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Dressed Perturbation Theory for the Quark Propagator

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Resumo

A aproximação "rainbow-ladder" (arcoíris-escada) é um esquema de truncamento não-perturbativo simples para cálculos não-perturbativos na cromodinâmica quântica (QCD). Ela fornece resultados satisfatórios para os estados fundamentais de mésons pseudo-escalares e vetoriais, mas para estados excitados e estados exóticos, como também para mésons pesados-leves, este esquema de truncamento fornece resultados menos precisos. Na presente dissertação propomos um novo método que vai além do truncamento "rainbow-ladder", que denominamos "dressed perturbation theory" (teoria de perturbação vestida). O princípio norteador do método é modificar a parte quadrática da ação da QCD pela adição de termos de auto-energia para os quarks e glúons e efetuar teoria de perturbação com a parte de interação da ação, subtraída das mesmas auto-energias. Ao invés de propagadores não-interagentes dos tradicionais cálculos de QCD perturbativa, o novo esquema perturbativo é implementado com propagadores vestidos. As auto-energias empregadas nos propagadores vestidos podem ser tomadas da aproximação "ladder", ou de simulações de QCD na rede. Nossa preocupação principal nesta dissertação é a aplicação do novo formalismo ao propagador dos quarks. As correções não-triviais de ordem mais baixa são correções de vértice. Resultados numéricos são obtidos para a função de massa dos quarks e propriedades do pión no limite quirial para um ansatz simples para a auto-energia do glúon.

Palavras chaves: Cromodinâmica quântica; simetria quirial; quebra espontânea de simetria; modelo de Nambu-Jona-Lasinio; equações de Schwinger-Dyson; propagador dos quarks, propagador dos glúons.

Areas: Física de Hadrons, Física de partículas .

Abstract

The rainbow-ladder approximation is a simple nonperturbative truncation scheme for nonperturbative calculations in Quantum Chromodynamics (QCD). It provides satisfactory results for the ground-states of light pseudoscalar and vector mesons, but for excited and exotic states, as well as for heavy-light mesons, this truncation provides less accurate results. In the present dissertation we have proposed a novel approach to go beyond the rainbow-ladder truncation, which we name dressed perturbation theory. The guiding principle of the approach is to modify the quadratic part of the QCD action by the addition of self-energy terms for the quarks and gluons, and perform perturbative calculations with the interaction action subtracted by the same self-energies. Instead of noninteracting propagators of traditional perturbative QCD calculations, the novel perturbative scheme is performed with dressed propagators. The self-energies used in the dressed propagators can be taken from the ladder approximation, or from lattice QCD simulations. Our main concern in the dissertation is the application of the formalism to the quark propagator. The lowest nontrivial order corrections are two-loop vertex corrections. Numerical results are obtained for quark mass function and pion properties in the chiral limit for a simple ansatz for the gluon self-energy.

Key words: Quantum chromodynamics; chiral symmetry; spontaneous symmetry breaking; Nambu-Jona-Lasinio model; Schwinger-Dyson equations; quark propagator; gluon propagator.

Areas: Hadron Physics, Particle Physics.

Índice

1	INTRODUCTION	vi
2	QUANTUM CHROMODYNAMICS	1
2.1	The Lagrangian of QCD	7
2.2	Internal symmetries of QCD	8
2.2.1	Isospin Symmetry	9
2.2.2	Chiral Symmetry	10
2.3	Dynamical Chiral Symmetry Breaking	11
2.3.1	The Nambu–Jona-Lasinio model	12
2.4	Feynman Rules for the QCD Lagrangian	22
2.5	Summary	24
3	DYSON-SCHWINGER EQUATIONS	26
3.1	Derivation of DSEs in QED	31
3.1.1	DSE for the photon propagator	31
3.1.2	DSE for the electron propagator	34
3.1.3	DSE for the electron-photon vertex function	36
3.2	DSE for the quark propagator	37
3.2.1	The rainbow approximation and confining contact interaction	39
3.3	Summary	42
4	THE BETHE-SALPETER EQUATION	44
4.1	Derivation of the BSE	45
4.2	The BSE in Momentum Space	49
4.3	The Homogeneous BSE	50
4.4	Normalization condition	52
4.5	The Ladder Approximation	54
4.6	Structure of the Meson BSA	54

4.7	Application for the pion	58
4.7.1	Leptonic decay constant	59
4.7.2	The pion BSE amplitude	60
4.7.3	Numerical results	65
4.8	Summary	65
5	DRESSED PERTURBATION THEORY	67
5.1	Perturbation theory with dressed propagators	67
5.2	Application: quark propagator, contact interaction	75
5.2.1	Abelian contribution	77
5.2.2	Non-Abelian contribution	81
5.3	Results and Discussions	82
6	CONCLUSIONS AND OUTLOOK	87
A	NOTATIONS AND CONVENTIONS	90
A.1	Minkowski Space Conventions	90
A.2	Dirac Matrices	91
A.3	Euclidean Space Conventions	93
B	EUCLIDEAN QCD	95
C	PROPER TIME REGULARIZATION	97
D	The Ward-Takahashi Identity	99
D.1	Ward-Takahashi Identity in QED	99
D.2	Slavnov-Taylor Identities	101
	Bibliography	102

Chapter 1

INTRODUCTION

It is a well established fact that hadronic processes at large momentum transfers are well described with perturbative calculations in quantum chromodynamics (QCD). The basic reason for the applicability of perturbative calculations in a theory meant to describe the strong interactions is asymptotic freedom [1], in that the interaction strength of QCD becomes weak at the small distances probed with large-momentum transfer processes. The smallness of the coupling is a prerequisite for a calculation based on an expansion in the coupling constant. Moreover, the calculations can be implemented in a systematic and well controlled fashion. But at small momentum transfers, or large distances, the QCD coupling is not small and one reaches the realm of strong QCD, where perturbation theory as an expansion in the coupling constant is not valid. Moreover, at such scales, the prominent nonperturbative phenomena of color confinement and dynamical chiral symmetry breaking set in. Confinement refers to the experimental fact that quarks and gluons are confined within hadrons; they do not appear in nature as asymptotic states like electrons and photons, for example. Dynamical chiral symmetry breaking refers to the phenomenon of mass generation, in that from essentially massless quarks, hadrons masses are generated by the fundamental quark-gluon interaction.

According to the standard model of particle physics, the masses of the quarks and leptons arise from the Higgs mechanism [2,3]. However, the main contribution to the mass of the visible matter of the universe has a different origin. Indeed, the “usual matter” resides in nuclei composed of nucleons, the protons and neutrons. But the issue here is that one cannot explain the hadrons masses as arising solely from the masses of the quarks in the QCD Lagrangian which are generated by the Higgs mechanism. Consider, for example, the proton, which is composed of three

light quarks (two u and one d). The mass of the u quark is of the order of 3 MeV and of the d is of the order of 5 MeV, so that summing up the masses of the three quarks gives less than 15 MeV, while the proton has a mass around 1000 MeV. Thus, the Higgs mechanism, responsible for masses of the quarks u and d that appear in the QCD Lagrangian, is irrelevant for expanding the origin of mass of the visible matter in the universe.

The quark model, which ultimately was responsible for the establishment of QCD, is built on the premise that a proton is made up of three quarks with a mass of the order of 300 MeV each. Within the same quark model, on the other hand, it is extremely difficult to accommodate the pion, which has a mass of the order of 140 MeV, as a bound state of a quark and an antiquark of 300 MeV each. The fundamental question is: How does one make an almost massless (on the scale of the hadron masses) pion from such massive quarks? Here, the explanation comes from the phenomenon of dynamical chiral symmetry breaking mentioned above. The connection between the masses of the QCD Lagrangian, known as current quark masses, and those of the quark model, known as constituent quarks, is a direct result of the complicated structure of the strong interaction which binds quarks and gluons into hadrons. Hence, understanding of our “massive world” requires understanding the strong interactions at the confinement scale is one of the most important open problems in contemporary physics and is one of the main goals of hadron physics.

Confinement of quarks and gluons, dynamical chiral symmetry breaking (DCSB) and the formation of bound states, as already mentioned, are nonperturbative phenomena. Two important nonperturbative, first principles methods for the study of these phenomena are well known; lattice QCD and Dyson-Schwinger equations (DSEs). Lattice QCD was invented by K. Wilson in 1974 [4]; it is based on the formulation of the theory on a discrete space-time Euclidean lattice, which lends the calculations of correlation functions and observables amenable to numerical simulations with Monte Carlo methods in supercomputers. It has been a successful source of qualitative and quantitative information about QCD [5]. As a major achievement one can mention the calculation of mass of the proton within an error of less than 2 percent [6]. The second approach is based on functional methods in the continuum for correlation functions and bound-state equations [7,8], described

by Dyson-Schwinger and Bethe-Salpeter equations [9,10,11]. The phenomenon of DCSB, reflected e.g. in the smallness of the pion mass, is readily accessible with a well-controlled approximation scheme [12]. Also, the pion weak decay constant f_π is a key observable related to DCSB and is equally well described within the same scheme.

Since the DSEs are a set of infinite coupled integral equations, one needs to introduce a truncation scheme. This also applies to the BSEs, because they require e.g. the four-point scattering kernel which, in turn, is part of the hierarchy of DSEs. The best known and most studied truncation scheme is the rainbow-ladder approximation [13,14,15,16]. It preserves various symmetries of QCD, such as chiral symmetry, Lorentz invariance and, in addition, satisfies renormalization group invariance. However, there are significant problems associated with the rainbow-ladder approximation such as, the loss of gauge covariance, which is a direct consequence of the violation of a Ward identity [9]. In the context of the Bethe-Salpeter equations, it describes well light-quark $q\bar{q}$ states, but has difficulties with heavy-light systems [17]. Such difficulties impact, e.g. determination of charmed-hadron couplings, a subject of immense current interest [19].

The problems with the rainbow-ladder scheme call for improvements. The present dissertation is an attempt in this direction. We propose a novel approach to go beyond the rainbow-ladder truncation, which we name *dressed perturbation theory*. The idea of the approach is to modify the quadratic part of the QCD action by the addition of self-energy terms for the quarks and gluons, and perform perturbative calculations with the interaction action subtracted by the same self-energies. Instead of noninteracting propagators of traditional perturbative QCD calculations, the novel perturbative scheme is performed with dressed propagators. The self-energies used in the dressed propagators can be taken from the ladder approximation, or from lattice QCD simulations. Dressed perturbation theory borrows elements from similar schemes, like the linear δ expansion [20], the optimized perturbation theory [21] and screened perturbation theory [22]. Essentially, in all of these approaches the basic idea consists in adding a constant-mass term in the quadratic action of the theory and subtract the same mass term from the original interaction Lagrangian, and perform a perturbative calculation with the modified interaction Lagrangian. The linear δ expansion, for example, has been successfully

applied to many different problems in field theory [23,24,25], statistical physics [26], quantum mechanics [27], relativistic nuclear models [28,29] and also in the Nambu–Jona-Lasinio model [30] for DCSB in QCD [31]. Corrections to correlation functions and observables beyond zeroth order are calculated with the modified interactions; they involve propagators with a mass that is different from the mass of the original Lagrangian. The approaches differ in the way the added mass term is determined. In dressed perturbation theory, the added and subtracted term is not a constant-mass term; it is a self-energy that is not constant. Moreover, these self-energies can be taken from a rainbow-ladder approximation, or from lattice QCD simulations [32].

Our main concern in the present dissertation is the application of the novel formalism to the quark propagator. As we will show, the lowest nontrivial order contribution to the propagator are two-loop vertex corrections. Since this is the first application of this novel formalism, we obtain explicit numerical results using a simple model for the gluon propagator. This is to gain insight into the formalism and set up our codes for more realistic situations. In addition, in view of the limited aims of the present work, we will not consider vertex corrections to the Bethe-Salpeter equation. In doing so, it should be kept in mind that our results for the pion properties are not entirely consistent, since we are correcting only the quark propagator in the equation, but not the scattering kernel. Numerical results are obtained for the quark mass function and pion properties in the chiral limit for a simple ansatz for the gluon self-energy. Although the formalism is much more general than the simple application made here, we believe that the application made serves the purpose of showing how the formalism can be applied to more general and realistic settings.

The dissertation is organized in the following manner. In the next chapter we start with a historical background of QCD and then we review its basic elements. Here we write down the basic Lagrangian of QCD describing its complete structure and subsequently we discuss with some details two important symmetries that are characteristic of this Lagrangian. After of this we discuss the mechanism of dynamical mass generation as a consequence of dynamical chiral symmetry breaking, in particular we review the Nambu-Jona-Lasinio model. We end the chapter writing down the Feynman rules for QCD. The third chapter is devoted to the Dyson-Schwinger equations (DSEs), where we show the general procedure of their derivation in field

theory. In particular, we derive the DSEs for the photon and electron propagators in quantum electrodynamics. Then, we write down the corresponding DSEs for QCD, where we examine the gap equation for a contact interaction. In chapter four we review in detail the Bethe-Salpeter equation, which allows to study relativistic two-body bound state. We show the derivation of the inhomogeneous and homogeneous Bethe-Salpeter equations and its normalization condition. Then, we write down the homogeneous Bethe-Salpeter equation for mesons, with special interest in pseudoscalar particles. Specifically, we review the pion Bethe-Salpeter equation in the rainbow-ladder truncation with a contact interaction ansatz for the gluon propagator. We evaluate the matrix element giving the leptonic decay constant of the pion in terms of its Bethe-Salpeter amplitude. In the chapter five we introduce the dressed perturbation theory. Using the approach of dressed perturbation theory, we calculate the first nontrivial correction to the quark propagator in terms of the dressed quark and gluon propagators. Next, we use contact interaction ansatz for the gluon propagator and evaluate explicitly the corrections to the quark propagator. With this we present explicit results for the dressed-quark mass function, $M(p^2)$ and the wave-function renormalization $Z(p^2)$. We then use the new quark propagator to calculate some physical quantities in the chiral limit, such as quark condensate and the pion decay constant. Finally, in the last chapter of the dissertation we present our conclusions and discuss future directions. The dissertation also contains four appendices, where supplemental material is collected.

Chapter 2

QUANTUM CHROMODYNAMICS

Quantum Chromodynamics (QCD) is the fundamental theory that describes the strong interactions; it is part of the Standard Model of Fundamental Particles and Interactions [2]. The basic degrees of freedom of QCD are quarks and gluons [33], but only their bound states, the hadrons, are observed in particle detectors. The construction of QCD is the culmination of a long series of attempts to construct a theory for the strong interactions based on the tenets of quantum field theory. The first attempt was led by Yukawa in 1935 [34]. In the field theory proposed by Yukawa, the interaction of the nucleons was supposed to be mediated by a bosonic (meson) field whose quanta would represent a new kind of particle with nonzero mass. The requirement for a nonzero mass for the meson is because the force is known to be of short range, a feature that is an important difference from QED, where the interaction is of long range and the particles mediating the interactions are massless (photons). In 1947 the existence of the postulated meson was confirmed with the discovery* of the pion (π -meson) in cosmic rays [35]. The pion was seen in three charge states, π^+ , π^0 and π^- , which form an isospin triplet.

The Lagrangian of the meson-nucleon theory takes the form

$$\mathcal{L}_{Yukawa} = i\bar{N}\not{\partial}N + \frac{1}{2}(\partial_\mu\phi)^2 - m\bar{N}N - \frac{\mu^2}{2}\phi^2 - ig\bar{N}\gamma_5\tau_i\phi_iN, \quad (2.1)$$

where τ_i , $i = 1, 2, 3$ are the Pauli matrices and g is the coupling constant. In the language of group theory, one has that the nucleons form an isospin doublet, the fundamental representation of the isospin symmetry group $SU(2)$, and the triplet of pions are in the adjoint (or regular) representation of the group. Since after the

*The π -meson was discovered by the group lead by Cecil Powell, who in 1950 won the Nobel Prize in Physics, for his development of the photographic method of studying nuclear processes.

discovery of the pion many more hadrons[†] were discovered in cosmic rays and also in accelerator experiments. Extension of the Lagrangian (2.1) to include the newly discovered particles soon lead to theoretical difficulties. The main problem of the theory was that the coupling constant g was rather big

$$\frac{g^2}{4\pi} \simeq 14, \quad (2.2)$$

this is ~ 1000 times larger than the QED coupling, $\alpha = e^2/4\pi \sim 1/137$, so that perturbative methods based on a series expansion in g become inapplicable and nonperturbative methods need to be employed. The problem with this is the absence of analytical calculation methods in quantum field theory to treat nonperturbative phenomena that are at the same time systematic and allow approximations with controllable errors. This was the situation in the past and is still the case today. In fact, this situation led many physicists in the 1960's to oppose the use of quantum field theory for the strong interactions and advocate new methods, like S-matrix methods – for a short historical perspective, see Ref. [36]. We will come back to this shortly ahead.

The fact that many particles and resonances were discovered in the fifties led to the question of how to explain this wide range of new particles discovered in accelerator experiments. During the 1950s and 1960s, Gell-Mann and others postulated an organizational principle to put order in this new zoo of strongly interacting particles, similarly as done by Mendeleev for the chemical elements [37]. Besides the then familiar isospin quantum number, it was necessary to introduce an additional quantum number, strangeness (S), to accommodate the newly discovered Λ and K particle: they were found to be produced copiously but to decay slowly [38]. S is a conserved quantum number in the strong interaction but is violated in decay by weak interaction, as $\Lambda \rightarrow N + \pi$, or $\Lambda \rightarrow p + e^- + \bar{\nu}$. With this, it was found that all known hadrons at that time could be grouped into octet and decuplet representations of an internal symmetry group, the $SU(3)$ flavor symmetry group. This organization is known as the “eightfold way”.

The eightfold way refers to the classification of the lowest-lying pseudoscalar mesons and spin-1/2 baryons within octets in $SU(3)$ -flavor space (u, d, s). The same

[†]The word hadron is derived from the Greek hadros, which means “bulky”. Particles that interact by the strong interaction are called hadrons. There are two families of hadrons, (integer spin) mesons and (half-integer spin) baryons.

principles of the eightfold way allow to group the ground state spin-3/2 baryons into a decuplet - Fig.2.1 is a pictorial representation of the classification. The most famous prediction of the scheme was the existence of the hyperon (Ω^-), that it would have a mass of 1680MeV, strangeness-3 and electric charge -1; this prediction was soon confirmed by a particle accelerator group at Brookhaven National Laboratory in the USA [39].

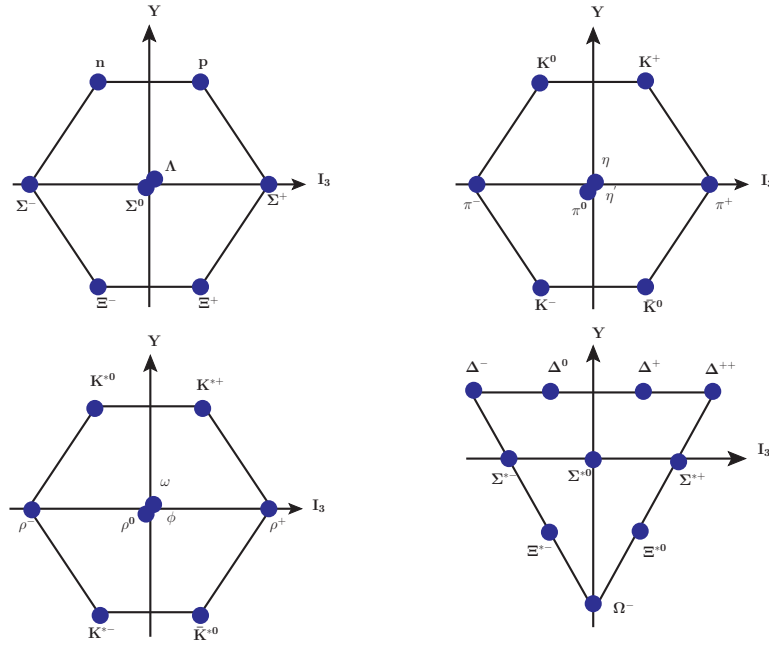


Figure 2.1: Hadrons in SU(3) representations within the “eightfold way”.

Only octets, singlets and decuplets have been observed in experiments, but no triplet, which corresponds to the fundamental representation of SU(3). This fact led Gell-Mann to propose that the hadrons are composed of even more elementary constituents, which called quarks. The quarks come in three flavors, up (u), down (d) and strange (s), which forming a pattern based

$$\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{s} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad (2.3)$$

and the anti-quarks

$$\bar{\mathbf{u}} = (1 \ 0 \ 0), \quad \bar{\mathbf{s}} = (0 \ 1 \ 0), \quad \bar{\mathbf{s}} = (0 \ 0 \ 1). \quad (2.4)$$

The quarks and anti-quarks form a triangular eightfold way pattern, as indicated in Fig. 2.2. The pion π^+ , for example, can be represented as

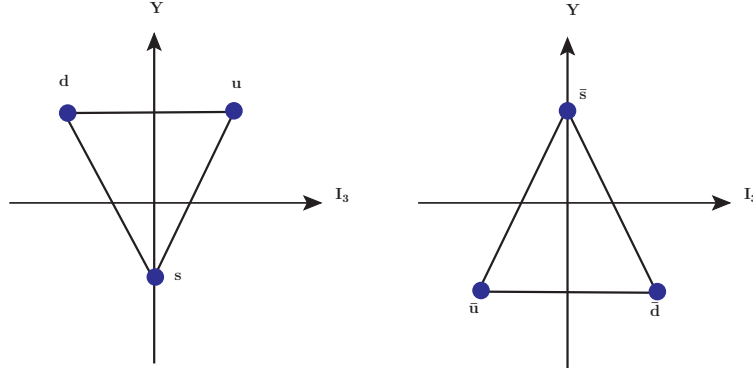


Figure 2.2: Triplet of quarks and antitriplet of anti-quarks.

$$\pi^+ = \mathbf{u} \otimes \bar{\mathbf{d}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (2.5)$$

and all mesons can be built from the bases (2.3) and (2.4). The basic reduction of products of representation that are important for the quark model are

$$\text{Barion} = \mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}, \quad (2.6)$$

$$\text{Meson} = \mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}. \quad (2.7)$$

On the other hand, since the electric charges of mesons and baryons were well known to be integer multiples of the elementary electric charge e , it was necessary to postulate that the quarks had fractional electric charges, namely

$$Q_u = \frac{2}{3}e; \quad Q_d = -\frac{1}{3}e; \quad Q_s = -\frac{1}{3}e. \quad (2.8)$$

The properties of the u, d, s quarks are summarized in the following table. Here, B is the baryon number, I isospin, I_3 third component of isospin, S strangeness and Q electric charge.

In the following years, new hadrons were discovered that required the introduction of new flavors. Today, there are six flavors known: u (up), d (down), s (strange), c (charm), b (bottom), and t (top). The quark model was very successful in prediction of new hadronic states, and in explaining the strengths of

q	B	I	I_3	S	Q
u	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{2}{3}$
d	$\frac{1}{3}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{3}$
s	$\frac{1}{3}$	0	0	-1	$\frac{2}{3}$

Table 2.1: Properties of the quarks

electromagnetic and weak-interaction transitions among hadrons. However, the dynamic nature of the forces binding together the quarks in hadrons remained completely unknown. Progress has been made with the nonrelativistic constituent quark model [40]: baryons are bound states of three quarks and mesons of a quark and an antiquark, the quarks and antiquarks are fermions of spin-1/2 with a mass of the order of 330 MeV, and are confined by a phenomenological potential to be used in a nonrelativistic Schrödinger equation. If in addition spin-dependent forces are included in the potential, a very decent description of the low-lying hadron states are reproduced by such a simple model. However, the simple model had two fundamental problems. The first was that free particles with fractional electric charges were never seen in detectors. The second difficulty was of theoretical nature. According to the eightfold way, the particle Δ^{++} is made of three quarks u in an S state with the same spin orientation, namely

$$|\Delta^{++}\rangle = |u_{\uparrow}u_{\uparrow}u_{\uparrow}\rangle, \quad (2.9)$$

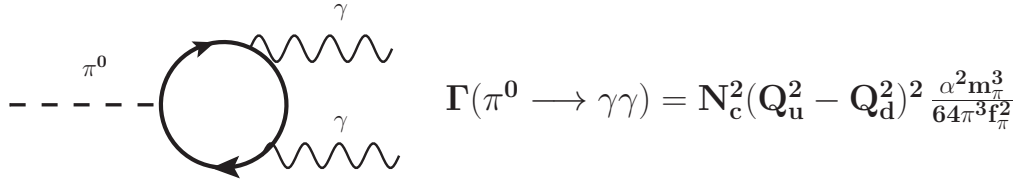
clearly a highly symmetric configuration. However Δ^{++} is a fermion, it must have an overall antisymmetric wave function under exchange of the quark spin and flavor quantum number. So that the Pauli principle appears to be violated. While the “solution” for the first problem came with the confinement hypothesis, that is, the nature of the forces that bind quarks into hadrons are such that they are permanently confined to the interior of hadrons, the solution to the second problem came in 1965, fourteen years after the discovery of the Δ^{++} : a solution was proposed by postulating a new quantum number for each flavor of quark, named color charge, associated with the group $SU(3)$ [41,42]. This color $SU(3)$ symmetry is exact, i.e. not broken like the flavor symmetry. So the Pauli principle is not violated because we can write the

wave function for the baryons in the antisymmetric form

$$|\Delta^{++}\rangle = \epsilon^{ijk} |u_{i\uparrow} u_{j\uparrow} u_{k\uparrow}\rangle , \quad (2.10)$$

where $i, j, k = 1, 2, 3$ are the color indices of the fundamental representation of $SU(3)$. Experimental confirmation of number of colors were provided first by measurements of the decay width of $\pi^0 \rightarrow \gamma\gamma$, which is proportional to N_c^2 [33], see Fig. 2.3, with N_c being the number colors. The second confirmation was given by the famous ratio of e^+e^- collisions [33]:

$$R = \frac{\sigma(e^+e^- \rightarrow q\bar{q})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} = \sum_{flavor} N_c Q_q^2 . \quad (2.11)$$



$$\Gamma(\pi^0 \rightarrow \gamma\gamma) = N_c^2 (Q_u^2 - Q_d^2)^2 \frac{\alpha^2 m_\pi^3}{64\pi^3 f_\pi^2}$$

Figure 2.3: Decay width of $\pi^0 \rightarrow \gamma\gamma$.

It turned out that the color degree of freedom plays a fundamental role in the formulation of a fundamental theory for the strong interactions. The field theory on whose basis QCD was built, Yang-Mills theory, had actually been invented in 1954 [43]. Yang and Mills generalized the principle of local invariance to the non-Abelian gauge group $SU(2)$ and wrote the Lagrangian

$$\mathcal{L}_{YM} = -\frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu} . \quad (2.12)$$

Yang and Mills added a term of mass and the Lagrangian (2.12) became

$$\mathcal{L}_{YM} = -\frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu} - \frac{1}{2} m^2 (A_\mu^a)^2 , \quad (2.13)$$

and tried to associate the resulting massive vector with the ρ mesons. This attempt did not work. The Yang-Mills theory was not considered to be a realistic physical theory during the following 15 years. The first successful application the Yang-Mills theory was the unification of the weak and electromagnetic interactions, by Weinberg and Salam in 1967 [44,45].

The Yang-Mills theory describes not only the physics of the weak interaction but also the strong interaction. Salam, Fritzsche, Gell-Mann, Leutwyler, and Weinberg suggested that the quarks interact with each other by exchange of gluons, massless vector particles belonging to the octet (adjunct) representation of SU(3). The gluons also interact among themselves, as is dictated by the Yang-Mills Lagrangian with the SU(3) color gauge group. In the next section we will discuss this Lagrangian.

2.1 The Lagrangian of QCD

The Lagrangian density of QCD in Minkowski space, with metric signature $(+ - - -)$, is given by

$$\mathcal{L}_{QCD} = \sum_j^{N_j} \bar{\psi}_j^c (i\gamma^\mu D_\mu^{cc'} - \delta^{cc'} m_j) \psi_j^{c'} - \frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu} , \quad (2.14)$$

where ψ_j are the quark fields, with $j = 1, \dots, N_j$ being the quark flavor index ($\psi_j : u, d, s, c, b, \dots$), the indices $c, c' = 1, 2, 3$ and $a = 1, \dots, 8$ are the color indices in the fundamental and adjoint representation of SU(3) respectively, γ^μ are the Dirac matrices, $\mu, \nu = 0, \dots, 3$ are the Lorenz indices, and D_μ is the covariant derivative (minimal coupling):

$$D_\mu^{cc'} = \delta^{cc'} \partial_\mu - ig t_a^{cc'} A_\mu^a , \quad (2.15)$$

with g the strong coupling constant, A_μ^a are the gauge fields, gluons, which are the carrier of the SU(3) colors forces, and $F_{\mu\nu}^a$ (2.14) is the field strength tensor:

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc} A_\mu^b A_\nu^c , \quad (2.16)$$

where the f^{abc} are the structure constants of SU(3). The Lagrangian in (2.14) is invariant under global and local gauge transformations. Under a local gauge transformation, the quarks and gluons fields transform as

$$\psi \longrightarrow \psi' = U(\theta)\psi , \quad (2.17)$$

$$A_\mu^a \longrightarrow A_\mu'^a = U(\theta) A_\mu^a U^{-1}(\theta) + \frac{i}{g} U(\theta) \partial_\mu U^{-1}(\theta) , \quad (2.18)$$

where

$$U(\theta) = \exp(i\theta^a \cdot t_a) . \quad (2.19)$$

where θ^a are the space-time dependent parameters. In addition, $t_a \equiv \lambda_a/2$ where λ_a are the Hermitian Gell-Mann's matrices (generators of $SU(3)$). These matrices satisfy

$$\text{tr}(\lambda_a \lambda_b) = 2\delta_{ab} , \quad [\lambda_a, \lambda_b] = 2if_{abc}\lambda_c , \quad (2.20)$$

which lead trivially to

$$\text{tr}(t_a t_b) = \delta_{ab}/2 , \quad [t_a, t_b] = if_{abc}t_c . \quad (2.21)$$

and also

$$(t_a)^{cd}(t_a)^{de} = C_F \delta_{ce} , \quad f^{abc}f^{ebc} = C_A \delta_{ae} , \quad (2.22)$$

where $C_F = (N_c^2 - 1)/2N_c$ is the Casimir invariant associated with gluon emission from a quark, and $C_A = N_c = 3$ is the color-factor associated with gluon emission from a gluon. An important feature of QCD, which is different from QED, is that

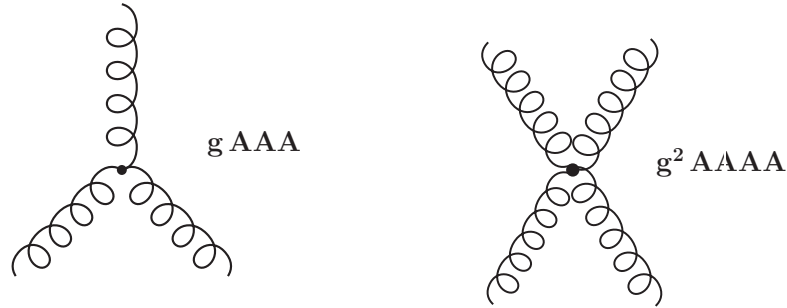


Figure 2.4: Self-interactions in QCD.

the gauge fields are self-interacting: the term $gf^{abc}A_\mu^b A_\nu^c$ in Eq. (2.16) generates the self-interactions pictured in Fig. 2.4.

2.2 Internal symmetries of QCD

The QCD Lagrangian (2.14) has a set of well-known strong-interaction symmetries. These include all the flavor symmetries of quark model, charge conjugation and parity. In the next sections we will describe two important symmetries of QCD for the purposes of the present dissertation: isospin and chiral symmetries.

2.2.1 Isospin Symmetry

Let us for the moment consider the QCD Lagrangian for the quarks of flavor u and d only, and assume that their masses are equal, $m_u = m_d = m$. Hence, the Lagrangian (2.14) can be represented as

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}^a F_a^{\mu\nu} , \quad (2.23)$$

where we can write the quark fields as a doublet of isospin[‡], namely

$$\psi = \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix} , \quad (2.24)$$

which means that they are in the fundamental representation of the SU(2) group. It is straightforward to verify that (2.23) is invariant under the transformation

$$\psi(x) \longrightarrow \psi'(x) = U(\theta)\psi(x) , \quad (2.25)$$

where $U(\theta)$ is given by

$$U(\theta) = \exp\left(\frac{-i\tau \cdot \theta}{2}\right) , \quad (2.26)$$

with $\tau = (\tau_1, \tau_2, \tau_3)$ being the Pauli matrices, satisfying

$$\left[\frac{\tau_i}{2}, \frac{\tau_j}{2}\right] = i\epsilon_{ijk}\frac{\tau_k}{2} \quad i, j, k = 1, 2, 3. \quad (2.27)$$

and $\theta = (\theta_1, \theta_2, \theta_3)$ are constant parameters of the SU(2) transformation.

Now, if we consider the six flavors, (2.14) is invariant under SU(6) global transformations in the (fictitious) limit that the quarks masses are equal. As known experimentally, flavor symmetry is broken in nature, as hadrons of different flavors have different masses. The only source of flavor symmetry breaking in the QCD Lagrangian is the mass term, with the masses of the quarks of different flavors being different. In phenomena involving only the u , d and s quarks (the light quark sector), the flavor symmetry is an approximate symmetry, while when the c , b and t quarks are involved (the heavy quark sector), the symmetry is severely broken.

[‡]Isospin was introduced by Heisenberg in 1932 and Wigner in 1937 [46] to explain symmetries of the then newly discovered neutron. Proton and neutron have similar masses, but different electric charges, positive and zero, respectively. So that proton and neutron are considered as different states of the same particle of isospin 1/2, with isospin projections $I_3 = +1/2$ and $I_3 = -1/2$ respectively.

2.2.2 Chiral Symmetry

Let us to consider the quark part of the QCD Lagrangian in Eq. (2.14)

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi. \quad (2.28)$$

Here, m is a diagonal matrix that contains the masses of the six different quark flavors; the values of these masses are generated in the electroweak sector of the standard model [2]. It is important here to mention that m is the current quark mass. The name is due to the fact these are the masses considered in the context of current algebra [3], a pre-QCD formalism used to related different matrix elements of different processes involving quark currents. These masses are renormalization scheme and scale dependent (in the same way as the electron mass is); their values extracted using information from several sources at the scale of 2 GeV in the $\bar{\text{MS}}$ scheme are [47]: $m_u = 2.15(15)$ MeV and $m_d = 4.70(20)$ MeV. The current quark masses should not be confused with the constituent quark masses of the nonrelativistic quark model, whose values, in the u and d sector, for example, are of the order of 300 MeV, as mentioned previously. A good deal of the present dissertation deals with the understanding of the dynamical connection between these two kinds of quark masses.

Let us to consider the chiral limit, $m_u = m_d = m_s = 0$, of the three light-quark sector. It is not difficult to show that in this limit, the Lagrangian (2.28) is symmetric under the global unitary transformation $SU(3) \otimes SU(3)$:

$$\psi \longrightarrow e^{i\lambda_a\theta_V^a}\psi, \quad \bar{\psi} \longrightarrow e^{i\gamma_5\lambda_a\theta_A^a}\bar{\psi}, \quad (2.29)$$

where θ_V^a and θ_A^a are constants and λ_a are the generators of the $SU(3)$ flavor group. From Noether's theorem, this symmetry generates the conserved vector and axial currents:

$$\mathcal{V}_a^\mu(x) = \bar{\psi}(x)\gamma^\mu t_a \psi(x), \quad (2.30)$$

$$\mathcal{A}_a^\mu(x) = \bar{\psi}(x)\gamma^\mu \gamma_5 t_a \psi(x), \quad (2.31)$$

and the corresponding charges are given by

$$Q_a = \int d^3x \mathcal{V}_a^0(x), \quad (2.32)$$

$$Q_{a5} = \int d^3x \mathcal{A}_a^0(x). \quad (2.33)$$

Also, in this limit of zero quark masses, the Lagrangian can be split into a sum of the Lagrangians for the left and right-handed Weyl-spinors:

$$\psi_L = \frac{1 - \gamma_5}{2} \psi, \quad \psi_R = \frac{1 + \gamma_5}{2} \psi, \quad (2.34)$$

$$\bar{\psi}_L = \bar{\psi} \frac{1 + \gamma_5}{2}, \quad \bar{\psi}_R = \bar{\psi} \frac{1 - \gamma_5}{2}. \quad (2.35)$$

But in the presence of a non-vanishing mass matrix m , this separation into two independent Lagrangians is not possible anymore; the mass term spoils the invariance under the second type of transformation in (2.29). One finds that the divergences of these currents are given by:

$$\partial_\mu \mathcal{V}_a^\mu(x) = \bar{\psi}[t_a, m]\psi, \quad (2.36)$$

$$\partial_\mu \mathcal{A}_a^\mu(x) = \bar{\psi}(x)\{\gamma_5 t_a, m\}\psi(x). \quad (2.37)$$

Hence, one can see that the vector current (2.36) is conserved for the case of identical current-quark mass, which describes the flavor symmetry limit in the light quark sector of QCD. The axial vector current (2.37) is not conserved if we have a non-vanishing current-quark mass in the Lagrangian of QCD. This situation is known as explicit chiral symmetry breaking, to be differentiated with the dynamical chiral symmetry, which will discuss next.

2.3 Dynamical Chiral Symmetry Breaking

Dynamical chiral symmetry breaking (DCSB) is a phenomenon associated with the sector of the light quarks u and d of QCD. Historically, DCSB was invoked by Nambu [48] to explain why the pion mass is much lighter than the other hadrons like the proton, neutron, ρ and ω mesons, etc. Nambu's argument goes as the following: there a transformation associated with a baryon field ψ that leaves the Hamiltonian (or Lagrangian) of the ψ field invariant, but the symmetry is broken by the interactions, that is, the symmetry is broken dynamically. The invariance is the chiral transformation in Eq. (2.29). There are two important consequences of dynamical breaking in the limit of exact symmetry: (1) the ψ field acquires a mass, and (2) there is in the spectrum of the theory of a zero mass boson with the quantum numbers of the generator of the broken symmetry - in the case of chiral symmetry, the generator of the chiral rotation has odd parity. In the limit of exact

chiral symmetry, the boson would be massless - this is the Goldstone theorem [49,50], and the massless boson is the Goldstone boson. The chiral boson can be identified with the pion, it is a pseudoscalar particle; the fact that the pion is not massless is because the symmetry is only approximate, in a sense to be explained shortly.

These ideas are nicely illustrated by Nambu–Jona-Lasinio (NJL) model [30], which we will discuss in the next subsection.

2.3.1 The Nambu–Jona-Lasinio model

The NJL model was constructed in the pre-QCD era using an analogy with the BCS theory of superconductivity [51]; its basic degrees of freedom are nucleons and pions. Currently, the model is reinterpreted in terms of quark degrees of freedom - for reviews on the earlier applications of the model in the context of QCD, see Ref. [52,53]. For two flavors, the Lagrangian density of the model is given by

$$\mathcal{L}_{NJL} = \mathcal{L}_0 + \mathcal{L}_{int} , \quad (2.38)$$

where

$$\mathcal{L}_0 = \bar{\psi}(i\gamma^\mu\partial_\mu - m_0)\psi , \quad (2.39)$$

and

$$\mathcal{L}_{int} = G[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\tau\psi)^2] , \quad (2.40)$$

with

$$\psi = \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix} , \quad (2.41)$$

where m_0 is the bare mass, G is the coupling constant and τ are the Pauli matrices. It is not difficult to show that this Lagrangian is chiral invariant when $m_0 = 0$: the kinetic term is invariant and, although neither of the separate terms in (2.40) is invariant, their sum is invariant.

One strategy to investigate DCSB in the model is the one of the original papers of Nambu and Jona-Lasinio [30]: (1) check if the interaction generates a mass term in the quark propagator, (2) check if this solution leads to a vacuum energy that is less than the vacuum energy calculated with a propagator without a mass term. Since the model cannot be solved analytically, an approximation scheme needs to be used. However, the approximation scheme cannot be based on a series expansion in

the constant G (perturbation theory). This is because perturbation theory would not lead to a quark propagator with a mass term when $m_0 = 0$, as the quark self-energy corrections calculated perturbatively are proportional to the Lagrangian mass which, in the chiral limit, is equal to zero. To see this, let us consider the self-energy in the lowest order in G . In momentum space, the quark self-energy $\Sigma(p)$ is defined as:

$$S^{-1}(p) = [S^{(0)}(p)]^{-1} - \Sigma(p). \quad (2.42)$$

where $[S^{(0)}(p)]^{-1} = \not{p} - m_0$ is the noninteracting propagator. In first order in G , the self-energy is given by

$$\begin{aligned} \Sigma^{(1)}(p) &= 2iG \int \frac{d^4k}{(2\pi)^4} \left[\text{Tr}[S^{(0)}(k)] - S^{(0)}(k) - \gamma_5 \text{Tr}[S^{(0)}(k)\gamma_5] + \gamma_5 S^{(0)}(k)\gamma_5 \right] \\ &= 8iG \left(N_c N_f + \frac{1}{2} \right) \int \frac{d^4k}{(2\pi)^4} \frac{m_0}{p^2 - m_0^2 + i\epsilon}, \end{aligned} \quad (2.43)$$

where $N_c (= 3)$ is the number of colors and $N_f (= 2)$ the number of flavors. When $m = 0$, one has

$$\Sigma^{(1)}(p) = 0, \quad (2.44)$$

and this entails

$$S^{(1)}(p) = S^{(0)}(p) = \frac{1}{\not{p} + i\epsilon}. \quad (2.45)$$

Equally, in all orders in perturbation theory, the self-energy will continue to be zero. On the other hand, starting with $m_0 \neq 0$, this m_0 must be small on the hadronic scale so that the concept of an approximate symmetry makes sense. But in this case, although the perturbative solution would generate self-energy corrections, these are not sizable enough to lead to good chiral parameters, like the pion decay constant and e.g. the masses of the ρ , ω , etc.

The nonperturbative approach of Nambu and Jona-Lasinio is based on the following argument: since the generation of mass is the result of the effects of the self-interaction among fermions, subtracting a self-energy term from the interaction should weaken the difference should be small. Let us formulate this in more precise terms. In general, the self-energy is a two-point function, i.e. in coordinate space its dependence on the space coordinates is such that $\Sigma(x, y) = \Sigma(x - y)$ due to translation invariance. In momentum space, it is a function of a single variable, $\Sigma(p)$. If one approximates $\Sigma(p)$ by a constant, i.e. $\Sigma(p) = \Sigma_0$, in coordinate space

one would have $\Sigma(x, y) = \Sigma(x - y) = \Sigma_0 \delta^4(x - y)$. Next, we add and subtract $\bar{\psi} \Sigma_0 \psi$ in the NJL Lagrangian (2.38), so that:

$$\begin{aligned} \mathcal{L}_{NJL} &= \mathcal{L}_0 + \mathcal{L}_{int} \\ &= \bar{\psi}(i\gamma^\mu \partial_\mu - m_0)\psi - \bar{\psi} \Sigma_0 \psi + G[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\tau\psi)^2] + \bar{\psi} \Sigma_0 \psi \\ &= \bar{\mathcal{L}}_0 + \bar{\mathcal{L}}_{int} , \end{aligned} \quad (2.46)$$

where

$$\bar{\mathcal{L}}_0 = \bar{\psi}(i\gamma^\mu \partial_\mu - m_0 - \Sigma_0)\psi = \bar{\psi}(i\gamma^\mu \partial_\mu - M)\psi, \quad (2.47)$$

with

$$M = m_0 + \Sigma_0, \quad (2.48)$$

and

$$\bar{\mathcal{L}}_{int} = G[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\tau\psi)^2] + \bar{\psi} \Sigma_0 \psi. \quad (2.49)$$

Therefore, if Σ_0 captures the main effect of the interaction, $\bar{\mathcal{L}}_{int}$ should be small and, therefore, amenable to perturbation theory. One extreme approximation would be to take the self-energy calculated at first order in $\bar{\mathcal{L}}_{int}$ equal to zero. This would lead to

$$0 = 2iG \int \frac{d^4k}{(2\pi)^4} \{ \text{Tr}[\bar{S}^{(0)}(k)] - \bar{S}^{(0)}(k) - \gamma_5 \text{Tr}[\bar{S}^{(0)}(k)\gamma_5] + \gamma_5 \bar{S}^{(0)}(k)\gamma_5 \} - \Sigma_0, \quad (2.50)$$

where $\bar{S}^{(0)}$ is the propagator corresponding to the “noninteracting” Lagrangian $\bar{\mathcal{L}}_0$ in Eq. (2.47):

$$\bar{S}^{(0)}(p) = \frac{1}{\not{p} - M + i\epsilon}. \quad (2.51)$$

Therefore, Eq. (2.50) leads to the self-consistent equation, known as the gap equation:

$$\Sigma_0 = 8iG \left(N_c N_f + \frac{1}{2} \right) \int \frac{d^4p}{(2\pi)^4} \frac{M}{p^2 - M^2 + i\epsilon}, \quad (2.52)$$

or, since $\Sigma_0 = M - m_0$:

$$M = m_0 + 8iG \left(N_c N_f + \frac{1}{2} \right) \int \frac{d^4p}{(2\pi)^4} \frac{M}{p^2 - M^2 + i\epsilon}. \quad (2.53)$$

The combination of terms $N_c N_f + 1/2$ comes from the traced and non-traced terms in Eq. (2.50) - in the language of many-body physics, they are the Hartree and Fock terms, respectively. It is common practice [52] to neglect the Fock term on the basis that it is $1/N_c$ suppressed as compared to the Hartree term. Although not very relevant for phenomenology for our purposes here, we follow this practice here simply to simplify the presentation of the equations.

An important feature of the model is its nonrenormalizability, i.e. ultraviolet divergences cannot be eliminated with a finite number of counter-terms. Therefore, in addition to the parameters m_0 and G that appear explicitly in the Lagrangian, another parameter is needed to regulate loop integrals. Specifically, a parameter Λ is introduced as an ultraviolet cutoff: the integral over the p_0 in Eq. (2.53) can be done explicitly and the remaining three-dimensional integral over $\mathbf{p}$ is divergent and is then regulated with Λ as (we consider here $m_0 = 0$)

$$M = 4GN_c N_f \int_0^\Lambda \frac{d^3k}{(2\pi)^3} \frac{M}{\sqrt{\mathbf{p}^2 + M^2}} . \quad (2.54)$$

Clearly, $M = 0$ is a solution of this equation. But $M \neq 0$ solutions also exist, if the coupling G exceeds a critical value: assuming $M \neq 0$ and evaluating integral, one obtains

$$\frac{\pi^2}{N_c N_f} = G\Lambda^2 \left[\left(1 + \frac{M^2}{\Lambda^2} \right)^{1/2} - \frac{M^2}{\Lambda^2} \sinh^{-1} \left(\frac{\Lambda}{M} \right) \right] , \quad (2.55)$$

which can be shown numerically that has solutions for $M \neq 0$ for $G > G_c$, where $G_c \Lambda^2 = \pi^2 / N_c N_f$. This entails that chiral symmetry is broken and, as shown shortly ahead, that Goldstone modes (pseudoscalar massless particles) are present in the spectrum. One says that M is generated dynamically and it is named constituent quark mass.

Next, we discuss the quark-antiquark bound state with the quantum numbers of the pion and show that it has zero mass. In order to motivate how one can calculate easily the pion mass, we start with a heuristic discussion of the exchange of an elementary pion in nucleon-nucleon and quark-quark scattering - we follow closely the presentation in Ref. [52].

The interaction Lagrangian that describes the coupling between the pions and nucleons is given by

$$\mathcal{L}_{\pi NN} = iG_{\pi NN} \bar{\psi}_N (\gamma_5 \boldsymbol{\tau} \cdot \boldsymbol{\pi}) \psi_N , \quad (2.56)$$

where $\boldsymbol{\pi}$ and ψ_N are respectively the pion and nucleon field operators, and $G_{\pi NN}$ is the coupling constant. This model can be taken over into the quark picture, namely

$$\mathcal{L}_{\pi NN} \longrightarrow \mathcal{L}_{\pi qq} = ig_{\pi qq} \bar{\psi}(\gamma_5 \boldsymbol{\tau} \cdot \boldsymbol{\pi}) \psi . \quad (2.57)$$

One may decompose

$$\boldsymbol{\tau} \cdot \boldsymbol{\pi} = \tau^{(+)} \pi^{(+)} + \tau^{(-)} \pi^{(-)} + \tau^{(3)} \pi^{(3)} , \quad (2.58)$$

with

$$\tau^{(\pm)} = \frac{1}{\sqrt{2}}(\tau_1 \pm \tau_2) , \quad \pi^{(\pm)} = \frac{1}{\sqrt{2}}(\pi_1 \mp \pi_2) . \quad (2.59)$$

The diagram representing the scattering of a u and a d quark via the exchange of a π^+ meson is depicted in Fig. 2.5. In terms of momentum-space Dirac spinors and pion propagator the diagram can be written as

$$[\bar{d}'(i\gamma_5 \tau^{(-)})u] \frac{i(ig_{\pi qq})^2}{k^2 - m_\pi^2} [\bar{u}'(i\gamma_5 \tau^{(+)})d] , \quad (2.60)$$

where k is the four momentum of the exchanged pion - the difference of the momenta of the u and d quarks.

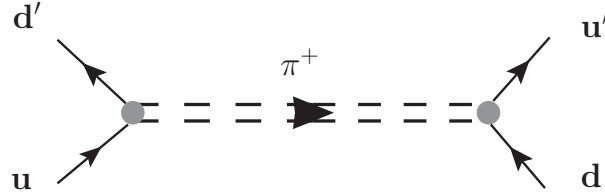


Figure 2.5: Scattering diagram $(ud) \longrightarrow (d'u')$. The arrows represent the Dirac spinors, the dots represent the $\gamma_5 \tau^{(\pm)}$ and the double-dashed line represents the pion propagator.

A detailed and precise derivation of such an expression can be obtained using the Lehmann-Symazik-Zimmermann (LSZ) reduction formulas [8] applied to the matrix element $\langle u'd'; out | ud; in \rangle$. An effective vertex interaction, $U_{eff}(k^2)$, for the $(ud) \rightarrow (u'd')$ process via the exchange of a $\pi^{(+)}$ is defined by amputating the external legs, by removing the Dirac spinors from the amplitude in Eq. (2.60), leaving in the way the product of isospin and gamma matrices and the pion propagator - see Fig. 2.6:

$$i U_{eff}(k^2) = (i\gamma_5) \tau^{(-)} \frac{i(ig_{\pi qq})^2}{k^2 - m_\pi^2} (i\gamma_5) \tau^{(+)} . \quad (2.61)$$



Figure 2.6: Effective quark-pion interaction. The dots represent the $\gamma_5 \tau^{(\pm)}$ and the dashed line represents the pion propagator.

The aim now is to obtain such an interaction within the NJL model. There will one main difference with the elementary pion case, in that the pion propagator representing the exchange of an elementary pion will be replaced by the exchange of a chain of quark-antiquark pairs with the quantum numbers of the pion. The term $(\bar{\psi} i \gamma_5 \tau \psi)^2$ in the NJL Lagrangian is responsible for exciting the pseudoscalar mode to be identified as the pion. The derivation of the matrix element for scattering process $(ud) \rightarrow (u'd')$ proceeds in a similar way as in the elementary pion case by using LSZ reduction formulas in the matrix element $\langle u'd'; out | ud; in \rangle$, with the interaction in Eq. (2.57) replaced by the NJL interaction $(\bar{\psi} i \gamma_5 \tau \psi)^2$. It is not difficult to show that the effective interaction $i U_{eff}(k^2)$ can be expressed to leading order in N_c as an infinite sum of terms as shown pictorially in Fig. 2.7. In the language of many-body physics, this known as the random-phase approximation (RPA).

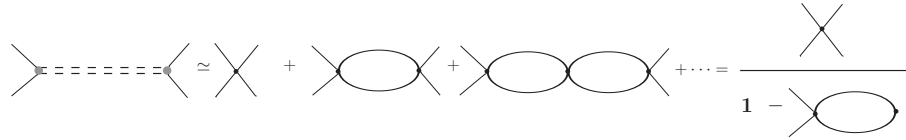


Figure 2.7: The effective ud interaction to leading order in N_c .

In order to obtain a pole, one has to sum this infinite series. Explicit evaluation leads to

$$i U_{ij}(k^2) = (i \gamma_5 T_i) \left[\frac{2iG}{1 - 2G \Pi_{ps}(k^2)} \right] (i \gamma_5 T_j) , \quad (2.62)$$

where $\Pi_{ps}(k^2)$ is the proper polarization:

$$-i \Pi_{ps}(k^2) = - \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \{ (i \gamma_5 T_i) [i \bar{S}(p + k/2)] (i \gamma_5 T_j) [i \bar{S}(p - k/2)] \} , \quad (2.63)$$

wherein for the channel of π^0 we have $T_i = T_j = \tau_3$, and $T_i = \tau^{(\pm)}$, $T_j = \tau^{(\mp)}$ for the π^+ or π^- channels, respectively. The expression for $\Pi_{ps}(k^2)$ is represented pictorially in Fig. 2.8.

$$-i\Pi_{ps}(k^2) = i\gamma_5 T_i \text{ --- } \text{---} i\gamma_5 T_j$$

Figure 2.8: The proper polarization.

Eq. (2.62) shows clearly that a pole at a four-momentum square $k^2 = m_{ps}^2$ in the scattering amplitude is obtained whenever[§]

$$1 - 2G\Pi_{ps}(k^2 = m_{ps}^2) = 0 . \quad (2.64)$$

Moreover, Laurent expansion about the pole leads to

$$i U_{ij}(k^2) \simeq (i\gamma_5)T_i \left(\frac{a_{-1}}{k^2 - m_\pi^2} \right) (i\gamma_5)T_j , \quad (2.65)$$

with a_{-1} being the residue at the pole, given explicitly by

$$\begin{aligned} a_{-1} &= 2iG \lim_{k^2 \rightarrow m_\pi^2} \left[\frac{k^2 - m_\pi^2}{1 - 2G\Pi_{ps}(k^2)} \right] = 2iG \frac{1}{-2G \left(d\Pi_{ps}(k^2)/dk^2 \right) \Big|_{k^2=m_\pi^2}} \\ &= -i \left(\frac{d\Pi_{ps}(k^2)}{dk^2} \right)^{-1} \Big|_{k^2 \rightarrow m_\pi^2} , \end{aligned} \quad (2.66)$$

so that the effective interaction for the exchange of a pion can be written as

$$i U_{ij}(k^2) \simeq (i\gamma_5)T_i \frac{-i \left[(d\Pi_{ps}(k^2)/dk^2) \Big|_{k^2=m_\pi^2} \right]^{-1}}{k^2 - m_\pi^2} (i\gamma_5)T_j . \quad (2.67)$$

Comparison with Eq. (2.61) leads to the identification:

$$g_{\pi qq}^2 = \left[\frac{d\Pi_{ps}(k^2)}{dk^2} \Big|_{k^2=m_\pi^2} \right]^{-1} . \quad (2.68)$$

Next, we show that in the chiral limit the $m_{ps}^2 = 0$, while for $m_0 \neq 0$ one obtains $m_{ps}^2 \sim m_0$. Let us consider the proper polarization given by Eq. (2.63). We use Eq. (2.51) for $\bar{S}$ in that equation to obtain

$$-i\Pi_{ps}(k^2) = - \int \frac{d^4p}{(2\pi)^4} \frac{\text{Tr}[\gamma_5 T_i (\not{p} + \not{k}/2 + M) \gamma_5 T_i (\not{p} - \not{k}/2 + M)]}{[(p + k/2)^2 - M^2 + i\epsilon][(p - k/2)^2 - M^2 + i\epsilon]} . \quad (2.69)$$

[§]This equation was first derived by Nambu and Jona-Lasinio [30], who examined a Bethe-Salpeter equation for the vertex function in the ladder approximation. The Bethe-Salpeter equation will be treated in Chapter 4 of this dissertation.

Using the properties of the gamma matrices and evaluating the traces, see Appendix A, we find

$$-i\Pi_{ps}(k^2) = -4N_c N_f \int \frac{d^4 p}{(2\pi)^4} \frac{[M^2 - p^2 + k^2/4]}{[(p+k/2)^2 - M^2 + i\epsilon][(p-k/2)^2 - M^2 + i\epsilon]} . \quad (2.70)$$

Now writing the denominator in the above in terms of partial fractions, one can write

$$\begin{aligned} -i\Pi_{ps}(k^2) &= 2N_c N_f \left\{ \int \frac{d^4 p}{(2\pi)^4} \left[\frac{1}{(p+k/2)^2 - M^2 + i\epsilon} + \frac{1}{(p-k/2)^2 - M^2} \right] \right. \\ &\quad \left. - \int \frac{d^4 p}{(2\pi)^4} \frac{1}{[(p+k/2)^2 - M^2 + i\epsilon][(p-k/2)^2 - M^2 + i\epsilon]} \right\} \\ &= 4N_c N_f \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 - M^2 + i\epsilon} - 2N_c N_f k^2 I(k^2) , \end{aligned} \quad (2.71)$$

with $I(k^2)$ given by

$$I(k^2) = \int \frac{d^4 p}{(2\pi)^4} \frac{1}{[(p+k/2)^2 - M^2 + i\epsilon][(p-k/2)^2 - M^2 + i\epsilon]} . \quad (2.72)$$

Note that in order to obtain the first term in (2.71) we made variable shifts in the integrands. Such an operation is delicate when dealing with divergent integrals, as it can lead to breaking of symmetries, in particular of the chiral symmetry [54]. Therefore, implicit here is the hypothesis that one is using a symmetry-preserving regularization scheme – for a detailed discussion on this subject, see Ref. [54]. We will come back to this issue in the next Chapters.

From the gap equation (2.53), with the Fock term $1/2$ omitted, one can write the first integral in (2.71) as

$$4N_c N_f \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 - M^2 + i\epsilon} = \frac{M - m_0}{2iGM} , \quad (2.73)$$

so that Eq. (2.71) can be rewritten as

$$1 - 2G\Pi_{ps}(k^2) = \frac{m_0}{M} + 4iGN_c N_f k^2 I(k^2) . \quad (2.74)$$

From the pole condition (2.64), one has then

$$m_{ps}^2 = -\frac{m_0}{M} \frac{1}{4iGN_c N_f I(m_{ps}^2)} . \quad (2.75)$$

Clearly, when $m_0 = 0$, one has that there is a zero mass pole $m_{ps} = 0$ in the pseudoscalar channel - in the chiral limit, we have therefore found the Goldstone

boson. For $m_0 \neq 0$, in order to find m_{ps}^2 one has to solve the transcendental equation (2.75). However, assuming m_0 small, and writing $I(m_{ps}^2) = I(0) + m_{ps}^2 I'(0) + \dots$, it is clear that $m_{ps}^2 \sim m_0$.

To conclude this part of the discussion, we remark that the existence of a zero mass pole in the pseudoscalar channel is equivalent to the gap equation in the chiral limit. This can be shown as follows. If one expresses the pole condition (2.64) in terms of (2.71) in the chiral limit, we have

$$1 - 2G\Pi_{ps}(0) = 0 \rightarrow \frac{1}{2G} = \Pi_{ps}(0) = 4iN_c N_f \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 - M^2 + i\epsilon}. \quad (2.76)$$

Or, equivalently:

$$1 = 8iGN_c N_f \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 - M^2 + i\epsilon}, \quad (2.77)$$

which is precisely the gap equation for $M \neq 0$ in Eq. (2.53) in the chiral limit (omitting the Fock term $1/2$). That is, as said above, the condition for the existence of a zero mass pole in the pseudoscalar channel is equivalent to the gap equation.

Another important physical quantity related to DCSB is the pion decay constant f_π , which is measured experimentally from the decay $\pi^- \rightarrow \mu^- + \nu_\mu$; its experimental value is $f_\pi \simeq 93$ MeV. This value is directly related to the value of the constituent quark mass M . The pion decay constant is defined via the matrix element

$$\langle 0 | A_{5\mu}^a(0) | \pi^b(k) \rangle = ik_\mu f_\pi \delta^{ab}, \quad (2.78)$$

where $A_{5\mu}^a(0)$ is the axial-vector current and a and b are isospin indices. This equation is pictorially represented in Fig. 2.9.

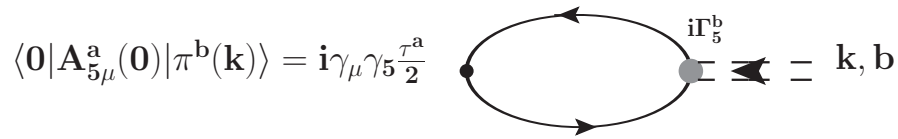


Figure 2.9: The axial-vector transition amplitude.

From this diagram, one finds

$$ik_\mu f_\pi \delta^{ab} = - \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \{ [i\gamma_\mu \gamma_5 (\tau^a/2)] [i\bar{S}(p + k/2)] [i\Gamma_5^b] [i\bar{S}(p - k/2)] \}, \quad (2.79)$$

wherein $\Gamma_5^b = g_{\pi qq} \gamma_5 \tau^b$ - a more formal discussion on this will appear in the next Chapter. Evaluation of the traces leads to

$$ik_\mu f_\pi = g_{\pi qq} k_\mu 4N_c M I(k^2), \quad (2.80)$$

where we have used (2.72). Now using the relation given by (2.74) and (2.68), one finds

$$g_{\pi qq}^{-2} = -2iN_cN_f I(0) , \quad (2.81)$$

where we have taken $k^2 = 0$. The pion decay constant can be calculated with the squared of (2.80) at $k^2 = 0$ and using (2.81), to obtain

$$f_\pi^2 = -8 \frac{N_c}{N_f} M^2 I(0) . \quad (2.82)$$

Another quantity commonly used to quantify DCSB is the quark condensate $\langle \bar{\psi}^f \psi^f \rangle = \langle \Omega | \bar{\psi}^f \psi^f | \Omega \rangle$, where f indicates flavor and $|\Omega\rangle$ is the vacuum state. This quantity can be expressed in terms of the quark propagator of flavor f :

$$\begin{aligned} \langle \bar{\psi}^f(x) \psi^f(x) \rangle &= \sum_{\alpha,c} \langle \Omega | \bar{\psi}_\alpha^{c,f}(x) \psi_\alpha^{c,f}(x) | \Omega \rangle = \lim_{y \rightarrow x^+} \sum_{\alpha,c} \langle \Omega | \bar{\psi}_\alpha^{c,f}(y) \psi_\alpha^{c,f}(x) | \Omega \rangle \\ &= \lim_{y \rightarrow x^+} \sum_{\alpha,c} \langle \Omega | T[\bar{\psi}_\alpha^{c,f}(y) \psi_\alpha^{c,f}(x)] | \Omega \rangle \\ &= - \lim_{y \rightarrow x^+} \sum_{\alpha,c} \langle \Omega | T[\psi_\alpha^{c,f}(x) \bar{\psi}_\alpha^{c,f}(y)] | \Omega \rangle \\ &= -i \lim_{y \rightarrow x^+} \sum_{\alpha,c} S_{\alpha,\alpha}^{c,f}(x-y) = - \lim_{y \rightarrow x^+} \text{Tr} S^f(x-y) = -i \text{Tr} S^f(0) \\ &= -i \int \frac{d^4 p}{(2\pi)^4} \text{Tr} S^f(p) , \end{aligned} \quad (2.83)$$

where the trace is over color and spinor indices. For the case of the NJL quark propagator (2.51):

$$\begin{aligned} \langle \bar{\psi}^f \psi^f \rangle &= -i \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \frac{1}{\not{p} - M^f + i\epsilon} = -i \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \frac{\not{p} + M^f}{p^2 - (M^f)^2 + i\epsilon} \\ &= -12i \int \frac{d^4 p}{(2\pi)^4} \frac{M^f}{p^2 - (M^f)^2 + i\epsilon} . \end{aligned} \quad (2.84)$$

Clearly, as $M = 0$, the quark condensate is also zero. One summarizes this as:

$$\langle \bar{\psi} \psi \rangle = \begin{cases} 0 & \text{chiral symmetry is not broken.} \\ \neq 0 & \text{chiral symmetry is dynamically broken.} \end{cases} \quad (2.85)$$

The last sentence requires clarification: it is strictly true for m_0^f ; when one has an explicitly symmetry breaking current quark mass, $m_0^f \neq 0$, one usually defines the

quark condensate as

$$\langle \bar{\psi}^f \psi^f \rangle = -i \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \left[S^f(p) - \frac{1}{\not{p} - m_0^f + i\epsilon} \right], \quad (2.86)$$

which then measures the extent the interactions dress the current quark mass, i.e. the effect of the interactions such that $m_0^f \rightarrow M^f$. Of course, it should be clear that the linkage of DCSB with the quark condensate is useful for the light-quark sector of QCD. We will come back to this issues in the forthcoming Chapters.

It is important to mention that the quark condensate is not a physically measurable quantity. However, traditionally it has been extracted in studies of hadron masses using QCD sum rules [55]. The value extracted (assuming SU(2) isospin symmetry) in these studies is $\langle \bar{u}u \rangle = \langle \bar{d}d \rangle \simeq -(250 \text{ MeV})^3$.

We conclude this Section mentioning that the vacuum expectation value of the energy density calculated with a DCSB NJL propagator is less than the one calculated with a propagator with zero quark mass. Specifically, let $\langle T_m^{00} \rangle$ indicate the vacuum energy density calculated with a quark propagator with mass m ; then, it is not difficult to show that [52]

$$\Delta\epsilon \equiv \langle T_M^{00} \rangle - \langle T_0^{00} \rangle \simeq -\frac{\Lambda^4}{8\pi^2} \frac{M^2}{\Lambda^2} \left(1 - \frac{\pi^2}{GN_c N_f \Lambda^2} \right), \quad (2.87)$$

for $M/\Lambda \ll 1$. Therefore, if M is obtained from the gap equation, the last term in this equation is less than unity and $\Delta\epsilon < 0$.

2.4 Feynman Rules for the QCD Lagrangian

The Feynman rules for the QCD Lagrangian (2.14) can be derived from the a generating functional over the fields ψ , $\bar{\psi}$ and A_μ^a , namely

$$Z[\bar{\eta}, \eta, J_\mu] = \int \mathcal{D}(\bar{\psi}, \psi, A) \exp \left(iS[\bar{\psi}, \psi, A_\mu] + i \int d^4 x [\bar{\psi}\eta + \bar{\eta}\psi + A_\mu^a J_\mu^a] \right), \quad (2.88)$$

where $\bar{\eta}$, η and J^μ are, the source field for the quark, antiquark and gluon respectively, and we have defined

$$S[\bar{\psi}, \psi, A_\mu] = \int d^4 x \left[\bar{\psi}(x) (i\gamma^\mu D_\mu - m) \psi(x) - \frac{1}{4} F_{\mu\nu}^a(x) F_a^{\mu\nu}(x) - \frac{1}{2\xi} (\partial_\mu A_\mu^a)^2 \right]. \quad (2.89)$$

where we have introduced the gauge fixing term via Faddeev-Popov quantization [56]. Therefore, at zero external sources the generating functional (2.88) becomes

$$Z = \int \mathcal{D}(\bar{\psi}, \psi, A) e^{iS_0 + iS_I}, \quad (2.90)$$

where S_0 and S_I are given by

$$\begin{aligned}
S_0 &= \int d^4x d^4y \left[\bar{\psi}(x) (i\gamma^\mu \partial_\mu - m + i\epsilon) \delta^4(x-y) \psi(y) \right. \\
&\quad \left. + \frac{1}{2} A^{a\mu}(x) \delta^{ab} \left[\partial^2 g_{\mu\nu} - \left(1 - \frac{1}{\xi}\right) \partial_\mu \partial_\nu \right] \delta^4(x-y) A^{b\nu}(y) \right] \\
&= \int d^4x d^4y \left[\bar{\psi}(x) S_0^{-1}(x-y) \psi(y) + \frac{1}{2} A^{a\mu}(x) (D_0^{-1})_{\mu\nu}^{ab}(x-y) A^{b\nu}(y) \right] , \quad (2.91)
\end{aligned}$$

$$\begin{aligned}
S_I &= \int d^4x \left[-g A_\mu^a \bar{\psi}(x) \gamma^\mu t_a \psi(x) - g f^{abc} \partial_\mu A_\nu^a(x) A^{b\mu}(x) A^{c\nu}(x) \right. \\
&\quad \left. - \frac{1}{4} g^2 f^{abc} f^{ade} A_\mu^b(x) A_\nu^c A^{d\mu}(x) A^{e\nu}(x) \right] . \quad (2.92)
\end{aligned}$$

In Eq. (2.91) we have defined

$$S_0^{-1}(x-y) = (i\gamma^\mu \partial_\mu - m + i\epsilon) \delta^4(x-y), \quad (2.93)$$

which satisfies

$$\int d^4z S_0^{-1}(x-z) S_0(z-y) = \delta^4(x-y) . \quad (2.94)$$

Taking Fourier transformation[¶] and using translational invariance, we have

$$(\gamma^\mu p_\mu - m + i\epsilon) S_0(p) = 1 , \quad (2.95)$$

which implies that

$$S_0(p) = \frac{1}{\not{p} - m + i\epsilon} , \quad (2.96)$$

which is precisely the free-fermion propagator. In Eq. (2.91), we also have defined

$$(D_0^{-1})_{\mu\nu}^{ab}(x-y) = \delta^{ab} \left[\partial^2 g_{\mu\nu} - \left(1 - \frac{1}{\xi}\right) \partial_\mu \partial_\nu \right] \delta^4(x-y) , \quad (2.97)$$

which satisfies

$$\int d^4z (D_0^{-1})_{\mu\nu}^{ab}(x-z) (D_0)^{\nu\lambda}_{bc}(z-y) = g_\mu^\lambda \delta_c^a \delta^4(x-y) , \quad (2.98)$$

whose solution leads to

$$(D_0)^{ab}_{\mu\nu}(k) = -\frac{\delta^{ab}}{k^2 + i\epsilon} \left[g_{\mu\nu} - \left(1 - \xi\right) \frac{k_\mu k_\nu}{k^2} \right] , \quad (2.99)$$

this is the free gluon propagator in momentum space.

[¶]In Appendix A, are shown the conventions used in this thesis.

On the other hand, the first term in Eq. (2.92) gives the quark-gluon vertex

$$-ig(t^a)_{ij}\gamma^\mu. \quad (2.100)$$

If we expand the term e^{iS_I} in power series in g , and making wick contraction with external gauge particles, the second term in (2.92) leads to the three gluon vertex. The term $(\partial_\mu A_\nu^a)$ contributes with a factor $(-ik_k)$, whose contribution has the form

$$-igf^{abc}(-ik_\mu)g^{\mu\rho}, \quad (2.101)$$

and there are $3!$ possible contractions. Thus, one finds that

$$g f^{abc}(2\pi)^4\delta(k_1 + k_2 + k_3)N_{\mu\nu\rho}(k_1, k_2, k_3), \quad (2.102)$$

where

$$N_{\mu\nu\rho}(k_1, k_2, k_3) = (k_1 - k_2)_\rho g_{\mu\nu} + (k_2 - k_3)_\mu g_{\nu\rho} + (k_3 - k_1)_\nu g_{\mu\rho}. \quad (2.103)$$

The last term in (2.92) gives the four gluon vertex. One possible contraction gives the contribution

$$-ig^2 f^{abc}f^{ecd}g^{\mu\rho}g^{\mu\sigma}. \quad (2.104)$$

There are $4!$ possible contractions. Then, we obtain that

$$-ig^2(2\pi)^4\delta(k_1 + k_2 + k_3 + k_4)N_{\mu\nu\rho\sigma}(k_1, k_2, k_3, k_4), \quad (2.105)$$

where

$$N_{\mu\nu\rho\sigma}(k_1, k_2, k_3, k_4) = f^{eab}f^{ecd}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \quad (2.106)$$

$$+ f^{eac}f^{edb}(g_{\mu\sigma}g_{\rho\nu} - g_{\mu\nu}g_{\rho\sigma}) \quad (2.107)$$

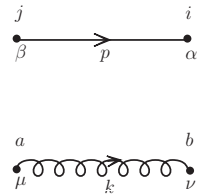
$$+ f^{ead}f^{ebc}(g_{\mu\nu}g_{\sigma\rho} - g_{\mu\rho}g_{\sigma\nu}). \quad (2.108)$$

In the Fig. 2.10 and Fig. 2.11 are depicted the Feynman rules in Minkowski space.

2.5 Summary

We presented a study of QCD, where we review a little its history and general properties. We saw that the mass term in the Lagrangean of QCD is very important, since for the case of consider the limit when the quarks masses are equal, this

Lagrangian is invariant under $SU(6)$ global transformation and for the case of non-vanishing current-quark mass it is not invariant under a type of transformation called chiral transformation. In order to understand the the phenomenon of dynamical chiral symmetric breaking, we review the Nambu-Jona-Lasinio model, where this phenomenon has as a consequence the dynamical mass generation and the appearance of Goldstone modes, which are identified as the pions. We also study the pion in the framework of the Nambu-Jona-Lasinio model, where we show that in the chiral limit, this can be identified as the Goldstone boson. In addition we saw that the equation (2.64), which is a Bethe-Salpeter equation in this limit becomes the Gap equation. Finally we write the Feynman rules of QCD.

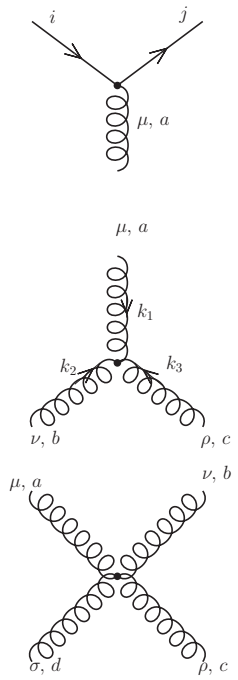


The figure shows two Feynman diagrams. The top diagram is a quark propagator, represented by a straight line with arrows indicating the direction of quark number flow. It has indices j, β at the left end and i, α at the right end, with momentum p flowing from left to right. The bottom diagram is a gluon propagator, represented by a wavy line. It has indices a, μ at the left end and b, ν at the right end, with momentum k flowing from left to right.

$$iS_{\alpha\beta}^{ij}(p) = \left(\frac{i\delta^{ij}}{\not{p} - m + i\epsilon} \right)_{\alpha\beta}$$

$$i(D)_{\mu\nu}^{ab}(k) = -i \frac{\delta^{ab}}{k^2 + i\epsilon} [g_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{k^2}]$$

Figure 2.10: Quark and gluon propagators.



The figure shows three Feynman diagrams representing QCD vertices. The top diagram is a quark-gluon vertex, where a quark line splits into a quark and a gluon. The quark indices are i, j and the gluon indices are μ, a . The middle diagram is a three-gluon vertex, where three gluon lines meet at a point. The gluon indices are μ, a , ν, b , and ρ, c , and the momenta are k_1, k_2, k_3 . The bottom diagram is a four-gluon vertex, where four gluon lines meet at a point. The gluon indices are μ, a , ν, b , σ, d , and ρ, c , and the momenta are k_1, k_2, k_3, k_4 .

$$-ig(t^a)_{ij}\gamma^\mu$$

$$g f^{abc} N_{\mu\nu\rho}(k_1, k_2, k_3),$$

$$N_{\mu\nu\rho}(k_1, k_2, k_3) = (k_1 - k_2)_\rho g_{\mu\nu} + (k_2 - k_3)_\mu g_{\nu\rho} + (k_3 - k_1)_\nu g_{\mu\rho}$$

$$-ig^2 N_{\mu\nu\rho\sigma}(k_1, k_2, k_3, k_4),$$

$$N_{\mu\nu\rho\sigma}(k_1, k_2, k_3, k_4) = f^{eab} f^{ecd} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) + f^{eac} f^{edb} (g_{\mu\sigma} g_{\rho\nu} - g_{\mu\nu} g_{\rho\sigma}) + f^{ead} f^{ebc} (g_{\mu\nu} g_{\sigma\rho} - g_{\mu\rho} g_{\sigma\nu})$$

Figure 2.11: QCD Feynman rules in Minkowski space.

Chapter 3

DYSON-SCHWINGER EQUATIONS

In quantum field theory, the DSEs are nonperturbative equivalents of the Lagrange equations of motion. The fact that, from the field equations of a quantum field theory, one can derive a system of coupled integral equations interrelating all the Green's functions of the theory [57,58,59]. These set of infinity coupled equations are called Dyson-Schwinger equations. Then solving these equations provides a solution of the theory. Because a field theory is completely defined when all of n-point functions are well known, namely, all cross-sections can be constructed. The DSEs are a particularly well suited for use in QCD.

On the other hand, DCSB profoundly affects the character of the hadron spectrum. However, its understanding is best wanted via a Euclidean formulation of quantum field theory. In addition, Lattice QCD is formulated in Euclidean space. The main reason is that the term $\int d^4x \mathcal{L}$ in the generating functional of the theory oscillates very violently and integrals do not converge. The Euclidean-QCD action defines a probability measure, for which many numerical simulation algorithms are available. Here we will work in Euclidean space.

A possible way to derivate the DSEs for a given quantum field theory is based on a simple trick with functional integrals. To explain the basic idea of the trick, let us consider the generating functional of correlation functions in the quantum field theory of a real scalar field with an action $S[\phi]$ in Euclidean space:

$$Z[J] = \int \mathcal{D}\phi e^{-S[\phi] + J \cdot \phi} , \quad (3.1)$$

where J is an external source, and $J \cdot \phi$ denotes

$$J \cdot \phi = \int d^4x J(x) \phi(x) .$$

Correlation functions of the field ϕ (in the presence of the external source) are

simply functional derivatives of $Z[J]$ with respect to J . For example, the two-point correlation function

$$\langle \phi(x)\phi(y) \rangle = \int \mathcal{D}\phi \phi(x)\phi(y) e^{-S[\phi]} = \frac{\delta^2 Z[J]}{\delta J(x)\delta J(y)} \Big|_{J=0}. \quad (3.2)$$

$Z[J]$ is equivalent to the partition function in statistical mechanics, with J playing the role of an external magnetic field.

Now, for ordinary integrals, one has that

$$\int_{-\infty}^{\infty} dx \frac{d}{dx} f(x) = f(\infty) - f(-\infty) = 0, \quad (3.3)$$

so long as $f(\infty) = f(-\infty)$, which includes $f(\infty) = f(-\infty) = 0$. Let $I(a)$ be the integral

$$I(a) = \int dx e^{ax}. \quad (3.4)$$

One has:

$$\int dx x e^{ax} = \int dx \frac{d}{da} e^{ax} = \frac{d}{da} \int dx e^{ax} = \frac{d}{da} I(a), \quad (3.5)$$

$$\int dx f(x) e^{ax} = \int dx f\left(\frac{d}{da}\right) e^{ax} = f\left(\frac{d}{da}\right) \int dx e^{ax} = f\left(\frac{d}{da}\right) I(a). \quad (3.6)$$

The generalization to functional integrals of these results is as follows. First, the equivalent to Eq. (3.3) is

$$\int \mathcal{D}\phi \frac{\delta}{\delta \phi(x)} e^{-S[\phi] + J \cdot \phi} = 0. \quad (3.7)$$

Writing explicitly, we have

$$\int \mathcal{D}\phi \left(-\frac{\delta S[\phi]}{\delta \phi(x)} + J(x) \right) e^{-S[\phi] + J \cdot \phi} = 0. \quad (3.8)$$

The equivalent of Eq. (3.6) is, then:

$$\left(\frac{\delta S}{\delta \phi} \left[\frac{\delta}{\delta J(x)} \right] - J(x) \right) Z[J] = 0. \quad (3.9)$$

This equation is a compact form of equations of motion for correlation functions (or Euclidean Green's functions) - which are the Dyson-Schwinger equations: an infinite set of relations between correlation functions obtained by expanding in powers of the source $J(x)$, and setting $J(x) = 0$ at the end.

Equivalent to the Helmholtz free energy in statistical mechanics, is the functional $W[J]$, defined as

$$W[J] = \ln Z[J] \rightarrow Z[J] = e^{W[J]}. \quad (3.10)$$

$W[J]$ is the generating functional of connected correlation functions. For example, the connected two point function:

$$\begin{aligned} \langle \phi(x)\phi(y) \rangle_c &\equiv \Delta(x-y) = \frac{1}{Z} \int \mathcal{D}\phi \, \phi(x)\phi(y) e^{-S[\phi]} \\ &= \left. \frac{\delta^2 \ln Z[J]}{\delta J(x)\delta J(y)} \right|_{J=0} \\ &= \left. \frac{\delta^2 W[J]}{\delta J(x)\delta J(y)} \right|_{J=0} . \end{aligned} \quad (3.11)$$

From Eq. (3.9), one can write:

$$e^{-W[J]} \frac{\delta S}{\delta \phi} \left[\frac{\delta}{\delta J(x)} \right] e^{W[J]} = J(x), \quad (3.12)$$

or

$$\frac{\delta S}{\delta \phi} \left[\frac{\delta W}{\delta J} + \frac{\delta}{\delta J} \right] - J(x) = 0. \quad (3.13)$$

This is a compact form of equations of motion for connected correlation functions. While Eq. (3.9) generates the equations of motion for the *full* Green's function, Eq. (3.13) generates the *connected* Green's functions.

It is convenient to deal with generating functionals that do not depend on the external source; as in statistical mechanics, such generating functionals are obtained via a Legendre transform. This leads to a generating functional of irreducible functions (or proper vertices, or proper Green's functions). The appropriate Legendre transform is:

$$\Gamma[\phi^c] = \phi^c \cdot J - W[J] = \phi^c \cdot J - \ln Z[J], \quad (3.14)$$

with the field $\phi^c(x)$ being conjugate to the source J , defined as

$$\phi^c(x) = \phi^c(x, J) = \frac{\delta W[J]}{\delta J(x)}. \quad (3.15)$$

The index 'c' is for c-number; it is important to notice that ϕ^c should not be confused with the ϕ field in the functional integral (3.1). Note also that the Legendre transform can be constructed if one can express the source as a function of the field ϕ^c : $J(x) = J(x, \phi^c)$. When $J = 0$, one has that ϕ^c is the vacuum expectation value of ϕ ; in the absence of spontaneous symmetry breaking, $\phi^c(x, J)|_{J=0} = 0$. With minor modifications, the case of spontaneous symmetry breaking can also be handled with this formalism [8].

Note that:

$$J(x) = \frac{\delta\Gamma[\phi]}{\delta\phi^c(x)}, \quad (3.16)$$

since

$$\begin{aligned} \frac{\delta\Gamma[\phi]}{\delta\phi^c(x)} &= \int d^4x' \left[\frac{\delta\phi^c(x')}{\delta\phi^c(x)} J(x') + \phi^c(x') \frac{\delta J(x')}{\delta\phi^c(x)} \right] - \int d^4x' \frac{\delta W[J]}{\delta J(x')} \frac{\delta J(x')}{\delta\phi^c(x)} \\ &= J(x) + \int d^4x' \left[\phi^c(x') - \frac{\delta W[J]}{\delta J(x')} \right] \frac{\delta J(x')}{\delta\phi^c(x)} \\ &= J(x), \end{aligned} \quad (3.17)$$

where we used Eq. (3.15) in the last integral. Equations for the proper functions can be derived from Eq. (3.13): inserting in this equation the definition of Γ , Eq. (3.14), one obtains:

$$\begin{aligned} J(x) &= e^{-\phi^c \cdot J + \Gamma[\phi^c]} \frac{\delta S}{\delta\phi} \left[\frac{\delta}{\delta J(x)} \right] e^{\phi^c \cdot J - \Gamma[\phi^c]} \\ &= \frac{\delta S}{\delta\phi} \left[\phi^c + J \cdot \frac{\delta\phi^c}{\delta J} - \frac{\delta\Gamma[\phi^c]}{\delta\phi^c} \cdot \frac{\delta\phi^c}{\delta J} + \frac{\delta\phi^c}{\delta J} \cdot \frac{\delta}{\delta\phi^c} \right] \\ &= \frac{\delta S}{\delta\phi} \left[\phi^c + \frac{\delta\phi^c}{\delta J} \cdot \frac{\delta}{\delta\phi^c} \right], \end{aligned} \quad (3.18)$$

where we used Eq. (3.16). Next, from Eq. (3.15), one obtains:

$$\frac{\delta\phi^c(x)}{\delta J(x')} = \frac{\delta^2 W}{\delta J(x') \delta J(x)}. \quad (3.19)$$

Also:

$$\begin{aligned} \delta(x - x') &= \frac{\delta\phi^c(x)}{\delta\phi^c(x')} = \int d^4z \frac{\delta\phi^c(x)}{\delta J(z)} \frac{\delta J(z)}{\delta\phi^c(x')} \\ &= \int d^4z \frac{\delta^2\Gamma}{\delta\phi^c(x') \delta\phi^c(z)} \frac{\delta^2 W}{\delta J(z) \delta J(x)}, \end{aligned} \quad (3.20)$$

from where one concludes that

$$\frac{\delta^2 W}{\delta J(x) \delta J(x')} = \left[\frac{\delta^2\Gamma}{\delta\phi^c(x) \delta\phi^c(x')} \right]^{-1}. \quad (3.21)$$

This implies that, from Eq. (3.11):

$$\left. \frac{\delta^2\Gamma}{\delta\phi^c(x) \delta\phi^c(y)} \right|_{\phi^c=0} = \Delta^{-1}(x - y). \quad (3.22)$$

Therefore, DSEs for the proper Green's functions (or proper vertices) can be derived by from

$$\frac{\delta\Gamma}{\delta\phi^c} - \frac{\delta S}{\delta\phi} \left[\phi^c - \left(\frac{\delta^2\Gamma}{\delta\phi^c\delta\phi^c} \right)^{-1} \frac{\delta}{\delta\phi^c} \right] = 0. \quad (3.23)$$

In summary, equations (3.9), (3.13) and (3.23) provide a means to derive DSEs for n -point correlation functions (vacuum expectation values of products of fields) by taking n -th derivatives with respect to external sources J or conjugate fields ϕ^c of appropriate generating functionals, and setting all sources or conjugate fields to zero afterwards.

Note that in order to avoid overloading the notation, the index ' c ' in $\phi^c(x)$ can be dropped in the equations above once one is clear about their meaning and their different roles played in the equations above. This will be followed in the next section, when we discuss QED and QCD.

We note that our derivations of the different DSEs do not consider renormalization issues, a crucial aspect in any field-theoretic calculation. We also do not consider ghosts fields, an extremely important aspect of gauge fixing in non-Abelian gauge field theories when dealing with covariant gauges. The simple model application of the dressed perturbation theory we are considering in this dissertation, which concentrates on the DSE for the quark propagator for a given contact interaction ansatz for the gluon propagator, these issues do not appear - the DSE equation for the quark propagator has the same structure as the DSE for the electron propagator. The application discussed here is a first step toward a more elaborate approach, in which the gluon propagator is also corrected via dressed perturbation theory. In this latter case, renormalization and ghosts fields must be considered.

In the next section, we show with some detail how the use of the tricks discussed above allows to obtain the DSEs for the photon and electron propagators, and for electron-photon vertex function in QED. In Section 3.2 we simply present the DSE for the quark propagator, without detailed derivations. The reason for that is that, as said in the previous paragraph, we are considering only the DSE for quark propagator, whose structure is similar to the DSE for the electron.

3.1 Derivation of DSEs in QED

Although our interests are in the DSE for the quark propagator in QCD, we discuss initially DSEs in QED. The motivations is to avoid complications related to color indices and the non-Abelian nature of the gauge fields. We start with the derivation of the DSE for the photon propagator in the next subsection, and in the following subsections we derive the DSE for the electron propagator and for the electron-photon vertex function.

3.1.1 DSE for the photon propagator

The generating functional for QED in Euclidean space has the form [8]

$$Z[\bar{\eta}, \eta, J_\mu] = \int \mathcal{D}(\bar{\psi}, \psi, A) \exp \left(-S[\bar{\psi}, \psi, A_\mu] + \int d^4x [\bar{\psi}\eta + \bar{\eta}\psi + A_\mu J^\mu] \right), \quad (3.24)$$

with $S[\bar{\psi}, \psi, A_\mu]$ being the QED Euclidean action:

$$\begin{aligned} S[\bar{\psi}, \psi, A_\mu] = & \int d^4x \left\{ \bar{\psi}(x) (\gamma \cdot \partial + m + ie \gamma \cdot A) \psi(x) \right. \\ & \left. + \frac{1}{2} A_\mu(x) \left[-\delta_{\mu\nu} \partial^2 + \left(1 - \frac{1}{\xi} \right) \partial_\mu \partial_\nu \right] A_\nu(x) \right\}, \quad (3.25) \end{aligned}$$

where ξ is the covariant-gauge fixing parameter. We use the fact that the functional integral of a total functional derivative is zero given appropriate boundary conditions – see previous section:

$$\begin{aligned} 0 &= \int \mathcal{D}(\bar{\psi}, \psi, A_\mu) \frac{\delta}{\delta A_\mu(x)} \exp \left(-S[\bar{\psi}, \psi, A_\mu] + \int d^4x [\bar{\psi}_j \eta_j + \bar{\eta}_j \psi_j + A_\mu J^\mu] \right) \\ &= \left(-\frac{\delta S}{\delta A_\mu(x)} \left[\frac{\delta}{\delta J}, \frac{\delta}{\delta \bar{\eta}}, -\frac{\delta}{\delta \eta} \right] + J_\mu(x) \right) Z[\bar{\eta}, \eta, J_\mu]. \quad (3.26) \end{aligned}$$

From (3.25), one obtains

$$-\frac{\delta S}{\delta A_\mu(x)} = -ie \bar{\psi}(x) \gamma_\mu \psi(x) - \left[-\delta_{\mu\nu} \partial^2 + \left(1 - \frac{1}{\xi} \right) \partial_\mu \partial_\nu \right] A_\nu(x) \quad (3.27)$$

Next, introducing the generating functional of connected Green's functions $W[\bar{\eta}, \eta, J_\mu]$:

$$Z[\bar{\eta}, \eta, J_\mu] = e^{-W[\bar{\eta}, \eta, J_\mu]}, \quad (3.28)$$

it is advantageous to introduce the generating functional for one-particle-irreducible (1PI) Green's functions $\Gamma[\bar{\psi}, \psi, A_\mu]$ via the Legendre transformation

$$W[\bar{\eta}, \eta, J_\mu] = \Gamma[\bar{\psi}, \psi, A_\mu] - \int d^4x [\bar{\psi}_j \eta_j + \bar{\eta}_j \psi_j + A_\mu J^\mu], \quad (3.29)$$

with

$$\frac{\delta W}{\delta J_\mu} = -A_\mu, \quad \frac{\delta W}{\delta \bar{\eta}} = -\psi, \quad \frac{\delta W}{\delta \eta} = \bar{\psi} \quad (3.30)$$

and

$$\frac{\delta \Gamma}{\delta A_\mu} = J_\mu, \quad \frac{\delta \Gamma}{\delta \bar{\psi}} = -\bar{\eta}, \quad \frac{\delta \Gamma}{\delta \psi} = \eta \quad (3.31)$$

Note that J_μ is equivalent to the external source considered in the previous section, and η and $\bar{\eta}$ are Grassmann c-number external sources. In addition, as remarked above, we are dropping the index ‘c’ on c-number field variables.

In the present case, this allows to write (3.26) as

$$\begin{aligned} \left. \frac{\delta \Gamma}{\delta A_\mu(x)} \right|_{\bar{\psi}=\psi=0} &= \left[-\delta_{\mu\nu} \partial^2 + \left(1 - \frac{1}{\xi}\right) \partial_\mu \partial_\nu \right] A_\nu(x) \\ &+ ie \text{Tr} \left[\gamma_\mu \left(\left. \frac{\delta^2 \Gamma}{\delta \bar{\psi}(x) \delta \psi(x)} \right|_{\bar{\psi}=\psi=0} \right)^{-1} \right], \end{aligned} \quad (3.32)$$

since $W[\bar{\eta}, \eta, J_\mu]$ depends only on powers of pairs of $\bar{\eta}$ and η , which implies that setting $\bar{\eta} = \eta = 0$ after differentiating W will only give a nonzero result for equal numbers of $\bar{\eta}$ and η derivatives. In (3.32), one has

$$\left(\left. \frac{\delta^2 \Gamma}{\delta \bar{\psi}(x) \delta \psi(y)} \right|_{\bar{\psi}=\psi=0} \right)^{-1} = S(x, y, [A_\mu]), \quad (3.33)$$

where $S(x, y, [A_\mu])|_{A_\mu=0} = S(x, y)$ is the electron propagator. A next step in the derivation of the DSE for the inverse of the photon propagator is to act with $\delta/\delta A_\nu$ on (3.33) and set $A_\mu(x) = 0$:

$$\begin{aligned} D_{\mu\nu}^{-1}(x, y) &= \left. \frac{\delta^2 \Gamma}{\delta A_\mu(x) \delta A_\nu(y)} \right|_{\substack{A_\mu=0 \\ \bar{\psi}=\psi=0}} \\ &= \left[-\delta_{\mu\nu} \partial^2 + \left(1 - \frac{1}{\xi}\right) \partial_\mu \partial_\nu \right] \delta^4(x - y) \\ &+ ie \text{Tr} \left[\gamma_\mu \frac{\delta}{\delta A_\mu(x)} \left(\left. \frac{\delta^2 \Gamma}{\delta \bar{\psi}(x) \delta \psi(x)} \right|_{\bar{\psi}=\psi=0} \right)^{-1} \right]. \end{aligned} \quad (3.34)$$

Now using the relation for finite dimensional matrices

$$-\frac{dA(x)^{-1}}{dx} = A(x)^{-1} \frac{dA(x)}{dx} A(x)^{-1}, \quad (3.35)$$

one easily shows that Eq. (3.34) becomes

$$\begin{aligned} D_{\mu\nu}^{-1}(x, y) &= \left[-\delta_{\mu\nu} \partial^2 + \left(1 - \frac{1}{\xi}\right) \partial_\mu \partial_\nu \right] \delta^4(x - y) \\ &- ie \int d^4u d^4w \text{Tr} \left[\gamma_\mu S(x, w) \frac{\delta}{\delta A_\nu(y)} \frac{\delta^2 \Gamma}{\delta \bar{\psi}(u) \delta \psi(w)} S(w, x) \right], \end{aligned} \quad (3.36)$$

where we have used (3.33) and set $A_\mu = 0$ in that equation. Finally, one can write this (3.36) as

$$D_{\mu\nu}^{-1}(x, y) = (D_0)_{\mu\nu}^{-1}(x, y) + \Pi_{\mu\nu}(x, y) , \quad (3.37)$$

where $(D_0)_{\mu\nu}^{-1}(x, y)$ is the inverse of the noninteracting photon propagator:

$$(D_0)_{\mu\nu}^{-1}(x, y) = \left[-\delta_{\mu\nu} \partial^2 + \left(1 - \frac{1}{\xi} \right) \partial_\mu \partial_\nu \right] \delta^4(x - y) , \quad (3.38)$$

and $\Pi_{\mu\nu}(x, y)$ is the photon polarization tensor:

$$\Pi_{\mu\nu}(x, y) = -e^2 \int d^4u d^4w \text{Tr}[\gamma_\mu S(x, w) \Gamma_\nu(y; u, w) S(w, x)] , \quad (3.39)$$

where

$$e\Gamma_\mu(x, y; z) = \frac{\delta}{\delta A_\nu(x)} \frac{\delta^2 \Gamma}{\delta \bar{\psi}^f(x) \delta \psi^f(y)} \Big|_{\substack{A_\mu=0 \\ \psi=\bar{\psi}=0}} , \quad (3.40)$$

which is the proper fermion-gauge-boson vertex. Eq. (3.37) is the DSE for the inverse of photon propagator; it is depicted in Fig. 3.1.

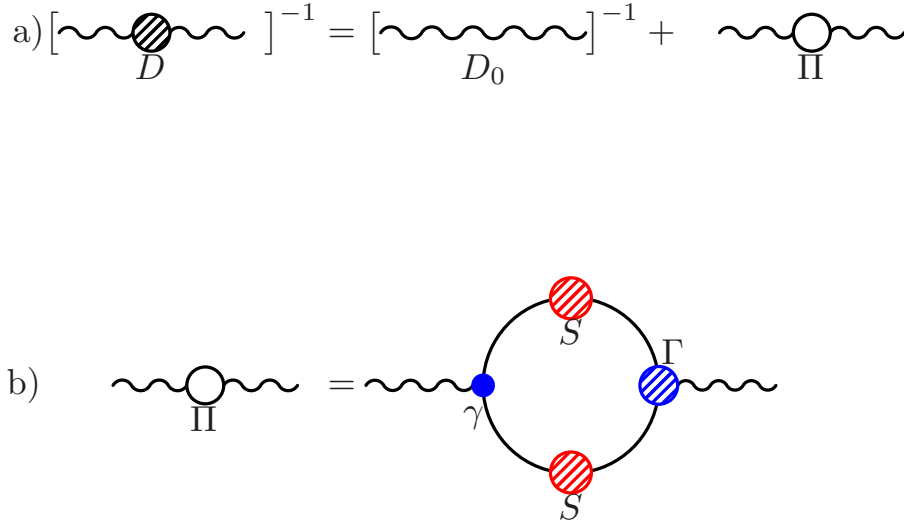


Figure 3.1: a) Dyson-Schwinger equation for the inverse of photon propagator. b) Photon self-energy.

For explicit analytic or numeric calculations, it is more convenient to work in momentum space. Taking Fourier transform of (3.37) and using translational invariance we have

$$D_{\mu\nu}^{-1}(k) = (D_0)_{\mu\nu}^{-1}(k) + \Pi_{\mu\nu}(k) , \quad (3.41)$$

where

$$(D_0)_{\mu\nu}^{-1}(k) = k^2 \delta_{\mu\nu} - \left(1 - \frac{1}{\xi}\right) k_\mu k_\nu . \quad (3.42)$$

The Ward-Takahashi identity, $k_\mu \Pi^{\mu\nu}(k^2) = 0$, implies that

$$\Pi^{\mu\nu}(k^2) = (\delta_{\mu\nu} k^2 - k_\mu k_\nu) \Pi(k^2) , \quad (3.43)$$

so that we can write (3.41) in the form

$$D_{\mu\nu}^{-1}(k) = \delta_{\mu\nu} (\Pi(k^2) + 1) k^2 - k_\mu k_\nu \left[\left(1 - \frac{1}{\xi}\right) + \Pi(k^2) \right] . \quad (3.44)$$

Now, using the fact that the most general form for photon propagator is

$$D_{\mu\nu}(k) = A(k) \delta_{\mu\nu} + B(k) k_\mu k_\nu , \quad (3.45)$$

and that

$$D_{\mu\nu}^{-1}(k) D^{\nu\lambda}(k) = \delta_\mu^\lambda , \quad (3.46)$$

one finds that

$$A(k) = \frac{1}{k^2(1 + \Pi(k^2))}, \quad B(k) = \frac{[(1 + \Pi(k^2))\xi - 1]}{k^4[1 + \Pi(k^2)]} , \quad (3.47)$$

and, therefore:

$$D_{\mu\nu}(k) = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \frac{1}{k^2[1 + \Pi(k^2)]} + \xi \frac{k_\mu k_\nu}{k^4} . \quad (3.48)$$

In momentum space, the DSE for the photon propagator in momentum space is

$$D_{\mu\nu}(k) = (D_0)_{\mu\nu}(k) - (D_0)_{\mu\tau}(k) \Pi_{\tau\rho}(k) D_{\rho\nu}(k) . \quad (3.49)$$

Fig. 3.2 presents a pictorial representation of this DSE.

To finalize this discussion, we present the the photon's self-energy (3.39) in momentum space:

$$\Pi_{\mu\nu}(k) = -e^2 \int \frac{d^4 p}{(2\pi)^4} \text{Tr} [\gamma_\mu S(p) \Gamma_\nu(p, p+k) S(p+k)] . \quad (3.50)$$

3.1.2 DSE for the electron propagator

The starting point is to perform the functional derivative with respect to $\bar{\psi}(x)$, instead of A_μ as in Eq. (3.26), so that

$$\left[\left(\gamma \cdot \partial + m + ie \gamma^\mu \frac{\delta}{\delta J^\mu(x)} \right) \frac{\delta}{\delta \bar{\eta}(x)} - \eta(x) \right] Z[\bar{\eta}, \eta, J] = 0 . \quad (3.51)$$

$$\text{wavy line with } D \text{ circle} = \text{wavy line } D_0 - \text{wavy line with } D_0 \Pi \text{ circle} \text{ and } D \text{ circle}$$

Figure 3.2: Dyson-Schwinger equation for the photon propagator.

When one is interested in the two point Green's function, as is the case here, one takes a functional derivative with respect to $\eta(y)$ and set $\bar{\eta} = \eta = 0$ to obtain

$$\delta^4(x-y)Z[0,0,J] - \left(\gamma \cdot \partial + m + ie\gamma^\mu \frac{\delta}{\delta J^\mu(x)} \right) S(x,y;[J]) Z[0,0,J] = 0, \quad (3.52)$$

where $S(x,y;[J])$ is here defined in terms of W , instead of Γ , as in Eq. (3.33):

$$S(x,y;[J]) = \frac{\delta^2 W}{\delta \bar{\eta}(x) \delta \eta(y)}, \quad (3.53)$$

and describes the propagation in the presence of the source J - the electron propagator is $S(x,y) = S(x,y;[J])|_{J=0}$. If we use the definition of the connected Green's function via (3.28) and then apply the chain rule one can write (3.52) as

$$e^{-W[\bar{\eta},\eta,J]} \left[\delta^4(x-y) - \left(\gamma \cdot \partial + m - ie\gamma^\mu \frac{\delta W}{\delta J^\mu(x)} + ie\gamma^\mu \frac{\delta}{\delta J^\mu(x)} \right) S(x,y;[J]) \right] = 0. \quad (3.54)$$

We can now divide by e^{-W} and also using that $A_\mu = -\delta W / \delta J^\mu$ we can rewrite the last equation in the form

$$\delta^4(x-y) - \left(\gamma \cdot \partial + m + ie\gamma^\mu A_\mu(x;[J]) + ie\gamma^\mu \frac{\delta}{\delta J^\mu(x)} \right) S(x,y;[J]) = 0. \quad (3.55)$$

Now we evaluate the term $\delta S(x,y;[J]) / \delta J^\mu(x)$ as follows:

$$\begin{aligned} \frac{\delta}{\delta J^\mu(x)} S(x,y;[J]) &= \int d^4 z \frac{\delta A_\nu(z)}{\delta J^\mu(x)} \frac{\delta}{\delta A_\mu(z)} \left(\frac{\delta^2 \Gamma}{\delta \bar{\psi}(x) \delta \psi(y)} \Big|_{\bar{\psi}=\psi=0} \right)^{-1} \\ &= -e \int d^4 z d^4 u d^4 v \frac{\delta A_\nu(z)}{\delta J^\mu(x)} S(x,u) \Gamma_\nu(u,v;z) S(v,y) \\ &= e \int d^4 z d^4 u d^4 v D_{\mu\nu}(x,z) S(x,u) \Gamma_\nu(u,v;z) S(v,y). \end{aligned} \quad (3.56)$$

Here we used (3.33), (3.35) and in the last line we set $J = 0$ and used $D_{\mu\nu}(x,z) = -\delta A_\mu(z) / \delta J^\mu(x)$. Now in the absence of external sources, $A_\mu(x;[J])_{J=0} = 0$, and then (3.55) can be rewritten as

$$\begin{aligned} S^{-1}(x,y) &= (\gamma \cdot \partial + m) \delta^4(x-y) \\ &+ e^2 \int d^4 z d^4 u D_{\mu\nu}(x,z) \gamma_\mu S(x,u) \Gamma_\nu(u,y;z) S(v,y), \end{aligned} \quad (3.57)$$

where we have multiplied by $S^{-1}(y, y')$ and integrated with respect to y' . This is the DSE for the electron propagator in the coordinate representation.

The corresponding equation in momentum space is obtained via Fourier transform; it can be written as

$$\begin{aligned} S^{-1}(p) &= i\gamma \cdot p + m + e^2 \int \frac{d^4 q}{(2\pi)^4} \gamma_\mu D_{\mu\nu}(p - q) S(q) \Gamma_\nu(q, p) \\ &= i\gamma \cdot p + m + \Sigma(p) \end{aligned} \quad (3.58)$$

where

$$\Sigma(p) = e^2 \int \frac{d^4 q}{(2\pi)^4} \gamma_\mu D_{\mu\nu}(p - q) S(q) \Gamma_\nu(q, p) , \quad (3.59)$$

is the electron self-energy. Figs. 3.3 and 3.4 are pictorial representations of the DSE and self-energy, respectively.

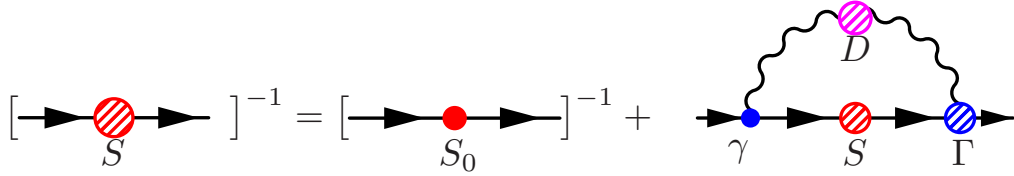


Figure 3.3: DSE for the electron propagator.

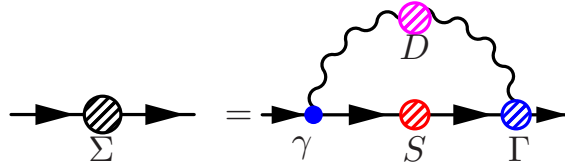


Figure 3.4: Electron self-energy.

3.1.3 DSE for the electron-photon vertex function

The DSE for the electron-photon vertex, Γ_μ , can be derived in a similar way. In momentum space it has the form [9]

$$\Gamma_\mu(q, p) = \gamma_\mu + \int \frac{d^4 l}{(2\pi)^4} S(q + l) \Gamma_\mu(q + l, p + l) S(p + l) K(p + l, q + l, l) , \quad (3.60)$$

where K is the fermion-antifermion scattering kernel. The diagrammatic representation of (3.59) is given in Fig. 3.5.

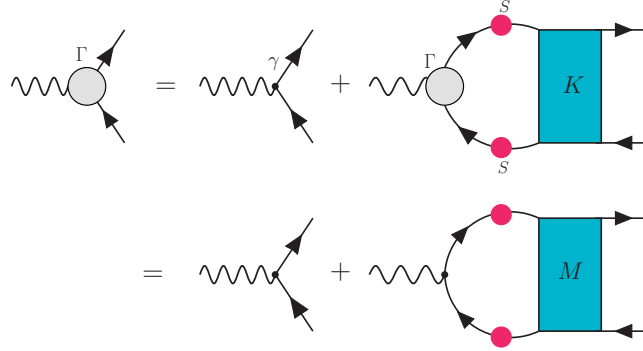


Figure 3.5: DSE for the fermion-photon vertex function.

From Fig. 3.5 we have

$$M = K + K(SS)K + K(SS)K(SS)K + \cdots = K + K(SS)M, \quad (3.61)$$

where M is the fermion-antifermion scattering amplitude (a 4-point function), which itself satisfies an integral equation. The amplitude M is 1PI with respect to the fermion lines and does not contain any fermion-antifermion annihilation contributions. To the lowest order in perturbation theory $M = K = \gamma_\mu(D_0)_{\mu\nu}\gamma_\nu$, with D_0 being the bare photon propagator. When one approximates M by iterating the lowest order contribution for K - i.e. $K = \gamma_\mu(D_0)_{\mu\nu}\gamma_\nu$ - together with S replaced by S_0 - the noninteracting propagator - one obtains the so-called *ladder approximation*, which will be discussed in Section 4.5 of Chapter 4.

3.2 DSE for the quark propagator

The derivation of DSEs in QCD is similar as in QED, but here it is much more involved due to the non-Abelian structure of the theory. The non-Abelian structure introduces, in addition to the quark-gluon vertex (which is similar to the electron-photon vertex), three- and four-gluon vertices and the ghost-gluon vertex. As explained above, we will not make any detailed discussion about ghosts or derivation of QCD DSE equations, we simply quote results. Having chosen to do so, it is important to call attention that the full strength of dressed perturbation theory can only be appreciated when considering the DSE equations for the gluon and vertex functions which, in turn, require also the DSE for ghosts when working in covariant gauges.

Since our work concentrates on the quark propagator, we present the explicit

expression of its DSE. As for the electron, it is convenient to define a self-energy for the quark via the quark propagator as

$$S^{-1}(p) = S_0^{-1}(p) + \Sigma(p), \quad (3.62)$$

where $S_0^{-1}(p) = i\gamma \cdot p + m_q$ is the noninteracting propagator and

$$\Sigma(p) = g^2 \int \frac{d^4 q}{(2\pi)^4} \frac{\lambda^a}{2} \gamma_\mu D_{\mu\nu}^{ab}(p-q) S(q) \Gamma_\nu^b(q, p), \quad (3.63)$$

where $D_{\mu\nu}^{ab}(p-q) = \delta^{ab} D_{\mu\nu}(p-q)$ is the full gluon propagator, g is the coupling constant, and $\Gamma_\nu^a(q, p) = \frac{\lambda^a}{2} \Gamma_\nu(q, p)$ is the full quark-gluon vertex - notations and conventions for color indices and SU(3) matrices are the same as in the previous chapter. As in the case of QED, (3.63) is an integral equation for Σ , as it appears also in the integral via the quark propagator - Fig. 3.6 is a pictorial representation of this DSE.

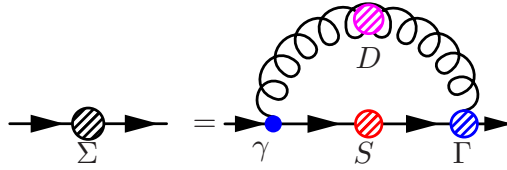


Figure 3.6: Dyson-schwinger equation for the quark self-energy.

The DSE for the gluon propagator $D_{\mu\nu}^{ab}(p)$ is depicted in Fig. 3.7. One of the graphs involves the full ghost propagator which, in turn, has its own DSE. In addition, one sees the appearance of the three-gluon vertex, in addition to the quark-gluon vertex. As one proceeds to correlation functions of higher order (expectation values of products of more and more fields), new structures will appear which also satisfy their own DSEs. Obviously, it is impossible to solve this intricate, infinite chain of coupled correlation functions and truncations and approximation schemes must be invoked. The dressed perturbation theory proposed here is one of such schemes. The rainbow approximation is a classic approximation scheme and we will briefly discuss in the next subsection.

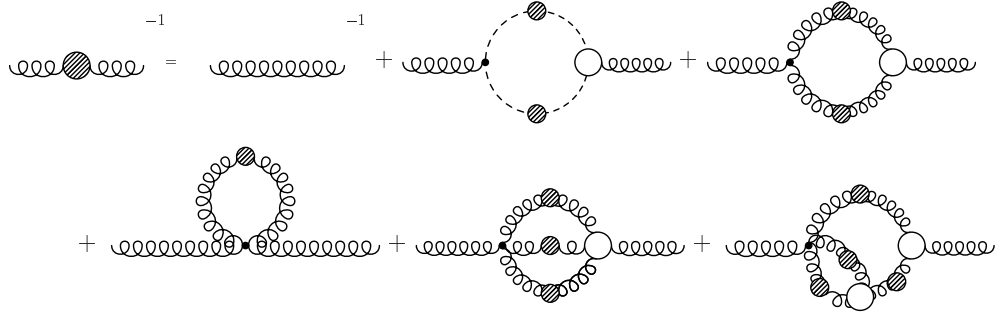


Figure 3.7: Dyson-Schwinger equation for the gluon propagator. Figure taken from Ref. [60].

3.2.1 The rainbow approximation and confining contact interaction

The rainbow approximation is the name attributed to the simultaneous use of both the bare vertex and free gluon propagator [9] in Eq. (3.63), so that

$$S^{-1}(p) = i\gamma \cdot p + m + e^2 \int \frac{d^4 q}{(2\pi)^4} D_0^{\mu\nu}(p-q) \gamma_\mu S(q) \gamma_\nu, \quad (3.64)$$

where D_0 is the noninteracting gluon propagator. If one iterates the right hand side of this equation, one obtains an expansion that is illustrated in Fig. 3.8 - the figure motivates the name rainbow for the approximation scheme. It is important to note that the integral equation sums this entire class of diagrams.

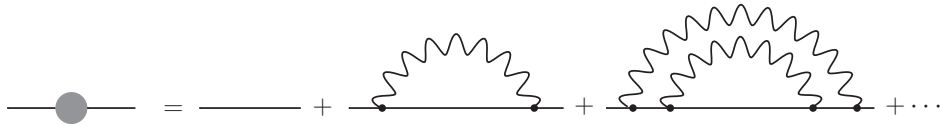


Figure 3.8: Fermion self-energy in the rainbow approximation.

Our perturbative expansion, to be discussed in Chapter 5, requires a starting ansatz (zeroth order approximation) for the quark and gluon self-energies. As for the quark self energy, the one obtained from the rainbow approximation could be one possibility - this would imply in a zero gluon self-energy and, of course, no confinement would be modeled at zeroth order. Since our interests are in the infrared sector of QCD, this might not be a good starting point. Instead of the noninteracting gluon propagator, one alternative is to employ a confining contact interaction [61], see Fig. 3.9, that we shall describe in the following.

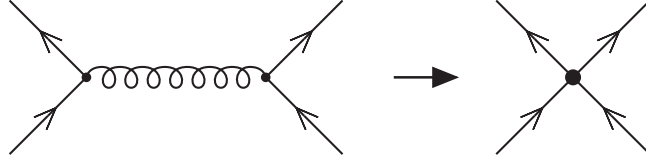


Figure 3.9: Contact interaction.

The starting point is the use in the DSE for the quark propagator of (here we follow very closely the notation and line of argumentation of Ref. [61])

$$g^2 D^{\mu\nu}(p) \Gamma_\nu^a = \frac{4\pi\alpha_{IR}}{m_G^2} t^a \gamma_\mu , \quad (3.65)$$

for the product of the gluon propagator and the quark-gluon vertex in the ladder quark DSE in Eq. (3.64). This is to be understood as being an ansatz for the infrared sector and, therefore, not being suitable for describing short-distance effects. Since the most general form of the dressed quark propagator is

$$S(p) = \frac{1}{i\gamma \cdot p A(p^2) + B(p^2)} = \frac{-i\gamma \cdot p A(p^2) + B(p^2)}{p^2 A^2(p^2) + B^2(p^2)} , \quad (3.66)$$

where $A(p^2)$ and $B(p^2)$ are Lorentz scalar functions, one has that the DSE becomes

$$\begin{aligned} i\gamma \cdot p A(p^2) + B(p^2) &= i\gamma \cdot p + m \\ &+ \frac{16\pi\alpha_{IR}}{3m_G^2} \int \frac{d^4 q}{(2\pi)^4} \gamma_\mu \frac{-i\gamma \cdot q A(q^2) + B(q^2)}{q^2 A^2(q^2) + B^2(q^2)} \gamma_\mu , \end{aligned} \quad (3.67)$$

where we used the fact that

$$\frac{\lambda^a}{2} \frac{\lambda^a}{2} = \frac{4}{3} . \quad (3.68)$$

Multiplying (3.67) this by $-i\gamma \cdot p$ and subsequently taking the trace over Dirac indices, one finds easily that

$$p^2 A(p^2) = p^2 + \frac{32\pi\alpha_{IR}}{3m_G^2} \int \frac{d^4 q}{(2\pi)^4} \frac{(p \cdot q) A(q^2)}{q^2 A^2(q^2) + B^2(q^2)} . \quad (3.69)$$

In the integral above, the angular integral vanishes, since

$$\int d^4 q f(q^2) p \cdot q \sim \int dq q^3 f(q^2) p q \int_0^\pi d\theta \sin^2 \theta \cos \theta = 0 , \quad (3.70)$$

which implies in

$$A(p^2) = 1 . \quad (3.71)$$

Using this in (3.67) and taking the trace, one obtains

$$B(p^2) = m + 4 \frac{16\pi\alpha_{IR}}{3m_G^2} \int \frac{d^4q}{(2\pi)^4} \frac{B(q^2)}{q^2 + B^2(q^2)}. \quad (3.72)$$

Since the right hand side of this equation is independent of p , one has that B is independent of p . Defining

$$B(p^2) = M, \quad (3.73)$$

one has that

$$S^{-1}(p) = i\gamma \cdot p + M, \quad (3.74)$$

where

$$M = m + \frac{16\pi\alpha_{IR}}{3m_G^2} \int \frac{d^4q}{(2\pi)^4} \frac{4M}{q^2 + M^2}. \quad (3.75)$$

This is precisely the same as in the NJL model; this is Eq. (2.53) transcribed to Euclidean space, with the identification $G = 16\pi\alpha_{IR}/(39m_G^2)$, for $N_c = 3$ and $N_f = 2$. As with the NJL model, one needs an ultraviolet regularization scheme to obtain physical results.

So far, no confining property is present in the ansatz, since quarks can go on shell for $s = -M^2$. Elimination of the mass shell can be achieved via a proper-time regularization scheme [62], in that in addition of an ultraviolet regulator, an infrared cutoff is introduced in a way that no poles appear in the quark propagator on the real s -axis - if the quark propagator does not have such poles, quarks cannot propagate asymptotically and there are no quark-antiquark thresholds. This will be discussed next.

First, we rewrite (3.75) as

$$M = m + M \frac{4\alpha_{IR}}{3\pi m_G^2} \int_0^\infty ds s \frac{1}{s + M^2}. \quad (3.76)$$

Here we have introduced the variable $s = q^2$ and integrated over the solid angle. Proper time regularization starts from the identity:

$$\frac{1}{s + M^2} = \int_0^\infty d\tau e^{-\tau(s+M^2)}. \quad (3.77)$$

Regularization of the ultraviolet corresponds to cut off the high- s part of the integrand in (3.76). This is equivalent to cut off the low- τ values in the integral (3.77), as:

$$\int_{\tau_{UV}^2}^\infty d\tau e^{-\tau(s+M^2)} = \frac{e^{-(s+M^2)\tau_{UV}^2}}{s + M^2}. \quad (3.78)$$

Clearly, when $s \rightarrow \infty$, the integrand in (3.76) goes to zero exponentially. Now, in order to eliminate the mass shell in the propagator, one introduces an infrared regulator, cutting off the high- τ values in (3.77). Specifically:

$$\int_{\tau_{UV}^2}^{\tau_{IR}^2} d\tau e^{-\tau(s+M^2)} = \frac{e^{-(s+M^2)\tau_{UV}^2} - e^{-(s+M^2)\tau_{IR}^2}}{s + M^2}, \quad (3.79)$$

which has no poles when $s \rightarrow -M^2$. Using this in the integral in (3.76), one can write

$$\begin{aligned} \int_0^\infty ds s \frac{1}{s + M^2} &= \int_0^\infty ds s \int_{\tau_{UV}^2}^{\tau_{IR}^2} d\tau e^{-\tau(s+M^2)} = \int_{\tau_{UV}^2}^{\tau_{IR}^2} d\tau \frac{1}{\tau} \int_0^\infty ds e^{-\tau(s+M^2)} \\ &= - \int_{\tau_{UV}^2}^{\tau_{IR}^2} d\tau \frac{1}{\tau^2} e^{-\tau(s+M^2)} \Big|_0^\infty = \int_{\tau_{UV}^2}^{\tau_{IR}^2} d\tau \tau^{-2} e^{-\tau M^2}. \end{aligned} \quad (3.80)$$

Making the change of variable $t = \tau M^2$, we can rewrite the above as

$$\begin{aligned} \int_0^\infty ds s \frac{1}{s + M^2} &= M^2 \int_{\tau_{UV}^2 M^2}^{\tau_{IR}^2 M^2} dt t^{-2} e^{-t} = M^2 \left(\int_{\tau_{UV}^2 M^2}^\infty dt t^{-2} e^{-t} - \int_{\tau_{IR}^2 M^2}^\infty dt t^{-2} e^{-t} \right) \\ &= M^2 [\Gamma(-1, \tau_{UV}^2 M^2) - \Gamma(-1, \tau_{IR}^2 M^2)], \end{aligned} \quad (3.81)$$

where $\Gamma(\alpha, y)$ is the incomplete gamma function:

$$\Gamma(\alpha, y) = \int_y^\infty dt t^{\alpha-1} e^{-t}. \quad (3.82)$$

Thus, the DSE (3.76) (the gap equation), using the notation of Re. [61], can be written as

$$M = m + M \frac{4\alpha_{IR}}{3\pi m_G^2} \mathcal{C}(M^2; \tau_{IR}, \tau_{UV}), \quad (3.83)$$

with

$$\mathcal{C}(M^2; \tau_{IR}^2, \tau_{UV}^2) = M^2 [\Gamma(-1, \tau_{UV}^2 M^2) - \Gamma(-1, \tau_{IR}^2 M^2)]. \quad (3.84)$$

Numerical solutions of the gap equation will be discussed in the next section, when we will also discuss properties of the pion in the same model.

3.3 Summary

In the present chapter we presented the Dyson-Schwinger equations for QED and QCD. We have discussed a procedure to obtain DSEs in a field theory, in particular we obtained explicitly the photon and electron propagators in QED, and discussed

via graphic representation the electron-photon vertex. Next we considered the quark propagator in QCD and discussed the rainbow approximation, an approximation in for the DSE that uses the bare quark-gluon vertex and the noninteracting gluon propagator. Motivated by our future applications, we used a contact quark-gluon interaction with a proper-time regularization that leads to a solution of the DSE for the quark propagator has no poles on the real axis, modeling in this way the absence of a mass shell for the quarks.

Chapter 4

THE BETHE-SALPETER EQUATION

In non-relativistic quantum mechanics, bound states are described by normalized wave functions, solutions of the many-particle Schrödinger equation. A proper relativistic treatment is based on the BetheSalpeter equation (BSE), which is an integral equation to study bound and scattering states in a relativistic quantum field theory (RQFT). This equation is the relativistic analog of the Lippmann-Schwinger equation in quantum mechanics [63]. The BSE was first proposed by Yochiro Nambu in 1950 [64], but without derivation. The general form of the BSE was derived in 1951 by Bethe and Salpeter [65] on the basis of Feynman-graphical considerations. In 1951, Gell-Mann and Low established its field-theoretical foundation [66]. In this same year, Schwinger independently proposed also the BSE, who used the functional derivative formalism [67], and one year later Kita, who employed the S-matrix-theoretical procedure [68]. Due to its mathematical difficulty, in the first decade of its history the theoretical studies of the equation were not very intense.

The Bethe-Salpeter equation can be written down in any field theory, and for any subsector of the Fock space of constituents in a bound system. In addition the BSE is nonperturbative, and this fact is important, because bound states are not be accessible in perturbation theory. The first application of the BSE was in condensed matter physics, performed by Hanke and Sham in 1980 [69]; they studied the many-particle effects in the optical spectrum of a semiconductor. Today, the applications of BSE have been successful not only in condensed matter physics, but in hadron physics as well.

4.1 Derivation of the BSE

In the relativistic approach, bound states and resonances are identified by the occurrence of poles in Green functions. For simplicity of presentation, we consider a model with two scalar particles ϕ_1 and ϕ_2 exchanging a third scalar particle Φ .

The generating functional in Euclidean space for this theory is

$$Z[j_1, j_2, J] = \int \mathcal{D}(\phi_1, \phi_2, \Phi) \exp \left(-S + \sum_{i=1}^2 j_i \cdot \phi_i + J \cdot \Phi \right), \quad (4.1)$$

where

$$S = \int d^4x \left[\frac{1}{2} \sum_{i=1}^2 (\partial_\mu \phi_i \partial_\mu \phi_i + m_i^2 \phi_i^2) + \frac{1}{2} (\partial_\mu \Phi \partial_\mu \Phi + M^2 \Phi^2) + \frac{1}{2} g (\phi_1^2 \Phi + \phi_2^2 \Phi) \right]. \quad (4.2)$$

As seen in the last chapter, DSEs for one-particle propagator can be obtained taking the functional derivative with respect to ϕ_1 , to obtain

$$\left(-\partial^2 + m_1^2 + g \frac{\delta}{\delta J(x)} \right) \frac{\delta Z[j_1, j_2, J]}{\delta j_1(x)} - j_1(x) Z[j_1, j_2, J] = 0, \quad (4.3)$$

where we have performed the substitutions $\phi_1 \rightarrow \delta/\delta j_1$ and $\Phi \rightarrow \delta/\delta J$.

In an analogous manner as in QED, one can easily find that

$$(-\partial^2 + m_1^2) S_1(x, y) + \int d^4 z_2 \Sigma_1(x, z_2) S_1(z_2, y) = \delta^4(x - y), \quad (4.4)$$

with

$$\Sigma_1(x, z_2) = -g^2 \int d^4 z d^4 z_1 \Delta_\Phi(x, z) S_1(x, z_1) \Lambda_1(z_1, z_2; z), \quad (4.5)$$

whose diagrammatic representation is

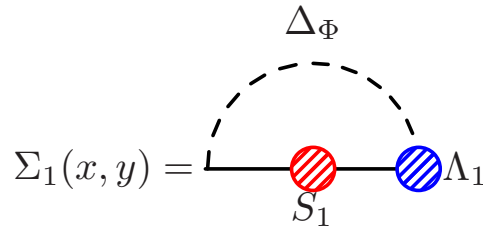


Figure 4.1: Self-energy. Here the solid and dashed lines are the ϕ and Φ propagators respectively.

where

$$S_1(x, y) = \left(\frac{\delta \Gamma_1}{\delta \phi_1(x) \delta \phi_1(y)} \right)^{-1}, \quad (4.6)$$

$$\Delta_{\Phi}(x, y) = \frac{\delta\Phi(y)}{\delta J(x)} , \quad (4.7)$$

$$g\Lambda_1(x, y; z) = \frac{\delta}{\delta\Phi(z)} \frac{\delta\Gamma}{\delta\phi_1(x)\delta\phi_1(y)} , \quad (4.8)$$

and Λ_1 being the connected, irreducible vertex function for this theory.

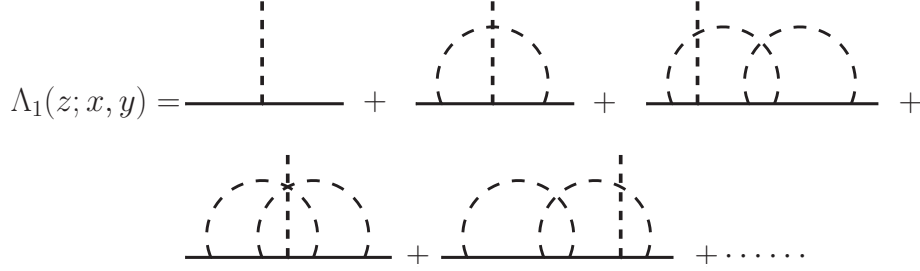


Figure 4.2: 1PI three-point function.

Equation (4.4) can be written as

$$(-\partial^2 + m_1^2 + \Sigma_1)S_1(x, y) = \delta^4(x - y) , \quad (4.9)$$

where we used the notation

$$\Sigma_1 S_1(x, y) = \int d^4 z_2 \Sigma_1(x, z_2) S_1(z_2, y) .$$

We are interested in the propagation of two particles, namely, the four-point Green function $G(x_1, y_1; x_2, y_2)$. Then, taking three functional derivatives with respect to j_1 and j_2 into (4.3), and after to set j_1 and j_2 at zero, one finds

$$\left(-\partial^2 + m_1^2 + g \frac{\delta}{\delta J(x)} \right) G(x_1, y_1; x_2, y_2) = \delta^4(x_1 - y_1) S_2(x_2, y_2) , \quad (4.10)$$

which can be written in the form

$$\begin{aligned} \left(-\partial^2 + m_1 + \Sigma_1 \right) G(x_1, y_1; x_2, y_2) &= \delta^4(x_1 - y_1) S_2(x_2, y_2) \\ &+ \left(\Sigma_1 - g \frac{\delta}{\delta J(x)} \right) G(x_1, y_1; x_2, y_2) . \end{aligned} \quad (4.11)$$

Multiplying by $S_1(z_1, x_1)$ and using (4.9) and integrating over x_1 , one finds that

$$\begin{aligned} G(x_1, y_1; x_2, y_2) &= S_1(x_1, y_1) S_2(x_2, y_2) \\ &+ \int d^4 z_1 S_1(x_1, z_1) \left(\Sigma_1 - g \frac{\delta}{\delta J(z_1)} \right) G(z_1, y_1; x_2, y_2) . \end{aligned} \quad (4.12)$$

The last term on the right side of (4.12) may best understood from a graphical analysis. One can see that the functional derivative of the four-point function with respect to external sources generates the two particle irreducible kernel. To see this, let us to consider a simpler case, for example

$$\frac{\delta S_1(x, y, j)}{\delta J(x)}.$$

First, we expand $S_1(x, y, j)$ graphically as in Fig.4.3.

$$S_1(x, y, J) = \text{---} + \text{---} \overset{\times}{\uparrow} \text{---} + \text{---} \text{---} \text{---} + \text{---} \overset{\times}{\uparrow} \text{---} \overset{\times}{\uparrow} \text{---} +$$

$$+ \overset{\times}{\uparrow} \text{---} \text{---} \text{---} + \text{---} \overset{\times}{\uparrow} \text{---} \text{---} \text{---} + \text{---} \text{---} \overset{\times}{\uparrow} \text{---} + \text{---} \text{---} \text{---} \text{---} \text{---}$$

$$+ \overset{\times}{\uparrow} \text{---} \text{---} \text{---} \text{---} \text{---} + \text{---} \overset{\times}{\uparrow} \text{---} \text{---} \text{---} \text{---} \text{---} + \dots$$

$$+ \text{---} \text{---} \text{---} \text{---} \text{---} + \text{---} \overset{\times}{\uparrow} \text{---} \text{---} \text{---} \text{---} \text{---} + \text{---} \overset{\times}{\uparrow} \text{---} \text{---} \text{---} \text{---} \text{---} + \dots$$

Figure 4.3: Diagrammatic expansion of $S_1(x, y, J)$. Here $\times$ represents to the source J .

Now to take the functional derivative with respect to $J(x)$ is straightforward. First, we take one of the j lines from each graph and attach them to the point x , and then set $J = 0$, which will eliminate any graph with more than one source of interaction, see Fig.4.4.

$$\frac{\delta S_1(x, y, J)}{\delta J(x)} \Big|_{J=0} = \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} \text{---} \text{---}$$

$$+ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} + \dots$$

$$+ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} + \dots$$

We see that in this case the functional derivative with respect to J generates the self-energy terms. Similarly for the case of the four-point function, one has

$$\begin{aligned}
 &= \left[\text{---} \overset{\text{dashed}}{\cap} \text{---} + \text{---} \overset{\text{dashed}}{\cap} \overset{\text{dashed}}{\cap} \text{---} + \text{---} \overset{\text{dashed}}{\cap} \overset{\text{dashed}}{\cap} \overset{\text{dashed}}{\cap} \text{---} + \dots \right] \times \\
 &\left[\text{---} + \text{---} \overset{\text{dashed}}{\cap} \text{---} + \text{---} \overset{\text{dashed}}{\cap} \overset{\text{dashed}}{\cap} \text{---} + \text{---} \overset{\text{dashed}}{\cap} \overset{\text{dashed}}{\cap} \overset{\text{dashed}}{\cap} \text{---} \right. \\
 &\quad \left. + \text{---} \overset{\text{dashed}}{\cap} \text{---} + \dots \right] = \Sigma_1 S_1
 \end{aligned}$$

 Figure 4.4: Functional derivative of $S_1(x, y; J)$ with respect to J .

$$G(x_1, y_1; x_2, y_2; J) = \text{---} \text{---} \text{---} \text{---} = \text{---} + \text{---} \Lambda_1 \text{---} + \text{---} \Lambda_1 \text{---} + \text{---} \Lambda_1 \Lambda_1 \text{---} + \text{---} \Lambda_1 \Lambda_1 \text{---} + \text{---} \Lambda_1 \Lambda_1 \text{---} + \dots$$

 Figure 4.5: Diagrammatic expansion of $G(x_1, y_1; x_2, y_2 : J)$.

After taking the functional derivative with respect to J , we find

$$S_1(x_1, z_1) \frac{\delta}{\delta J(z_1)} G(x_1, y_1; x_2, y_2) = \text{---} \Sigma_1 \text{---} G + \text{---} K \text{---} G$$

 Figure 4.6: Functional derivative of G with respect to J .

From this figure we see that the first term vanishes when inserted into (4.12), so

that one finds

$$\begin{aligned}
 G(x_1, y_1; x_2, y_2) &= S_1(x_1, y_1)S_2(x_2, y_2) + \int d^4 z_1 d^4 z_2 d^4 z_3 d^4 z_4 S_1(x_1, z_1)S_2(x_2, z_2) \\
 &\times K(z_1, z_3; z_2, z_4)G(z_3, y_1; z_4, y_2) .
 \end{aligned}
 \tag{4.13}$$

This is the inhomogeneous Bethe-Salpeter equation; it is depicted in Fig. 4.7.

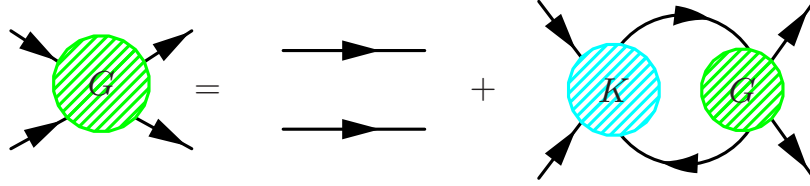


Figure 4.7: The inhomogeneous Bethe-Salpeter equation.

4.2 The BSE in Momentum Space

For actual calculations, it is more useful to work in momentum space. Hence, we will rewrite the BSE (4.13) in momentum space. First, we define relative space-time coordinates, namely

$$x = x_1 - x_2 , \quad \text{and} \quad y = y_1 - y_2 . \tag{4.14}$$

This choice is motivated by the translational-invariance of the theory. Now we may choose two positive quantities η_1 and η_2 such that

$$X = \eta_1 x_1 + \eta_2 x_2 , \quad x_1 = X + \eta_2 x_2 , \quad x_2 = X - \eta_1 x_1 , \tag{4.15}$$

with

$$\eta_1 + \eta_2 = 1 . \tag{4.16}$$

And analogously,

$$Y = \eta_1 y_1 + \eta_2 y_2 , \quad y_1 = Y + \eta_2 y_2 , \quad y_2 = Y - \eta_1 y_1 . \tag{4.17}$$

Now let p denote the conjugate variable to x . Then, according to (4.15) we have

$$P = p_1 + p_2 , \quad p = \eta_2 p_1 - \eta_1 p_2 , \tag{4.18}$$

where

$$p_1 = \eta_1 P + p, \quad p_2 = \eta_2 P - p. \quad (4.19)$$

Here P is the total momentum of the two particles ϕ_1 and ϕ_2 , p their relative momentum, and $p_{1,2}$ their individual momenta. Then, after taking Fourier transformation, we obtain

$$\begin{aligned} G(P; q, q') &= S_1(\eta_1 P + p) S_2(\eta_2 P - p) \delta^4(q - q') \\ &+ S_1(\eta_1 P + p) S_2(\eta_2 P - p) \int d^4 q'' K(P; q, q'') G(P; q, q''). \end{aligned} \quad (4.20)$$

4.3 The Homogeneous BSE

Here we will discuss bound states. It is useful to define a bound-state amplitude known as the Bethe-Salpeter Amplitude (BSA). In order to obtain the homogeneous BSE, we start with the definition of the four-point Green's function:

$$G(x_1, y_1; x_2, y_2) = \langle 0 | T \{ \psi_1(x_1) \psi_2(x_2) \bar{\psi}_1(y_1) \bar{\psi}_2(y_2) \} | 0 \rangle, \quad (4.21)$$

with $x_1^0, x_2^0 > y_1^0, y_2^0$. Then, we can write G as

$$G(x_1, y_1; x_2, y_2) = \langle 0 | T \{ \psi_1(x_1) \psi_2(x_2) \} T \{ \bar{\psi}_1(y_1) \bar{\psi}_2(y_2) \} | 0 \rangle, \quad (4.22)$$

Now, we insert a complete set of states $|P, \alpha\rangle$ into the middle of the right-hand side of (4.22). With this, we have

$$G(x_1, y_1; x_2, y_2) = \sum_{P, \alpha} \langle 0 | T \{ \psi_1(x_1) \psi_2(x_2) \} | P, \alpha \rangle \langle P, \alpha | T \{ \bar{\psi}_1(y_1) \bar{\psi}_2(y_2) \} | 0 \rangle, \quad (4.23)$$

where P is the total momentum of the state and α all other quantum numbers. Then, the contribution from a particular bound state B is

$$G(x_1, y_1; x_2, y_2) = \sum_P \langle 0 | T \{ \psi_1(x_1) \psi_2(x_2) \} | P_B \rangle \langle P_B | T \{ \bar{\psi}_1(y_1) \bar{\psi}_2(y_2) \} | 0 \rangle. \quad (4.24)$$

Now, we define the called Bethe-Salpeter wave function as:

$$\chi_B(x_1, x_2; P_B) = \langle 0 | T \{ \psi_1(x_1) \psi_2(x_2) \} | P_B \rangle, \quad (4.25)$$

and its conjugate

$$\bar{\chi}_B(x_1, x_2; P_B) = \langle P_B | T \{ \bar{\psi}_1(x_1) \bar{\psi}_2(x_2) \} | 0 \rangle. \quad (4.26)$$

Because of the translational invariance of the theory, we can write

$$\chi_B(x_1, x_2; P_B) = e^{-iP_B \cdot X} \chi(x, P_B) , \quad (4.27)$$

$$\bar{\chi}_B(x_1, x_2; P_B) = e^{iP_B \cdot X} \bar{\chi}(x, P_B) . \quad (4.28)$$

With this, Eq. (4.24) can be written as

$$\begin{aligned} G(x_1, y_1; x_2, y_2) &= \int \frac{d^4 P}{(2\pi)^3} \chi_B(x_1, x_2; P_B) \bar{\chi}_B(y_1, y_2; P_B) \theta(P^0) \delta(P^2 - M^2) \theta(X^0 - Y^0) \\ &= \int \frac{d^3 P}{(2\pi)^3 2P_B^0} \theta(X^0 - Y^0) \chi_B(x; P_B) \bar{\chi}_B(y; P_B) e^{-iP_B^0(X^0 - Y^0) + i\mathbf{P}(\mathbf{X} - \mathbf{Y})} , \end{aligned} \quad (4.29)$$

where we have used the fact that the on-mass-shell condition $P^2 = M^2$ holds for the bound state with mass M , then the sum over momentum states can be written as the integral over the mass shell in momentum space. In the second line $\mathbf{P}_B = \mathbf{P}$, and we use the relation

$$\int \frac{d^3 P}{2P^0} = \int d^4 P \theta(P^0) \delta(P^2 - M^2) . \quad (4.30)$$

Now, we employ the identity

$$\theta(z) = \frac{i}{2\pi} \int dk \frac{e^{-ikz}}{k + i\epsilon} . \quad (4.31)$$

Then, after a transformation $k = P^0 - P_B^0$, Eq. (4.29) can be easily rewritten as

$$G(x_1, y_1; x_2, y_2) = i \int \frac{d^4 P}{(2\pi)^4} \chi_B(x; P_B) \bar{\chi}_B(y; P_B) \frac{e^{-iP \cdot (X - Y)}}{2P_B^0(P^0 - P_B^0 + i\epsilon)} . \quad (4.32)$$

Now, after taking Fourier transform of the last equation, one has [70,71]

$$G(P, q, q') = i \frac{\chi_B(P, q) \bar{\chi}_B(P, q')|_{P^2=M^2}}{2P_B^0(P^0 - P_B^0 + i\epsilon)} + R , \quad (4.33)$$

where R are regular terms in P^2 (has no poles at $P^2 = M^2$). Now, by adding the contribution from the anti-particle states, we obtain in Euclidean space

$$G(P, q, q') = \frac{\chi(P, q) \bar{\chi}(P, q')|_{P^2=-M^2}}{P^2 + M^2} + R . \quad (4.34)$$

The diagrammatic representation of this equation is depicted in Fig.4.8.

In order to obtain the homogeneous BSE, we rewrite (4.20) in a symbolic notation

$$G = S_1 S_2 + S_1 S_2 K G . \quad (4.35)$$

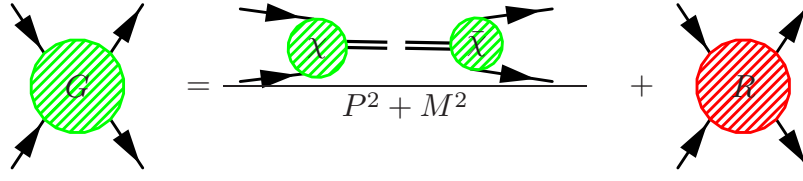


Figure 4.8: Pole contribution in the total momentum of the four-point Green's function at on-shell momenta P , with $P^2 = -M^2$.

If we insert (4.33) into (4.35), we have

$$\frac{\chi_B \bar{\chi}_B}{P^2 + M^2} + R = S_1 S_2 + S_1 S_2 K \left(\frac{\chi_B \bar{\chi}_B}{P^2 + M^2} + R \right). \quad (4.36)$$

Comparing the residue at the pole $P^2 = -M^2$ (multiply the equation by $(P^2 + M^2)$ and afterwards take the limit $P^2 = -M^2$), yields

$$\chi = S_1 S_2 K \chi, \quad (4.37)$$

or

$$\chi(P, q) = S_1(\eta_1 P + p) S_2(\eta_2 P - p) \int \frac{d^4 q'}{(2\pi)^4} K(P; q, q') \chi(P, q'). \quad (4.38)$$

Multiplying by $S_1^{-1} S_2^{-1}$ and setting $\chi = S_1 \Gamma S_2$, one obtains the homogeneous BSE for the Bethe-Salpeter Amplitude $\Gamma(P, q)$, namely

$$\Gamma(P, q) = \int \frac{d^4 q'}{(2\pi)^4} K(P; q, q') S_1(\eta_1 P + q') \Gamma(P, q') S_2(\eta_2 P - q'). \quad (4.39)$$

In diagrammatic representation, this equation has the form

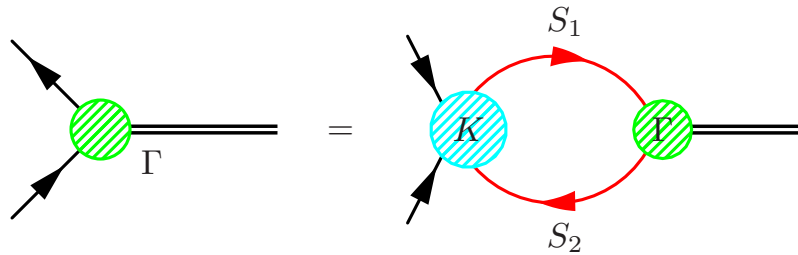


Figure 4.9: Homogeneous Bethe-Salpeter equation.

4.4 Normalization condition

The Bethe-Salpeter wave function must necessarily satisfy a normalization condition similar to the normalization of the wave function in quantum mechanics. From

the inhomogeneous BSE (4.35), one can derive the corresponding normalization condition [8,70]. First, we define D as the inverse of the product $S_1 S_2$, namely

$$D^{-1} = S_1 S_2 . \quad (4.40)$$

If we replace this into expression for G (4.35), one easily finds

$$(D - K)G = \mathbf{1} , \quad (4.41)$$

whose formal solution is

$$G = (D - K)^{-1} . \quad (4.42)$$

Now using the relation

$$\frac{\partial \mathcal{M}^{-1}(\lambda)}{\partial \lambda} = \mathcal{M}^{-1}(\lambda) \frac{\partial \mathcal{M}(\lambda)}{\partial \lambda} \mathcal{M}^{-1}(\lambda) , \quad (4.43)$$

in (4.42), which is valid for any invertible operator $\mathcal{M}$, we obtain

$$\begin{aligned} \frac{\partial G}{\partial \lambda} &= -(D - K)^{-1} \left(\frac{\partial D}{\partial \lambda} - \frac{\partial K}{\partial \lambda} \right) (D - K)^{-1} \\ &= -G \left(\frac{\partial D}{\partial \lambda} - \frac{\partial K}{\partial \lambda} \right) G . \end{aligned} \quad (4.44)$$

Again, we use pole terms for G (4.33) and equate the corresponding residue into (4.44), one finds

$$\frac{\partial M^2}{\partial \lambda} = -\bar{\chi}_B \left(\frac{\partial D}{\partial \lambda} - \frac{\partial K}{\partial \lambda} \right) \chi_B . \quad (4.45)$$

Now, replace $\lambda \rightarrow P_\mu$, to obtain

$$2P_\mu = \frac{\partial}{\partial P_\mu} \bar{\chi} (D - K) \chi , \quad (4.46)$$

or

$$\begin{aligned} 2P_\mu &= \frac{\partial}{\partial P_\mu} \int \frac{d^4 q}{(2\pi)^4} \frac{d^4 q'}{(2\pi)^4} \bar{\chi}(P, q) \left[S_1^{-1}(\eta_1 P + q) S_2^{-1}(\eta_2 P - q) (2\pi)^4 \delta^4(q - q') \right. \\ &\quad \left. - K(P; q, q') \right] \chi(P, q) . \end{aligned} \quad (4.47)$$

This is the sought normalization for χ .

Similarly, one can obtain an analogous canonical normalization condition for $\Gamma(P, q)$. This equation and (4.38) are the basic ingredients to study relativistic two-body bound states.

4.5 The Ladder Approximation

In The BSE (4.39), $K(p, p'; P)$ is the two-particle irreducible kernel, which is a sum of infinitely many terms where each diagram cannot be separated. In order to be tractable, we must truncate the sum after a finite number of diagrams. The Ladder Approximation consists in retain only simple diagrams of a single ϕ exchange in the expansion of K , in other words, it is the lowest order in the expression of this kernel. For this case, the ladder approximation for the kernel K is given by the propagator of the exchanged particle Φ , multiplied by one coupling constant g for each vertex, namely:

$$K_{ladder}(x_1, x_2; y_1, y_2) = -g^2 \Delta_\Phi(x_1, y_1) \delta^4(x_1 - x_2) \delta^4(y_1 - y_2) . \quad (4.48)$$

The BSE in the ladder approximation has the form depicted in Fig. 4.10.

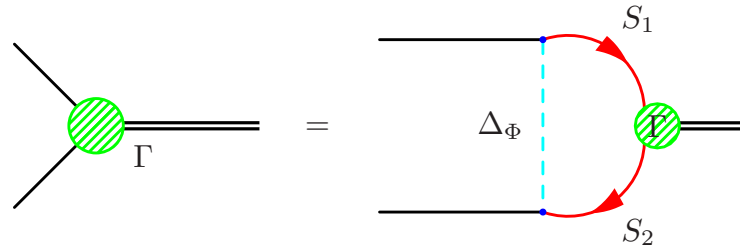


Figure 4.10: Homogeneous Bethe-Salpeter equation in the ladder approximation.

In QCD, the ladder approximation realizes DCSB and has proven to be an important tool to understand dynamical mass generation in QCD. [9]. Also it has been used very intensively in the calculation of hadron masses [17].

4.6 Structure of the Meson BSA

Mesons are hadrons, and they are subject to the strong interaction. They have nonzero masses and integer spin; they carry neither baryon nor lepton numbers. The most interesting meson to study is the pion, since it is the Goldstone mode arising from the dynamical breaking of chiral symmetry in QCD. In this section we follow closely the discussion given in [18]

We have that $\Gamma(P, q)$ depends on two independent momenta, the total momentum of the bound state and the relative momentum q of the meson's constituents. Now

one has three four vectors to build the amplitude: γ_μ , q_μ and P_μ . In order to understand how to build the amplitude, we must characterize each meson by one set of quantum numbers:

$$J^{PC} \rightarrow \begin{cases} J & \text{is the total angular momentum.} \\ P & \text{is the parity.} \\ C & \text{is the charge conjugation.} \end{cases}$$

In order to construct the amplitude, we look for all possible scalar products of these three four-vectors (use them as basis elements in Dirac space). First, we find that $(\gamma_\mu \gamma^\mu, q^2, P^2, q \cdot P) \sim \mathbf{1}$ in Dirac space. Since we have $\mathbf{1}$ as a basis element, we can ignore these. Therefore the remaining possibilities are: $\not{q} = \gamma \cdot q$ and $\not{P} = \gamma \cdot P$, which are Lorentz scalar or pseudoscalar by construction ($\gamma_5 \not{q}$ and $\gamma_5 \not{P}$). In addition, these scalar products have positive parity.

Next we examine charge conjugation. We know that charge conjugate inverts all the quantum numbers including relative momentum*. However, the total momentum P remains unchanged. The charge conjugate of an amplitude is given by

$$\Gamma(P; q) \rightarrow \bar{\Gamma}(P; q) = (C\Gamma(P; -q)C^{-1})^T, \quad (4.49)$$

where the superscript T denotes matrix transposition. Using this and with $C\gamma^\mu C^{-1} = -(\gamma^\mu)^T$, we find easily that $\mathbf{1}$ and $\not{q}$ have positive charge conjugation, while $\not{P}$ has negative charge conjugation. With all this information we can investigate and analyze the property of any amplitude.

For simplicity we start with a meson with spin zero, namely $J = 0$. There are two possibilities for parity: positive parity (scalar) or negative parity (pseudoscalar). We begin with one particle with positive parity. We are interested in finding the possibilities to build an amplitude which corresponds to $J^P = 0^+$. In terms of γ_μ , q_μ and P_μ , we see that only γ_μ introduces a Dirac structure different from $\mathbf{1}$. We need find all possible and linearly independent Dirac structures obtainable from the vectors given here. These structures are called Dirac covariants, and once found, any function multiplying them can be only depend on scalar products of the four

*Since we have that under charge conjugation particle $\rightarrow$ antiparticle, one has that the definition of relative coordinate (4.14) $x_1 \rightarrow x_2$ and $x_2 \rightarrow x_1$, leads to $x \rightarrow -x$ and therefore $p \rightarrow -p$.

vectors q_μ and P_μ . In other words, we can write the BSA as

$$\Gamma(P, q) = \sum_{i=1}^{N_J} T_i(\gamma; P, q) F_i(P^2; q^2; q \cdot P) , \quad (4.50)$$

where T_i are the Dirac covariants, F_i the Lorentz-invariant functions and N_J the number of covariants depending on the spin J of the meson. Now the goal here is to arrive at a linearly independent and complete set of covariants. Then, we will build an orthogonal basis in Dirac space. As previously seen, the products $(\gamma_\mu \gamma^\mu, q^2, P^2, q \cdot P) \sim \mathbf{1}$, so there are our first basis elements.

Now to see orthogonality, we need a scalar product, which in this case is defined as

$$\langle A|B \rangle = \text{Tr}(A \cdot B) . \quad (4.51)$$

We have that the remaining two scalar products from our set of four-vectors are $\not{q}$ and $\not{P}$. Therefore we can see that $\not{q}$ and $\not{P}$ are orthogonal to $\mathbf{1}$, namely

$$\text{Tr}(\mathbf{1} \cdot \not{q}) = \text{Tr}(\not{q}) = 0 = \text{Tr}(\mathbf{1} \cdot \not{P}) = \text{Tr}(\not{P}) . \quad (4.52)$$

On the other hand, we see that

$$\text{Tr}(\not{q} \cdot \not{P}) = 4q \cdot P \mathbf{1} , \quad (4.53)$$

which indicates that they are not orthogonal to each other. However, these two terms can be made orthogonal by the small modification of one of them, namely

$$\not{q} \rightarrow \not{q} - \not{P} \frac{q \cdot P}{P^2} , \quad (4.54)$$

where the last term in the right-hand side is a transversal projection with respect to the total momentum P . Therefore one has

$$\text{Tr} \left[\left(\not{q} - \not{P} \frac{q \cdot P}{P^2} \right) \cdot \not{P} \right] = 0 . \quad (4.55)$$

So far, we have that $\not{q}\not{q}$, $\not{P}\not{P}$ and multiplying $\mathbf{1}$ with either $\not{q}$ or $\not{P}$, do not introduce anything new. The next possibility is $\not{q}\not{P}$. One finds that the construction $\not{q}\not{P} - \not{P}\not{q} = [\not{q}, \not{P}]$ is orthogonal to the three elements that we already have, namely

$$\begin{aligned} \text{Tr}(\mathbf{1} \cdot [\not{q}, \not{P}]) &= 0 , \\ \text{Tr} \left[\left(\not{q} - \not{P} \frac{q \cdot P}{P^2} \right) \cdot [\not{q}, \not{P}] \right] &= 0 , \\ \text{Tr}(\not{P} \cdot [\not{q}, \not{P}]) &= 0 . \end{aligned} \quad (4.56)$$

It is not possible to find any additional linearly independent covariants, since any products of $[\not{q}, \not{P}]$ with $\not{q}$ or $\not{P}$ or itself can be reduced to already known covariants via the Clifford algebra, $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$. Hence, the basis in Dirac space for a scalar meson with $J^P = 0^+$ can be given by

$$T_1 = \mathbf{1} , \quad T_2 = i\not{P} , \quad T_3 = i\not{q} , \quad T_4 = [\not{q}, \not{P}] . \quad (4.57)$$

We note that a pseudoscalar-meson set of covariants can be constructed from here by multiplication of each covariant by γ_5 , since this object has negative parity, so the result is

$$T_1 = i\gamma_5 , \quad T_2 = \gamma_5 \not{P} , \quad T_3 = \gamma_5 \not{q} , \quad T_4 = i\gamma_5 [\not{q}, \not{P}] . \quad (4.58)$$

Now let us to consider the charge conjugation of each of the covariants for the scalar and pseudoscalar cases. We saw that $\mathbf{1}$ and $\not{q}$ have positive charge conjugation, but $\not{P}$ has negative charge conjugation. We can see that the covariant $[\not{q}, \not{P}]$ has positive charge conjugation, namely

$$\begin{aligned} (C[\not{q}, \not{P}]C^{-1})^T &= (C(\not{q}, \not{P} - \not{P}, \not{q})C^{-1})^T \\ &= (C\not{q}\not{P}C^{-1})^T - (C\not{P}\not{q}C^{-1})^T \\ &= (C\not{q}C^{-1}C\not{P}C^{-1})^T - (C\not{P}C^{-1}C\not{q}C^{-1})^T \\ &= -\not{P}\not{q} - (\not{q}\not{P}) = [\not{q}, \not{P}] . \end{aligned} \quad (4.59)$$

Then, we can write the BSA for a scalar meson with positive charge conjugation, $J^{PC} = 0^{++}$ as

$$\begin{aligned} \Gamma_s(P, q) &= [\mathbf{1}F_1(P^2; q^2; q \cdot P) + i\not{P}F_2(P^2; q^2; q \cdot P) \\ &\quad + i\not{q}F_3(P^2; q^2; q \cdot P) + [\not{q}, \not{P}]F_4(P^2; q^2; q \cdot P)] . \end{aligned} \quad (4.60)$$

Here we have charge conjugations of T_1, T_2, T_3 and T_4 being $+, -, +$ and $+$ and $+, -, +$ and $+$ for the Lorentz-invariant functions F_1, F_2, F_3 and F_4 . Now, for the pseudoscalar covariants one can similarly find that:

$$\begin{aligned} (C\gamma_5\not{P}C^{-1})^T &= (C\gamma_5C^{-1}C\not{P}C^{-1})^T = -\not{P}\gamma_5 = \gamma_5\not{P} , \\ (C\gamma_5\not{q}C^{-1})^T &= (C\gamma_5C^{-1}C\not{q}C^{-1})^T = \not{q}\gamma_5 = -\gamma_5\not{q} , \\ (C\gamma_5[\not{q}, \not{P}]C^{-1})^T &= \gamma_5[\not{q}, \not{P}] . \end{aligned} \quad (4.61)$$

Thus, in order to have a pseudoscalar meson with $J^{PC} = 0^{-+}$, one needs charge conjugations $+$, $+$, $-$ and $+$ as well for F_1 , F_2 , F_3 and F_4 . Then, the BSA for the pseudoscalar meson can be written as

$$\begin{aligned}\Gamma_{ps}(P, q) = & \gamma_5[iF_1(P^2; q^2; q \cdot P) + \not{P}F_2(P^2; q^2; q \cdot P) \\ & + \not{q}F_3(P^2; q^2; q \cdot P) + i[\not{q}, \not{P}]F_4(P^2; q^2; q \cdot P)] .\end{aligned}\quad (4.62)$$

4.7 Application for the pion

As previously mentioned, n -particle bound states are identified by the occurrence of poles in the n -point functions; so that a meson is identified as a pole in the quark-antiquark Green's function.

For the case of pseudoscalar particles, the homogeneous BSE is given by [72]

$$[\Gamma^j(k; P)]_{tu} = \int \frac{d^4q}{(2\pi)^4} [\chi^j(q; P)]_{sr} K_{tu}^{rs}(q, k; P) , \quad (4.63)$$

where

$$\chi^j(q; P) = S(q_+) \Gamma^j(q; P) S(q_-) , \quad (4.64)$$

with $q_+ = q + \eta_P P$ and $q_- = q - (1 - \eta_P)P$. In (4.63), k is the relative momentum and P the total momentum of the bound state, $r, \dots, u$ represent color, flavor and Dirac indices, S is the dressed-quark propagator, and K is the fully-amputated quark-antiquark scattering kernel.

In (4.63), $\Gamma(k; P)$ is the meson Bethe-Salpeter amplitude, normalized canonically according to [70,72]

$$\begin{aligned}2\delta^{ij}P_\mu = & N_c \frac{\partial}{\partial P_\mu} \left\{ \int \frac{d^4q}{(2\pi)^4} \text{Tr}[\bar{\Gamma}^i(q; -P) S(q_+) \Gamma^j(q; P) S(q_-)] \right. \\ & \left. + \int \frac{d^4q}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} \text{Tr}[\bar{\chi}^i(q; -P) K(q, k; P) \chi^j(q; P)] \right\} ,\end{aligned}\quad (4.65)$$

where $\bar{\Gamma}(q; P)^T = C^{-1} \Gamma(-q; P) C$ defines the corresponding anti-meson amplitude. The ladder approximation consists in using in (4.63) for the kernel K the one-gluon exchange, see Fig. 4.11 , namely

$$K_{tu}^{rs}(q, k; P) \longrightarrow -4\pi\alpha((k - q)^2) D_{\mu\nu}^{\text{free}}(k - q) \left(\gamma_\mu \frac{\lambda^a}{2} \right)_{tr} \left(\gamma_\nu \frac{\lambda^a}{2} \right)_{su} , \quad (4.66)$$

The BSE in the ladder approximation has the form depicted in Fig.4.10. where

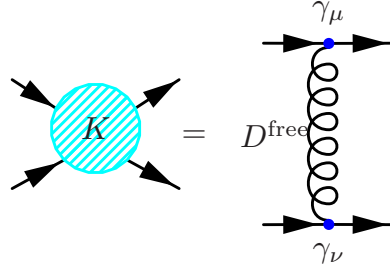


Figure 4.11: The ladder approximation for the scattering kernel [17].

$\alpha((k-p)^2)$ is an effective running coupling. The truncation (4.66) correspond to the first term in a systematic expansion of the kernel K . Now, the truncation for the quark DSE is

$$g^2 D_{\mu\nu}(k-q) \Gamma_\nu^i(k,p) \longrightarrow 4\pi\alpha((k-q)^2) D_{\mu\nu}^{\text{free}}(k-q) \gamma_\mu \frac{\lambda^i}{2}. \quad (4.67)$$

The simultaneous use of (4.66) together with (4.67) is called rainbow-ladder truncation. This truncation respect Lorentz invariance, chiral symmetry and renormalization group invariance, which are relevant symmetry of QCD.

In ladder truncation, the kernel K of the BSE is independent of P , therefore $\partial K / \partial P = 0$ and the normalization condition (4.65) becomes

$$2\delta^{ij} P_\mu = N_c \frac{\partial}{\partial P_\mu} \int \frac{d^4 q}{(2\pi)^4} \text{Tr}[\bar{\Gamma}^i(q; -P) S(q_+) \Gamma^j(q; P) S(q_-)] . \quad (4.68)$$

4.7.1 Leptonic decay constant

The pseudoscalar meson decay constant, f_P , is described by the matrix element [73]

$$\langle 0 | \bar{\Psi}(0) (T^P)^t \gamma_\mu \gamma_5 \Psi(0) | \Phi(P) \rangle = f_P P_\mu , \quad (4.69)$$

where $|\Phi(P)\rangle$ is the pseudoscalar meson bound state vector, T^P are the matrices acting in flavor space and Ψ is the Dirac field operator.

The matrix element can be written in terms of the BSA. We can write the Bethe-Salpeter wave function (4.64) as

$$(2\pi)^4 \delta^4(q-k) \chi(q, P) = \int d^4 x d^4 y e^{-iP \cdot (\eta_P x + (1-\eta_P) y)} e^{-i(k \cdot x - q \cdot y)} \langle 0 | \Psi(x) \bar{\Psi}(y) | \Phi(P) \rangle . \quad (4.70)$$

If we multiply both sides of this equation by $(T^P)^t \gamma_5 \gamma \cdot P$, evaluating the integrals over q and k and using (4.69) one finds [74]:

$$P^2 f_P = N_c \int \frac{d^4 q}{(2\pi)^4} \text{Tr}[(T^P)^t \gamma_5 \gamma \cdot P S(q_+) \Gamma_P^C(q, P) S(q_-)] , \quad (4.71)$$

which gives the relation between the BSA and the leptonic decay constant of a pseudoscalar meson. Here $\Gamma^C(q; P)$ is normalized canonically according to (4.68).

4.7.2 The pion BSE amplitude

We are interested in the pion, since it is a two body bound state problem and its understanding plays a key role in strong interaction physics. To study the pion as a bound state, it is necessary to understand its Bethe-Salpeter amplitude. As we saw previously, for the case of pseudoscalar meson, it has the general structure [70]:

$$\begin{aligned} \Gamma_\pi^j(k; P) &= \tau^j \gamma_5 [iE_\pi(k; P) + \not{P} F_\pi(k; P) \\ &+ \not{k} G_\pi(k; P) + \sigma_{\mu\nu} k_\mu P_\nu H_\pi(k; P)] , \end{aligned} \quad (4.72)$$

and satisfies the homogeneous BSE (4.63). In (4.72), τ^j are the Pauli matrices.

If we consider the rainbow-ladder truncation and using the contact interaction (3.65), we can rewrite the homogeneous BSE (4.63) as [61,75]

$$\Gamma_\pi(P) = -\frac{16\pi}{3} \frac{\alpha_{IR}}{m_G^2} \int \frac{d^4 q}{(2\pi)^4} \gamma_\mu S(q+P) \Gamma_\pi(P) S(q) \gamma_\mu , \quad (4.73)$$

where the BSA in this case (contact interaction) has not depend on relative momentum k . Thus, Eq. (4.72) can be rewritten as

$$\Gamma_\pi(P) = \gamma_5 \left[iE_\pi(P) + \frac{1}{M} \gamma \cdot P F_\pi(P) \right] . \quad (4.74)$$

With this, (4.73) can be written as

$$E_\pi = -\frac{4\pi}{3} \frac{\alpha_{IR}}{m_G^2} \int \frac{d^4 q}{(2\pi)^4} \frac{\text{Tr}[\gamma_5 \gamma_\mu (-i\not{q} + M) \gamma_5 E_\pi(-i\not{q} + M) \gamma_\mu]}{(q^2 + M^2)^2} , \quad (4.75)$$

where we have used (3.66) with (3.71) and $P = 0$. After evaluating the traces, one finds

$$E_\pi = \frac{4\pi}{3} \frac{16\alpha_{IR}}{m_G^2} \int \frac{d^4 q}{(2\pi)^4} \frac{E_\pi}{q^2 + M^2} . \quad (4.76)$$

We can see that this is equivalent to the gap equation (3.75) (in the chiral limit). Therefore there is a massless solution, with $E_\pi = M$, where M is the constituent quark mass, and the pion can be identified as the Goldstone boson.

On the other hand, an important fact that must be checked is the preserving of the vector and axial-vector Ward-Takahashi identities; as they play a crucial role when computing properties of the pion, since chiral symmetry and its explicit

dynamical breaking are expressed in the axial-vector Ward-Takahashi identities (AV-WTI). In the chiral limit, the AV-WTI, see Appendix D, is

$$-iP_\mu \Gamma_{5\mu}(k_+, k) = S^{-1}(k_+) \gamma_5 + \gamma_5 S^{-1}(k) , \quad (4.77)$$

where $\Gamma_{5\mu}(k_+, k)$ is the axial-vector vertex, given by

$$\Gamma_{5\mu}(k_+, k) = \gamma_5 \gamma_\mu - \frac{16\pi}{3} \frac{\alpha_{IR}}{m_G^2} \int \frac{d^4 q}{(2\pi)^4} \gamma_\alpha \chi_{5\mu}(q_+, q) \gamma_\alpha , \quad (4.78)$$

where $k_+ = k + P$ and

$$\chi_{5\mu}(q_+, q) = S(q_+) \Gamma_{5\mu}(q_+, q) S(q) . \quad (4.79)$$

Hence, since we have freedom with the regularization of divergences, one can choose one that maintains (4.77). We contract (4.78) with $-iP_\mu$ and use (4.77) to obtain

$$\begin{aligned} S^{-1}(k_+) \gamma_5 + \gamma_5 S^{-1}(k) &= -i\gamma_5 \not{P} \\ &- \frac{16\pi}{3} \frac{\alpha_{IR}}{m_G^2} \int \frac{d^4 q}{(2\pi)^4} (\gamma_\alpha \gamma_5 S(q) \gamma_\alpha + \gamma_\alpha S(q_+) \gamma_5 \gamma_\alpha) , \end{aligned} \quad (4.80)$$

which can be written as

$$\begin{aligned} 2\gamma_5 M + i\gamma_5 \not{P} &= -i\gamma_5 \not{P} \\ &+ \frac{16\pi}{3} \frac{\alpha_{IR}}{m_G^2} \int \frac{d^4 q}{(2\pi)^4} \gamma_5 \left[\gamma_\alpha \frac{(-i\not{q} + M)}{q^2 + M^2} \gamma_\alpha + \gamma_\alpha \frac{(i\not{q}_+ + M)}{q_+^2 + M^2} \gamma_\alpha \right] . \end{aligned} \quad (4.81)$$

Taking the trace over gamma matrices, one finds

$$M = \frac{8\pi}{3} \frac{4\alpha_{IR} M}{m_G^2} \int \frac{d^4 q}{(2\pi)^4} \left[\frac{1}{q^2 + M^2} + \frac{1}{q_+^2 + M^2} \right] , \quad (4.82)$$

$$0 = \int \frac{d^4 q}{(2\pi)^4} \left[\frac{P \cdot q_+}{q_+^2 + M^2} - \frac{P \cdot q}{q^2 + M^2} \right] . \quad (4.83)$$

One can show easily that for $P^2 = 0$, the Eq. (4.82) becomes the chiral gap equation (3.75). Also, Eq. (4.83) can be rewritten as

$$0 = \int \frac{d^4 q}{(2\pi)^4} \frac{q^2(1 - 2\cos^2 \theta) + M^2}{(q^2 + M^2)^2} , \quad (4.84)$$

which after doing the angular integration, one finds

$$0 = \int \frac{d^4 q}{(2\pi)^4} \frac{\frac{1}{2}q^2 + M^2}{(q^2 + M^2)^2} . \quad (4.85)$$

This equation states that the axial-vector Ward-Takahashi identity is satisfied if, and only if, the model is regularized so as to ensure there are no quadratic or logarithmic divergences.

Now, using (4.74) and again taking trace over gamma matrices, one can write (4.73) in the form

$$\begin{bmatrix} E_\pi(P) \\ F_\pi(P) \end{bmatrix} = \frac{4\pi\alpha_{\text{IR}}}{3m_G^2} \begin{bmatrix} \mathcal{I}_\pi^{EE} & \mathcal{I}_\pi^{EF} \\ \mathcal{I}_\pi^{FE} & \mathcal{I}_\pi^{FF} \end{bmatrix} \begin{bmatrix} E_\pi(P) \\ F_\pi(P) \end{bmatrix}, \quad (4.86)$$

where

$$\mathcal{I}_\pi^{EE} = - \int \frac{d^4q}{(2\pi)^4} \text{Tr}[\gamma_5 \gamma_\mu S(q+P) \gamma_5 S(q) \gamma_\mu], \quad (4.87)$$

$$\mathcal{I}_\pi^{EF} = \frac{i}{M} \int \frac{d^4q}{(2\pi)^4} \text{Tr}[\gamma_5 \gamma_\mu S(q+P) \gamma_5 \not{P} S(q) \gamma_\mu], \quad (4.88)$$

$$\mathcal{I}_\pi^{FE} = \frac{iM}{P^2} \int \frac{d^4q}{(2\pi)^4} \text{Tr}[\gamma_5 \not{P} \gamma_\mu S(q+P) \gamma_5 S(q) \gamma_\mu], \quad (4.89)$$

$$\mathcal{I}_\pi^{FF} = \frac{1}{P^2} \int \frac{d^4q}{(2\pi)^4} \text{Tr}[\gamma_5 \not{P} \gamma_\mu S(q+P) \gamma_5 \not{P} S(q) \gamma_\mu]. \quad (4.90)$$

Using the fact that for any function $f(p^2)$

$$f((p+k)^2) = f(p^2) + 2k \cdot p \frac{df(p^2)}{dp^2} + O(k^2), \quad (4.91)$$

and after evaluating the traces in (4.87) one finds

$$\mathcal{I}_\pi^{EE} = 16 \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 + M^2} - 16 \int \frac{d^4q}{(2\pi)^4} \frac{P \cdot q}{(q^2 + M^2)^2}. \quad (4.92)$$

The last integral does not contribute, since the angular integration is zero. Thus, we obtain

$$\mathcal{I}_\pi^{EE} = 16 \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 + M^2} = \frac{1}{\pi^2} \mathcal{K}_\pi^{EE}, \quad (4.93)$$

where have defined

$$\mathcal{K}_\pi^{EE} = \mathcal{C}(M^2; \tau_{IR}^2, \tau_{UV}^2), \quad (4.94)$$

where $\mathcal{C}(M^2; \tau_{IR}^2, \tau_{UV}^2)$ is given by (3.84). Now (4.88), becomes

$$\begin{aligned} \mathcal{I}_\pi^{EF} &= 16P^2 \int \frac{d^4q}{(2\pi)^4} \frac{1}{(q^2 + M^2)^2} \\ &= \frac{P^2}{\pi^2 M^2} \mathcal{C}_1(M^2; \tau_{IR}^2, \tau_{UV}^2) \\ &= \frac{1}{\pi^2} \mathcal{K}_\pi^{EF}, \end{aligned} \quad (4.95)$$

where we have defined

$$\begin{aligned}\mathcal{K}_\pi^{EF} &= \frac{P^2}{M^2} \mathcal{C}_1(M^2; \tau_{IR}^2, \tau_{UV}^2) \\ &= \frac{P^2}{M^2} \left[-M^2 \frac{d}{dM^2} \mathcal{C}(M^2; \tau_{IR}^2, \tau_{UV}^2) \right].\end{aligned}\quad (4.96)$$

So that with $P^2 = 0$, one has

$$\mathcal{K}_\pi^{EF} = 0. \quad (4.97)$$

Then, (4.89) gives

$$\begin{aligned}\mathcal{I}_\pi^{FE} &= 8M^2 \int \frac{d^4q}{(2\pi)^4} \frac{1}{(q^2 + M^2)^2} \\ &= \frac{1}{2\pi^2} \mathcal{C}_1(M^2; \tau_{IR}^2, \tau_{UV}^2) = \frac{1}{\pi^2} \mathcal{K}_\pi^{FE},\end{aligned}\quad (4.98)$$

where

$$\mathcal{K}_\pi^{FE} = \frac{1}{2} \mathcal{C}_1(M^2; \tau_{IR}, \tau_{UV}). \quad (4.99)$$

Finally, (4.90) gives

$$\begin{aligned}\mathcal{I}_\pi^{FF} &= \frac{8}{P^2} \int \frac{d^4q}{(2\pi)^4} \frac{P^2 q^2 - 2(P \cdot q)^2 - M^2 P^2}{(q^2 + M^2)^2} \\ &= 8 \int \frac{d^4q}{(2\pi)^4} \frac{q^2(1 - 2\cos^2\theta) - M^2 P^2}{(q^2 + M^2)^2} \\ &= 8 \int \frac{d^4q}{(2\pi)^4} \frac{\frac{1}{2}q^2 + M^2}{(q^2 + M^2)^2} - 16M^2 \int \frac{d^4q}{(2\pi)^4} \frac{1}{(q^2 + M^2)^2}.\end{aligned}\quad (4.100)$$

Using the chiral identity (4.85), we have

$$\begin{aligned}\mathcal{I}_\pi^{FF} &= -16M^2 \int \frac{d^4q}{(2\pi)^4} \frac{1}{(q^2 + M^2)^2} \\ &= -\frac{1}{\pi^2} \mathcal{C}_1(M^2; \tau_{IR}^2, \tau_{UV}^2) \\ &= \frac{1}{\pi^2} \mathcal{K}_\pi^{FF},\end{aligned}\quad (4.101)$$

where

$$\mathcal{K}_\pi^{FF} = -2\mathcal{K}_\pi^{EF} = -\mathcal{C}_1(M^2; \tau_{IR}^2, \tau_{UV}^2). \quad (4.102)$$

Now, we can rewrite (4.86) as

$$\begin{bmatrix} E_\pi(P) \\ F_\pi(P) \end{bmatrix} = \frac{4\alpha_{\text{IR}}}{3\pi m_G^2} \begin{bmatrix} \mathcal{K}_\pi^{EE} & \mathcal{K}_\pi^{EF} \\ \mathcal{K}_\pi^{FE} & \mathcal{K}_\pi^{FF} \end{bmatrix} \begin{bmatrix} E_\pi(P) \\ F_\pi(P) \end{bmatrix}, \quad (4.103)$$

whose solution gives the pion's chiral-limit Bethe-Salpeter amplitude.

In addition, from Eq. (4.71) we can write the pion decay constant in terms of E_π and F_π as

$$P_\mu f_\pi = N_c \int \frac{d^4 q}{(2\pi)^4} \text{Tr}[S(q+P)\Gamma_\pi^C(q;P)S(q)\gamma_5 P_\mu] , \quad (4.104)$$

where

$$\begin{aligned} \Gamma_\pi^C(P) &= \gamma_5 \left[iE_\pi^C(P) + \frac{1}{M} \gamma \cdot P F_\pi^C(P) \right] \\ &= \frac{1}{\mathcal{N}} \Gamma_\pi(P) . \end{aligned} \quad (4.105)$$

Here, $\mathcal{N}$ is the canonical normalization constant for the pion Bethe-Salpeter amplitude. Using this, and evaluating the traces over gamma matrices, one easily finds

$$f_\pi = \frac{4N_c}{M} \left[E_\pi^C M^2 \int \frac{d^4 q}{(2\pi)^4} \frac{1}{(q^2 + M^2)^2} + F_\pi^C \int \frac{d^4 q}{(2\pi)^4} \frac{\frac{1}{2}q^2 - M^2}{(q^2 + M^2)^2} \right] , \quad (4.106)$$

which can be written as

$$f_\pi^0 = \frac{N_c}{4\pi^2 M} [E_\pi^C - 2F_\pi^C] \mathcal{C}_1(M^2; \tau_{IR}^2, \tau_{UV}^2) , \quad (4.107)$$

where again we have used (4.85).

On the other hand, the normalization condition for the pion

$$P_\mu = N_c \int \frac{d^4 q}{(2\pi)^4} \text{Tr} \left[\Gamma_\pi^C(-P) \frac{\partial S(q+P)}{\partial P_\mu} \Gamma_\pi^C(P) S(q) \right] , \quad (4.108)$$

which, in the chiral limit, becomes:

$$1 = \frac{N_c}{4\pi^2 M^2} E_\pi^C [E_\pi^C - 2F_\pi^C] \mathcal{C}_1(M^2; \tau_{IR}^2, \tau_{UV}^2) . \quad (4.109)$$

Now, taking the ratio between (4.107) and (4.109), we have

$$E_\pi^C = \frac{E_\pi}{\mathcal{N}_0} = \frac{M}{f_\pi^0} . \quad (4.110)$$

So that in the chiral limit, the canonical normalization constant for the pion Bethe-Salpeter amplitude is equivalent to its leptonic decay constant ($\mathcal{N}_0 = f_\pi^0$).

Once the AV-WTI (4.77) is preserved, then, near the mass shell of the chiral limit pion, the solution of (4.78) has the form [74]

$$\Gamma_{5\mu}(k_+, k) = \gamma_5 \gamma_\mu F_R(P) + \frac{P_\mu}{P^2} 2f_\pi^0 \Gamma_\pi^C(P) . \quad (4.111)$$

Contracting with $-iP_\mu$, one readily finds

$$f_\pi^0 E_\pi^C = M , \quad 2 \frac{F_\pi^C}{E_\pi^C} + F_R = 1 , \quad (4.112)$$

which are the known Goldberger-Treiman relations [72].

4.7.3 Numerical results

From (4.103), one has

$$\begin{aligned} F_\pi &= \frac{2\alpha_{IR}}{3\pi m_G^2} \mathcal{C}_1(M^2; \tau_{IR}^2, \tau_{UV}^2) [E_\pi - 2F_\pi] , \\ &= \frac{2\alpha_{IR}}{3\pi m_G^2} \mathcal{C}_1 \frac{E_\pi}{\left(1 + \frac{4\alpha_{IR}\mathcal{C}_1}{3\pi m_G^2}\right)} . \end{aligned} \quad (4.113)$$

With this, we can write (4.107) as

$$(f_\pi^0)^2 = \frac{N_c}{4\pi^2} \frac{\mathcal{C}_1}{\left(1 + \frac{4\alpha_{IR}\mathcal{C}_1}{3\pi m_G^2}\right)} , \quad (4.114)$$

where we have used unit normalization and that in the chiral limit $E_\pi = M/f_\pi^0$.

Another important physics quantity is the vacuum quark condensate κ_π [76], which in the chiral limit, $\kappa_\pi \longrightarrow \kappa_\pi^0 = -\langle \bar{q}q \rangle^{1/3}$ and which can be calculated by (2.83). Thus, one finds easily in Euclidean space that

$$\begin{aligned} \kappa_\pi^0 &= \frac{N_c}{4\pi^2} M \int d^2p^2 \frac{p^2}{p^2 + M^2} \\ &= \frac{N_c}{4\pi^2} M \mathcal{C}(M^2; \tau_{IR}^2, \tau_{UV}^2). \end{aligned} \quad (4.115)$$

m	M	$\kappa_\pi^0 = -\langle \bar{q}q \rangle^{1/3}$	f_π^0	$f_\pi^0 _{F_\pi \rightarrow 0}$
0	0.358	0.241	0.100	0.118

Table 4.1: Results in GeV, obtained with [77] $m_G = 0.8$ GeV, $\tau_{IR}^2 = (0.24 \text{ GeV})^{-2}$, and $\tau_{UV}^2 = (0.905 \text{ GeV})^{-2}$.

In the Table.4.1 are shown the numerical values of M , κ_π^0 and f_π^0 [61]. This values were calculated using (3.83), (4.115) and (4.107). We can see that when we consider a nonzero $F_\pi(P)$, it has an important impact on the value of f_π , since its value improve.

4.8 Summary

We have explored the Bethe-Salpeter equation, which is a very important tool for the study of the relativistic two-body problem. We have derived its inhomogeneous

and homogeneous forms, where this last one was obtained considering the existence of a bound state by the occurrence of a pole in the four-point Green's function. We saw that the homogeneous Bethe-Salpeter equation is an eigenvalue problem, where the eigenvector associated with each eigenvalue is the Bethe-Salpeter amplitude, which is normalized according to Eq. (4.46). In addition, we saw the form of the BSE for the case of the ladder truncation. Also we have learned how to build the BSA for the case of scalar and pseudoscalar mesons. In order to illustrate the use of BSE, we considered the particular case of pseudoscalar particles, in special the pion. We show that the pseudoscalar decay constant can be written in terms of the pseudoscalar, E_π , and pseudo vector amplitudes, F_π . Finally, we studied the homogeneous BSE for the case of a contact interaction. We show that the BSE becomes the gap equation in the chiral limit. We found that the canonical normalization constant of the pion BSA is equivalent to its leptonic decay constant when both are evaluated in the chiral limit and a symmetry preserved regularization is employed. Furthermore, we saw the important measurable impact of F_π on the value of f_π .

Chapter 5

DRESSED PERTURBATION THEORY

As mentioned in the introduction of this thesis, the rainbow-ladder approximation is the simplest truncation that satisfies the requirement of to preserve chiral symmetry and other symmetries of the theory. The chiral symmetry, is expressed via the axial-vector Ward-Takahashi identity, which ensures that pions may be identified as the pseudo-Goldstone bosons of the theory. The rainbow-ladder truncation has been successful, since for example it reproduced the masses of the of light-mesons, vectors, leptonic decay constant, electromagnetic form factors and pion charge radius [61,77]. Unfortunately, the rainbow-ladder describes only pure $q\bar{q}$ states, namely no flavor mixing, no exotics, no decay channels and no variety in attraction/repulsion. In this chapter we will try to go beyond this approximation including Yang-Mills corrections through dressed perturbation theory scheme.

5.1 Perturbation theory with dressed propagators

The approach we implement is inspired in the method employed by Nambu and Jona-Lasinio in their original seminal papers [30]. It also has similarities with the linear δ expansion (LDE) [20] and screened perturbation theory (SPT) [22]. The LDE, in particular, has been used very recently in a study of the thermodynamics of the $\lambda\phi^4$ theory [25] and in the past it has been applied in the of relativistic many-body theories [28,29]. The basic idea is to modify the action of the theory by addition and subtraction of self-energy terms, as done in the Lagrangian of the NJL model, Eq. (2.46).

In order to implement this in QCD, let us to consider the QCD generating functional (at zero external sources) in Euclidean space (to minimise potential confusion in the notation between actions and quark propagators we will denote actions by

calligraphic letters):

$$Z = \int \mathcal{D}(\bar{\psi}, \psi, A) e^{-\mathcal{S}} = \int \mathcal{D}(\bar{\psi}, \psi, A) e^{-\mathcal{S}_0 - \mathcal{S}_I} , \quad (5.1)$$

where the quadratic action $\mathcal{S}_0$ is given by

$$\mathcal{S}_0 = \int d^4x d^4y \left[\bar{\psi}(x) S_0^{-1}(x-y) \psi(y) + \frac{1}{2} A_\mu^a(x) (D_0^{-1})_{\mu\nu}^{ab}(x-y) A_\nu^b(y) \right] , \quad (5.2)$$

with

$$S_0^{-1}(x-y) = (\gamma^\mu \partial_\mu + m) \delta^4(x-y) , \quad (5.3)$$

$$(D_0^{-1})_{\mu\nu}^{ab}(x-y) = \delta^{ab} \left[-\delta_{\mu\nu} \partial^2 + \left(1 - \frac{1}{\xi}\right) \partial_\mu \partial_\nu \right] \delta^4(x-y) , \quad (5.4)$$

and the interaction action $\mathcal{S}_I$ is

$$\begin{aligned} \mathcal{S}_I = & \int d^4x \left[ig \bar{\psi}(x) t^a \gamma \cdot A^a \psi(x) - g f^{abc} \partial_\mu A_\nu^a(x) A^{b\mu}(x) A^{c\nu}(x) \right. \\ & \left. + \frac{1}{4} g^2 f^{abc} f^{ade} A_\mu^b(x) A_\nu^c A^{d\mu}(x) A^{e\nu}(x) \right] . \end{aligned} \quad (5.5)$$

In ordinary perturbation theory, correlation functions can be calculated via

$$\langle O(\bar{\psi}, \psi, A) \rangle = \frac{\int \mathcal{D}(\bar{\psi}, \psi, A) e^{-\mathcal{S}_0} O(\bar{\psi}, \psi, A) [1 - \mathcal{S}_I + \frac{1}{2}(\mathcal{S}_I)^2 + \dots]}{\int \mathcal{D}(\bar{\psi}, \psi, A) e^{-\mathcal{S}_0} [1 - \mathcal{S}_I + \frac{1}{2}(\mathcal{S}_I)^2 + \dots]} , \quad (5.6)$$

Now we add to $\mathcal{S}_0$ and simultaneously subtract from $\mathcal{S}_I$ quadratic terms in quark and gluon fields involving quark and gluon self-energies $\int d^4x d^4y \bar{\psi}(x) \Sigma_0(x-y) \psi(y)$ and $\int d^4x d^4y A_\mu(x) \Pi_{0\mu\nu}(x-y) A_\nu(y)$ such that one obtains a new quadratic action $\bar{\mathcal{S}}_0$ and a modified interaction action $\bar{\mathcal{S}}_I$:

$$\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_I \rightarrow \mathcal{S} = \bar{\mathcal{S}}_0 + \bar{\mathcal{S}}_I , \quad (5.7)$$

with

$$\begin{aligned} \bar{\mathcal{S}}_0 = & \int d^4x d^4y \bar{\psi}(x) [S_0^{-1}(x-y) + \Sigma_0(x-y)] \psi(y) \\ & + \frac{1}{2} \int d^4x d^4y A_\mu^a(x) [(D_0^{-1})_{\mu\nu}^{ab}(x-y) + \Pi_{0\mu\nu}^{ab}(x-y)] A_\nu^b(y) , \end{aligned} \quad (5.8)$$

$$\begin{aligned} \bar{\mathcal{S}}_I = & \mathcal{S}_I \\ & - \int d^4x d^4y \bar{\psi}(x) \Sigma_0(x-y) \psi(y) - \frac{1}{2} \int d^4x d^4y A_\mu^a(x) \Pi_{0\mu\nu}^{ab}(x-y) A_\nu^b(y) . \end{aligned} \quad (5.9)$$

As in the case of the LDE or SPT, the idea now is to perform perturbation theory in $\bar{\mathcal{S}}_I$, with the self-energies Σ_0 and Π_0 specified through some criterium. A judiciously chosen criterion, which takes into account the envisaged applications, is crucial for the success of the program. As previous experience has demonstrated, a great deal of the low-energy QCD phenomenology is well described by the rainbow-ladder approximation using a model gluon propagator. Therefore, it seems that if one chooses the zeroth order to such rainbow-ladder scheme for propagators and bound-state amplitudes, higher order terms in the perturbative expansion might be able to capture effects left out at zeroth order. Of course, such an expectation can only be verified with detailed calculations. But, once the zeroth order is fixed, corrections can be calculated in a systematic and controlled way.

In order to organize the new perturbation theory, we introduce a book keeping parameter δ , which we put $\delta = 1$ at the end of the calculation. Specifically, we write

$$\begin{aligned} \bar{\mathcal{S}}_I(\delta) = & \delta \int d^4x \left[ig \bar{\psi}(x) t^a \gamma \cdot A^a \psi(x) - g f^{abc} \partial_\mu A_\nu^a(x) A_\mu^b(x) A_\nu^c(x) \right. \\ & \left. + \delta g^2 \frac{1}{4} f^{abc} f^{ade} A_\mu^b(x) A_\nu^c A_\mu^d(x) A_\nu^e(x) \right] \\ & - \delta^2 \int d^4x d^4y \left[\bar{\psi}(x) \Sigma_0(x-y) \psi(y) - \frac{1}{2} A_\mu^a(x) \Pi_{0\mu\nu}^{ab}(x-y) A_\nu^b(y) \right], \quad (5.10) \end{aligned}$$

and calculate correlation functions or bound-state amplitudes as a power series in δ . For example, the quark propagator (two-point correlation function), which is our main concern in the present dissertation, is calculated as

$$S_{\alpha\beta}^{cc'}(x-y) = \frac{\int \mathcal{D}(\bar{\psi}, \psi, A) e^{-\bar{\mathcal{S}}_0} \psi_\alpha^c(x) \bar{\psi}_\beta^{c'}(y) \{1 - \bar{\mathcal{S}}_I(\delta) + \frac{1}{2} [\bar{\mathcal{S}}_I(\delta)]^2 + \dots\}}{\int \mathcal{D}(\bar{\psi}, \psi, A) e^{-\bar{\mathcal{S}}_0} \{1 - \bar{\mathcal{S}}_I(\delta) + \frac{1}{2} [\bar{\mathcal{S}}_I(\delta)]^2 + \dots\}}. \quad (5.11)$$

As in ordinary perturbation theory, terms in the denominator cancel disconnected diagrams arising from the numerator. Now, it is very important to note that the different terms in $\mathcal{S}_I(\delta)$ are not all of the same order in δ . This means that in the expansion of the exponential $e^{-\mathcal{S}_I(\delta)} = \sum_n [\mathcal{S}_I(\delta)]^n / n!$, a power $[\mathcal{S}_I(\delta)]^n$ has terms of the same order in δ than, say $[\mathcal{S}_I(\delta)]^{n\pm 2}$. The first term in the expansion (5.11), of zeroth-order in δ , gives

$$\begin{aligned} S_{\alpha\beta}^{cc(0)}(x-y) &= \langle \psi_\alpha^c(x) \bar{\psi}_\beta^{c'}(y) \rangle_0 = \delta^{cc'} \overline{\psi_\alpha^c(x) \bar{\psi}_\beta^{c'}(y)} \\ &= \delta^{cc'} \int \frac{d^4p}{(2\pi)^4} e^{ip \cdot (x-y)} \left[\frac{1}{i\gamma \cdot p + m + \Sigma_0(p)} \right]_{\alpha\beta}, \quad (5.12) \end{aligned}$$

where $\Sigma_0(p)$ is the Fourier transform of $\Sigma_0(x - x')$. From now on, we represent the zeroth-order propagator by $\bar{S}^{(0)}$, so that

$$[\bar{S}^{(0)}(p)]^{-1} = i\gamma \cdot p + m + \Sigma_0(p) . \quad (5.13)$$

In the same way, the zeroth-order gluon propagator is

$$\bar{D}_{\mu\nu}^{(0)ab}(p) = \delta^{ab} \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \frac{1}{p^2[1 + \Pi_0(p^2)]} + \xi \frac{p_\mu p_\nu}{p^4} , \quad (5.14)$$

with $\Pi_0(p^2)$ defined via

$$\Pi_{0\mu\nu}^{ab}(p^2) = \delta^{ab}(\delta_{\mu\nu}p^2 - p_\mu p_\nu)\Pi_0(p^2) , \quad (5.15)$$

where $\Pi_{0\mu\nu}^{ab}(p^2)$ is the Fourier transform of $\Pi_{0\mu\nu}^{ab}(x - x')$.

By construction, the zeroth-order propagators depend on the added self-energies. As we shall show next, one way one can start at zeroth order with the rainbow-ladder approximation is to follow the method of Nambu and Jona-Lasinio, in that Σ_0 is determined by requiring that the lowest nontrivial order in δ gives zero for the correction of quark self-energy, for a given zeroth order gluon propagator.

Beyond the zeroth-order, the first non trivial contribution to the quark self-energy is $\mathcal{O}(\delta^2)$, as the functional integrals in Eq. (5.11) of the $\mathcal{O}(\delta)$ term gives zero. Explicitly, the first nonzero correction

$$\Sigma_2(p) = \frac{4}{3}g^2 \int \frac{d^4q}{(2\pi)^4} \bar{D}_{\mu\nu}^{(0)}(p - q) \gamma_\mu \bar{S}^{(0)}(q) \gamma_\nu - \Sigma_0(p) , \quad (5.16)$$

where $\bar{S}^{(0)}(p)$ is given in Eq. (5.13). Fig. 5.1 is a pictorial representation of this correction.

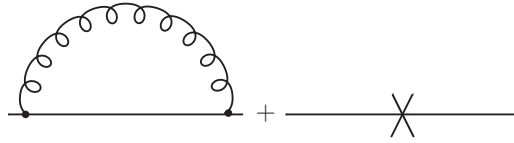


Figure 5.1: Diagrammatic representation of $\Sigma_2(p)$. The solid and curly lines represent dressed propagators. The X represents Σ_0 .

In order to start with a self-consistent zeroth order contribution, one requires that this first nontrivial correction be zero. This requirement leads to the familiar rainbow expression for the quark self-energy:

$$\Sigma_0(p) = \frac{4}{3}g^2 \int \frac{d^4q}{(2\pi)^4} \bar{D}_{\mu\nu}^{(0)}(p - q) \gamma_\mu \bar{S}^{(0)}(q) \gamma_\nu , \quad (5.17)$$

for a given zeroth order gluon propagator $\bar{D}^{(0)}$.

In a similar fashion, the first nontrivial correction to the gluon self-energy is $\mathcal{O}(\delta^2)$. Suppressing color indices, it is given by - see Fig. 5.2 for a pictorial representation:

$$\begin{aligned}
[\Pi_{\mu\nu}(p)]_2 &= -g^2 \int \frac{d^4 q}{(2\pi)^4} \text{Tr}_D [\gamma_\mu \bar{S}^{(0)}(p+q) \gamma_\nu \bar{S}(q)] \\
&+ g^2 \frac{3}{2} \int \frac{d^4 q}{(2\pi)^4} N_{\mu\rho\sigma}(p, -(p+q), q) \bar{D}_{\sigma\sigma'}^{(0)}(q) \bar{D}_{\rho\rho'}^{(0)}(p+q) N_{\sigma'\rho'\nu}(-q, p+q, -p) \\
&+ \text{CONST}_{tad} - \Pi_{0\mu\nu}(p) ,
\end{aligned} \tag{5.18}$$

where Tr_D means trace over Dirac indices, CONST_{tad} is the tadpole contribution, which is a constant (third graph in Fig. 5.2), and

$$N_{\mu\nu\rho}(k, p, q) = -i(k-p)_\rho \delta_{\mu\nu} - i(p-q)_\mu \delta_{\nu\rho} - i(q-k)_\nu \delta_{\mu\rho}. \tag{5.19}$$

Again, to start with a self-consistent zeroth order, one requires that $[\Pi_{\mu\nu}(p)]_2$ vanishes and this conditions determines $\Pi_{0\mu\nu}$ via:

$$\begin{aligned}
\Pi_{0\mu\nu}(p) &= -g^2 \int \frac{d^4 q}{(2\pi)^4} \text{Tr}_D [\gamma_\mu \bar{S}^{(0)}(p+q) \gamma_\nu \bar{S}(q)] \\
&+ g^2 \frac{3}{2} \int \frac{d^4 q}{(2\pi)^4} N_{\mu\rho\sigma}(p, -(p+q), q) \bar{D}_{\sigma\sigma'}^{(0)}(q) \bar{D}_{\rho\rho'}^{(0)}(p+q) N_{\sigma'\rho'\nu}(-q, p+q, -p) \\
&+ \text{CONST}_{tad} .
\end{aligned} \tag{5.20}$$

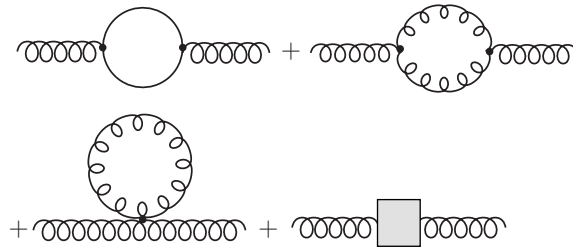


Figure 5.2: Diagrammatic representation of $[\Pi_{\mu\nu}(p)]_2$. The solid and curly lines represent dressed propagators. The box represents $\Pi_{0\mu\nu}$.

Note that we are not considering the contributions from ghost fields, which are an essential ingredient in covariant gauges. They do not enter in the DSE for the quark propagator, our main concern in this dissertation, and for this reason we will

not discuss their effects here. There is no problem to include them; actually one can include a self-energy term in the quadratic action and subtract this same term from the interaction term and perform perturbative calculations in the same manner as we are proposing for the quark and gluon fields.

Next, let us consider the contributions of higher order to the quark self-energy. First, it is not difficult to see that all odd-power contributions $\mathcal{O}(\delta^{2n+1})$, $n = 0, 1, 2, \dots$, vanish trivially. For $n = 0$, this fact has been already used above. The reason for the vanishing of all odd powers of δ is that these terms come with an odd number of gluon fields A_μ^a in the path integral in (5.11) and, since the action is quadratic in these fields (and in the absence of spontaneous symmetry breaking), the path integral vanishes trivially. Therefore, the expansion of the quark self-energy in powers of δ is of the form:

$$\Sigma = \sum_{n=0} \delta^{2n} \Sigma_{2n} = \Sigma_0 + \delta^2 \Sigma_2 + \delta^4 \Sigma_4 + \dots \quad (5.21)$$

The $\mathcal{O}(\delta^2)$ is zero, $\Sigma_2 = 0$, because of the self-consistency condition for the zeroth-order quark self-energy (5.17).

Let us then examine the first nonzero contribution to the self-energy; it is of $\mathcal{O}(\delta^4)$. Fig. 5.3 shows all graphs contributing to Σ_4 - only the 1PI terms are shown.

Schematically, the contributions to Σ_4 can be grouped into a form to make explicit cancellations due to the self-consistency requirements $\Sigma_2 = 0$ and $[\Pi_{\mu\nu}]_2 = 0$ - see Fig. 5.4:

$$\begin{aligned} \Sigma_4 = & \lambda_1 g^2 \bar{S}^{(0)} \bar{D}^{(0)} \Pi_2 \bar{D}^{(0)} + g^2 \lambda_2 \bar{S}^{(0)} \bar{D}^{(0)} [\bar{S}^{(0)} \Sigma_2 \bar{S}^{(0)}] \\ & + \lambda_3 g^4 \bar{S}^{(0)} \bar{D}^{(0)} \bar{S}^{(0)} \bar{D}^{(0)} \bar{S}^{(0)} + \lambda_4 g^4 \bar{S}^{(0)} \bar{D}^{(0)} N \bar{D}^{(0)} \bar{S}^{(0)} \bar{D}^{(0)}, \quad (5.22) \end{aligned}$$

where λ_i , $i = 1, \dots, 4$ are numbers related to combinatorial factors and color indices, and N is the three-gluon vertex. The first two terms are zero, because Σ_2 and Π_2 are zero in view of the self-consistency conditions for the zeroth-order quark and gluon self-energies. The remaining, nonzero contributions are the vertex corrections, last two diagrams in Fig. 5.4.

We denote these vertex corrections by Σ^A and Σ^{NA} , where the superscript A means that the vertex correction is like an Abelian one and NA means that it is of

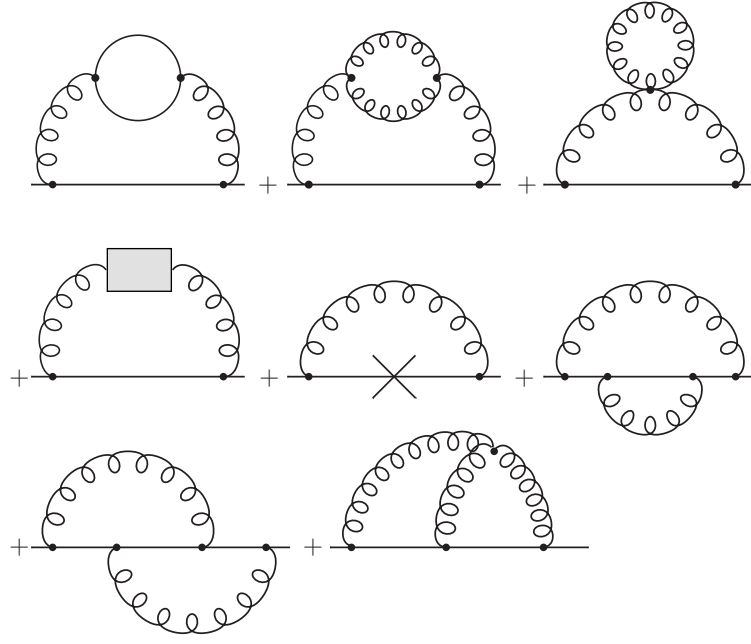


Figure 5.3: Diagrams contributing to $\mathcal{O}(\delta^4)$, denoted by Σ_4 . The solid and curly lines represent dressed propagators.

non-Abelian nature. Their explicit expressions are:

$$\begin{aligned} \Sigma^A(p) &= -g^4 F^A \int_{k_1} \int_{k_2} \bar{S}^{(0)}(p - k_1) \gamma_\mu \bar{D}_{\mu\nu}^{(0)}(k_1) \gamma_\rho \bar{S}^{(0)}(p - k_1 - k_2) \gamma_\nu \\ &\quad \times \bar{S}^{(0)}(p - k_2) \bar{D}_{\rho\sigma}^{(0)}(k_2) \gamma_\sigma, \end{aligned} \quad (5.23)$$

$$\begin{aligned} \Sigma^{NA}(p) &= -g^4 F^{NA} \int_q \int_l \gamma_\mu \bar{D}_{\mu\gamma}^{(0)}(p - q) \bar{S}^{(0)}(q) \gamma_\nu \bar{D}_{\nu\sigma}^{(0)}(q - l) \bar{S}^{(0)}(l) \gamma_\lambda \\ &\quad \times N_{\gamma\sigma\rho}(p - q, q - l, l - p) \bar{D}_{\lambda\rho}^{(0)}(l - p), \end{aligned} \quad (5.24)$$

where we are using the notation

$$\int_k = \int \frac{d^4 k}{(2\pi)^4}, \quad (5.25)$$

the color factors F^A and F^{NA} are given by

$$F^A = t^a t^{a'} t^a t^{a'} = -\frac{N_c^2 - 1}{4N_c^2}, \quad (5.26)$$

$$F^{NA} = t^a t^b t^c f^{abc} = i \frac{N_c^2 - 1}{4}. \quad (5.27)$$

and $N_{\gamma\sigma\rho}(p - q, q - l, l - p)$ can be read off Eq. (5.19). The momentum routing used in the loop integrals for Σ^A and Σ^{NA} is as indicated in Fig. 5.5 .

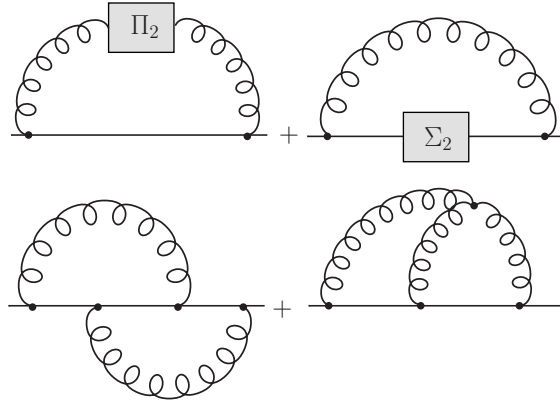


Figure 5.4: Contributions to Σ_4 grouped in way to make explicit cancellations due to the self-consistency requirements $\Sigma_2 = 0$ and $[\Pi_{\mu\nu}]_2 = 0$. The solid and curly lines represent dressed propagators.

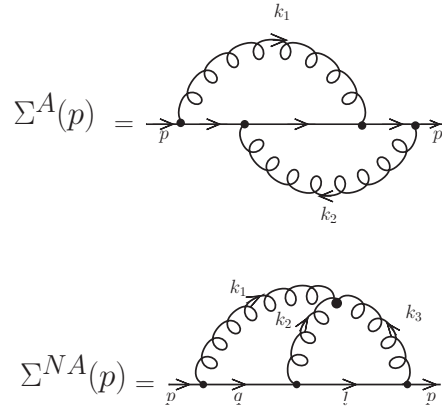


Figure 5.5: Abelian and non-Abelian contributions to the quark self-energy. The solid and curly lines represent dressed propagators.

Before we embark upon explicit calculations, we examine the different dependence in the number of colors N_c of the vertex correction. Fig. 5.6 is a pictorial representation of the corrections to the bare $ig\gamma^\mu t^a$ vertex. Our convention is that the vertex Γ_μ^a is the sum of the bare vertex plus corrections:

$$i\Gamma_\mu^a(p+k, p) = ig\gamma^\mu t^a + i\Gamma_\mu^{Aa}(p+k, p) + i\Gamma_\mu^{NAa}(p+k, p) , \quad (5.28)$$

where the explicit expressions for Γ_μ^{Aa} and Γ_μ^{NAa} are:

$$\Gamma_\mu^{Aa}(p+k, p) = -g^3 t^b t^a t^b \int_q \gamma_\sigma \bar{D}_{\sigma\rho}^{(0)}(p-q) \bar{S}^{(0)}(q+k) \gamma_\mu \bar{S}^{(0)}(q) \gamma_\rho, \quad (5.29)$$

$$\begin{aligned} \Gamma_\mu^{NAa}(p+k, p) &= -g^3 f^{abc} t^b t^c \int_q \gamma_\sigma \bar{S}^{(0)}(p-q) \gamma_\rho \bar{D}_{\sigma\sigma'}^{(0)}(q+k) \\ &\quad \times N_{\sigma'\rho'\mu}(q+k, -q, -k) \bar{D}_{\rho'\rho}^{(0)}(q). \end{aligned} \quad (5.30)$$

The color factors are such that

$$t^b t^a t^b = -\frac{1}{2N_c} t^a, \quad f^{abc} t^b t^c = i \frac{N_c}{2} t^a. \quad (5.31)$$

The relationship of these factors with the F^A and F^{NA} in the self-energies is obtained with the use of the relation

$$t^a t^a = \frac{N_c^2 - 1}{2N_c}. \quad (5.32)$$

One notices that the contributions have opposite sign (notice that there is an $-i$ in the definition of the triple-gluon vertex $N_{\mu\nu\sigma}$). Moreover, the non-Abelian correction is a factor N_c^2 bigger than the Abelian correction. Thus, the non-Abelian correction is sizeable dominant, a feature already noticed previously in Refs. [17,78,79]. Another important difference between these diagrams is the quark mass dependence due to the quark propagators. The Abelian diagram contains two quark propagators, while the non-Abelian has one, and this will also have an impact in the self-energy of the quark [78], as we will show in the following.

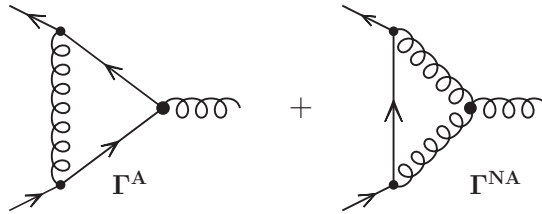


Figure 5.6: The one-loop corrections to the quark-gluon vertex.

5.2 Application: quark propagator, contact interaction

In the present section we apply the formalism discussed above for the quark propagator. As mentioned in the Introduction, since this is the first application of this

novel formalism, we feel that it would be appropriate to start with a simple model to gain insight and set up our codes for more realistic situations. In view of the limited aims of the present work, we will not consider the corrections to the Bethe-Salpeter equation. In doing so, it should be kept in mind that our results for the pion properties are not entirely consistent, since we are correcting only the quark propagator in the equation, but not the scattering kernel. One consequence of the lack of self-consistency is that Ward identities are not strictly obeyed.

With this aim, and noticing that, in practice, the two-loop quark-gluon vertex corrections given by Eqs. (5.23) and (5.24) always involve the product of a gluon propagator and a quark-gluon vertex, it is natural to choose the contact interaction discussed in Chapter 3, more precisely Eq. (3.65). In this sense, our approach using such a zeroth order ansatz can be viewed as going beyond the dressing of a bare vertex, as originally formulated. Along with this ansatz, we impose the same infrared-ultraviolet regulation discussed previously.

We write the dressed-quark propagator in the chiral limit as

$$S^{-1}(p) = [\bar{S}^{(0)}(p)]^{-1} + \Sigma_4(p) = i\gamma \cdot p + M + \Sigma_4(p) , \quad (5.33)$$

where $\Sigma_4(p)$ is the $\mathcal{O}(\delta^4)$ correction discussed in the previous section. Next, we decompose $\Sigma_4(p)$ into a Lorentz-vector and -vector pieces, namely

$$\Sigma_4(p) = i\gamma \cdot p \Sigma_V(p^2) + \Sigma_S(p^2) . \quad (5.34)$$

Therefore, we can rewrite the dressed-quark propagator in the form

$$S(p) = \frac{Z(p^2)}{i\gamma \cdot p + M(p^2)} , \quad (5.35)$$

where $Z(p^2)$ is the wave-function renormalization function:

$$Z(p^2) = \frac{1}{1 + \Sigma_V(p^2)} , \quad (5.36)$$

and $M(p^2)$ is the momentum-dependent quark mass function:

$$M(p^2) = \frac{M + \Sigma_S(p^2)}{1 + \Sigma_V(p^2)} . \quad (5.37)$$

Clearly, in the absence of corrections, $M(p^2) = M$ and $Z(p^2) = 1$, and $S(p) = S^{(0)}(p)$.

From the expressions in Eqs. (5.23) and (5.24) for the Abelian and non-Abelian contributions to the self-energy, one separates the vector and scalar contributions taking appropriate traces:

$$\Sigma_S(p^2) = \frac{1}{4} \text{Tr}_D[\Sigma_4(p)] , \quad (5.38)$$

$$\Sigma_V(p^2) = -\frac{1}{4p^2} \text{Tr}_D[i\gamma \cdot p \Sigma_4(p)] , \quad (5.39)$$

where Tr_D means trace over Dirac indices. In the following subsections we obtain explicit expressions for the separate contributions.

5.2.1 Abelian contribution

We start with Abelian contribution $\Sigma^A(p)$, Eq. (5.23). Using Eq. (3.65) for $\bar{D}_{\mu\nu}^{(0)}$ in that equation, one obtains:

$$\begin{aligned} \Sigma^A(p) &= -F^A \left(\frac{4\pi\alpha_{IR}}{m_G^2} \right)^2 \int_q \int_l \gamma_\mu \bar{S}^{(0)}(q) \gamma_\rho \bar{S}^{(0)}(q+l-p) \gamma_\mu \bar{S}^{(0)}(l) \gamma_\rho \\ &= -F^A \left(\frac{4\pi\alpha_{IR}}{m_G^2} \right)^2 \int_q \int_l \frac{\mathcal{H}(l, q, p)}{(q^2 + M^2)[(q+l-p)^2 + M^2](l^2 + M^2)} , \end{aligned} \quad (5.40)$$

where we have made the changes of variables $q = p - k_1$ and $l = p - k_2$, and

$$\mathcal{H}(l, q, p) = \gamma_\mu (-i\gamma \cdot q + M) \gamma_\rho [-i\gamma \cdot (q+l-p) + M] \gamma_\mu (-i\gamma \cdot l + M) \gamma_\rho . \quad (5.41)$$

Here we used $\bar{S}^{(0)}(p)$ as :

$$\bar{S}^{(0)}(p) = \frac{-i\gamma \cdot p + M}{p^2 + M^2} . \quad (5.42)$$

After evaluating the traces, and using Eqs. (5.38) and (5.39) to separate the scalar and vector components of Σ^A , one can write:

$$\Sigma_S^A(p^2) = -4F^A \left(\frac{4\pi\alpha_{IR}}{m_G^2} \right)^2 M (3I_1 + 2I_2 - I_3 - I_4 - 4M^2 I_5) , \quad (5.43)$$

$$\Sigma_V^A(p^2) = +\frac{4F^A}{p^2} \left(\frac{4\pi\alpha_{IR}}{m_G^2} \right)^2 (4I_6 - 2p^2 I_1 + 2M^2 I_3 + 2M^2 I_4 - M^2 p^2 I_5) , \quad (5.44)$$

where I_i , $i = 1, \dots, 6$ are the following integrals:

$$I_1 = \int_q \int_l \frac{(q \cdot l)}{(q^2 + M^2)[(q + l - p)^2 + M^2](l^2 + M^2)} , \quad (5.45)$$

$$I_2 = \int_q \int_l \frac{1}{[(q + l - p)^2 + M^2](l^2 + M^2)} , \quad (5.46)$$

$$I_3 = \int_q \int_l \frac{(p \cdot l)}{(q^2 + M^2)[(q + l - p)^2 + M^2](l^2 + M^2)} , \quad (5.47)$$

$$I_4 = \int_q \int_l \frac{(p \cdot q)}{(q^2 + M^2)[(q + l - p)^2 + M^2](l^2 + M^2)} , \quad (5.48)$$

$$I_5 = \int_q \int_l \frac{1}{(q^2 + M^2)[(q + l - p)^2 + M^2](l^2 + M^2)} , \quad (5.49)$$

$$I_6 = \int_q \int_l \frac{(q \cdot l)(q \cdot p)}{(q^2 + M^2)[(q + l - p)^2 + M^2](l^2 + M^2)} . \quad (5.50)$$

As mentioned above, we use the infrared-ultraviolet regularization of Refs. [62] and [77], discussed in Chapter 3. Specifically, the quark propagator (5.42) is rewritten as

$$\bar{S}^{(0)}(p) = \frac{-i\gamma \cdot p + M}{p^2 + M^2} \left[e^{-(p^2 + M^2)\tau_{UV}^2} - e^{-(p^2 + M^2)\tau_{IR}^2} \right] . \quad (5.51)$$

This guarantees that there will be no ultraviolet divergences and removes the pole at $p^2 = -M^2$ - Eq. (5.51) has poles for complex values of p^2 . This last feature guarantees that there will be no quark-antiquark thresholds, a feature that simulates confinement. Note that in using such a replacement in loop integrals, we have in mind more realistic situations in which $\bar{S}^{(0)}(p)$ has the confining feature. In doing so, we depart from the procedure of Ref. [77], in which the integrand of a loop integral is manipulated using Feynman parametrization and only the remaining momentum integrals are regularized - as done in the integral for f_π discussed in Chapter 4. Here, we are using the propagator regularized, as it would be if one would use for $\bar{S}^{(0)}$ a propagator resulting from a rainbow approximation or from a lattice simulation.

We start considering integrals I_2 and I_5 , since they can be integrated straightforwardly. Using the regularization (5.51) and integrating on the variable q , and

subsequently on l , we obtain

$$\begin{aligned} I_2 &= \left(\frac{1}{4\pi}\right)^4 \left(\int_{\tau_{UV}^2}^{\tau_{IR}^2} d\alpha \frac{e^{-M^2\alpha}}{\alpha^2}\right)^2 \\ &= \left(\frac{M}{4\pi}\right)^4 [\Gamma(-1, \tau_{UV}^2 M^2) - \Gamma(-1, \tau_{IR}^2 M^2)]^2, \end{aligned} \quad (5.52)$$

$$\begin{aligned} I_5 &= \frac{1}{16} \left(\frac{M}{4\pi^2}\right)^2 \int_{M^2\tau_{UV}^2}^{M^2\tau_{IR}^2} d\tilde{\alpha}_1 d\tilde{\alpha}_2 d\tilde{\alpha}_3 \frac{1}{[\tilde{\alpha}_2\tilde{\alpha}_3 + \tilde{\alpha}_1(\tilde{\alpha}_2 + \tilde{\alpha}_3)]^2} \\ &\quad \times \exp \left[-\frac{\tilde{p}^2\tilde{\alpha}_1\tilde{\alpha}_2\tilde{\alpha}_3}{\tilde{\alpha}_2\tilde{\alpha}_3 + \tilde{\alpha}_1(\tilde{\alpha}_2 + \tilde{\alpha}_3)} - (\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3) \right], \end{aligned} \quad (5.53)$$

where we have introduced the dimensionless quantities $\tilde{\alpha} = M^2\alpha$ and $\tilde{p} = p/M$, and used that

$$\int_{\tau_{UV}^2}^{\tau_{IR}^2} d\alpha \frac{e^{-M^2\alpha}}{\alpha^2} = M^2 [\Gamma(-1, \tau_{UV}^2 M^2) - \Gamma(-1, \tau_{IR}^2 M^2)], \quad (5.54)$$

with $\Gamma(\alpha, y)$ being the incomplete gamma function defined previously in Eq. (3.82).

In order to calculate I_1 , I_3 , I_4 and I_6 , let us to consider the following generic integral

$$\begin{aligned} I &= \int_q \int_l \int_0^\infty d\alpha_1 e^{-\alpha_1(q^2+M^2)} \int_0^\infty d\alpha_2 e^{-\alpha_2(q^2+l^2+\lambda_1 q \cdot l - \lambda_2 q \cdot p - \lambda_3 p \cdot l + p^2 + M^2)} \\ &\quad \times \int_0^\infty d\alpha_3 e^{-\alpha_3(l^2+M^2)}. \end{aligned} \quad (5.55)$$

One sees that I_1 , I_3 and I_4 are given by taking appropriate derivatives with respect to λ_1 , λ_2 and λ_3 , respectively, and at the end set $\lambda_1 = \lambda_2 = \lambda_3 = 2$:

$$I_1 = -\frac{1}{\alpha_2} \frac{\partial I}{\partial \lambda_1} \Big|_{\lambda_i=2}, \quad (5.56)$$

$$I_3 = +\frac{1}{\alpha_2} \frac{\partial I}{\partial \lambda_3} \Big|_{\lambda_i=2}, \quad (5.57)$$

$$I_4 = +\frac{1}{\alpha_2} \frac{\partial I}{\partial \lambda_2} \Big|_{\lambda_i=2}, \quad (5.58)$$

$$I_6 = -\frac{1}{\alpha_2} \frac{\partial^2 I}{\partial \lambda_1 \partial \lambda_2} \Big|_{\lambda_i=2}. \quad (5.59)$$

Thus, we need to calculate I in terms of these three λ -parameters. Integrating (5.55) over q and l , we obtain

$$I = \left(\frac{M}{4\pi^2}\right)^2 \int_{M^2\tau_{UV}^2}^{M^2\tau_{IR}^2} d\tilde{\alpha}_1 d\tilde{\alpha}_2 d\tilde{\alpha}_3 \mathcal{A}(\tilde{\alpha}_i) e^{B(\tilde{\alpha}_i)\tilde{p}^2}, \quad (5.60)$$

where we have defined $\mathcal{A}(\tilde{\alpha}_i) = \mathcal{A}(\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3)$ to be

$$\mathcal{A}(\tilde{\alpha}_i) = \frac{e^{-(\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3)}}{[\tilde{\alpha}_2^2 \lambda_1^2 - 4(\tilde{\alpha}_1 + \tilde{\alpha}_2)(\tilde{\alpha}_2 + \tilde{\alpha}_3)]^2}, \quad (5.61)$$

and $\mathcal{B}(\tilde{\alpha}_i) = \mathcal{B}(\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3)$:

$$\mathcal{B}(\tilde{\alpha}_i) = \frac{\tilde{\alpha}_2^2 \lambda_2^2 - 4\tilde{\alpha}_2(\tilde{\alpha}_1 + \tilde{\alpha}_2)}{4(\tilde{\alpha}_1 + \tilde{\alpha}_2)} - \frac{(2\tilde{\alpha}_2 \lambda_3(\tilde{\alpha}_1 + \tilde{\alpha}_2) - \tilde{\alpha}_2^2 \lambda_1 \lambda_2)^2}{4(\tilde{\alpha}_1 + \tilde{\alpha}_2)[\tilde{\alpha}_2^2 \lambda_1^2 - 4(\tilde{\alpha}_1 + \tilde{\alpha}_2)(\tilde{\alpha}_2 + \tilde{\alpha}_3)]}. \quad (5.62)$$

Thus, after taking the derivatives and simplifying, we obtain

$$\begin{aligned} I_1 &= -\frac{1}{16} \frac{M^4}{(4\pi^2)^2} \int_{M^2 \tau^2}^{M^2 \tau_{IR}^2} d\tilde{\alpha}_1 d\tilde{\alpha}_2 d\tilde{\alpha}_3 \frac{\tilde{\alpha}_2 \{ \tilde{\alpha}_1 [\tilde{\alpha}_2 (2 - \tilde{p}^2 \tilde{\alpha}_3) + 2\tilde{\alpha}_3] + 2\tilde{\alpha}_2 \tilde{\alpha}_3 \}}{[\tilde{\alpha}_2 \tilde{\alpha}_3 + \tilde{\alpha}_1 (\tilde{\alpha}_2 + \tilde{\alpha}_3)]^4} \\ &\quad \times \exp \left[-\frac{\tilde{p}^2 \tilde{\alpha}_1 \tilde{\alpha}_2 \tilde{\alpha}_3}{\tilde{\alpha}_2 \tilde{\alpha}_3 + \tilde{\alpha}_1 (\tilde{\alpha}_2 + \tilde{\alpha}_3)} - (\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3) \right], \end{aligned} \quad (5.63)$$

$$\begin{aligned} I_3 &= \frac{\tilde{p}^2}{16} \frac{M^4}{(4\pi^2)^2} \int_{M^2 \tau_{UV}^2}^{M^2 \tau_{IR}^2} d\tilde{\alpha}_1 d\tilde{\alpha}_2 d\tilde{\alpha}_3 \frac{\tilde{\alpha}_1 \tilde{\alpha}_2}{[\tilde{\alpha}_2 \tilde{\alpha}_3 + \tilde{\alpha}_1 (\tilde{\alpha}_2 + \tilde{\alpha}_3)]^3} \\ &\quad \times \exp \left[-\frac{\tilde{p}^2 \tilde{\alpha}_1 \tilde{\alpha}_2 \tilde{\alpha}_3}{\tilde{\alpha}_2 \tilde{\alpha}_3 + \tilde{\alpha}_1 (\tilde{\alpha}_2 + \tilde{\alpha}_3)} - (\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3) \right], \end{aligned} \quad (5.64)$$

$$I_4 = I_3. \quad (5.65)$$

Replacing these results in the expressions for $\Sigma_S^A(p^2)$ and $\Sigma_V^A(p^2)$, Eqs. (5.43) and (5.44), one can write:

$$\Sigma_S^A(p^2) = -\frac{M^5}{9(4\pi^2)^2} \left(\frac{4\pi\alpha_{IR}}{m_G^2} \right)^2 \left[\mathcal{G}_S - \int_{M^2 \tau_{UV}^2}^{M^2 \tau_{IR}^2} d\tilde{\alpha}_1 d\tilde{\alpha}_2 d\tilde{\alpha}_3 \mathcal{F}_S(\tilde{\alpha}_i) \right], \quad (5.66)$$

$$\Sigma_V^A(p^2) = -\frac{M^4}{9(4\pi^2)^2} \left(\frac{4\pi\alpha_{IR}}{m_G^2} \right)^2 \int_{M^2 \tau_{UV}^2}^{M^2 \tau_{IR}^2} d\tilde{\alpha}_1 d\tilde{\alpha}_2 d\tilde{\alpha}_3 \mathcal{F}_V(\tilde{\alpha}_i), \quad (5.67)$$

where we have defined

$$\mathcal{G}_S = [\Gamma(-1, \tau_{UV}^2 M^2) - \Gamma(-1, \tau_{IR}^2 M^2)]^2, \quad (5.68)$$

$$\mathcal{F}_S(\tilde{\alpha}_i) = [\tilde{p}^2 A_S(\tilde{\alpha}_i) + B_S(\tilde{\alpha}_i)] e^{-a(\tilde{\alpha}_i) \tilde{p}^2 - b(\tilde{\alpha}_i)}, \quad (5.69)$$

$$\mathcal{F}_V(\tilde{\alpha}_i) = [\tilde{p}^2 A_V(\tilde{\alpha}_i) + B_V(\tilde{\alpha}_i)] e^{-a(\tilde{\alpha}_i) \tilde{p}^2 - b(\tilde{\alpha}_i)}, \quad (5.70)$$

with

$$A_S(\tilde{\alpha}_i) = \frac{\tilde{\alpha}_2 \tilde{\alpha}_3 [2(\tilde{\alpha}_1 + \tilde{\alpha}_2) \tilde{\alpha}_3 - \tilde{\alpha}_1 \tilde{\alpha}_2]}{2[\tilde{\alpha}_2 \tilde{\alpha}_3 + \tilde{\alpha}_1(\tilde{\alpha}_2 + \tilde{\alpha}_3)]^4}, \quad (5.71)$$

$$B_S(\tilde{\alpha}_i) = \frac{(2\tilde{\alpha}_1 + 3)\tilde{\alpha}_2 + 2(\tilde{\alpha}_1 + \tilde{\alpha}_2)\tilde{\alpha}_3}{[\tilde{\alpha}_2 \tilde{\alpha}_3 + \tilde{\alpha}_1(\tilde{\alpha}_2 + \tilde{\alpha}_3)]^3}, \quad (5.72)$$

$$A_V(\tilde{\alpha}_i) = \frac{\tilde{p}^2 \tilde{\alpha}_1 \tilde{\alpha}_2 [\tilde{\alpha}_3(\tilde{\alpha}_3 + 1)\tilde{\alpha}_2^2 + 3\tilde{\alpha}_1 \tilde{\alpha}_3(\tilde{\alpha}_2 + \tilde{\alpha}_3)\tilde{\alpha}_2 + 2\tilde{\alpha}_1^2(\tilde{\alpha}_2 + \tilde{\alpha}_3)^2]}{[\tilde{\alpha}_2 \tilde{\alpha}_3 + \tilde{\alpha}_1(\tilde{\alpha}_2 + \tilde{\alpha}_3)]^5}, \quad (5.73)$$

$$B_V(\tilde{\alpha}_i) = -\frac{(\tilde{\alpha}_3 - 2)\tilde{\alpha}_3 \tilde{\alpha}_2^2 + 2\tilde{\alpha}_1(\tilde{\alpha}_3 - 3)(\tilde{\alpha}_2 + \tilde{\alpha}_3)\tilde{\alpha}_2 + \tilde{\alpha}_1^2(\tilde{\alpha}_2 + \tilde{\alpha}_3)^2}{2[\tilde{\alpha}_2 \tilde{\alpha}_3 + \tilde{\alpha}_1(\tilde{\alpha}_2 + \tilde{\alpha}_3)]^4}, \quad (5.74)$$

$$a(\tilde{\alpha}_i) = \frac{\tilde{p}^2 \tilde{\alpha}_1 \tilde{\alpha}_2 \tilde{\alpha}_3}{\tilde{\alpha}_2 \tilde{\alpha}_3 + \tilde{\alpha}_1(\tilde{\alpha}_2 + \tilde{\alpha}_3)}, \quad b(\tilde{\alpha}_i) = \tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3. \quad (5.75)$$

5.2.2 Non-Abelian contribution

Now, let us to consider the non-Abelian contribution $\Sigma^{NA}(p)$:

$$\begin{aligned} \Sigma^{NA}(p) &= -g^4 F^{NA} \left(\frac{4\pi\alpha_{IR}}{m_G^2} \right)^2 \int_q \int_l \gamma_\mu \bar{S}^{(0)}(q) \gamma_\nu \bar{S}^{(0)}(l) \gamma_\lambda \\ &\quad \times N_{\mu\nu\lambda}(p - q, q - l, l - p). \end{aligned} \quad (5.76)$$

Separating the scalar and vector parts according to Eqs. (5.38) and (5.39), one obtains:

$$\Sigma_S^{NA}(p^2) = \frac{12M}{m_G^2} \left(\frac{4\pi\alpha_{IR}}{m_G^2} \right)^2 \int_q \int_l \frac{(4q^2 - p \cdot q + 2l \cdot q + l \cdot p)}{(q^2 + M^2)(l^2 + M^2)}, \quad (5.77)$$

$$\Sigma_V^{NA}(p^2) = \frac{6M^2}{m_G^2} \left(\frac{4\pi\alpha_{IR}}{m_G^2} \right)^2 \int_q \int_l \frac{1}{(q^2 + M^2)(l^2 + M^2)}. \quad (5.78)$$

Using the regularization and performing the integrals, the final result for the non-Abelian contribution can be written as

$$\begin{aligned} \Sigma_S^{NA}(p^2) &= \frac{6M^7}{m_G^2 (4\pi^2)^2} \left(\frac{4\pi\alpha_{IR}}{m_G^2} \right)^2 [\Gamma(-1, \tau_{UV}^2 M^2) - \Gamma(-1, \tau_{IR}^2 M^2)] \\ &\quad \times [\Gamma(-2, \tau_{UV}^2 M^2) - \Gamma(-2, \tau_{IR}^2 M^2)], \end{aligned} \quad (5.79)$$

$$\Sigma_V^{NA}(p^2) = \frac{3M^6}{2m_G^2 (4\pi^2)^2} \left(\frac{4\pi\alpha_{IR}}{m_G^2} \right)^2 [\Gamma(-1, \tau_{UV}^2 M^2) - \Gamma(-1, \tau_{IR}^2 M^2)]^2. \quad (5.80)$$

Having obtained the analytical results for Σ_4 , we are ready to discuss numerical results. This will be done in the next section.

5.3 Results and Discussions

In this section we present and discuss our numerical results. We use the same values of parameters employed in Chapter 4 and which have been taken from Ref. [77]. Specifically, the values of the parameters α_{IR} and m_G defining the ansatz gluon propagator, Eq. (3.65), are

$$\frac{\alpha_{IR}}{\pi} = 0.93, \quad m_G = 0.8 \text{ GeV} . \quad (5.81)$$

The infrared and ultraviolet cutoff parameters, τ_{IR}^2 and τ_{UV}^2 , are:

$$\tau_{IR}^2 = (0.24 \text{ GeV})^{-2}, \quad \tau_{UV}^2 = (0.905 \text{ GeV})^{-2} . \quad (5.82)$$

With this set of parameters, the solution of Eq (5.17) for the zeroth-order self-energy $\Sigma_0(p)$ for the ansatz (3.65) for $\bar{D}_{\mu\nu}^{(0)}$, is $\Sigma_0(p) = M$, with $M = 358 \text{ MeV}$. In addition, the values for the quark condensate, or in-pion condensate in the chiral limit [77] κ_π , and the pion decay constant f_π are: $\kappa_\pi = 241 \text{ MeV}$ and $f_\pi = 93 \text{ MeV}$ - see first row in Table 5.1. Next, we examine the numerical impact of the vertex corrections to the self-energy.

In Fig. 5.7 we show the behavior of $\Sigma_S(p^2)$ as a function of the Euclidean momentum p when the Abelian and non-Abelian contribution are considered independently and in combination. Recall that in the absence of corrections, there is only a scalar component in the self-energy and it is given by $M = 0.358 \text{ GeV}$. From the figure one sees that the Abelian contribution gives a negative contribution to the scalar component (line labeled with the letter “A”), while the non-Abelian contribution gives a positive contribution (line labeled with “NA”). Moreover, the non-Abelian correction is much larger in absolute value than the Abelian correction, a feature due to the $N_c^2 = 9$ factor mentioned earlier. As a result, when both contributions are taking in combination (line labeled as “A+NA”), the correction is almost entirely given by the non-Abelian part. Another feature one observes in Fig. 5.7 is the weak momentum dependence of both contributions to Σ_S . This weak momentum dependence is due to the relatively small value of momentum ultraviolet cutoff employed in our regularization scheme. The low-momentum cutoff has the effect of leaving little phase space for quark propagation in the loop integrals defining $\Sigma_S(p^2)$ - as we shall see next, the same is basically true for $\Sigma_V(p^2)$.

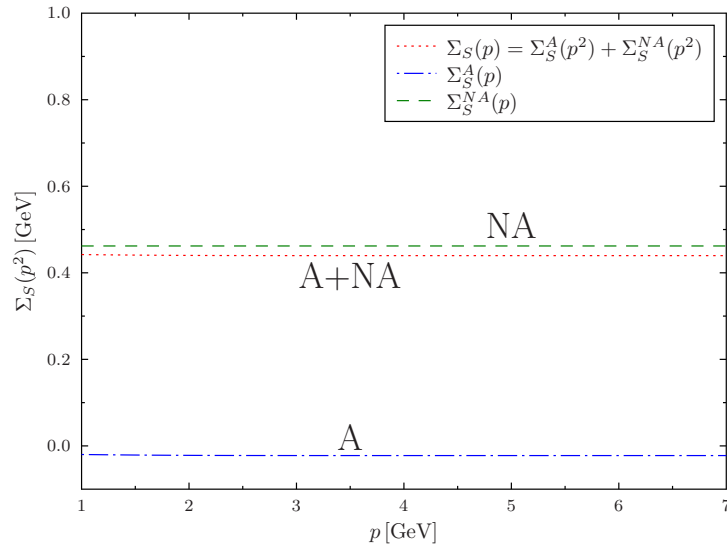


Figure 5.7: The scalar component $\Sigma_S(p^2)$ of the quark self-energy as function of the Euclidean momentum p . Labels are as: “A” for the Abelian contribution, “NA” for the non-Abelian contribution, and “A+NA” the sum of both contributions.

Fig. 5.8 presents the corresponding results for Σ_V . In absence of corrections, there is no vector component in the self-energy. From Fig. 5.8 one sees the effect of the Abelian correction is also very small, almost zero; while the non-Abelian correction is clearly dominant. There is, however, a weak momentum dependence for small p in the abelian contribution, and almost none in the non-Abelian correction.

For completeness, we show in Fig. 5.9 the quark wave-function renormalisation function $Z(p^2)$, defined in Eq. (5.36) - in the absence corrections, $Z(p^2) = 1$. Consistent with the result for $\Sigma_V(p^2)$, the impact of the Abelian contribution on $Z(p^2)$ is very small, and the momentum dependence of $Z(p^2)$ is very weak.

The results for the quark mass function $M(p^2)$, defined in Eq. (5.37), are shown in Fig. 5.10. Clearly seen in the figure is the general trend already discussed above, that the non-Abelian correction is dominant over the Abelian-like. In addition, one sees that the non-Abelian contribution increases the value of the quark mass compared to the value of zeroth-order quark mass M , a feature characteristic of an attractive interaction; in this sense, the Abelian-like correction is repulsive.

It is worth mentioning that our results are in agreement with results of the literature, in particular with Ref. [79]. In that reference, the author has examined corrections beyond the rainbow approximation for the quark propagator due to the

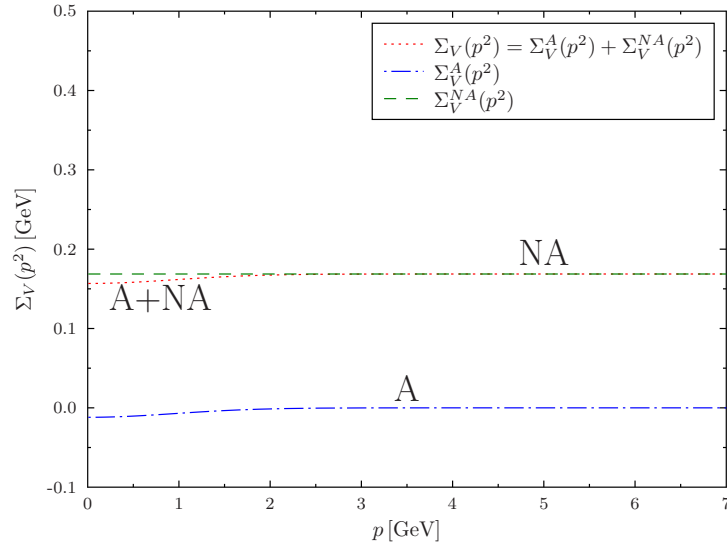


Figure 5.8: The vector component $\Sigma_V(p^2)$ of the quark self-energy as function of the Euclidean momentum p . Labelling the same as in Fig. 5.7.

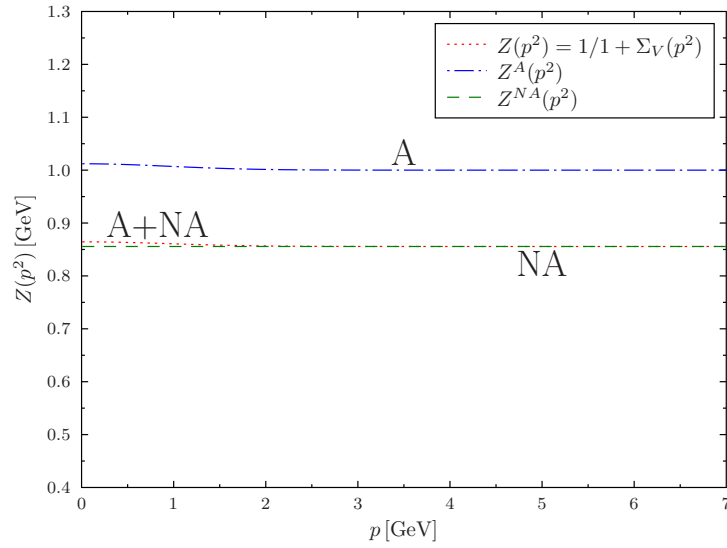


Figure 5.9: The wave-function renormalization $Z(p^2)$; labelling same as previous two figures.

same vertex corrections discussed here. The main difference with our calculation is that here we are using the zeroth-order quark propagator $\bar{S}^{(0)}$, while there the author used the self-consistent quark propagator. Ref. [79] observed the same pattern in the behavior of the quark mass function as we observed. Moreover, the size of the Abelian-like and non-Abelian corrections obtained for small values of p in that

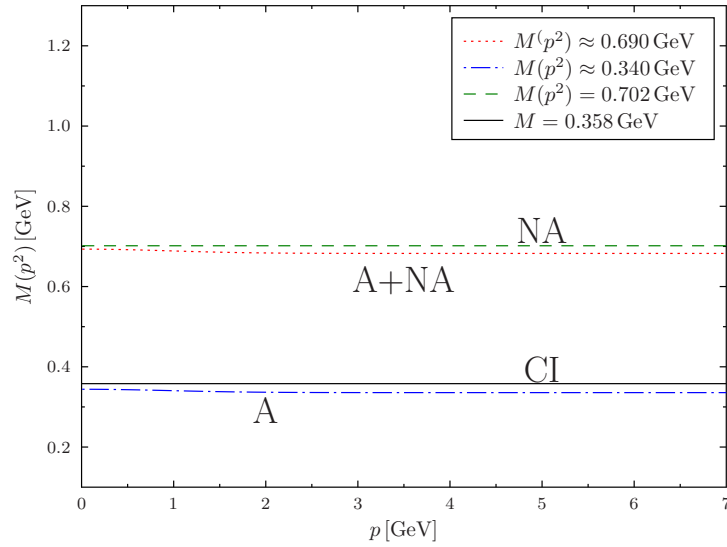


Figure 5.10: Quark mass function $M(p^2)$ for the zeroth-order approximation (CI), Abelian (A), non-Abelian (NA) and Abelian + Non-Abelian (A+NA). Shown in the legends are the values of $M(p^2)$ for large p , where the quark-mass function reaches a constant value.

reference is very much similar to what we obtained. One difference is the momentum dependence, as we used a contact interaction and Ref. [79] used a model interaction that is strongly momentum dependent.

Now we come to our final results. Since the the mass function is essentially momentum-independent over the whole momentum range, it is natural to approximate it as a constant. $Z(p^2)$ is also essentially constant and approximately equal to unity. In taking these quantities as constants, one could of course readjust the values of the parameters α_{IR} and m_G to reproduce the same values for physical quantities as obtained with the zeroth order propagator. However, we will not proceed that way; rather, we will examine the effect of the corrections on physical quantities using the original parameters. We use the quark propagator in the ladder Bethe-Salpeter equation for the pion and obtain the pion decay constant f_π and in-pion quark condensate κ_π . In doing so, it is important to call attention that *we are not* including consistently the vertex corrections in the scattering kernel in the BSE. In doing so, we are not consistently respecting the axial-vector Ward identity. However, we do impose the ward identity, Eq. (4.77):

$$-iP_\mu \Gamma_{5\mu}(k_+, k) = S^{-1}(k_+) \gamma_5 + \gamma_5 S^{-1}(k) , \quad (5.83)$$

where $\Gamma_{5\mu}(k_+, k)$ obtained with the contact interaction

$$\Gamma_{5\mu}(k_+, k) = \gamma_5 \gamma_\mu - \frac{16\pi}{3} \frac{\alpha_{IR}}{m_G^2} \int \frac{d^4 q}{(2\pi)^4} \gamma_\alpha \chi_{5\mu}(q_+, q) \gamma_\alpha, \quad (5.84)$$

where $k_+ = k + P$ and

$$\chi_{5\mu}(q_+, q) = S(q_+) \Gamma_{5\mu}(q_+, q) S(q), \quad (5.85)$$

and the quark mass is a constant in the quark propagator. The implication of this is that

$$\int \frac{d^4 q}{(2\pi)^4} \frac{\frac{1}{2}q^2 + M^2}{(q^2 + M^2)^2} = 0 \quad (5.86)$$

must be imposed.

In the Table. 5.1 we summarize our numerical results; we present the for the quark mass function, pion decay constant, in-pion quark condensate, the BSE amplitudes E_π and F_π . The first observation is that despite the quark mass is increased substantially by the vertex corrections, pion observables are not affected much. The reason is, of course, that the pion is protected by chiral symmetry, in the sense that once the axial-vector Ward-Takahashi identity is imposed via the vanishing of the integral in Eq. (5.86), pion properties are essentially independent of the value of the quark mass within the contact interaction setting.

Correction	E_π	F_π	M	κ_π	f_π	$f_\pi _{F_\pi \rightarrow 0}$
$0^{th} - order$	3.567	0.458	0.358	0.241	0.100	0.118
A	3.451	0.427	0.340	0.239	0.098	0.117
NA	6.452	0.974	0.702	0.229	0.109	0.110
A+NA	6.327	0.959	0.690	0.230	0.109	0.111

Table 5.1: Results (in GeV) obtained in the chiral limit for the quark mass and pion observables.

Chapter 6

CONCLUSIONS AND OUTLOOK

In this dissertation we have introduced a novel approach, that we named *dressed perturbation theory*, to treat Dyson-Schwinger and Bethe-Salpeter equations beyond the commonly used rainbow-ladder truncation scheme. The approach shares similarities with the linear δ expansion [20], the optimized perturbation theory [21] and screened perturbation theory [22]. The main idea of all these methods, including dressed perturbation theory, is to modify the noninteracting part of the QCD action, $\mathcal{S}_0$, by adding terms quadratic in quark and gluon fields, and implement perturbative calculations with a modified interaction. The modified interaction is obtained from the original QCD action, $\mathcal{S}_I$, subtracted with the same quadratic term added to $\mathcal{S}_0$. This gives rise to a new quadratic action, $\bar{\mathcal{S}}^{(0)}$, and a new interaction, $\mathcal{S}_I$. Instead of noninteracting propagators of traditional perturbative QCD calculations, dressed perturbation theory is performed with dressed propagators. The most significant and important difference of dressed perturbation theory with its congeners is the momentum dependence of added quadratic terms and the way they are determined; while in dressed perturbation theory the added and subtracted term is not a constant-mass term, it is a momentum-dependent self-energy. Moreover, these self-energies can, in principle, be taken from e.g. rainbow-ladder approximation, or from lattice QCD simulations.

While we have discussed the method in great generality, explicit results were obtained for the quark propagator only starting with quark and gluon self-energies obtained self-consistently with bare vertices. The self-consistency is obtained by imposing that the lowest nontrivial order correction vanishes. At this point it is important to address some issues not touched in the text so far. First, taken as said, the departing point consists in using zeroth order quark and gluon self-energies de-

terminated self-consistently from Eqs. (5.17) and (5.20). Now, if one takes, e.g. both zeroth order self-energies from fits to lattice data or other approximation schemes, it should be clear that Eqs. (5.17) and (5.20) cannot not be satisfied simultaneously. However, if one uses e.g. a gluon self-energy from elsewhere, the equation for the zeroth order quark self-energy (5.17) certainly can be satisfied. Another issue not discussed so far is the one of renormalization. Dressed perturbation theory loops in general will introduce ultraviolet divergences and a novel renormalization scheme must be devised. This is particularly important when one wants to use e.g. gluon self-energies from elsewhere, as issues related to scale dependence of effective gluon masses [80] certainly will appear in the calculation of vertex corrections for the quark propagator. Another very important issue is the practical application of the method to calculate loop integrals when one uses nontrivial quark and gluon self energies. For example, if one uses gluon propagators from lattice simulations, most certainly loop calculations will not be feasible analytically, as they will be given as numerical tables or given in terms of non-linear fitting formulas that most certainly will not lead to common integrals obtained in perturbative calculations. Still another potential complication is preservation of symmetries, as most of the regularization schemes used in the literature have the potential of breaking fundamental symmetries, like gauge and Lorentz symmetries. However, all these issues can be tackled simultaneously [81] with use of a novel method advocated by P. Tandy and collaborators [82], in that quark and gluon propagators can be parametrized in terms of functions that resemble free propagators with complex masses and residues. Specifically, it is not difficult to fit a generic quark propagator in the form

$$S(p) = \frac{1}{i\gamma \cdot p A(p^2) + B(p^2)} = \sum_{k=1}^N \left(\frac{z_k}{i\gamma \cdot p + m_k} + \frac{z_k^*}{i\gamma \cdot p + m_k^*} \right), \quad (6.1)$$

where z_k and m_k are complex numbers. The same is true for generic gluon propagators. When such representations are used in loop integrals, usual Feynman parametrization tricks can be used and integrations can be performed with no additional difficulties beyond those of usual perturbative calculations.

Regarding ultraviolet divergences, a symmetry preserving regularization scheme we envisage [81] to use the method employed recently [54] in the context of the Nambu–Jona-Lasinio model. The method allows to separate from a generic loop integral terms that are finite, purely divergent, and symmetry-offending. The symmetry-

offending terms are integrals of the kind of Eq. (5.86), which must vanish. The great advantage of the method we advocate is that it does not require any explicit regulator, like a cutoff, and is fully consistent with dimensional regularization of ordinary perturbation theory. Another great advantage of the method is that it can deal with different masses running in loops, in way that all divergences appear in terms of an arbitrary mass scale which not necessarily is one of the quark masses. This fact can be of great help when dealing with heavy-light systems [17], since it avoids problems that arise when one deals with renormalization of amplitudes where widely different mass scales are running in loops.

As an explicit application of the dressed perturbation theory we are proposing, we investigated the impact of the two-loop quark-gluon vertex corrections on the dressed quark mass function for a given model contact interaction. We obtained, in accord with the literature, that the non-Abelian correction is dominant over the Abelian-like, and that the non-Abelian contribution increases the value of the quark mass compared to the value of zeroth-order quark mass M , a feature characteristic of an attractive interaction, while the Abelian-like correction is weakly repulsive.

To conclude, our immediate goal is the application of dressed perturbation theory to heavy-light systems, like D and D^* mesons. In particular, we envisage to use the symmetry preserving regularization scheme of Ref. [54] to investigate the impact of the two-loop quark-gluon vertex corrections on the electroweak decay constants of these mesons. As mentioned in the Introduction, the rainbow-ladder approximation has difficulties to describe these mesons [17].

Appendix A

NOTATIONS AND CONVENTIONS

In this appendix we will show the notations and conventions used in this work. We will start with Minkowski space conventions and later we shall show how to go from this to the Euclidean space.

A.1 Minkowski Space Conventions

In Minkowski space the space coordinates are denoted by

$$x^\mu = (x^0, x^1, x^2, x^3) \equiv (t, x, y, z) = (t, \mathbf{x}) , \quad (\text{A.1})$$

this is a contravariant four-vector. Now the covariant four-vector is obtained by changing the sign of the spatial components of the contravariant vector, namely

$$x_\mu = (x_0, x_1, x_2, x_3) \equiv (t, -x, -y, -z) = (t, -\mathbf{x}) , \quad (\text{A.2})$$

which we can rewrite in form

$$x_\mu = g_{\mu\nu} x_\nu , \quad (\text{A.3})$$

where we have introduced the metric tensor $g_{\mu\nu}$, which is defined as

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \quad (\text{A.4})$$

Now we define the product of two four-vector a and b as

$$a \cdot b = g_{\mu\nu} a^\mu b^\nu = a_\mu b^\mu . \quad (\text{A.5})$$

The Poincar-invariant length of any vector is

$$x^2 = x \cdot x = t^2 - \mathbf{x}^2 . \quad (\text{A.6})$$

Similarly the momentum four-vector is defined as

$$p^\mu = (p^0, p^1, p^2, p^3) \equiv (E, p_x, p_y, p_z) = (E, \mathbf{p}) , \quad (\text{A.7})$$

and

$$(p, q) = E_p E_q - \mathbf{p} \cdot \mathbf{q} , \quad (\text{A.8})$$

$$(x, p) = t E - \mathbf{x} \cdot \mathbf{p} . \quad (\text{A.9})$$

The momentum operator is defined as

$$p^\mu := i \frac{\partial}{\partial x_\mu} = \left(i \frac{\partial}{\partial t}, -i \nabla \right) := i \nabla^\mu , \quad (\text{A.10})$$

which as a contravariant four-vector, namely

$$p_\mu p^\mu := -\partial^2 = -\frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x^\mu} . \quad (\text{A.11})$$

Now in our conventions, we defined the Fourier transform as

$$f(x, y) = \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot (x-y)} f(p) . \quad (\text{A.12})$$

A.2 Dirac Matrices

On the other hand, we introduce now the Dirac matrices, which are indispensable to describe particles with spin.

The Dirac matrices are defined by the Clifford Algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\mathbf{1}_{4 \times 4} g^{\mu\nu} , \quad (\text{A.13})$$

where $\mathbf{1}_{4 \times 4}$ is the 4×4 identity matrix. The common $\times 4$ representation of Dirac matrices is

$$\boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\tau} \\ -\boldsymbol{\tau} & 0 \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} \mathbf{1}_{2 \times 2} & 0 \\ 0 & -\mathbf{1}_{2 \times 2} \end{pmatrix}, \quad (\text{A.14})$$

where $\mathbf{1}_{2 \times 2}$ is the 2×2 identity matrix and $\boldsymbol{\tau} = (\tau^1, \tau^2, \tau^3)$ are the Pauli matrices, defined as

$$\tau^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (\text{A.15})$$

One can see easily that the gamma matrices have the properties

$$\gamma_0^\dagger = \gamma_0, \quad \boldsymbol{\gamma}^\dagger = -\boldsymbol{\gamma} . \quad (\text{A.16})$$

Now we introduce

$$\gamma^5 = \gamma_5 = i \gamma^1 \gamma^2 \gamma^3 \gamma^4 = \frac{i}{4!} \epsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma , \quad (\text{A.17})$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the completely antisymmetric Levi-Civita tensor in four dimensions.

In the Pauli-Dirac representation, γ_5 is given by

$$\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \quad (\text{A.18})$$

Some of its properties are

$$\{\gamma_5, \gamma^\mu\} = 0 , \quad \gamma_5^\dagger = \gamma_5 , \quad (\text{A.19})$$

$$[\gamma_5, \sigma^{\mu\nu}] = 0 , \quad (\text{A.20})$$

where

$$\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu] . \quad (\text{A.21})$$

The following identities are important in evaluating the cross-sections for decay and scattering processes

$$\text{Tr}[\gamma_5] = 0 , \quad (\text{A.22})$$

$$\text{Tr}[\mathbf{1}] = 0 , \quad (\text{A.23})$$

$$\text{Tr}[\not{a}\not{b}] = 4(a \cdot b) , \quad (\text{A.24})$$

$$\begin{aligned} \text{Tr}[\not{a}_1 \not{a}_2 \not{a}_3 \not{a}_4] &= 4[(a_1 \cdot a_2)(a_3 \cdot a_4) - \\ &\quad (a_1 \cdot a_3)(a_2 \cdot a_4) + (a_1 \cdot a_4)(a_2 \cdot a_3)] , \end{aligned} \quad (\text{A.25})$$

$$\text{Tr}[\not{a}_1 \cdots \not{a}_n] = 0 , \quad \text{for } n \text{ odd} , \quad (\text{A.26})$$

$$\text{Tr}[\gamma_5 \not{a}\not{b}] = 0 , \quad (\text{A.27})$$

$$\text{Tr}[\gamma_5 \not{a}_1 \not{a}_2 \not{a}_3 \not{a}_4] = 4i \epsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma , \quad (\text{A.28})$$

$$\gamma_\mu \not{a} \gamma^\mu = -2\not{a} , \quad (\text{A.29})$$

$$\gamma_\mu \not{a} \not{b} \gamma^\mu = 4(a \cdot b) , \quad (\text{A.30})$$

$$\gamma_\mu \not{a} \not{b} \not{c} \gamma^\mu = -2\not{c} \not{a} \not{b} , \quad (\text{A.31})$$

here we have used the “slash” notation, $\not{a} = \gamma^\mu a_\mu$.

A.3 Euclidean Space Conventions

In Euclidean space we define the Fourier transform as

$$f(x, y) = \int \frac{d^4 p}{(2\pi)^4} e^{ip \cdot (x-y)} f(p) . \quad (\text{A.32})$$

Now in order to go from Minkowski space to Euclidean space we consider the free fermion field. The action for free Dirac fields is

$$S_0^\psi = \int d^4 x \bar{\psi}(x) (i\not{\partial} - m + i\eta) \psi(x) \quad (\text{A.33})$$

$$= \int_{-\infty}^{\infty} dt \int d^3 x \bar{\psi}(x) (i\not{\partial} - m + i\eta) \psi(x) . \quad (\text{A.34})$$

First we make a change of variables and introduce a Euclidean time: $t \rightarrow -it^E$ (Wick rotation). In order to apply this we must first see its effect on $i\not{\partial}$ and $\not{A}$, namely

$$\begin{aligned} i\not{\partial} &= ig_{\mu\nu} \gamma^\mu \partial^\nu = i\gamma^0 \frac{\partial}{\partial t} + i\gamma^i \frac{\partial}{\partial x^i} \xrightarrow{M \rightarrow E} i\gamma^0 \frac{\partial}{\partial(-it^E)} + i\gamma^i \frac{\partial}{\partial x^i} \\ &= -\gamma^0 \frac{\partial}{\partial(t^E)} - (-i\gamma^i) \frac{\partial}{\partial x^i} \\ &= -\gamma_\mu^E \partial^E , \end{aligned} \quad (\text{A.35})$$

$$\not{A} = g_{\mu\nu} \gamma^\mu A^\nu = \gamma^0 A^0 - \gamma^i A^i \xrightarrow{M \rightarrow E} -i\gamma^E \cdot A^E \quad (\text{A.36})$$

where we can defined the Euclidean Dirac matrices as

$$\gamma_4^E = \gamma^0 ; \quad \gamma_i^E = -i\gamma^i , \quad i = 1, 2, 3. \quad (\text{A.37})$$

These matrices are hermitian and satisfy the algebra

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} , \quad (\text{A.38})$$

where $\delta_{\mu\nu}$ is the Kronecker delta. For four-vectors A, B we define

$$A \cdot B = \delta_{\mu\nu} A_\mu B_\nu = \sum_{i=1}^4 A_i B_i , \quad (\text{A.39})$$

we have also

$$\gamma_5 = -\gamma_1 \gamma_2 \gamma_3 \gamma_4 ; \quad (\text{A.40})$$

$$\text{tr}[\gamma_5 \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma] = -4 \epsilon_{\mu\nu\rho\sigma} . \quad (\text{A.41})$$

In Eucliden the Dirac matrices have the form

$$\gamma = \begin{pmatrix} 0 & -i\vec{\tau} \\ i\vec{\tau} & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} \tau^0 & 0 \\ 0 & -\tau^0 \end{pmatrix}. \quad (\text{A.42})$$

Appendix B

EUCLIDEAN QCD

Using the conventions defined in Appendix A, is straightforward to find that the generating functional of QCD in Euclidean space is

$$Z[\bar{\eta}, \eta, J_\mu, \bar{\zeta}, \zeta] = \int \mathcal{D}(\bar{\psi}, \psi, A, \bar{\omega}, \omega) \exp \left[-S[\bar{\psi}, \psi, A_\mu] - S_{gh}[\bar{\omega}, \omega, A] \right. \\ \left. + \int d^4x (\bar{\psi}\eta + \bar{\eta}\psi + A_\mu^a J_\mu^a + \bar{\omega}\zeta + \bar{\zeta}\omega) \right], \quad (\text{B.1})$$

where

$$S[\bar{\psi}, \psi, A_\mu] = \int d^4x [\bar{\psi}(x) (\gamma \cdot \partial + m - ig t^a \gamma \cdot A^a) \psi(x) \\ + \frac{1}{4} F_{\mu\nu}^a(x) F_{\mu\nu}^a(x) + \frac{1}{2\xi} \partial \cdot A^a(x) \partial \cdot A^a(x)] \quad (\text{B.2})$$

$$S_{gh}[\bar{\omega}, \omega, A] = \int d^4x (-\partial_\mu \bar{\omega}^a \partial_\mu \bar{\omega}^a - gf^{abc} \partial_\mu \bar{\omega}^a \omega^b A_\mu^c) \quad (\text{B.3})$$

In Eq. (B.2) $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc} A_\mu^b A_\nu^c$. Note that, in order to maintain gauge invariance and unitarity in covariant gauges, we have introduced the ghost fields $\bar{\omega}, \omega$. Then, the QCD Feynman rules in Euclidean are given by

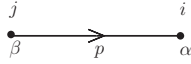
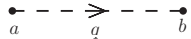
	$S_{\alpha\beta}(p) = \left(\frac{\delta^{ij}}{i\gamma \cdot p + m} \right)_{\alpha\beta}$
	$D_{\mu\nu}^{ab}(k) = \frac{\delta^{ab}}{k^2} \left[\delta_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{k^2} \right]$
	$D^{ab}(q) = -\frac{\delta^{ab}}{q^2}$

Figure B.1: Quark, gluon and ghost propagators.

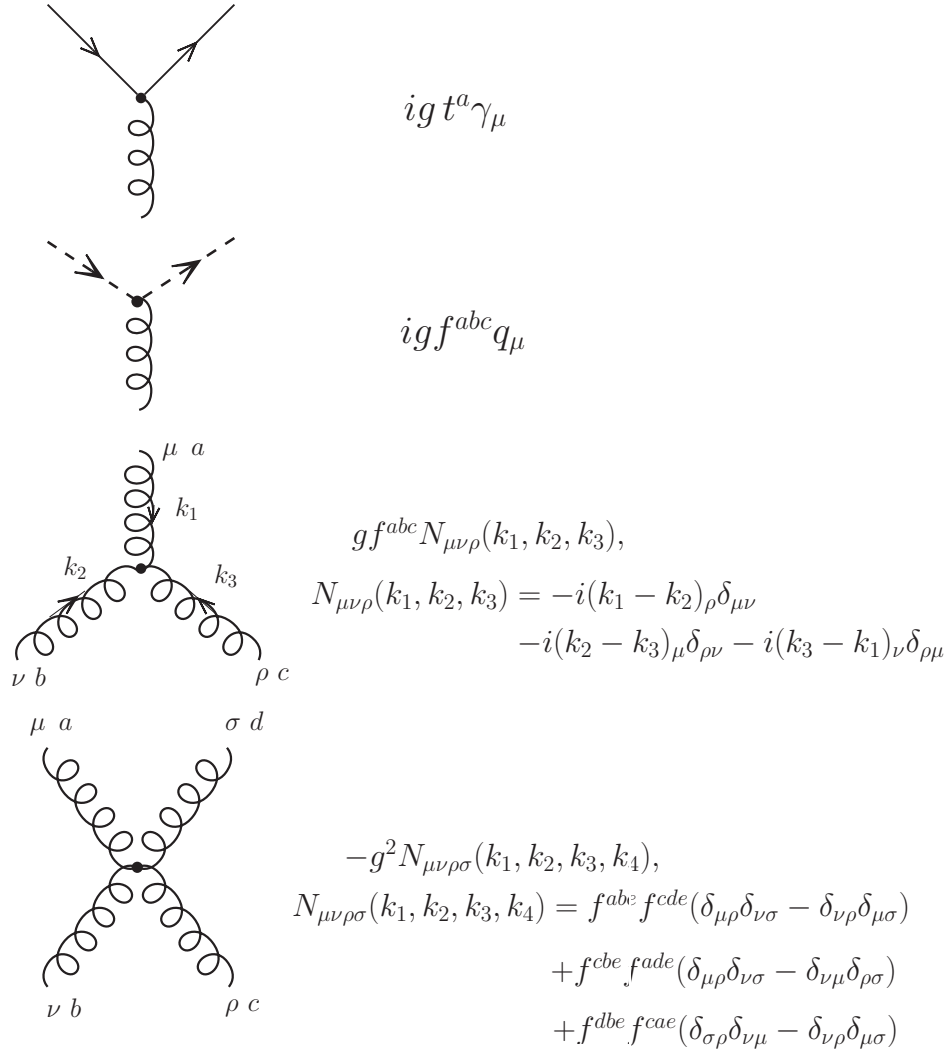


Figure B.2: QCD Feynman rules in Euclidean space.

Appendix C

PROPER TIME REGULARIZATION

The regularization in proper time is very simple. This consist in rewrite all the propagators as Gaussian integrals. For simplicity we consider the tadpole diagram (ϕ^4 -theory)

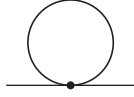


Figure C.1: Tadpole Diagram.

Its integral expression is

$$I = \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + m^2} . \quad (\text{C.1})$$

We now write

$$\frac{1}{q^2 + m^2} = \int_0^\infty d\alpha e^{-\alpha(q^2 + m^2)} , \quad (\text{C.2})$$

where the variable α is a regularizator and it is called proper time. So that (C.1) can be rewrite as

$$I = \int \frac{d^D q}{(2\pi)^D} \int_0^\infty d\alpha e^{-\alpha(q^2 + m^2)} . \quad (\text{C.3})$$

Now the integral over the proper time can be exchanged with the momentum integral, leading to

$$I = \int_0^\infty d\alpha e^{-\alpha m^2} \int \frac{d^D q}{(2\pi)^D} e^{-\alpha q^2} . \quad (\text{C.4})$$

The D-dimensional Gaussian integral on the right-hand side can easily be done and yields

$$\int \frac{d^D q}{(2\pi)^D} e^{-\alpha q^2} = \left(\int \frac{dq}{(2\pi)^D} e^{-\alpha q^2} \right)^D = \left(\frac{1}{4\pi\alpha} \right)^{D/2} , \quad (\text{C.5})$$

so that (C.4) becomes

$$I = \frac{1}{(4\pi)^{D/2}} \int_0^\infty d\alpha \frac{e^{-\alpha m^2}}{\alpha^{D/2}} = \frac{(m^2)^{D/2-1}}{(4\pi)^{D/2}} \Gamma(1 - D/2) \quad (\text{C.6})$$

where we have used the integral definition of the Gamma function

$$\Gamma(z) = \int_0^\infty d\alpha \alpha^{z-1} e^{-\alpha} . \quad (\text{C.7})$$

Taking derivatives $(\partial/\partial m^2)^{n-1}$ on both sides of I , we obtain in general

$$I = \int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2 + m^2)^n} = \frac{\Gamma(1 - D/2)}{(4\pi)^n \Gamma(n)} \left(\frac{m^2}{4\pi} \right)^{\frac{D}{2} - n} . \quad (\text{C.8})$$

Appendix D

The Ward-Takahashi Identity

D.1 Ward-Takahashi Identity in QED

The Ward-Takahashi Identity play an importante role in the study of gauge theories, especially in the implementation of consitent non-perturbative truncations of the corresponding DSE.

Using the path integral formalism we can obtain the called Ward-Takahashi Identity (WTI)[83,84]. We start with the generating functional (3.24)

$$Z[\bar{\eta}, \eta, J] = \int \mathcal{D}(\bar{\psi}, \psi, A) \exp \left(-S[\bar{\psi}, \psi, A_\mu] + \int d^4x [\bar{\psi}\eta + \bar{\eta}\psi + A_\mu J^\mu] \right), \quad (\text{D.1})$$

being S the QED action (3.25). Let us to consider an infinitesimal local gauge invariance transformation

$$\begin{aligned} \delta\psi(x) &= ie\epsilon(x)\psi(x) \\ \delta A_\mu(x) &= \partial_\mu\epsilon(x). \end{aligned} \quad (\text{D.2})$$

If we consider this as just a change of integration variables, the path integral should not change. On the other hand, by varying the functional, the gauge fixing term $(\partial_\mu A^\mu)^2$ and the source terms will change, so to first order in $\epsilon(x)$ one has

$$0 = \delta Z = \int \mathcal{D}(\bar{\psi}, \psi, A) e^{-S} \int d^4x \left[ie\epsilon(\bar{\eta}\psi - \bar{\psi}\eta) + (\partial_\mu\epsilon)J_\mu - \frac{1}{\xi}(\partial_\mu A^\mu)\partial^2\epsilon \right], \quad (\text{D.3})$$

or performing the functional derivative with respecto to $\epsilon(x)$

$$\left[-\partial_\mu J_\mu + ie \left(\bar{\eta} \frac{\delta}{\delta \bar{\eta}} - \eta \frac{\delta}{\delta \eta} \right) - \frac{1}{\xi} \partial^2 \partial_\mu \frac{\delta}{\delta J_\mu} \right] Z[\bar{\eta}, \eta, J] = 0. \quad (\text{D.4})$$

Now we can rewrite this in terms of 1PI Green's functions Γ

$$\partial_\mu \frac{\delta \Gamma}{\delta A_\mu} + ie \left(\frac{\delta \Gamma}{\delta \psi} \psi + \bar{\psi} \frac{\delta \Gamma}{\delta \bar{\psi}} \right) + \frac{1}{\xi} \partial^2 \partial_\mu A_\mu = 0. \quad (\text{D.5})$$

By taking functional derivatives with respect to ψ and $\bar{\psi}$, and setting the field A , ψ and $\bar{\psi}$ equal to zero, we obtain

$$\delta_x^\mu \frac{\delta^3 \Gamma}{\delta \bar{\psi}(z) \delta \psi(y) \delta A_\mu(x)} = ie \left[-\delta^4(x-y) \frac{\delta^2 \Gamma}{\delta \bar{\psi}(z) \delta \psi(x)} + \delta^4(x-z) \frac{\delta^2 \Gamma}{\delta \bar{\psi}(y) \delta \psi(x)} \right]$$

$$\delta_x^\mu \Gamma_\mu(x, y; z) = ie [\delta^4(x-y) S^{-1}(x-z) - \delta^4(x-z) S^{-1}(x-y)] , \quad (\text{D.6})$$

which in momentum space this relation becomes

$$k^\mu \Gamma_\mu(k; q_1, q_2) = ie [S^{-1}(q_2) - S^{-1}(q_1)] , \quad (\text{D.7})$$

which is the Ward-Takahashi identity.

On the other hand, if we take the functional derivative with respect to A_μ and again setting A , ψ and $\bar{\psi}$ equal to zero, we have

$$\begin{aligned} \frac{\delta}{\delta x_\mu} \frac{\delta^2}{\delta A_\mu(x) \delta A_\nu(y)} \Big|_{A=\psi=\bar{\psi}=0} + \frac{1}{\xi} \frac{\partial^2}{\partial x_\mu^2} \frac{\partial}{\partial x_\nu} \delta^4(x-y) &= 0 \\ \frac{\delta}{\delta x_\mu} D_{\mu\nu}^{-1}(x-y) + \frac{1}{\xi} \frac{\partial^2}{\partial x_\mu^2} \frac{\partial}{\partial x_\nu} \delta^4(x-y) &= 0 , \end{aligned} \quad (\text{D.8})$$

where $D_{\mu\nu}^{-1}$ is the inverse of the photon propagator. Taking the Fourier transform, we obtain

$$k^\mu \left[\frac{k^2}{\xi} \delta_{\mu\nu} - D_{\mu\nu}^{-1}(k) \right] = 0 . \quad (\text{D.9})$$

On the other hand, we have

$$D_{\mu\nu}^{-1}(k) = (D_0)_{\mu\nu}^{-1}(k) + \Pi_{\mu\nu}(k) , \quad (\text{D.10})$$

with

$$(D_0)_{\mu\nu}^{-1}(k) = k^2 \delta_{\mu\nu} - \left(1 - \frac{1}{\xi} \right) k_\mu k_\nu , \quad (\text{D.11})$$

then, we have

$$k^\mu (D_0)_{\mu\nu}^{-1}(k) = \frac{k^2}{\xi} k_\nu . \quad (\text{D.12})$$

Replacing it in (D.9) , we obtain

$$k^\mu \Pi_{\mu\nu}(k) = 0 . \quad (\text{D.13})$$

That is, the 1PI corrections to the photon propagator are transverse. That means that we can write them in terms of a single scalar function, namely

$$\Pi_{\mu\nu}(k^2) = (\delta_{\mu\nu} k^2 - k_\mu k_\nu) \Pi(k^2) . \quad (\text{D.14})$$

D.2 Slavnov-Taylor Identities

The Slavnov-Taylor identities (STI) are a non-Abelian generalization of the WTI and are a consequences of local gauge invariance [85,86]. The STI for the gluon propagator can be written as

$$k^\mu \Pi_{\mu\nu}^{ab}(k) = 0 , \quad (\text{D.15})$$

where $\Pi_{\mu\nu}^{ab}(k) = \delta^{ab} \Pi_{\mu\nu}(k)$, with

$$\Pi_{\mu\nu}(k^2) = (\delta_{\mu\nu} k^2 - k_\mu k_\nu) \Pi(k^2) . \quad (\text{D.16})$$

Equivalently, we can write [87]

$$k^\mu k^\nu D_{\mu\nu}^{ab}(k) = \xi \delta^{ab} . \quad (\text{D.17})$$

On the other hand, in the chiral limit, QCD is a chirally symmetric theory and has a STI for the proper vertex

$$-ik^\mu \Gamma_{5\mu}(p', p) = S^{-1}(p') \gamma_5 + \gamma_5 S^{-1}(p) . \quad (\text{D.18})$$

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