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$\mathbb{D}_n$-forced symmetry breaking of $\mathbb{O}(2)$-equivariant problems

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We use singularity theory to classify forced symmetry-breaking bifurcation problems

$$f(z, \lambda, \mu) = f_1(z, \lambda) + \mu f_2(z, \lambda, \mu) = 0,$$

where f_1 is $\mathbb{O}(2)$-equivariant and f_2 is $\mathbb{D}_n$-equivariant with the orthogonal group actions on $z \in \mathbb{R}^2$. Forced symmetry breaking occurs when the symmetry of the equation changes when parameters are varied. We explicitly apply our results to the branching of subharmonic solutions in a model periodic perturbation of an autonomous equation and sketch further applications.

1. Introduction

In a previous work [6], we developed a singularity theory approach for the study of the bifurcation equation used, for instance, to find periodic solutions of

$$\ddot{\theta} + g(\theta, \lambda)\theta + \mu h(t, \theta, \lambda, \mu) = 0, \tag{1.1}$$

where h is a non-autonomous T-periodic perturbation. The bifurcation problem corresponds to *forced symmetry breaking* in $\mathbb{O}(2)$-equivariant equations when the symmetry of the equation changes when the parameter μ varies. This is to be contrasted with *spontaneous symmetry breaking* when it is the symmetry of solutions that changes even if the symmetry of the equation is kept constant.

In this work we study bifurcation problems when the symmetry breaking is from $\mathbb{O}(2)$ to one of its compact subgroups $\mathbb{D}_n$. The forcing in (1.1) by an even T/n-periodic function h is a typical example when the linear mode of the autonomous part is of period T (cf. [8]). In § 3, we apply our abstract results to the existence of periodic orbits of the model ordinary differential equation

$$\ddot{\theta} + \theta + g(\theta, \lambda)\theta + \mu h(t) = 0, \tag{1.2}$$

where h is $2\pi/n$-periodic and g, h may have some additional symmetry properties. We extend some results of [8] and of previous papers in its references. In particular, we show that the important coefficients of the bifurcation equation are products of some nonlinear combination of the derivatives of g and of some integral of h. Using singularity theory, we can study systematically what happens when a generic coefficient is zero. We use a Lyapunov–Schmid reduction to get the bifurcation equation (details are in the appendix),

$$f(z, \lambda, \mu) = f_1(z, \lambda) + \mu f_2(z, \lambda, \mu), \tag{1.3}$$

where f_1 is $\mathbb{O}(2)$-equivariant in $z \in \mathbb{R}^2$ and f_2 is only $\mathbb{D}_n$-equivariant. We also discuss briefly some other applications. In § 2 we give the classification of the generic and topological codimension-1 bifurcation map germs and analyse their bifurcation diagrams and symmetry-breaking scenario in § 4. To be able to prove our classification theorem, we present in § 5 a general theory of unfoldings, finite determinacy and the recognition problem for bifurcation map germs of the type of (1.3). The proofs and calculations are made explicit in § 6.

1.1. Singularity theory

Singularity theory is about the systematic study of nonlinear maps using groups of change of coordinates that preserve their qualitative features, here the equivariance and the zero-sets. We use a group of changes of coordinates $\mathcal{K}_{(\lambda,\mu)}^{\mathbb{O}(2),\mathbb{D}_n}$ that are $\mathbb{O}(2)$-equivariant when $\mu = 0$, but only $\mathbb{D}_n$-equivariant when $\mu \neq 0$ (cf. § 5.2.1). The essential ideas of the theory were advanced in [10], but not much action was taken on it, the main theory drifting instead towards the very rich field of spontaneous symmetry breaking. The double equivariant structure of $\mathcal{K}_{(\lambda,\mu)}^{\mathbb{O}(2),\mathbb{D}_n}$ is transported to the tangent spaces. They are modules over the $\mathbb{O}(2)$-equivariant functions when $\mu = 0$ and over the $\mathbb{D}_n$-invariant functions when $\mu \neq 0$. This is a somewhat unusual case. But, fundamentally, in his general framework, Damon did define in [2] the necessary extended concepts to deal with this new situation. The first point is to make sure that a version of the preparation theorem applies, so we could work with the algebraic structure of the tangent spaces. For the usual theory ($\mathcal{K}_\lambda^\Gamma$), the ring of invariant functions has a structure of differentiable (DA)-algebra (and so the preparation theorem holds true). When Γ is a continuous Lie group, the ring of symmetry-breaking invariant functions is not a DA-algebra, but Damon in [2] showed that the main properties of DA-algebras can be extended to this situation of so-called *extended DA-algebras*. And so $\mathcal{K}_{(\lambda,\mu)}^{\mathbb{O}(2),\mathbb{D}_n}$ is a geometric subgroup of contact equivalences that satisfies the abstract theorems of [2]. We thus get the theories for universal unfoldings and determinacy with estimates of the higher-order terms $\mathcal{P}(f)$, terms we can discard in the contact class of the normal form of f.

We use extensively *topological* contact equivalence (contact equivalence with continuous changes of coordinates, cf. [4]) because it is efficient to use in our context. The smooth classification can generate many moduli that are irrelevant for the study of both of the bifurcation diagrams and their universal unfoldings. We recall in § 5.5 the principal results we need for it. The use of topological equivalence is made simpler by the fact that the terms in our germs of the lowest degree in the filtrations have an homogeneous structure and are of finite codimension.

Table 1.

case	universal unfolding/normal form	C^∞-cod
I_1^n	$(\epsilon_1 u + \delta_1\lambda)z + \mu(\alpha + av + \delta_4\lambda + \delta_5\mu)\bar{z}^{n-1}$	2
II_1^n	$(\epsilon_1 u + \delta_2\lambda^2 + \delta_6\mu + \alpha)z + \mu\epsilon\bar{z}^{n-1}$	1
III_1^3	$(\epsilon_2 u^2 + \alpha u + \delta_1\lambda + \xi_0\mu u + \xi_1\mu v + \xi_2\mu^2 u)z + \mu\epsilon\bar{z}^2$	2–4
III_1^n	$(\epsilon_2 u^2 + \alpha u + \delta_1\lambda + d\mu u + \xi_1\mu v + \xi_2\mu^2 u)z + \mu\epsilon\bar{z}^{n-1}$	2–4

Table 2.

case	universal unfolding/normal form	C^∞-cod
I_1^4	$(\epsilon_1 u + \delta_1\lambda)z + \mu(\alpha + a\Delta + \delta_4\lambda + \delta_5\mu)\delta\bar{z}$	2
II_1^4	$(\epsilon_1 u + \delta_2\lambda^2 + \delta_6\mu + \alpha)z + \mu\epsilon\delta\bar{z}$	1
III_1^4	$(\epsilon_2 u^2 + \alpha u + \delta_1\lambda + c\mu u + \xi_3\mu\Delta + \xi_2\mu^2 u)z + \mu\epsilon\delta\bar{z}$	3–4

2. Classification theorem

The state variable is $z = (x, y) \in \mathbb{R}^2$ and the distinguished bifurcation parameters are $\lambda, \mu \in \mathbb{R}$. We identify $\mathbb{R}^2$ with $\mathbb{C}$ using $(x, y) \mapsto z = x + \mathrm{i}y$. The derivatives are denoted by subscripts, f_x for $(\partial f/\partial x)\dots$, and the superscript 'o' denotes the value of any function at the origin, $f^{\mathrm{o}} = f(0)$, $f_x^{\mathrm{o}} = f_x(0)$, etc. We consider the usual orthogonal action of $\mathbb{O}(2)$ on the plane, generated by $\theta : z \mapsto \mathrm{e}^{\mathrm{i}\theta}z$, $\theta \in \mathbb{S}^1$, and $\kappa : z \mapsto \bar{z}$. The dihedral subgroup $\mathbb{D}_n \subset \mathbb{O}(2)$ acts as the restriction of the action of $\mathbb{O}(2)$.

From [11, 15], any map in (1.3) can be written uniquely as

$$f(z, \lambda, \mu) = [p(u, \lambda) + \mu r(u, v, \lambda, \mu)]z + \mu s(u, v, \lambda, \mu)\bar{z}^{n-1}, \tag{2.1}$$

where $\bar{u}(z) = z\bar{z}$ and $\bar{v} = \frac{1}{2}(z^n + \bar{z}^n)$. For $n = 4$, we use different coordinates that have been widely used in the literature. Let $\delta = -\frac{1}{2}(z^2 + \bar{z}^2)$ and $\Delta = \delta^2$, and so

$$f(z, \lambda, \mu) = [p(u, \lambda) + \mu r(u, \Delta, \lambda, \mu)]z + \mu s(u, \Delta, \lambda, \mu)\delta\bar{z}. \tag{2.2}$$

The smooth classification involve many moduli, parameters that are invariant under smooth change of coordinates but not under continuous ones, and so becomes rapidly complicated (cf. [6], for instance). But the topological type of a germ *and its universal unfolding* is the main information of practical importance for their study. Here we classify them up to topological codimension 1.

THEOREM 2.1.

(a) (*Generic normal form.*) *The generic case* I_0^3 *is given by*

$$f(z, \lambda, \mu) = \begin{cases} (\epsilon_1 u + \delta_1\lambda)z + \mu\epsilon\bar{z}^{n-1}, & n \neq 4, \\ (\epsilon_1 u + \delta_1\lambda)z + \mu\epsilon\delta\bar{z}, & n = 4, \end{cases} \tag{2.3}$$

where $\epsilon = \operatorname{sgn} s^{\mathrm{o}}$. *The smooth codimension of* f *is equal to 0.*

(b) *The diagrams of topological codimension 1 are given in tables 1 and 2 for* $n \neq 4$ *and* $n = 4$, *respectively.*

The coefficients ϵ, ϵ_i, δ_i are ± 1 and the modal parameters satisfy $a, d \neq 0$ and $c \neq \epsilon, 0$. The coefficients ξ_i, $0 \leqslant i \leqslant 3$, are topologically irrelevant, but they determine the smooth codimension. In the name of the cases, the Roman numeral corresponds to the classification of the $\mathbb{O}(2)$-equivariant part p (cf. § 6.1.1), the subscript to the topological codimension of f and the superscript to the dihedral group.

Proof. The proof is to be found in § 6.4. □

3. Periodic perturbation of autonomous equations

We study the bifurcation from the trivial branch of 2π-periodic solutions of

$$F(\theta, \lambda, \mu, t) = \ddot{\theta} + \theta + g(\theta, \lambda)\theta + \mu h(t) = 0, \tag{3.1}$$

where h is $2\pi/n$-periodic in t and $g(0, 0) = 0$. To avoid unnecessary details, we have assumed that the origin is the trivial solution for all λ and that the period of h is minimal. Hence the kernel of the linearization at 0 is generated by $\cos t$ and $\sin t$. We derive a reduced bifurcation equation f, given in (A 4), via the classical Lyapunov–Schmidt method, as described in the appendix.

3.1. Symmetries

When $\mu = 0$, F is $\mathbb{O}(2)$-equivariant with the action generated by the phase shifts $(\psi\theta)(t) = \theta(t + \psi)$, $\psi \in [0, 2\pi)$, and the reversor $(\kappa\theta)(t) = \theta(-t)$. Usually, when $\mu \neq 0$, only the discrete phase shifts $2k\pi/n$, $0 \leqslant k \leqslant n - 1$, remain, leading to a forced symmetry breaking from $\mathbb{O}(2)$ to $\mathbb{Z}_n$. There are cases when a reflection symmetry is kept in (3.1). For instance, when h is an even function of t, κ will clearly remain a symmetry of (3.1). On the other hand, when h is an odd function of t and g is *even* in θ, $(\kappa_2\theta)(t) = -\theta(-t)$ will now act as a reversor. In both cases, the reversor and the phase shifts form a semi-direct product isomorphic to $\mathbb{D}_n$. When g is even in θ, we have an additional action of $\mathbb{Z}_2$ via $(\kappa_3\theta)(t) = -\theta(t)$ if we also act on the parameter μ, that is, $F(\kappa_3\theta, \lambda, -\mu, t) = \kappa_3 F(\theta, \lambda, \mu, t)$. This copy of $\mathbb{Z}_2$ commutes with the previous symmetries generating $\mathbb{D}_n$.

3.2. Previous work

Building from specific examples in [12, 13], it was shown in [14] that when $n = 1$ and

$$\int_0^{2\pi} h(t) \cos t \, \mathrm{d}t \neq 0,$$

all small bifurcating 2π-periodic solutions of (3.1) are still even in t. When $n \geqslant 2$, a similar result has been shown: when a certain coefficient ρ is non-zero (a generic condition), all the small 2π-periodic solutions are even in t. In terms of symmetries, this means that if $\rho \neq 0$, the only small solutions of (3.1) are $\mathbb{Z}_2$-invariant. As a consequence of lemma 5.2 of [9], $\rho \neq 0$ if and only if $s^{\circ} \neq 0$. In that case, it is clear that there cannot be branches of solutions with trivial isotropy even in any unfolding (perturbation). In [8], the special case when h is *odd-harmonic*, that is, $h(t + \pi/n) = -h(t)$, has been investigated. When n is even, h is actually of period

π, and so the theory works for solutions of half the period. Hence the problem is $\mathbb{D}_{2n}$-equivariant, not $\mathbb{D}_n$, which explains why one finds generically twice the number of branches. Moreover, when g is even, the additional action of $\mathbb{Z}_2$ induced by κ_3 means that there is a combined action on state and parameter space, that is, the bifurcation equation f will depend on μ^2. It is straightforward to adapt the theory of parameter symmetry developed in [5] to this situation.

3.3. Normal forms

The coefficients we need to monitor for the $\mathbb{O}(2)$-equivariant part of f are more or less well known. For example, $\delta_i = \operatorname{sgn} g^{\circ}_{\lambda^i}$, $i = 1, 2$, and $\epsilon_1 = \operatorname{sgn}(9g^{\circ}_{\theta\theta} - 20{g^{\circ}_{\theta}}^2)$. The coefficients of the bifurcation equations corresponding to the more degenerate cases of theorem 2.1 are the product of a term that depends only on some derivatives of g and a term that depends only some Fourier coefficients of h.

PROPOSITION 3.1.

(a) *The coefficient* s_0 *is equal to*

$$S_n\left(\int_0^{2\pi} \cos(nt)h(t)\,\mathrm{d}t\right),$$

where S_n *is some nonlinear combination of* $g^{\circ}_{\theta^j}$, $1 \leqslant j \leqslant n-1$. *When* $n = 3$, *for instance,* S_3 *is a non-zero multiple of* $3g^{\circ}_{\theta\theta} + 4{g^{\circ}_{\theta}}^2$.

(b) *The parameter* δ_4 *is equal to the sign of*

$$E_n\left(\int_0^{2\pi} \cos(nt)h(t)\,\mathrm{d}t\right),$$

where E_n *is some nonlinear combination of derivatives of* g.

(c) *The parameter* δ_5 *is equal to the sign of*

$$F_n\left(\int_0^{2\pi} \cos(nt)h(t)^2\,\mathrm{d}t\right),$$

where F_n *is some nonlinear combination of derivatives of* g.

(d) *The modal parameter* a *is equal to*

$$A_n\left(\int_0^{2\pi} \cos(2nt)h(t)\,\mathrm{d}t\right),$$

where A_n *is some nonlinear combination of derivatives of* g.

(e) *The modal parameters* c *(respectively,* d*) are equal to the product of* C_n *(respectively,* D_n*), with* $\int_0^{2\pi} h(t)\,\mathrm{d}t$, *where* C_n *and* D_n *are some nonlinear combination of derivatives of* g.

Proof. We shall only show how to get part (a). The other cases follow a similar line of proof. Note that s_0 is the coefficient of $\mu\bar{z}^{n-1}$ in f and so we only need to evaluate the powers of $\bar{z}$ in

$$\partial_\mu \mathit{\Pi}\biggl[\sum_{k=1}^{n-1}\biggl(\sum_{|l|=n-1} c_k^{(l)} g^{\mathrm{o}}_{\theta^k} w^{(l)}\chi(z)^{n-1}\biggr)\biggr],$$

where $(l)=\{l_1,\dots,l_k\}$, $l_j\geqslant 1$, $l_1+\cdots+l_k=n-1$, the $c_k^{(l)}$ are some real coefficients and $w^1(\chi(z))=\chi(z)$, with $w^{(l)}=w^{l_1}\chi(z)^{l_1}\cdots w^{l_k}\chi(z)^{l_k}$. We evaluate

$$\mathit{\Pi}\biggl[\sum_{k=1}^{n-1}\biggl(\sum_{|l|=n-1} c_k^{(l)} g^{\mathrm{o}}_{\theta^{k+1}} w_v^{(l)}\chi(z)^{n-1}w_\mu\biggr)\biggr]+\mathit{\Pi}\biggl[\sum_{(l,j)\in L_k^{n-1}} d_k^{(l,j)} g^{\mathrm{o}}_{\theta^k} w_{v\mu}^{(l,j)}\chi(z)^{n-1}\biggr],$$

where $(l,j)=\{l_1,\dots,l_k,j_1,\dots,j_k\}$,

$$w_{u\mu}^{(l,j)}\chi(z)^{n-1}=w_{u^{l_1}\mu^{j_1}}\chi(z)^{l_1}w_{u^{l_k}\mu^{j_k}}\chi(z)^{l_k},$$

the $d_k^{(l,j)}$ are some real coefficients and

$$L_k^{n-1}=\biggl\{(l,j)\biggm|\sum_{m=1}^{k} j_m=1,\ \sum_{m=1}^{k} l_m=n-1,\ l_m,j_m\geqslant 0,\ l_m+j_m\geqslant 1\biggr\}.$$

The terms in $\bar{z}$ of the derivatives $w_{v^j}\chi(z)^j$ are equal to $a_j\mathrm{e}^{\mathrm{i}jt}\bar{z}^j$ for some $a_j\in\mathbb{R}$, depending on the derivatives of g in θ. And so the relevant terms in s^{o} become

$$\mathit{\Pi}\biggl[A\mathrm{e}^{\mathrm{i}(n-1)t}w_\mu\bar{z}^{n-1}+\sum_{k=1}^{n-1}A_k\mathrm{e}^{\mathrm{i}(n-1-k)t}\bar{z}^{(n-1-k)}w_{v^k\mu}\chi(z)^k\biggr]$$

for some $A,A_k\in\mathbb{R}$. We need to evaluate

$$\int_0^{2\pi}\mathrm{e}^{\mathrm{i}(n-k)t}w_{v^k\mu}\chi(z)^{(n-1-k)}\,\mathrm{d}t\quad\text{for } 1\leqslant k\leqslant n-1.$$

The equation satisfied by $w_{v^k\mu}$ is (for some $b_j\in\mathbb{R}$)

$$\mathcal{L}w_{v^k\mu}\chi(z)^k+\sum_{j=0}^{k-1}b_jQ\mathrm{e}^{\mathrm{i}(k-j)t}\bar{z}^{k-j}w_{v^j\mu}\chi(z)^j=0.$$

Multiplying this equation by $\mathrm{e}^{\mathrm{i}(n-k)t}$ and integrating, we get

$$\int_0^{2\pi}\mathrm{e}^{\mathrm{i}(n-k)t}\mathcal{L}w_{v^k\mu}\chi(z)^k\,\mathrm{d}t+\sum_{j=0}^{k-1}b_j\int_0^{2\pi}\mathrm{e}^{\mathrm{i}(n-j)t}Qw_{v^j\mu}\chi(z)^j\,\mathrm{d}t=0.$$

Integrating by parts the first term to eliminate the double derivative, we get

$$[1-(n-k)^2]\int_0^{2\pi}\mathrm{e}^{\mathrm{i}(n-k)t}\mathcal{L}w_{v^k\mu}\chi(z)^k\,\mathrm{d}t+\sum_{j=0}^{k-1}b_j\int_0^{2\pi}\mathrm{e}^{\mathrm{i}(n-j)t}Qw_{v^j\mu}\chi(z)^j\,\mathrm{d}t=0.$$

When $k=0$,

$$\int_0^{2\pi} \mathrm{e}^{\mathrm{i}nt} w_\mu \,\mathrm{d}t = \int_0^{2\pi} \mathrm{e}^{\mathrm{i}nt} h(t)\,\mathrm{d}t.$$

By induction on k, we get that all these integrals are multiples of $\int_0^{2\pi} \mathrm{e}^{\mathrm{i}nt} h(t)\,\mathrm{d}t$. □

3.4. Explicit examples of (3.1)

In this subsection we give conditions for examples with $n=3$ of each of the cases mentioned in theorem 2.1 and illustrated in § 4. Note the special structure of the interaction between the derivatives of g and the Fourier coefficients of h.

First, the generic diagram I_0^3 occurs exactly when each of the following quantities are non-zero:

$$\epsilon_1 = \operatorname{sgn}(9g^{\mathrm{o}}_{\theta\theta} - 20 g^{\mathrm{o}2}_{\theta}), \quad \delta_1 = g^{\mathrm{o}}_{\lambda} \quad \text{and} \quad \epsilon_0 = \operatorname{sgn}\left[(3g^{\mathrm{o}}_{\theta\theta} + 4g^{\mathrm{o}2}_{\theta}) \int_0^{2\pi} h(t)\cos 3t\,\mathrm{d}t\right].$$

Of the three codimension-1 normal forms, case II_1^3 occurs when $g^{\mathrm{o}}_{\lambda} = 0$ with $\epsilon_1\epsilon_0 \neq 0$. Moreover, we need

$$\delta_2 = \operatorname{sgn}(g^{\mathrm{o}}_{\lambda\lambda}) \quad \text{and} \quad \delta_6 = -\operatorname{sgn}\left(g^{\mathrm{o}}_{\theta} \int_0^{2\pi} h(t)\,\mathrm{d}t\right)$$

to be non-zero.

For the last two examples, to simplify the calculations, we assume that g is a cubic polynomial. Then case III_1^3 follows when

$$9g^{\mathrm{o}}_{\theta\theta} = 20 g^{\mathrm{o}2}_{\theta} \quad \text{and} \quad \epsilon_2 = \operatorname{sgn}\left(\tfrac{9}{16} g^{\mathrm{o}\,2}_{\theta\theta} + \tfrac{357}{4} g_{\theta\theta} g^{\mathrm{o}2}_{\theta} - 55 g^{\mathrm{o}4}_{\theta}\right) \neq 0.$$

Introducing the condition for $\epsilon_1 = 0$, we find that $\epsilon_2 = 1$. A typical example of a nonlinearity giving rise to the universal unfolding of case III_1^3 would be $g(\theta,\alpha) = 10\theta^2 + (3+\alpha)\theta$ and h such that

$$\int_0^{2\pi} h(t)\cos 3t\,\mathrm{d}t \neq 0.$$

Finally, we get case I_1^3 with its full symmetry-breaking solutions. To get $s^{\mathrm{o}} = 0$, we cannot assume that

$$\int_0^{2\pi} h(t)\cos 3t\,\mathrm{d}t = 0,$$

because in that case δ_4 would also be 0. We assume then that $g^{\mathrm{o}}_{\theta\theta} = -\frac{4}{3} g^{\mathrm{o}2}_{\theta}$. In this case, we need that $\epsilon_1\delta_1\delta_4\delta_5 \neq 0$, where

$$\delta_5 = \operatorname{sgn}\left[g^{\mathrm{o}}_{\theta}\left(226 \int_0^{2\pi} h(t)^2 \cos 3t\,\mathrm{d}t + 13\left(\int_0^{2\pi} h(t)\cos 3t\,\mathrm{d}t\right)\left(\int_0^{2\pi} h(t)\,\mathrm{d}t\right)\right)\right]$$

and

$$\delta_4 = \operatorname{sgn}\left(g^{\mathrm{o}}_{\lambda} \int_0^{2\pi} h(t)\cos 3t\,\mathrm{d}t\right).$$

The value of δ_4 follows directly from the calculation of

$$\pi f_{\lambda\mu zz} = g^{\mathrm{o}}_{\lambda}\left(\tfrac{3}{256} g^{\mathrm{o}}_{\theta\theta} + \tfrac{11}{288} g^{\mathrm{o}2}_{\theta}\right) \int_0^{2\pi} h(t)\cos 3t\,\mathrm{d}t$$

and δ_5 from

$$\pi f_{\mu\mu zz} = \frac{-1}{96} g_\theta^{\circ}(125 g_{\theta\theta}^{\circ} + 16 g_\theta^{\circ 2}) \int_0^{2\pi} h(t)^2 \cos 3t \, \mathrm{d}t$$

after a change of coordinate in (λ, μ). A typical example would be with $g(\theta, \alpha) = 6\theta^2 - (3+\alpha)\theta$ and h such that $\delta_4\delta_5 \neq 0$.

3.5. Partial differential equations and finite-element methods with circular geometries

Examples in the literature leading to our framework have involved partial differential equations (PDEs) in circular geometries [8,14]. When $\mu = 0$, the autonomous PDE has a two-dimensional kernel at the bifurcation point and then, when $\mu \neq 0$, the nonlinearity is perturbed to a n-rotational and reflective non-autonomous function. This is basically the same situation as in (1.2). Another type of problem involves the perturbation of the domain. For instance, perturb the boundary of the unit disc to $r(\phi) = 1 + \mu h(\phi)$, where h is an even function of period $2\pi/n$. If the unperturbed PDE problem has a two-dimensional kernel with eigenfunctions without any symmetry, the perturbed bifurcation equation is of the type (1.3). We can rescale the domain to the unit disc and get a perturbed PDE in terms involving derivatives, like in the linear part of a semilinear PDE. For instance, with the Laplacian operator $\mathcal{L}v = \Delta v$, the perturbed linearization is $\mathcal{L}_0 + \mu\mathcal{L}_\mu + \cdots$, where $\mathcal{L}_0$ is the usual Laplacian in polar coordinates and

$$\mathcal{L}_\mu v = \frac{1}{r}(\ddot{h} - h)v_r + \frac{2}{r}\dot{h} v_{r\phi} - \frac{2}{r^2} h v_{\phi\phi}.$$

One could also consider another variation on the theme. Consider a thin elastic shell of revolution with $\mathbb{O}(2)$-symmetry and consider a numerical approximation by a finite-element method. There are many strategies available to carry out such a programme [16]. One is to fix a number of sectors, say n, and refine the triangulation in them, of size proportional to μ. Near an exact bifurcation from the primary branch with a full two-dimensional kernel generated by non-symmetric modes, the numerical scheme follows a bifurcation equation of type (1.3). For instance, when a path following the solution branches, the number n is usually fixed. In practice, non-symmetric connecting branches can be important because they influence the evaluation of the strength of the structure. This problem could also be considered as a domain perturbation because the numerical solutions will have support on a domain with only $\mathbb{D}_n$-symmetry. Note though that our analysis does not apply to the important problem of the numerical approximation of the spontaneous symmetry breaking from $\mathbb{O}(2)$-symmetric solution branches in complex elastic problems.

3.6. Variational problems

In [1] the bifurcation of period-n points ($n \geqslant 3$) of $\mathbb{SO}(2)$-equivariant symplectic maps in $\mathbb{R}^4$ was considered when the four multipliers of a fixed point collide at an nth root of unity with opposite signature. Those maps are of interest because discrete or continuous rotational symmetries are the only ones compatible with such bifurcation. A typical example is the time-T map of a periodically forced Lagrange top. In [1] we develop a Lagrangian approach via invariant action functionals. The families of n-cycles riding on the $\mathbb{SO}(2)$-orbits in phase space satisfy the first part of the normal form II_1^n. Perturbing the action functional by a non-$\mathbb{SO}(2)$-invariant but reversible term, we would get the full normal form II_1^n (in [1] such germs were called *collision normal forms*).

When the bifurcation equation is the gradient of a parametrized functional, this structure is not preserved by a generic contact equivalence. But contact equivalence still induces an *equivalence relation* on the set of gradient bifurcation problems and their unfoldings. A theory using a path formulation has been developed in [1] for finite groups. For $\mathbb{O}(2)$, it is not completely clear how to extend those ideas, more so to the $(\mathbb{O}(2), \mathbb{D}_n)$-symmetry breaking. In [1] we showed how to define for a general group a space of gradient deformations. This could be readily extended to our setting with a forced symmetry-breaking term. But here no more powerful theory is needed. The reasoning goes as follows. Our normal forms and their universal unfoldings are already *gradients*. It means that if any gradient bifurcation problem, $\nabla_z g(z, \lambda, \mu)$ say, satisfies the recognition conditions to be in one of our classes, then it is contact equivalent to the representative normal form, which *is* a gradient. Moreover, the universal unfolding is *also* a gradient, so any gradient deformation of $\nabla_z g$ reduces into a (gradient) deformation of the gradient normal form.

3.7. Bifurcations from orbits of solutions under perturbations

A series of previous work have been concerned with the symmetry of bifurcating solutions of (1.3). In the $\mathbb{O}(2)$-equivariant situation, one can sometimes show that, under some hypotheses, *all* bifurcating solutions have at least isotropy $\mathbb{Z}_2$ (cases I_0^3, II_1^3, III_1^3). Part of our analysis of § 3 is to look at what happens in the simplest situation when this does not work (case I_1^3). The singularity theory approach provides a framework to study and classify systematically what can happen.

4. Classification and bifurcation diagrams

4.1. Solution sets and stability

The solution set of one of the bifurcation problem can be split into several pieces. Their description follows (with their stability).

4.1.1. *Trivial solution and* $\mathbb{O}(2)$*-equivariant branches*

The trivial solution $z = 0$ exists for all values of the parameters. The determinant of the linearization at $z = 0$ is given by $[p(0, \lambda) + r(0, 0, \lambda, \mu)]^2$ and the trace by $2[p(0, \lambda) + r(0, 0, \lambda, \mu)]$. We have steady-state bifurcations from the trivial branch when $(p + \mu r)(0, \lambda, \mu) = 0$. When $\mu = 0$, we have the usual $\mathbb{O}(2)$-bifurcation set, with solutions given by $p(u, \lambda) = 0$. By symmetry, one eigenvalue is always 0 and the sign of the other one is given by the sign of p_u along the branches of solutions.

4.1.2. *The rest of the bifurcation diagram when* $n \neq 4$

When $\mu \neq 0$, we have the solutions coming from the fixed-point subspaces of the isotropy subgroups of $\mathbb{D}_n$, when $n \neq 4$. Those subgroups are well known (cf. [11]): $\mathbb{Z}_2^{\kappa}$, $\mathbb{Z}_2^{\zeta\kappa}$ (not conjugate to $\mathbb{Z}_2^{\kappa}$ when n is even) and the trivial isotropy $\mathbf{1}$.

(1) The isotropy subgroup is $\mathbb{Z}_2^{\kappa}$ (when $y = 0$). The solutions are given by the equation $p(x^2, \lambda) + \mu r(x^2, x^n, \lambda, \mu) + \mu s(x^2, x^n, \lambda, \mu)x^{n-2} = 0$.

The eigenvalues along the branches of solutions are $\lambda_1 = -n\mu x^{n-2}s$ and $\lambda_2 = 2x^2 p_u + 2\mu x^2 r_u + n\mu x^n r_v + (n-2)\mu x^{n-2}s + 2\mu x^n s_u + n\mu s_v x^{2n-2}$.

(2) The isotropy subgroup is $\mathbb{Z}_2^{\zeta\kappa}$ (when $y = \cot(\mathrm{i}\pi/n)x$) when n is even and the solutions are given by $p(x^2, \lambda) + \mu r(x^2, -x^n, \lambda, \mu) - \mu s(x^2, -x^n, \lambda, \mu)x^{n-2} = 0$.

The eigenvalues along the branches of solutions are $\lambda_1 = n\mu x^{n-2}s$ and $\lambda_2 = 2x^2 p_u + 2\mu x^2 r_u - n\mu x^n r_v - (n-2)\mu x^{n-2}s - 2\mu x^n s_u + n\mu x^{2n-2}s_v$.

(3) For the trivial isotropy (when $\mathrm{Im}(z^n) \neq 0$), the solutions are given by the system of equations $p(u, \lambda) + \mu r(u, v, \lambda, \mu) = s(u, v, \lambda, \mu) = 0$.

The determinant of the linearization is $\frac{1}{2}n\mu(z^n - \bar{z}^n)^2[p_u s_v + \mu(r_u s_v - s_u r_v)]$ and the trace is $2up_u + \mu[2ur_u + (nr_v + 2s_u)v + nu^{n-1}s_v]$.

4.1.3. *The rest of the bifurcation diagram when $n = 4$*

When $n = 4$ and $\mu \neq 0$, we have the same classes but different formulae.

(1) For the isotropy subgroup $\mathbb{Z}_2^{\kappa}$ ($y = 0$), the solutions are given by the equation $p(x^2, \lambda) + \mu r(x^2, x^4, \lambda, \mu) - \mu s(x^2, x^4, \lambda, \mu)x^2 = 0$.

The eigenvalues along the branches of solutions are $\lambda_1 = 2\mu x^2 s$ and $\lambda_2 = 2x^2(p_u + \mu r_u - \mu s + \mu x^2(2r_\Delta - s_u) - 2\mu x^4 s_\Delta)$.

(2) For the isotropy subgroup $\mathbb{Z}_2^{\zeta\kappa}$ (when $y = x$), the solutions are given by the equation $p(2x^2, \lambda) + \mu r(2x^2, \lambda, \mu) = 0$.

The eigenvalues along the branches of solutions are $\lambda_1 = -2\mu x^2 s$ and $\lambda_2 = 2x^2[p_u + \mu(r_u - s)]$

(3) When $\mathrm{Im}(z^4) \neq 0$ (trivial isotropy), the solutions are given by the system of equations $p(u, \lambda) + \mu r(u, \Delta, \lambda, \mu) = s(u, \Delta, \lambda, \mu) = 0$.

The sign of the determinant of the linearization along the solutions is given by the sign of $-\mu[p_u s_\Delta + \mu(r_u s_\Delta - r_\Delta s_u)]$ and the sign of the trace is the sign of $u(p_u + \mu r_u) + 2\mu\Delta r_\Delta - \mu\Delta s_u - 2\mu u\Delta s_\Delta$.

Remark 4.1. When $\mu \neq 0$, the first eigenvalue is a perturbation of the zero eigenvalue from the $\mathbb{O}(2)$-bifurcation problem.

Remark 4.2. When n is even, we consider solutions coming from the two classes of conjugate subgroups $\mathbb{Z}_2^{\kappa}$ and $\mathbb{Z}_2^{\zeta\kappa}$, but we only have to analyse the solution orbit for $x > 0$.

4.2. Generic diagrams

In the generic diagrams, figures 1–3, the $\mathbb{D}_n$-equivariant perturbation selects two particular orbits of $\mathbb{Z}_2$-invariant solutions from the original circle of $\mathbb{O}(2)$-invariant solutions, one stable and the other unstable. No additional secondary bifurcation occurs.

Figure 1. Generic bifurcation for $n = 3$ when $\epsilon = \epsilon_1 = -1$.

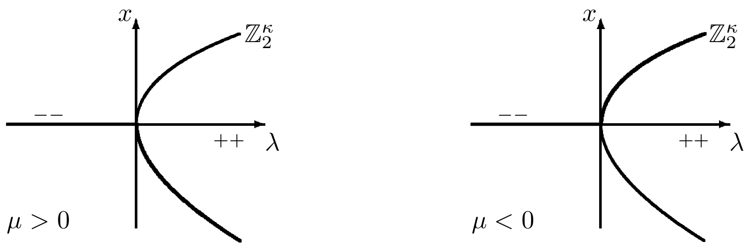

Figure 2. Generic bifurcation for n odd when $n \geqslant 5$ and $\epsilon = \epsilon_1 = -1$.

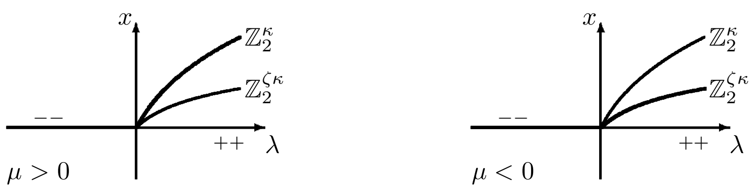

Figure 3. Generic bifurcation for n even when $\epsilon = \epsilon_1 = -1$. The stability assignments correspond to $n = 4$. The isotropy groups of the branches are interchanged for $n \geqslant 6$.

In the following set of pictures, figures 1–15, we use thick lines to denote branches of stable solutions. Stability corresponds to both eigenvalues being negative. When n is even, we distinguish between branches of solutions with isotropy of class $\mathbb{Z}_2^\kappa$ and $\mathbb{Z}_2^{\zeta\kappa}$. In this case, we only portray the positive half of the branches.

4.3. Case $\mathbf{I}_1^n$

In the following pictures, figures 4–9, we assume that the modal parameter a is negative. The *transition varieties* are curves in the (λ, μ)-space that are a projection of the bifurcation points of $f^{-1}(0)$. They only depend on the sign of α because we consider only diagrams of top-cod 1. In figure 4 the cuspidal wedge corresponds to the appearance of **1**-branches via symmetry breaking from the $\mathbb{Z}_2^\kappa$-branches. The turning points on the $\mathbb{Z}_2^\kappa$-branches correspond to the quartic curve. Those turning points correspond to the stability recovery of the symmetry-breaking branches whose bifurcation is transcritical when $n = 3$. The μ-axis represents the bifurcation from the trivial branch and the λ-axis the unperturbed case.

We have drawn horizontal sections A, B, C and D to portray the bifurcation diagrams. Although convenient for the description of the zero-set, note that those slices are not preserved by the equivalence. It is only the respective positions of the two-dimensional regions that are diffeomorphic via the equivalence relation.

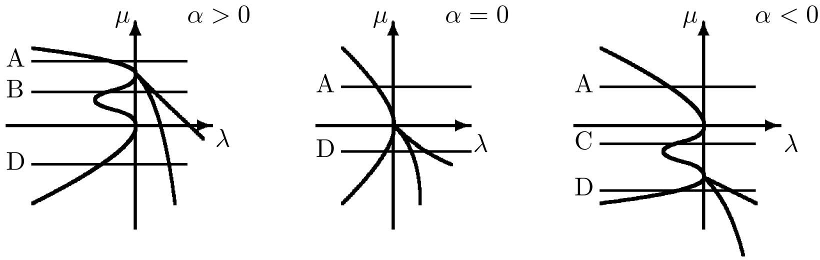

Figure 4. Transition varieties for case I_1^3 with $\epsilon_1 = -\delta_1 = \delta_4 = \delta_5 = -1$.

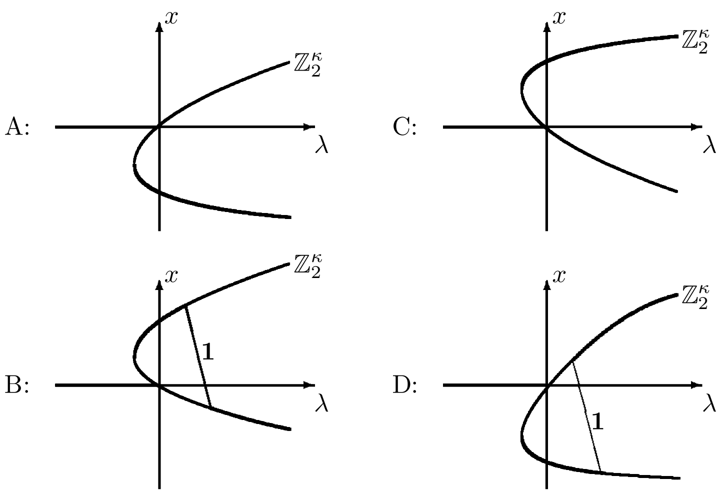

Figure 5. Bifurcation diagrams for case I_1^3 (cf. figure 4).

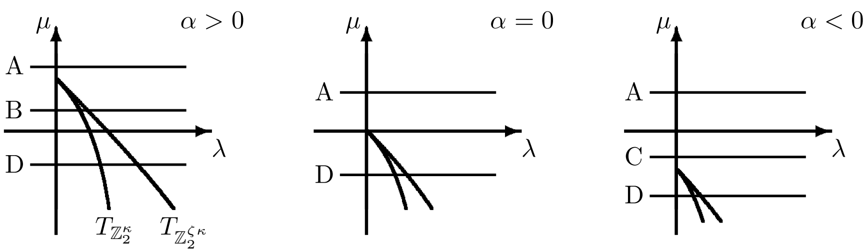

Figure 6. Transition varieties for case I_1^4 with $\epsilon_1 = -\delta_1 = \delta_4 = \delta_5 = -1$.

Note that our representation of the **1**-branch is somewhat misleading. In reality, the **1**-branch connects the positive x half of each of the three parabola-shaped $\mathbb{Z}_2^\kappa$-branches to the negative x half of the next branch because the stability on those branches differ between the origin and the larger values. The **1**-branch is stable for $\mu > 0$ and unstable if $\mu > 0$. Note that the sign of μ controls what branches are ultimately stable at the end of the symmetry-breaking process.

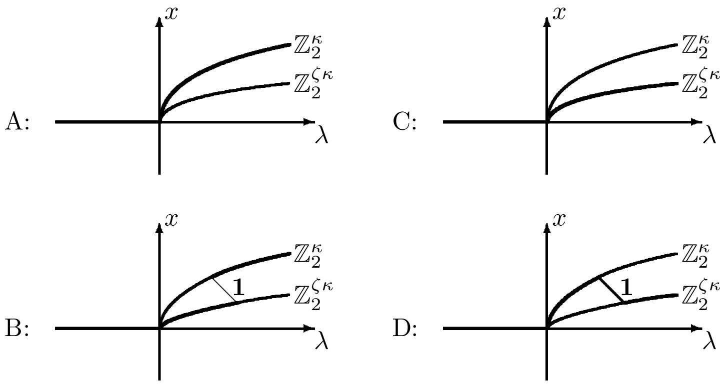

Figure 7. Bifurcation diagrams for case I_1^4 (cf. figure 6).

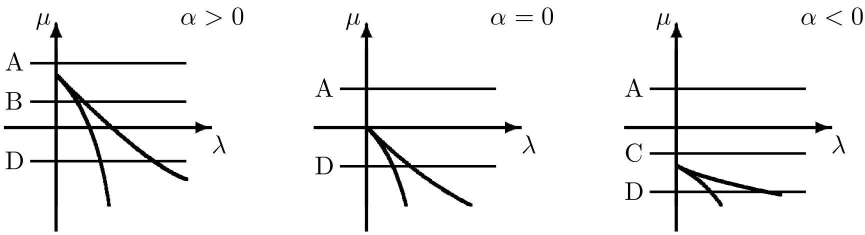

Figure 8. Transition varieties for case I_1^n when $n \geqslant 5$ with $\epsilon_1 = -\delta_1 = \delta_4 = \delta_5 = -1$.

The μ-axis is the bifurcation from the trivial branch, as $T_{\mathbb{Z}_2^\kappa}$ and $T_{\mathbb{Z}_2^{\zeta\kappa}}$ correspond to secondary bifurcations of **1**-branches from the $\mathbb{Z}_2^\kappa$ or $\mathbb{Z}_2^{\zeta\kappa}$-branches, respectively.

When $n \geqslant 5$, the **1**-branch wedge is again 'opening' like when $n = 3$.

When n is even, the bifurcation diagrams can be obtained from figure 7 by exchanging A with C and B with D.

4.4. Case $\mathbf{II}_1^n$

In case II_1^n, like in the next case III_1^n, $s^o \neq 0$, so we have no **1**-branches. It is the bifurcation structure of the $\mathbb{Z}_2$-branches that is affected by the perturbation. A particular effect is that the sign of μ determines what half (when n is odd) or what branch ($\mathbb{Z}_2^\kappa$ or $\mathbb{Z}_2^{\zeta\kappa}$ when n is even) is stable.

The λ-axis still corresponds to the unperturbed problem and the parabola to the bifurcation of the $\mathbb{Z}_2^\kappa/\mathbb{Z}_2^{\zeta\kappa}$-branches from the trivial solution. The bifurcating branches are either ellipses or hyperbolas, depending on the sign of $\epsilon_1\delta_2$. In figure 11 we describe the elliptic case $\epsilon_1\delta_2 = 1$. When $\epsilon_1\delta_2 = -1$, the ellipses are transformed in hyperbolas. As previously, the sign of $\epsilon\mu$ (when n is even) or $\epsilon\mu x$ (when n is odd) determine the sign of the zero eigenvalue of the unperturbed problem.

To find the bifurcation diagrams when $n \geqslant 6$ and n is even, we can exchange the diagrams B with C when $n = 4$.

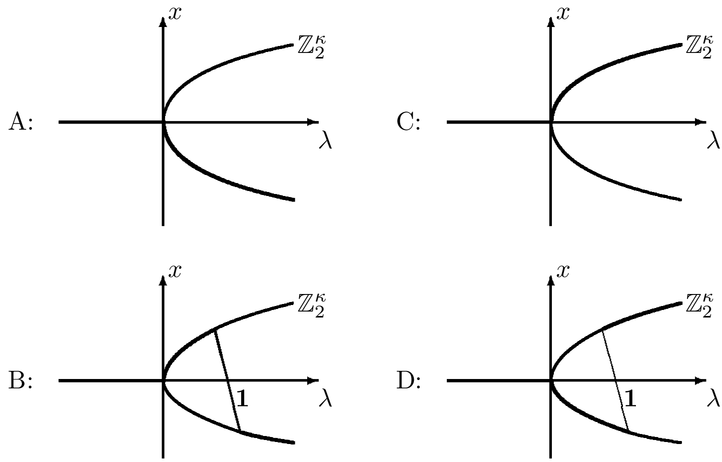

Figure 9. Bifurcation diagrams for case I_1^n when n is odd (cf. figure 8).

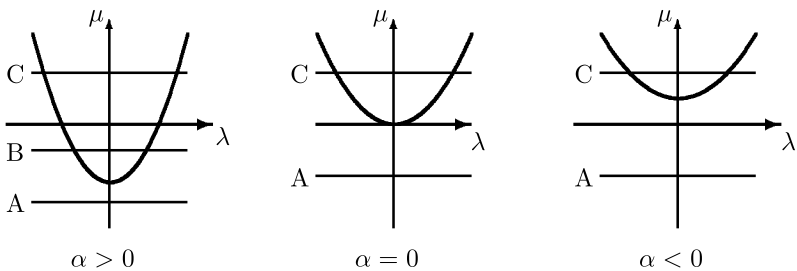

Figure 10. Transition varieties for case II_1^n with $\epsilon = \epsilon_1 = -1$ and $\delta_2 = -\delta_6 = -1$.

4.5. Case $\mathbf{III}_1^n$

The λ-axis corresponds to the unperturbed problem and the μ-axis to the bifurcation from the trivial branch. The structure of the turning points on the $\mathbb{Z}_2^\kappa$-branches is more complicated, developing a swallow-tail-like transition variety.

When $n = 4$, we need to distinguish between the three intervals for the modal parameter c separated by $c = 0$ and $c = \epsilon$. As previously, the transition varieties are composed of the λ- and μ-axes. The half branches of parabola correspond to the turning points on the $\mathbb{Z}_2^\kappa$- or $\mathbb{Z}_2^{\zeta\kappa}$-branches.

When $n \geqslant 6$ is even, the transition varieties can be read from figure 14 by exchanging the diagrams A with C, B with E and D with F. When $d < 0$, we get the pictures for $c < -1$ and when $d > 0$ those for $c > 0$. When $n \geqslant 5$ is odd, the transition varieties are like for n even and we can read the diagrams from figure 15, where the $\mathbb{Z}_2^\kappa$-half branch is for $x > 0$ and the $\mathbb{Z}_2^{\zeta\kappa}$-half branch is for $x < 0$.

5. $(\mathbb{O}(2), \mathbb{D}_n)$-symmetry-breaking theory

In this section we recall the fundamental results and techniques from the general theory of unfoldings, finite determinacy and the recognition problem for the bifurcation problems with forced symmetry breaking (cf. [2, 6]).

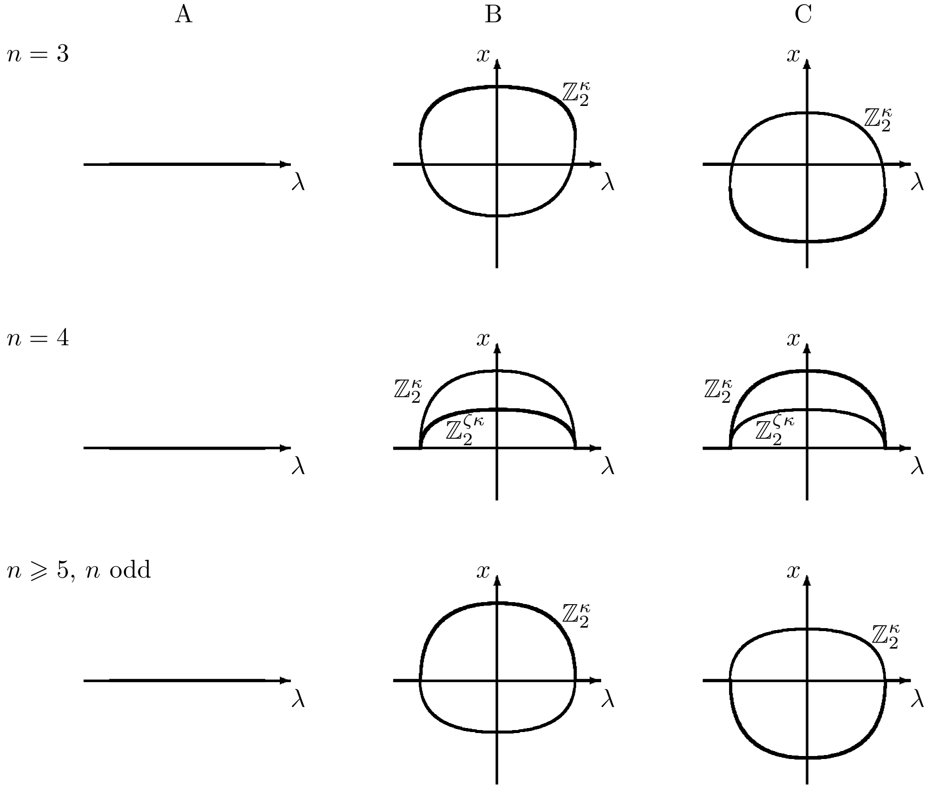

Figure 11. Bifurcation diagrams for case II_1^n (cf. figure 10).

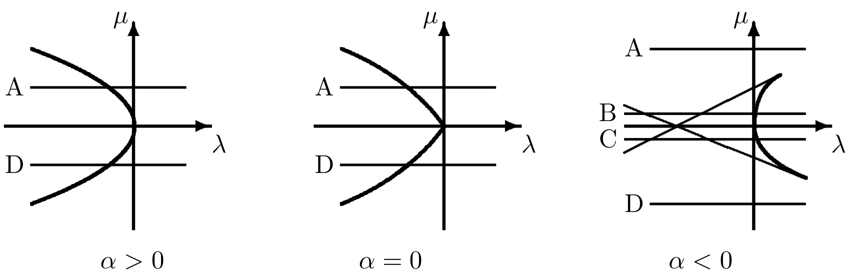

Figure 12. Transition varieties for case III_1^3 with $\epsilon_2 = -\delta_1 = -\epsilon = 1$.

5.1. Notation and preliminary definitions

For any set of variables, $a \in \mathbb{R}^m$, we denote by $\mathcal{E}_a$ the ring of smooth germs $f : (\mathbb{R}^m, 0) \to \mathbb{R}$ and by $\mathcal{M}_a$ its maximal ideal. We denote by $\vec{\mathcal{E}}_a$ the $\mathcal{E}_a$-module of smooth germs $g : (\mathbb{R}^m, 0) \to \mathbb{R}^2$.

5.1.1. *Invariant functions*

Let $\mathcal{E}_{(z,\lambda)}^{\mathbb{O}(2)}$ be the ring of smooth $\mathbb{O}(2)$-invariant germs $h : (\mathbb{R}^{2+1}, 0) \to \mathbb{R}$ such that

$$h(\gamma z, \lambda) = \gamma h(z, \lambda) \quad \forall \gamma \in \mathbb{O}(2),$$

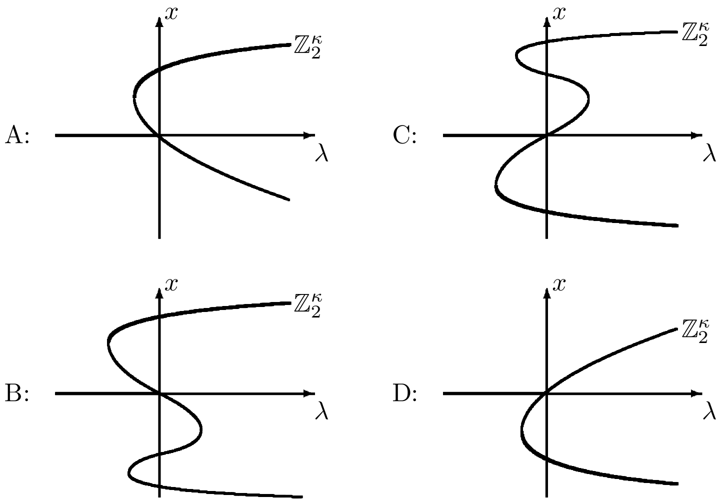

Figure 13. Bifurcation diagrams for case III_1^3 (cf. figure 12).

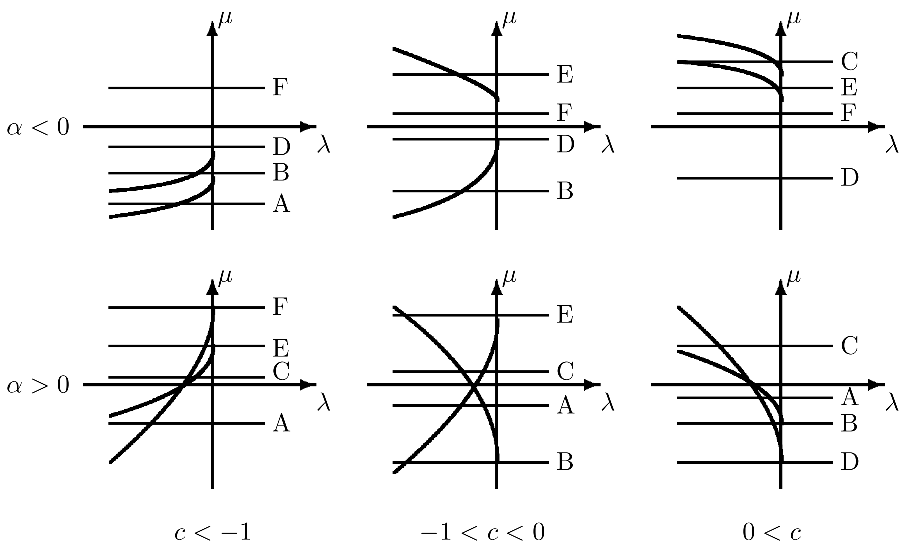

Figure 14. Transition varieties for case III_1^n when $n \geqslant 4$ with $\epsilon_2 = -\delta_1 = \epsilon = -1$.

and $\mathcal{M}_{(z,\lambda)}^{\mathbb{O}(2)}$ its maximal ideal. From [15], $\mathcal{E}_{(z,\lambda)}^{\mathbb{O}(2)}$ is the pullback by

$$\bar{u}(z) = z\bar{z}$$

of $\mathcal{E}_{(u,\lambda)}$. Similarly, the subset $\mathcal{E}_{(z,\lambda,\mu)}^{\mathbb{D}_n}$ of $\mathbb{D}_n$-invariant germs of $\mathcal{E}_{(z,\lambda,\mu)}$ is $\bar{p}^*\mathcal{E}_{(u,v,\lambda,\mu)}$, where

$$\bar{p} = (\bar{u}, \bar{v}), \quad \text{with } \bar{v} = \tfrac{1}{2}(z^n + \bar{z}^n).$$

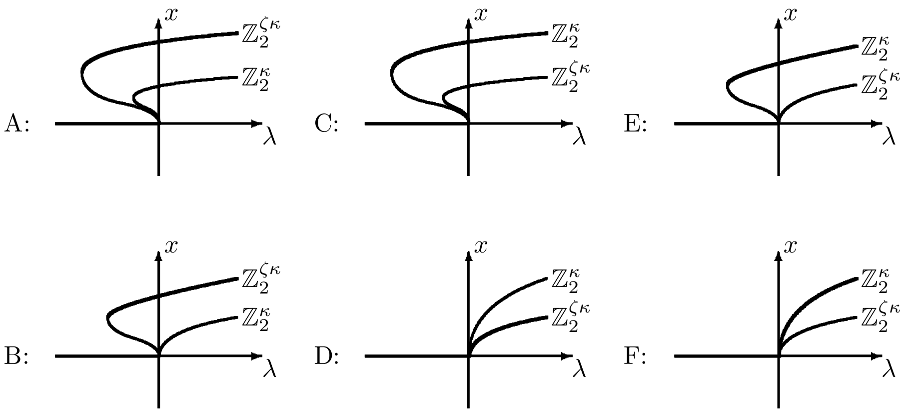

Figure 15. Bifurcation diagrams for case III_1^4 (cf. figure 14).

For $n = 4$, let $\bar{p} = (\bar{u}, \Delta)$, where $\Delta = \delta^2$, with $\delta = -\frac{1}{2}(z^2 + \bar{z}^2)$. Then

$$\mathcal{E}^{\mathbb{D}_4}_{(z,\lambda,\mu)} = \bar{p}^* \mathcal{E}_{(u,\Delta,\lambda,\mu)}.$$

Finally, we define the local ring $\mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)}$ of germs $h : (\mathbb{R}^{2+2}, 0) \to \mathbb{R}$ of the form

$$h = h_1 + \mu h_2,$$

with $h_1 \in \mathcal{E}^{\mathbb{O}(2)}_{(z,\lambda)}$ and $h_2 \in \mathcal{E}^{\mathbb{D}_n}_{(z,\lambda,\mu)}$ of maximal ideal

$$\mathcal{M}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)} = \mathcal{M}^{\mathbb{O}(2)}_{(z,\lambda)} + \mu \mathcal{E}^{\mathbb{D}_n}_{(z,\lambda,\mu)}.$$

We also define $\mathcal{E}_\Lambda = \mathcal{E}_\lambda + \mu \mathcal{E}_{(\lambda,\mu)}$.

5.1.2. *Equivariant maps*

Our first space of mappings is the $\mathcal{E}^{\mathbb{O}(2)}_{(z,\lambda)}$-module $\vec{\mathcal{E}}^{\mathbb{O}(2)}_{(z,\lambda)}$ of $\mathbb{O}(2)$-equivariant maps $f_1 \in \vec{\mathcal{E}}_{(z,\lambda)}$ such that

$$f_1(\gamma z, \lambda) = \gamma f_1(z, \lambda) \quad \forall \gamma \in \mathbb{O}(2),$$

generated over $\mathcal{E}^{\mathbb{O}(2)}_{(z,\lambda)}$ by the polynomials map z. We define also $\vec{\mathcal{E}}^{\mathbb{D}_n}_{(z,\lambda,\mu)}$ as the $\mathcal{E}^{\mathbb{D}_n}_{(z,\lambda,\mu)}$-module of $\mathbb{D}_n$-equivariant maps $f_2 \in \vec{\mathcal{E}}_{(z,\lambda,\mu)}$. Its generators are the polynomial maps z and $\bar{z}^{n-1}$ (when $n \neq 4$), or by z and $\delta \bar{z}$ (when $n = 4$). Our space of bifurcation germs is the $\mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)}$-module

$$\vec{\mathcal{E}}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)} = \vec{\mathcal{E}}^{\mathbb{O}(2)}_{(z,\lambda)} + \mu \vec{\mathcal{E}}^{\mathbb{D}_n}_{(z,\lambda,\mu)}.$$

We also need the $\mathcal{E}_\Lambda$-modules

$$\boldsymbol{\Lambda}_\Lambda = \{\Lambda = (\Lambda_1, \Lambda_2) \in \vec{\mathcal{E}}_{(\lambda,\mu)} \mid \Lambda_1 \in \mathcal{E}_\Lambda,\ \Lambda_2 = \mu \Lambda_{22},\ \Lambda_{22} \in \mathcal{E}_{(\lambda,\mu)}\}$$

and

$$\vec{\mathcal{M}}_{(\lambda,\mu)} = \{\Lambda \in \vec{\mathcal{E}}_{(\lambda,\mu)} \mid \Lambda^{\circ} = 0\}.$$

5.2. Contact equivalence

The rings of invariants $\mathcal{E}^{\mathbb{O}(2)}_{(z,\lambda)}$ and $\mathcal{E}^{\mathbb{D}_n}_{(z,\lambda,\mu)}$ are DA-algebras and $\mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)}$ is only an extended DA-algebra (cf. [2,6]). Those important algebraic properties are needed to use the basic theory of Damon [2] to establish the unfolding and finite determinacy theorems. Next we introduce some equivalence relations.

5.2.1. $\mathcal{K}^{\mathbb{O}(2),\mathbb{D}_n}_{(\lambda,\mu)}$-*equivalence*

Let $\boldsymbol{M}^{\mathbb{O}(2)}_{(z,\lambda)}$ be the $\mathcal{E}^{\mathbb{O}(2)}_{(z,\lambda)}$-module of smooth $\mathbb{O}(2)$-equivariant matrix-valued maps,

$$T_1(\gamma z,\lambda)\gamma = \gamma T_1(z,\lambda) \quad \forall \gamma \in \mathbb{O}(2).$$

It is generated by

$$S_1(z,\lambda)\omega = \omega \quad \text{and} \quad S_2(z,\lambda)\omega = z^2\bar{\omega}.$$

Now let $\boldsymbol{M}^{\mathbb{D}_n}_{(z,\lambda,\mu)}$ be the $\mathcal{E}^{\mathbb{D}_n}_{(z,\lambda,\mu)}$-module of smooth $\mathbb{D}_n$-equivariant matrix-valued maps, generated by S_1, S_2, $S_3(z,\lambda,\mu)\omega = \bar{z}^{n-2}\bar{\omega}$ and $S_4(z,\lambda,\mu)\omega = z^n\omega$. Then let

$$\boldsymbol{M}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)} = \boldsymbol{M}^{\mathbb{O}(2)}_{(z,\lambda)} + \mu \boldsymbol{M}^{\mathbb{D}_n}_{(z,\lambda,\mu)}.$$

We denote by $\mathrm{GL}^{\mathbb{O}(2)}_{(z,\lambda)}$, $\mathrm{GL}^{\mathbb{D}_n}_{(z,\lambda,\mu)}$ and by $\mathrm{GL}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)}$ the corresponding subsets of matrix functions with value in $\mathrm{GL}_2(\mathbb{R})$.

Our equivalences must preserve the $\mathbb{O}(2)$-equivariance of the $(\mu = 0)$-slice as well as the $\mathbb{D}_n$-equivariance of the total germ. A simple group of transformations is obtained by combining $\mathcal{K}^{\mathbb{O}(2)}_{(z,\lambda)}$ and $\mathcal{K}^{\mathbb{D}_n}_{(z,\lambda,\mu)}$ in the group $\hat{K}(\mathbb{O}(2),\mathbb{D}_n)$, defined as the connected component of the identity in

$$\{(T,X,\Lambda) \mid T \in \boldsymbol{M}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)},\ X \in \mathcal{E}^{\mathbb{O}(2),\ \mathbb{D}_n}_{(z,\lambda,\mu)}, \Lambda \in \mathcal{M}_{(\lambda,\mu)}\}.$$

But equivalences that leave germs in $\mathcal{E}^{\mathbb{O}(2)}_{(z,\lambda)}$ pointwise invariant can provide a fundamental contribution in the μ-dependent part of $\mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)}$. This is the key point to finite codimension with continuous symmetry groups as mentioned in [10]. There are two groups of such equivalences (cf. [6]). First, the group $\hat{M}(\mathbb{O}(2),\mathbb{D}_n)$ of matrix-valued maps $M : (\mathbb{R}^{2+2},0) \to \mathbb{SO}(2)$ such that $M \in \mathrm{GL}^{\mathbb{D}_n}_{(z,\lambda,\mu)}$. It is the isotropy subgroup in $\mathrm{GL}^{\mathbb{D}_n}_{(z,\lambda,\mu)}$ of the elements in $\mathcal{E}^{\mathbb{O}(2)}_{(z,\lambda)}$ under the conjugation $T \mapsto T^{-1}fT$. Second, $\hat{S}(\mathbb{O}(2),\mathbb{D}_n)$ is the component of the identity in

$$\{S \in \mathrm{GL}^{\mathbb{D}_n}_{(z,\lambda,\mu)} \mid S(z,\lambda,\mu)z = z\}.$$

In § 5.3.2 we show that only $\hat{S}(\mathbb{O}(2),\mathbb{D}_n)$ brings about any new reduction of the normal forms or their unfoldings. Note that we can always assume that the changes of coordinates are $\mathbb{D}_n$-equivariant.

So, finally, the product

$$\mathcal{K}^{\mathbb{O}(2),\mathbb{D}_n}_{(\lambda,\mu)} = \hat{S}(\mathbb{O}(2),\mathbb{D}_n) * \hat{K}(\mathbb{O}(2),\mathbb{D}_n)$$

forms a group of changes of coordinates that acts on $\mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)}$ by conjugation. For $(T,X,\Lambda) \in \hat{K}(\mathbb{O}(2),\mathbb{D}_n)$, $(T,X,\Lambda)f = Tf(X,\Lambda)$ and for $S \in \hat{S}(\mathbb{O}(2),\mathbb{D}_n)$,

$Sf = S(f_1 + f_2) = f_1 + \mu S f_2$. Note that $\hat{K}(\mathbb{O}(2), \mathbb{D}_n)$ is a semi-direct product and, although $\hat{S}(\mathbb{O}(2), \mathbb{D}_n)$ is not normal in $\mathcal{K}^{\mathbb{O}(2),\mathbb{D}_n}_{(\lambda,\mu)}$, we showed in [6] that we can use the changes of coordinates in $\mathcal{K}^{\mathbb{O}(2),\mathbb{D}_n}_{(\lambda,\mu)}$ in any order we may wish because any element in $\mathcal{K}^{\mathbb{O}(2),\mathbb{D}_n}_{(\lambda,\mu)}$ can be written as a product of only two elements, one in each of $\hat{S}(\mathbb{O}(2), \mathbb{D}_n)$ and $\hat{K}(\mathbb{O}(2), \mathbb{D}_n)$.

5.2.2. $\mathcal{K}^{\mathbb{O}(2),\mathbb{D}_n}_{un}(k)$-*equivalence*

For $\beta \in \mathbb{R}^k$ we extend the previous definitions to their β-parametrized versions $\mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu,\beta)}$, $\boldsymbol{M}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu,\beta)}$ and $\mathcal{M}_{(\lambda,\mu,\beta)}$. Perturbations of any $f \in \mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)}$ are described by (k-parameter) *unfoldings* of f, map germs $F \in \mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu,\beta)}$ such that

$$F(z,\lambda,\mu,0) = f(z,\lambda,\mu) \quad \text{and} \quad F(z,\lambda,0,\beta) \in \mathcal{E}^{\mathbb{O}(2)}_{(z,\lambda,\beta)}.$$

We denote by $\mathcal{K}^{\mathbb{O}(2),\mathbb{D}_n}_{un}(k)$ the extension of $\mathcal{K}^{\mathbb{O}(2),\mathbb{D}_n}_{(\lambda,\mu)}$ formed of elements

$$(T, X, \Lambda, \Phi) \in \boldsymbol{M}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu,\beta)} \times \mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu,\beta)} \times \mathcal{M}_{(\lambda,\mu,\beta)} \times \mathcal{M}_{\beta,\beta}$$

such that (T, X, Λ) is a k parameter unfolding of an element of $\mathcal{K}^{\mathbb{O}(2),\mathbb{D}_n}_{(\lambda,\mu)}$ and Φ is a diffeomorphism germ. We say that $F, G \in \mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu,\beta)}$ are $\mathcal{K}^{\mathbb{O}(2),\mathbb{D}_n}_{un}(k)$-*equivalent* if they belong to the same $\mathcal{K}^{\mathbb{O}(2),\mathbb{D}_n}_{un}(k)$-orbit.

5.3. Tangent spaces

To get the fundamental results of the theory, we need first to introduce the algebraic structure of extended tangent spaces that will underpin the main calculations we need to exploit the results of the abstract theory.

5.3.1. *Extended tangent spaces*

We define the $\mathcal{K}^{\mathbb{O}(2),\mathbb{D}_n}_{(\lambda,\mu)}$-*extended tangent space* $\mathcal{T}^{\mathbb{O}(2),\mathbb{D}_n}_{\mathrm{e}}(f)$ of $f \in \mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)}$ as

$$\{T_1 f_1 + f_{1z} X_1 + f_{1\lambda}\Lambda_{11} + \mu(T_1 f_2 + f_{2z} X_1 + f_{2\lambda}\Lambda_{11}) \tag{5.1}$$

$$+ \mu(T_2 f + f_z X_2 + f_\lambda \Lambda_{12} + f_2 \Lambda_{22} + f_{2\mu}\Lambda_{22} + \hat{s}(f_2))\}, \tag{5.2}$$

where

$$T = T_1 + \mu T_2 \in \boldsymbol{M}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)},$$
$$X = X_1 + \mu X_2 \in \mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)},$$
$$\Lambda = (\Lambda_1, \mu\Lambda_{22}) \in \mathcal{E}_{(\lambda,\mu)}$$

and $\hat{s}(\mathbb{O}(2), \mathbb{D}_n)$ is the tangent space of $\hat{S}(\mathbb{O}(2), \mathbb{D}_n)$m as described in the next subsection. The extended tangent space $\mathcal{T}^{\mathbb{O}(2),\mathbb{D}_n}_{\mathrm{e}}(f)$ is a sum of finitely generated modules over the system of rings $\{\mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)}, \mathcal{E}_{(\lambda,\mu)}\}$ (cf. [2]). The *extended normal space* to f is defined by

$$\mathcal{N}^{\mathbb{O}(2),\mathbb{D}_n}_{\mathrm{e}}(f) = \mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)} / \mathcal{T}^{\mathbb{O}(2),\mathbb{D}_n}_{\mathrm{e}}(f)$$

and the $(\mathbb{O}(2),\mathbb{D}_n)$-*codimension* of $f\in\mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)}$ is

$$\mathrm{cod}^{\mathbb{O}(2)}_{\mathbb{D}_n}(f)=\dim_{\mathbb{R}}\mathcal{N}^{\mathbb{O}(2),\mathbb{D}_n}_{\mathrm{e}}(f).$$

The first part of the tangent space (5.1) represents the usual well-known extended tangent space in $\mathcal{E}^{\mathbb{O}(2)}_{(z,\lambda)}$, image of the tangent map $\phi^{f_1}_1 : k^{\mathbb{O}(2)}_\lambda\to\mathcal{E}^{\mathbb{O}(2)}_{(z,\lambda)}$,

$$(T_1,X_1,\Lambda_{11})\mapsto T_1f_1+f_{1z}X_1+f_{1\lambda}\Lambda_{11}.$$

By definition, for fixed f_1, the kernel of $\phi^{f_1}_1$, the module of syzygies of its image, denoted by $\Phi^{f_1}_1$, does not act on f_1, so its elements can make effective independent contributions to the μ-dependent part of the tangent space when applied to f_2. Let $\zeta\in\Phi^{f_1}_1$. Then $\zeta(f)=\mu\zeta(f_2)$. In general, $\Phi^{f_1}_1$ is the sum of $\mathcal{E}_\lambda$- and $\mathcal{E}_{(u,\lambda)}$-modules. We calculate it in § 6.1.2 for our normal forms. Therefore, the calculations for the extended tangent space $\mathcal{T}^{\mathbb{O}(2),\mathbb{D}_n}_{\mathrm{e}}(f)$ of

$$f=hz+\mu\begin{bmatrix}r\\ s\end{bmatrix}$$

is split into two parts: the first in $\mathcal{E}^{\mathbb{O}(2)}_{(z,\lambda)}$,

$$\mathcal{T}_{\mathrm{e}}(h)\approx\langle h,uh_u\rangle_{\mathcal{E}_{(u,\lambda)}}+\langle h_\lambda\rangle_{\mathcal{E}_\lambda}, \tag{5.3}$$

and the second (the principal calculation) in $\mathcal{E}^{\mathbb{D}_n}_{(z,\lambda,\mu)}$, with generators as defined in (5.2) with the contribution of $\Phi^{f_1}_1$. We can say that

$$\mathcal{N}^{\mathbb{O}(2),\Sigma}_{\mathrm{e}}(f)\equiv\mathcal{N}^{\mathbb{O}(2)}_{\mathrm{e}}(f_1)\oplus(\mathcal{E}_{(z,\lambda,\mu)}/\mathcal{M}),$$

where $\mathcal{M}$ represents all the elements of the T_2, X_2, Λ_{22}, $\hat{s}$ and $\Phi^{f_1}_1$ contributions.

5.3.2. *The groups* $\hat{S}(\mathbb{O}(2),\mathbb{D}_n)$, $\hat{M}(\mathbb{O}(2),\mathbb{D}_n)$ *and their tangent spaces*

Here we describe the groups $\hat{S}(\mathbb{O}(2),\mathbb{D}_n)$, $\hat{M}(\mathbb{O}(2),\mathbb{D}_n)$ and show that only the tangent space $\hat{s}(\mathbb{O}(2),\mathbb{D}_n)$ brings about new information to the tangent spaces.

Proposition 5.1. *Let*

$$\begin{aligned}U(z)\omega&=z\bar{\omega}-\bar{z}\omega,\\ V(z)\omega&=z^2\bar{\omega}-u\omega,\\ W(z)\omega&=u\bar{z}^{n-2}\bar{\omega}-\bar{z}^n\omega.\end{aligned}$$

For our examples, an element $S\in\hat{S}(\mathbb{O}(2),\mathbb{D}_n)$ *is equal to*

$$S(z,\lambda,\mu)\omega=\omega+\tfrac{1}{4}h_1(u,v,\lambda,\mu)V(z)\omega+\tfrac{1}{4}h_2(u,v,\lambda,\mu)W(z)\omega$$

for real-valued h_1, h_2 *and* $\hat{s}(\mathbb{O}(2),\mathbb{D}_n)$ *is generated over* $\mathcal{E}^{\mathbb{D}_n}_{(z,\lambda,\mu)}$ *by* V *and* W.

Moreover, $s\in\hat{s}(\mathbb{O}(2),\mathbb{D}_n)$ *acts via simple matrix multiplication* $f\mapsto sf$.

Proof. From [6], $\hat{S}(\mathbb{O}(2),\mathbf{1})$ is generated over the complex-valued functions by U. In complex notation, $M\in\hat{S}(\mathbb{O}(2),\mathbf{1})$ is equal to $M(z,\lambda,\mu)\omega=\omega+\frac{1}{2}h(z,\lambda,\mu)U(z)\omega$

for some complex-valued h. To find the elements $S \in \hat{S}(\mathbb{O}(2), \mathbb{D}_n)$, we need to average M over $\mathbb{D}_n$ (with the measure of $\mathbb{D}_n$ equal to 1), that is,

$$S(z, \lambda, \mu)\omega = \int_{\mathbb{D}_n} \sigma^{-1} M(\sigma z, \lambda, \mu)\sigma\omega \,\mathrm{d}\sigma.$$

Recall that $\mathbb{D}_n$ is $\mathbb{Z}_n \cup \kappa\mathbb{Z}_n$, where $\kappa\mathbb{Z}_n = \{\kappa\sigma | \sigma \in \mathbb{Z}_n\}$. We get

$$\begin{aligned} S(z, \lambda, \mu)\omega &= \frac{1}{2}\int_{\mathbb{Z}_n} \sigma^{-1} M(\sigma z, \lambda, \mu)\sigma\omega \,\mathrm{d}\sigma + \frac{1}{2}\int_{\mathbb{Z}_n} \sigma^{-1}\kappa M(\sigma^{-1}\bar{z}, \lambda, \mu)\sigma^{-1}\bar{\omega}\,\mathrm{d}\sigma \\ &= \omega + \frac{1}{4}\left[\int_{\mathbb{Z}_n} \sigma^{-1}(h(\sigma z, \lambda, \mu) + \overline{h(\sigma^{-1}\bar{z}, \lambda, \mu)})\,\mathrm{d}\sigma\right] U(z)\omega. \end{aligned} \tag{5.4}$$

The function g defined as

$$g(z, \lambda, \mu) = \int_{\mathbb{Z}_n} \sigma^{-1}[h(\sigma z, \lambda, \mu) + \overline{h(\sigma^{-1}\bar{z}, \lambda, \mu)}]\,\mathrm{d}\sigma$$

is $\mathbb{D}_n$-equivariant because, for any $\nu \in \mathbb{Z}_n$,

$$g(\nu z, \lambda, \mu) = \int_{\mathbb{Z}_n} \nu\xi^{-1}[h(\xi z, \lambda, \mu) + \overline{h(\xi^{-1}\bar{z}, \lambda, \mu)}]\,\mathrm{d}\xi = \nu g(z, \lambda, \mu) \quad (\xi = \sigma\nu)$$

and, similarly, $g(\kappa\nu z, \lambda, \mu) = \kappa\nu g(z, \lambda, \mu)$. Therefore, the space of h is generated over $\mathcal{E}^{\mathbb{D}_n}_{(z,\lambda,\mu)}$ by z and $\bar{z}^{n-1}$, and so the conclusion. To compute the tangent space $\hat{s}(\mathbb{O}(2), \mathbb{D}_n)$, we differentiate all possible paths $t \mapsto S(t)$ with values in $\hat{S}(\mathbb{O}(2), \mathbb{D}_n)$ passing through the identity. We get the matrices S without their identity part ω and they act by simple multiplication. □

Now we show that the contribution of $\hat{M}(\mathbb{O}(2), \mathbb{D}_n)$ can be ignored.

Proposition 5.2. *The group* $\hat{M}(\mathbb{O}(2), \mathbb{D}_n)$ *is equal to* $\{\mathrm{e}^{\mathrm{i}w\theta(u,v,\lambda,\mu)} \mid \theta \in \mathcal{E}^{\mathbb{D}_n}_{(z,\lambda,\mu)}\}$ *and its tangent space* $\hat{m}(\mathbb{O}(2), \mathbb{D}_n)$ *is generated over* $\mathcal{E}^{\mathbb{D}_n}_{(z,\lambda,\mu)}$ *by* iw. *Moreover, its action on* $\mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)}$ *is already in* $\mathcal{T}^{\mathbb{O}(2),\mathbb{D}_n}_{\mathrm{e}}(f)$.

Proof. An element $M \in \hat{M}(\mathbb{O}(2), \mathbb{D}_n)$ is represented by $\mathrm{e}^{\mathrm{i}\theta(z,\lambda,\mu)}$ for some real-valued function θ such that M is $\mathbb{D}_n$-equivariant. So θ should be $\mathbb{Z}_n$-invariant,

$$\theta(z, \lambda, \mu) = \theta_1(u, v, \lambda, \mu) + w\theta_2(u, v, \lambda, \mu),$$

and with the reflection $\theta(z, \lambda, \mu) + \theta(\bar{z}, \lambda, \mu) = 0$. The symmetry condition becomes

$$\theta_1(u, v, \lambda, \mu) + w\theta_2(u, v, \lambda, \mu) + \theta_1(u, v, \lambda, \mu) - w\theta_2(u, v, \lambda, \mu) = 0.$$

And so $\theta_1 \equiv 0$, which means that

$$\hat{M}(\mathbb{O}(2), \mathbb{D}_n) = \{\mathrm{e}^{\mathrm{i}w\theta(u,v,\lambda,\mu)} \mid \theta \in \mathcal{E}^{\mathbb{D}_n}_{(z,\lambda,\mu)}\}.$$

To compute the tangent spaces $\hat{m}(\mathbb{O}(2), \Sigma)$, we differentiate in time the paths $t \mapsto M(t)$ passing through the identity with values in $\hat{M}(\mathbb{O}(2), \mathbb{D}_n)$. Recall that $M \in \hat{M}(\mathbb{O}(2), \mathbb{D}_n)$ acts as $f \mapsto \mathrm{e}^{-\mathrm{i}\theta} f(\mathrm{e}^{\mathrm{i}\theta} z)$ for some function $\theta(z, \lambda, \mu)$. Therefore, when we differentiate a path in $\hat{M}(\mathbb{O}(2), \mathbb{D}_n)$, we get an element of $\hat{m}(\mathbb{O}(2), \mathbb{D}_n)$ of the form iθ, for some function $\theta(z, \lambda, \mu)$, which acts as $f \mapsto -\mathrm{i}\theta f + f_z \mathrm{i}\theta z$. It is then a straightforward, albeit lengthy, calculation to show that $-\mathrm{i}\theta f + f_z \mathrm{i}\theta z$ is in the extended $\mathcal{K}^{\mathbb{O}(2),\mathbb{D}_n}_{(\lambda,\mu)}$-tangent space of f, as defined in (5.1). □

5.4. The unfolding and determinacy theories

The unfolding and finite determinacy theorems are a rewording of the results in the general theory of [2] (see also [6]). Let $F \in \mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu,\beta)}$ be an unfoldings of $f \in \mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)}$ with k parameters, and let $G \in \mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu,\alpha)}$ be an unfolding of $f \in \mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)}$ with r parameters. We say that G *maps into* F, or G *factors through* F, if there exist $T \in \boldsymbol{M}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\alpha)}$, $X \in \mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu,\alpha)}$, $\Lambda \in \mathcal{E}_{(\lambda,\mu,\alpha)}$ and $A : (\mathbb{R}^r, 0) \to (\mathbb{R}^k, 0)$ satisfying $T(z,\lambda,\mu,0) = \boldsymbol{I}_2$, $X(z,\lambda,\mu,0) = z$ and $\Lambda(\lambda,\mu,0) = (\lambda,\mu)$, such that

$$G(z,\lambda,\mu,\alpha) = T(z,\lambda,\mu,\alpha)F(X(z,\lambda,\mu,\alpha),\Lambda(\lambda,\mu,\alpha),A(\alpha)).$$

The unfolding F is called *versal* if any unfolding G of f maps into F. If F is versal and has minimal number of parameters, it is called *universal*. First we rewrite the abstract theorems of [2].

THEOREM 5.3 (unfolding theorem [2]). *Let $f \in \mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)}$ and $F \in \mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu,\alpha)}$ be an unfolding of f with k parameters, $\alpha = (\alpha_1 \dots \alpha_k)$. Then we have the following.*

(1) *F is versal if and only if $\mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)} = \mathcal{T}^{\mathbb{O}(2),\mathbb{D}_n}_e(f) + \langle F_{\alpha_i}(\cdot,\cdot,0)\rangle_{\mathbb{R}}$.*

(2) *Two versal unfoldings of a germ in $\mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)}$ are equivalent as unfoldings if and only if they have the same number of unfolding parameters.*

(3) *Let $W \subset \mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)}$ be a finite-dimensional complement of $\mathcal{N}^{\mathbb{O}(2),\mathbb{D}_n}_e(f)$ as a vector space, that is,*

$$\mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)} = \mathcal{T}^{\mathbb{O}(2),\mathbb{D}_n}_e(f) \oplus W.$$

Let $k = \dim W = \mathrm{cod}^{\mathbb{O}(2)}_{\mathbb{D}_n}(f)$ be a basis for W and $\{p_i\}_{i=1}^k$ be a basis for W. Then a universal unfolding of g is

$$F(z,\lambda,\mu,\alpha) = f(z,\lambda,\mu) + \sum_{j=1}^{k} \alpha_j p_j(z,\lambda,\mu).$$

(4) *If f and $g \in \mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)}$ are two $\mathcal{K}^{\mathbb{O}(2)}_\lambda$-equivalent germs of finite codimension and F and $G \in \mathcal{E}^{\mathbb{O}(2)}_{(z,\lambda,\alpha)}$, with $\alpha = (\alpha_1 \dots \alpha_k)$, are two universal unfoldings of f and g, respectively, then F and G are $\mathcal{K}^{\mathbb{O}(2)}_{un}(k)$-equivalent.*

We denote by $j^k(f)$ the Taylor polynomial of order k (or *k-jet*) of f. A germ $f \in \mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)}$ is k-$\mathcal{K}^{\mathbb{O}(2),\mathbb{D}_n}_{(\lambda,\mu)}$-*determined* if every germ $g \in \mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)}$ with $j^k(g) = j^k(f)$ is $\mathcal{K}^{\mathbb{O}(2),\mathbb{D}_n}_{(\lambda,\mu)}$-equivalent to f. A germ is *finitely* $\mathcal{K}^{\mathbb{O}(2),\mathbb{D}_n}_{(\lambda,\mu)}$-*determined* if it is k-$\mathcal{K}^{\mathbb{O}(2),\mathbb{D}_n}_{(\lambda,\mu)}$-determined for some integer k. There is a close relationship between being finitely determined and being of finite codimension.

THEOREM 5.4 (finite determinacy theorem [2]). *A germ $f \in \mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)}$ is finitely $\mathcal{K}^{\mathbb{O}(2),\mathbb{D}_n}_{(\lambda,\mu)}$-determined if and only if $\mathrm{cod}^{\mathbb{O}(2)}_{\mathbb{D}_n}(f)$ is finite.*

5.4.1. *The recognition problem, higher-order terms*

The *recognition problem* seeks conditions for a germ $g \in \mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)}$ to be $\mathcal{K}^{\mathbb{O}(2),\mathbb{D}_n}_{(\lambda,\mu)}$-equivalent to a given normal form. It is solved by characterizing explicitly the $\mathcal{K}^{\mathbb{O}(2),\mathbb{D}_n}_{(\lambda,\mu)}$-equivalence class in terms of a finite number of polynomial equalities and inequalities to be satisfied by the Taylor coefficients of the elements of that class. Let $\Phi = (T, X, \Lambda) \in \mathcal{K}^{\mathbb{O}(2),\mathbb{D}_n}_{(\lambda,\mu)}$ and consider the mapping $f \mapsto \Phi(f) = T \cdot f \circ (X, \Lambda)$. A subspace $M \subset \mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)}$ is *intrinsic* if $\Phi(f) \in M$ for all $f \in M$ and all $\Phi \in \mathcal{K}^{\mathbb{O}(2),\mathbb{D}_n}_{(\lambda,\mu)}$. If $V \subset \mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)}$ then the *intrinsic part* of V, denoted by $\operatorname{Itr} V$, is the largest intrinsic subspace of $\mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)}$ contained in V. Note that any power of the maximal ideal is intrinsic. Let $f \in \mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)}$. The 'perturbation term' $p \in \mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)}$ is of *higher order* with respect to f if $g + p$ is $\mathcal{K}^{\mathbb{O}(2),\mathbb{D}_n}_{(\lambda,\mu)}$-equivalent to f for every g, that is, $\mathcal{K}^{\mathbb{O}(2),\mathbb{D}_n}_{(\lambda,\mu)}$-equivalent to f. By definition, such a perturbation cannot enter into a solution of the recognition problem for f. We denote by $\mathcal{P}(f)$ the set of all higher order terms of f, that is, the set of $p \in \mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)}$ such that $g+p$ is equivalent to f for all g in the orbit of f. With the usual proof (cf. [7]) one gets the following characterization.

Proposition 5.5. *The set* $\mathcal{P}(f)$ *is an intrinsic* $\mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)}$-*submodule of* $\mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)}$.

To evaluate $\mathcal{P}(f)$, we introduce the subgroup $\mathcal{U}^{\mathbb{O}(2),\mathbb{D}_n}_{(\lambda,\mu)}$ of $\mathcal{K}^{\mathbb{O}(2),\mathbb{D}_n}_{(\lambda,\mu)}$ of unipotent equivalences. Let $\mathcal{N}_1$ be a maximal unipotent subgroup of $L^{\circ}_{\mathbb{O}(2)}(n)$, the connected component of the identity in the subset of $GL(n)$ of $\mathbb{O}(2)$-equivariant matrices and

$$\mathcal{N}_2 = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \,\middle|\, a \in \mathbb{R} \right\}.$$

If the group acts absolutely irreducibly, the corresponding $\mathcal{N}_i$ is trivial. In particular, $\mathcal{N}_1$ is trivial. Consider now the projection maps π_1 sending $(T, X, \Lambda) \in \hat{K}(\mathbb{O}(2), \mathbb{D}_n)$ onto $(T^{\circ}, X^{\circ}_x, \Lambda^{\circ}_{(\lambda,\mu)})$ and π_2 sending $S \in \hat{S}(\mathbb{O}(2), \mathbb{D}_n)$ onto S°. We define $\mathcal{U}^{\mathbb{O}(2),\mathbb{D}_n}_{(\lambda,\mu)}$ as being generated by the combinations of the inverse images $\pi_1^{-1}(\mathcal{N}_1, \mathcal{N}_1, \mathcal{N}_2)$, $\pi_2^{-1}(\mathcal{N}_2)$. It is a normal subgroup of $\mathcal{K}^{\mathbb{O}(2),\mathbb{D}_n}_{(\lambda,\mu)}$ consisting of unipotent diffeomorphisms, and is called the subgroup of *unipotent* $\mathcal{K}^{\mathbb{O}(2),\mathbb{D}_n}_{(\lambda,\mu)}$-equivalences. Its associated tangent space at $f \in \mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)}$ is

$$\begin{aligned} \mathcal{T}\mathcal{U}^{\mathbb{O}(2),\mathbb{D}_n}_{(\lambda,\mu)}(f) = \{ & T_1 f_1 + f_{1x} X_1 + f_{1\lambda} \Lambda_{11} + \mu(T_1 f_2 + f_{2z} X_1 + f_{2\lambda} \Lambda_{11}) \\ & + \mu(T_2 f + f_z X_2 + f_\lambda \Lambda_{12} + f_2 \Lambda_{22}) + f_{2\mu} \Lambda_{22} + \widehat{us}(f_2)\}, \end{aligned}$$

where $T \in \boldsymbol{M}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)}$ with $T^{\circ} = 0$, $X \in \mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)}$ with $X^{\circ} = 0 = X^{\circ}_x$ and $\Lambda_{11} \in \mathcal{M}^2_\lambda$, $\Lambda_{12} \in \mathcal{E}_{(\lambda,\mu)}$, $\Lambda_{22} \in \mathcal{M}_{(\lambda,\mu)}$ and $\widehat{us}$ is the tangent space of the kernel of π_2.

The structure of the unipotent tangent space of f is similar to that of the extended tangent space, but there is an important point to note: that we are interested in the entirety of the unipotent tangent space to be able to estimate its maximal intrinsic subspace (hence $\mathcal{P}(f)$). The contributions by T_2, X_2 and $\hat{s}$ are the same as previously, and for the parameters and Φ contributions we have

$$\left\langle \begin{bmatrix} xh_\lambda + \mu r_\lambda \\ yh_\lambda + \mu s_\lambda \end{bmatrix} \right\rangle_{\mathcal{E}_{(\lambda,\mu)}} + \left\langle \begin{bmatrix} r + \mu r_\mu \\ s + \mu s_\mu \end{bmatrix} \right\rangle_{\mathcal{M}_{(\lambda,\mu)}} + \Phi_2^{f_1}(f),$$

where $\Phi_2^{f_1}$ is the kernel of $\phi_2^{f_1} : u_\lambda^{\mathbb{O}(2)} \to \mathcal{E}_{(z,\lambda)}^{\mathbb{O}(2)}$,

$$(T_1, X_1, \Lambda_{11}) \mapsto T_1 f_1 + f_{1z} X_1 + f_{1\lambda} \Lambda_{11}.$$

In our cases, $\Phi_2^{f_1}$ can be larger than $\mathcal{M}_{(u,\lambda,\mu)} \cdot \Phi_1^{f_1}$ (cf. proposition 6.1). Following the proof of theorem 1.17 in [7, p. 108], we get the following proposition.

PROPOSITION 5.6. *Let $f \in \mathcal{E}_{(z,\lambda,\mu)}^{\mathbb{O}(2),\mathbb{D}_n}$ be of finite $\mathcal{K}_{(\lambda,\mu)}^{\mathbb{O}(2),\mathbb{D}_n}$-codimension. Then we have the following.*

(1) $\mathcal{P}(f) \supset \operatorname{Itr} \mathcal{TU}_{(\lambda,\mu)}^{\mathbb{O}(2),\mathbb{D}_n}(f)$.

(2) *If $p \in \operatorname{Itr} \mathcal{TU}_{(\lambda,\mu)}^{\mathbb{O}(2),\mathbb{D}_n}(f)$, then $f + p$ is $\mathcal{K}_{(\lambda,\mu)}^{\mathbb{O}(2),\mathbb{D}_n}$-equivalent to f.*

5.5. Topological equivalence

In this section we present the main results we need to carry out the classification of theorem 2.1 and check out the topological versality of the unfoldings to eliminate the moduli in our normal forms and their unfoldings. Our cases use the different ideas and theorems given in [3,4] to get sufficient conditions for topological versality.

In cases I_1^n and III_1^n ($n \geqslant 5$), we use the (*restricted*) *action* of the subgroup of $\mathcal{K}_{(\lambda,\mu)}^{\mathbb{O}(2),\mathbb{D}_n}$ that fixes some components $\tilde{f}$ of the versal unfolding F of f to prove the topological results. We decompose

$$\mathcal{E}_{(z,\lambda,\mu,\beta)}^{\mathbb{O}(2),\mathbb{D}_n} = \tilde{f} + \mathcal{F}_{un}$$

and consider the subgroup $\mathcal{K}_{un}$ of $\mathcal{K}_{(\lambda,\mu)}^{\mathbb{O}(2),\mathbb{D}_n}$ acting on $\mathcal{E}_{(z,\lambda,\mu,\beta)}^{\mathbb{O}(2),\mathbb{D}_n}$ fixing $\tilde{f}$. So $\mathcal{T}_{\mathrm{e}}\mathcal{K}_{un}$ acts on $\mathcal{T}_{e,un}^{\mathbb{O}(2),\mathbb{D}_n}(F) \simeq \mathcal{T}_{\mathrm{e}}\mathcal{F}_{un}$. The stratification is given by the invariants of the group $\mathbb{D}_n$.

Suppose we have a set of weights (a, b, c) for (z, λ, μ), with $a \in \mathbb{N}^n$. Let $f \in \mathcal{E}_{(z,\lambda,\mu)}^{\mathbb{O}(2),\mathbb{D}_n}$, the *weight* of f, $wt(f)$, is the minimum weight of the non-zero monomials in the Taylor series expansion of f. The *initial part* of f, denoted by f_{in}, is the sum of those monomials of degree exactly $wt(f)$. We say that two germs $f, g \in \mathcal{E}_{(z,\lambda,\mu)}^{\mathbb{O}(2),\mathbb{D}_n}$ are *topologically* $\mathcal{K}_{(\lambda,\mu)}^{\mathbb{O}(2),\mathbb{D}_n}$-equivalent if the changes of coordinates involved in $\mathcal{K}_{(\lambda,\mu)}^{\mathbb{O}(2),\mathbb{D}_n}$ are only homeomorphisms. Similarly, we define the notion of *topological* mapping between unfoldings, which induces the notion of *topological* $\mathcal{K}_{(\lambda,\mu)}^{\mathbb{O}(2),\mathbb{D}_n}$-versality. A germ f is *semiweighted homogeneous* if f_{in} has finite codimension.

Let

$$F(z, \lambda, \mu, \alpha) = f(z, \lambda, \mu) + \sum_{i=1}^{\operatorname{cod}_{\mathbb{D}_n}^{\mathbb{O}(2)}} \alpha_i p_i(z, \lambda, \mu),$$

with $\{p_i\}_{i=1}^{\mathrm{cod}_{\mathbb{D}_n}^{\mathbb{O}(2)}}$ projecting onto a weighted homogeneous basis of $\mathcal{N}_{\mathrm{e}}^{\mathbb{O}(2),\mathbb{D}_n}(f)$. To discuss the topological versality of the versal unfolding F of f, we construct the following sub-unfolding. Let

$$F_{\mathrm{in}}(z,\lambda,\mu,\beta) = f(z,\lambda,\mu) + \sum_{i=1}^{J} \beta_i p_i(z,\lambda,\mu),$$

where $wt(p_i) < wt(f)$, F is *versal in positive weight* if the following vector space is of finite dimension:

$$\frac{\mathcal{E}_{(z,\lambda,\mu,\beta)}^{\mathbb{O}(2),\mathbb{D}_n}}{\mathcal{T}_{\mathrm{e},un}^{\mathbb{O}(2),\mathbb{D}_n}(F_{\mathrm{in}}) + \langle p_1 \dots p_J\rangle_{\mathcal{E}_\beta}}. \tag{5.5}$$

THEOREM 5.7 (topological determinacy [4]). *Let $f \in \mathcal{E}_{(z,\lambda,\mu)}^{\mathbb{O}(2),\mathbb{D}_n}$ be semiweighted homogeneous. Then any $g \in \mathcal{E}_{(z,\lambda,\mu)}^{\mathbb{O}(2),\mathbb{D}_n}$ such that $g_{\mathrm{in}} = f_{\mathrm{in}}$ is topologically $\mathcal{K}_{(\lambda,\mu)}^{\mathbb{O}(2),\mathbb{D}_n}$-equivalent to f.*

THEOREM 5.8 (topological versality I [4]). *Let $f \in \mathcal{E}_{(z,\lambda,\mu)}^{\mathbb{O}(2),\mathbb{D}_n}$ be semiweighted homogeneous. Let F be an unfolding of f-versal in positive weight. Then F is topologically $\mathcal{K}_{(\lambda,\mu)}^{\mathbb{O}(2),\mathbb{D}_n}$-versal. Moreover, F is topologically $\mathcal{K}_{(\lambda,\mu)}^{\mathbb{O}(2),\mathbb{D}_n}$-trivial along the directions of weight larger or equal to $wt(f)$.*

For the next theorem, we let $f \in \mathcal{E}_{(z,\lambda,\mu)}^{\mathbb{O}(2),\mathbb{D}_n}$ be weighted homogeneous such that $\mathrm{cod}_{\mathbb{D}_n}^{\mathbb{O}(2)}(f) < \infty$. Let $m \geqslant 0$ and $\{\phi_i\}_{i=1}^{r}$ project to a basis for the normal space $\mathcal{N}_{\mathrm{e}}^{\mathbb{O}(2),\mathbb{D}_n}(f)$ of f having weights less than m and $\{\bar{\phi}_i\}_{i=1}^{r}$ project to the same basis having weights greater than or equal to m. In addition, we suppose that the images of $\{\phi_i\}_{i=1}^{r}$ and $\{\bar{\phi}_i\}_{i=1}^{r}$ in $\mathcal{T}_{e,un}\mathcal{K}_{(\lambda,\mu)}^{\mathbb{O}(2),\mathbb{D}_n}$ are weighted homogeneous. Let F a $\mathcal{K}_{(\lambda,\mu)}^{\mathbb{O}(2),\mathbb{D}_n}$-versal unfolding of f and F_{in} as before. The Euler vector field is defined by

$$e = wt(z)z(F_{\mathrm{in}})_z + wt(\lambda)\lambda(F_{\mathrm{in}})_\lambda + wt(\mu)\mu(F_{\mathrm{in}})_\mu - wt(f)f.$$

By the generalized preparation theorem, we may write

$$g(\lambda,\mu)e = \sum_{i=1}^{s} h_{i,k}(\alpha)\bar{\phi}_i \mod (\langle \phi_i\rangle_{\mathcal{E}_\alpha} + \mathcal{T}_{\mathrm{e},un}^{\mathbb{O}(2),\mathbb{D}_n}(F)), \tag{5.6}$$

with polynomial coefficients $h_{ik}(\alpha)$.

THEOREM 5.9 (topological versality II [4]). *Suppose that, considered as a complex matrix, there is a k such that the $(k \times s)$-matrix $M = (h_{ik}(\alpha))$ has rank s on $\mathbb{C}^\alpha - \{0\}$. Then F is topologically $\mathcal{K}_{(\lambda,\mu)}^{\mathbb{O}(2),\mathbb{D}_n}$-versal.*

6. Proofs

6.1. Calculations for the $(\mathbb{O}(2), \mathbb{D}_n)$-symmetry-breaking theory

First we look at the $\mathbb{O}(2)$-part of the problem. The results can be found in [11] or are easily computed directly.

6.1.1. $\mathbb{O}(2)$-*theory in* $\mathcal{E}^{\mathbb{O}(2)}_{(z,\lambda)}$

We identify $\mathcal{E}^{\mathbb{O}(2)}_{(z,\lambda)}$ and $\mathcal{E}_{(u,\lambda)}$ via $f(z,\lambda) = p(u,\lambda)z$. Thus

$$\mathcal{T}^{\mathbb{O}(2)}_{\mathrm{e}}(f) \approx \langle p, up_u\rangle_{\mathcal{E}_{(u,\lambda)}} + \langle p_\lambda\rangle_{\mathcal{E}_\lambda}$$

and

$$\mathcal{P}(f) \approx \mathrm{Itr}\{\langle uh, \lambda h, u^2 h_u, u\lambda h_u\rangle_{\mathcal{E}_{(u,\lambda)}} + \langle \lambda^2 h_\lambda\rangle_{\mathcal{E}_\lambda}\},$$

where $\mathrm{Itr}(V)$ represents the intrinsic part of the vector space V. We shall be concerned with two classes of normal forms in $\mathcal{E}^{\mathbb{O}(2)}_{(z,\lambda)}$ (of codimension $m-1$),

$$\begin{aligned} \mathrm{I}_m\colon\quad & p^i_m(u,\lambda)z = (\epsilon_1 u + \delta_m \lambda^m)z,\\ \mathrm{II}_m\colon\quad & p^{ii}_m(u,\lambda)z = (\epsilon_m u^m + \delta_1 \lambda)z, \end{aligned}$$

where $m = 1, 2, 3$, $\epsilon_m = \operatorname{sgn} p^{\mathrm{o}}_{u^m}$, $\delta_m = \operatorname{sgn} p^{\mathrm{o}}_{\lambda^m}$ (for both classes). The universal unfoldings for the previous germs are given by

$$H^i_m(u,\lambda,\alpha)z = \left(\epsilon_1 u + \delta_m\lambda^m + \sum_{i=0}^{m-2}\alpha_i\lambda^i\right)z$$

or

$$H^{ii}_m(u,\lambda,\alpha)z = \left(\epsilon_m u^m + \delta_1\lambda + \sum_{i=1}^{m-2}\alpha_i u^i\right)z.$$

6.1.2. *Determination of* $\Phi^{f_1}_1$ *and* $\Phi^{f_1}_2$

Recall that $\Phi^{f_1}_2 \subset \Phi^{f_1}_1 \subset k^{\mathbb{O}(2)}_\lambda$ and, as such, any element $(T_1, X_1, \Lambda_1) \in \Phi^{f_1}_1$ can be represented by germs (A, B, C, D), where $A, B, C \in \mathcal{E}_{(u,\lambda)}$ and $D \in \mathcal{E}_\lambda$ because $T_1(z,\lambda)w = Aw + Bz^2\bar{w}$, $X_1(z,\lambda) = Cz$ and $\Lambda_1(\lambda) = D\lambda$. The next proposition gives explicitly the generators of $\Phi^{f_1}_1$.

Proposition 6.1 (cf. [6]).

(1) *For a normal form of the type* $h^{ii}_m(u,\lambda)z = (\epsilon_m u^m + \delta_1\lambda)z$, $\Phi^{f_1}_1$ *is generated over* $\mathcal{E}_{(u,\lambda)}$ *by* $(-u, 1, 0, 0)$ (*already in* $\hat{s}$), *by*

$$\Phi_{11} = (-(2m+1)\delta_1 u^m - \epsilon_m\lambda, 0, \epsilon_m\lambda + \delta_1 u^m, 0)$$

and over $\mathcal{E}_\lambda$ *by* $\Phi_{12} = (-(2m+1), 0, 1, 2m)$.

(2) *For a normal form of the type* $h^i_m(u,\lambda)z = (\epsilon_1 u + \delta_m\lambda^m)z$, $\Phi^{f_1}_1$ *is generated over* $\mathcal{E}_{(u,\lambda)}$ *by* $(-u, 1, 0, 0)$ (*already in* $\hat{s}$), *by*

$$\Phi_{11} = (3\delta_m u + \epsilon_1\lambda^m, 0, -\epsilon_1\lambda^m - \delta_m u, 0)$$

and over $\mathcal{E}_\lambda$ *by* $\Phi_{12} = (-3m, 0, m, 2)$.

(3) *For our normal forms* h_i, $1 \leqslant i \leqslant 6$, *the part of* $\Phi^{f_1}_2$, *not in* $\hat{s}$ *is generated over* $\mathcal{E}_{(u,\lambda)}$ *by* Φ_{11} *and over* $\mathcal{E}_\lambda$ *by* $\lambda\Phi_{12}$.

REMARK 6.2. The generators of $\Phi_1^{f_1}$, $\Phi_2^{f_1}$ act on $f_2 = (r, s)$ as

$$\mu \begin{bmatrix} Ar + uBr + 2vBs + Cr + 2uCr_u + nvCr_v + D\lambda r_\lambda \\ As - uBs + (n-1)Cs + 2uCs_u + nvCr_v + D\lambda s_\lambda \end{bmatrix}. \tag{6.1}$$

6.2. Explicit tangent spaces

6.2.1. *Tangent space for $\mathbb{D}_n$-equivariant maps ($n \neq 4$)*

Recall that the general $\mathbb{D}_n$-equivariant map is of the form (2.1) when $n \neq 4$,

$$f(z, \lambda, \mu) = [p(u, \lambda) + \mu r(u, v, \lambda, \mu)]z + \mu s(u, v, \lambda, \mu)\bar{z}^{n-1}.$$

The generators of the extended tangent space in $\mu\mathcal{E}^{\mathbb{D}_n}_{(z,\lambda,\mu)}$ are (in z, $\bar{z}^{n-1}$)

$$\begin{aligned} &\left\langle \begin{bmatrix} p + \mu r \\ \mu s \end{bmatrix}, \begin{bmatrix} \mu u^{n-2} s \\ p + \mu r \end{bmatrix}, \begin{bmatrix} (v^2 - u^n) r_v \\ (v^2 - u^n) s_v \end{bmatrix}, \begin{bmatrix} 2up_u + \mu(2ur_u + nvr_v) \\ \mu((n-2)s + 2us_u + nvs_v) \end{bmatrix}, \right. \\ &\qquad \left. \begin{bmatrix} vs \\ -us \end{bmatrix}, \begin{bmatrix} u^{n-1} s \\ -vs \end{bmatrix}, \begin{bmatrix} 2vp_u + \mu(2vr_u + nu^{n-1} r_v + (n-2)u^{n-2} s) \\ \mu(2vs_u + nu^{n-1} s_v) \end{bmatrix} \right\rangle_{\mathcal{E}_{(u,v,\lambda,\mu)}} \\ &\qquad\qquad + \left\langle \begin{bmatrix} p_\lambda + \mu r_\lambda \\ \mu s_\lambda \end{bmatrix}, \begin{bmatrix} r + \mu r_\mu \\ s + \mu s_\mu \end{bmatrix} \right\rangle_{\mathcal{E}_{(\lambda,\mu)}} + \Phi_1^{f_1}(f_2), \end{aligned} \tag{6.2}$$

where $\Phi_1^{f_1}(f_2)$ is the image of the action of the elements of $\Phi_1^{f_1}$ on f (see (6.1)).

6.2.2. *Tangent space for $\mathbb{D}_4$-equivariant maps*

The general $\mathbb{D}_4$-equivariant map is of the general form (2.2),

$$f(z, \lambda, \mu) = [p(u, \lambda) + \mu r(u, \Delta, \lambda, \mu)]z + \mu s(u, \Delta, \lambda, \mu)\delta\bar{z}.$$

The generators of the extended tangent space in $\mu\mathcal{E}^{\mathbb{D}_4}_{(z,\lambda,\mu)}$ are (in z, $\delta\bar{z}$)

$$\begin{aligned} &\left\langle \begin{bmatrix} p + \mu r \\ \mu s \end{bmatrix}, \begin{bmatrix} 0 \\ p + \mu(r - us) \end{bmatrix}, \begin{bmatrix} \Delta s \\ us \end{bmatrix}, \begin{bmatrix} 0 \\ (u^2 - \Delta)s \end{bmatrix}, \begin{bmatrix} \Delta(\Delta - u^2) r_\Delta \\ \Delta(\Delta - u^2) s_\Delta \end{bmatrix}, \right. \\ &\qquad \left. \begin{bmatrix} up_u + \mu(ur_u + 2\Delta r_\Delta) \\ \mu(s + us_u + 2\Delta s_\Delta) \end{bmatrix}, \begin{bmatrix} \Delta p_u + \mu(\Delta r_u + 2u\Delta r_\Delta) \\ \mu(us + \Delta s_u + 2u\Delta s_\Delta) \end{bmatrix} \right\rangle_{\mathcal{E}_{(u,\Delta,\lambda,\mu)}} \\ &\qquad\qquad + \left\langle \begin{bmatrix} p_\lambda + \mu r_\lambda \\ \mu s_\lambda \end{bmatrix}, \begin{bmatrix} r + \mu r_\mu \\ s + \mu s_\mu \end{bmatrix} \right\rangle_{\mathcal{E}_{(\lambda,\mu)}} + \Phi_1^{f_1}(f_2). \end{aligned} \tag{6.3}$$

6.3. Pre-normal forms

For the classification of the $\mathbb{D}_n$-forced symmetry breaking, it is useful to get simplified *a priori* forms for the bifurcation germs.

PROPOSITION 6.3.

(a) *When $\epsilon = s^{\circ} \neq 0$, f can be cast into*

$$f(z, \Lambda) = (p(u, \lambda) + \mu(r_1(u, \lambda, \mu) + vr_2(u, \lambda, \mu)))z + \mu\epsilon\bar{z}^{n-1}. \tag{6.4}$$

(b) *If* $s^{\rm o} = 0$ *and* $p_u^{\rm o} \neq 0$, *one can change coordinates so that* r *depends on* (λ, μ) *only and so* f *can be cast into*

$$f(z, \Lambda) = (\epsilon_1 u + \delta_m \lambda^m + \mu r(\lambda, \mu))z + \mu s(v, \lambda, \mu)\bar{z}^{n-1}. \tag{6.5}$$

(c) *If* $p_\lambda^{\rm o} \neq 0$, f *can be cast into*

$$f(z, \Lambda) = (\epsilon_m u^m + \delta_1 \lambda + \mu r(u, v, \mu))z + \mu s(u, v, \mu)\bar{z}^{n-1}. \tag{6.6}$$

Proof. (a) We use $T(z, \lambda, \mu)\omega = (1 + \mu t(u, v, \lambda, \mu))\omega$ for some $t \in \mathcal{E}^{\mathbb{D}_n}_{(z,\lambda,\mu)}$ to be determined to get rid of the higher-order terms in μ. Thus

$$(T \cdot f)(z, \lambda, \mu) = (p(u, \lambda) + \mu\hat{r}(u, v, \lambda, \mu))z + \mu(s(u, v, \lambda, \mu) + \mu t(u, v, \lambda, \mu)s(u, v, \lambda, \mu))\bar{z}^{n-1}.$$

We want to find t such that $s(u, v, \lambda, \mu) + \mu t(u, v, \lambda, \mu)s(u, v, \lambda, \mu) = \hat{s}(u, v, \lambda)$, where $\hat{s}$ is to be determined (so eliminating μ from s). As $\hat{s}^{\rm o} \neq 0$, we can write $[s(u, v, \lambda, \mu)]^{-1} = b(u, v, \lambda)(1 + \mu\hat{a}(u, v, \lambda, \mu))$ and take $t(u, v, \lambda) = \hat{a}(u, v, \lambda, \mu)$, so

$$f_{pn1}(z, \Lambda) = (p(u, \lambda) + \mu\hat{r}(u, v, \lambda, \mu))z + \mu\hat{s}(u, v, \lambda)\bar{z}^{n-1}.$$

In a second step, we use $\hat{S}(\mathbb{O}(2), \mathbb{D}_n)$ to get rid of the terms in u and v in $\hat{s}$. From proposition 5.1, $S(z, \lambda)\omega = \omega + \frac{1}{4}h_1(u, v, \lambda)V(z)\omega + \frac{1}{4}h_2(u, v, \lambda)W(z)\omega$, and so

$$S \cdot [(p + \mu\hat{r})z + \mu\hat{s}\bar{z}^{n-1}] = (p + \mu\hat{r})z + \mu\hat{s}\bar{z}^{n-1} + \tfrac{1}{4}h_1(2\mu v\hat{s}z - 2\mu u\hat{s}\bar{z}^{n-1}) + \tfrac{1}{4}h_2 \cdot (2\mu u^{n-1}\hat{s}z - 2\mu v\hat{s}\bar{z}^{n-1}).$$

We want to find h_1, h_2 and determine h such that

$$\hat{s}(u, v, \lambda) - \tfrac{1}{2}uh_1(u, v, \lambda)\hat{s}(u, v, \lambda) - \tfrac{1}{2}vh_2(u, v, \lambda)\hat{s}(u, v, \lambda) = h(\lambda).$$

As $s^{\rm o} \neq 0$, we can write

$$[\hat{s}(u, v, \lambda)]^{-1} = f(\lambda)(1 + \hat{a_1}(u, v, \lambda)u + \hat{a_2}(u, v, \lambda)v).$$

Taking $-\frac{1}{2}h_1(u, v, \lambda) = \hat{a_1}(u, v, \lambda)$ and $-\frac{1}{2}h_2(u, v, \lambda) = \hat{a_2}(u, v, \lambda)$, we get h as required. So we end up with the following pre-normal form:

$$f_{pn2}(z, \Lambda) = (p(u, \lambda) + \mu\hat{r}(u, v, \lambda, \mu))z + \mu\hat{\hat{s}}(\lambda)\bar{z}^{n-1}.$$

Because $\hat{\hat{s}}^{\rm o} \neq 0$, we can transform $\mu\hat{\hat{s}}$ into $\epsilon\mu$ via a change of coordinates in μ and find $f_{pn3}(z, \Lambda) = (p(u, \lambda) + \mu\hat{r}(u, v, \lambda, \mu))z + \mu\epsilon\bar{z}^{n-1}$ with $\epsilon = \operatorname{sgn} s^{\rm o}$. Again, we use $\hat{S}(\mathbb{O}(2), \mathbb{D}_n)$ to change $\hat{r}$. Take

$$S(z, \lambda, \mu)\omega = \omega + \tfrac{1}{4}h_1(u, v, \lambda, \mu)vV(z)\omega - \tfrac{1}{4}h_1(u, v, \lambda, \mu)uW(z)\omega.$$

We get $S \cdot f_{pn3} = pz + \mu(\hat{r} + \frac{1}{2}h_1\epsilon v^2 - \frac{1}{2}h_1\epsilon u^n)z + \mu\epsilon\bar{z}^{n-1}$.

We want to find h_1 that solves the equation

$$\hat{r}(u, v, \lambda, \mu) + \tfrac{1}{2}h_1(u, v, \lambda, \mu)\epsilon v^2 - \tfrac{1}{2}h_1(u, v, \lambda, \mu)\epsilon u^n = r_1(u, \lambda, \mu) + vr_2(u, \lambda, \mu), \tag{6.7}$$

where r_1, r_2 are to be determined. First, we may write

$$\hat{r}(u,v,\lambda,\mu) = R_1(u,v^2,\lambda,\mu) + vR_2(u,v^2,\lambda,\mu)$$

and now we decompose R_1 in the following way,

$$R_1(u,v^2,\lambda,\mu) = R_{11}(u,v^2-u^n,\lambda,\mu) + R_{12}(u,\lambda,\mu),$$

where $R_{11}(u,0,\lambda,\mu)=0$, and so

$$R_1(u,v^2,\lambda,\mu) = (v^2-u^n)\hat{R}_{11}(u,v^2-u^n,\lambda,\mu) + R_{12}(u,\lambda,\mu).$$

Decomposing R_2 in the same way and replacing in (6.7), we can find h_1.

The pre-normal form given by (b) is obtained as follows. To get rid of higher-order terms in u from r and s, we use

$$T(z,\lambda,\mu)\omega = (1+\mu t_1(u,v,\lambda,\mu))\omega + \mu t_2(u,v,\lambda,\mu)\bar{z}^{n-2}\bar{\omega}$$

for some $t_1, t_2 \in \mathcal{E}^{\mathbb{D}_n}_{(z,\lambda,\mu)}$ to be determined. From the $\mathbb{O}(2)$-classification, we take

$$p(u,\lambda) = \epsilon_1 u + \delta_m \lambda^m,$$

and applying T on f, we get

$$(\epsilon_1 u + \delta_m\lambda^m)z + \mu(r + t_1\cdot(\epsilon_1 u + \delta_m\lambda^m + \mu r) + \mu t_2 s u^{n-2})z \\ + \mu(s + \mu t_1 s + t_2\cdot(\epsilon_1 u + \delta_m\lambda^m + \mu r))\bar{z}^{n-1}.$$

We want to find t_1, t_2 and determine $g, h \in \mathcal{E}^{\mathbb{D}_n}_{(z,\lambda,\mu)}$ such that

$$t_1(u,v,\lambda,\mu)(\epsilon_1 u + \delta_m\lambda^m + \mu r(u,v,\lambda,\mu)) + \hat{r}(u,v,\lambda,\mu) = g(v,\lambda,\mu), \quad (6.8)$$

$$t_2(u,v,\lambda,\mu)(\epsilon_1 u + \delta_m\lambda^m + \mu r(u,v,\lambda,\mu)) + \hat{s}(u,v,\lambda,\mu) = h(v,\lambda,\mu). \quad (6.9)$$

Both equations can be solved in the same way. We just show how to handle (6.8). Let $F(u,v,\lambda,\mu) = \epsilon_1 u + \delta_m\lambda^m + \mu r(u,v,\lambda,\mu)$. From the implicit function theorem, there is a unique solution $u = \bar{u}(v,\lambda,\mu)$ to the equation $F(u,v,\lambda,\mu)=0$. Consider the following module homomorphism, $\phi : \mathcal{E}^{\mathbb{D}_n}_{(z,\lambda,\mu)} \to \mathcal{E}_{(v,\lambda,\mu)}$, $f \mapsto \phi(f)$, such that

$$\phi(f)(u,v,\lambda,\mu) = f(\bar{u}(v,\lambda,\mu),v,\lambda,\mu).$$

Let $f \in \ker\phi$. As f and F have the same zero set of solutions, they are $\mathcal{C}$-equivalent and so

$$\ker\phi = \langle \epsilon_1 u + \delta_m\lambda^m + \mu r(u,v,\lambda,\mu)\rangle.$$

As ϕ is surjective, $\mathcal{E}^{\mathbb{D}_n}_{(z,\lambda,\mu)}/\ker\phi$ is isomorphic to $\mathcal{E}_{(v,\lambda,\mu)}$. Hence there exist $t_1 \in \mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)}$ and $g \in \mathcal{E}_{(v,\lambda,\mu)}$ such that

$$t_1(u,v,\lambda,\mu)(\epsilon_1 u + \delta_m\lambda^m + \mu r(u,v,\lambda,\mu)) + \hat{r}(u,v,\lambda,\mu) = g(v,\lambda,\mu),$$

which is what we need to solve (6.8). Therefore, we get the pre-normal form

$$f_{pn1}(z,\Lambda) = (\epsilon_1 u + \delta_m\lambda^m + \mu r(v,\lambda,\mu))z + \mu s(v,\lambda,\mu)\bar{z}^{n-1}.$$

With a lengthy calculation and now using elements of the group $\hat{M}(\mathbb{O}(2),\mathbb{D}_n)$, we can eliminate v from r and get the required normal form.

The case under (c) works similarly to the first part of the proof of case (b). $\square$

Corollary 6.4.

(a) *When* $p(u,\lambda)=\epsilon_1 u+\delta_1\lambda$, f *can be cast into*

$$f(z,\Lambda)=(\epsilon_1 u+\delta_1\lambda)z+\mu s(v,\lambda,\mu)\bar{z}^{n-1}. \tag{6.10}$$

(b) *If* $p(u,\lambda)=\epsilon_1 u+\delta_m\lambda^m$ *and* $s^{\mathrm{o}}\neq 0$, f *can be cast into*

$$f(z,\Lambda)=(\epsilon_1 u+\delta_m\lambda^m+\mu r(\lambda,\mu))z+\mu\epsilon\bar{z}^{n-1}. \tag{6.11}$$

Proof. Both cases follow from an even application of the same techniques. □

6.4. Proof of the classification theorem

In this part, we deal with some classification results. Note that the smooth classification can involve many moduli and becomes rapidly complicated (cf. [6], for instance). Moreover, the main information of practical importance for the study of the bifurcation diagrams is the topological type of germs and unfoldings. So our result is about the topological codimension-1 classification of the germs.

Proof. (a) From proposition 6.3 (a), we get the following pre-normal form:

$$f(z,\lambda,\mu)=(\epsilon_1 u+\delta_1\lambda)z+\mu r(\mu)z+\mu\epsilon\bar{z}^{n-1}.$$

Making the change of coordinates $\lambda\mapsto\lambda-(\mu/\delta_1)r(\mu)$ and $\mu\mapsto\mu$, we get (2.3). The generators of the extended tangent space (ignoring $\Phi_1^{f_1}(f_2)$) are

$$\left\langle\begin{bmatrix}\epsilon_1 u+\delta_1\lambda\\ \epsilon\mu\end{bmatrix},\begin{bmatrix}\epsilon\mu u^{n-2}\\ \epsilon_1 u+\delta_1\lambda\end{bmatrix},\begin{bmatrix}v\\ -u\end{bmatrix},\right.$$
$$\left.\begin{bmatrix}u^{n-1}\\ -v\end{bmatrix},\begin{bmatrix}2\epsilon_1 u\\ (n-2)\mu\epsilon\end{bmatrix},\begin{bmatrix}2\epsilon_1 v+\epsilon(n-2)\mu u^{n-2}\\ 0\end{bmatrix}\right\rangle_{\mathcal{E}_{(u,v,\lambda,\mu)}}+\left\langle\begin{bmatrix}\delta_1\\ 0\end{bmatrix},\begin{bmatrix}0\\ \epsilon\end{bmatrix}\right\rangle_{\mathcal{E}_{\lambda,\mu}}.$$

Simplifying the generators modulo $\mathcal{T}_e^{\mathbb{O}(2),\mathbb{D}_n}(f)$, we get that $\mathcal{N}_e^{\mathbb{O}(2),\mathbb{D}_n}(f)=\{0\}$ and hence the codimension of the germ is 0.

(b) (I_1^n.) From corollary 6.4 (a), the pre-normal form is

$$f(z,\lambda,\mu)=\begin{cases}(\epsilon_1 u+\delta_1\lambda)z+\mu s(v,\lambda,\mu)\bar{z}^{n-1}, & n\neq 4,\\ (\epsilon_1 u+\delta_1\lambda)z+\mu s(v,\lambda,\mu)\delta\bar{z}, & n=4.\end{cases} \tag{6.12}$$

The generators of the extended tangent space are

$$\left\langle\begin{bmatrix}\epsilon_1 u+\delta_1\lambda\\ \mu(av+\delta_4\lambda+\delta_5\mu)\end{bmatrix},\begin{bmatrix}\mu u^{n-2}(av+\delta_4\lambda+\delta_5\mu)\\ \epsilon_1 u+\delta_1\lambda\end{bmatrix},\begin{bmatrix}0\\ a(v^2-u^n)\end{bmatrix},\right.$$
$$\begin{bmatrix}v(av+\delta_4\lambda+\delta_5\mu)\\ -u(av+\delta_4\lambda+\delta_5\mu)\end{bmatrix},\begin{bmatrix}2\epsilon_1 v+(n-2)\mu u^{n-2}(av+\delta_4\lambda+\delta_5\mu)\\ na\mu u^{n-1}\end{bmatrix},$$
$$\left.\begin{bmatrix}u^{n-1}(av+\delta_4\lambda+\delta_5\mu)\\ -v(av+\delta_4\lambda+\delta_5\mu)\end{bmatrix},\begin{bmatrix}2\epsilon_1 u\\ (n-2)\mu(av+\delta_4\lambda+\delta_5\mu)+nav\mu\end{bmatrix}\right\rangle_{\mathcal{E}_{(u,v,\lambda,\mu)}}$$
$$+\left\langle\begin{bmatrix}\delta_1\\ \delta_4\mu\end{bmatrix},\begin{bmatrix}-2\delta_1\delta_4\delta_5\mu\\ av+\delta_4\lambda\end{bmatrix}\right\rangle_{\mathcal{E}_{(\lambda,\mu)}}+\left\langle\begin{bmatrix}(4-n)\delta_1\delta_4\delta_5\mu\\ 2(n-2)av+(n-2)\delta_4\lambda\end{bmatrix}\right\rangle_{\mathcal{E}_{\lambda}}.$$

To compute the normal space, let

$$M = \left\langle \begin{bmatrix} \epsilon_1 u + \delta_1\lambda \\ \mu(av + \delta_4\lambda + \delta_5\mu) \end{bmatrix}, \begin{bmatrix} \mu u^{n-2}(av + \delta_4\lambda + \delta_5\mu) \\ \epsilon_1 u + \delta_1\lambda \end{bmatrix}, \begin{bmatrix} 0 \\ a(v^2 - u^n) \end{bmatrix}, \begin{bmatrix} v(av + \delta_4\lambda + \delta_5\mu) \\ -u(av + \delta_4\lambda + \delta_5\mu) \end{bmatrix}, \begin{bmatrix} 2\epsilon_1 v + (n-2)\mu u^{n-2}(av + \delta_4\lambda + \delta_5\mu) \\ nau^{n-1}\mu \end{bmatrix}, \begin{bmatrix} u^{n-1}(av + \delta_4\lambda + \delta_5\mu) \\ -v(av + \delta_4\lambda + \delta_5\mu) \end{bmatrix}, \begin{bmatrix} 2\epsilon_1 u \\ (n-2)\mu(av + \delta_4\lambda + \delta_5\mu) + nav\mu \end{bmatrix} \right\rangle_{\mathcal{E}_{(u,v,\lambda,\mu)}},$$

and we consider the quotient

$$\mathcal{N}_e^{\mathbb{O}(2),\mathbb{D}_n}(f) = \frac{\mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)}/M}{\mathcal{T}_{\mathrm{e}}^{\mathbb{O}(2),\mathbb{D}_n}(f)/M}.$$

Let $N = \mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)}/M$. After simplification, $N/\mathcal{M}_{(\lambda,\mu)} \cdot N$ is generated over $\mathbb{R}$ by $(1,0), (0,1)$ and $(0,v)$, and so N is equal to the quotient of the $\mathcal{E}_{(\lambda,\mu)}$-module generated by the same elements by

$$\left\langle \begin{bmatrix} \delta_1 \\ \delta_4\mu \end{bmatrix}, \begin{bmatrix} 0 \\ \lambda\mu \end{bmatrix}, \begin{bmatrix} 0 \\ -\lambda^2 + 2\mu^2 \end{bmatrix}, \begin{bmatrix} -2\delta_1\lambda \\ -n\delta_4\lambda\mu + (-3n+4)\delta_5\mu^2) \end{bmatrix} \right\rangle_{\mathcal{E}_{(\lambda,\mu)}} + \left\langle \begin{bmatrix} 0 \\ (2-n)\delta_4\lambda + (-3n+4)\delta_5\mu \end{bmatrix} \right\rangle_{\mathcal{E}_\lambda}$$

to get $\alpha\bar{z}^{n-1}$ as the unfolding term and a as a modal parameter.

For the case II_1^n, we take $p(u,\lambda) = \epsilon_1 u + \delta_2\lambda^2$ and assume that $r^{\mathrm{o}} \neq 0$ and $s^{\mathrm{o}} \neq 0$. From corollary 6.4, the pre-normal form is

$$f(z,\lambda,\mu) = \begin{cases} (\epsilon_1 u + \delta_2\lambda^2 + \delta_6\mu)z + \mu\epsilon\bar{z}^{n-1}, & n \neq 4, \\ (\epsilon_1 u + \delta_2\lambda^2 + \delta_6\mu)z + \mu\epsilon\delta\bar{z}, & n = 4. \end{cases} \tag{6.13}$$

Computing the tangent space as in I_1^n, we get αz as an unfolding parameter.

For the case III_1^n, we consider $p(u,\lambda) = \epsilon_2 u^2 + \delta_1\lambda$ and $s^{\mathrm{o}} \neq 0$. The pre-normal form is (6.4). We remark that when $n = 8$, the minimum C^∞-codimension is again 3.

To prove that all the germs in the classification have topological codimension 1, we follow § 5.5. For the case I_1^n, we consider a subgroup $\mathcal{K}_{un}$ that fixes the $(p+\mu r)$-component in the unfolding of the normal forms. In this case,

$$\mathcal{E}^{\mathbb{O}(2),\mathbb{D}_n}_{(z,\lambda,\mu)} = (p + \mu r) + \mathcal{F}_{un}, \quad \text{with } \mathcal{F}_{un} = \{\mu s\bar{z}^{n-1} \mid s = s(v,\lambda,\mu,\alpha)\}.$$

The tangent space $\mathcal{T}_{e,un}^{\mathbb{O}(2),\mathbb{D}_n}(F_{\mathrm{in}})$ related to the restricted action is in $\mu\mathcal{E}^{\mathbb{D}_n}_{(z,\lambda,\mu)}$, and some of its generators over $\mathcal{E}_{(\lambda,\mu,\alpha)}$ are

$$(n-4)s + 2\delta_4\lambda + nav,$$
$$2u + 2\epsilon_1 u + \delta_1\lambda - (n-2)\mu^2 u^{n-3}s^2 - na\mu^2 v\mu^{n-3}s,$$
$$-4\epsilon_1 us - 2na\mu u^{n-1}s + \epsilon_1(n-2)^2\mu^2 u^{n-3}s^3 + \epsilon_1 na(n-2)\mu^2 u^{n-3}vs^2,$$
$$2\epsilon_1 vs + (n-2)\mu u^{n-2}s^2 + na\mu vu^{n-2}s,$$
$$s + c\mu$$

and finally 1, but over $\mathcal{E}_\alpha$. Those generators are enough to show that f_{in} has finite codimension, the unfolding is versal in positive weight and so the result follows from theorem 5.8.

The case II_2^n is obvious and we apply theorem 5.9 for case III_1^n. Let

$$F_{in}(z,\lambda,\mu,\alpha) = (\epsilon_2 u^2 + \delta_1\lambda + \alpha u)z + \epsilon\mu\bar{z}^2,$$

$\phi_1 = uz$, $\bar{\phi}_1 = \mu uz$, $\bar{\phi}_2 = \mu vz$ and $\bar{\phi}_3 = \mu^2 uz$. The Euler vector field is $e = -2\alpha uz$ and we decompose the following vectors, modulo $\langle \phi_1^{f_1}\rangle_{\mathcal{E}_\alpha} + \mathcal{T}_{\text{e},un}^{\mathbb{O}(2),\mathbb{D}_n}(F_{in})$. We get

$$\begin{aligned} \mu e &= -2\alpha\bar{\phi}_1, \\ \mu^2 e &= -2\alpha\bar{\phi}_3, \\ \lambda e &= \tilde{a}_1(\alpha)\bar{\phi}_1 + \tilde{a}_2(\alpha)\bar{\phi}_2 + \tilde{a}_3(\alpha)\bar{\phi}_3, \end{aligned}$$

with $\tilde{a}_2(0) \neq 0$. The matrix M has rank 3 if and only if $\alpha \neq 0$, therefore ξ_1 and ξ_2 are topologically irrelevant. Note that the germ in III_1^4 is weighted homogeneous and we prove that the topological codimension is 1 by constructing the Euler vector field and using theorem 5.9 as in the previous case.

When $n \geqslant 5$, the germ is semi-weighted homogeneous and the proof is similar to case I_1^n. We consider a subgroup $\mathcal{K}_{un}$, which fixes the p- and μs-components in the unfolding of the normal forms. In this case,

$$\mathcal{E}_{(z,\lambda,\mu)}^{\mathbb{O}(2),\mathbb{D}_n} = (p + \mu s) + \mathcal{F}_{un},$$

with

$$\mathcal{F}_{un} = \{\mu(r_1 + vr_2) \mid r_1 = r_1(u,\lambda,\mu,\alpha) \text{ and } r_2 = r_2(u,\lambda,\mu,\alpha)\}.$$

The tangent space $\mathcal{T}_{e,un}^{\mathbb{O}(2),\mathbb{D}_n}(F_{\text{in}})$ related to the restricted action is in $\mu\mathcal{E}_{(u,\lambda,\mu,\alpha)} \cdot \langle(1,0),(v,0)\rangle$. □

Appendix A. Lyapunov–Schmidt reduction

Let $C^{2,\text{per}} \subset C^{0,\text{per}}$ be the spaces of twice continuously differentiable (respectively, continuous) complex-valued 2π-periodic functions. The map $F : C^{2,\text{per}} \to C^{0,\text{per}}$ of (3.1) has a bifurcation at $(0,0,0)$ because its linearization at the origin,

$$\mathcal{L}\phi = F_\theta(0,0,0,\cdot)\phi = \ddot{\phi} + \phi,$$

has a non-trivial kernel. From the Lyapunov–Schmidt reduction technique, we get the bifurcation equations for (3.1). Consider a projection $P : C^{0,\text{per}} \to \ker\mathcal{L}$ and define $Q = I - P$. The projections define a splitting on $C^{0,\text{per}}$ by $\theta = \Phi \oplus \Psi$, where $\Phi \in \ker\mathcal{L}$ and $\Phi \in \operatorname{Im}\mathcal{L}$. Equation (3.1) is equivalent to

$$PF(\theta,\lambda,\mu,\cdot) = 0, \tag{A 1}$$

$$QF(\theta,\lambda,\mu,\cdot) = 0. \tag{A 2}$$

Define $H(\Phi,\Psi,\lambda,\mu) = QF(\Phi+\Psi,\lambda,\mu,\cdot)$. Using the implicit function theorem, we can solve $H = 0$ for Ψ as a function of (Φ,λ,μ). We denote by $\tilde{w}(\Phi,\lambda,\mu)$ this solution. Equation (3.1) is now equivalent to the *reduced bifurcation equation*

$$f(\Phi,\lambda,\mu) = PF(\Phi + \tilde{w}(\Phi,\lambda,\mu),\lambda,\mu,\cdot) = 0. \tag{A 3}$$

Now we introduce complex coordinates. We identify $\ker\mathcal{L} = \langle e^{it}, e^{-it}\rangle_{\mathbb{R}}$, with $\mathbb{C}$ via $\chi : \mathbb{C} \to \ker\mathcal{L}$ defined as $\chi(z) = \frac{1}{2}(ze^{-it} + \bar{z}e^{it})$. Define $\Pi : C^{0,\mathrm{per}} \to \mathbb{C}$ by

$$\Pi(\theta) = \frac{1}{\pi}\int_0^{2\pi} e^{it}\theta(t)\,dt.$$

Note that Π is the left inverse of χ. We define the projector P as $P(\theta) = \chi(\Pi(\theta))$. Using those coordinates, we get

$$f(z,\lambda,\mu) = \Pi[F(\chi(z) + \tilde{w}(\chi(z),\lambda,\mu),\lambda,\mu)] = 0. \tag{A 4}$$

We find the derivatives of f using Taylor series expansions. The chain rule applied to $F(v + w(v,\Lambda),\Lambda,\cdot)$ gives the following derivatives up to order 2:

$$\begin{aligned}
\partial_\lambda&: \quad F_v w_\lambda + F_\lambda,\\
\partial_\mu&: \quad F_v w_\mu + F_\mu,\\
\partial_{zz}&: \quad F_{vv}(I + w_v)^2 + F_v w_{vv},\\
\partial_{z\lambda}&: \quad F_{vv}(I + w_v)w_\lambda + F_{v\lambda}(I + w_v) + F_v w_{v\lambda},\\
\partial_{\lambda\lambda}&: \quad F_{vv}(w_\lambda)^2 + 2F_{v\lambda}w_\lambda + F_{\lambda\lambda} + F_v w_{\lambda\lambda},\\
\partial_{\mu z}&: \quad F_{vv}(I + w_v)w_\mu + F_{v\mu}(I + w_v) + F_v w_{v\mu},\\
\partial_{\mu\lambda}&: \quad F_{vv}w_\lambda w_\mu + F_{v\lambda}w_\mu + F_{v\mu}w_\lambda + F_{\lambda\mu} + F_v w_{\lambda\mu}.
\end{aligned}$$

The other derivatives are obtained by additional differentiation.

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References

1 T. J. Bridges and J. E. Furter. *Singularity theory and equivariant symplectic maps*. Lecture Notes in Mathematics, vol. 1558 (Springer, 1993).

2 J. Damon. The unfolding and determinacy theorems for subgroups of $\mathcal{A}$ and $\mathcal{K}$. *Mem. Am. Math. Soc.* **306** (1984), 1–88.

3 J. Damon. Topological triviality and versality for subgroups of $\mathcal{A}$ and $\mathcal{K}$. *Mem. Am. Math. Soc.* **389** (1988), 1–106.

4 J. Damon. Topological triviality and versality for subgroups of $\mathcal{A}$ and $\mathcal{K}$. II. Sufficient conditions and applications. *Nonlinearity* **5** (1992), 373–412.

5 J. E. Furter, A. M. Sitta and I. N. Stewart. Algebraic path formulation for equivariant bifurcation problems. *Math. Proc. Camb. Phil. Soc.* **124** (1998), 275–304.

6 J. E. Furter, M. A. Ruas and A. M. Sitta. Singularity theory and forced symmetry breaking in equations. (Submitted.)

7 T. Gaffney. Some new results in the classification theory of bifurcation problems. In *Multiparameter bifurcation theory* (ed. M. Golubitsky and J. Guckenheimer). Contemporary Mathematics, vol. 56 (Providence, RI: American Mathematical Society, 1986).

8 L. F. Galante and H. M. Rodrigues. On bifurcation and symmetry of solutions of nonlinear D_m-equivariant equations. *Dynam. Syst. Applic.* **2** (1993), 75–99.

9 L. F. Galante and H. M. Rodrigues. On bifurcation and symmetry of solutions of symmetric nonlinear equations with odd-harmonic forcings. *J. Math. Analysis Applic.* **196** (1995), 526–553.

10 M. Golubitsky and D. G. Schaeffer. A discussion of symmetry and symmetry breaking. In *Proc. Symp. Pure Mathematics*, vol. 40 (Providence, RI: American Mathematical Society, 1983).

11 M. Golubitsky, I. N. Stewart and D. G. Schaeffer. *Singularities and groups in bifurcation theory II*. Applied Mathematical Science, vol. 69 (Springer, 1988).

12 J. K. Hale and H. M. Rodrigues. Bifurcations in the Duffing equation with independent parameters. I. *Proc. R. Soc. Edinb.* A **77** (1977), 57–65.

13 J. K. Hale and H. M. Rodrigues. Bifurcations in the Duffing equation with independent parameters. II. *Proc. R. Soc. Edinb.* A **79** (1978), 317–326.

14 H. M. Rodrigues and A. Vanderbauwhede. Symmetric perturbations of nonlinear equations: symmetry of small solutions. *Nonlin. Analysis* **2** (1978), 27–46.

15 G. Schwartz. Smooth functions invariant under the action of a compact Lie group. *Topology* **14** (1975), 63–68.

16 J. C. Wohlever. Some computational aspects of a group theoretic finite element approach to the buckling and postbuckling analyses of plates and shells of revolution. *Comput. Meth. Appl. Mech. Engng* **170** (1999), 373–406.

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