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Massive spin-3/2 models in $D = 2 + 1$

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ABSTRACT

We have demonstrated that the fermionic part of the so called "New Topologically Massive Supergravity", which is of third order in derivatives, is classically equivalent to self-dual models of lower order in derivatives, forming then a sequence of self-dual descriptions $SD(i)$, with $i = 1, 2, 3$ meaning the order in derivatives of each description. We have connected all the models by symmetry arguments through a Noether Gauge Embedment approach. This is completely equivalent to what happens in the bosonic cases of spins 1, 2 and 3, so some discussion about the similarities is made along the work. An analogue version of the Fierz–Pauli theory is suggested. Then through the NGE approach a fourth order model is obtained, which in our point of view would be the analogue version of the linearized New Massive Gravity theory for spin-3/2 fermionic particles.

1. Introduction

The so called self-dual models have equivalent equations of motion, such equivalence persists even at the quantum level when we are able in to construct a master action which interpolate among the alternative descriptions. In general they differ each other by an order in derivatives and invariance under gauge transformations. The simplest example we can give refers to the well known equivalence between the self-dual SD [1] and the Maxwell–Chern–Simons models MCS [2]. In this case both of them describe a single massive spin-1 particle in $D = 2 + 1$ dimensions. Notice that the MCS model is gauge invariant while the self-dual is not, due to the presence of a Proca mass term.

Once such alternative descriptions also occurs for particles with higher spins, in the last years we have generalized the proof of equivalence we have observed in the case of spin-1 for spin-2 [3,4] and spin-3 [5,6] particles in $D = 2 + 1$. In particular we have learned with these bosonic examples that beyond the self-dual models which already exists one can find new models by means of systematic dualization procedures. Indeed in the case of spin-2 particles a parallel interest is in game. We have noticed that this dualization procedures can take us to the linearized versions of gravitational models. As an example the linearized version of the New Massive Gravity NMG model [7] can be obtained by mean of a generalized soldering of self-dual models [8,9].

In this work we have used the Noether Gauge Embedment NGE approach in order to establish the equivalence among self-dual models which describes massive spin-3/2 particles in $D = 2 + 1$ dimensions. In our initial scenario we have three self-dual descriptions of first $SD(1)$, second $SD(2)$ [10,11] and third order $SD(3)$ [12] in derivatives. The $SD(1)$ and the $SD(2)$ are already connected via a master action [10]. Here we will show that they can be connected by mean of symmetry arguments obtaining the second one by gauge embedding the first one. By the same technique we show that the $SD(3)$ can also be obtained in a second round of NGE connecting it with the previous ones. As in the spin-2 case we have also a parallel interest. In fact the $SD(3)$ model we have reached here consists of the fermionic part of the so called "New Topologically Massive Supergravity" NTMS. This may possibly suggest us new ways of understanding the building blocks of supergravity theories.

The self-dual models describes only a single propagation of helicity $+3/2$ or $-3/2$. In the fourth section of this work we have suggested a dublet model which describes both helicities $\pm 3/2$. We study the Fierz–Pauli conditions to this model in order to guarantee that it propagates only two degrees of freedom. The model we suggest is analogue to the Fierz–Pauli model, so it is non gauge invariant due to the presence of a mass term. Applying the NGE approach we obtain a fourth order model with same particle content, which seems to be an analogue version of the NMG theory at the linearized level. As well as the spin-2 particles have special appeal due to the fact they are closely related with gravitational

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models, here we think that the supergravity extension can also be considered in the future.

We start the next section by verifying the Fierz–Pauli conditions to the vector-spinor, counting then the number of degrees of freedom of the first-order self-dual model.

2. The first order self-dual model

The first order self-dual model introduced in [10] is given by:

$$S_{SD(1)} = \int d^3x \left(-\epsilon^{\mu\nu\alpha}\bar{\psi}_\mu \partial_\nu \psi_\alpha + m\epsilon^{\mu\nu\alpha}\bar{\psi}_\mu \gamma_\nu \psi_\alpha\right), \tag{1}$$

where the fields ψ_μ and $\bar{\psi}_\mu$ are Majorana two component vector-spinors. The Greek indices corresponds to the space–time components, while the spinorial indices has been suppressed. Then, it can be concluded that each vector-spinor field has six independent components in three space–time dimensions. The gamma matrices are indeed the Pauli matrices in agreement with [2], satisfying $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$ and $\gamma_\mu\gamma_\nu = \eta_{\mu\nu} + \epsilon_{\mu\nu\alpha}\gamma^\alpha$. Our metric is mostly plus $(-,+,+)$, and additional properties will be given along the work. About the model, we notice that it is quite similar to the bosonic cases of spins 1 [1], 2 [13] and 3 [14]. As in those models, the Chern–Simons like term is invariant under the gauge symmetries $\delta\psi_\mu = \partial_\mu\Lambda$ and $\delta\bar{\psi}_\mu = \partial_\mu\bar{\Lambda}$ where the $\Lambda's$ are arbitrary spinor fields. The same symmetries are broken by the mass term, and this suggest us to implement the NGE approach. However, first we would like to check that the model (1) satisfy all the Fierz–Pauli conditions i.e., from the equations of motion the field must be gamma-traceless $\gamma_\mu\psi^\mu = 0$ transverse $\partial_\mu\psi^\mu = 0$ and we have to be able to obtain a Klein–Gordon equation. All we have for the field ψ_μ one can demonstrate to the self-adjoint field, then let us consider the equations of motion with respect to the self-adjoint field:

$$\epsilon^{\mu\nu\alpha}\partial_\nu\psi_\alpha - m\epsilon^{\mu\nu\alpha}\gamma_\nu\psi_\alpha = 0. \tag{2}$$

One can notice that once the gamma matrices are constants, by applying ∂_μ on (2) we have $\epsilon^{\mu\nu\alpha}\gamma_\nu\partial_\mu\psi_\alpha = 0$. With this result in hand one can verify two Fierz–Pauli conditions. First notice that after applying γ_μ on (2) and using some gamma properties we can have $\gamma_\mu\psi^\mu = 0$. Then by rewriting $\epsilon^{\mu\nu\alpha}\gamma_\nu$ and using the fact that the field is now gamma-traceless we can demonstrate that it is also transverse $\partial^\mu\psi_\mu = 0$.

By multiplying the equation (2) by $\epsilon_{\mu\lambda\sigma}$ we have obtained:

$$-\partial_\lambda\psi_\sigma + \partial_\sigma\psi_\lambda + m\gamma_\lambda\psi_\sigma - m\gamma_\sigma\psi_\lambda = 0. \tag{3}$$

From here, we take ∂^λ on (3), which leaves us with $\Box\psi_\sigma - m\gamma_\lambda\partial^\lambda\psi_\sigma = 0$. On the other hand by applying γ^λ on (3) one obtain the Majorana equation:

$$(\gamma_\lambda\partial^\lambda - m)\psi_\sigma = 0. \tag{4}$$

With all together we have obtained the Klein–Gordon equation:

$$(\Box - m^2)\psi_\sigma = 0, \tag{5}$$

which finally completes the Fierz–Pauli conditions. This set of "constraints" are telling us that from the six independent components we left with only two of them. In order to have only one degree of freedom, we need one more constraint which indeed exists and tell us about the spin. This is the Pauli–Lubanski condition and we have to be able in deriving it from the equations of motion. To construct the Pauli–Lubanski operator we have to find out a spin-generator for spin-3/2 particles, which for example can be adapted from the reference [15], where the authors have provided an expression to the spin-generator in $D = 3+1$ dimensions which is an antisymmetric tensor $S_{\mu\nu}$. We can now adapt that result by noticing that in $D = 2+1$ dimensions any antisymmetric tensor can be rewritten as $S_{\mu\nu} = \epsilon_{\mu\nu\alpha}S^\alpha$, where S^α is precisely the object we are looking for. In this case we have:

$$(S^\alpha)^{\mu\nu} = i\epsilon^{\mu\alpha\nu} + i\frac{\gamma^\alpha}{2}\eta^{\mu\nu}. \tag{6}$$

Where we have verified that the spin generator for the spin-3/2 particle is in fact the sum of the spin-1 generator with the spin-1/2 generator, both given by [16]. Once we know that $P_\mu = i\partial_\mu$, it is possible to show that with the help of the Majorana equation (4), the equation of motion (2) can be rewritten as:

$$[(P.S)^{\mu\nu} + sm\eta^{\mu\nu}]\tilde{\psi}_\nu = 0, \tag{7}$$

where s is precisely 3/2. Notice that we have used a tilde variable, which means that this equation is valid only for the transverse and gamma-traceless field. Besides, it is straightforward to demonstrate that S_μ satisfy a Lee algebra since it is the sum of the spin-1 and 1/2 parts.

In the next section, we give symmetry arguments to find out a second-order self-dual model from the first order one, and then a third order-self-dual model from the second one.

3. The Noether Gauge Embedment – singlets

3.1. From $SD(1)$ to $SD(2)$

Since the first order self-dual model is non gauge invariant under $\delta\psi_\mu = \partial_\mu\Lambda$ and $\delta\bar{\psi}_\mu = \partial_\mu\bar{\Lambda}$, we may use the NGE approach to turn it gauge invariant. In consequence, by making that, the gauge invariant model becomes second-order in derivative. At the end of this section we would like to be able to establish the classical equivalence between the first and the second order self-dual models. In order to observe such equivalence, we are going to add source terms j_μ and $\bar{j}_\mu$ that will give us a dual map between the models. Then we start by rewritten (1) as:

$$S_{SD(1)} = \int d^3x \left(-\epsilon^{\mu\nu\alpha}\bar{\psi}_\mu\partial_\nu\psi_\alpha + m\epsilon^{\mu\nu\alpha}\bar{\psi}_\mu\gamma_\nu\psi_\alpha + \bar{\psi}_\mu j^\mu + \bar{j}^\mu\psi_\mu\right). \tag{8}$$

From (8) we obtain the Euler vector-spinors:

$$K^\mu = -\epsilon^{\mu\nu\alpha}\partial_\nu\psi_\alpha + m\epsilon^{\mu\nu\alpha}\gamma_\nu\psi_\alpha + j^\mu \tag{9}$$

$$\bar{K}^\mu = -\epsilon^{\mu\nu\alpha}\partial_\nu\bar{\psi}_\alpha + m\epsilon^{\mu\nu\alpha}\bar{\psi}_\nu\gamma_\alpha + \bar{j}^\mu, \tag{10}$$

and introduce a first-iterated action given by:

$$S_1 = S_{SD(1)} + \int d^3x \left(\bar{a}_\mu K^\mu + \bar{K}^\mu a_\mu\right), \tag{11}$$

where $\bar{a}_\mu$ and a_μ are auxiliary fields. By taking the gauge variation of (11) and choosing properly $\delta a_\mu = -\partial_\mu\Lambda$ and $\delta\bar{a}_\mu = -\partial_\mu\bar{\Lambda}$ it is straightforward to demonstrate that we can have:

$$\delta_{\Lambda,\bar{\Lambda}}S_1 = \int d^3x\, \delta\left(-m\epsilon^{\mu\nu\alpha}\bar{a}_\mu\gamma_\nu a_\alpha\right). \tag{12}$$

Automatically from (12) we have a gauge invariant model given by:

$$S_2 = S_{SD(1)} + \int d^3x \left(\bar{a}_\mu K^\mu + \bar{K}^\mu a_\mu + m\epsilon^{\mu\nu\alpha}\bar{a}_\mu\gamma_\nu a_\alpha\right). \tag{13}$$

Then by eliminating the auxiliary fields with the help of their equations of motion we have after some manipulation:

$$a^\mu = -\frac{\gamma_\nu\gamma^\mu K^\nu}{2m} \quad ; \quad \bar{a}^\mu = -\frac{\bar{K}^\nu\gamma^\mu\gamma_\nu}{2m}, \tag{14}$$

which can be substituted back in (13) giving us:

$$S_2 = S_{SD(1)} - \int d^3x \left(\frac{\bar{K}^\mu \gamma_\nu \gamma_\mu K^\nu}{2m} \right). \tag{15}$$

In order to have the second order model we need to substitute the Euler vector-spinors given by (9) and (10) in (15). Then we have:

$$S_{SD(2)} = \int d^3x \Big[\epsilon^{\mu\nu\alpha} \bar{\psi}_\mu \partial_\nu \psi_\alpha - \frac{1}{2m} \bar{f}^\mu(\psi) \gamma_\nu \gamma_\mu f^\nu(\psi) + \bar{F}_\mu j^\mu + \bar{j}^\mu F_\mu + \mathcal{O}(j^2) \Big], \tag{16}$$

where in agreement with [10] we have used the transverse object $f^\mu(\psi) \equiv \epsilon^{\mu\alpha\beta} \partial_\alpha \psi_\beta$ which is the Rarita–Schwinger field strength. It is straightforward to check that from the equations of motion (2) with the help of the dual map $\psi_\mu \to F_\mu = \gamma_\nu \gamma_\mu f^\nu/2m$ one can obtain exactly the equations of motion which comes from the second order self-dual model. Here F_μ is the dual field which is gauge invariant. Similarly, the same happens for the self-adjoint field. Besides, the second order model we have obtained here is precisely the same the author has been introduced in [11], which is the fermionic part of the so called Topologically Massive Supergravity TMS. After all, we have noticed that there are quadratic terms in the source represented by $\mathcal{O}(j^2)$. From now on they will be completely neglected along the work.[1]

We have learned with the bosonic cases of spins 1 and 2 that the number of self-dual models we have, seems to be dependent of the spin s we are describing. Notice that for spin-1 we have two self-dual models. For spin-2 we have four self-dual models [3]. A rule for the number of self-dual models of the type $2s$ seems to be present. Apparently this kind of rule is broken for the spin-3 case [5], where we have found just four self-dual models so far. Sounds like an interesting question if it would be possible to obtain a third order self-dual model from the second order model we just obtained here. In the next section we investigate what gauge symmetry could be implemented in a second round of NGE approach.

3.2. From $SD(2)$ to $SD(3)$

As also observed in [10] the second order term in (16) is invariant under the gauge symmetries $\delta\psi_\mu = \gamma_\mu \xi$ and $\delta\bar{\psi}_\mu = \gamma_\mu \bar{\xi}$. But the same symmetries are broken by the first order term. As we have done before we can systematically impose the gauge symmetry to the model, and in order to do this we derive the following Euler vector-spinors from (16) given by:

$$K^\mu = \epsilon^{\mu\alpha\beta} \partial_\alpha \psi_\beta - \frac{1}{2m} \epsilon^{\mu\alpha\beta} \epsilon^{\nu\lambda\sigma} \gamma_\nu \gamma_\beta \partial_\alpha \partial_\lambda \psi_\sigma + \frac{1}{2m} \epsilon^{\mu\alpha\beta} \gamma_\nu \gamma_\beta \partial_\alpha j^\nu \tag{17}$$

$$\bar{K}^\mu = \epsilon^{\mu\alpha\beta} \partial_\alpha \bar{\psi}_\beta - \frac{1}{2m} \epsilon^{\mu\alpha\beta} \epsilon^{\nu\lambda\sigma} \partial_\alpha \partial_\lambda \bar{\psi}_\sigma \gamma_\beta \gamma_\nu + \frac{1}{2m} \epsilon^{\mu\alpha\beta} \partial_\alpha \bar{j}^\nu \gamma_\beta \gamma_\nu. \tag{18}$$

Then, with the help of auxiliary fields we propose the first iterated action given by:

$$S_{(1)} = S_{SD(2)} + \int d^3x \left(\bar{a}_\mu K^\mu + \bar{K}^\mu a_\mu \right). \tag{19}$$

By choosing properly the gauge symmetries for the auxiliary fields satisfying $\delta_\xi \psi_\mu = -\delta_\xi a_\mu$ and $\delta_{\bar{\xi}} \bar{\psi}_\mu = -\delta_{\bar{\xi}} \bar{a}_\mu$ we end up with the second iterated gauge invariant action:

$$S_{(2)} = S_{(1)} + \int d^3x \, \epsilon^{\mu\alpha\beta} \bar{a}_\mu \partial_\alpha a_\beta. \tag{20}$$

Although the equations of motion for a_μ and $\bar{a}_\mu$ are not algebraic as in the last case, they can still be eliminated by noticing that $K^\mu = \epsilon^{\mu\alpha\beta} \partial_\alpha \Phi_\beta$ and $\bar{K}^\mu = \epsilon^{\mu\alpha\beta} \partial_\alpha \bar{\Phi}_\beta$ with Φ_β and $\bar{\Phi}_\beta$ given by:

$$\Phi_\beta = \psi_\beta - \frac{1}{2m} \epsilon^{\nu\lambda\sigma} \gamma_\nu \gamma_\beta \partial_\lambda \psi_\sigma \tag{21}$$

$$\bar{\Phi}_\beta = \bar{\psi}_\beta - \frac{1}{2m} \epsilon^{\nu\lambda\sigma} \partial_\lambda \bar{\psi}_\sigma \gamma_\beta \gamma_\nu. \tag{22}$$

Then, the expression (20) can be put in the following way:

$$S_{(2)} = S_{SD(2)} + \int d^3x \left[\epsilon^{\mu\alpha\beta} (\bar{a}_\mu + \bar{\Phi}_\mu) \partial_\alpha (a_\beta + \Phi_\beta) - \epsilon^{\mu\alpha\beta} \bar{\Phi}_\mu \partial_\alpha \Phi_\beta \right]. \tag{23}$$

After the shifts $\bar{a}_\mu \to \bar{a}_\mu - \bar{\Phi}_\mu$ and $a_\mu \to a_\mu - \Phi_\mu$ we end up with a completely decoupled term in the auxiliary fields which can be dropped out. The final result after substituting back the definitions (21) and (22) is automatically gauge invariant and is third order in derivatives. After some manipulations we have:

$$S_{SD(3)} = \int d^3x \Big[\frac{1}{2m} \bar{f}(\psi)^\mu \gamma_\nu \gamma_\mu f^\nu(\psi) - \frac{1}{4m^2} \epsilon^{\alpha\lambda\beta} \bar{f}^\mu(\psi) \gamma_\alpha \gamma_\mu \gamma_\nu \gamma_\beta \partial_\lambda f^\nu(\psi) + \bar{G}_\mu j^\mu + \bar{j}^\mu G_\mu \Big] \tag{24}$$

The third order self-dual model we found here is invariant under a larger set of gauge transformations, i.e. $\delta\bar{\psi}_\mu = \partial_\mu \bar{\Lambda} + \gamma_\mu \bar{\xi}$ and $\delta\psi_\mu = \partial_\mu \Lambda + \gamma_\mu \xi$. This is precisely the fermionic part of the quadratic approximation of the “New Topologically Massive Supergravity” NTMS introduced in [12]. In order to compare the notation we have used here to the one the authors have used in that work, one can first verify that the Rarita–Schwinger field strength is written as $f^\mu(\psi) = \mathscr{R}^\mu_{(\mathrm{lin})}$. It is straightforward to check that the second order term in (24) is given by $\bar{\psi}_\mu \mathscr{C}^\mu_{(\mathrm{lin})}$, where the authors have defined a second order “Cottino tensor” $\mathscr{C}^\mu_{(\mathrm{lin})} = \gamma^\nu \partial_\nu \mathscr{R}^\mu_{(\mathrm{lin})} + \epsilon^{\mu\nu\alpha} \partial_\nu \mathscr{R}_{\alpha(\mathrm{lin})}$. Using these definitions it is not difficult to show that the third order term of (24) corresponds to $\bar{\psi}_\mu (\gamma^\nu \partial_\nu) \mathscr{C}^\mu_{(\mathrm{lin})}$.

The classical equivalence between the first and the third order self-dual models can be checked with the help of the gauge invariant dual fields G_μ and $\bar{G}_\mu$ given by:

$$G_\mu(\psi) = \frac{1}{4m^2} \epsilon^{\nu\alpha\beta} \gamma_\nu \gamma_\mu \gamma_\sigma \gamma_\beta \partial_\alpha f^\sigma(\psi)$$

$$\bar{G}_\mu(\bar{\psi}) = \frac{1}{4m^2} \epsilon^{\nu\alpha\beta} \partial_\alpha \bar{f}^\sigma(\psi) \gamma_\beta \gamma_\sigma \gamma_\mu \gamma_\nu \tag{25}$$

in other words by substituting $\psi_\mu \to G_\mu(\psi)$ in (2) we have exactly the equations of motion of the third order self-dual model with respect to $\bar{\psi}_\mu$. With this result we have showed that the third order self-dual model of [12] is in fact part of a natural sequence of self-dual models in a way quite similar to the bosonic cases of spins 1, 2 and 3.

By interpreting the second order term in (24) as a analogue of the Maxwell or Einstein–Hilbert terms one could ask what would be the analogue version of the Proca or Fierz–Pauli theories for the spin-3/2 case. In the next section we investigate this issue closely.

[1] Such terms give rise to contact terms when, through a master action, we compare correlation functions between the fields and their duals. The contact terms are proportional to $\sim \delta_{\mu\nu}\delta(x-y)$, then for the purposes of this work they are not important at all, see [4] for example.

4. The Noether Gauge Embedment – dublets

The Proca theory in terms of the vector gauge field A_μ, is non gauge invariant under $\delta A_\mu = \partial_\mu \Lambda$. This is due to the presence of the Proca mass term. However one could ask what would we find if we gauge embedding the missing symmetry through the NGE approach. The answer is well known, and the gauge invariant higher derivative model we obtain is the Podolski theory, which have ghosts in its spectrum, see the discussion of the section 2.1 of [3]. One might be discouraged in trying the same with the spin-2 models. The Fierz–Pauli theory is also non gauge invariant under reparametrizations of the type $\delta h_{\mu\nu} = \partial_\mu \Lambda_\nu + \partial_\nu \Lambda_\mu$. However when we gauge embedding the missing symmetry we obtain the linearized version of the New Massive Gravity model. The explanation on why in the spin-1 case we obtain a model with ghosts and in the spin-2 case we obtain a ghost-free model is better understood by means of the master action technique. We have observed in [3] that because the Maxwell term is non trivial it may not be used as a mixing term in order to construct a master action, this kind of thing does not happen with the linearized Einstein–Hilbert term which is (in three dimensions) free of particle content.

Here, the second order term like the Einstein–Hilbert term, is also free of particle content. This suggest us that we can construct an analogue version of the Fierz–Pauli model (non gauge invariant) and then to implement the NGE approach in order to obtain a fourth order model, free of ghosts like NMG.

The analogue version of the Fierz–Pauli model for spin-3/2 particles must describe a dublet of massive spins +3/2 and −3/2. Let us suggest the following combination:

$$S = \int d^3x \left[-\frac{1}{4} \bar{f}^\mu(\psi) \gamma_\nu \gamma_\mu f^\nu(\psi) - \frac{m^2}{2} \epsilon^{\mu\nu\beta} \bar{\psi}_\mu \gamma_\nu \psi_\beta \right]. \quad (26)$$

In order to verify that this model describes the correct number of degrees of freedom, we take the equations of motion with respect to $\bar{\psi}_\mu$, in order study the Fierz–Pauli conditions:

$$\epsilon^{\mu\alpha\beta} \epsilon^{\nu\lambda\sigma} \gamma_\nu \gamma_\beta \partial_\alpha \partial_\lambda \psi_\sigma - 2m^2 \epsilon^{\mu\alpha\beta} \gamma_\alpha \psi_\beta = 0. \quad (27)$$

First we notice that by applying ∂_μ in (27) we have $\epsilon^{\alpha\mu\beta} \gamma_\alpha \partial_\mu \psi_\beta = 0$. On the other hand if we apply γ_μ in (27) we conclude that the field is gamma-traceless $\gamma^\mu \psi_\mu = 0$, which together with the previous result give us $\partial^\mu \psi_\mu = 0$. Back in the equations of motion with both constraints (gamma-traceless and transversality), after several rearrangements using the gamma properties, we may deduce that in fact (27) reduces to the Klein–Gordon equation:

$$(\Box - m^2)\psi^\mu = 0. \quad (28)$$

Since the equations of motion are second-order in derivatives, by this time we do not have a Pauli–Lubanski equation, and we end up with two degrees of freedom, corresponding to the positive +3/2 and negative −3/2 helicities. Then one may conclude that the model describes the correct number of degrees of freedom.

Adding source terms $\bar{j}^\mu \psi_\mu$ and $\bar{\psi}_\mu j^\mu$ to the action (27), we can now rewrite (26) as the Euler vector-spinors:

$$K^\beta = \frac{1}{4} \epsilon^{\beta\mu\lambda} \gamma_\nu \gamma_\mu \partial_\lambda f^\nu(\psi) - \frac{m^2}{2} \epsilon^{\beta\mu\lambda} \gamma_\mu \psi_\lambda + j^\beta \quad (29)$$

$$\bar{K}^\beta = -\frac{1}{4} \epsilon^{\beta\mu\lambda} \partial_\mu \bar{f}^\nu(\psi) \gamma_\lambda \gamma_\nu - \frac{m^2}{2} \epsilon^{\beta\mu\lambda} \bar{\psi}_\mu \gamma_\lambda + \bar{j}^\beta \quad (30)$$

which can be used in order to construct a first iterated action given by:

$$S^{(1)} = S + \int d^3x \left(\bar{K}^\beta a_\beta + \bar{a}_\beta K^\beta \right). \quad (31)$$

Making proper choices for the variations of the auxiliary fields such that $\delta a_\mu = -\delta\psi_\mu$ and $\delta\bar{a}_\mu = -\delta\bar{\psi}_\mu$ we can demonstrate that we have at the end:

$$\delta S^{(1)} = \int d^3x\, \delta \left(\frac{m^2}{2} \epsilon^{\lambda\nu\beta} \bar{a}_\lambda \gamma_\nu a_\beta \right). \quad (32)$$

Then we have automatically a gauge invariant action given by:

$$S^{(2)} = S^{(1)} - \int d^3x \left(\frac{m^2}{2} \epsilon^{\lambda\nu\beta} \bar{a}_\lambda \gamma_\nu \delta a_\beta \right) \quad (33)$$

Solving the equations of motion for the auxiliary fields and plugging back the result in (33), we have after some manipulations:

$$S^{(2)} = S + \int d^3x \frac{4}{m^2} \bar{K}^\alpha \gamma_\sigma \gamma_\alpha K^\sigma. \quad (34)$$

Which can be rewritten in terms of the original fields if we substitute K_μ and $\bar{K}_\mu$ from (29) and (30), giving us the fourth order model:

$$S^{(2)} = \int d^3x \left[-\frac{1}{16m^2} \partial_\lambda \bar{f}^\mu(F) \epsilon^{\lambda\nu\alpha} \gamma_\nu \gamma_\mu \gamma_\sigma \gamma_\alpha \gamma_\beta \gamma_\rho \epsilon^{\rho\theta\sigma} \partial_\theta f^\beta(F) + \frac{1}{4} \bar{f}^\mu(F) \gamma_\nu \gamma_\mu f^\nu(F) + \frac{1}{2m^2} \bar{j}_\mu G^\mu(\psi) - \frac{1}{2m^2} \bar{G}_\mu(\psi) j^\mu \right] \quad (35)$$

The dual equivalence between the fourth order theory and the second order one can be established thanks to the dual fields $G_\mu(\psi)$ and $\bar{G}_\mu(\psi)$ which are given by:

$$\psi_\alpha \to G_\alpha(\psi) = \epsilon^{\rho\theta\sigma} \gamma_\sigma \gamma_\alpha \gamma_\beta \gamma_\rho \partial_\theta f^\beta(\psi) \quad (36)$$

$$\bar{\psi}_\alpha \to \bar{G}_\alpha(\psi) = \epsilon^{\lambda\nu\alpha} \partial_\lambda \bar{f}^\mu(F) \gamma_\nu \gamma_\mu \gamma_\sigma \gamma_\alpha \quad (37)$$

The fourth order theory we have obtained here can be related to the second order one through the dual maps $\psi_\alpha \to G_\alpha(\psi)$ and $\bar{\psi}_\alpha \to \bar{G}_\alpha(\psi)$ and the equivalence can be understood at the level of the equations of motion. From a certain point of view, once we interpret the second order self-dual model as the analogue version of spin-3/2 to the Fierz–Pauli theory, the fourth order model we have obtained here can be understood as a spin-3/2 version of the NMG theory at the linearized level. Once the second order term is trivial, i.e. free of particle content, one could also construct a master action interpolating between the second and the fourth order model, and it guarantee that the fourth order model is free of ghosts.

5. Conclusion

From the statistical point of view we have generalized the NGE approach to the case of fermionic fields, in particular we have analyzed the case of massive spin-3/2 particles, which are described by a vector-spinor field. We have been counting the number of degrees of freedom in the first order self-dual model by studying the Fierz–Pauli conditions, showing then from the equations of motion one can obtain the gamma traceless and the transversality constraints. Besides, we have obtained a Pauli–Lubanski equation and a Klein–Gordon equation.

We have demonstrated that the recent third order model proposed by [12] is in fact equivalent at least in the level of the equations of motion to the two self-dual models of first and second order in derivatives proposed by [10]. Dual maps connecting the three self-dual versions are given, and we can transit among the equations of motion of the models. It is interesting to notice that the third order term suggested in [12] is generated by means of a systematic procedure. A proof of equivalence at the quantum level must be verified by comparing the correlation functions derived from a master action, quite similar to what we have done for example in [4]. In general we have always observed that once we have the classical equivalence verified through the NGE approach, one can observe the quantum equivalence through the master action technique.

Taking the Fierz–Pauli model as an inspiration, we suggested a non gauge invariant second order model which by means of the study of the Fierz–Pauli conditions, describe two degrees of freedom, corresponding to the pair of helicities $\pm 3/2$. By gauge embedding this model we have obtained a fourth order model, which would be in our point of view the analogue version of the linearized New Massive Gravity. We argue that, the fourth order model must be ghost free, once the second order term like the Einstein–Hilbert term is free of particle content and may be used in order to construct a master action interpolating between the models.

It would be interesting to connect the three self-dual models we have studied here by means of a unique master action. In order to construct master actions one need to prove that the mixing terms are free of particle contents. In a work in progress we have analyzed the constraints structure of such terms in order to prove that they are free of particle content and then to construct a unique master action interpolating among the three self-dual models and also between the dublet models. We have also verified that it is possible to obtain the second order dublet model by generalizing soldering two second order self-dual models. We think that the fourth order dublet model consists in fact of the soldering of two third order self-dual models similarly to what happens in the spin-2 case. Generalizations to higher rank "spinorial" fields, like the self-dual descriptions of spin-5/2, may be also connected through the NGE approach. In that case, similar to what happens in the spin-3 case, auxiliary fields are needed in order to remove spurious degrees of freedom, what brings some difficulties to the procedures.

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