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On a symmetry in strong distributions

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Dedicated to Professor Haakon Waadeland on the occasion of his 70th birthday

Abstract

A strong Stieltjes distribution $d\psi(t)$ is called symmetric if it satisfies the property

$$t^{\omega}\, d\psi(\beta^2/t) = -(\beta^2/t)^{\omega}\, d\psi(t), \quad \text{for } t \in (a,b) \subseteq (0,\infty), 2\omega \in \mathbb{Z}, \text{ and } \beta > 0.$$

In this article some consequences of symmetry on the moments, the orthogonal L-polynomials and the quadrature formulae associated with the distribution are given.

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1. Introduction

The connections between moment problems, corresponding continued fractions, Padé approximants and orthogonal polynomials are classical [1]. The histories of each topic are intertwined, though each is now often taught without recourse to any of the other three. History has repeated itself with the comparatively recent advent of two-point Padé approximants, continued fractions that correspond to two series, strong moment problems, and the associated orthogonal Laurent polynomials. These are topics which have been the subject of substantial interest, and some of the published results have dealt with particular moment distributions which are symmetrical in some sense. In this article the authors consider a symmetry in strong distributions that encompasses many particular ones but

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extends these earlier results. In addition, unlike much of the earlier work, the authors consider the whole of the associated Padé table rather than a single row in it, and hence they are able to exhibit many further consequences of the symmetry in the distribution.

Let $0 < \beta < b \leqslant \infty$, set $a = \beta^2/b$ and let $\psi(t)$ be a real, bounded and nondecreasing function defined on (a,b), with infinitely many points of increase in (a,b) such that the moments

$$\mu_m = \int_a^b t^m \, \mathrm{d}\psi(t), \quad m = 0, \pm 1, \pm 2, \ldots, \tag{1.1}$$

all exist. Then $\psi(t)$ is a strong distribution function on (a,b). As $(a,b) \subseteq (0,\infty)$, $\psi(t)$ is called a strong Stieltjes distribution.

In this work we consider the strong Stieltjes distributions that satisfy the following symmetric property

$$\frac{\mathrm{d}\psi(t)}{t^\omega} = -\frac{\mathrm{d}\psi(\beta^2/t)}{(\beta^2/t)^\omega}, \quad t \in (a,b), \quad 2\omega \in \mathbb{Z}. \tag{1.2}$$

Here we denote these strong symmetric Stieltjes distributions as $S^3[\omega, \beta, b]$ distributions.

These distributions have been studied by Sri Ranga [10] and by Sri Ranga et al. [11], for the cases $w = 1/2$ and $w = 0$, where they are denoted by $ScS(a,b)$, and $S\bar{c}S(a,b)$, respectively. They were also treated by Common and McCabe [3] for the case $w = 0$ and $\beta = 1$.

When $\mathrm{d}\psi(t)$ can be given in the form $w(t)\,\mathrm{d}t$ then the property (1.2) becomes

$$t^{1-\omega} w(t) = (\beta^2/t)^{1-\omega} w(\beta^2/t), \quad t \in (a,b), \quad 2\omega \in \mathbb{Z}.$$

By substituting t for β^2/t in Eq. (1.1) we get

$$\mu_m = -\int_a^b \beta^{2m} t^{-m} \, \mathrm{d}\psi(\beta^2/t).$$

Hence, using Eq. (1.2) we can conclude that the moments of an $S^3[\omega, \beta, b]$ distribution satisfy the relation

$$\mu_m = \beta^{2(m+\omega)} \mu_{-m-2\omega}, \quad m \in \mathbb{Z}. \tag{1.3}$$

From Eq. (1.2), since

$$\frac{t^m \, \mathrm{d}\psi(t)}{t^{\omega+m}} = -\frac{(\beta^2/t)^m \, \mathrm{d}\psi(\beta^2/t)}{(\beta^2/t)^{\omega+m}}, \quad t \in (a,b), \quad 2\omega \in \mathbb{Z},$$

we note that for $m \in \mathbb{Z}$, $t^m \, \mathrm{d}\psi(t)$ is an $S^3[\omega + m, \beta, b]$ distribution. Furthermore, multiplying both sides of Eq. (1.2) by $(t+\beta)/\sqrt{t}$, we get

$$\frac{(t+\beta)\,\mathrm{d}\psi(t)}{t^{\omega+1/2}} = -\frac{(\beta^2/t+\beta)\,\mathrm{d}\psi(\beta^2/t)}{(\beta^2/t)^{\omega+1/2}}, \quad t \in (a,b), \ 2\omega \in \mathbb{Z}.$$

Hence, $(t+\beta)\,\mathrm{d}\psi(t)$ is an $S^3[\omega + 1/2, \beta, b]$ distribution.

It is well known that if $\psi(t)$ is a strong Stieltjes distribution then the associated Hankel determinants $H_n^{(m)}$, given by

$$H_n^{(m)} = \begin{vmatrix} \mu_m & \mu_{m+1} & \cdots & \mu_{m+n-1} \\ \mu_{m+1} & \mu_{m+2} & \cdots & \mu_{m+n} \\ \vdots & \vdots & \cdots & \vdots \\ \mu_{m+n-1} & \mu_{m+n} & \cdots & \mu_{m+2n-2} \end{vmatrix} \tag{1.4}$$

are positive, for $m = 0, \pm 1, \pm 2, \ldots,$ and $n \geqslant 1$.

For an $S^3[\omega, \beta, b]$ distribution the Hankel determinants satisfy

$$H_n^{(m)} = \beta^{2n(m+\omega+n-1)} H_n^{(-m-2\omega-2n+2)}, \quad m = 0, \pm 1, \pm 2, \ldots, \quad n \geqslant 1.$$

The proof of this result follows by substituting Eq. (1.3) in Eq. (1.4) and using properties of determinants.

Sri Ranga [10] studied the sequence of monic polynomials $B_n^{(0)}(z)$, $n = 0, 1, 2\ldots$, defined by the orthogonality conditions

$$\int_a^b t^{-n+s} B_n^{(0)}(t)\, \mathrm{d}\psi(t) = 0, \quad 0 \leqslant s \leqslant n-1, \tag{1.5}$$

where $\psi(t)$ satisfies Eq. (1.1). These polynomials are related to the orthogonal Laurent polynomials connected with the strong Stieltjes moment problem treated by Jones et al. [5,7]. For example, the exact relation between the orthogonal Laurent polynomials, $Q_n(z)$, given in [5, p.109] and the polynomials $B_n^{(0)}(z)$ is

$$Q_n(z) = z^{-[\![\frac{n+1}{2}]\!]} \frac{B_n^{(0)}(z)}{B_n^{(0)}(0)}.$$

In Section 2 we study the polynomials $B_n^{(0)}(z)$, and others, that satisfy Eq. (1.5) and in Section 3 we present some inversive properties for $\mathrm{d}\psi(t)$ that satisfies Eq. (1.2). We also consider the quadrature formula involving the zeros of these polynomials in Section 4. Finally, some examples are presented in Section 5.

2. The polynomials $B_n^{(r)}(z)$

Analogous to Eq. (1.5), we define sequences of monic polynomials $B_n^{(r)}(z)$, $n = 0, 1, 2, \ldots$ and $r = 0, \pm 1, \pm 2, \ldots,$ by

$$\int_a^b t^{-n+s+r} B_n^{(r)}(t)\, \mathrm{d}\psi(t) = 0, \quad 0 \leqslant s \leqslant n-1. \tag{2.1}$$

One can see that the polynomials $B_n^{(r)}(z)$ satisfy Eq. (1.5) for the distribution $t^r\, \mathrm{d}\psi(t)$. Thus, the results related for $B_n^{(0)}(z)$, given by Sri Ranga [10], can be extended to $B_n^{(r)}(z)$, as follows.

If we define $\rho_{n,k}^{(r)}$, for $n = 0, 1, 2, \ldots,$ $r = 0, \pm 1, \pm 2, \ldots,$ and $k \in \mathbb{Z}$, by

$$\rho_{n,k}^{(r)} = \int_a^b t^k B_n^{(r)}(t)\, \mathrm{d}\psi(t) \tag{2.2}$$

then from Eq. (2.1),

$$\rho^{(r)}_{n,-n+s+r} = 0, \quad \text{for } 0 \leqslant s \leqslant n-1. \tag{2.3}$$

Also,

$$\rho^{(r)}_{n,r} = \frac{H^{(r-n)}_{n+1}}{H^{(r-n)}_{n}} > 0 \quad \text{and} \quad (-1)^n \rho^{(r)}_{n,r-(n+1)} = \frac{H^{(r-(n+1))}_{n+1}}{H^{(r-n)}_{n}} > 0.$$

The polynomials $B^{(r)}_n(z)$ satisfy the three-term recurrence relation

$$B^{(r)}_{n+1}(z) = (z - \beta^{(r)}_{n+1}) B^{(r)}_n(z) - \alpha^{(r)}_{n+1} z B^{(r)}_{n-1}(z), \quad n \geqslant 1, \tag{2.4}$$

where

$$\alpha^{(r)}_{n+1} = \frac{\rho^{(r)}_{n,r}}{\rho^{(r)}_{n-1,r}} \quad \text{and } \beta^{(r)}_{n+1} = -\alpha^{(r)}_{n+1} \frac{\rho^{(r)}_{n-1,r-n}}{\rho^{(r)}_{n,r-(n+1)}}, \quad n \geqslant 1,$$

and $B^{(r)}_0(z) = 1$, $B^{(r)}_1(z) = z - \beta^{(r)}_1$, where, $\beta^{(r)}_1 = \mu_r/\mu_{r-1}$. The zeros of $B^{(r)}_n(z)$ are real, and distinct, and they all lie inside (a, b).

It is clear from Eq. (2.4), using known results regarding continued fractions and three term recurrence relations (see, e.g., [6]), that the polynomial $B^{(r)}_n(z)$ is the denominator of the nth convergent of the M-fraction

$$\frac{\mu_r}{z - \beta^{(r)}_1} - \frac{\alpha^{(r)}_2 z}{z - \beta^{(r)}_2} - \frac{\alpha^{(r)}_3 z}{z - \beta^{(r)}_3} - \frac{\alpha^{(r)}_4 z}{z - \beta^{(r)}_4} - \cdots. \tag{2.5}$$

The numerator polynomials, $C^{(r)}_n(z)$, of the convergents of the M-fraction (2.5) satisfy the same recurrence relation as $B^{(r)}_n(z)$, but with different starting values, namely $C^{(r)}_0(z) = 0$, $C^{(r)}_1(z) = \mu_r$.

Furthermore, the continued fraction (2.5) corresponds to the two series

$$\frac{\mu_r}{z} + \frac{\mu_{r+1}}{z^2} + \frac{\mu_{r+2}}{z^3} + \frac{\mu_{r+3}}{z^4} + \frac{\mu_{r+4}}{z^5} + \cdots, \tag{2.6}$$

$$-\mu_{r-1} - \mu_{r-2} z - \mu_{r-3} z^2 - \mu_{r-4} z^3 - \cdots, \tag{2.7}$$

i.e., the nth convergent of the continued fraction, when expanded accordingly, agrees with n terms of each of the two above series, from standard results in moment theory (see [6]).

The coefficients $\alpha^{(r)}_n$ and $\beta^{(r)}_n$ can be obtained from the quotient difference relations (see [8]),

$$\alpha^{(r)}_{n+1} = \beta^{(r+1)}_n + \alpha^{(r+1)}_n - \beta^{(r)}_n \quad \text{and} \quad \beta^{(r)}_{n+1} = \frac{\alpha^{(r)}_{n+1} \beta^{(r-1)}_n}{\alpha^{(r-1)}_{n+1}}, \quad n \geqslant 1, \tag{2.8}$$

with $\alpha^{(r)}_1 = 0$ and $\beta^{(r)}_1 = \mu_r/\mu_{r-1}$, for all values of r.

We now consider the polynomials given by $B^{(0)}_n(z) + \tau_{n,1} B^{(0)}_{n-1}(z)$, where $\tau_{n,1} \in \mathbb{R}$. It is easily shown that

$$\int_a^b t^{-n+s} (B^{(0)}_n(t) + \tau_{n,1} B^{(0)}_{n-1}(t)) \, \mathrm{d}\psi(t) = 0, \quad 1 \leqslant s \leqslant n-1.$$

However, if we choose $\tau_{n,1} = -\alpha_{n+1}^{(0)}$ then the above integral also vanishes when $s=n$. In view of Eq. (2.1), we know that

$$\int_a^b t^{-n+s} B_n^{(1)}(t)\,\mathrm{d}\psi(t) = 0, \quad 1 \leqslant s \leqslant n,$$

and as both polynomials are monic it follows that $B_n^{(1)}(z) = B_n^{(0)}(z) - \alpha_{n+1}^{(0)} B_{n-1}^{(0)}(z)$.

Generalising, we consider the polynomials given by $B_n^{(r-1)}(z) + \tau_{n,r} B_{n-1}^{(r-1)}(z)$, $\tau_{n,r} \in \mathbb{R}$, and in a similar way we see that

$$B_n^{(r)}(z) = B_n^{(r-1)}(z) - \alpha_{n+1}^{(r-1)} B_{n-1}^{(r-1)}(z), \quad r = 0, \pm 1, \pm 2, \ldots. \tag{2.9}$$

This result can alternatively be established from properties of the two-point Padé table for the series (2.6) and (2.7).

We now consider the real monic polynomials

$$B_n(\lambda_{n,1}^{(r)}, \ldots, \lambda_{n,r}^{(r)}; z) = B_n^{(0)}(z) + \lambda_{n,1}^{(r)} B_{n-1}^{(0)}(z) + \cdots + \lambda_{n,r}^{(r)} B_{n-r}^{(0)}(z), \tag{2.10}$$

where $\lambda_{n,1}^{(r)}, \ldots, \lambda_{n,r}^{(r)} \in \mathbb{R}$, and $1 \leqslant r \leqslant n$, $n = 1, 2, \ldots$.

By using the definitions (2.2) we then have

$$\int_a^b t^{-n+s} B_n(\lambda_{n,1}^{(r)}, \ldots, \lambda_{n,r}^{(r)}; t)\mathrm{d}\psi(t) = \rho_{n,-n+s}^{(0)} + \lambda_{n,1}^{(r)} \rho_{n-1,-n+s}^{(0)} + \cdots + \lambda_{n,r}^{(r)} \rho_{n-r,-n+s}^{(0)}.$$

Hence, from Eq. (2.3), the orthogonality conditions

$$\int_a^b t^{-n+s} B_n(\lambda_{n,1}^{(r)}, \ldots, \lambda_{n,r}^{(r)}; t)\,\mathrm{d}\psi(t) = 0, \quad r \leqslant s \leqslant n-1,$$

are satisfied. The polynomials $B_n(\lambda_{n,1}^{(r)}, \ldots, \lambda_{n,r}^{(r)}; z)$ have at least $n-r$ zeros of odd multiplicity inside (a,b). The proof of this result is very simple. Assume that $B_n(\lambda_{n,1}^{(r)}, \ldots, \lambda_{n,r}^{(r)}; z)$ changes sign exactly m times in (a,b) at the points $z_1, z_2, \ldots, z_m$. Consider the polynomial $\sigma(z) = (z-z_1)(z-z_2)\cdots(z-z_m)$, we then have

$$\int_a^b t^{-n+r} \sigma(z) B_n(\lambda_{n,1}^{(r)}, \ldots, \lambda_{n,r}^{(r)}; t)\,\mathrm{d}\psi(t) \neq 0.$$

This is a contradiction of the above result unless $m \geqslant n-r$.

Setting $\rho_{n,-n+s}^{(0)} + \lambda_{n,1}^{(r)} \rho_{n-1,-n+s}^{(0)} + \cdots + \lambda_{n,r}^{(r)} \rho_{n-r,-n+s}^{(0)} = 0$, for $n \leqslant s \leqslant n+r-1$, we obtain the $r \times r$ linear system

$$\begin{pmatrix} \rho_{n-1,0}^{(0)} & \rho_{n-2,0}^{(0)} & \cdots & \rho_{n-r,0}^{(0)} \\ \rho_{n-1,1}^{(0)} & \rho_{n-2,1}^{(0)} & \cdots & \rho_{n-r,1}^{(0)} \\ \cdots & \cdots & \cdots & \cdots \\ \rho_{n-1,r-1}^{(0)} & \rho_{n-2,r-1}^{(0)} & \cdots & \rho_{n-r,r-1}^{(0)} \end{pmatrix} \begin{pmatrix} \lambda_{n,1}^{(r)} \\ \lambda_{n,2}^{(r)} \\ \vdots \\ \lambda_{n,r}^{(r)} \end{pmatrix} = \begin{pmatrix} -\rho_{n,0}^{(0)} \\ -\rho_{n,1}^{(0)} \\ \vdots \\ -\rho_{n,r-1}^{(0)} \end{pmatrix}.$$

Suppose $\hat{\lambda}_{n,1}^{(r)}, \ldots, \hat{\lambda}_{n,r}^{(r)}$ is a solution of the above system, then

$$\int_a^b t^{-n+s} B_n(\hat{\lambda}_{n,1}^{(r)}, \ldots, \hat{\lambda}_{n,r}^{(r)}; t)\mathrm{d}\psi(t) = 0, \quad r \leqslant s \leqslant n+r-1.$$

From Eq. (2.1) we know that these conditions define a unique monic polynomial. The system thus has a unique solution and $B_n^{(r)}(z) = B_n(\hat{\lambda}_{n,1}^{(r)}, \ldots, \hat{\lambda}_{n,r}^{(r)}; z)$.

Now, from Eq. (2.9), we know that

$$B_n^{(r)}(z) = B_n^{(r-1)}(z) - \alpha_{n+1}^{(r-1)} B_{n-1}^{(r-1)}(z),$$

and using Eq. (2.9) again we have

$$\begin{aligned} B_n^{(r)}(z) &= B_n^{(r-2)}(z) - [\alpha_{n+1}^{(r-1)} + \alpha_{n+1}^{(r-2)}] B_{n-1}^{(r-2)}(z) + \alpha_{n+1}^{(r-1)} \alpha_n^{(r-2)} B_{n-2}^{(r-2)}(z), \\ B_n^{(r)}(z) &= B_n^{(r-3)}(z) - [\alpha_{n+1}^{(r-1)} + \alpha_{n+1}^{(r-2)} + \alpha_{n+1}^{(r-3)}] B_{n-1}^{(r-3)}(z) \\ &\quad + [\alpha_{n+1}^{(r-1)} \alpha_n^{(r-2)} + \alpha_{n+1}^{(r-1)} \alpha_n^{(r-3)} + \alpha_{n+1}^{(r-2)} \alpha_n^{(r-3)}] B_{n-2}^{(r-3)}(z) \\ &\quad - \alpha_{n+1}^{(r-1)} \alpha_n^{(r-2)} \alpha_{n-1}^{(r-3)} B_{n-3}^{(r-3)}(z). \end{aligned}$$

Continuing in this way we see that

$$\begin{aligned} B_n^{(r)}(z) &= B_n^{(0)}(z) - [\alpha_{n+1}^{(r-1)} + \alpha_{n+1}^{(r-2)} + \cdots + \alpha_{n+1}^{(0)}] B_{n-1}^{(0)}(z) \\ &\quad + [\alpha_{n+1}^{(r-1)} \alpha_n^{(r-2)} + \alpha_{n+1}^{(r-1)} \alpha_n^{(r-3)} + \cdots + \alpha_{n+1}^{(1)} \alpha_n^{(0)}] B_{n-2}^{(0)}(z) \\ &\quad - [\alpha_{n+1}^{(r-1)} \alpha_n^{(r-2)} \alpha_{n-1}^{(r-3)} + \alpha_{n+1}^{(r-1)} \alpha_n^{(r-2)} \alpha_{n-1}^{(r-4)} + \cdots + \alpha_{n+1}^{(2)} \alpha_n^{(1)} \alpha_{n-1}^{(0)}] B_{n-3}^{(0)}(z) \\ &\quad + \cdots (-1)^r \alpha_{n+1}^{(r-1)} \alpha_n^{(r-2)} \cdots \alpha_{n+2-r}^{(0)} B_{n-r}^{(0)}(z). \end{aligned}$$

Since $B_n^{(r)}(z) = B_n(\hat{\lambda}_{n,1}^{(r)}, \ldots, \hat{\lambda}_{n,r}^{(r)}; z)$ we obtain

$$\hat{\lambda}_{n,i}^{(r)} = (-1)^i \sum_1^p \prod_{k=1}^i \alpha_{n+2-k}^{(m_k)}, \quad i = 1, 2, \ldots, r, \quad p = \binom{r}{i}, \tag{2.11}$$

where the summation is over the $\binom{r}{i}$ ways of choosing the i integers m_k satisfying $0 \leqslant m_i < m_{i-1} < \cdots < m_1 \leqslant r-1$, $r \leqslant n$.

3. Inversive properties from an $S^3[\omega, \beta, b]$ distribution

Now we give some results to show the behavior of the polynomials $B_n^{(r)}(z)$ and its coefficients in the recurrence relation, when $\mathrm{d}\psi(t)$ is an $S^3[\omega, \beta, b]$ distribution.

Theorem 3.1. *Let* $\mathrm{d}\psi(t)$ *be an* $S^3[\omega, \beta, b]$ *distribution with* $0 < \beta < b \leqslant \infty$, $a = \beta^2/b$ *and* $2\omega \in \mathbb{Z}$. *Then for* $n \geqslant 0$ *and* $j = 1 - 2\omega$,

$$\frac{z^n B_n^{(l)}(\beta^2/z)}{B_n^{(l)}(0)} = B_n^{(j-l)}(z), \quad l = 0, \pm 1, \pm 2, \ldots. \tag{3.1}$$

Proof. From Eq. (2.1)

$$\int_a^b u^{-n+s+l} B_n^{(l)}(u) \mathrm{d}\psi(u) = 0, \quad 0 \leqslant s \leqslant n-1, \quad l = 0, \pm 1, \pm 2, \ldots.$$

Setting $u=\beta^2/t$, dividing by $B_n^{(l)}(0)$ and using Eq. (1.2) we get

$$\int_a^b t^{-s-2\omega-l}\left(\frac{t^n B_n^{(l)}(\beta^2/t)}{B_n^{(l)}(0)}\right)\mathrm{d}\psi(t)=0,\quad 0\leqslant s\leqslant n-1,\quad l=0,\pm1,\pm2,\ldots.$$

Since $j=1-2\omega$, the substitution of $-s-2\omega-l$ by $-n+s+j-l$ then yields

$$\int_a^b t^{-n+s+j-l}\left(\frac{t^n B_n^{(l)}(\beta^2/t)}{B_n^{(l)}(0)}\right)\mathrm{d}\psi(t)=0,\quad 0\leqslant s\leqslant n-1,\quad l=0,\pm1,\pm2,\ldots.$$

Using Eq. (2.1) for $B_n^{(j-l)}(z)$ and the fact that both polynomials are monic, we see that Eq. (3.1) holds. □

The following corollary brings out the particular details of the symmetry in Eq. (3.1) when 2ω is odd or even.

Corollary 3.2. *Let* $\mathrm{d}\psi(t)$ *be an* $S^3[\omega,\beta,b]$ *distribution with* $0<\beta<b\leqslant\infty$, $a=\beta^2/b$ *and* $2\omega\in\mathbb{Z}$. *Then for* $n\geqslant0$ *and*

(a) *for* 2ω *odd and* $j=-\omega+1/2$,

$$\frac{z^n B_n^{(j+l)}(\beta^2/z)}{B_n^{(j+l)}(0)}=B_n^{(j-l)}(z),\quad l=0,\pm1,\pm2,\ldots,$$

(b) *for* 2ω *even and* $j=-\omega$,

$$\frac{z^n B_n^{(j+l)}(\beta^2/z)}{B_n^{(j+l)}(0)}=B_n^{(j+1-l)}(z),\quad l=0,\pm1,\pm2,\ldots.$$

If we denote the zeros of $B_n^{(r)}(z)$ by $z_{n,1}^{(r)},z_{n,2}^{(r)},\ldots,z_{n,n}^{(r)}$, in increasing order, from Eq. (3.1) one can see that for $j=1-2\omega$ the zeros of $B_n^{(l)}(z)$ and the zeros of $B_n^{(j-l)}(z)$ satisfy the following relation

$$z_{n,i}^{(l)}=\frac{\beta^2}{z_{n,n+1-i}^{(j-l)}},\quad i=1,2,\ldots,n,\quad l=0,\pm1,\pm2,\ldots.\tag{3.2}$$

Theorem 3.3. *Let* $\mathrm{d}\psi(t)$ *be an* $S^3[\omega,\beta,b]$ *distribution with* $0<\beta<b\leqslant\infty$, $a=\beta^2/b$ *and* $2\omega\in\mathbb{Z}$. *Then for* $j=1-2\omega$,

$$\beta_n^{(l)}\beta_n^{(j-l)}=\beta^2,\quad \textit{and}\quad \frac{\alpha_{n+1}^{(j-l)}}{\alpha_{n+1}^{(l)}}=\frac{\beta_n^{(j-l)}}{\beta_{n+1}^{(l)}},\quad n\geqslant1,\quad l=0,\pm1,\pm2,\ldots\ .\tag{3.3}$$

Proof. From the recurrence relation (2.4) for $l=0,\pm1,\pm2,\ldots,$ we have,

$$B_{n+1}^{(l)}(z)=(z-\beta_{n+1}^{(l)})B_n^{(l)}(z)-\alpha_{n+1}^{(l)}zB_{n-1}^{(l)}(z),\quad n\geqslant1$$

with $B_0^{(l)}(z)=1$, $B_1^{(l)}(z)=z-\beta_1^{(l)}$.

Replacing z by β^2/z and multipling by z^{n+1} we obtain

$$z^{n+1}B_{n+1}^{(l)}(\beta^2/z)=(\beta^2-z\beta_{n+1}^{(l)})z^nB_n^{(l)}(\beta^2/z)-\alpha_{n+1}^{(l)}\beta^2zz^{n-1}B_{n-1}^{(l)}(\beta^2/z),\quad n\geqslant1,$$

and dividing by $B_{n+1}^{(l)}(0) = -\beta_{n+1}^{(l)} B_n^{(l)}(0) = \beta_{n+1}^{(l)}\beta_n^{(l)} B_{n-1}^{(l)}(0)$ then yields

$$\frac{z^{n+1}B_{n+1}^{(l)}(\beta^2/z)}{B_{n+1}^{(l)}(0)} = \left(z - \frac{\beta^2}{\beta_{n+1}^{(l)}}\right)\frac{z^n B_n^{(l)}(\beta^2/z)}{B_n^{(l)}(0)} - \frac{\alpha_{n+1}^{(l)}\beta^2}{\beta_{n+1}^{(l)}\beta_n^{(l)}} z \frac{z^{n-1}B_{n-1}^{(l)}(\beta^2/z)}{B_{n-1}^{(l)}(0)}, \quad n \geqslant 1.$$

Then, using Eq. (3.1),

$$B_{n+1}^{(j-l)}(z) = \left(z - \frac{\beta^2}{\beta_{n+1}^{(l)}}\right) B_n^{(j-l)}(z) - \frac{\alpha_{n+1}^{(l)}\beta^2}{\beta_{n+1}^{(l)}\beta_n^{(l)}} z B_{n-1}^{(j-l)}(z), \quad n \geqslant 1, \tag{3.4}$$

with $B_0^{(j-l)}(z) = 1$, $B_1^{(j-l)}(z) = z - \beta^2/\beta_1^{(l)}$.

The result follows when substituting r by $j - l$ in Eq. (2.4) and comparing with Eq. (3.4). □

Again the following corollary is a restatement of the symmetry in Eq. (3.3), where one can see the behavior of the symmetry when 2ω is odd or even.

Corollary 3.4. *Let* $d\psi(t)$ *be an* $S^3[\omega, \beta, b]$ *distribution with* $0 < \beta < b \leqslant \infty$, $a = \beta^2/b$ *and* $2\omega \in \mathbb{Z}$. *Then*

(a) *for* 2ω *odd and* $j = -\omega + 1/2$,

$$\beta_n^{(j+l)}\beta_n^{(j-l)} = \beta^2 \quad \textit{and} \quad \frac{\alpha_{n+1}^{(j+l)}}{\alpha_{n+1}^{(j-l)}} = \frac{\beta_n^{(j+l)}}{\beta_{n+1}^{(j-l)}}, \quad n \geqslant 1, \quad l = 0, \pm 1, \pm 2, \ldots,$$

consequently, when $l = 0$, $\beta_n^{(j)} = \beta$, $n \geqslant 1$.

(b) *for* 2ω *even and* $j = -\omega$

$$\beta_n^{(j+l)}\beta_n^{(j+1-l)} = \beta^2 \quad \textit{and} \quad \frac{\alpha_{n+1}^{(j+l)}}{\alpha_{n+1}^{(j+1-l)}} = \frac{\beta_n^{(j+l)}}{\beta_{n+1}^{(j+1-l)}}, \quad n \geqslant 1, \quad l = 0, \pm 1, \pm 2, \ldots.$$

4. Quadrature formula

Given $f(t)$, a real valued function, we consider the quadrature formula of the form

$$\int_a^b f(t)\, d\psi(t) = \sum_{i=1}^n w_{n,i}^{(r)} f(z_{n,i}^{(r)}) + \mathbb{E}_n(f), \tag{4.1}$$

where the knots $z_{n,i}^{(r)}$, $i = 1, \ldots, n$ are the zeros of the polynomial $B_n^{(r)}(z)$ and the weights are given by

$$w_{n,i}^{(r)} = \frac{(z_{n,i}^{(r)})^{n-r}}{B_n^{(r)\prime}(z_{n,i}^{(r)})} \int_a^b \frac{t^{-n+r} B_n^{(r)}(t)}{t - z_{n,i}^{(r)}}\, d\psi(t), \quad \text{for } i = 1, 2, \ldots, n. \tag{4.2}$$

From Eq. (2.1) it follows that $\mathbb{E}_n(f) = 0$ whenever $z^{n-r} f(z) \in \mathbb{P}_{2n-1}$. We can prove that the weights $w_{n,i}^{(r)}$, $i = 1, 2, \ldots, n$, are positive by considering the special cases where $f(t) = t^{-n+r}\{B_n^{(r)}(t)/t - z_{n,i}^{(r)}\}^2$, $i = 1, 2, \ldots, n$.

The associated polynomial $C_n^{(r)}(z)$ is of degree $n-1$, and can be defined from the polynomial $B_n^{(r)}(z)$ by

$$C_n^{(r)}(z) = \int_a^b t^r \frac{B_n^{(r)}(z) - B_n^{(r)}(t)}{z-t}\, \mathrm{d}\psi(t), \quad n \geqslant 0. \tag{4.3}$$

From Eq. (4.3)

$$C_n^{(r)}(z) = \int_a^b t^{r-p} \frac{t^p B_n^{(r)}(z) - t^p B_n^{(r)}(t)}{z-t}\, \mathrm{d}\psi(t).$$

Adding and subtracting $z^p B_n^{(r)}(t)$ in the numerator of the integrand we get

$$C_n^{(r)}(z) = \int_a^b t^{r-p} \frac{t^p B_n^{(r)}(z) - z^p B_n^{(r)}(t)}{z-t} \mathrm{d}\psi(t) + \int_a^b t^{r-p} \left(\frac{z^p - t^p}{z-t}\right) B_n^{(r)}(t)\, \mathrm{d}\psi(t),$$

and using Eq. (2.1) we can also write the polynomial $C_n^{(r)}(z)$, as

$$C_n^{(r)}(z) = \int_a^b t^{r-p} \frac{t^p B_n^{(r)}(z) - z^p B_n^{(r)}(t)}{z-t}\, \mathrm{d}\psi(t), \quad 0 \leqslant p \leqslant n, \quad n \geqslant 0. \tag{4.4}$$

Then using Eqs. (4.2) and (4.4) with $p=n$ and $z = z_{n,i}^{(r)}$, $i = 1, 2, \ldots, n$, we can easily prove that the weights $w_{n,i}^{(r)}$, $i = 1, 2, \ldots, n$, are given by

$$w_{n,i}^{(r)} = (z_{n,i}^{(r)})^{-r} \frac{C_n^{(r)}(z_{n,i}^{(r)})}{B_n^{(r)\prime}(z_{n,i}^{(r)})}. \tag{4.5}$$

Finally, we prove two further consequences of the symmetry.

Theorem 4.1. *Let $\psi(t)$ be an $S^3[\omega, \beta, b]$ distribution with $0 < \beta < b \leqslant \infty$, $a = \beta^2/b$ and $2\omega \in \mathbb{Z}$. Then for $n \geqslant 1$ and $j = 1 - 2\omega$,*

$$\frac{w_{n,i}^{(l)}}{(z_{n,i}^{(l)})^{\omega}} = \frac{w_{n,n+1-i}^{(j-l)}}{(z_{n,n+1-i}^{(j-l)})^{\omega}}, \quad l = 0, \pm 1, \pm 2, \ldots . \tag{4.6}$$

Proof. From Eq. (3.1) we see that for $l = 0, \pm 1, \pm 2, \ldots$,

$$B_n^{(l)\prime}(\beta^2/z) = \beta^{-2} B_n^{(l)}(0)[n z^{-n+1} B_n^{(j-l)}(z) - z^{-n+2} B_n^{(j-l)\prime}(z)], \tag{4.7}$$

and using Eqs. (4.4) and (3.1) we can deduce that

$$C_n^{(l)}(\beta^2/z) = -\beta^{2(l-1+\omega)} z^{-n+1} B_n^{(l)}(0) C_n^{(j-l)}(z). \tag{4.8}$$

From Eqs. (3.2) and (4.5) we have

$$w_{n,i}^{(l)} = (z_{n,i}^{(l)})^{-l} \frac{C_n^{(l)}(z_{n,i}^{(l)})}{B_n^{(l)\prime}(z_{n,i}^{(l)})} = (z_{n,i}^{(l)})^{-l} \frac{C_n^{(l)}(\beta^2/z_{n,n+1-i}^{(j-l)})}{B_n^{(l)\prime}(\beta^2/z_{n,n+1-i}^{(j-l)})},$$

and substituting z by $z_{n,n+1-i}^{(j-l)}$ in Eqs. (4.7) and (4.8) we get

$$w_{n,i}^{(l)} = (z_{n,i}^{(l)})^{-l} \frac{\beta^{2(l+\omega)} C_n^{(j-l)}(z_{n,n+1-i}^{(j-l)})}{z_{n,n+1-i}^{(j-l)} B_n^{(j-l)\prime}(z_{n,n+1-i}^{(j-l)})}.$$

Since $\beta^2 = z_{n,i}^{(l)} z_{n,n+1-i}^{(j-l)}$ then

$$w_{n,i}^{(l)} = \frac{(z_{n,i}^{(l)})^{-l}(z_{n,i}^{(l)})^{l+\omega}(z_{n,n+1-i}^{(j-l)})^{l+\omega} C_n^{(j-l)}(z_{n,n+1-i}^{(j-l)})}{z_{n,n+1-i}^{(j-l)} B_n^{(j-l)\prime}(z_{n,n+1-i}^{(j-l)})} = \frac{(z_{n,i}^{(l)})^{\omega}(z_{n,n+1-i}^{(j-l)})^{l+2\omega} C_n^{(j-l)}(z_{n,n+1-i}^{(j-l)})}{(z_{n,n+1-i}^{(j-l)})^{1+\omega} B_n^{(j-l)\prime}(z_{n,n+1-i}^{(j-l)})}.$$

With $j = 1-2\omega$ and using again Eq. (4.5),

$$w_{n,i}^{(l)} = \frac{(z_{n,i}^{(l)})^{\omega}(z_{n,n+1-i}^{(j-l)})^{1-(j-l)} C_n^{(j-l)}(z_{n,n+1-i}^{(j-l)})}{(z_{n,n+1-i}^{(j-l)})^{1+\omega} B_n^{(j-l)\prime}(z_{n,n+1-i}^{(j-l)})} = \frac{(z_{n,i}^{(l)})^{\omega}}{(z_{n,n+1-i}^{(j-l)})^{\omega}} w_{n,n+1-i}^{(j-l)},$$

which completes the proof. □

Theorem 4.2. *Let $\psi(t)$ be an $S^3[\omega,\beta,b]$ distribution with $0<\beta<b\leqslant\infty$, $a=\beta^2/b$ and $2\omega\in\mathbb{Z}$. Then for $n\geqslant 1$ and $j=1-2\omega$,*

$$\frac{z^{n-1}C_n^{(l)}(\beta^2/z)}{C_n^{(l)}(0)} = \frac{C_n^{(j-l)}(z)}{\mu_{j-l}}, \quad l=0,\pm1,\pm2,\ldots, \quad n\geqslant 1. \tag{4.9}$$

Proof. From Eq. (4.8) with $j=1-2\omega$, we can write

$$\frac{z^{n-1}C_n^{(l)}(\beta^2/z)}{C_n^{(l)}(0)} = -\beta^{2l-1-j}\frac{B_n^{(l)}(0)}{C_n^{(l)}(0)} C_n^{(j-l)}(z), \quad l=0,\pm1,\pm2,\ldots\,.$$

As

$$B_n^{(l)}(0) = (-1)^n \beta_n^{(l)}\beta_{n-1}^{(l)}\cdots\beta_2^{(l)}\beta_1^{(l)} \quad \text{with } \beta_1^{(l)} = \mu_l/\mu_{l-1},$$

$$C_n^{(l)}(0) = (-1)^{n-1}\beta_n^{(l)}\beta_{n-1}^{(l)}\cdots\beta_2^{(l)}\mu_l,$$

and using Eq. (1.3), the result holds. □

As before, the results of Theorems 4.1 and 4.2 can be divided according to the two cases where $j=-\omega+1/2$ when 2ω is odd and $j=-\omega$ when 2ω is even.

5. Examples

(1) The generalized log-normal distribution defined by

$$d\psi_p(t) = \frac{\sqrt{q}}{2\lambda\sqrt{\pi}} t^p \exp\left[-\left(\frac{\ln(t)}{2\lambda}\right)^2\right] dt,$$

in $(0,\infty)$, with $0<q<1$, $q=e^{-2\lambda^2}$, is an $S^3[\omega,\beta,b]$ distribution with $\beta=q^{\omega-p-1}$.

This example has been considered by many authors (see [2,4] for example). It was presented as the first explicit example of orthogonal Laurent polynomials by Pastro in [9]. In the symmetric case, it has been considered as an $S^3[0,1,\infty]$ with $p=-1$ by Sri Ranga, de Andrade and McCabe [11] and by Common and McCabe [3]. It has also been treated as an $S^3[1/2,q^{-1/2},\infty]$ with $p=0$ by Sri Ranga in [10].

Let us first consider the classical log-normal distribution where $p=0$. Then $\beta=q^{\omega-1}$ and $d\psi_0(t)$ is an $S^3[w,q^{\omega-1},\infty]$ distribution. The coefficients $\beta_n^{(r)}$, $n\geqslant 0$, are constants for r fixed. In fact, using the equations (2.8) and the moments

$$\mu_m=q^{-(m+m^2/2)},\quad m=0,\pm1,\pm2,\ldots,$$

we calculate the coefficients

$$\beta_n^{(r)}=q^{-\left(\frac{1}{2}+r\right)}\quad\text{and}\quad\alpha_{n+1}^{(r)}=q^{-\left(\frac{1}{2}+r\right)}(q^{-n}-1),\quad n\geqslant 1.$$

We can prove by mathematical induction and by using properties of the q-binomial coefficients that

$$B_n^{(r)}(z)=\sum_{j=0}^{n}(-1)^j q^{-j\left(\frac{1}{2}+r\right)}q^{j(j-n)}\begin{bmatrix}n\\j\end{bmatrix}z^{n-j},\quad n\geqslant 0,$$

where $\begin{bmatrix}n\\j\end{bmatrix}$ are the q-binomial coefficients given, for $n\geqslant 0$, by

$$\begin{bmatrix}n\\j\end{bmatrix}=\frac{\prod_{k=1}^{n}(1-q^k)}{\prod_{k=1}^{j}(1-q^k)\prod_{k=1}^{n-j}(1-q^k)},\quad 1\leqslant j\leqslant n-1,\quad \begin{bmatrix}n\\0\end{bmatrix}=\begin{bmatrix}n\\n\end{bmatrix}=1.$$

It is easy to see from Theorem 3.1 that for $s=0,\pm1,\pm2,\ldots$

$$\frac{z^nB_n^{(l)}(q^s/z)}{B_n^{(l)}(0)}=B_n^{(-s-1-l)}(z),\quad l=0,\pm1,\pm2,\ldots.$$

The generalized log-normal distribution can be given, in terms of the classical one, by $d\psi_p(t)=t^p\,d\psi_0(t)$. The moments for the generalized log-normal distribution are $\mu_m^p=\mu_{m+p}=q^{(1-(1+m+p)^2)/2}$, $m=0,\pm1,\pm2,\ldots$.

Consequently, the coefficients $\beta_n^{(p,r)}$, $\alpha_{n+1}^{(p,r)}$, $n\geqslant 1$ and the polynomials $B_n^{(p,r)}(z)$, $n\geqslant 0$, for the generalized log-normal distribution are, respectively, the coefficients $\beta_n^{(r+p)}$, $\alpha_{n+1}^{(r+p)}$, $n\geqslant 1$ and the polynomials $B_n^{(r+p)}(z)$, $n\geqslant 0$, for the classical distribution.

Other examples of $S^3[\omega,\beta,b]$ distribution:

(2) The distribution

$$d\psi(t)=\frac{(1+\beta/t)^{1-2\omega}}{\sqrt{b-t}\sqrt{t-a}}\,dt,$$

defined on (a,b), where $0<\beta<b<\infty$, with $\beta=\sqrt{ab}$ is an $S^3[\omega,\beta,b]$ distribution. Special cases of this distribution were considered by Sri Ranga et al. in [10–13].

(3) The distribution

$$d\psi(t)=t^{\omega-1}\,dt,$$

again defined on (a,b), where $0<\beta<b<\infty$, with $\beta=\sqrt{ab}$ is an $S^3[\omega,\beta,b]$ distribution.

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