polynomials systems on the unit circle can be found in the monographs [8–12]. More recent surveys in [13–17].

If f is a polynomial of degree n then the reverse polynomial f^* is $z^n\overline{f(1/\bar{z})}$, that is

$$f^*(z) = \sum_{k=0}^{n} \bar{a}_k z^{n-k} \quad \text{if } f(z) = \sum_{k=0}^{n} a_k z^k \quad \text{and} \quad a_n \neq 0.$$

The sequence $(\phi_n)_{n=0}^{\infty}$ of orthonormal polynomials on the unit circle (OPUC for short) satisfy the recurrence relations [8, (11.4.6), (11.4.7)],

$$\begin{aligned} \kappa_n z\phi_n(z) &= \kappa_{n+1}\phi_{n+1}(z) - \phi_{n+1}(0)\phi_{n+1}^*(z), \\ \kappa_n\phi_{n+1}(z) &= \kappa_{n+1}z\phi_n(z) + \phi_{n+1}(0)\phi_n^*(z). \end{aligned} \tag{1.3}$$

In terms of the monic polynomials, the above recursion also can be expressed as [14, (1.5.10), (1.5.40)]

$$\Phi_{n+1}(z) = z\Phi_n(z) - \bar{\alpha}_n\Phi_n^*(z), \tag{1.4}$$

$$z\Phi_n(z) = \rho_n^{-2}\Phi_{n+1}(z) + \bar{\alpha}_n\Phi_{n+1}^*(z), \tag{1.5}$$

where $\alpha_n = -\overline{\Phi_{n+1}(0)}$ and $\rho_n = \kappa_n/\kappa_{n+1}$. The coefficients (α_n) are called the recursion coefficients or the Geronimus coefficients. Simon [14] makes a strong case and calls these coefficients as Verblunsky coefficients.

Consider the *hypergeometric equation*

$$z(1-z)y'' + (c-(a+b+1)z)y' - aby = 0, \tag{1.6}$$

we denote by ${}_2F_1(a,b;c;z)$ the Gauss hypergeometric series or hypergeometric function of the variable z with parameters a,b,c; cf. [18, p.56], defined as

$${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}, \quad |z| < 1,$$

where $c \neq 0, -1, -2, \ldots$, which is a solution to (1.6), holomorphic at $z=0$ and can be extended appropriately by analytic continuation, see please [18, Section 2.1.4]. Here $(a)_k = a(a+1)\cdots(a+k-1)$, $k \in \mathbb{N}$; $(a)_0 = 1$ denotes the Pochhammer symbol. All the solutions to (1.6) can be expressed in terms of the hypergeometric series (also expressible in terms of the Riemann P function, if we consider the point of view of Riemann for the hypergeometric equation).

We will follow [18, p.57] in supplementing the definition of the hypergeometric series for the case $c = -m, m \in \mathbb{N} \cup \{0\}$. If $a = -n$ or $b = -n$, where $n \in \mathbb{N} \cup \{0\}$ and if $c = -m$ where $m = n, n+1, n+2, \ldots$, then

$${}_2F_1(-n,b;-m;z) = \sum_{k=0}^{n} \frac{(-n)_k(b)_k}{(-m)_k}\frac{z^k}{k!}, \tag{1.7}$$

$${}_2F_1(a,-n;-m;z) = \sum_{k=0}^{n} \frac{(a)_k(-n)_k}{(-m)_k}\frac{z^k}{k!}. \tag{1.8}$$

If $c = -n, n \in \mathbb{N} \cup \{0\}$ and $-a, -b \notin \{n, n-1, \dots, 0\}$, we define

$$
\begin{aligned}
{}_2\mathbb{F}_1(a, b; -n; z) &= \sum_{k=0}^{\infty} \frac{(a+n+1)_k (b+n+1)_k}{(n+2)_k} \frac{z^k}{k!} \\
&= z^{n+1} {}_2F_1(a+n+1, b+n+1; n+2; z),
\end{aligned} \tag{1.9}
$$

which is also holomorphic at $z=0$ and a solution to (1.6).

We unify our notation by using the symbol ${}_2\mathbb{F}_1(a, b; c; z)$ to referring to ${}_2F_1(a, b; c; z)$ or (1.7)–(1.9) when the parameters a,b,c assume the values specified. Notice that ${}_2\mathbb{F}_1$ is a polynomial whenever $-a$ or $-b$ is a non-negative integer.

In a similar way, we denote by ${}_1F_1$ the confluent hypergeometric function or Kummer's series

$$
{}_1F_1(a; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} \frac{z^k}{k!},
$$

where $c \neq 0, -1, -2, \dots$, which is holomorphic at $z=0$ and is a solution of the *confluent hypergeometric equation*, cf. [18, p.248]

$$
zy'' + (c-z)y' - ay = 0.
$$

If $a=-n$, where $n \in \mathbb{N} \cup \{0\}$ and if $c=-m$ where $m = n, n+1, n+2, \dots$, then

$$
{}_1F_1(-n; -m; z) = \sum_{k=0}^{n} \frac{(-n)_k}{(-m)_k} \frac{z^k}{k!}. \tag{1.10}
$$

We define

$$
{}_1\mathbb{F}_1(a; -n; z) = \sum_{k=0}^{\infty} \frac{(a+n+1)_k}{(n+2)_k} \frac{z^k}{k!} = z^{n+1} {}_1F_1(a+n+1; n+2; z), \tag{1.11}
$$

when $c = -n,\ n \in \mathbb{N} \cup \{0\}$ and $-a \notin \{n, n-1, \dots, 0\}$.

We will use the symbol ${}_1\mathbb{F}_1(a; c; z)$ to refer to ${}_1F_1(a; c; z)$ or (1.10) and (1.11) when the parameters a,c assume the values specified.

Let us consider the differential equation

$$
A_n(z)y'' + B_n(z)y' + C_n y = 0, \tag{1.12}
$$

where $A_n(z) = a_{2,n}z^2 + a_{1,n}z + a_{0,n}$ with $a_{2,n} \neq 0,\ a_{1,n}^2 - 4a_{2,n}a_{0,n} \neq 0,\ \forall n \in \mathbb{N}$ or $a_{2,n} = 0,\ a_{1,n} \neq 0,\ \forall n \in \mathbb{N},\ B_n(z) = b_{1,n} + b_{0,n}z$ and $(a_{i,n})_{n\in\mathbb{N}},\ i = 0, 1, 2;\ (b_{i,n})_{n\in\mathbb{N}},\ i = 0, 1;$ $(C_n)_{n\in\mathbb{N}}$ are sequences of complex numbers. Under a linear complex change in the variable z, the differential equation (1.12) can be transformed to

$$
\theta(z)y'' + (c_n - b_n z)y' + \lambda_n y = 0, \tag{1.13}
$$

where $(b_n)_{n\in\mathbb{N}},\ (c_n)_{n\in\mathbb{N}}$ and $(\lambda_n)_{n\in\mathbb{N}}$ are sequences of complex numbers and

$$
\theta(z) = \begin{cases} z(1-z), & \text{if } a_{2,n} \neq 0,\ \ a_{1,n}^2 - 4a_{2,n}a_{0,n} \neq 0,\ \ \forall n \in \mathbb{N}, \\ z, & \text{if } a_{2,n} = 0,\ \ a_{1,n} \neq 0,\ \ \forall n \in \mathbb{N}. \end{cases}
$$

We are interested in those differential equations of the form (1.13) for which there exists a unique sequence $(\phi_n)_{n=0}^{\infty}$ of OPUC with ϕ_n solving (1.13).

It is clear that for every $n \in \mathbb{N}$ fixed, the existence and uniqueness of a polynomial solution of degree n to the differential equation (1.13) will depend on the values assumed by the parameters b_n, c_n and λ_n. This study is done in the following proposition, whose proof will be given in Section 2.

Proposition 1.1: *Let $n \in \mathbb{N}$ be fixed. There exists a unique polynomial solution of degree n for the differential equation (1.13) if and only if*

$$\lambda_n = \begin{cases} n(n-1+b_n), & \theta(z) = z(1-z), \\ nb_n, & \theta(z) = z, \end{cases} \tag{1.14}$$

and

$$b_n \neq \begin{cases} -2n+2, -2n+3, \cdots, -n, -n+1, & \theta(z) = z(1-z), \\ 0, & \theta(z) = z. \end{cases} \tag{1.15}$$

This solution is

$$\pi_n(z) = const. \begin{cases} {}_2\mathbb{F}_1(-n, b_n+n-1; c_n; z), & \text{if } \theta(z) = z(1-z), \\ {}_1\mathbb{F}_1(-n; c_n; b_n z), & \text{if } \theta(z) = z. \end{cases}$$

By virtue of the above proposition, we study up to a complex linear change in the variable z, those sequences of OPUC satisfying a second-order differential equation of the form

$$y'' + P_n(z)y' + Q_n(z)y = 0, \tag{1.16}$$

where

$$P_n(z) = \frac{c_n - b_n z}{\theta(z)}, \quad Q_n(z) = \frac{n(n-1+b_n)}{\theta(z)} \quad \text{if } \theta(z) = z(1-z),$$
$$P_n(z) = \frac{c_n - z}{\theta(z)}, \quad Q_n(z) = \frac{n}{\theta(z)} \quad \text{if } \theta(z) = z,$$

and $b_n \notin \{-2n+2, -2n+3, \ldots, -n, -n+1\}$ for $\theta(z) = z(1-z)$.

1.1.1. Raising and lowering operators for orthogonal polynomials on the unit circle.

Let w be a weight function supported on a subset of the unit circle and assume that w is normalized by

$$\int_{|\xi|=1} w(\xi) \frac{\mathrm{d}\xi}{\mathrm{i}\xi} = 1,$$

and define the external field v, cf. [19]

$$w(z) = \mathrm{e}^{-v(z)}.$$

Ismail and Witte [20] derived raising and lowering operators $L_{1,n}$ and $L_{2,n}$ whose expression is given by Ismail and Witte [20, (2.21)]. Under the assumptions that w is differentiable in a neighbourhood of the unit circle and the Verblunsky coefficients do not vanish, these lowering and raising operators define a pair of second-order differential equations such that

$$\phi_n'' + \mathcal{P}_{1,n}\phi_n' + \mathcal{Q}_{1,n}\phi_n = 0,$$
$$\phi_n'' + \mathcal{P}_{2,n}\phi_n' + \mathcal{Q}_{2,n}\phi_n = 0,$$

with coefficients given by Ismail and Witte [20, (2.22),(3.8)] and depending on functional coefficients $\mathcal{A}_n$ and $\mathcal{B}_n$

$$\mathcal{A}_n(z) = n\frac{\kappa_{n-1}}{\kappa_n} + \mathrm{i}\frac{\kappa_{n-1}}{\phi_n(0)} z \int_{|\xi|=1} \frac{v'(z) - v'(\xi)}{z-\xi}\phi_n(\xi)\overline{\phi_n^*(\xi)}w(\xi)\,\mathrm{d}\xi,$$
$$\phi_n'(z) = \mathcal{A}_n(z)\phi_{n-1}(z) - \mathcal{B}_n(z)\phi_n(z).$$

The authors proved that, cf. [20, Theorem 3.1 & Remark 3], if v is a meromorphic function on the unit disk then

$$\mathcal{P}_{1,n} = \mathcal{P}_{2,n}, \tag{1.17}$$

therefore, $\mathcal{Q}_{1,n} = \mathcal{Q}_{2,n}$.

From Ismail and Witte [20, Example 1], for the orthonormal circular Jacobi polynomials ϕ_n

$$\phi_n(z) = \frac{(a)_n}{\sqrt{n!(2a+1)_n}}\,{}_2F_1(-n, a+1; -a+1-n; z),$$

relation (1.17) gives

$$\phi_n'' + \left(\frac{1-n-a}{z} - \frac{2a+1}{1-z}\right)\phi_n' + \frac{n(1+a)}{z(1-z)}\phi_n = 0,$$

that is, by taking the weight associated with the circular Jacobi polynomials, the relation (1.17) defines a linear differential equation of the form (1.16).

It is reasonable to ask if we drop the differentiability condition over the weight and allow $\mathcal{A}_n$ and $\mathcal{B}_n$ varying in a wider class of functions, does the relation (1.17) give other classes of OPUC satisfying a linear differential equation of the form (1.13)? In Theorem 1.6, we give a necessary condition over $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ for this to happen. In particular, we obtain the system of OPUC defined by R_n which is orthogonal with respect to ω. Note that this weight is not in general a meromorphic function on the unit disk.

Before we enunciate the results in this article, we introduce some auxiliary notation used in the sequel.

We denote by Ω_θ the region given by

$$\Omega_\theta = \begin{cases} D^*(0,\epsilon_0) \cup D^*(1,\epsilon_1), & \theta(z) = z(1-z), \\ D^*(0,\epsilon_0), & \theta(z) = z, \end{cases}$$

here $D^*(0,\epsilon_0)$ and $D^*(1,\epsilon_1)$ are punctured open disks of respective radius ϵ_0 and ϵ_1 sufficiently small and with their respective centres at the points 0 and 1.

Let $(\phi_n)_{n=0}^{\infty}$ be a sequence of orthogonal polynomials with respect to a Borel measure supported on the unit circle satisfying (1.16). We denote by $\{m_n\}_{n\in\mathbb{N}}$ the set of indices corresponding to non-null Verblunsky coefficients. Define $A_0 = B_0 = 0$, for each $n \in \mathbb{N}$ fixed, denote by $A_n(z)$ and $B_n(z)$ for $z \in \Omega_\theta$ a solution of

$$\phi_n'(z) = A_n(z)\phi_{n-1}(z) - B_n(z)\phi_n(z), \tag{1.18}$$

in particular the functions $\mathcal{A}_n$ and $\mathcal{B}_n$ are a special case.

Before we enunciate the main result, we need some auxiliary propositions whose proof will be given in Section 2.

Proposition 1.2: *Let $(\phi_n)_{n=0}^{\infty}$ be the sequence of orthogonal polynomials with respect to a Borel measure supported on the unit circle and suppose that $(\phi_n)_{n=0}^{\infty}$ satisfies* (1.16), *then $\phi_n(z) = z^n$, $\forall n \in \mathbb{N}$ if and only if there exists $n_0 \in \mathbb{N}$ such that $\phi_{n_0}(z) = z^{n_0}$.*

Proposition 1.3: *Let $(\phi_n)_{n=0}^{\infty}$ be a sequence of OPUC satisfying* (1.16) *with $\phi_1(z) \neq z$, then $\{m_n\}_{n\in\mathbb{N}}$ is infinite and consecutive Verblunsky coefficients cannot vanish simultaneously.*

Proposition 1.4: *Let $(\phi_n)_{n=0}^{\infty}$ be a sequence of OPUC such that $\phi_1(z) \neq z$ and A_n, B_n satisfying* (1.18)*. Then for $n \geq 1$*

$$\phi_{m_n}'' + P_{1,m_n}\phi_{m_n}' + Q_{1,m_n}\phi_{m_n} = 0, \tag{1.19}$$

$$\phi_{m_n}'' + P_{2,m_n}\phi_{m_n}' + Q_{2,m_n}\phi_{m_n} = 0, \tag{1.20}$$

where

$$\begin{aligned} P_{1,m_n} &= B_{m_n} + B_{m_{n-1}} - \frac{A_{m_n}'}{A_{m_n}} + \frac{m_{n-1}}{z} - \frac{m_n}{z} + \frac{1}{z} - \frac{\kappa_{m_{n-1}}}{\kappa_{m_{n-1}-1}}\frac{A_{m_{n-1}}}{z} \\ &\quad - z^{m_n - m_{n-1} - 1}\frac{\kappa_{m_n}}{\kappa_{m_{n-1}-1}}\frac{\phi_{m_{n-1}}(0)}{\phi_{m_n}(0)}A_{m_{n-1}}, \\ Q_{1,m_n} &= A_{m_n}\left(\frac{B_{m_n}}{A_{m_n}}\right)' + B_{m_{n-1}}B_{m_n} - \frac{\kappa_{m_{n-1}}}{\kappa_{m_{n-1}-1}}\frac{A_{m_{n-1}}B_{m_n}}{z} \\ &\quad - \frac{\kappa_{m_n}}{\kappa_{m_{n-1}-1}}\frac{\phi_{m_{n-1}}(0)}{\phi_{m_n}(0)}z^{-m_{n-1}+m_n-1}A_{m_{n-1}}B_{m_n} \\ &\quad + \frac{\kappa_{m_n-1}}{\kappa_{m_{n-1}-1}}\frac{\phi_{m_{n-1}}(0)}{\phi_{m_n}(0)}z^{-m_{n-1}+m_n-2}A_{m_{n-1}}A_{m_n} + \frac{m_{n-1} - m_n + 1}{z}B_{m_n}, \end{aligned}$$

and

$$P_{2,m_n} = B_{m_{n+1}} + B_{m_n} - \frac{A_{m_n}'}{A_{m_n}} - \frac{\kappa_{m_n}}{\kappa_{m_n-1}}\frac{A_{m_n}}{z} - z^{m_{n+1}-m_n-1}\frac{\kappa_{m_{n+1}}}{\kappa_{m_n-1}}\frac{\phi_{m_n}(0)}{\phi_{m_{n+1}}(0)}A_{m_n} + \frac{1}{z}$$

$$
\begin{aligned}
Q_{2,m_n} = {} & A_{m_n}\left(\frac{B_{m_n}}{A_{m_n}}\right)' + B_{m_n}B_{m_{n+1}} - \frac{\kappa_{m_n}}{\kappa_{m_n-1}}\frac{A_{m_n}B_{m_{n+1}}}{z} \\
& - \frac{\kappa_{m_{n+1}}}{\kappa_{m_n-1}}\frac{\phi_{m_n}(0)}{\phi_{m_{n+1}}(0)} z^{-m_n+m_{n+1}-1}A_{m_n}B_{m_{n+1}} \\
& + \frac{\phi_{m_n}(0)}{\phi_{m_{n+1}}(0)}\frac{\kappa_{m_{n+1}-1}}{\kappa_{m_n-1}} z^{-m_n+m_{n+1}-2}A_{m_n}A_{m_{n+1}} \\
& + \frac{B_{m_n}}{z} - \frac{\kappa_{m_{n+1}}}{\kappa_{m_n-1}}\frac{\phi_{m_n}(0)}{\phi_{m_{n+1}}(0)}(m_{n+1}-m_n) z^{-m_n+m_{n+1}-2}A_{m_n}.
\end{aligned}
$$

Proposition 1.5: *Let $(\phi_n)_{n=0}^{\infty}$ be a sequence of OPUC satisfying* (1.16), $\phi_1(z) \neq z$. *Then there exist functions A_{m_n} and B_{m_n} analytic in Ω_θ satisfying* (1.18) *such that for $n \geq 1$*

$$P_{1,m_n} = P_{2,m_n}, \tag{1.21}$$

$$Q_{1,m_n} = Q_{2,m_n}, \tag{1.22}$$

where P_{1,m_n}, Q_{1,m_n} and P_{2,m_n}, Q_{2,m_n} are as in Proposition 1.4.

By analogy with (1.17), let $\mathfrak{P}_{m_n}$ be the common value in (1.21). In the present article, we prove the following theorem.

Theorem 1.6: *Let $(\phi_n)_{n=0}^{\infty}$ be a sequence of orthonormal polynomials with respect to a positive Borel measure on the unit circle satisfying* (1.16). *Then the following statements hold:*

(a) *If $c_1 = 0$ the whole sequence reduces to $(z^n)_{n=0}^{\infty}$.*
(b) *If $P_{m_n} = \mathfrak{P}_{m_n}$, $\forall n \in \mathbb{N}$, then there exist $p_n, q_n,\ r_n \in \mathbb{Z}$, $p_1, q_1 = 0$, $c_1 \neq 0$, $b_1 + p_n \notin \{-n+1, -n+2, \ldots, 0\}$ and γ_n an appropriate non-null complex constant such that*

$$\phi_n(z) = \gamma_n \begin{cases} {}_2\mathbb{F}_1(-n, b_1 + p_n; c_1 + q_n; z), & \text{if } \theta(z) = z(1-z), \\ {}_1\mathbb{F}_1(-n; c_1 + r_n; z), & \text{if } \theta(z) = z, \end{cases} \tag{1.23}$$

The above theorem gives as a particular case, the sequence $(R_n)_{n=0}^{\infty}$ which is orthogonal with respect to ω when ω is positive by choosing $p_n = 0, q_n = 1 - n$ and $b_1 = \alpha + \beta + 1$, $c_1 = \beta - \alpha$; $\alpha \in \mathbb{R}$, $\alpha > -\frac{1}{2}$ and $i\beta \in \mathbb{R}$. It seems plausible to conjecture that for $\theta(z) = z(1-z)$ the family $(R_n)_{n=0}^{\infty}$ is the unique sequence of orthogonal polynomials on the unit circle satisfying (1.16). Furthermore, when $\theta(z) = z$ also there exists a unique sequence and is given by $\phi_n(z) = z^n$.

As a restatement of the above theorem, we obtain

Corollary 1.7: *With the same hypothesis of Theorem* 1.6,

(a) *If $c_1 = 0$ the whole sequence reduces to $(z^n)_{n=0}^{\infty}$.*
(b) *If $P_{m_n} = \mathfrak{P}_{m_n}$, $\forall n \in \mathbb{N}$, then $b_n - b_1,\ c_n - c_1 \in \mathbb{Z}$.*

2. Proof of the results

The sketch of the proof for Theorem 1.6 is as follows. We start by obtaining a pair of second-order differential equations in terms of two functional coefficients A_n, B_n satisfied for an arbitrary sequence of orthogonal polynomials for a measure supported on the unit circle. This will be done in Proposition 1.4. Later, in Proposition 1.5, we unify the differential equations obtained in Proposition 1.4. Under the assumption that the sequence of orthogonal polynomials is of hypergeometric or confluent hypergeometric type and the differential equation, they satisfy is equal to the differential equations obtained in Proposition 1.4, we obtain a system of equations with A_n and B_n as functional variables. By analysing the singular points of A_n, we obtain the possible sequences of orthogonal polynomials.

Proof of Proposition* 1.1*: For any $n \in \mathbb{N}$, let $y_n(z) = \sum_{k=0}^{n} a_k z^k$ be a monic polynomial. Then y_n satisfies (1.13) if and only if

$$\begin{aligned} (k+1)(c_n+k)a_{k+1} + (\lambda_n - \eta_k)a_k = 0, \quad 0 \le k \le n, \\ a_{n+1} = 0, \end{aligned} \tag{2.1}$$

where

$$\eta_k = \begin{cases} k(k-1+b_n), & \theta(z) = z(1-z), \\ kb_n, & \theta(z) = z. \end{cases}$$

From (2.1), $a_0, \dots, a_{n-1}$ is uniquely determined if and only if $\lambda_n - \eta_k \neq 0$, $\forall 0 \le k \le n-1$ and $\lambda_n = \eta_n$.

For $k = n$, we have

$$\lambda_n = \begin{cases} n(n-1+b_n), & \theta(z) = z(1-z), \\ nb_n, & \theta(z) = z. \end{cases} \tag{2.2}$$

For the case $\theta(z) = z$, it is straightforward that relation (2.2) and $b_n \neq 0$ is the necessary and sufficient conditions for (1.13) to have a unique polynomial solution.

Consider now the case $\theta(z) = z(1-z)$. The condition $\lambda_n - \eta_k \neq 0$, $\forall 0 \le k \le n-1$ is equivalent to saying that $b_n \notin \{-2n+2, -2n+3, \dots, -n, -n+1\}$ and this completes the proof of the existence and uniqueness of a polynomial solution of degree n for (1.13).

Assume now that λ_n is given as in (1.14). It follows from the theory for the hypergeometric (confluent hypergeometric) equation, see [18, p.56, 248] that a holomorphic solution in a neighbourhood of $z=0$ for (1.13) is given by

$$\pi_n(z) = const. \begin{cases} {}_2\mathbb{F}_1(-n, b_n+n-1; c_n; z), & \text{if } \theta(z) = z(1-z), \\ {}_1\mathbb{F}_1(-n; c_n; b_n z), & \text{if } \theta(z) = z, \end{cases}$$

the condition $b_n \notin \{-2n+2, -2n+3, \dots, -n, -n+1\}$ for $\theta(z) = z(1-z)$ implies that $b_n + n - 1 \notin \{0, -1, \dots, -n+1\}$, therefore, from the existence and uniqueness condition it follows that π_n is the polynomial solution of degree n. ∎

Proof of Proposition* 1.2*: Assume that there exists $n_0 \in \mathbb{N}$ such that $\Phi_{n_0}(z) = z^{n_0}$. From the recurrence formula for the monic polynomials (1.4), it is straightforward that $\Phi_n(z) =$

$z^n, n \leq n_0$. A straightforward argument by induction shows that $\Phi_n(z) = z^n,\ n \geq n_0$. Indeed, for $n = n_0 + 1$, from the recurrence formula for the monic polynomials (1.4)

$$\Phi_{n_0+1}(z) = z^{n_0+1} + \Phi_{n_0+1}(0), \quad |\Phi_{n_0+1}(0)| < 1. \tag{2.3}$$

From Proposition 1.1

$$\Phi_{n_0+1}(z) = const. \begin{cases} {}_2\mathbb{F}_1(-n_0 - 1, b_{n_0+1} + n_0; c_{n_0+1}; z), & \text{if } \theta(z) = z(1-z), \\ {}_1\mathbb{F}_1(-n_0 - 1; c_{n_0+1}; z), & \text{if } \theta(z) = z, \end{cases} \tag{2.4}$$

therefore, from (1.9) and (1.11) we have that $\Phi_{n_0+1}(0) = 0$, hence $\phi_n(z) = z^n$, $\forall n \in \mathbb{N}$.
The converse implication is trivial. ■

Proof: Let us assume that $\{m_n\}_{n\in\mathbb{N}}$ is finite. Denote $\mu = \max\{m_n\}_{n\in\mathbb{N}}$, by hypothesis $\phi_1(z) \neq z$, hence $\mu \geq 1$. Since ϕ_μ satisfies (1.16), from Proposition 1.1, it follows that

$$\phi_\mu(z) = \gamma_\mu \begin{cases} {}_2F_1(-\mu, b_\mu + \mu - 1; c_\mu; z), & \theta(z) = z(1-z), \\ {}_1F_1(-\mu; c_\mu; z), & \theta(z) = z, \end{cases} \tag{2.5}$$

where $\gamma_\mu \in \mathbb{C} \setminus \{0\}$ is an appropriate constant and $c_\mu \notin \{-\mu + 1, \ldots, 0\}$.

From the condition $\phi_n(0) = 0$ for $n > \mu$, we have that $\kappa_\mu = \kappa_n$, therefore from (1.3) and (2.5)

$$\phi_n(z) = z^{n-\mu}\phi_\mu(z). \tag{2.6}$$

If $c_n \in \mathbb{N} \cup \{0\},\ c_n < n$, then

$$\begin{aligned} {}_2\mathbb{F}_1(-n, b_n + n - 1; -c_n; z) &= z^{c_n+1}{}_2F_1(-n + c_n + 1, b_n + c_n + n; c_n + 2; z), \\ {}_1\mathbb{F}_1(-n; -c_n; z) &= z^{c_n+1}{}_1F_1(-n + c_n + 1; c_n + 2; z). \end{aligned} \tag{2.7}$$

Since ϕ_n satisfies (1.16) and $\phi_n(0) = 0$, for $n > \mu$, Proposition 1.1 gives

$$\phi_n(z) = const_n \begin{cases} {}_2\mathbb{F}_1(-n, b_n + n - 1; -c_n; z), & \theta(z) = z(1-z), \\ {}_1\mathbb{F}_1(-n; -c_n; z), & \theta(z) = z, \end{cases} \tag{2.8}$$

where $c_n \in \mathbb{N} \cup \{0\},\ c_n < n$, therefore from (2.7) and (2.8)

$$\phi_n(z) = const_n z^{c_n+1} \begin{cases} {}_2F_1(-n + c_n + 1, b_n + c_n + n; c_n + 2; z), & \theta(z) = z(1-z), \\ {}_1F_1(-n + c_n + 1; c_n + 2; z), & \theta(z) = z. \end{cases} \tag{2.9}$$

Since the order of the zero at $z=0$ in the relations (2.6) and (2.9) must coincide, we obtain $c_n = n - \mu - 1$. From (2.6), $\lim_{z\to 0}(\phi_n(z)/z^{n-\mu}) = \gamma_\mu$ and from (2.9) with $c_n = n - \mu - 1$ it follows that $const_n = \gamma_\mu$.

We prove now that (2.6) gives a contradiction, unless $n = \mu + 1$. As a consequence, we have that consecutive Verblunsky coefficients cannot vanish simultaneously.

Indeed, by comparing the coefficients of the expressions (2.6) and (2.9) that define the polynomial ϕ_n, we find that for $1 \leq k \leq n$ and $n > \mu$

$$\frac{(b_n + 2n - \mu - 1)_k}{(n - \mu + 1)_k} = \frac{(b_\mu + \mu - 1)_k}{(c_\mu)_k}, \quad \theta(z) = z(1 - z),$$
$$(n - \mu + 1)_k = (c_\mu)_k, \quad \theta(z) = z. \tag{2.10}$$

Consider the case $\theta(z) = z(1 - z)$. Relation (2.10) for $k = 1$ and $k = 2$ gives

$$\frac{b_n + 2n - \mu - 1}{n - \mu + 1} = \frac{b_\mu + \mu - 1}{c_\mu}, \tag{2.11}$$

$$\frac{(b_n + 2n - \mu - 1)(b_n + 2n - \mu)}{(n - \mu + 1)(n - \mu + 2)} = \frac{(b_\mu + \mu - 1)(b_\mu + \mu)}{c_\mu(c_\mu + 1)}. \tag{2.12}$$

From (2.11) and (2.12)

$$\frac{b_n + 2n - \mu}{n - \mu + 2} = \frac{b_\mu + \mu}{c_\mu + 1}. \tag{2.13}$$

From relations (2.11) and (2.13)

$$\frac{(c_\mu - n + \mu - 1)(-b_\mu + c_\mu - \mu + 1)}{c_\mu(c_\mu + 1)} = 0, \tag{2.14}$$

it follows that

$$b_\mu = c_\mu - \mu + 1, \tag{2.15}$$

$$c_\mu = n - \mu + 1. \tag{2.16}$$

The alternative (2.15) is not possible. Indeed, by substituting the value b_μ obtained in (2.15) into (2.5) we find that

$$\phi_\mu(z) = \gamma_\mu {}_2F_1(-\mu, c_\mu; c_\mu; z),$$

which is impossible, since $|\Phi_\mu(0)| = 1$.

Since c_μ is fixed, relation (2.16) is only possible for $n = \mu + 1$, it follows that (2.6) only holds for $n = \mu + 1$ and in this case $c_\mu = 2, b_{\mu+1} = b_\mu - 2$.

Consider now the case $\theta(z) = z$. Relation (2.10) gives $c_\mu = n - \mu + 1,\ n > \mu$, that is, relation (2.16) is only possible for $n = \mu + 1$. We conclude that (2.6) only holds for $n = \mu + 1$ with $c_\mu = 2$ and this completes the proof of the lemma. ■

Proof: From Proposition 1.3, the set $\{m_n\}_{n\in\mathbb{N}}$ is infinite, therefore, from the recurrence relations (1.3), for $n \geq 2$

$$\kappa_{m_{n-1}-1} z\phi_{m_{n-1}-1}(z) = \kappa_{m_{n-1}}\phi_{m_{n-1}}(z) - \phi_{m_{n-1}}(0)\phi^*_{m_{n-1}}(z), \tag{2.17}$$

$$\kappa_{m_n-1}\phi_{m_n}(z) = z\kappa_{m_n}\phi_{m_n-1}(z) + \phi_{m_n}(0)\phi^*_{m_n-1}(z). \tag{2.18}$$

Since $\{m_n\}_{n\in\mathbb{N}}$ is the set of indices for which $\phi_{m_n}(0) \neq 0$, then $\kappa_{m_{n-1}} = \kappa_{m_n-1}$, hence

$$\phi_{m_n-1}(z) = z^{m_n-m_{n-1}-1}\phi_{m_{n-1}}(z), \tag{2.19}$$

$$\phi^*_{m_n-1}(z) = \phi^*_{m_{n-1}}(z). \tag{2.20}$$

Relations (2.17)–(2.20) give

$$\begin{aligned} &z(\kappa_{m_{n-1}-1}\phi_{m_n}(0)\phi_{m_{n-1}-1}(z) - \phi_{m_{n-1}}(0)\kappa_{m_n}z^{m_n-m_{n-1}-1}\phi_{m_{n-1}}(z)) \\ &\quad = \kappa_{m_{n-1}}\phi_{m_n}(0)\phi_{m_{n-1}}(z) - \kappa_{m_n-1}\phi_{m_{n-1}}(0)\phi_{m_n}(z). \end{aligned} \tag{2.21}$$

Let A_n, B_n be such that (1.18) holds. Define the operators

$$T_{m_n,1} = \frac{\mathrm{d}}{\mathrm{d}z} + B_{m_n}(z),$$

$$T_{m_n,2} = -\frac{\mathrm{d}}{\mathrm{d}z} - B_{m_{n-1}}(z) + A_{m_{n-1}}(z)\frac{\kappa_{m_n}}{\kappa_{m_{n-1}-1}}\frac{\phi_{m_{n-1}}(0)}{\phi_{m_n}(0)}z^{m_n-m_{n-1}-1} + \frac{A_{m_{n-1}}(z)}{z}\frac{\kappa_{m_{n-1}}}{\kappa_{m_{n-1}-1}}.$$

From (2.21) and (1.18), the operators $T_{m_n,1}$ and $T_{m_n,2}$ are anhilation and creation operators in the sense that they satisfy

$$T_{m_n,1}[\phi_{m_n}(z)] = A_{m_n}(z)z^{m_n-m_{n-1}-1}\phi_{m_{n-1}}(z), \tag{2.22}$$

$$T_{m_n,2}[\phi_{m_{n-1}}(z)] = \frac{A_{m_{n-1}}(z)}{z}\frac{\kappa_{m_n-1}}{\kappa_{m_{n-1}-1}}\frac{\phi_{m_{n-1}}(0)}{\phi_{m_n}(0)}\phi_{m_n}(z), \tag{2.23}$$

Relations (2.22) and (2.23) give that ϕ_n satisfy the differential equations

$$\begin{aligned} &T_{m_n,2}\left[\frac{1}{A_{m_n}(z)z^{m_n-m_{n-1}-1}}T_{m_n,1}[\phi_{m_n}(z)]\right] \\ &\quad = \frac{A_{m_{n-1}}(z)}{z}\frac{\kappa_{m_n-1}}{\kappa_{m_{n-1}-1}}\frac{\phi_{m_{n-1}}(0)}{\phi_{m_n}(0)}\phi_{m_n}(z), \end{aligned} \tag{2.24}$$

$$\begin{aligned} &T_{m_{n+1},1}\left[\frac{z}{A_{m_n}(z)}T_{m_{n+1},2}[\phi_{m_n}(z)]\right] \\ &\quad = A_{m_{n+1}}(z)z^{m_{n+1}-m_n-1}\frac{\kappa_{m_{n+1}-1}}{\kappa_{m_n-1}}\frac{\phi_{m_n}(0)}{\phi_{m_{n+1}}(0)}\phi_{m_n}(z). \end{aligned} \tag{2.25}$$

By expanding (2.24) and (2.25) we obtain the coefficients of the differential equation for $n \geq 2$. From the definition of $A_0 = B_0 = 0$ we find that the lemma also holds for $n=1$ and this completes the proof. ■

Lemma 2.1: *The Riccati equation*

$$y' + P_1 y - y^2 - Q_1 = 0, \tag{2.26}$$

has an analytic solution in Ω_θ.

Proof: The change of variable

$$y = \frac{v'}{v}, \tag{2.27}$$

transforms the differential equation (2.26) in the second-order differential equation

$$v'' - P_1 v' + Q_1 v = 0. \tag{2.28}$$

Let us consider the case $\theta(z) = z(1-z)$ and express (2.28) as the hypergeometric differential equation

$$z(1-z)v'' + (-c_1 - (\eta_1 + \eta_2 + 1)z)v' - \eta_1\eta_2 v = 0, \tag{2.29}$$

where η_1, η_2 are the solutions of the equation

$$\eta^2 + (b_1 + 1)\eta - b_1 = 0.$$

From the theory of hypergeometric functions [18, Section 2.1.1 (3)], we find that the function

$$u(z) = \begin{cases} {}_2F_1(\eta_1, \eta_2; -c_1; z), & c_1 \notin \mathbb{N} \cup \{0\}, \\ z^{c_1+1}{}_2F_1(\eta_1 + c_1 + 1, \eta_2 + c_1 + 1; c_1 + 2; z), & c_1 \in \mathbb{N} \cup \{0\}, \end{cases} \tag{2.30}$$

is an holomorphic solution to (2.29) at $z = 0$, therefore, from (2.27) we have that there exists an analytic solution for the Equation (2.26) of the form

$$y(z) = \frac{u'(z)}{u(z)}, \quad z \in D^*(0, \epsilon_0), \tag{2.31}$$

where ϵ_0 is sufficiently small.

For the point $z = 1$ we consider [18, Section 2.9 (5)], that is

$$u(z) = \begin{cases} {}_2F_1(\eta_1, \eta_2; c_1 - b_1; 1-z), & b_1 - c_1 \notin \mathbb{N} \cup \{0\}, \\ (1-z)^{b_1 - c_1 + 1}{}_2F_1(-\eta_2 - c_1, -\eta_1 - c_1; b_1 - c_1 + 2; 1-z), & b_1 - c_1 \in \mathbb{N}, \end{cases}$$

which is holomorphic in a neighbourhood of $z = 1$. Applying now (2.27) we obtain the lemma for $\theta(z) = z(1-z)$.

The analysis for the case $\theta(z) = z$ is similar. ■

Lemma 2.2: *There exist analytic functions A_1, B_1 in Ω_θ such that the following system holds*

$$P_1(z) = -\frac{A_1'(z)}{A_1(z)} + B_1(z), \tag{2.32}$$

$$Q_1(z) = B_1'(z) - B_1(z)\frac{A_1'(z)}{A_1(z)}, \tag{2.33}$$

$$\kappa_1 = A_1(z) - B_1(z)\phi_1(z). \tag{2.34}$$

Proof: From (1.16)

$$\kappa_1 P_1(z) + Q_1(z)\phi_1(z) = 0.$$

From this last relation, it follows that the substitution of (2.34) into (2.32) or (2.33) gives the Riccati equation

$$B_1' + B_1 P_1 - B_1^2 - Q_1 = 0. \tag{2.35}$$

From Lemma 2.1, there exists an analytic solution for (2.35) in Ω_θ. The statement for the function A_1 follows immediately from (2.34). ∎

Proof of Proposition 1.5***:*** For $n=1$ we obtain the system defined by (2.32)–(2.34) and from Lemma 2.2, there exists analytic functions in Ω_θ, A_1 and B_1 such that this system holds.

For $n \geq 1$, let us write

$$P_{1,m_n}(z) = \Gamma_{m_n}(z) - \frac{A'_{m_n}(z)}{A_{m_n}(z)}, \tag{2.36}$$

where

$$\begin{aligned}\Gamma_{m_n}(z) = B_{m_n}(z) + B_{m_{n-1}}(z) - \frac{\kappa_{m_{n-1}}}{\kappa_{m_{n-1}-1}} \frac{A_{m_{n-1}}(z)}{z} \\ - z^{m_n - m_{n-1} - 1} \frac{\kappa_{m_n}}{\kappa_{m_{n-1}-1}} \frac{\phi_{m_{n-1}}(0)}{\phi_{m_n}(0)} A_{m_{n-1}}(z) + \frac{m_{n-1}}{z} - \frac{m_n}{z} + \frac{1}{z},\end{aligned} \tag{2.37}$$

and $\Gamma_1 = B_1$.

Note that if (1.18) and (1.21) hold then (1.22) also holds. Indeed, since ϕ_{m_n} satisfies (1.19) and (1.20)

$$\begin{aligned}\phi''_{m_n} + P_{1,m_n}\phi'_{m_n} + Q_{1,m_n}\phi_{m_n} = 0, \\ \phi''_{m_n} + P_{1,m_n}\phi'_{m_n} + Q_{2,m_n}\phi_{m_n} = 0,\end{aligned}$$

the difference between these two relations gives $Q_{1,m_n} = Q_{2,m_n}$, hence, Equation (1.22) is redundant.

From the expressions for P_{1,m_n} and P_{2,m_n} given in Proposition 1.4 we find that the relation $P_{1,m_n} = P_{2,m_n}$ in (1.21) holds if and only if Γ_{m_n} satisfies the difference equation

$$\begin{aligned}\Gamma_{m_{n+1}}(z) = \Gamma_{m_n}(z) + \frac{m_n}{z} - \frac{m_{n+1}}{z}, \\ \Gamma_1(z) = B_1(z).\end{aligned} \tag{2.38}$$

By hypothesis $\phi_1(z) \neq z$. Hence, from Proposition 1.2, $m_1 = 1$, it follows that the solution to the difference Equation (2.38) for $n \geq 1$ is

$$\Gamma_{m_n}(z) = B_1(z) + \frac{1 - m_n}{z}. \tag{2.39}$$

From (2.36) and (2.38) we have that (1.21) holds. Therefore, for $n \geq 1$, if Γ_{m_n} is given by (2.39), we define the analytic functions in Ω_θ, A_{m_n} and B_{m_n} as the unique solution of

(1.18) and (2.37). We have that (1.18), (1.21) and (1.22) hold and this completes the proof of the lemma. ■

From the preceding lemmas we now are able to give a proof of Theorem 1.6.

Proof of Theorem 1.6**:** Statement (a) follows immediately from Proposition 1.2.

To prove (b) note that we have two cases, the set of indices $\{m_n\}_{n\in\mathbb{N}}$ for which $\phi_{m_n}(0) \neq 0$ is finite or infinite. Results in Propositions 1.2 and 1.3 show that the first case implies that the whole sequence reduces to $\phi_n(z) = z^n$.

Let us analyse the second case.

From Proposition 1.5 there exist A_{m_n} and B_{m_n} such that $P_{1,m_n} = P_{2,m_n}$. From hypothesis $P_{m_n} = P_{1,m_n} = P_{2,m_n}$. Therefore, from (1.21), (2.36) and (2.39)

$$P_{m_n}(z) = \frac{1-m_n}{z} - \frac{A'_{m_n}(z)}{A_{m_n}(z)} + B_1(z). \tag{2.40}$$

We have that $\phi_1(z) \neq z$ and from (2.40) of Proposition 1.5

$$\begin{aligned} P_1(z) &= -\frac{A'_1(z)}{A_1(z)} + B_1(z), \\ P_{m_n}(z) &= \frac{1-m_n}{z} - \frac{A'_{m_n}(z)}{A_{m_n}(z)} + B_1(z), \quad n > 1. \end{aligned}$$

Therefore, $A'_{m_n}/A_{m_n} = A'_1/A_1 + \pi_n$, where

$$\pi_n(z) = \begin{cases} \dfrac{c_1 - c_{m_n} + 1 - m_n - (b_1 - b_{m_n} + 1 - m_n)z}{\theta(z)}, & \theta(z) = z(1-z), \\ \dfrac{c_1 - c_{m_n} + 1 - m_n}{\theta(z)}, & \theta(z) = z. \end{cases}$$

That is

$$A_{m_n} = A_1 \tau_n, \tag{2.41}$$

where

$$\tau_n(z) = c \begin{cases} z^{u_{m_n}}(1-z)^{v_{m_n}}, & \theta(z) = z(1-z), \\ z^{u_{m_n}}, & \theta(z) = z, \end{cases}$$

here $c \in \mathbb{C} \setminus \{0\}$ is an appropriate constant and

$$u_{m_n} = c_1 - c_{m_n} + 1 - m_n \tag{2.42}$$

$$v_{m_n} = b_1 - b_{m_n} - (c_1 - c_{m_n}). \tag{2.43}$$

From Proposition 1.5 and from the assumption of the theorem, there exists B_1 analytic in Ω_θ, hence, from Proposition 1.5, the function A_{m_n} is analytic in Ω_θ. Therefore, from (2.41), (2.42) and (2.43), we have $u_{m_n}, v_{m_n} \in \mathbb{Z}$, $\forall n \geq 2$.

Now, from (2.42) and (2.43)

$$\phi_{m_n}(z) = \gamma_{m_n} {}_2F_1(-m_n, b_1 + p_{m_n}; c_1 + q_{m_n}; z),$$

where $p_{m_n}, q_{m_n} \in \mathbb{Z}$; $p_1 = 0, q_1 = 0$, $c_1 \neq 0$, $b_1 + p_{m_n} \notin \{-m_n + 1, -m_n + 2, \ldots, 0\}$ and γ_{m_n} an appropriate complex constant.

For the indices k_n such that $\phi_{k_n}(0) = 0$

$$\phi_{k_n}(z) = \gamma_{k_n} {}_2\mathbb{F}_1(-k_n, b_1 + p_{k_n}; c_1 + q_{k_n}; z),$$

where $p_{k_n}, q_{k_n} \in \mathbb{Z}$; $c_1 + q_{k_n} = 0$, $b_1 + p_{k_n} \notin \{-k_n + 1, -k_n + 2, \ldots, 0\}$ and γ_{k_n} an appropriate complex constant, and this completes the proof of theorem. ■

Corollary 1.7 follows immediately from relation (1.23).

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