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Exploring Teleparallel Dark Energy

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Resumo

Nesta tese fazemos um estudo da dinâmica cosmológica da energia escura do Universo, usando o contexto do *Equivalente Teleparalelo da Relatividade Geral*, também conhecido como Gravitação Teleparalela. Recentemente têm sido propostos diversos modelos com acoplamento não-minimal entre um campo escalar (quintessência ou taquion) e a gravitação, usando o contexto do teleparalelismo, motivados por construções semelhantes no contexto da Relatividade Geral. Generalizando e estudando a dinâmica desses modêlos, encontramos soluções interessantes cosmologicamente viáveis. Em seguida, propomos um novo modêlo no qual a derivada do campo escalar que representa a energia escura se acopla não-minimalmente com a *torção vetorial*. Esse tipo de acoplamento é inspirado no fato que, no contexto teleparalelo um campo escalar se acopla com a torção através de sua derivada, a qual é um campo vetorial. Diferentemente dos modelos mais antigos de energia escura teleparalela, este novo modêlo é conceitualmente consistente com os preceitos da teoria teleparalela, e não possui equivalente no contexto da Relatividade Geral.

Palavras chave: Gravitação teleparalela, energia escura, acoplamento não-minimal

Áreas do conhecimento: Gravitação, cosmologia, energia escura

Abstract

In the present thesis we study the cosmological dynamics of the dark energy component of the Universe in the context of the *Teleparallel Equivalent of General Relativity*, also known as Teleparallel Gravity. It has been proposed recently the existence of a nonminimal coupling between a scalar field (quintessence or tachyon) and gravity in the framework of Teleparallel Gravity, motivated by similar constructions in the context of General Relativity. By generalizing and studying the dynamics of these models, we find interesting and viable cosmological solutions. Then we propose a new model in which the four-derivative of the scalar field of dark energy is nonminimally coupled to *vector torsion*. This kind of coupling is grounded on the fact that in Teleparallel Gravity a scalar field as a particle of spin zero couples to torsion through its four-derivative, which is a vector field. Differently from the old teleparallel dark energy scenario, this new model is conceptually consistent with the precepts of Teleparallel Gravity, and has no equivalent in the General Relativity context.

Keywords: Teleparallel gravity, dark energy, nonminimal coupling

Areas of knowledge: Gravitation, cosmology, dark energy

To my brothers and mom.

Contents

Chapter 1

Introduction to Teleparallel Gravity

A crucial concept of gravitation is that the metric tensor itself defines neither curvature nor torsion. Curvature and torsion are properties of connections, and many different connections, with different curvature and torsion tensors, can be defined on the very same metric spacetime [1]. A general Lorentz connection has 24 independent components, and thus it is seen that any gravitational theory in which the source is the 10 components symmetric energy-momentum tensor, will not be able to determine uniquely the connection. The teleparallel connection and the Levi-Civita (or Christoffel) connection are the only two choices respecting the correct number of degrees of freedom of gravitation. The teleparallel connection can be considered a kind of "dual" of the Levi-Civita connection in the sense that it is a connection with vanishing torsion, but non-vanishing curvature [1].

The Levi-Civita connection is the most intuitive from the point of view of universality. In this case gravitation can be easily understood by incorporating the Levi-Civita curvature in the definition of spacetime, in such a way that all (spinless) particles, independently of their masses and constitutions, will follow a geodesic of the (supposed) curved spacetime. However, like the other fundamental interactions of Nature, gravitation can be described in terms of a gauge theory, just Teleparallel Gravity (TG), which attributes gravitation to torsion, but not through a geometrization: it acts as a force. In consequence, there are no geodesics in TG, but only force equations quite analogous to the Lorentz force equation of electrodynamics.

In addition to the gauge description of TG, universality of free fall makes it possible a second, geometrized description, based on the equivalence principle, just General Relativity (GR). As the sole universal interaction, it is the only one to allow also a geometrical interpretation, and hence two alternative descriptions. From this point of view, curvature and torsion are simply alternative ways of representing the same gravitational field, accounting for the same degrees of freedom of gravity. Although equivalent to general relativity, TG introduces new concepts into both classical and quantum gravity. So, TG can provide a new way to look at the universe and thus it can also lead to a reappraisal of the entire cosmology [1].

1.1 Teleparallel lagrangian and field equations

Following [1], we briefly review the key ingredients of TG, a gauge theory for the translation group, which is fully equivalent to Einstein's general relativity. The tetrad field, the dynamical variable, forms an orthonormal basis for the tangent Minkowski space at each point x^μ of the spacetime and provides a relation between the tangent space metric η_{ab} and the spacetime metric $g_{\mu\nu}$ through

$$g_{\mu\nu} = h^a{}_\mu \, h^b{}_\nu \, \eta_{ab}, \tag{1.1}$$

where $h^a{}_\mu$ and the inverse $h_a{}^\mu$ are the components of the tetrad field, which satisfy the orthogonality conditions $h^a{}_\mu \, h_a{}^\nu = \delta^\nu_\mu$ and $h^a{}_\mu \, h_b{}^\mu = \delta^a_b$ (see Appendix A). The tetrad field can be divided into a trivial part (non-gravitational) and a non-trivial part that is a translational gauge potential, which represents the gravitational field. We use the metric signature convention $(+,-,-,-)$ throughout and natural units.

As a gauge theory for the translation group, the gravitational action of TG can be written as

$$S_G = \frac{1}{16\,\pi\, G} \int tr(T \wedge \star T), \tag{1.2}$$

where

$$T = \frac{1}{2} \, T^a{}_{\mu\nu} \, P_a \, dx^\mu \wedge \, dx^\nu \tag{1.3}$$

is the torsion 2-form, and

$$\star T = \frac{1}{2} (\star T^a{}_{\rho\sigma}) \, P_a \, dx^\rho \wedge \, dx^\sigma \tag{1.4}$$

is the corresponding dual form.

By using the definition of the dual torsion

$$\star T^\rho{}_{\mu\nu} = h \, \epsilon_{\mu\nu\alpha\beta} \, S^{\rho\alpha\beta}, \tag{1.5}$$

(with $h = \sqrt{-g}$) and since that $tr(P_a\, P_b) = \delta_{ab}$, the action functional reduce to

$$S = \frac{1}{2\,\kappa^2} \int dx^4\, h\, T, \tag{1.6}$$

($\kappa^2 = 8\,\pi\, G$) where the torsion scalar T is defined as

$$T \equiv S_\rho^{\;\mu\nu}\, T^\rho_{\;\mu\nu}, \tag{1.7}$$

the torsion tensor is

$$T^\rho_{\;\mu\nu} \equiv h_a^{\;\rho}\, T^a_{\;\mu\nu} = h_a^{\;\rho}\, \left(\partial_\mu h^a_{\;\nu} - \partial_\nu h^a_{\;\mu} + A^a_{\;e\mu}\, h^e_{\;\nu} - A^a_{\;e\nu}\, h^e_{\;\mu}\right), \tag{1.8}$$

the superpotential

$$S_\rho^{\;\mu\nu} \equiv \frac{1}{2} \left(K^{\mu\nu}_{\;\;\;\rho} + \delta^\mu_\rho\, T^{\theta\nu}_{\;\;\;\theta} - \delta^\nu_\rho\, T^{\theta\mu}_{\;\;\;\theta}\right), \tag{1.9}$$

and

$$K^{\mu\nu}_{\;\;\;\rho} \equiv -\frac{1}{2} \left(T^{\mu\nu}_{\;\;\;\rho} - T^{\nu\mu}_{\;\;\;\rho} - T_\rho^{\;\mu\nu}\right), \tag{1.10}$$

the contortion tensor. The spin connection of TG $A^a_{\;e\mu}$, which represents only inertial effects, is given by

$$A^a_{\;e\mu} = \Lambda^a_{\;d}(x)\, \partial_\mu \Lambda_e^{\;d}(x), \tag{1.11}$$

where $\Lambda^a_{\;b}(x^\mu)$ is a local (point-dependent) Lorentz transformation (see Appendix B). For this connection the curvature vanishes identically

$$R^a_{\;b\nu\mu} = \partial_\nu A^a_{\;b\mu} - \partial_\mu A^a_{\;b\nu} + A^a_{\;e\nu}\, A^e_{\;b\mu} - A^a_{\;e\mu}\, A^e_{\;b\nu} = 0, \tag{1.12}$$

but the torsion (1.8) is non-vanishing. This connection can be considered a kind of "dual" of the GR connection, which is a connection with vanishing torsion, but non-vanishing curvature. Also, it is important to note that curvature and torsion are properties of connections, and not of space-time. Many different connections, each one with different curvature and torsion, can be defined on the very same metric space-time. The teleparallel connection and the GR connection are the only two choices respecting the correct number of degrees of freedom of gravitation. The linear connection corresponding to the spin connection $A^e_{\;b\nu}$ is

$$\Gamma^\rho_{\;\nu\mu} = h_a^{\;\rho} \left(\partial_\mu h^a_{\;\nu} + A^a_{\;b\mu} h^b_{\;\nu}\right), \tag{1.13}$$

which is called Weitzenböck connection. This connection is related to the Levi-Civita connection $\bar{\Gamma}^\rho_{\;\mu\nu}$ of GR by

$$\Gamma^\rho_{\;\mu\nu} = \bar{\Gamma}^\rho_{\;\mu\nu} + K^\rho_{\;\mu\nu}. \tag{1.14}$$

In the presence of a source matter, the variation of the teleparallel action (1.6) with respect to the tetrad fields yields the teleparallel version of the gravitational field equation

$$G_a{}^{\nu} \equiv h^{-1}\,\partial_\mu\,(h\,S_a{}^{\nu\mu}) + h_a{}^{\lambda}\,T^{\rho}{}_{\mu\lambda}\,S_\rho{}^{\nu\mu} + \frac{1}{4}\,h_a{}^{\nu}\,T - h_c{}^{\rho}\,A^c{}_{a\sigma}\,S_\rho{}^{\nu\sigma} = \frac{\kappa^2}{2}\,h_a{}^{\rho}\,\Theta_\rho{}^{\nu}, \tag{1.15}$$

where

$$\Theta_a{}^{\nu} = -\frac{1}{h}\,\frac{\delta S_m}{\delta h^a{}_\nu} \tag{1.16}$$

and $\Theta_\rho{}^{\nu} \equiv h^a{}_\rho\,\Theta_a{}^{\nu}$ stands for the symmetric matter energy-momentum tensor. Using (1.14), through a lengthy but straightforward calculation, it can be shown that the gravitational field equation (1.15) is equivalent to the Einstein's field equation. In this equation also can be identified the gauge current, which in this case represents the Noether energy-momentum density of gravitation [1]

$$j_a{}^{\nu} \equiv -\frac{1}{h}\,\frac{\partial \mathcal{L}_G}{\partial h^a{}_\nu} = \frac{1}{\kappa^2}\left(-h_a{}^{\lambda}\,T^{\rho}{}_{\mu\lambda}\,S_\rho{}^{\nu\mu} - \frac{1}{4}\,h_a{}^{\nu}\,T + h_c{}^{\rho}\,A^c{}_{a\sigma}\,S_\rho{}^{\nu\sigma}\right). \tag{1.17}$$

Given the nature of TG as a field theory of gravitation, it is possible to define a energy momentum tensor for the gravitational field, which is represented by the first two terms of (1.17) and by excluding the inertial effects (the last term) [1, 4].

Now, like in special relativity, it is possible to define the preferred class of inertial frames, in the context of TG it is also possible to define a preferred class of frames: the class that reduces to the inertial class in the absence of gravitation. In this preferred class of frames, the spin connection of TG vanishes everywhere

$$A^a{}_{b\mu} = 0, \tag{1.18}$$

and the Weitzenböck connection assumes the form

$$\Gamma^{\rho}{}_{\nu\mu} = h_a{}^{\rho}\,\partial_\mu h^a{}_\nu. \tag{1.19}$$

Of course, once a class of frames is specified, TG is no longer "manifestly" Lorentz invariant. For this reason, it is sometimes said that TG is not Lorentz invariant; that is, however, a wrong interpretation. This would be equivalent to write down Maxwell theory in a specific gauge, and then to conclude that it is not gauge invariant. Anyway, in the same way that one can choose to work with electromagnetism in a specific gauge, one can also choose to work with TG in a specific class of frames [1].

From TG to cosmology, for a flat FRW background metric, the tetrad has the form

$$h^a{}_\mu(t) = \mathrm{diag}(1, a(t), a(t), a(t)), \tag{1.20}$$

where $a(t)$ is the scale factor. In the preferred class of inertial frames (1.18), the tetrad choice (1.20) is an exact solution of the field equation (1.15), and it is seen that the corresponding Friedmann equations are identical to the GR ones both at the background and perturbation levels [1, 2, 5]. Also, when one introduces a scalar field as source of dark energy (for example quintessence field), in the case of minimal coupling, there is no difference whether it is added in either of the two theories. However, this does not happen in the non-minimal case. In the non-minimal case the additional scalar sector is coupled to gravity, with the torsion scalar in TG and with the curvature scalar in GR, and thus the resulting coupled equations do not coincide, which implies that the resulting theories are completely different [7, 8]. This can be better understood if we take into account that the lagrangian of TG is equivalent to the lagrangian of GR up to a divergence.

Chapter 2

Teleparallel Dark Energy

Since the discovery of the universe accelerated expansion, this issue has been one of the most active fields in modern cosmology. This result emerges from cosmic observations of Supernovae Ia (SNe Ia) [9], cosmic microwave background (CMB) radiation [10], large scale structure (LSS) [11], baryon acoustic oscillations (BAO) [12], and weak lensing [13].

The simplest explanation for the dark energy—the name given to the agent responsible for such accelerated expansion—is provided by a cosmological constant. However, this scenario is plagued by a severe fine tuning problem associated with its energy scale and the coincidence problem [14, 16, 17]. There are two main approaches to explain such behavior, apart from the simple consideration of a cosmological constant. One is to modify the gravitational sector by generalizing the Einstein-Hilbert action of GR, which gives rise to the so-called $F(R)$ theories [14, 15] (For the teleparallel gravity generalization, the so-called $F(T)$ theory, see Refs. [16, 30, 31]). The other approach consists in considering a scalar field with a dynamically varying equation of state, like for example quintessence dark energy models. Scalar fields naturally arise in particle physics (including string theory), and these fields can act as candidates for dark energy. So far a wide variety of scalar-field dark energy models have been proposed, such as quintessence, phantoms, k-essence, amongst many others [16, 18].

Usually, dark energy models are based on scalar fields minimally coupled to gravity, as well as a possible coupling of the field to a background fluid (dark matter). Also, a possible coupling between the scalar field and the Ricci scalar R is not to be excluded in the context of generalized Einstein gravity theories. For example, a quintessence field coupled to gravity and called "extended quintessence" was proposed in Ref. [21] (see also Refs. [22, 23]). K-essence models non-minimally coupled to gravity were studied in Refs. [26, 27]. It is also important to note that the first application of the non-minimal coupling to inflationary cosmology was done in Ref. [20]. Also, the application of non-minimally coupled models to quantum cosmology was considered in Ref. [24]. Finally, an important model of the non-minimally coupled Higgs field has been considered in Ref. [25]. Other dark energy models using covariant versions with non-minimal coupling can also be found in the literature [29].

In analogy to a similar construction in GR, it was proposed in Ref. [5] a non-minimal coupling between quintessence and gravity in the framework of TG, motivated by a similar construction in the context of GR. This theory has a rich structure, and has been called "teleparallel dark energy" (TDE); its dynamics was studied later in Refs. [6–8]. However, no scaling attractor was found.

Here we generalize the non-minimal coupling function $\phi^2 \to f(\phi)$ and we define $\alpha \equiv f_{,\phi}/\sqrt{f}$ such that α is constant for $f(\phi) \propto \phi^2$, but it can also be a dynamically changing quantity $\alpha(\phi)$. For dynamically changing α we show the existence of scaling attractors with accelerated expansion in agreement with observations. As long as the scaling solution is an attractor, for any generic initial conditions the system would sooner or later enter the scaling regime in which the dark energy density is comparable to the matter energy density with accelerated expansion of the universe, alleviating in this the fine tuning problem of dark energy and the coincidence problem [16–18].

2.1 Quintessence field in general relativity

As is known [21–23], one can generalize quintessence by including a non-minimal coupling between quintessence and gravity. The relevant action reads

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{2\,\kappa^2} + \frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi - V(\phi) + \xi\; f(\phi)\,R \right] + S_m, \tag{2.1}$$

where R is the Ricci scalar, S_m is the matter action. Also, the parameter ξ is a dimensionless constant and $f(\phi)$ (with units of $mass^2$ and positive) is an arbitrary function of ϕ responsible for non-minimal coupling. In [32–35] adopted $f \backsim \phi^2$ in the context inflation, other forms have been considered in [36, 37] (for a more general form that includes derivatives see [38]). When including a non-minimal coupling in quintessence, the corresponding Friedmann equations are preserved. On the other hand, the pressure density and energy density of quintessence are changed to

$$p_\phi = \frac{1}{2}\left(1 + 4\,\xi\, f_{,\phi\phi}\right)\dot{\phi}^2 - V(\phi) + 4\,\xi\left(f(\phi) + 3\,\xi\, f_{,\phi}^2\right)\dot{H} - 2\,\xi\, f_{,\phi}\,\dot{\phi}\, H - 2\,\xi\, f_{,\phi}\, V_{,\phi} + 6\,\xi\left(f(\phi) + 4\,\xi\, f_{,\phi}^2\right)H^2, \quad (2.2)$$

$$\rho_\phi = \frac{1}{2}\,\dot{\phi}^2 + V(\phi) - 6\,\xi\, H\, f_{,\phi}\,\dot{\phi} - 6\,\xi\, H^2\, f(\phi), \quad (2.3)$$

where $V_{,\phi} \equiv dV/d\phi$, $f_{,\phi} \equiv df/d\phi$ and H is the Hubble parameter. Also, a dot denotes differentiation with respect to the cosmic time t. For find p_ϕ and ρ_ϕ we have used the equation of motion

$$\ddot{\phi} + 3\,H\,\dot{\phi} - \xi\, R\, f_{,\phi} + V_{,\phi} = 0, \quad (2.4)$$

calculated from the action (2.1), and the useful expression $R = 6\left(\dot{H} + 2\,H^2\right)$ in the flat Friedmann-Lemaitre-Robertson-Walker (FLRW) geometry, see [22, 23].

2.2 Interacting teleparallel dark energy

In what follows we study the action of TDE, in which a non-minimal coupling between an additional scalar sector and the torsion scalar of TG is considered. Also, we consider a possible interaction between TDE and dark matter, that is, “interacting teleparallel dark energy”, since there is no physical argument to exclude the possible interaction between them. We will work in the preferred class of frames (1.18). The action will simply read

$$S = \int d^4x\, h\left[\frac{T}{2\,\kappa^2} + \frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi - V(\phi) + \xi\, f(\phi)\, T\right] + S_m, \quad (2.5)$$

and the variation with respect to the tetrad field yields the coupled field equation

$$2\left(\frac{1}{\kappa^2} + 2\,\xi\, f(\phi)\right)\left[h^{-1}\, h^a{}_\alpha\,\partial_\sigma\left(h\, h_a{}^\tau\, S_\tau{}^{\rho\sigma}\right) + T^\tau{}_{\nu\alpha}\, S_\tau{}^{\rho\nu} + \frac{T}{4}\,\delta_\alpha^\rho\right] + 4\,\xi\, S_\alpha{}^{\rho\sigma}\, f_{,\phi}\,\partial_\sigma\phi + \left[\frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi - V(\phi)\right]\delta_\alpha^\rho - \partial_\alpha\phi\,\partial^\rho\phi = \Theta_\alpha{}^\rho. \quad (2.6)$$

In flat FLRW geometry (1.20), the non-zero components of the torsion tensor, contortion tensor and the superpotential are

$$T^i{}_{0j} = -T^i{}_{j0} = H\,\delta^i_j, \quad K^{0i}{}_j = -K^{i0}{}_j = H\,\delta^i_j, \quad S_j{}^{i0} = -S_j{}^{0i} = H\,\delta^i_j, \quad (2.7)$$

where $i, j = 1, 2, 3$. Therefore, imposing the flat FLRW geometry (1.20) in (2.6), we obtain the Friedmann equations with

$$\rho_\phi = \frac{1}{2}\,\dot{\phi}^2 + V(\phi) - 6\,\xi\, H^2\, f(\phi), \quad (2.8)$$

the scalar energy density and

$$p_\phi = \frac{1}{2}\,\dot{\phi}^2 - V(\phi) + 4\,\xi\,H\,f_{,\phi}\,\dot{\phi} + 6\,\xi\,\left(3\,H^2 + 2\,\dot{H}\right)\,f(\phi), \tag{2.9}$$

the pressure density of field. Here we also use the useful relation $T = -6\,H^2$, which arises for flat FLRW geometry.

In this same background, the variation of the action (2.5) with respect to scalar field yields the field equation

$$\ddot{\phi} + 3\,H\,\dot{\phi} + V_{,\phi} + 6\,\xi\,f_{,\phi}\,H^2 = -\sigma. \tag{2.10}$$

where σ is the scalar charge corresponds to coupling between TDE and dark matter, and it is defined by the relation $\delta S_m/\delta\phi = -h\,\sigma$ [16, 18, 19]. Rewriting (2.10) in terms of ρ_ϕ and p_ϕ we find the continuity equation for the field

$$\dot{\rho}_\phi + 3\,H\,\rho_\phi\,(1 + \omega_\phi) = -Q, \tag{2.11}$$

whereas for matter

$$\dot{\rho}_m + 3\,H\,\rho_m\,(1 + \omega_m) = Q, \tag{2.12}$$

where $\omega_\phi \equiv p_\phi/\rho_\phi$ and $\omega_m \equiv p_m/\rho_m = const \geq 0$ are the equation-of-state parameter of TDE and dark matter, respectively. The parameter $Q \equiv \dot{\phi}\,\sigma$ indicates the coupling between the two components. Also, we will define the barotropic index $\gamma \equiv 1 + \omega_m$ such that $1 \leq \gamma < 2$.

2.3 Dynamical system

Of particular importance in the investigation of cosmological scenarios are those solutions in which the energy density of the scalar field mimics the background fluid energy density, $\rho_\phi/\rho_m = C$, with C a nonzero constant. Cosmological solutions which satisfy this condition are called "scaling solutions" and the cosmological coincidence problem can be alleviated in most dark energy models via scaling attractors (it is defined later) [15, 16]. To study the cosmological dynamics of the model, we introduce the followings dimensionless variables:

$$x \equiv \frac{\kappa\,\dot{\phi}}{\sqrt{6}\,H} \qquad y \equiv \frac{\kappa\,\sqrt{V}}{\sqrt{3}\,H}, \qquad u \equiv \kappa\,\sqrt{f}, \qquad \lambda \equiv -\frac{V_{,\phi}}{\kappa\,V}, \qquad \alpha \equiv \frac{f_{,\phi}}{\sqrt{f}}. \tag{2.13}$$

Using these variables we define

$$s \equiv -\frac{\dot{H}}{H^2} = \frac{-3\,\gamma\,y^2 + 3\,(2-\gamma)\,x^2 + 4\,\sqrt{6}\,\alpha\,\xi\,u\,x}{2\,(2\,\xi\,u^2 + 1)} + \frac{3\,\gamma}{2}, \tag{2.14}$$

and the Eqs. (2.11) and (2.12) can be rewritten as a dynamical system of ordinary differential equations (ODE), namely

$$x' = \frac{\sqrt{6}}{2}\,\lambda\, y^2 - \sqrt{6}\,\alpha\,\xi\, u + x\,\left(s-3\right) - \hat{Q}, \tag{2.15}$$

$$y' = \left(s - \frac{\sqrt{6}\,\lambda\, x}{2}\right)\, y, \tag{2.16}$$

$$u' = \frac{\sqrt{6}\,\alpha\, x}{2}, \tag{2.17}$$

$$\lambda' = -\sqrt{6}\,\left(\Gamma - 1\right)\,\lambda^2\, x, \tag{2.18}$$

$$\alpha' = \sqrt{6}\,\left(\Pi - \frac{1}{2}\right)\,\frac{\alpha^2\, x}{u}. \tag{2.19}$$

In these equations primes denote derivative with respect to the so-called e-folding time $N \equiv \ln a$. Also, we define

$$\hat{Q} \equiv \frac{\kappa\, Q}{\sqrt{6}\, H^2\,\dot{\phi}}, \qquad \Pi \equiv \frac{f\, f_{,\phi\phi}}{f_{,\phi}^2}, \qquad \Gamma \equiv \frac{V\, V_{,\phi\phi}}{V_{,\phi}^2}. \tag{2.20}$$

In terms of these dimensionless variables, the fractional energy densities $\Omega \equiv (\kappa^2 \rho)/(3H^2)$ for the scalar field and background matter are given by

$$\Omega_\phi = y^2 + x^2 - 2\,\xi\, u^2, \qquad \Omega_m = 1 - \Omega_\phi, \tag{2.21}$$

respectively. Also, the equation of state of the field $\omega_\phi = p_\phi/\rho_\phi$ reads

$$\omega_\phi = \frac{x^2 - y^2 + 4\sqrt{\frac{2}{3}}\,\alpha\,\xi\, u\, x + 2\left(1 - \frac{2}{3}\, s\right)\,\xi\, u^2}{(y^2 + x^2) - 2\,\xi\, u^2}. \tag{2.22}$$

On the other hand, the effective equation of state $\omega_{eff} = \left(p_m + p_\phi\right) / \left(\rho_m + \rho_\phi\right)$ is given by

$$\omega_{eff} = -\,\gamma\, y^2 - \left(\gamma - 2\right)\, x^2 + 4\sqrt{\frac{2}{3}}\,\alpha\,\xi\, u\, x + 2\,\xi\left(\gamma - \frac{2}{3}\, s\right)\, u^2 + \gamma - 1, \tag{2.23}$$

and the accelerated expansion occurs for $\omega_{eff} < -1/3$.

Here, we should emphasize that the dynamical system of ODE (2.15)-(2.19) are not a dynamical autonomous system unless the parameters Γ and Π are know.
From now on we concentrate on exponential scalar-field potential of the form $V(\phi) =$

$V_0\, e^{-\lambda\kappa\phi}$, such that λ is a dimensionless constant, that is, $\Gamma = 1$ (equivalently, we could consider potentials satisfying $\lambda \equiv -V_{,\phi}/\kappa\, V \approx const$, which is valid for arbitrary but nearly flat potentials [39–41]). In addition to the fact that exponential potentials can give rise to an accelerated expansion (a power-law expansion $a(t) \propto t^p$ with $p > 1$), they possess cosmological scaling solutions [16, 18]. On the other hand, for $f(\phi) \propto \phi^2$ or equivalently $\Pi = 1/2$ then $\alpha \equiv f_{,\phi}/\sqrt{f} = const$. Also, for a general coupling function $u \equiv \kappa\sqrt{f(\phi)}$, with inverse function $\phi = f^{-1}(u^2/\kappa^2)$, then $\alpha(\phi)$ and $\Pi(\phi)$ can be expressed in terms of u (this approach is similar to that followed in the case of quintessence in GR with potential beyond exponential potential, see Ref. [42]). Therefore, two situations may arise; one where α is a constant and another where α depends on u. In both cases, we have a three-dimensional autonomous system (2.15)-(2.17).

Once identified the autonomous system we can obtain the fixed points or critical points (x_c, y_c, u_c) by imposing the conditions $x_c' = y_c' = u_c' = 0$. From the definition (2.13), x_c, y_c, u_c should be real, with $y_c \geq 0$ and $u_c \geq 0$. To study the stability of the critical points, we substitute linear perturbations, $x \to x_c + \delta x$, $y \to y_c + \delta y$, and $u \to u_c + \delta u$ about the critical point (x_c, y_c, u_c) into the autonomous system (2.15)-(2.17) and linearize them. The eigenvalues of the perturbations matrix $\mathcal{M}$, namely, μ_1, μ_2 and μ_3, determine the conditions of stability of the critical points. One generally uses the following classification [15, 16]: (i) Stable node: $\mu_1 < 0$, $\mu_2 < 0$ and $\mu_3 < 0$. (ii) Unstable node: $\mu_1 > 0$, $\mu_2 > 0$ and $\mu_3 > 0$. (iii) Saddle point: one or two of the three eigenvalues are positive and the other negative. (iv) Stable spiral: The determinant of the matrix $\mathcal{M}$ is negative and the real parts of μ_1, μ_2 and μ_3 are negative. A critical point is an attractor in the cases (i) and (iv), but it is not so in the cases (ii) and (iii). The universe will eventually enter these attractor solutions regardless of the initial conditions (see Appendix C).

Now we are going to study the autonomous dynamical system (2.15)-(2.17), first for $\alpha = const \neq 0$ and then for dynamically changing $\alpha(u)$, such that $\alpha(u) \to \alpha(u_c) = 0$ when the system falls into the critical point (x_c, y_c, u_c). Also, the fact that the energy density of TDE Ω_ϕ is of the same order as of dark matter Ω_m in the present universe, suggests that there may be some relation or interaction between them. Several different forms of the coupling between dark energy and dark matter have been proposed. One possibility usually studied is to consider an interaction of the form $Q = \beta\,\kappa\,\rho_m\,\dot{\phi}$ with β a dimensionless constant, see Refs. [16, 18, 19]. A second approach is to introduce an interaction of the form $Q = \Upsilon\,\rho_m$ with the normalization of Υ in terms of the Hubble parameter H, i.e. $\Upsilon/H = \tau$, where τ is a dimensionless constant [18].

Here we consider only the first possibility ($Q = \beta\,\kappa\,\rho_m\,\dot{\phi}$), for both, constant α and dynamically changing α.

2.4 Constant α

2.4.1 Critical points

In this section we consider a non-minimal coupling function $f(\phi) \propto \phi^2$ such that $\alpha = const \neq 0$. From Eq. (2.20), for the coupling $Q = \beta\,\kappa\,\rho_m\,\dot{\phi}$ it is easy to find that

$$\hat{Q} = \frac{\sqrt{6}\,\beta\,\Omega_m}{2}, \tag{2.24}$$

with Ω_m given in the equation (2.21). Substituting (2.24) in (2.15) we can obtain the critical points of the autonomous system (2.15)-(2.17) for constant α. The critical points are presented in the Table 2.1, and these are a generalization of the fixed points that were found in [6, 7]. In Table 2.2 we summarize the stability properties (to be studied below), and conditions for acceleration and existence for each point. The parameters $q_\pm$ are defined as

$$q_\pm = -\alpha\,\xi \pm \sqrt{\xi\,\left(\alpha^2\,\xi - 2\,\beta^2\right)}, \tag{2.25}$$

and the parameters $v_\pm$ as

$$v_\pm = \alpha\,\xi \pm \sqrt{\xi\,\left(\alpha^2\,\xi - 2\,\lambda^2\right)}. \tag{2.26}$$

The point I.a is a matter-dominated solution ($\Omega_m = 1$) with equation of state type cosmological constant $\omega_\phi = -1$, that exists for all values of ξ, λ and $\beta = 0$. Point I.b is a scaling solution and exists solely for $\xi < 0$, $\alpha > 0$ and $\beta > 0$, since in this case the condition $0 \leq \Omega_\phi \leq 1$ is satisfied. The critical point I.c is also a scaling solution which there exists for $\xi < 0$, but contrary to I.b, with $\alpha < 0$ and $\beta < 0$. Accelerated expansion never occurs for these three points because $\omega_{eff} > -1/3$. Points I.d and I.e both correspond to dark-energy-dominated de Sitter solutions with $\Omega_\phi = 1$ and $\omega_\phi = \omega_{eff} = -1$. From (2.26), the fixed point I.d exists for:

$$\xi \geq 2\,\lambda^2/\alpha^2 > 0 \quad \text{and} \quad \lambda/\alpha > 0 \quad \text{or} \quad \xi < 0, \quad \alpha < 0 \quad \text{and} \quad \lambda > 0. \tag{2.27}$$

By the other hand, the point I.e exists for

$$\xi \geq 2\,\lambda^2/\alpha^2 > 0 \quad \text{and} \quad \lambda/\alpha > 0 \quad \text{or} \quad \xi < 0, \quad \alpha > 0 \quad \text{and} \quad \lambda < 0. \tag{2.28}$$

We note that the points I.d and I.e exist irrespective of the presence of the coupling β, since $\Omega_\phi = 1$ in these cases.

Table 2.1: Critical points for the autonomous system (2.15)-(2.17) for constant $\alpha \neq 0$.

Name	x_c	y_c	u_c	Ω_ϕ	ω_ϕ	ω_{eff}
I.a	0	0	0	0	-1	$\gamma-1$
I.b	0	0	$\frac{q_-}{2\beta\xi}$	$-\frac{q_-^2}{2\beta^2\xi}$	$\gamma-1$	$\gamma-1$
I.c	0	0	$\frac{q_+}{2\beta\xi}$	$-\frac{q_+^2}{2\beta^2\xi}$	$\gamma-1$	$\gamma-1$
I.d	0	$\sqrt{\frac{\alpha v_-}{\lambda^2}}$	$\frac{v_-}{2\lambda\xi}$	1	-1	-1
I.e	0	$\sqrt{\frac{\alpha v_+}{\lambda^2}}$	$\frac{v_+}{2\lambda\xi}$	1	-1	-1

Table 2.2: Stability properties, and conditions for acceleration and existence of the fixed points in Table 2.1.

Name	Stability	Acceleration	Existence
I.a	Saddle	No	$\beta = 0$
I.b	Saddle	No	$\xi < 0$, $\alpha > 0$ and $\beta > 0$
I.c	Saddle	No	$\xi < 0$, $\alpha < 0$ and $\beta < 0$
I.d	S. node or S. spiral, or Saddle	All values	(2.27)
I.e	S. node or S. spiral, or Saddle	All values	(2.28)

2.4.2 Stability

Substituting the linear perturbations, $x \to x_c+\delta x$, $y \to y_c+\delta y$, and $u \to u_c+\delta u$ into the autonomous system (2.15)-(2.17) and linearize them, we find the follows components for the matrix of linear perturbations $\mathcal{M}$

$$\mathcal{M}_{11} = \frac{-3\,\gamma\,y_c^2 + 9\,\left(2-\gamma\right)\,x_c^2 + 8\,\sqrt{6}\,\alpha\,\xi\,u_c\,x_c}{2\,\left(2\,\xi\,u_c^2+1\right)} - \frac{3\,\left(2-\gamma\right)}{2} - \frac{\partial \hat{Q}}{\partial x}|_{x_c,y_c,u_c}, \tag{2.29}$$

$$\mathcal{M}_{12} = \sqrt{6}\,\lambda\,y_c - \frac{3\,\gamma\,x_c\,y_c}{2\,\xi\,u_c^2+1} - \frac{\partial \hat{Q}}{\partial y}|_{x_c,y_c,u_c}, \tag{2.30}$$

$$\mathcal{M}_{13} = \frac{6\,\xi\,u_c\,x_c\,\left(\gamma\,y_c^2 - (2-\gamma)\,x_c^2\right) + 4\,\sqrt{6}\,\alpha\,\xi\,x_c^2}{\left(2\,\xi\,u_c^2+1\right)^2} - \frac{2\,\sqrt{6}\,\alpha\,\xi\,x_c^2}{2\,\xi\,u_c^2+1} - \sqrt{6}\,\alpha\,\xi - \frac{\partial\hat{Q}}{\partial u}|_{x_c,y_c,u_c}, \tag{2.31}$$

$$\mathcal{M}_{21} = \left(\frac{3\,(2-\gamma)\,x_c + 2\,\sqrt{6}\,\alpha\,\xi\,u_c}{2\,\xi\,u_c^2+1} - \frac{\sqrt{6}\,\lambda}{2}\right) y_c, \tag{2.32}$$

$$\mathcal{M}_{22} = \frac{-9\,\gamma\,y_c^2 + 3\,(2-\gamma)\,x_c^2 + 4\,\sqrt{6}\,\alpha\,\xi\,u_c\,x_c}{2\,\left(2\,\xi\,u_c^2+1\right)} - \frac{\sqrt{6}\,\lambda\,x_c - 3\,\gamma}{2}, \tag{2.33}$$

$$\mathcal{M}_{23} = \frac{6\,\xi\,u_c\,y_c\,\left(\gamma\,y_c^2 + (2-\gamma)\,x_c^2\right) + 4\,\sqrt{6}\,\alpha\,\xi\,x_c\,y_c}{\left(2\,\xi\,u_c^2+1\right)^2} - \frac{2\,\sqrt{6}\,\alpha\,\xi\,x_c\,y_c}{2\,\xi\,u_c^2+1}, \tag{2.34}$$

$$\mathcal{M}_{31} = \frac{\sqrt{6}\,\alpha}{2}, \qquad \mathcal{M}_{32} = 0, \qquad \mathcal{M}_{33} = 0. \tag{2.35}$$

The eigenvalues of the matrix $\mathcal{M}$, for each critical point, are as follows:

- Point I.a:

$$\mu_{1,2} = \frac{3\,(2-\gamma)\,\left(-1 \pm \sqrt{1 - \frac{16\,\alpha^2\,\xi}{3\,(2-\gamma)^2}}\right)}{4}, \qquad \mu_3 = \frac{3\,\gamma}{2}. \tag{2.36}$$

- Point I.b:

$$\mu_{1,2} = \frac{3\,(2-\gamma)\,\left(-1 \pm \sqrt{1 + \frac{16\,\alpha\,\sqrt{\xi\,(\alpha^2\,\xi - 2\,\beta^2)}}{3\,(\gamma-2)^2}}\right)}{4}, \qquad \mu_3 = \frac{3\,\gamma}{2}. \tag{2.37}$$

- Point I.c:

$$\mu_{1,2} = \frac{3\,(2-\gamma)\,\left(-1 \pm \sqrt{1 - \frac{16\,\alpha\,\sqrt{\xi\,(\alpha^2\,\xi - 2\,\beta^2)}}{3\,(\gamma-2)^2}}\right)}{4}, \qquad \mu_3 = \frac{3\,\gamma}{2}. \tag{2.38}$$

- Point I.d:

$$\mu_{1,2} = \frac{3\,\left(-1 \pm \sqrt{1 - \frac{4}{3}\,\alpha\,\sqrt{\xi\,\left(\alpha^2\,\xi - 2\,\lambda^2\right)}}\right)}{2}, \qquad \mu_3 = -3\,\gamma. \tag{2.39}$$

- Point I.e:

$$\mu_{1,2} = \frac{3\left(-1 \pm \sqrt{1+\frac{4}{3}\,\alpha\,\sqrt{\xi\,(\alpha^2\,\xi - 2\,\lambda^2)}}\right)}{2}, \qquad \mu_3 = -3\,\gamma. \tag{2.40}$$

The points I.a, I.b, and I.c are saddle points in any case, since $\mu_2 < 0$ and $\mu_3 > 0$. For $\xi < 0$ and $\alpha < 0$ or $\xi > 2\,\lambda^2/\alpha^2$ and $\alpha < 0$ point I.d is a saddle point. On the other hand, for

$$\frac{2\,\lambda^2}{\alpha^2} < \xi \leq \frac{\lambda^2}{\alpha^2}\left(1+\sqrt{1+\frac{9}{16\,\lambda^4}}\right), \tag{2.41}$$

and $\alpha > 0$ it is a stable node. Also, when $\xi > \frac{\lambda^2}{\alpha^2}\left(1+\sqrt{1+\frac{9}{16\,\lambda^4}}\right)$ then μ_1 and μ_2 are complex with real part negative and $\det(\mathcal{M}) = -9\,\alpha\,\gamma\,\sqrt{\xi\,(\alpha^2\,\xi - 2\,\lambda^2)} < 0$ for $\alpha > 0$. So, in this case point I.d is a stable spiral.

Finally, point I.e is a saddle point for $\xi < 0$ and $\alpha > 0$ or $\xi > 2\,\lambda^2/\alpha^2$ and $\alpha > 0$. On the other hand, for ξ as in (2.41) and $\alpha < 0$, it is a stable node. When $\xi > \frac{\lambda^2}{\alpha^2}\left(1+\sqrt{1+\frac{9}{16\,\lambda^4}}\right)$ then μ_1 and μ_2 are complex with real part negative and $\det(\mathcal{M}) = 9\,\alpha\,\gamma\,\sqrt{\xi\,(\alpha^2\,\xi - 2\,\lambda^2)} < 0$ for $\alpha < 0$. In this case, point I.e is a stable spiral.

So, for constant α, $\xi > \frac{2\,\lambda^2}{\alpha^2}$ and $\beta = 0$, the universe enters the matter-dominated solution I.a with $\omega_\phi = -1$ and $\omega_{eff} = 0$ for $\gamma = 1$ (non-relativistic dark matter), but since this solution is unstable, finally the system falls into the attractor I.d or I.e (dark-energy-dominated de Sitter solution) with $\Omega_\phi = 1$ and $\omega_\phi = \omega_{eff} = -1$. Moreover, for $\beta \neq 0$ and $\xi < 0$ the system enters in the unstable scaling solution I.b or I.c, but, in this case the de Sitter solutions I.d or I.e are not late-time attractors of the system. Also, for constant α no scaling attractor was found, even when we allow the possible interaction between TDE and dark matter.

2.5 Dynamically changing α

Now let us consider a general function of non-minimal coupling $f(\phi)$ such that α depends on u and $\alpha(u) \to \alpha(u_c) = 0$ when $(x, y, u) \to (x_c, y_c, u_c)$ ((x_c, y_c, u_c) is a fixed point of the system). The field ϕ rolls down toward $\pm\infty$ ($x > 0$ or $x < 0$) with $f(\phi) \to u_c^2/\kappa^2$ when $(x, y, u) \to (x_c, y_c, u_c)$. In this section, for simplicity, we shall consider the situation with positive values of λ and β.

2.5.1 Critical points

The critical points of the autonomous system (2.15)-(2.17) (with α replaced by $\alpha(u)$) are presented in Table 2.3. In Table 2.3 we define $\hat{\xi} \equiv 2\,\xi\,u_c^2 + 1$ and $\zeta \equiv \lambda + \beta > 0$. In Table 2.4 we summarize the stability properties, and conditions for acceleration and existence for each point.

Point II.a corresponds to a scaling solution. The condition $0 \leq \Omega_\phi \leq 1$ is satisfied for $\beta < \sqrt{3/8}\,(2-\gamma)$ if

$$\frac{3\,(2-\gamma)^2}{4\,\beta^2}\left(1+\sqrt{1-\frac{8\,\beta^2}{3\,(2-\gamma)^2}}\right) \leq \hat{\xi} \leq \frac{3\,(2-\gamma)^2}{2\,\beta^2}, \tag{2.42}$$

or

$$0 \leq \hat{\xi} \leq \frac{3\,(2-\gamma)^2}{4\,\beta^2}\left(1-\sqrt{1-\frac{8\,\beta^2}{3\,(2-\gamma)^2}}\right) < \frac{3\,(2-\gamma)^2}{2\,\beta^2}. \tag{2.43}$$

On the other hand, for $\beta \geq \sqrt{3/8}\,(2-\gamma)$ it is required merely that

$$0 \leq \hat{\xi} \leq \frac{3\,(2-\gamma)^2}{2\,\beta^2}. \tag{2.44}$$

The equation of state for this solution is given by

$$\omega_\phi = \frac{2\,\beta^2\,(2-\gamma)\,\hat{\xi}}{\left(\hat{\xi}-1\right)\left(2\,\beta^2\left(\hat{\xi}-1\right)-3\,(\gamma-4)\,\gamma+4\,(\beta^2-3)\right)+2\,\beta^2} + \gamma - 1, \tag{2.45}$$

and for $\hat{\xi} \approx 1$ ($\xi \approx 0$), and $\beta \neq 0$, we have that $\omega_\phi \approx 1$ (stiff matter).

The condition of accelerated expansion is given by

$$\hat{\xi} < -\frac{(2-\gamma)\,(3\,\gamma-2)}{2\,\beta^2} < 0. \tag{2.46}$$

Therefore, the accelerated expansion no occurs in this solution since necessary $\hat{\xi}$ is positive to ensure $0 \leq \Omega_\phi \leq 1$.

Points II.b and II.c exists for $\hat{\xi} \geq 0$. Both points are scalar-field dominant solutions ($\Omega_\phi = 1$), but without accelerated expansion because $\omega_{eff} = 1$.

Point II.d is also a scaling solution. From the condition $0 \leq \Omega_\phi \leq 1$ we have the following constraint

$$\frac{3\,\gamma}{\lambda\,\zeta} \leq \hat{\xi} \leq \frac{3\,\gamma}{\lambda\,\zeta} + \frac{\zeta}{\lambda}. \tag{2.47}$$

The equation of state of field for this solution is given by

$$\omega_\phi = \frac{\beta\,\zeta\,\gamma}{\beta\,\lambda\left(\hat{\xi}-2\right)+\lambda^2\left(\hat{\xi}-1\right)-3\,\gamma-\beta^2} + \gamma - 1, \tag{2.48}$$

Table 2.3: Critical points of the autonomous system (2.15)-(2.17) for dynamically changing $\alpha(u)$ such that $\alpha(u) \to \alpha(u_c) = 0$ and $u_c \geq 0$. We define $\hat{\xi} \equiv 2\,\xi\,u_c^2 + 1$ and $\zeta \equiv \lambda + \beta$.

Name	x_c	y_c	u_c	Ω_ϕ	ω_ϕ	ω_{eff}
II.a	$-\frac{\sqrt{6}\,\beta\,\hat{\xi}}{3\,(2-\gamma)}$	0	u_c	$\frac{2\,\beta^2\,\hat{\xi}^2}{3\,(\gamma-2)^2} + 1 - \hat{\xi}$	(2.45)	$\frac{2\,\beta^2\,\hat{\xi}}{3\,(2-\gamma)} + \gamma - 1$
II.b	$-\sqrt{\hat{\xi}}$	0	u_c	1	1	1
II.c	$\sqrt{\hat{\xi}}$	0	u_c	1	1	1
II.d	$\frac{\sqrt{6}\,\gamma}{2\,\zeta}$	$\frac{\sqrt{\beta\,\zeta\,\hat{\xi}+\frac{3}{2}\,(2-\gamma)\,\gamma}}{\zeta}$	u_c	$-\frac{\lambda\,\hat{\xi}}{\zeta} + \frac{3\,\gamma}{\zeta^2} + 1$	(2.48)	$\frac{\lambda\,(\gamma-1)-\beta}{\zeta}$
II.e	$\frac{\lambda\,\hat{\xi}}{\sqrt{6}}$	$\sqrt{\hat{\xi}\,\left(1 - \frac{\lambda^2\,\hat{\xi}}{6}\right)}$	u_c	1	$\frac{\lambda^2\,\hat{\xi}}{3} - 1$	$\frac{\lambda^2\,\hat{\xi}}{3} - 1$

and for $\hat{\xi} \approx 1$, $\lambda << \beta$ ($\beta >> 1$) we have that $\omega_\phi \approx -1$ (De Sitter solution). The universe exhibits an accelerated expansion independently of $\hat{\xi}$ for

$$\lambda < \frac{2\,\beta}{3\,(\gamma - 1) + 1}. \tag{2.49}$$

In the case of $\gamma = 1$ the accelerated expansion occurs for $\lambda < 2\,\beta$.

Point II.e is a scalar-field dominant solution ($\Omega_\phi = 1$) that exists for

$$0 \leq \hat{\xi} \leq \frac{6}{\lambda^2}, \tag{2.50}$$

and gives an accelerated expansion at late times for

$$\hat{\xi} < \frac{2}{\lambda^2}. \tag{2.51}$$

Given that $\hat{\xi} \geq 0$ is necessary for the existence of this solution, then in Table 2.3 we have that $\omega_\phi = \omega_{eff} = \lambda^2\,\hat{\xi}/3 - 1 \geq -1$.

2.5.2 Stability

For dynamically changing $\alpha(u)$, such that $\alpha(u) \to \alpha(u_c) = 0$, the components of the matrix of perturbation $\mathcal{M}$ are written as

$$\mathcal{M}_{11} = \frac{3\,\left(3\,(2-\gamma)\,x_c^2 - \gamma\,y_c^2\right)}{2\,\hat{\xi}} - \frac{3\,(2-\gamma)}{2} - \frac{\partial \hat{Q}}{\partial x}|_{x_c,y_c,u_c}, \tag{2.52}$$

Table 2.4: Stability properties, and conditions for acceleration and existence of the fixed points in Table 2.3. As in Table 2.3 we define $\hat{\xi} \equiv 2\,\xi\,u_c^2 + 1$ and $\zeta \equiv \lambda + \beta$.

Name	Stability	Acceleration	Existence
II.a	Saddle	No	(2.42), (2.43), (2.44)
II.b	U. node or Saddle	No	$\hat{\xi} \geq 0$
II.c	U. node or Saddle	No	$\hat{\xi} \geq 0$
II.d	S. node or S. spiral or Saddle	$\lambda < \frac{2\beta}{3(\gamma-1)+1}$	$\frac{3\gamma}{\lambda\zeta} \leq \hat{\xi} \leq \frac{3\gamma}{\lambda\zeta} + \frac{\zeta}{\lambda}$
II.e	S. node or Saddle	$\hat{\xi} < \frac{2}{\lambda^2}$	$0 \leq \hat{\xi} \leq \frac{6}{\lambda^2}$

$$\mathcal{M}_{12} = \sqrt{6}\,\lambda\,y_c - \frac{3\,\gamma\,x_c\,y_c}{\hat{\xi}} - \frac{\partial \hat{Q}}{\partial y}|_{x_c,y_c,u_c}, \tag{2.53}$$

$$\mathcal{M}_{13} = \frac{6\,\xi\,u_c\,x_c\,\left(\gamma\,y_c^2 - (2-\gamma)\;x_c^2\right)}{\hat{\xi}^2} + \frac{2\,\sqrt{6}\,\xi\,\eta_c\,u_c\,x_c^2}{\hat{\xi}} - \sqrt{6}\,\xi\,\eta_c\,u_c - \frac{\partial \hat{Q}}{\partial u}|_{x_c,y_c,u_c}, \tag{2.54}$$

$$\mathcal{M}_{21} = \frac{3\;(2-\gamma)\;x_c\,y_c}{\hat{\xi}} - \frac{\sqrt{6}\,\lambda\,y_c}{2}, \tag{2.55}$$

$$\mathcal{M}_{22} = \frac{3\;\left((2-\gamma)\;x_c^2 - 3\,\gamma\,y_c^2\right)}{2\,\hat{\xi}} - \frac{\sqrt{6}\,\lambda\,x_c - 3\,\gamma}{2}, \tag{2.56}$$

$$\mathcal{M}_{23} = \frac{6\,\xi\,u_c\,y_c\,\left(\gamma\,y_c^2 - (2-\gamma)\;x_c^2\right)}{\hat{\xi}^2} + \frac{2\,\sqrt{6}\,\xi\,\eta_c\,u_c\,x_c\,y_c}{\hat{\xi}}, \tag{2.57}$$

$$\mathcal{M}_{31} = 0, \qquad \mathcal{M}_{32} = 0, \qquad \mathcal{M}_{33} = \frac{\sqrt{6}\,\eta_c\,x_c}{2}. \tag{2.58}$$

Here we define $\eta_c \equiv \frac{d\alpha(u)}{du}|_{u=u_c}$.

Using (2.24) in (2.52)-(2.54) we can calculate the components of $\mathcal{M}$ for each critical point and the corresponding eigenvalues:

- Point II.a:

$$\mu_1 = \frac{\beta\,\zeta\,\hat{\xi}}{2-\gamma} + \frac{3\,\gamma}{2}, \qquad \mu_2 = -\frac{\beta\,\eta_c\,\hat{\xi}}{2-\gamma}, \qquad \mu_3 = \frac{\beta^2\,\hat{\xi}}{2-\gamma} - \frac{3\;(2-\gamma)}{2}. \tag{2.59}$$

- Point II.b:

$$\mu_1 = -\frac{\sqrt{6}\,\eta_c\,\sqrt{\hat{\xi}}}{2}, \qquad \mu_2 = \frac{\sqrt{6}\,\left(\sqrt{6}+\lambda\,\sqrt{\hat{\xi}}\right)}{2}, \qquad \mu_3 = -\sqrt{6}\,\beta\,\sqrt{\hat{\xi}} + 3\,(2-\gamma)\,. \tag{2.60}$$

- Point II.c:

$$\mu_1 = \frac{\sqrt{6}\,\eta_c\,\sqrt{\hat{\xi}}}{2}, \qquad \mu_2 = \frac{\sqrt{6}\,\left(\sqrt{6}-\lambda\,\sqrt{\hat{\xi}}\right)}{2}, \qquad \mu_3 = \sqrt{6}\,\beta\,\sqrt{\hat{\xi}} + 3\,(2-\gamma)\,. \tag{2.61}$$

- Point II.d ($\gamma = 1$):

$$\mu_{1,2} = \frac{3\,(\lambda+2\,\beta)}{4\,\zeta}\left(-1 \pm \sqrt{1 - \frac{8\,\left(2\,\beta\,\zeta\,\hat{\xi}+3\right)\,\left(\zeta\,\lambda\,\hat{\xi}-3\right)}{3\,(\lambda+2\,\beta)^2\,\hat{\xi}}}\right), \qquad \mu_3 = \frac{3\,\eta_c}{2\,\zeta}. \tag{2.62}$$

- Point II.e:

$$\mu_1 = \lambda\,\zeta\,\hat{\xi} - 3\,\gamma, \qquad \mu_2 = \frac{\lambda^2\,\hat{\xi}-6}{2}, \qquad \mu_3 = \frac{\eta_c\,\lambda\,\hat{\xi}}{2}. \tag{2.63}$$

Since for the point II.a we have $0 \leq \hat{\xi} \leq 3\,(2-\gamma)^2/2\,\beta^2$ (Eqs. (2.42), (2.43) and (2.44)), then it is always a saddle point. The points II.b and II.c are either an unstable node or a saddle point. The point II.b is an unstable node for $0 < \hat{\xi} < 3\,(2-\gamma)^2/2\,\beta^2$ and $\eta_c < 0$, whereas it is a saddle point for $\eta_c > 0$ or $\hat{\xi} > 3\,(2-\gamma)^2/2\,\beta^2$. On the other hand, the point II.c is a unstable node if $0 < \hat{\xi} < 6/\lambda^2$ and $\eta_c > 0$, whereas for $\eta_c < 0$ or $\hat{\xi} > 6/\lambda^2$ it is a saddle point.

To study the properties of stability of the point II.d we shall consider the case where the background fluid is non-relativistic dark matter ($\gamma = 1$). So, we restrict ourselves to positive $\hat{\xi}$ as in constraint (2.47) with $\gamma = 1$, and $\lambda < 2\,\beta$ or equivalently $\zeta < 3\,\beta$.

The eigenvalues μ_1 and μ_2 are real and negatives if

$$\frac{3}{\zeta\,\lambda} < \hat{\xi} \leq \Delta + \frac{3}{\zeta\,\lambda}, \tag{2.64}$$

where

$$\Delta = \frac{3\,\left(\sqrt{\delta^2 + 64\,\beta\,\zeta\,(\lambda+2\,\beta)^2} + \delta\right)}{32\,\beta\,\zeta^2\,\lambda} > 0, \tag{2.65}$$

with $\delta = \left(-7\,\zeta^2 - 6\,\beta\,\zeta + \beta^2\right)$. Then, for $\eta_c < 0$ and $\hat{\xi}$ satisfying (2.64) the point II.d is a stable node. For $\eta_c > 0$ and $\hat{\xi}$ as in (2.64) it is saddle point. On the other hand, for

$$\Delta + \frac{3}{\zeta\,\lambda} < \hat{\xi} \leq \frac{3}{\lambda\,\zeta} + \frac{\zeta}{\lambda}, \tag{2.66}$$

μ_1 and μ_2 are complex with real part negative, and the determinant of the matrix $\mathcal{M}$ that is given by

$$\det(\mathcal{M}) = \frac{9\,\eta_c\,\left(2\,\beta\,\lambda\,\zeta^3\,\hat{\xi}^2 + 3\,\zeta^2\,(\lambda - 2\,\beta)\,\hat{\xi} - 9\,\zeta\right)}{4\,\zeta^4\,\hat{\xi}}, \tag{2.67}$$

is negative if in addition $\eta_c < 0$ (also in this case $\mu_3 < 0$). Therefore, for $\hat{\xi}$ as in (2.66) and $\eta_c < 0$, point II.d becomes a stable spiral. The constraint (2.64) is consistent with the physical condition (2.47) (it can be stronger or weaker). On the other hand, the constraint (2.66) is also consistent with (2.47) provided that $\Delta < \zeta/\lambda$. In any case, either is a stable node or a stable spiral, point II.d is a scaling attractor with accelerated expansion if in addition we have $\lambda < 2\,\beta$.

Point II.e is a stable node if

$$0 < \hat{\xi} < \frac{3\,\gamma}{\lambda\,\zeta} < \frac{6}{\lambda^2}, \tag{2.68}$$

and $\eta_c < 0$. Since $1 \leq \gamma < 2$ then $\frac{3\,\gamma}{\lambda\,\zeta} < \frac{6}{\lambda^2}$. On the other hand, it is a saddle point for

$$\frac{3\,\gamma}{\lambda\,\zeta} < \hat{\xi} < \frac{6}{\lambda^2}. \tag{2.69}$$

or $\eta_c > 0$. The point II.e is also an interesting solution, it is a scalar-field dominant solution ($\Omega_\phi = 1$) and can be attractor (Eq. (2.68)) with accelerated expansion (Eq. (2.51)) and such that $\omega_\phi = \omega_{eff} \geq -1$.

So, for dynamically changing α such that $\alpha(u) \to \alpha(u_c) = 0$, the fixed points II.d and II.e can attract the universe at late times and are equally viable solutions. But since we are interested in the scaling attractors, now we are going to concentrate on the solution II.d. It is a scaling solution that has accelerated expansion and at the same time is an attractor (stable node or stable spiral). For $\gamma = 1$, the accelerated expansion occurs, independently of $\hat{\xi}$, for $\lambda < 2\,\beta$ (Eq. (2.49)), and the physical condition $0 \leq \Omega_\phi \leq 1$ is ensures for $\hat{\xi}$ in accordance with (2.47). Also, for $\hat{\xi}$ in accordance with (2.64) and $\eta_c < 0$ point II.d is a stable node, or for $\hat{\xi}$ as in (2.66) and $\eta_c < 0$ it is a stable spiral. When point II.d is a stable node, the constraint (2.64) can be stronger or weaker that the physical condition (2.47). Numerically we find that exist $\lambda_*(\beta) < 2\,\beta$ such that for $\lambda < \lambda_*(\beta)$ then $\Delta > \zeta/\lambda$ and the constraint (2.64) is weaker that (2.47).

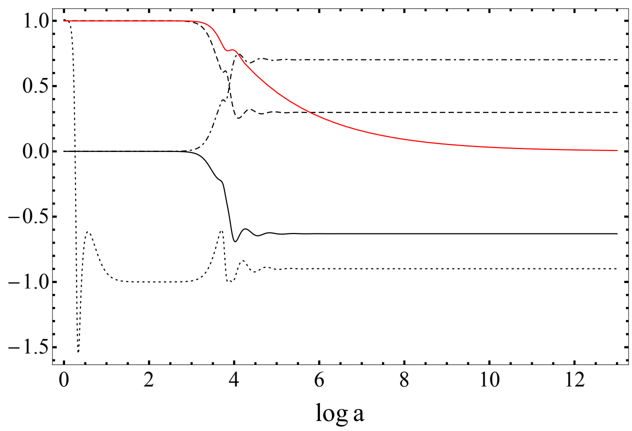

Figure 2.1: Evolution of Ω_m (dashed), Ω_ϕ (dotdashed), ω_ϕ (dotted), ω_{eff} (solid) and $\alpha(u)$ (red line, starting at $\alpha = 1$) with $\gamma = 1$, $\lambda = 2.4$ and $\xi = 3.5 \times 10^{-5}$, in the case where the coupling to dark matter β changes rapidly from $\beta_1 = 0$ to $\beta = 4.1$. We choose initial conditions $x_i = 5 \times 10^{-5}$, $y_i = 3.1 \times 10^{-6}$ and $u_i = 7 \times 10^{-4}$ with $\alpha(u) = u_c - u$. The universe exits from the matter-dominated solution I.a with constant α, $\omega_\phi = -1$, $\omega_{eff} = 0$ and approaches the scaling attractor II.d for dynamically changing $\alpha(u)$ with $\Omega_\phi \approx 0.70$, $\Omega_m \approx 0.30$, $\omega_\phi \approx -0.90$ and $\omega_{eff} \approx -0.63$. We note that ω_ϕ presents the phantom-divide crossing during cosmological evolution.

For example, for $\beta = 0.6$ then $\lambda_*(\beta) \approx 0.82$, and if $\lambda = 0.2$ the constraint (2.64) implies $18.75 < \hat{\xi} < 22.96$. Moreover, the physical condition (2.47) implies $18.75 < \hat{\xi} < 22.75$. Also, in this example, for $\hat{\xi} \approx 19.95$ we have $\omega_\phi \approx -1.07$ (Eq. (2.48)), $\Omega_\phi \approx 0.70$ and $\Omega_m \approx 0.30$ (with accelerated expansion since $\lambda < 2\,\beta$). However, this is not a suitable range for $\hat{\xi}$, since from solar system experiments (see for instance [22]), necessary $\hat{\xi} \approx 1$ (or $\xi \approx 0$). Finally, when point II.d is a stable spiral, that is, $\hat{\xi}$ in accordance with (2.66) and $\eta_c < 0$, the physical condition (2.47) is ensures whenever $\Delta < \zeta/\lambda$. For example, for $\lambda = 2.4$ and $\beta = 4.1$ we have that (2.66) implies $0.23 < \hat{\xi} < 2.90$, whereas the constraint (2.47) implies $0.19 < \hat{\xi} < 2.90$. In Fig. 2.1 we show the case when the system approaches the fixed point II.d in the above example for $\lambda = 2.4$, $\beta = 4.1$ and for instance $\hat{\xi} \approx 1.00064$ ($\xi \approx 3.5 \times 10^{-5}$ and $u_c = 1$). In this figure, by way of example, we consider the function $\alpha(u) = u_c - u$ with $\eta_c = -1$. Also, following [16, 43], we consider a non-linear coupling

$$\beta(u) = \frac{(\beta - \beta_1)\,\tanh\left(\frac{u-u_1}{b}\right) + \beta + \beta_1}{2}, \tag{2.70}$$

that changes between a small β_1 to a large β in order to that initially the system enters in the matter-dominated solution I.a with $\beta_1 = 0$ and eventually approaches the scaling attractor II.d with $\omega_\phi \approx -0.90$, $\Omega_\phi \approx 0.70$, $\Omega_m \approx 0.30$ and accelerated expansion ($\lambda < 2\,\beta$).

Chapter 3

Tachyonic Teleparallel Dark Energy

The tachyon field arising in the context of string theory provides an example of modified form of matter, which has been studied in applications to cosmology both as a source of early inflation and late-time speed-up of the cosmic expansion rate [44–46]. The dynamics of the tachyon is very different from the standard case (quintessence). As the lagrangian of quintessence generalizes the lagrangian of a non-relativistic particle, the lagrangian of the tachyon field generalizes the lagrangian of the relativistic particle [44]. In this regard the tachyon generalizes the quintessence field, and a non-minimal version in the context of TG was proposed in Ref. [47].

In this chapter we will be interested in the dynamics of "tachyonic teleparallel dark energy" (TTDE), as this model has been called [47]. Given the nature of the tachyon field, we can expect a richer structure than in the case of TDE. In fact, as we are going to see, an era ϕMDE (see Ref. [18]) is possible, but in order to have a viable cosmological evolution it is necessary to generalize the non-minimal coupling to a dynamically changing coefficient $\alpha \equiv f_{,\phi}/\sqrt{f}$, with $f(\phi)$ the general non-minimal coupling function [48].

3.1 Tachyon field in general relativity

The action for the tachyon scalar field minimally coupled with gravity is given by

$$S_\varphi = \int d^4x \, \sqrt{-g} \, \left[\frac{R}{2\,\kappa^2} - V(\varphi)\,\sqrt{1 - 2\,X} \right], \tag{3.1}$$

where $X = \frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi$. $V(\varphi)$ is the potential of the tachyon field, and the potential corresponding to scaling solutions (i.e., the field energy density ρ_φ is proportional to the fluid energy density ρ_m) is the inverse power-law type, $V(\varphi) \propto \varphi^{-2}$. Moreover, a remarkable feature of the stress tensor of the tachyon field is that it can be considered as the sum of a pressure-less dust component and a cosmological constant [44]. This means that the stress tensor can be thought of as made up of two components, one behaving like a pressure-less fluid (dark matter), while the other having a negative pressure (dark energy). This property is reflected in that when $\dot{\varphi}$ is small compared to unity (compared to $V(\varphi)$ in the case of quintessence), the tachyon field has equation of state $\omega_\varphi \to -1$ and mimic a cosmological constant, just like the quintessence field. But, when $\dot{\varphi} \to 1$ the tachyon field has equation of state $\omega_\varphi \approx 0$ and behaves like non-relativistic matter with $\rho_\varphi \propto a(t)^{-3}$ ($a(t)$ is the scale factor), whereas in the case of quintessence for $\dot{\varphi} >> V(\varphi)$, it has equation of state $\omega_\varphi \approx 1$ (stiff matter) leading to $\rho_\varphi \propto a(t)^{-6}$. So, the dynamics the tachyon field is very different from the standard field case, irrespective of the steepness of the tachyon potential the equation of state varies between 0 and -1, and the energy density behaves as $\rho_\varphi \propto a(t)^{-m}$ with $0 \leq m \leq 3$ [16].

A study of dynamical systems in Friedmann-Lemaitre-Robertson-Walker (FLRW) cosmology within phenomenological theories based on the effective tachyon action (3.1) can be found in [16, 45, 46]. In [46] was proposed perform a transformation of the form

$$\varphi \to \phi = \int d\varphi \sqrt{V(\varphi)} \Longleftrightarrow \partial\varphi = \frac{\partial\phi}{\sqrt{V(\phi)}}, \tag{3.2}$$

which allows to introduce normalized phase-space variables and in terms of these variables one can obtain a closed autonomous system of ordinary differential equations (ODE) out of the cosmological field equations written in terms of the transformed tachyon field ϕ, for a broad class of self-interaction potentials $V(\phi)$ (in [19] also was carried out a transformation of this type to study coupled dark energy in GR). Also, as we will show quite soon, the above field re-definition allows us to study a non-minimal coupling between tachyon field and TG in terms a closed autonomous system of ODE. We are going to concentrate on the inverse square potential $V(\varphi) \propto \varphi^{-2}$, that for the transformed field ϕ becomes $V(\phi) = V_0\, e^{-\lambda\,\kappa\,\phi}$, and λ is a constant.

3.2 Tachyonic teleparallel dark energy

In what follows we consider a non-minimal coupling between tachyon field and TG as was already considered in Ref. [47]. In order to have a closed autonomous system of ODE and study the dynamics of the model is required the transformation $\varphi \to \phi$ in accordance to (3.2). Under the transformation (3.2), the relevant action reads

$$S = \int d^4x\, h \left[\frac{T}{2\,\kappa^2} - V(\phi)\sqrt{1 - \frac{2\,X}{V(\phi)}} + \xi\, f(\phi)\, T \right] + S_m, \tag{3.3}$$

where $h \equiv \det(h^a{}_\mu) = \sqrt{-g}$ ($h^a{}_\mu$ are the orthonormal components of the tetrad), $T/2\,\kappa^2$ is the lagrangian of teleparallelism (T is the torsion scalar), S_m is the matter action, ξ is a dimensionless constant measuring the non-minimal coupling, and $f(\phi) > 0$ is the non-minimal coupling function with units of $mass^2$ that only depends of the transformed tachyon field ϕ (see Refs. [1, 8]). Varying the action (3.3) with respect to tetrad fields yields field equation

$$2\left(\frac{1}{\kappa^2} + 2\,\xi\, f(\phi)\right)\left[h^{-1}\, h^a{}_\alpha\, \partial_\sigma \left(h\, h_a{}^\tau\, S_\tau{}^{\rho\sigma}\right) + T^\tau{}_{\nu\alpha}\, S_\tau{}^{\rho\nu} + \frac{T}{4}\,\delta_\alpha^\rho\right]$$
$$+ 4\,\xi\, S_\alpha{}^{\rho\sigma}\, f_{,\phi}\, \partial_\sigma\phi - \mu^{-1}\, V(\phi)\, \delta_\alpha^\rho - \mu\, \partial_\alpha\phi\, \partial^\rho\phi = \Theta_\alpha^\rho. \tag{3.4}$$

where Θ_α^ρ stands for the symmetric matter energy-momentum tensor. Also, we define

$$\mu \equiv \frac{1}{\sqrt{1 - \frac{2\,X}{V}}}. \tag{3.5}$$

Imposing the flat FLRW geometry, we obtain the Friedmann equations with

$$\rho_\phi = \mu\, V\,(\phi) - 6\,\xi\, H^2\, f(\phi), \tag{3.6}$$

the scalar energy density and

$$p_\phi = -\mu^{-1}\, V(\phi) + 4\,\xi\, H\, f_{,\phi}\, \dot\phi + 2\,\xi\, \left(3\, H^2 + 2\, \dot H\right)\, f(\phi), \tag{3.7}$$

the pressure density of field. Here we also use the useful relation $T = -6\, H^2$, which arises for flat FLRW geometry.

The variation of the action with respect to the scalar field yields the motion equation

$$\Box\phi + \frac{\mu^2}{V}\left(\partial_\mu X - \frac{X}{V}\, V_{,\phi}\, \partial_\mu\phi\right)\partial^\mu\phi + \left(1 - \frac{X}{V}\right) V_{,\phi} - \xi\, \mu^{-1}\, f_{,\phi}\, T = 0; \tag{3.8}$$

where $V_{,\phi} = dV/d\phi$ and $\Box \equiv \partial_\mu\partial^\mu + h^{-1}\, \partial_\mu h\, \partial^\mu$. Consequently, in the flat FLRW background we find

$$\ddot\phi + 3\,\mu^{-2}\, H\, \dot\phi + \left(1 - \frac{3\,X}{V}\right) V_{,\phi} + 6\,\xi\, \mu^{-3}\, f_{,\phi}\, H^2 = 0. \tag{3.9}$$

Rewriting the equation of motion (3.9) in terms of scalar energy density and the pressure density of field we obtain

$$\dot{\rho}_\phi + 3\,H\,\rho_\phi\,(1+\omega_\phi) = 0, \tag{3.10}$$

whereas that for matter

$$\dot{\rho}_m + 3\,H\,\rho_m\,(1+\omega_m) = 0, \tag{3.11}$$

where $\omega_\phi \equiv p_\phi/\rho_\phi$ and $\omega_m \equiv p_m/\rho_m = const$ are the equation-of-state parameter of dark energy and dark matter, respectively. We also define the barotropic index $\gamma \equiv 1+\omega_m$, such that $0<\gamma<2$. On the other hand, we note that there is no coupling between dark energy and dark matter.

3.3 Phase-space analysis

In order to study the dynamics of the model it is convenient to introduce the following dimensionless variables

$$x \equiv \frac{\dot{\phi}}{\sqrt{V}}, \quad y \equiv \frac{\kappa\sqrt{V}}{\sqrt{3}\,H}, \quad u \equiv \kappa\,\sqrt{f}, \quad \alpha \equiv \frac{f_{,\phi}}{\sqrt{f}}, \quad \lambda \equiv -\frac{V_{,\phi}}{\kappa V}. \tag{3.12}$$

Using these variables we define

$$s \equiv -\frac{\dot{H}}{H^2} = \frac{4\sqrt{3}\,\alpha\,\xi\,u\,x\,y + 3\,\mu\,\left(x^2-\gamma\right)\,y^2}{2\,\left(2\,\xi\,u^2+1\right)} + \frac{3\,\gamma}{2}. \tag{3.13}$$

Also, using (3.12) the evolution equations (3.10) and (3.11) can be rewritten as a dynamical system of ODE, namely

$$x' = \frac{\sqrt{3}}{2}\,\left(\lambda\,x^2\,y + \lambda\,\left(2-3\,x^2\right)\,y - 4\,\alpha\,\xi\,u\,\mu^{-3}\,y^{-1} - 2\sqrt{3}\,x\,\mu^{-2}\right), \tag{3.14}$$

$$y' = \left(-\frac{\sqrt{3}\,\lambda}{2}\,x\,y + s\right)\,y, \tag{3.15}$$

$$u' = \frac{\sqrt{3}\,\alpha\,x\,y}{2}, \tag{3.16}$$

$$\lambda' = -\sqrt{3}\,\lambda^2\,x\,y\,\left(\Gamma-1\right), \tag{3.17}$$

$$\alpha' = \sqrt{3}\,\frac{x\,y\,\alpha^2}{u}\,\left(\Pi - \frac{1}{2}\right), \tag{3.18}$$

with $\mu = 1/\sqrt{1-x^2}$ and prime denotes derivative with respect to the so-called e-folding time $N \equiv \ln a$. Also, we define

$$\Pi \equiv \frac{f\, f_{,\phi\phi}}{f_{,\phi}^2}, \qquad \Gamma \equiv \frac{V\, V_{,\phi\phi}}{V_{,\phi}^2}. \tag{3.19}$$

The fractional energy densities $\Omega \equiv (\kappa^2\,\rho)/(3\,H^2)$ for the scalar field and background matter are given by

$$\Omega_\phi = \mu\, y^2 - 2\,\xi\, u^2, \qquad \Omega_m = 1 - \Omega_\phi. \tag{3.20}$$

The state equation of the field $\omega_\phi = p_\phi/\rho_\phi$ reads

$$\omega_\phi = \frac{-\,\mu^{-1}\, y^2 + 2\,\xi\, u\, \left(\frac{2\sqrt{3}}{3}\,\alpha\, x\, y + u\, \left(1 - \frac{2}{3}\, s\right)\right)}{\mu\, y^2 - 2\,\xi\, u^2}. \tag{3.21}$$

On the other hand, the effective equation of state $\omega_{eff} = (p_\phi + p_m)\,/\,(\rho_\phi + \rho_m)$ is given by

$$\omega_{eff} = \left(x^2 - \gamma\right)\mu\, y^2 + \frac{4\sqrt{3}}{3}\,\alpha\,\xi\, u\, x\, y + 2\,\left(\gamma - \frac{2}{3}\, s\right)\,\xi\, u^2 + \gamma - 1, \tag{3.22}$$

and the accelerated expansion of the universe occurs for $\omega_{eff} < -1/3$.

Once the parameters Γ and Π are known, the dynamical system (3.14)-(3.18) becomes an autonomous system and the dynamics can be analyzed in the usual way. Since we consider constant λ, this is equivalent to consider $\Gamma = 1$. On the other hand, for $f(\phi) \propto \phi^2$ or equivalently $\Pi = 1/2$ then $\alpha \equiv f_{,\phi}/\sqrt{f} = const \neq 0$. Moreover, following Ref. [8], for a general coupling function $u \equiv \kappa\,\sqrt{f(\phi)}$, with inverse function $\phi = f^{-1}(u^2/\kappa^2)$, $\alpha(\phi)$ and $\Pi(\phi)$ can be expressed in terms of u (this approach is similar to that followed in the case of quintessence in GR with potential beyond exponential potential [42]). Therefore, two situations may arise; one where α is a constant and another where α depends on u. In both cases, we have a three-dimensional autonomous system (3.14)-(3.16), and the fixed points or critical points (x_c, y_c, u_c) can be find by imposing the conditions $x'_c = y'_c = u'_c = 0$. From the definition (3.12), x_c, y_c, u_c should be real, with $x_c^2 \leq 1$, $y_c \geq 0$, and $u_c \geq 0$.

As in chapter 2, to study the stability of the critical point, we substitute linear perturbations, $x \to x_c + \delta x$, $y \to y_c + \delta y$, and $u \to u_c + \delta u$ about the critical point (x_c, y_c, u_c) into the autonomous system (3.14)-(3.16) and linearize them. The eigenvalues of the perturbations matrix $\mathcal{M}$, namely, τ_1, τ_2 and τ_3, determine the conditions of stability of the critical points.

In what follows we are going to study the three-dimensional autonomous dynamical system (3.14)-(3.16), first for $\alpha = const \neq 0$ and then for dynamically changing $\alpha(u)$, such that $\alpha(u) \to \alpha(u_c) = 0$ when the system falls into the critical point (x_c, y_c, u_c).

3.4 Constant α

3.4.1 Critical points

In this section we consider a non-minimal coupling function $f(\phi) \propto \phi^2$ such that $\alpha = const \neq 0$. The critical points of the autonomous system (3.14)-(3.16) are presented in Table 3.1. In Table 3.2 we summarize the stability properties (to be studied below), and conditions for acceleration and existence for each point. In Table 3.1 the variables $v_\pm$ are defined by

$$v_\pm = \left(\alpha\,\xi \pm \sqrt{\xi\,\left(\alpha^2\,\xi - 2\,\lambda^2\right)}\right). \tag{3.23}$$

The critical point I.a is a fluid dominant solution ($\Omega_m = 1$) that exists for all values of λ, ξ and α. The critical points I.b and I.c are both scaling solutions with $u_c \geq 0$, and the requirement of the condition $0 < \Omega_\phi < 1$ implies $0 < \hat{\xi} < 1$. The accelerated expansion occurs for these three points if $\omega_{eff} = \gamma - 1 < -1/3$, that is, for $\gamma < 2/3$. Points I.d and I.e both correspond to dark-energy-dominated de Sitter solutions with $\Omega_\phi = 1$ and $\omega_\phi = \omega_{eff} = -1$. From (3.23), the fixed point I.d exists for:

$$\xi \geq 2\,\lambda^2/\alpha^2 > 0 \quad \text{and} \quad \lambda/\alpha > 0 \quad \text{or} \quad \xi < 0, \quad \alpha < 0 \quad \text{and} \quad \lambda > 0. \tag{3.24}$$

By the other hand, the point I.e exists for

$$\xi \geq 2\,\lambda^2/\alpha^2 > 0 \quad \text{and} \quad \lambda/\alpha > 0 \quad \text{or} \quad \xi < 0, \quad \alpha > 0 \quad \text{and} \quad \lambda < 0. \tag{3.25}$$

3.4.2 Stability

Substituting the linear perturbations, $x \to x_c + \delta x$, $y \to y_c + \delta y$, and $u \to u_c + \delta u$ into the autonomous system (3.14)-(3.16) and linearize them, the components of the matrix of perturbations $\mathcal{M}$ are given by

$$\mathcal{M}_{11} = \sqrt{3}\,\left(-2\,\lambda\,x_c\,y_c - \sqrt{3}\left(1 - 3\,x_c^2\right) + 6\,\alpha\,\xi\,u_c\,x_c\,\mu_c^{-1}\,y_c^{-1}\right), \tag{3.26}$$

$$\mathcal{M}_{12} = \sqrt{3}\,\mu_c^{-2}\,\left(\lambda + 2\,\alpha\,\xi\,u_c\,\mu_c^{-1}\,y_c^{-2}\right), \tag{3.27}$$

$$\mathcal{M}_{13} = -2\,\sqrt{3}\,\alpha\,\xi\,\mu_c^{-3}\,y_c^{-1}, \tag{3.28}$$

Table 3.1: Critical points for the autonomous system (3.14)-(3.16) for constant $\alpha \neq 0$. We define $\hat{\xi} \equiv 1 + 2\,\xi\,u_c^2$ and $u_c \geq 0$.

Name	x_c	y_c	u_c	Ω_ϕ	ω_ϕ	ω_{eff}
I.a	0	0	0	0	-1	$\gamma - 1$
I.b	1	0	u_c	$1 - \hat{\xi}$	$\gamma - 1$	$\gamma - 1$
I.c	-1	0	u_c	$1 - \hat{\xi}$	$\gamma - 1$	$\gamma - 1$
I.d	0	$\sqrt{\frac{\alpha\,v_-}{\lambda^2}}$	$\frac{v_-}{2\,\lambda\,\xi}$	1	-1	-1
I.e	0	$\sqrt{\frac{\alpha\,v_+}{\lambda^2}}$	$\frac{v_+}{2\,\lambda\,\xi}$	1	-1	-1

Table 3.2: Stability properties, and conditions for acceleration and existence of the fixed points in Table 3.1.

Name	Stability	Acceleration	Existence
I.a	Unstable	$\gamma < 2/3$	All values
I.b	Saddle	$\gamma < 2/3$	$0 < \hat{\xi} < 1$
I.c	Saddle	$\gamma < 2/3$	$0 < \hat{\xi} < 1$
I.d	S. node or S. spiral, or Saddle	All values	(3.24)
I.e	S. node or S. spiral, or Saddle	All values	(3.25)

$$\mathcal{M}_{21} = \frac{y_c^2\,\left(-3\,\mu_c^3\,x_c\,y_c\,\left(x_c^2 + \gamma - 2\right) + 4\,\sqrt{3}\,\alpha\,\xi\,u_c\right)}{2\,\left(2\,\xi\,u_c^2 + 1\right)} - \frac{\sqrt{3}\,\lambda\,y_c^2}{2}, \tag{3.29}$$

$$\mathcal{M}_{22} = \frac{y_c\left(9\,\mu_c\,y_c\,\left(x_c^2 - \gamma\right) + 8\,\sqrt{3}\,\alpha\,\xi\,x_c\,u_c\right)}{2\,\left(2\,\xi\,u_c^2 + 1\right)} - \sqrt{3}\,\lambda\,x_c\,y_c + \frac{3\,\gamma}{2}, \tag{3.30}$$

$$\mathcal{M}_{23} = \frac{2\,\sqrt{3}\,\xi\,y_c^2\,\left(-\sqrt{3}\,\mu_c\,u_c\,y_c\,\left(x_c^2 - \gamma\right) + 2\,\alpha\,x_c\right)}{\left(2\,\xi\,u_c^2 + 1\right)^2} - \frac{2\,\sqrt{3}\,\alpha\,\xi\,x_c\,y_c^2}{2\,\xi\,u_c^2 + 1}, \tag{3.31}$$

$$\mathcal{M}_{31} = \frac{\sqrt{3}\,\alpha\,y_c}{2}, \qquad \mathcal{M}_{32} = \frac{\sqrt{3}\,\alpha\,x_c}{2}, \qquad \mathcal{M}_{33} = 0. \tag{3.32}$$

In the above we define $\mu_c = 1/\sqrt{1 - x_c^2}$.

Point I.a: The component $\mathcal{M}_{13}$ is divergent, which means that this point is unstable.

Points I.b and I.c: For both critical points the eigenvalues of $\mathcal{M}$ are given by

$$\tau_1 = \frac{3\,\gamma}{2}, \quad \tau_2 = -3, \qquad \tau_3 = 0. \tag{3.33}$$

Therefore these points are unstable.

Point I.d: For this point the eigenvalues are given by

$$\tau_{1,2} = \frac{3\left(-1 \pm \sqrt{1 - \frac{4}{3}\,\alpha\,\sqrt{\xi\,(\alpha^2\,\xi - 2\,\lambda^2)}}\right)}{2}, \qquad \tau_3 = -3\,\gamma. \tag{3.34}$$

It is a saddle point if $\xi < 0$ and $\alpha < 0$ or $\xi > 2\,\lambda^2/\alpha^2$ and $\alpha < 0$. On the other hand, for

$$\frac{2\,\lambda^2}{\alpha^2} < \xi \leq \frac{\lambda^2}{\alpha^2}\left(1 + \sqrt{1 + \frac{9}{16\,\lambda^4}}\right), \tag{3.35}$$

and $\alpha > 0$ it is a stable node. Also, when $\xi > \frac{\lambda^2}{\alpha^2}\left(1 + \sqrt{1 + \frac{9}{16\,\lambda^4}}\right)$ then τ_1 and τ_2 are complex with real part negative and $\det(\mathcal{M}) = -9\,\alpha\,\gamma\,\sqrt{\xi\,(\alpha^2\,\xi - 2\,\lambda^2)} < 0$ for $\alpha > 0$. Therefore, in this case it is a stable spiral.

Point I.e: Finally, for the point I.e, the eigenvalues are

$$\tau_{1,2} = \frac{3\left(-1 \pm \sqrt{1 + \frac{4}{3}\,\alpha\,\sqrt{\xi\,(\alpha^2\,\xi - 2\,\lambda^2)}}\right)}{2}, \qquad \tau_3 = -3\,\gamma. \tag{3.36}$$

This fixed point is a saddle point for $\xi < 0$ and $\alpha > 0$ or $\xi > 2\,\lambda^2/\alpha^2$ and $\alpha > 0$. On the other hand, for ξ as in (3.35) and $\alpha < 0$, it is a stable node. When $\xi > \frac{\lambda^2}{\alpha^2}\left(1 + \sqrt{1 + \frac{9}{16\,\lambda^4}}\right)$ then τ_1 and τ_2 are complex with real part negative and $\det(\mathcal{M}) = 9\,\alpha\,\gamma\,\sqrt{\xi\,(\alpha^2\,\xi - 2\,\lambda^2)} < 0$ for $\alpha < 0$. In this case, point I.e is a stable spiral.

The fixed points I.a, I.d and I.e are the same points that were found for TDE in Ref. [6–8]. The scaling solutions I.b and I.c are new solutions that are not present in TDE. Such as in TDE, in TTDE the universe is attracted for the dark-energy-dominated de Sitter solution I.d or I.e. However, unlike the former scenario, in TTDE the universe may present a phase ϕMDE, that is, the scaling solution I.b or I.c, in which it has some portions of the energy density of ϕ in the matter dominated era. This type of phase ϕMDE is also common in coupled dark energy in GR (see Refs. [16, 18, 19]). But since the scaling solutions I.b and I.c both require $-1/2\,u_c^2 < \xi < 0$ when $u_c > 0$, then the fixed points I.d and I.e are not achieved because in this case these are saddle points. To solve this problem is necessary to consider a dynamically changing α.

3.5 Dynamically changing α

Following Ref. [8], now we let us consider a general function of non-minimal coupling $f(\phi)$ such that α can be expressed in terms of u and $\alpha(u) \to \alpha(u_c) = 0$ when $(x, y, u) \to (x_c, y_c, u_c)$ (we note that (x_c, y_c, u_c) is a fixed point of the system). The field ϕ rolls down toward $\pm\infty$ ($x > 0$ or $x < 0$) with $f(\phi) \to u_c^2/\kappa^2$ when $(x, y, u) \to (x_c, y_c, u_c)$ (for simplicity and since we seek new solutions then we set $x_c \neq 0$ and $y_c \neq 0$). The fixed points are presented in Table 3.3. Also, we summarize the properties of the fixed points in Table 3.4. In Table 3.3 the parameter y_c is defined by

$$y_c = \sqrt{\frac{\hat{\xi}\left(-\lambda^2\,\hat{\xi} + \sqrt{\lambda^4\,\hat{\xi}^2 + 36}\right)}{6}}. \tag{3.37}$$

3.5.1 Critical points

Points II.a and II.b are scaling solutions in which the energy density of the scalar field decreases proportionally to that of the perfect fluid ($\omega_\phi = \omega_m$). The existence of these solutions requires the condition $0 < \gamma < 1$ or equivalently $-1 < \omega_m < 0$ as can be seen in the expression of x_c, y_c and Ω_ϕ. Also, for point II.a is required $\lambda < 0$ and for point II.b is required $\lambda > 0$. For both points, if $0 < \gamma < 1$, the condition $0 < \Omega_\phi < 1$ is ensured if

$$\frac{3\,\gamma}{\lambda^2\,\sqrt{1-\gamma}} < \hat{\xi} < \frac{3\,\gamma}{\lambda^2\,\sqrt{1-\gamma}} + 1. \tag{3.38}$$

The condition for accelerated expansion corresponds to $\gamma < 2/3$.

Point II.c is a scalar-field dominant solution ($\Omega_\phi = 1$) that gives an accelerated expansion at late times for $\lambda^2\,y_c^2 < 2$, or equivalently, this condition translates into

$$0 < \hat{\xi} < \frac{2\,\sqrt{3}}{\lambda^2}. \tag{3.39}$$

This point exists for $\hat{\xi} > 0$ and all values of λ.

3.5.2 Stability

For dynamically changing $\alpha(u)$ such that $\alpha(u) \to \alpha(u_c) = 0$, the components of the matrix of perturbation $\mathcal{M}$ are written as

$$\mathcal{M}_{11} = \sqrt{3}\,\left(-2\,\lambda\,x_c\,y_c + \sqrt{3}\,\left(3\,x_c^2 - 1\right)\right), \tag{3.40}$$

Table 3.3: Critical points of the autonomous system (3.14)-(3.16) for dynamically changing $\alpha(u)$ such that $\alpha(u) \to \alpha(u_c) = 0$ and $u_c \geq 0$. We define $\hat{\xi} \equiv 2\,\xi\,u_c^2 + 1$.

Name	x_c	y_c	u_c	Ω_ϕ	ω_ϕ	ω_{eff}
II.a	$-\sqrt{\gamma}$	$-\frac{\sqrt{3}\sqrt{\gamma}}{\lambda}$	u_c	$\frac{3\,\gamma}{\lambda^2\,\sqrt{1-\gamma}} + 1 - \hat{\xi}$	$\gamma - 1$	$\gamma - 1$
II.b	$\sqrt{\gamma}$	$\frac{\sqrt{3}\sqrt{\gamma}}{\lambda}$	u_c	$\frac{3\,\gamma}{\lambda^2\,\sqrt{1-\gamma}} + 1 - \hat{\xi}$	$\gamma - 1$	$\gamma - 1$
II.c	$\frac{\lambda\,y_c}{\sqrt{3}}$	y_c	u_c	1	$\frac{\lambda^2\,y_c^2}{3} - 1$	$\frac{\lambda^2\,y_c^2}{3} - 1$

Table 3.4: Stability properties, and conditions for acceleration and existence of the fixed points in Table 3.3.

Name	Stability	Acceleration	Existence
II.a	Stable node or stable spiral	$\gamma < 2/3$	(3.38) and $\lambda < 0$
II.b	Stable node or stable spiral	$\gamma < 2/3$	(3.38) and $\lambda > 0$
II.c	Stable node	$\hat{\xi} < \frac{2\sqrt{3}}{\lambda^2}$	$\hat{\xi} > 0$

$$\mathcal{M}_{12} = \sqrt{3}\,\lambda\,\mu_c^{-2}, \tag{3.41}$$

$$\mathcal{M}_{13} = -2\,\sqrt{3}\,\xi\,\eta_c\,u_c\,\mu_c^{-3}\,y_c^{-1}, \tag{3.42}$$

$$\mathcal{M}_{21} = -\frac{3\,\mu_c^3\,x_c\,y_c^3\,\left(x_c^2 + \gamma - 2\right)}{2\,\left(2\,\xi\,u_c^2 + 1\right)} - \frac{\sqrt{3}\,\lambda\,y_c^2}{2}, \tag{3.43}$$

$$\mathcal{M}_{22} = \frac{9\,\mu_c\,\left(x_c^2 - \gamma\right)\,y_c^2}{2\,\left(2\,\xi\,u_c^2 + 1\right)} - \sqrt{3}\,\lambda\,x_c\,y_c + \frac{3\,\gamma}{2}, \tag{3.44}$$

$$\mathcal{M}_{23} = -\frac{6\,\xi\,\mu_c\,u_c\,y_c^3\,\left(x_c^2 - \gamma\right)}{\left(2\,\xi\,u_c^2 + 1\right)^2} + \frac{2\,\sqrt{3}\,\xi\,\eta_c\,x_c\,y_c^2\,u_c}{2\,\xi\,u_c^2 + 1}, \tag{3.45}$$

$$\mathcal{M}_{31} = 0, \quad \mathcal{M}_{32} = 0, \quad \mathcal{M}_{33} = \frac{\sqrt{3}\,\eta_c\,x_c\,y_c}{2}. \tag{3.46}$$

Here η_c is defined by $\eta_c \equiv \frac{d\alpha(u)}{du}|_{u=u_c}$.

Points II.a and II.b: The eigenvalues are

$$\tau_{1,2} = \frac{3\left(\pm\sqrt{(2-\gamma)^2 + \frac{16\,\gamma\,(1-\gamma)}{\hat{\xi}}\left(\frac{3\,\gamma}{\lambda^2\sqrt{1-\gamma}} - \hat{\xi}\right)} - (2-\gamma)\right)}{4}, \qquad \tau_3 = \frac{3\,\eta_c\,\gamma}{2\,\lambda}. \tag{3.47}$$

Both points are stable node or stable spiral provided that $\Omega_\phi < 1$ and $\eta_c > 0$ (point II.a) or $\eta_c < 0$ (point II.b). In any case, both scaling solutions are not realistic solutions in applying to dark energy because of the condition $\gamma < 1$ or equivalently $\omega_m < 0$. This problem can be solved by considering a explicit coupling to dark matter. In this case, as was shown in Ref. [8] for interacting teleparallel dark energy, scaling attractors with accelerated expansion can be solutions of the system.

Point II.c: The eigenvalues are

$$\tau_1 = -3 + \frac{\lambda^2\,y_c^2}{2}, \qquad \tau_2 = -3\,\gamma + \lambda^2\,y_c^2, \qquad \tau_3 = \frac{\eta_c\,\lambda\,y_c^2}{2}, \tag{3.48}$$

with y_c given in Eq. (3.37). The eigenvalue τ_1 is always negative since $x_c^2 \leq 1$. In regard to τ_2, it is always negative provided that $\gamma \geq 1$. On the other hand, the eigenvalue τ_3 is negative if $\eta_c\,\lambda < 0$. So, for $\hat{\xi} > 0$ with $\gamma \geq 1$, $\lambda > 0$ and $\eta_c < 0$ (or $\lambda < 0$ and $\eta_c > 0$), then point II.c is a stable node.

Therefore, point II.c is a late-time attractor and a viable cosmological solution (scalar-field dominant solution) with accelerated expansion. Unlike the late-time attractors I.d and I.e for constant α, in this case the universe can enters in the scaling solutions I.b or I.c (phase ϕMDE) with constant α and eventually approaches the late-time attractor II.c for dynamically changing α, since in this case we can have $0 < \hat{\xi} < 1$ depending on the value of λ in (3.39). In Fig. 3.1 we show the case when the system approaches the fixed point II.c with $\gamma = 1$ (non-relativistic dark matter), $\lambda = 0.6$, $\xi = -3 \times 10^{-3}$, and following Ref. [8], by way of example we consider the function $\alpha(u) = u_c - u$ with $u_c = 1$ and $\eta_c = -1$. In this case Ω_ϕ grows to 0.7 at the present epoch $N' \approx 4$ and the system asymptotically evolves toward the values $\Omega_\phi = 1$, $\Omega_m = 0$ and $\omega_\phi = \omega_{eff} = -0.89 < -1/3$. Also, the universe undergoes a phase ϕMDE (scaling solution I.c) with $\Omega_\phi = 1 - \hat{\xi} \approx 0.04$ and $\omega_\phi = \omega_{eff} = 0$, before entering the late time attractor II.c.

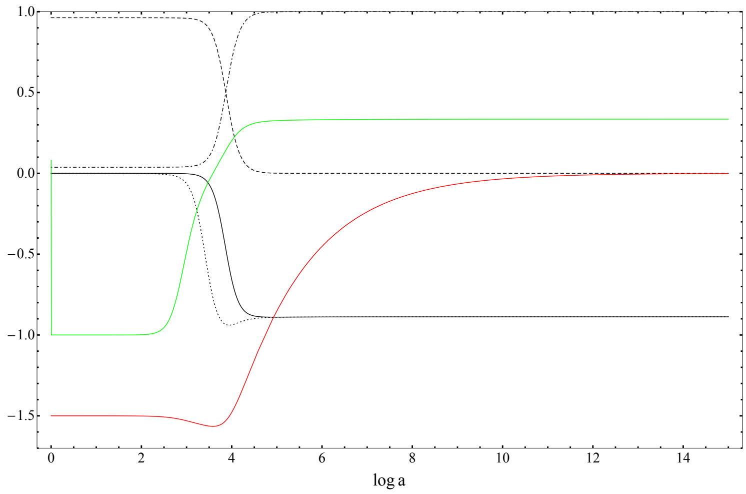

Figure 3.1: Evolution of Ω_m (dashed), Ω_ϕ (dotdashed), ω_ϕ (dotted), ω_{eff} (solid), x (green line, ending at $x_c \approx 0.34$) and $\alpha(u)$ (red line, starting at $\alpha = -1.5$) with $\gamma = 1$, $\lambda = 0.6$ and $\xi = -3 \times 10^{-3}$. We choose initial conditions $x_i = 0.1$, $y_i = 1.45 \times 10^{-6}$ and $u_i = 2.5$ with $\alpha(u) = u_c - u$. The universe exits from scaling solution I.c with constant $\alpha = -1.5$, $\Omega_\phi \approx 0.04$, $\omega_\phi = \omega_{eff} = 0$ and approaches the late-time attractor II.c for dynamically changing $\alpha(u)$ with $\Omega_\phi \approx 0.7$, $\Omega_m \approx 0.3$, $\omega_\phi \approx -0.9$ and accelerated expansion at the present epoch $N = \log a \approx 4$. The system asymptotically evolves toward the scalar-field dominant solution II.c with values $\Omega_\phi = 1$, $\Omega_m = 0$ and $\omega_\phi = \omega_{eff} = -0.89$.

Chapter 4

Growth of Matter Density Perturbations in TTDE

In general, the non-minimal coupling between the additional scalar field and gravity spoils the local Lorentz symmetry. To see this, we rewrite the field equation (3.4) in the form

$$\frac{2}{\kappa^2}\, G_a^{\ \rho} = \widetilde{\Theta}_a^{\ \rho} + \Theta_a^{\ \rho}, \tag{4.1}$$

where

$$\widetilde{\Theta}_a^{\ \rho} = -4\,\xi\, f(\phi)\, G_a^{\ \rho} - 4\,\xi\, h_a^{\ \mu}\, S_\mu^{\ \rho\sigma}\, f_{,\phi}\, \partial_\sigma \phi + \mu^{-1}\, V(\phi)\, \delta^\rho_{\ \mu} + \mu\, h_a^{\ \mu}\, \partial_\mu \phi\, \partial^\rho \phi, \tag{4.2}$$

is the energy-momentum tensor of the scalar field. The Local Lorentz invariance of the action of the scalar field $\delta\widetilde{S} = \int d^4x\, h\, \widetilde{\Theta}_\mu^{\ \nu}\, \delta\epsilon^\mu_{\ \nu} = 0$ with $\delta\epsilon^\mu_{\ \nu}$ an arbitrary antisymmetric (Lorentz) tensor requires that $\widetilde{\Theta}_\mu^{\ \nu} = h^a_{\ \mu}\, \widetilde{\Theta}_a^{\ \nu}$ be symmetric. But from equation (4.2) it is seen that

$$\widetilde{\Theta}_{[\mu\gamma]} = -4\,\xi\, f_{,\phi}\, \partial_\nu \phi\, S_{[\mu\gamma]}^{\ \ \ \ \nu}, \tag{4.3}$$

where $-4\, S_{[\mu\gamma]}^{\ \ \ \ \nu} = T^\nu_{\ \mu\gamma} + \delta^\nu_{\ \mu}\, T^\beta_{\ \gamma\beta} - \delta^\nu_{\ \gamma}\, T^\beta_{\ \mu\beta}$. However, this term is not relevant for cosmological dynamics, since in cosmological scales for the Friedmann solution the right side of (4.3) is zero and the local Lorentz invariance is kept. So, when we study the linear cosmological perturbations around the background solution, this term can still be neglected. Given the nature of TG as a field theory of gravitation, the appearance of this term is due to that the scalar field is coupled to the spin of the gravitational field. The spin of gravitational field is defined as [59] (see also Appendix D)

$$\sigma^{\mu}{}_{\alpha\beta} = -\frac{\partial \mathcal{L}_G}{\partial\partial_\mu h^a{}_\sigma}\,\frac{\delta h^a{}_\sigma}{\delta\epsilon^{\alpha\beta}}. \tag{4.4}$$

Since under a infitesimal Lorentz transformation $\Lambda_a{}^b = \delta_a{}^b + \delta\epsilon_a{}^b$ the tetrad field is transformed according to

$$\delta h^a{}_\sigma = h^a{}_\alpha\,\delta\epsilon^\alpha{}_\sigma, \tag{4.5}$$

we have that

$$\frac{\delta h^a{}_\sigma}{\delta\epsilon^{\alpha\beta}} = \frac{1}{2}\left(g_{\beta\sigma}\,h^a{}_\alpha - g_{\alpha\sigma}\,h^a{}_\beta\right). \tag{4.6}$$

and remembering that the gravitational lagrangian is $\mathcal{L}_G = \frac{h\,T}{2\,\kappa^2}$ thus

$$\frac{\partial \mathcal{L}_G}{\partial\partial_\mu h^a{}_\sigma} = -\frac{2\,h}{\kappa^2}\,S_a{}^{\sigma\mu}. \tag{4.7}$$

Using (4.6) and (4.7) in (4.4) we obtain

$$\sigma^{\mu}{}_{\alpha\beta} = \frac{h}{\kappa^2}\,S_{[\alpha\beta]}{}^{\mu} = -\frac{h}{4\,\kappa^2}\left(T^{\mu}{}_{\alpha\beta} + \delta^\mu_\alpha\,T^\tau{}_{\beta\tau} - \delta^\mu_\beta\,T^\tau{}_{\alpha\tau}\right). \tag{4.8}$$

From which we see that the right side of (4.3) is proportional to the coupling between the scalar field (through its four-derivative) and the spin tensor of the gravitational field. By using (4.8), the equation (4.3) can be rewritten as

$$\widetilde{\Theta}_{[\mu\nu]} = -\frac{4\,\kappa^2}{h}\,\xi\,f_{,\phi}\,\partial_\rho\phi\,\sigma^\rho{}_{\mu\nu}. \tag{4.9}$$

In the FLRW background (1.20) the only non-zero component is $\widetilde{\Theta}_{[0i]} \propto \partial_j\phi\,\sigma^j{}_{0i} \approx 0$ with $\sigma^i{}_{j0} = -\sigma^i{}_{0j} = \frac{a^3}{\kappa^2}\,H\,\delta^i{}_j$ and $\partial_i\phi \approx 0$ for the homogeneous scalar field.

Thus, since the Local Lorentz invariance is kept on the scale of importance to our study, we can choose to work in the preferred class of frames of TG and consider the perturbed tetrad field given by [51]

$$h^0{}_\mu = \delta^0{}_\mu\,(1+\psi) + a\,\delta^i{}_\mu\partial_i F + a\,\delta^i{}_\mu\,G_i, \quad h^i{}_\mu = a\,\delta^i{}_\mu\,(1-\varphi) + a\,\delta^j{}_\mu\left(\partial_j\partial^i B + \partial^i C_j + h^i{}_j\right), \tag{4.10}$$

where C_i and G_i are transverse vectors with $\partial^i C_i = \partial^i G_i = 0$. So, we can obtain the corresponding metric as:

$$g_{00} = 1 + 2\,\psi,\;\; g_{i0} = a\,\left(\partial_i F + G_i\right),\;\; g_{ij} = -a^2\,\left[(1-2\,\varphi)\,\delta_{ij} + h_{ij} + \partial_i\partial_j B + \partial_j C_i + \partial_i C_j\right], \tag{4.11}$$

which is the familiar form for the pertubed metric.

As mentioned above, once we set a class of frames, we are eliminating 6 degrees of freedom of the 16 degrees of freedom of the tetrad field and thus the number of degrees of freedom is equal to the same number of degrees of freedom of the metric in GR [1]. So, the 10 remaining components of the tetrad field are illustrated by the 4 scalar modes: ψ, φ, B, F, and 2 transverse vector modes: C_i and G_i, and a tranverse traceless tensor mode h_{ij}, respectively.

In the following, we focus on the cosmological implication of the scalar perturbations. We eliminate the variables F, B and C_i through the general coordinate transormations $x^\mu \to x^\mu + \epsilon^\mu(x)$ [51]. So, the perturbed tetrad is given by

$$h^0{}_\mu = \delta^0{}_\mu \, (1+\psi)\,, \qquad h^i{}_\mu = a\, \delta^i{}_\mu \, (1-\varphi)\,, \tag{4.12}$$

leading to the perturbed metric in the longitudinal gauge form

$$ds^2 = (1+2\,\psi)\; dt^2 - a^2\; (1-2\,\varphi)\; \delta_{ij}\, dx^i\, dx^j. \tag{4.13}$$

Using the perturbed tetrad (4.12), we obtain the torsion tensor and scalar to be

$$T^0_{\ 0i} = -\partial_i \psi, \qquad T^i_{\ 0j} = (H-\dot{\varphi})\; \delta^i{}_j, \quad T^i_{\ jk} = \delta^i{}_j\, \partial_k \varphi - \delta^i{}_k\, \partial_j \varphi. \tag{4.14}$$

and

$$T = -6\, H^2 + 12\, H\; (\dot{\varphi} + H\, \psi)\,. \tag{4.15}$$

On the other hand, we can also descompose the scalar field to homogeneous and perturbed parts: $\phi \to \phi(t) + \delta\phi(t,x)$. The scalar perturbations of the matter source $\Theta_\nu{}^\mu = h^a{}_\nu\, \Theta_a{}^\mu$ are denoted by the perfect fluid perturbations: $\Theta_0{}^0 = \rho_m + \delta\rho_m$, $\Theta_0{}^i = \rho_m\; (1+\omega_m)\; \partial^i \delta v$, $\Theta_i{}^0 = -a^2\, \rho_m\; (1+\omega_m)\; \partial_i \delta v$ and $\Theta_j{}^i = -\,(p_m + \delta p_m)\; \delta^i{}_j$, where δv is the velocity potential of the fluid ($\partial_i \delta v$ is the matter peculiar velocity) [17]. So, by substituting the above expressions to the field equation (4.1), we derive the perturbed gravitational equations:

$$\begin{aligned}
&\frac{2}{\kappa^2}\left(\frac{\partial^2\varphi}{a^2} - 3\,H\;(\dot{\varphi}+H\,\psi)\right) = -4\,\xi\, f(\phi)\,\left(\frac{\partial^2\varphi}{a^2} - 3\,H\;(\dot{\varphi}+H\,\psi)\right)\\
&- V(\phi)\,\left(1-\frac{2\,\dot{\phi}^2}{V}\right)\,\delta\mu - 6\,\xi\, H^2\, f_{,\phi}\,\delta\phi + \mu^{-1}\, V_{,\phi}\,\delta\phi - 2\,\mu\,\dot{\phi}^2\,\psi + 2\,\mu\,\dot{\phi}\,\dot{\delta\phi} + \delta\rho_m,
\end{aligned} \tag{4.16}$$

$$\frac{2}{\kappa^2}\,\partial^i(\dot\varphi+H\,\psi)=-4\,\xi\,f(\phi)\,\partial^i(\dot\varphi+H\,\psi)+2\,\xi\,H\,f_{,\phi}\,\partial^i\delta\phi-2\,\xi\,f_{,\phi}\,\dot\phi\,\partial^i\varphi+ \\ \mu\,\dot\phi\,\partial^i\delta\phi-a^2\,\rho_m\,(1+\omega_m)\,\partial^i\delta v, \quad (4.17)$$

$$\frac{2}{\kappa^2}\left(3\,H\,(\dot\varphi+H\,\psi)+H\,\dot\psi+2\,\dot H\,\psi+\ddot\varphi-\frac{\partial^2}{3\,a^2}(\varphi-\psi)\right)=\delta p_m+\mu^{-2}\,\delta\mu\,V(\phi) \\ -4\,\xi\,f(\phi)\left(3\,H\,(\dot\varphi+H\,\psi)+H\,\dot\psi+2\,\dot H\,\psi+\ddot\varphi-\frac{\partial^2}{3\,a^2}(\varphi-\psi)\right)-\mu^{-1}\,V_{,\phi}\,\delta\phi \\ +4\,\xi\,\left(\frac{3}{2}\,H^2+\dot H\right)\,f_{,\phi}\,\delta\phi+4\,\xi\,H\,\dot\phi\,f_{,\phi\phi}\,\delta\phi+4\,\xi\,H\,f_{,\phi}\,\dot{\delta\phi}-8\,\xi\,H\,f_{,\phi}\,\dot\phi\,\psi \\ -4\,\xi\,f_{,\phi}\,\dot\phi\,\dot\varphi, \quad (4.18)$$

$$\varphi=\psi. \quad (4.19)$$

In the last equation, it is seen that the total perturbed fluid remains a perfect fluid ($\delta\Theta_{T\;j}^{\;i}=0$ for $i\neq j$). In the above equations we have introduced

$$\delta\mu=\frac{\mu^3}{V}\,\left(-\psi\,\dot\phi^2+\dot{\delta\phi}\,\dot\phi-\frac{\dot\phi^2}{2\,V}\,V_{,\phi}\,\delta\phi\right). \quad (4.20)$$

Also, from equation (4.16) we find for the pertubed energy density

$$\delta\rho_\phi=-4\,\xi\,f(\phi)\,\left(\frac{\partial^2\varphi}{a^2}-3\,H\,(\dot\varphi+H\,\psi)\right)-V(\phi)\,\left(1-\frac{2\,\dot\phi^2}{V}\right)\,\delta\mu-6\,\xi\,H^2\,f_{,\phi}\,\delta\phi \\ +\mu^{-1}\,V_{,\phi}\,\delta\phi-2\,\mu\,\dot\phi^2\,\psi+2\,\mu\,\dot\phi\,\dot{\delta\phi}, \quad (4.21)$$

and from equation (4.18), the perturbed pressure density

$$\delta p_\phi=\mu^{-2}\,\delta\mu\,V(\phi)-4\,\xi\,f(\phi)\,\left(3\,H\,(\dot\varphi+H\,\psi)+H\,\dot\psi+2\,\dot H\,\psi+\ddot\varphi-\frac{\partial^2}{3\,a^2}(\varphi-\psi)\right) \\ -\mu^{-1}\,V_{,\phi}\,\delta\phi+4\,\xi\,\left(\frac{3}{2}\,H^2+\dot H\right)\,f_{,\phi}\,\delta\phi+4\,\xi\,H\,\dot\phi\,f_{,\phi\phi}\,\delta\phi+4\,\xi\,H\,f_{,\phi}\,\dot{\delta\phi} \\ -8\,\xi\,H\,f_{,\phi}\,\dot\phi\,\psi-4\,\xi\,f_{,\phi}\,\dot\phi\,\dot\varphi. \quad (4.22)$$

Similarly, by substituting the perturbed quantities in the scalar motion equation (3.8), we find the perturbed equation of motion for the scalar field

$$b_0\,\ddot{\delta\phi}+b_1\,\dot{\delta\phi}+b_2\,\delta\phi+b_3=0, \quad (4.23)$$

with

$$b_0 = \mu^2, \tag{4.24}$$

$$b_1 = 3\,H - \frac{6\,\xi\,H^2\,\mu\,f_{,\phi}\,\dot\phi}{V} - \frac{\mu^4\,V_{,\phi}\,\dot\phi}{V} + \frac{2\,\mu^4\,\dot\phi\,\ddot\phi}{V}, \tag{4.25}$$

$$b_2 = -\frac{\partial^2}{a^2} + \frac{3\,\xi\,H^2\,\mu\,f_{,\phi}\,V_{,\phi}\,\dot\phi^2}{V^2} + \frac{\mu^4\,V_{,\phi}^2\,\dot\phi^2}{2\,V^2} - \frac{\mu^4\,V_{,\phi}\,\dot\phi^2\,\ddot\phi}{V^2} + \frac{6\,\xi\,H^2\,f_{,\phi\phi}}{\mu} + \mu^2\left(1 - \frac{3\,\dot\phi^2}{2\,V}\right)V_{,\phi\phi}, \tag{4.26}$$

$$b_3 = -\psi\left(-\frac{\mu^4\,V_{,\phi}\,\dot\phi^2}{V} + 2\left(3\,H\,\dot\phi + \mu^4\,\ddot\phi\right)\right) + \dot\phi\left(-3\,\dot\varphi + \mu^2\left(1 - \frac{2\,\dot\phi^2}{V}\right)\dot\psi\right) + 6\,\xi\,H\,f_{,\phi}\,\mu^{-1}\left(\frac{H\,\mu^2\,\psi\,\dot\phi^2}{V} - 2\,(H\,\psi + \dot\varphi)\right). \tag{4.27}$$

4.1 Numerical results

Since that $\psi = \varphi$, we may use (4.18) and (4.23) to solve the evolutions for φ and $\delta\phi$, and thus obtain the result of the density perturbation δ_m through (4.16). As usual we will work in the Fourier space by expanding all perturbed quantities in Fourier modes, i.e. $\Psi(\vec{r}, t) \to e^{i\,\vec{k}\cdot\vec{r}}\,\Psi(t)$ with $k = ||\vec{k}||$.

During the matter dominated epoch where both ϕ and $\delta\phi$ are negligible, the equations (4.16) to (4.18) can be reduced to the same forms as those in GR. Of this form, since the gravitational potential does not vary in time during the matter-dominated era, we have as initial conditions $\varphi = \psi = const$. Moreover, from (4.16) we have $\delta_m(z \gg 1) = -2\,\psi$.

We solve the evolutions of the perturbed variables, along with the background variables, numerically as seen in Fig. 4.2 and Fig. 4.3. In Fig. 4.1 we shown the behavior of the background quantities. It is considered the system of background equations (3.14)-(3.16) with $\gamma = 1$, $\xi = -3 \times 10^{-3}$, $\lambda = 0.6$ and $\alpha(u) = u_c - u$ (with $u_c = 1$), for the initial values $u_i = 2.5$, $x_i = 10^{-8}$ and $y_i = 1.5 \times 10^{-6}$, where it is require the current matter density $\Omega_{m0} = 0.3$ and $\Omega_{\phi 0} = 0.7$.

In Fig. 4.2 we can see as the scalar field affects the gravitational potential. Once dark energy become important in the cosmic evolution, φ is no longer more constant.

It is seen that the effect of dark energy on the potential is dependent on the scale. Also, the perturbation of the field which is also dependent on the scale, becomes important only for large scale modes. For small scale modes $\delta\phi$ comes into damped oscillations and can be approximated as homogeneous. In Fig. 4.3 it is shown as the growing mode $\delta_m \propto a$, for small scale modes (sub-horizon scales), it is affects by the presence of dark energy. The growing of the matter density perturbation is reduced by the presence of dark energy.

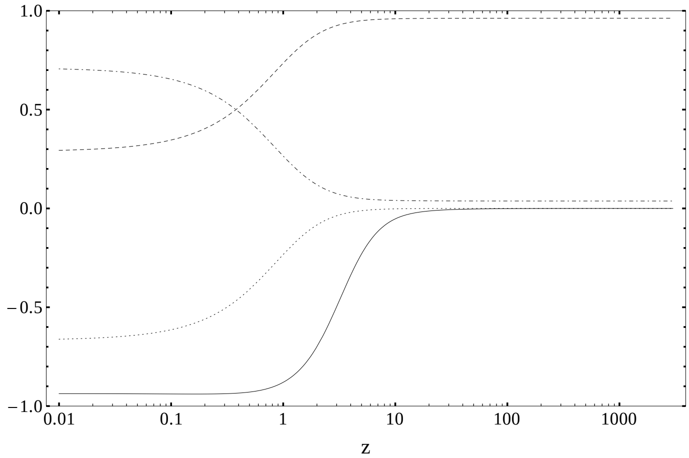

Figure 4.1: Evolution of Ω_m (dashed), Ω_ϕ (dotdashed), ω_ϕ (solid) and ω_{eff} (dotted) with $\gamma = 1$, $\lambda = 0.6$ and $\xi = -3\times10^{-3}$. We choose initial conditions $x_i = 10^{-8}$, $y_i = 1.5\times10^{-6}$ and $u_i = 2.5$ and by way of example we consider the function $\alpha(u) = u_c - u$ with $u_c = 1$. The universe exits from scaling solution I.c with constant $\alpha = -1.5$, $\Omega_\phi \approx 0.04$, $\omega_\phi = \omega_{eff} = 0$ and approaches the late-time attractor II.c for dynamically changing $\alpha(u)$ with $\Omega_\phi \approx 0.7$, $\Omega_m \approx 0.3$ and accelerated expansion at the present epoch $z = 0$.

4.2 Matter growth

Since, in the purely spacetime form, the conservation law of the source energy momentum tensor in TG coincides with the corresponding conservation law of GR [1] (see appendix E), thus the density perturbation of the dust-like source $\delta_m \equiv \delta\rho_m/\rho_m$ still satisfies the perturbation equation [51]

$$\ddot{\delta}_m + 2\,H\,\dot{\delta}_m = \frac{\partial^2}{a^2}\,\psi + 3\,\ddot{\varphi} + 6\,H\,\dot{\varphi}, \tag{4.28}$$

where we have used $p_m = \delta p_m = 0$.

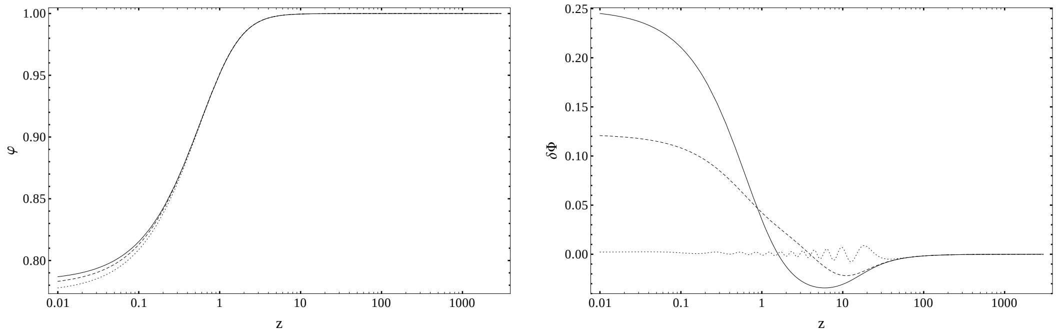

Figure 4.2: Evolution of gravitational potential φ and perturbation of the field $\delta\Phi \equiv \kappa\,\delta\phi$ with respect to the redshift z for $\xi = -3\times 10^{-3}$, $\lambda = 0.6$ and $\alpha(u) = u_c - u$ (with u_c normalized to 1). It is shown the behavior for various scales $k = 3\times 10^{-4}$ (solid), 1×10^{-3} (dashed) and $1\times 10^{-2} hMpc^{-1}$ (dotted) and the initial value of the potential φ_i is also normalized to unity while that the initial value of the perturbation of field is $\delta\phi_i = 0$ ($\delta\dot{\phi}_i = 0$) , both for $z \gg 1$.

By introducing the e-folding time $N \equiv \ln a$, the equation (4.28) can be rewritten as

$$\delta_m'' + \frac{1}{2}\,(1 - 3\,\omega_{eff})\;\delta_m' = -\frac{\psi}{\hat{\lambda}^2} + 3\,\varphi'' + \frac{3}{2}\,(1-3\,\omega_{eff})\;\varphi'. \tag{4.29}$$

where prime denote derivative with respect to N. In what follows we study the cosmological perturbations on sub-horizon scales (small scales) $\hat{\lambda} \equiv a\,H/k \ll 1$.

In this limit the last two terms in (4.29) can be neglect, so we have

$$\delta_m'' + \frac{1}{2}\,(1 - 3\,\omega_{eff})\;\delta_m' = -\frac{\psi}{\hat{\lambda}^2}. \tag{4.30}$$

In the sub-horizon limit the equation (4.16) can be rewritten as

$$-\frac{\varphi}{\hat{\lambda}^2} = \frac{3}{2}\,(\Omega_m\,\delta_m + \Omega_\phi\,\delta_\phi) = \frac{3}{2}\,\Omega_m\,\delta_m, \tag{4.31}$$

where we assumes that dark energy does not cluster such that $\delta_\phi \equiv \delta\rho_\phi/\rho_\phi = 0$. In fact, in the matter dominated era we have that $\Omega_\phi\,\delta_\phi \ll \Omega_m\,\delta_m$ and thus it can be neglected in (4.31). During scalar dominated era, as can be seen in Fig. 4.3, dark energy does not present cluster. So, by introducing the growth rate $\varrho \equiv \delta_m'/\delta_m$, and since $\psi = \varphi$, the equation (4.30) can be rewritten as

$$\varrho' + \frac{1}{2}\,(1-3\,\omega_{eff})\;\varrho + \varrho^2 - \frac{3}{2}\,\Omega_m = 0. \tag{4.32}$$

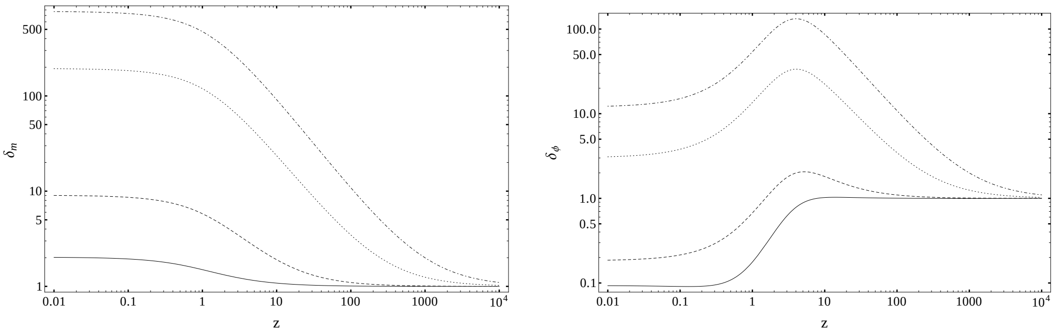

Figure 4.3: Evolution of the matter density perturbation δ_m with respect to the redshift z for various scales $k = 3\times 10^{-4}$ (solid), 1×10^{-3} (dashed), 5×10^{-3} (dotted) and $1\times 10^{-2} hMpc^{-1}$ (dotdashed), with the initial value normalized to unity for $z \gg 1$. It is considered $\xi = -3\times 10^{-3}$, $\lambda = 0.6$ and $\alpha(u) = u_c - u$ with $u_c = 1$.

The growth rate can be found numerically by integrating (4.32). However, a simple analytical solution exists if ω_{eff} and Ω_m are constants:

$$\varrho = -\frac{1}{4}\,(1-3\,\omega_{eff}) \pm \frac{1}{4}\sqrt{\left((1-3\,\omega_{eff})^2 + 24\,\Omega_m\right)}. \tag{4.33}$$

This particular case occurs on fixed points or critical points of the system. For the fixed points I.b and I.c, the solution "ϕMDE" in chapter 3, we have $\omega_{eff} = \gamma - 1 = 0$ and $\Omega_m = \hat{\xi}$ and (4.33) gives

$$\varrho_- = -\frac{1}{4}\left(1+\sqrt{1+24\,\hat{\xi}}\right), \qquad \varrho_+ = \frac{1}{4}\left(-1+\sqrt{1+24\,\hat{\xi}}\right), \tag{4.34}$$

for the decaying and growing modes respectively. In terms of the scale factor, we find for the growing solution

$$\delta_m(a) = C\,a^{\frac{1}{4}\left(-1+\sqrt{1+24\,\hat{\xi}}\right)} \approx C\,a^{1-\frac{3}{5}\,\Omega_\phi}, \tag{4.35}$$

with $\Omega_\phi = 1 - \hat{\xi} \ll 1$. In this case the presence of dark energy in a era "ϕMDE' reduces the standard growth rate for matter density perturbations ($\delta_m \propto a$ in sub-horizon scale) by a factor $a^{-\frac{3}{5}\,\Omega_\phi}$. On the other hand, for the scalar-field dominated solution II.c we have that $\Omega_\phi = 1$ and $\omega_{eff} = y_c^2\,\lambda^2/3 - 1$ and (4.33) gives

$$\varrho_- = -2 + \frac{y_c^2\,\lambda^2}{2} < 0 \qquad \varrho_+ = 0, \tag{4.36}$$

for $\lambda^2\, y_c^2 < 2$ (condition for accelerated expansion) and y_c as in (3.37). As is usual, the matter density perturbations are ”frozen“ ($\varrho_+ = 0$) during the accelerated expansion, which can also be checked in the Fig. 4.3.

Chapter 5

A Novel Teleparallel Dark Energy

The action in teleparallel dark energy models (quintessence, tachyonic) is given by

$$S=\int d^4x\, h\,\left[\frac{T}{2\,\kappa^2}+\mathcal{L}_\phi+\xi\, f(\phi)\, T\right]+S_m, \tag{5.1}$$

where $\mathcal{L}_\phi$ is the lagrangian of the scalar field, ξ is a dimensionless constant and $f(\phi)$ (with units of $mass^2$ and positive) is an arbitrary function of ϕ responsible for non-minimal coupling between the scalar field and gravity. This non-minimal coupling in the action (5.1) was motivated by an analogy with the non-minimally coupled scalar field in GR [5]. However, although equivalent with GR at the level of equations, TG is conceptually different. Furthermore, as it is also known [1], the scalar field as a particle of spin zero feels the torsion through its four derivative (which is a vector field). So, it may also be natural to consider a non-minimal coupling through the four derivative of the scalar field and the "vector torsion" [60].

5.1 The model

The torsion tensor can be decomposed into three components, irreducible under the global Lorentz group: there will be a vector

$$\mathcal{V}_\mu=T^{\nu}{}_{\nu\mu}, \tag{5.2}$$

an axial part

$$\mathcal{A}^\mu=\frac{1}{6}\,\epsilon^{\mu\nu\rho\sigma}\,T_{\nu\rho\sigma}, \tag{5.3}$$

and a purely tensor part

$$\mathcal{T}_{\lambda\mu\nu} = \frac{1}{2}\,(T_{\lambda\mu\nu} + T_{\mu\lambda\nu}) + \frac{1}{6}\,(g_{\nu\lambda}\,\mathcal{V}_{\mu} + g_{\nu\mu}\,\mathcal{V}_{\lambda}) - \frac{1}{3}\,g_{\lambda\mu}\,\mathcal{V}_{\nu}, \tag{5.4}$$

that is, a tensor with vanishing vector and axial parts. These components are usually called "vector torsion", "axial torsion" and "pure tensor torsion" [1]. When considering a scalar field, since it couples to curvature, it must also couple to torsion because the gravitational interaction can be described alternatively in terms of curvature or torsion. Also, since the scalar field interacts with torsion through its four derivative (vector field), when a non-minimal coupling to gravitation is allowed the four divergence of the scalar field can be coupled with the vector torsion. Thus, we consider the following action for the nonminimally coupled quintessence field

$$S = \int d^4x\, h\,\left[\frac{T}{2\,\kappa^2} + \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi) + \eta\,f(\phi)\,\partial_\mu\phi\,\mathcal{V}^\mu\right] + S_m(\psi_m, h^a{}_\rho), \tag{5.5}$$

where $h \equiv \det(h^a{}_\mu) = \sqrt{-g}$ ($h^a{}_\mu$ are the orthonormal components of the tetrad), $\mathcal{L}_G = \frac{h\,T}{2\,\kappa^2}$ is the lagrangian of teleparallelism and $S_m(\psi_m, h^a{}_\rho)$ is the matter action (see Refs. [1, 2, 8]). Moreover, the parameter η is a dimensionless constant and $f(\phi)$ is a arbitrary function of the scalar field with units of $mass$. It is interesting to note that under a conformal rescaling of the metric [53], the two models, TDE scenario [5] and the model in action (5.5), "new teleparallel dark energy" (NTDE), are mathematically related by defining a transformed scalar field, potential and adding an explicit coupling between the scalar field and matter (see Appendix F). However, although mathematically related by a conformal transformation, the two models are physically different (see for example Refs. [18, 19, 53, 54]). Here we do not consider an explicit coupling with matter, but this could be considered in future studies. On the other hand, it may be important to mention that the non-minimal coupling term in action (5.5) has appeared before in other contexts; metric-scalar gravity with torsion [55], conformally invariant teleparallel gravity [56] and $F(T)$ theories [57, 58], when studying conformal transformations and conformal symmetry. In what follows we calculate from the action (5.5) the total energy momentum tensor and the motion equation for the dark sector. The energy momentum tensor associated with the scalar field is calculated as

$$\begin{aligned}\Theta_a{}^\rho = -\frac{1}{h}\frac{\delta S_\phi}{\delta h^a{}_\rho} = -h_a{}^\rho\,\left(\frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi - V(\phi)\right) + \partial_a\phi\,\partial^\rho\phi + \\ \eta\,\left[f(\phi)\,\left(\mathcal{V}^\rho\,\partial_a\phi + \nabla_a\partial^\rho\phi - h_a{}^\rho\,\nabla_\mu\partial^\mu\phi\right) + f_{,\phi}\,\left(\partial_a\phi\,\partial^\rho\phi - h_a{}^\rho\,\partial_\mu\phi\,\partial^\mu\phi\right)\right],\end{aligned} \tag{5.6}$$

where ∇^μ is the covariant derivative in the teleparallel connection [1, 2]. The symmetric part is

$$\Theta_{(\mu\nu)} = -g_{\mu\nu}\left(\frac{1}{2}\,\partial_\epsilon\phi\,\partial^\epsilon\phi - V(\phi)\right) + \partial_\mu\phi\,\partial_\nu\phi +$$
$$\eta\,[f(\phi)\,\left(\mathcal{V}_{(\mu}\,\partial_{\nu)}\phi + \nabla_{(\mu}\partial_{\nu)}\phi - g_{\mu\nu}\,\nabla_\epsilon\partial^\epsilon\phi\right) + f_{,\phi}\,\left(\partial_\mu\phi\,\partial_\nu\phi - g_{\mu\nu}\,\partial_\epsilon\phi\,\partial^\epsilon\phi\right)], \quad (5.7)$$

while the anti-symmetric part is given by

$$\Theta_{[\mu\nu]} = \frac{2\,\kappa^2}{h}\,\eta\,f(\phi)\,\partial_\epsilon\phi\,\sigma^\epsilon{}_{\mu\nu}, \quad (5.8)$$

where $\sigma^\rho{}_{\mu\nu}$ is the *Spin Tensor* of the gravitational field, which is defined as [59]

$$\sigma^\mu{}_{\lambda\gamma} \equiv -\frac{\partial\mathcal{L}_G}{\partial\partial_\mu h^a{}_\sigma}\,\frac{\delta h^a{}_\sigma}{\delta\epsilon^{\lambda\gamma}} = \frac{h}{\kappa^2}\,S_{[\lambda\gamma]}{}^\mu = -\frac{h}{4\,\kappa^2}\left(T^\mu{}_{\lambda\gamma} + \delta^\mu_\gamma\,\mathcal{V}_\lambda - \delta^\mu_\lambda\,\mathcal{V}_\gamma\right), \quad (5.9)$$

where we have used

$$\frac{\delta h^a{}_\sigma}{\delta\epsilon^{\alpha\beta}} = \frac{1}{2}\,\left(g_{\beta\sigma}\,h^a{}_\alpha - g_{\alpha\sigma}\,h^a{}_\beta\right). \quad (5.10)$$

$\delta\epsilon^{\alpha\beta}$ is an infinitesimal anti-symmetric (Lorentz) tensor and $S_{[\lambda\gamma]}{}^\mu$ the anti-symmetric part of the superpotential [1]. However, the antisymmetric part (5.8) is not relevant on cosmological scales where there is homogeneity and isotropy. Varying the action with respect to the scalar field we find the motion equation

$$\nabla_\mu\partial^\mu\phi - \partial_\mu\phi\,\mathcal{V}^\mu + \eta\,f(\phi)\,\left(\nabla_\mu\mathcal{V}^\mu - \mathcal{V}_\mu\,\mathcal{V}^\mu\right) + V_{,\phi} = 0. \quad (5.11)$$

which is written in terms of the covariant derivative of the teleparallel connection. The first two terms reflect more once the fact already mentioned that the scalar field couples to torsion through its four derivative. We will now go to consider the spatially flat FLRW background (1.20) [8, 48], and we will study the cosmological implications for our model. As usual, we will work in the preferred class of frames where this background is a solution of the gravitational field equation (See Ref. [8]). By imposing the flat FLRW geometry (1.20) in (5.7) we obtain for the energy density

$$\rho_\phi = \frac{1}{2}\,\dot{\phi}^2 + V(\phi) - 3\,\eta\,f(\phi)\,H\,\dot{\phi}, \quad (5.12)$$

and for the pressure density

$$p_\phi = \frac{1}{2}\,\left(1 + 2\,\eta\,f_{,\phi}\right)\,\dot{\phi}^2 - V(\phi) + \eta\,f(\phi)\,\ddot{\phi}. \quad (5.13)$$

On the other hand, imposing the same background (1.20) in the motion equation (5.11) we find

$$\ddot{\phi} + 3\,H\,\dot{\phi} - 3\,\eta\,\left(\dot{H} + 3\,H^2\right)\,f(\phi) + V_{,\phi} = 0. \quad (5.14)$$

5.2 Cosmological dynamics

To study the cosmological dynamics of the model, we introduce the followings dimensionless variables:

$$x \equiv \frac{\kappa\,\dot{\phi}}{\sqrt{6}\,H}, \qquad y \equiv \frac{\kappa\,\sqrt{V}}{\sqrt{3}\,H}, \qquad u \equiv \kappa\,f, \qquad \lambda \equiv -\frac{V_{,\phi}}{\kappa\,V}, \qquad \alpha \equiv f_{,\phi}. \tag{5.15}$$

In terms of these dimensionless variables, the fractional energy densities Ω_ϕ and Ω_m for the scalar field and background matter are given by

$$\Omega_\phi \equiv \frac{\kappa^2\rho_\phi}{3H^2} = x^2 + y^2 - \sqrt{6}\,\eta\,u\,x, \qquad \Omega_m \equiv \frac{\kappa^2\rho_m}{3H^2} = 1 - \Omega_\phi, \tag{5.16}$$

respectively. By using the physical condition $0 \leq \Omega_\phi \leq 1$ in equation (5.16), the range in the phase space for the variables x, u and y is constrained. On the other hand, the equation of state of the field ω_ϕ reads

$$\omega_\phi \equiv \frac{p_\phi}{\rho_\phi} = \frac{(1+2\,\eta\,\alpha)\;x^2 - y^2 + \eta\,u\;\left(-\sqrt{6}\,x + \eta\,u\;(3-s) + \lambda\,y^2\right)}{x^2 + y^2 - \sqrt{6}\,\eta\,u\,x}. \tag{5.17}$$

The effective equation of state ω_{eff} is given by

$$\omega_{eff} \equiv \frac{p_m + p_\phi}{\rho_m + \rho_\phi} = (\gamma - 1)\;\left[1 - \left(x^2 + y^2 - \sqrt{6}\,\eta\,u\,x\right)\right] + (1+2\,\eta\,\alpha)\;x^2 - y^2 + \eta\,u\;\left(-\sqrt{6}\,x + \eta\,u\;(3-s) + \lambda\,y^2\right), \tag{5.18}$$

where we have defined $\gamma \equiv 1+\omega_m$ and the accelerated expansion occurs for $\omega_{eff} < -\frac{1}{3}$. Also, it is defined the parameter

$$s \equiv -\frac{\dot{H}}{H^2} = \left(\frac{2}{3} + \eta^2\,u^2\right)^{-1}\;[\gamma - (\gamma-1)\;\left(x^2 + y^2 - \sqrt{6}\,\eta\,u\,x\right) + (1+2\,\eta\,\alpha)\;x^2 + (\eta\,\lambda\,u - 1)\;y^2 + \sqrt{6}\,\eta\,u\;\left(\frac{\sqrt{6}}{2}\,\eta\,u - x\right)]. \tag{5.19}$$

The dynamical system of ordinary differential equations (ODE) for the model is written as

$$x' = (3-s)\;\left(-x + \frac{\sqrt{6}}{2}\,\eta\,u\right) + \frac{\sqrt{6}}{2}\,\lambda\,y^2, \tag{5.20}$$

$$y' = \left(s - \frac{\sqrt{6}\,\lambda\,x}{2}\right)\;y, \tag{5.21}$$

$$u' = \sqrt{6}\,\alpha\,x, \tag{5.22}$$

$$\lambda' = -\sqrt{6}\,(\Gamma - 1)\,\lambda^2\,x, \tag{5.23}$$

$$\alpha' = \sqrt{6}\,\Pi\,x. \tag{5.24}$$

In these equations primes denote derivative with respect to the so-called e-folding time $N \equiv \ln a$ [16, 17]. Also, we have defined the parameters

$$\Pi \equiv \kappa^{-1}\,f_{,\phi\phi}, \qquad \Gamma \equiv \frac{V\,V_{,\phi\phi}}{V_{,\phi}^2}. \tag{5.25}$$

From now we concentrate on exponential scalar field potential of the form $V(\phi) = V_0\,e^{-\lambda\,\kappa\,\phi}$, such that λ is a dimensionless constant, that is, $\Gamma = 1$ (equivalently, we could consider potentials satisfying $\lambda \equiv -\frac{V_{,\phi}}{\kappa\,V} \approx const$, which is valid for arbitrary but nearly flat potentials [39–41]). We study two simple cases: one where α is a constant and another where α depends on u. In the first case, it is considered a non-minimal coupling function $f(\phi) \propto \phi$ and thus $\alpha = const$ and $\Pi = 0$. Finally, we consider a dynamically changing $\alpha(u)$. If $u(\phi) \equiv \kappa\,f(\phi)$ is a general function, with inverse function $\phi(u) = f^{-1}(u/\kappa)$ thus $\alpha(\phi)$ and $\Pi(\phi)$ can be expressed in terms of u (see [8, 42, 48]). In this form, the dynamical system of (ODE) (5.20)-(5.24) is a dynamical autonomous system and we can obtain the fixed points or critical points (x_c, y_c, u_c) by imposing the conditions $x_c' = y_c' = u_c' = 0$. From the definition (5.15), x_c, y_c, u_c should be real, with $y_c \geq 0$. To study the stability of the critical points, we substitute linear perturbations, $x \to x_c + \delta x$, $y \to y_c + \delta y$, and $u \to u_c + \delta u$ around each critical point and linearize them. The eigenvalues of the perturbations matrix $\mathcal{M}$, namely, μ_1, μ_2 and μ_3, determine the conditions of stability of the critical points [15, 16]: (i) Stable node: $\mu_1 < 0$, $\mu_2 < 0$ and $\mu_3 < 0$. (ii) Unstable node: $\mu_1 > 0$, $\mu_2 > 0$ and $\mu_3 > 0$. (iii) Saddle point: one or two of the three eigenvalues are positive and the other negative. (iv) Stable spiral: The determinant of the matrix $\mathcal{M}$ is negative and the real parts of μ_1, μ_2 and μ_3 are negative. A critical point is an attractor in the cases (i) and (iv), but it is not so in the cases (ii) and (iii). The universe will eventually enter these attractor solutions regardless of the initial conditions.

5.3 Constant α

5.3.1 Critical points

In this section we consider a non-minimal coupling function $f(\phi) \propto \phi$ such that α is a constant. The critical points are presented in the Table 5.1, along with properties in Table 5.2. The critical point I.a is a matter-dominated solution ($\Omega_m = 1$) with equation of state type cosmological constant $\omega_\phi = -1$, that exists for all values η, λ

Table 5.1: Critical points for the autonomous system (5.20)-(5.24) when $f(\phi) \propto \phi$ and $\alpha = const.$

Name	x_c	y_c	u_c	Ω_ϕ	ω_ϕ	ω_{eff}
I.a	0	0	0	0	-1	$\gamma - 1$
I.b	0	0	u_c	0	$\eta\,\lambda\,u_c - 1$	1
I.c	0	1	$-\frac{\lambda}{3\,\eta}$	1	-1	-1

Table 5.2: Stability properties, and conditions for acceleration and existence for fixed points in Table 5.1.

Name	Stability	Acceleration	Existence
I.a	Saddle	No	All values
I.b	Unstable	No	$\omega_m = 1$
I.c	Stable node or Stable spiral, or Saddle	All values	All values

and γ. The point I.b is also a matter-dominated solution ($\Omega_m = 1$) that exists for all values $u_c \in \Re$ and $\omega_m = \omega_{eff} = 1$. On the other hand, the fixed point I.c correspond to a dark-energy-dominated de Sitter solution with $\Omega_\phi = 1$ and $\omega_\phi = \omega_{eff} = -1$. This point exists for all values of η and λ with accelerated expansion. It is a viable cosmological solution to describe the current accelerated expansion of our Universe.

5.3.2 Stability

Substituting the linear perturbations, $x \to x_c + \delta x$, $y \to y_c + \delta y$, and $u \to u_c + \delta u$ into the autonomous system (5.20)-(5.24) and linearize them, we find the follows components for the matrix of linear perturbations $\mathcal{M}$

$$\mathcal{M}_{11} = 3\,(2-\gamma) + \frac{\eta\,u_c\,\left(\lambda\,y_c^2 - \sqrt{6}\,(2\,\alpha\,\eta - 3\,\gamma + 6)\,x_c\right) - \gamma\,y_c^2 + 3\,(2\,\alpha\,\eta - \gamma + 2)\,x_c^2 - 3\,(2-\gamma)}{\eta^2\,u_c^2 + \frac{2}{3}}, \quad (5.26)$$

$$\mathcal{M}_{12} = \frac{\left(2\,(\eta\,\lambda\,u_c - \gamma)\,x_c + \sqrt{6}\,\gamma\,\eta\,u_c + \frac{2\,\sqrt{6}}{3}\,\lambda\right)\,y_c}{\eta^2\,u_c^2 + \frac{2}{3}}, \quad (5.27)$$

$$\mathcal{M}_{13} = -\frac{\eta\,\left(\left(2\,\lambda\,x_c+\sqrt{6}\,\gamma\right)\,y_c^2-\sqrt{6}\,\left(2\,\alpha\,\eta-3\,\gamma+6\right)\,x_c^2+\sqrt{6}\,\left(2-\gamma\right)\right)}{2\,\left(\eta^2\,u_c^2+\frac{2}{3}\right)}+$$
$$\frac{2\,\eta}{3}\left[\eta\,u_c\,\left(\left(3\,\gamma\,x_c-\sqrt{6}\,\lambda\right)\,y_c^2-3\,\left(2\,\alpha\,\eta-\gamma+2\right)\,x_c^3+9\,\left(2-\gamma\right)\,x_c\right)+\right.$$
$$\left.\left(2\,\lambda\,x_c+\sqrt{6}\,\gamma\right)\,y_c^2-\sqrt{6}\,\left(2\,\alpha\,\eta-3\,\gamma+6\right)\,x_c^2+\sqrt{6}\,\left(2-\gamma\right)\right]\,\left(\eta^2\,u_c^2+\frac{2}{3}\right)^{-2}, \quad (5.28)$$

$$\mathcal{M}_{21} = \frac{\left(2\,\left(2\,\alpha\,\eta-\gamma+2\right)\,x_c-\sqrt{6}\,\left(2-\gamma\right)\,\eta\,u_c\right)\,y_c}{\eta^2\,u_c^2+\frac{2}{3}}-\frac{\sqrt{6}\,\lambda\,y_c}{2}, \quad (5.29)$$

$$\mathcal{M}_{22} = \frac{\eta\,u_c\,\left(3\,\lambda\,y_c^2-\sqrt{6}\,\left(2-\gamma\right)\,x_c\right)-3\,\gamma\,y_c^2+\left(2\,\alpha\,\eta-\gamma+2\right)\,x_c^2+\gamma-2}{\eta^2\,u_c^2+\frac{2}{3}}-$$
$$\frac{\sqrt{6}\,\lambda\,x_c-6}{2}, \quad (5.30)$$

$$\mathcal{M}_{23} = -\frac{\eta\,y_c\,\left(\lambda\,y_c^2-\sqrt{6}\,\left(2-\gamma\right)\,x_c\right)}{\eta^2\,u_c^2+\frac{2}{3}}+$$
$$\frac{2\,\eta\,y_c\,\left(3\,\eta\,u_c\,\left(\gamma\,y_c^2+\left(-2\,\alpha\,\eta+\gamma-2\right)\,x_c^2-\gamma+2\right)+2\,\lambda\,y_c^2-2\,\sqrt{6}\,\left(2-\gamma\right)\,x_c\right)}{3\,\left(\eta^2\,u_c^2+\frac{2}{3}\right)^2}, \quad (5.31)$$

$$\mathcal{M}_{31} = \sqrt{6}\,\alpha, \qquad \mathcal{M}_{32} = 0, \qquad \mathcal{M}_{33} = 0. \quad (5.32)$$

The eigenvalues of the matrix $\mathcal{M}$, for each critical point, are as follows:

- Point I.a:

$$\mu_{1,2} = \frac{3}{4}\,\left(2-\gamma\right)\,\left(-1\pm\sqrt{1+\frac{8\,\eta\,\alpha}{2-\gamma}}\right), \qquad \mu_3 = \frac{3\,\gamma}{2}. \quad (5.33)$$

- Point I.b:

$$\mu_{1,2} = 0, \qquad \mu_3 = 3. \quad (5.34)$$

- Point I.c:

$$\mu_{1,2} = \frac{3}{2}\,\left(-1\pm\sqrt{1+\frac{24\,\eta\,\alpha}{\lambda^2+6}}\right), \qquad \mu_3 = -3\,\gamma. \quad (5.35)$$

The point I.a is a saddle point for $\eta\,\alpha \geq -\frac{2-\gamma}{8}$. The point I.b. is also unstable for all values. Finally, the critical point I.c is stable node for

$$-\frac{\lambda^2+6}{24} \leq \eta\,\alpha < 0. \quad (5.36)$$

For $\eta\,\alpha > 0$ it is a saddle point. On the other hand, when

$$\eta\,\alpha < -\frac{\lambda^2+6}{24}, \tag{5.37}$$

thus μ_1 and μ_2 are complex with real part negative. In this case, the determinant of the matrix of perturbations $\det\mathcal{M}|_{(x_c,y_c,u_c)} = \frac{162\,\gamma\,\eta\,\alpha}{\lambda^2+6}$ is negative and the point I.c is a stable spiral. So, we find that the fixed point I.c is a attractor (for conditions (5.36) or (5.37)) and a viable cosmological solution to explain the late time accelerated expansion. The universe will eventually enter this solution regardless of the initial conditions. In the Fig. 5.1 it is shown as the universe tends asymptotically to the dark-energy-dominated de Sitter solution I.c (passing through the matter-dominated solution I.a).

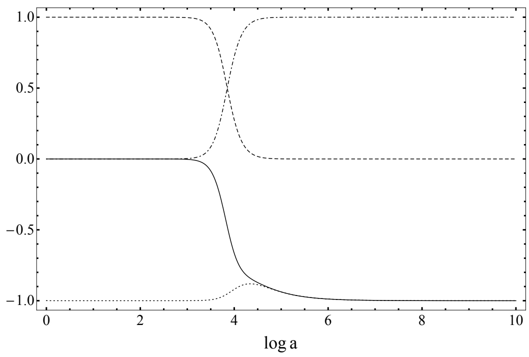

Figure 5.1: Evolution of Ω_m (dashed), Ω_ϕ (dotdashed), ω_ϕ (dotted) and ω_{eff} (solid) with $\gamma = 1$, $\lambda = 0.8$, $\alpha = 1$ and $\eta = -0.2$. It was considered the initial conditions $x_i = 1\times10^{-8}$, $y_i = 1.7\times10^{-6}$ and $u_i = 1\times10^{-7}$. The system asymptotically evolves toward the values $\Omega_\phi = 1$, $\Omega_m = 0$ and $\omega_\phi = \omega_{eff} = -1$. Also, we have $\Omega_\phi \approx 0.72$, $\Omega_m \approx 0.28$, $\omega_\phi \approx -0.92$ and $\omega_{eff} = -0.66$ at the present epoch $N = \log a \approx 4$.

5.4 Dynamically changing α

5.4.1 Critical points

Following Refs. [8, 42, 48], let us consider a general non-minimal coupling function $f(\phi)$ such that α can be expressed in terms of u and $\alpha(u) \to \alpha(u_c) = 0$ when $(x, y, u) \to (x_c, y_c, u_c)$ (we note that (x_c, y_c, u_c) is a fixed point of the system). The field ϕ rolls down toward $\pm\infty$ ($x > 0$ or $x < 0$) with $f(\phi) \to 1/\kappa$ and $u_c = 1$. The fixed points are presented in the Table 5.3, and the properties in Table 5.4. The point II.a is not

Table 5.3: Critical points for the autonomous system (5.20)-(5.24) for dynamically changing $\alpha(u) \to \alpha(u_c) = 0$ where $f(\phi) \to 1/\kappa$ and $u_c = 1$.

Name	x_c	y_c	Ω_ϕ	ω_ϕ	ω_{eff}
II.a	$\frac{\sqrt{6}\,\eta}{2}$	0	$-\frac{3\,\eta^2}{2}$	$\gamma-1$	$\gamma-1$
II.b	$\frac{\sqrt{6}\,\eta}{2} - \sqrt{1+\frac{3\,\eta^2}{2}}$	0	1	1	1
II.c	$\frac{\sqrt{6}\,\eta}{2} + \sqrt{1+\frac{3\,\eta^2}{2}}$	0	1	1	1
II.d	$\frac{\sqrt{6}\,\gamma}{2\,\lambda}$	$\frac{\sqrt{3}\,\sqrt{2-\gamma}\,\sqrt{\gamma-\eta\,\lambda}}{\sqrt{2}\,\lvert\lambda\rvert}$	$\frac{3\,(\gamma\,(2-\eta\,\lambda)-2\,\eta\,\lambda)}{2\,\lambda^2}$	$\gamma-1$	$\gamma-1$
II.e	$\frac{\sqrt{6}\,(3\,\eta+\lambda)}{3\,(2-\eta\,\lambda)}$	$\frac{\sqrt{6\,(1-\eta\,\lambda)-\lambda^2}\,\sqrt{\frac{2}{3}+\eta^2}}{2-\eta\,\lambda}$	1	(5.39)	(5.39)

Table 5.4: Stability properties, and conditions for acceleration and existence of the fixed points in Table 5.3.

Name	Stability	Acceleration	Existence
II.a	S. node or Saddle	No	No, if $0 \leq \Omega_\phi \leq 1$
II.b	U. node or Saddle ($\gamma = 1$)	No	All values
II.c	U. node or Saddle ($\gamma = 1$)	No	All values
II.d	S. node or Saddle or S. Spiral ($\gamma = 1$)	No	Eq. (5.38)
II.e	S. node or Saddle	$\eta\,\lambda < \frac{1}{2} - \frac{\lambda^2}{4}$	$\eta\,\lambda \leq 1 - \frac{\lambda^2}{6}$

realistic because $\Omega_\phi = -\frac{3\,\eta^2}{2} < 0$. The fixed points II.b and II.c are both scalar-field-dominated solutions with $\Omega_\phi = 1$ and equation of state type "stiff matter" $\omega_\phi = 1$. These points exist for all values of η and λ. The fixed point II.d is a scaling solution that exists for $\eta\,\lambda < \gamma < 2$. This point is a realistic solution when

$$\frac{2\,\left(3\,\gamma - \lambda^2\right)}{3\,(\gamma+2)} \leq \eta\,\lambda \leq \frac{2\,\gamma}{\gamma+2}, \tag{5.38}$$

since in this case $0 \leq \Omega_\phi \leq 1$. For nonrelativistic matter $\gamma = 1$ thus $\frac{2}{3} - \frac{2\,\lambda^2}{9} \leq \eta\,\lambda \leq \frac{2}{3}$. On the other hand, in the case of relativistic matter (radiation) $\gamma = 4/3$ we have that $\frac{4}{5} - \frac{\lambda^2}{5} \leq \eta\,\lambda \leq \frac{4}{5}$. Just like the above fixed points II.a-II.c, the point II.d is not viable to explain a late-time acceleration. However, since point II.d is a scaling solution, this can be used to provide the cosmological evolution in which the energy density of the scalar field decreases proportionally to that of the background fluid in either a radiation or matter-dominated era. Finally, the point II.e is also a scalar-field-dominated solution, but different to II.b and II.c, this point is a viable solution to explain the late-time cosmic acceleration. This point exists for $\eta\,\lambda \leq 1 - \frac{\lambda^2}{6}$. The equation of state is given by

$$\omega_\phi = \omega_{eff} = \frac{2\,\lambda^2 + 3\,(3\,\eta\,\lambda - 2)}{3\,(2 - \eta\,\lambda)}, \tag{5.39}$$

and the accelerated expansion occurs for $\eta\,\lambda < \frac{1}{2} - \frac{\lambda^2}{4}$. From (5.39) we have that $\omega_\phi \geq -1$ if $\eta\,\lambda \geq -\frac{\lambda^2}{3}$. Moreover, it is also possible that this solution to be phantom, that is $\omega_\phi < -1$, if $\eta\,\lambda < -\frac{\lambda^2}{3} < 0$. Is worth highlighting that unlike the dark energy models with phantom or ghost scalar field [16, 17], the present model is devoid of any quantum instability [61].

5.4.2 Stability

For dynamically changing $\alpha(u)$ the components of the matrix of perturbation $\mathcal{M}$ are written as

$$\mathcal{M}_{11} = \frac{3\,\left(\eta\,\left(\lambda\,y_c^2 - 3\,\sqrt{6}\,(2-\gamma)\,x_c\right) - \gamma\,y_c^2 + 3\,(2-\gamma)\,\left(x_c^2 - 1\right)\right)}{3\,\eta^2 + 2} + 3\,(2-\gamma)\,, \tag{5.40}$$

$$\mathcal{M}_{12} = \frac{\sqrt{6}\,\left(\sqrt{6}\,(\eta\,\lambda - \gamma)\,x_c + 2\,\lambda + 3\,\gamma\,\eta\right)\,y_c}{3\,\eta^2 + 2}, \tag{5.41}$$

$$\begin{aligned}\mathcal{M}_{13} = -\frac{1}{2}\,\left(3\,\eta^2 + 2\right)^{-1}\,[3\,\eta\,\Big(\Big(2\,\lambda\,x_c + \sqrt{6}\,\gamma\Big)\,y_c^2 - 4\,\tau_c\,x_c^3 - \sqrt{6}\,(2-\gamma)\,\left(3\,x_c^2 - 1\right)\Big) - \\ 4\,\Big(3\,\gamma\,x_c - \sqrt{6}\,\lambda\Big)\,y_c^2 - 4\,\sqrt{6}\,\tau_c\,x_c^2 + 12\,(2-\gamma)\,x_c\,\left(x_c^2 - 3\right)] + \\ 2\,\left(3\,\eta^2 + 2\right)^{-2}\,[3\,\eta\,\Big(\Big(2\,\lambda\,x_c + \sqrt{6}\,\gamma\Big)\,y_c^2 - \sqrt{6}\,(2-\gamma)\,\left(3\,x^2 - 1\right)\Big) - \\ 2\,\Big(3\,\gamma\,x_c - \sqrt{6}\,\lambda\Big)\,y_c^2 + 6\,(2-\gamma)\,x_c\,\left(x_c^2 - 3\right)] - \sqrt{6}\,\tau_c\,x_c^2,\end{aligned} \tag{5.42}$$

$$\mathcal{M}_{21} = \frac{3\,(2-\gamma)\,\left(2\,x_c - \sqrt{6}\,\eta\right)\,y_c}{3\,\eta^2 + 2} - \frac{\sqrt{6}\,\lambda\,y_c}{2}, \tag{5.43}$$

$$\mathcal{M}_{22} = \frac{3\,\left(\eta\,\left(3\,\lambda\,y_c^2 - \sqrt{6}\,(2-\gamma)\,x_c\right) - 3\,\gamma\,y_c^2 + (2-\gamma)\,\left(x_c^2 - 1\right)\right)}{3\,\eta^2 + 2} - \frac{\sqrt{6}\,\lambda\,x_c - 6}{2}, \tag{5.44}$$

$$\begin{aligned}\mathcal{M}_{23} = \frac{12\,y_c\,\left(\eta\,\left(\lambda\,y_c^2 - \sqrt{6}\,(2-\gamma)\,x_c\right) - \gamma\,y_c^2 + (2-\gamma)\,\left(x_c^2 - 1\right)\right)}{\left(3\,\eta^2 + 2\right)^2} + \\ \frac{3\,y_c\,\left(\eta\,\left(-\lambda\,y_c^2 + 2\,\tau_c\,x_c^2 + \sqrt{6}\,(2-\gamma)\,x_c\right) + 2\,\gamma\,y_c^2 - 2\,(2-\gamma)\,\left(x_c^2 - 1\right)\right)}{3\,\eta^2 + 2},\end{aligned} \tag{5.45}$$

$$\mathcal{M}_{31} = \mathcal{M}_{32} = 0, \qquad \mathcal{M}_{33} = \sqrt{6}\,\tau_c\,x_c. \tag{5.46}$$

Here τ_c is defined by $\tau_c \equiv \frac{d\alpha(u)}{du}|_{u=u_c}$. The eigenvalues of the matrix $\mathcal{M}$, for each critical point, are as follows:

- Point II.a:

$$\mu_1 = \frac{3\,(\gamma - \eta\,\lambda)}{2}, \qquad \mu_2 = -\frac{3\,(2-\gamma)}{2}, \qquad \mu_3 = 3\,\eta\,\tau_c. \tag{5.47}$$

- Point II.b ($\gamma = 1$):

$$\mu_1 = \frac{\lambda}{2}\left(\sqrt{9\,\eta^2+6} - \left(3\,\eta - \frac{6}{\lambda}\right)\right), \qquad \mu_2 = 3, \qquad \mu_3 = \tau_c\left(3\,\eta - \sqrt{9\,\eta^2+6}\right). \tag{5.48}$$

- Point II.c ($\gamma = 1$):

$$\mu_1 = -\frac{\lambda}{2}\left(\sqrt{9\,\eta^2+6} + 3\,\eta - \frac{6}{\lambda}\right), \qquad \mu_2 = 3, \qquad \mu_3 = \tau_c\left(3\,\eta + \sqrt{9\,\eta^2+6}\right). \tag{5.49}$$

- Point II.d ($\gamma = 1$):

$$\mu_{1,2} = \frac{3}{4}\left(-1 \pm \sqrt{1+Y}\right), \qquad \mu_3 = \frac{3\,\tau_c}{\lambda}, \qquad Y = \frac{8\,(\eta\,\lambda - 1)\,\left(2\,\lambda^2 + 9\,\eta\,\lambda - 6\right)}{(3\,\eta^2+2)\,\lambda^2}. \tag{5.50}$$

- Point II.e:

$$\mu_1 = \frac{3\,(\gamma+2)\,\eta\,\lambda + 2\,\left(\lambda^2 - 3\,\gamma\right)}{2-\eta\,\lambda}, \qquad \mu_2 = \frac{6\,\eta\,\lambda + \lambda^2 - 6}{2-\eta\,\lambda}, \qquad \mu_3 = \frac{2\,(\lambda + 3\,\eta)\,\tau_c}{2-\eta\,\lambda}. \tag{5.51}$$

The point II.a is a stable node for $\eta\,\lambda > \gamma$ and $\eta\,\tau_c < 0$. Meanwhile it is a saddle point for $\eta\,\lambda < \gamma$ and/or $\eta\,\tau_c > 0$. In any case, it is not realistic solution since $\Omega_\phi < 0$. When the background fluid is nonrelativistic matter $\gamma = 1$, the critical points II.b and II.c are unstable in any case. These points can be saddle point or unstable node. The scaling solution II.d, for $\gamma = 1$ and $\eta\,\lambda < 2/3$, is a stable node if

$$\frac{3\,(2 - 3\,\eta\,\lambda)}{2} < \lambda^2 \leq \frac{3\,(5\,\eta\,\lambda - 4)^2}{2\,(7 - 8\,\eta\,\lambda)} \quad \text{and} \quad \frac{\tau_c}{\lambda} < 0, \tag{5.52}$$

whereas for

$$\lambda^2 < \frac{3\,(2 - 3\,\eta\,\lambda)}{2} \quad \text{and/or} \quad \frac{\tau_c}{\lambda} > 0, \tag{5.53}$$

it is a saddle point. On the other hand, for

$$\lambda^2 > \frac{3\,(5\,\eta\,\lambda - 4)^2}{2\,(7 - 8\,\eta\,\lambda)}, \tag{5.54}$$

μ_1 and μ_2 are complex with real part negative. If in addition we have $\tau_c/\lambda < 0$ thus $\mu_3 < 0$. Since the determinant of the matrix of perturbations $\det \mathcal{M}|_{(x_c,y_c,u_c)} =$

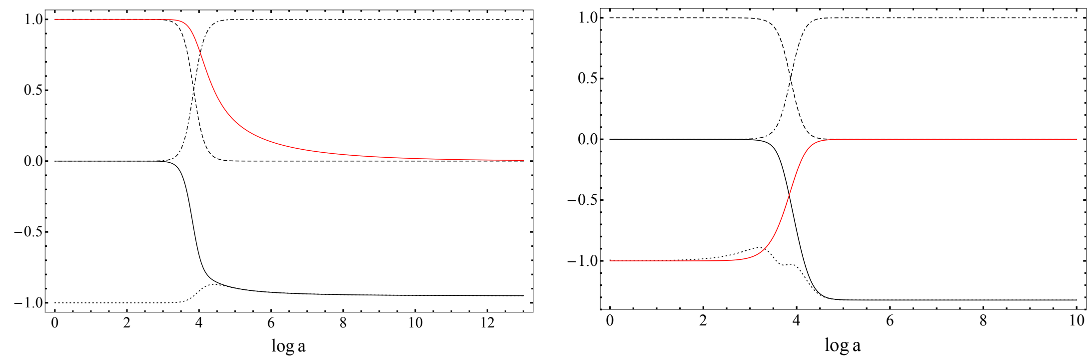

Figure 5.2: Evolution of Ω_m (dashed), Ω_ϕ (dotdashed), ω_ϕ (dotted) and ω_{eff} (solid), with $\gamma = 1$. In the left panel, for $\alpha(u) = 1 - u$ (red line, starting at $\alpha = 1$), we have chosen $\lambda = 0.8$ and $\eta = -0.2$, with initial conditions $x_i = 1 \times 10^{-8}$, $y_i = 1.7 \times 10^{-6}$ and $u_i = 1 \times 10^{-8}$. The present epoch ($N = \log a \approx 4$) corresponds to $\Omega_\phi \approx 0.72$, $\Omega_m \approx 0.28$, $\omega_\phi \approx -0.92$ and $\omega_{eff} \approx -0.66$. The Universe asymptotically evolves toward $\Omega_\phi = 1$ and $\omega_\phi = \omega_{eff} = -0.95$. In the right panel, for $\alpha(u) = -1 + u$ (red line, starting at $\alpha = -1$), we have chosen $\lambda = 0.3$ and $\eta = -1.4$, with initial conditions $x_i = 1 \times 10^{-9}$, $y_i = 2.16 \times 10^{-6}$ and $u_i = 1.1 \times 10^{-7}$. At the present time, also with $\Omega_\phi \approx 0.72$ and $\Omega_m \approx 0.28$, we have $\omega_\phi \approx -1.05$ and $\omega_{eff} \approx -0.75$. In this case the evolution converges to $\Omega_\phi = 1$ and $\omega_\phi = \omega_{eff} = -1.32$.

$-\frac{27}{16}\frac{\tau_c}{\lambda}\, Y$ is negative, in this case the point II.d is a stable spiral. Finally, for $\eta\,\lambda < 1 - \frac{\lambda^2}{6}$, the point II.e is a stable node if

$$\eta\,\lambda < \frac{2\,\left(3\,\gamma - \lambda^2\right)}{3\,(\gamma+2)} \qquad \text{and} \qquad (\lambda + 3\,\eta)\,\tau_c < 0, \tag{5.55}$$

otherwise it is a saddle point. Whenever accelerated expansion occurs, $\eta\,\lambda < \frac{1}{2} - \frac{\lambda^2}{4}$, this fixed point is a stable node and therefore a attractor. Just like the point I.c, the fixed point II.e is also a viable cosmological solution to explain the current phase of accelerated expansion. In Fig. 5.2 the Universe tends asymptotically to the solution II.e for dynamically changing $\alpha(u)$ with accelerated expansion, first passing through the matter-dominated solution I.a with constant α. In the left panel, we consider for simplicity the function $f(\phi) = \frac{1}{\kappa}\left(1 - e^{-\kappa\,\phi}\right)$ such that $\alpha(u) = 1 - u$ and $\tau_c = -1$. Similarly, in the right panel, it is considered the function $f(\phi) = \frac{1}{\kappa}\left(1 + e^{\kappa\,\phi}\right)$ such that $\alpha(u) = -1 + u$ and $\tau_c = 1$. Other functions may also be considered provided that $\alpha(u_c) = 0$ and $\tau_c < 0$ or $\tau_c > 0$ (according to (5.55)).

Chapter 6

Conclusions

6.1 Interacting teleparallel dark energy

It has been recently proposed [5] a non-minimal coupling ϕ^2 between quintessence and gravity in the framework of TG, motivated by a similar construction in the context of GR. The theory has been called "teleparallel dark energy" (TDE), and the cosmological dynamics of the model was studied later in [6–8], but no scaling attractor was found. In the chapter 2 we generalize the function of non-minimal coupling $\phi^2 \to f(\phi)$ and we define $\alpha \equiv f_{,\phi}/\sqrt{f}$ such that α is constant for $f(\phi) \propto \phi^2$, but it can also be a dynamically changing quantity $\alpha(\phi)$. Also, in both cases, we consider a possible interaction between TDE and dark matter, that is, "interacting teleparallel dark energy".

For constant $\alpha \neq 0$, the universe enters the matter-dominated solution I.a ($\Omega_m = 1$) with $\omega_\phi = -1$ and $\omega_{eff} = 0$, but since this solution is unstable, it will eventually fall into the attractor I.d or I.e (dark-energy-dominated de Sitter solution) with $\Omega_\phi = 1$ and $\omega_\phi = \omega_{eff} = -1$. So, for constant α no scaling attractor was found, even when we allow interaction between TDE and dark matter (see Table 2.1).

On the other hand, for dynamically changing $\alpha(\phi)$, and when it can be expressed in terms of $u \equiv \kappa\sqrt{f(\phi)}$ such that $\alpha(u) \to \alpha(u_c) = 0$ (we notice that (x_c, y_c, u_c) is a fixed point of the system), we show that interesting solutions exist (see Table 2.3), in particular scaling attractors with accelerated expansion. These are non-minimal generalizations of the fixed points presented in [16, 18, 19] for coupled dark energy in GR. The final attractor is either the scalar-field dominated solution II.e or the scaling attractor II.d, both with accelerated expansion. The scaling solution II.d can be a stable node or a stable spiral when the background fluid is non-relativistic dark matter ($\gamma = 1$). When point II.d is a stable spiral, that is, $\hat{\xi} \equiv 2\,\xi\, u_c^2 + 1$ as in constraint (2.66), and

$$\eta_c \equiv \frac{d\alpha(u)}{du}|_{u=u_c} < 0,$$

it is possible to find a range for $\hat{\xi}$ compatible with cosmological observations. For example, in Fig. 2.1 the system approaches point II.d with $\Omega_\phi \approx 0.70$, $\Omega_m \approx 0.30$, $\omega_\phi \approx -0.90$ and accelerated expansion, in agreement with observations. Also, we note that ω_ϕ presents the phantom-divide crossing during cosmological evolution, as has been stressed in Refs. [5, 6] for TDE. Since the scaling solution II.d attracts the universe at late times regardless of the initial conditions with Ω_ϕ comparable to Ω_m and accelerated expansion, the cosmological coincidence problem could be alleviated without fine-tunings [15, 16].

6.2 Tachyonic teleparallel dark energy

In Ref. [47] it was proposed a non-minimal coupling between a non-canonical scalar field (tachyon field) in the context of TG. Here, by studying the dynamics of this "tachyonic teleparallel dark energy model" (TTDE), we have found that, unlike TDE, in TTDE it is possible to have a phase ϕMDE, represented by the scaling solutions I.b and I.c of Table 3.1, which have some portions of the energy density of ϕ in the matter dominated era. The presence of this phase provides a distinguishable feature for matter density perturbations, as is the case of coupled dark energy in GR (see Refs. [16, 18, 19]). However, in order to allow the universe to enter the phase ϕMDE, and then to fall within a viable cosmologically late-time attractor with accelerated expansion, it is necessary that the non-minimal coupling be ruled by a dynamically changing coefficient $\alpha(\phi) \equiv f_{,\phi}/\sqrt{f}$, with $f(\phi)$ an arbitrary function of the scalar field ϕ. Following Ref. [8], we considered then that $\alpha(\phi)$ can be expressed in terms of the dimensionless parameter $u \equiv \kappa\sqrt{f(\phi)}$, such that $\alpha(u) \to \alpha(u_c) = 0$, with (x_c, y_c, u_c) a fixed point of the system. We have found the fixed points (see Table 3.3) that are non-minimal generalization of the fixed points presented in Ref. [16] for tachyon field in GR. The scalar-field dominant solution II.c is a late-time attractor with accelerated expansion, and ω_ϕ agrees with the observations. Also, it is possible in this case that the universe enters in the scaling solutions I.b or I.c (phase ϕMDE) for constant α and eventually approaches the late-time attractor II.c with accelerated expansion for dynamically changing $\alpha(u)$, as can be seen in Fig. 3.1. By studying linear perturbations, we find that the presence of dark energy in a era "ϕMDE' (I.b or I.c) reduces the standard growth rate for matter density perturbations ($\delta_m \propto a$ in sub-horizon scale) by a factor $a^{-\frac{3}{5}\Omega_\phi}$ with $\Omega_\phi = 1 - \hat{\xi} \ll 1$. On the other hand, for the scalar-field dominant solution II.c, as is usual, the matter density perturbations are "frozen" ($\varrho_+ = 0$) during the accelerated expansion, which can also be checked in the Fig. 4.3.

It should be noted that the formation of caustics in the field profile in the mass free space, for tachyon systems (Dirac-Born-Infeld systems) is an undesirable feature as it indicates the failure of physical theories to explain the evolution of the field in that particular region [49, 50]. As was shown in [50], in the FLRW expanding Universe the caustic formation in tachyon systems takes place for potentials decaying faster than $1/\varphi^2$ at infinity (for the untransformed field φ), where the dust-like solution is a late time attractor of the dynamics. On the other hand, in the case of inverse power-law potentials, $V(\varphi) = V_0/\varphi^n$, $0 \leq n \leq 2$, dark energy is a late time attractor of dynamics and they are free of caustics [50]. They may, therefore, be suitable for explaining the late time cosmic acceleration. So, since in the case of the model discussed, dark energy is a late time attractor of the dynamics, which gives rise to cosmic repulsion that compete with the tendency of caustic formation, we expect the model to be free of caustics and multivalued regions in the field profile.

6.3 A novel teleparallel dark energy

A novel model has been proposed, named "new teleparallel dark energy" (NTDE), in which it is allowed a nonminimal coupling between the scalar field of quintessence and gravity, represented by torsion. As is well-known [1, 2], the scalar field couples to torsion through its four-derivative—which is a vector field. It is then natural to consider a nonminimal coupling of the four-derivative of the scalar field and the vector part of torsion. It is important to note that unlike the TDE scenario of Ref. [5], which was proposed by following an analogy with the nonminimally coupled scalar field in GR, in the present model the nonminimal coupling is proposed according to a conceptually different description of the gravitational field in TG [1–3]. Let us stress that NTDE does not have an analogue in GR, since in GR the gravitational field can not be decomposed in a form analogous to the decomposition of torsion in TG (see Eqs. (5.2), (5.3), and (5.4)). It is also important to observe that once the nonminimal coupling is switched on, the scalar field becomes coupled, through its four-derivative, to the spin tensor of the gravitational field (see Eq. (5.8)). This is due to that in TG, besides a well-defined local energy-momentum density [1, 2, 4], the gravitational field also has a well-defined spin current density. The same happens in TDE, but until now had not been given an interpretation. In both models, the coupling to the gravitational spin tensor becomes negligible on cosmological scales.

By studying the dynamics of the model, we have found the critical points, presented in Tables 5.1 and 5.3. In Table 5.1 the final attractor of the Universe is a dark-energy-dominated de Sitter solution I.c, with $\Omega_\phi = 1$ and $\omega_\phi = \omega_{eff} = -1$. On the other hand, in Table 5.3 the final attractor is a scalar-field-dominated solution II.e, also with $\Omega_\phi = 1$, but in this case either $\omega_\phi = \omega_{eff} \geq -1$ or $\omega_\phi < -1$ in which case it represents a phantom-type solution. However, unlike the dark energy models with phantom (ghost) field [16, 17], here the phantom regime ($\omega_\phi < -1$) is described without the problem of quantum instability [61]. Additionally, unlike of the TDE scenario, here the phantom Universe is an attractor solution for the cosmological dynamics. The fixed points I.c and II.e are viable cosmological solutions to explain the current accelerated expansion of the Universe.

It is interesting to remark that the models TDE and NTDE are mathematically related through a conformal transformation. This can be seen by defining transformed scalar field and potential, and by adding an explicit coupling between the scalar field and matter. However, as already pointed out in the case of scalar-tensor theories and coupled dark energy in GR, where this type of mathematical relationship also occurs [18, 19, 53, 54], the two models are physically different. Furthermore, differently from coupled dark energy in GR, here we have taken the freedom to exclude a possible explicit coupling between the scalar field and matter (although this could be considered in a future work).

Appendix A

Linear Frames and Tetrads

Following Ref. [1], in this appendix and in the next we review briefly the definition and properties of linear frames, tetrads, connections and local Lorentz transformations in TG.

A general spacetime is a 4-dimensional differential manifold whose tangent space is, at any point, a Minkowski spacetime with metric $\eta_{ab} = diag\,(1, -1, -1, -1)$. Each one of these tangent spaces can be acted upon by the Poincaré group. Spacetime coordinates will be denoted by x^{μ}, whereas tangent space coordinates will be denoted by x^{a}. Such coordinates systems determine, on their domains of definition, local bases for vector fields, formed by the sets of gradients

$$\partial_{\mu} \equiv \frac{\partial}{\partial x^{\mu}}, \qquad \partial_{a} \equiv \frac{\partial}{\partial x^{a}}, \tag{A.1}$$

as well as bases dx^{μ} and dx^{a} for covector fields. These bases are dual, in the sense that

$$dx^{\mu}\left(\partial_{\nu}\right) = \delta^{\mu}_{\nu}, \qquad dx^{a}\left(\partial_{b}\right) = \delta^{a}_{b}. \tag{A.2}$$

A holonomic (or coordinate) base like $(e_a) = (\partial_a)$, related to a coordinate system, is very particular case of linear base. Any set of four linearly independent fields (e_a) will form another base, and will have a dual (e^a) whose members are such $e^a e_b = \delta^a_b$. These frame fields are the general linear bases for the tangent space at each point on the spacetime. On common domains, the members of a base can be written in terms of the members of the other. For example

$$e_a = e_a{}^{\mu}\,\partial_{\mu}, \qquad e^a = e^a{}_{\mu}\,dx^{\mu}, \tag{A.3}$$

and conversely

$$\partial_\mu = e^a{}_\mu \, e_a, \qquad dx^\mu = -e_a{}^\mu \, e^a, \tag{A.4}$$

and so for any pair of bases.

The spacetime metric $\mathbf{g}$, with components $g_{\mu\nu}$, in some dual holonomic base becomes

$$g = g_{\mu\nu} \, dx^\mu \otimes dx^\nu = g_{\mu\nu} \, dx^\mu \, dx^\nu. \tag{A.5}$$

So, a tetrad field

$$h_a = h_a{}^\mu \, \partial_\mu, \tag{A.6}$$

will be a linear base which relates $\mathbf{g}$ to the tangent-space metric

$$\eta = \eta_{ab} \, dx^a \otimes dx^b = \eta_{ab} \, dx^a \, dx^b, \tag{A.7}$$

by

$$\eta_{ab} = g(h_a, h_b) = g_{\mu\nu} h_a{}^\mu h_b{}^\nu. \tag{A.8}$$

This means that a tetrad field is a linear frame whose members h_a are pseudo-orthogonal by the pseudo-riemannian metric $g_{\mu\nu}$. The components of the dual base members $h_a = h^a{}_\mu dx^\mu$ satisfy

$$h^a{}_\mu \, h_a{}^\nu = \delta^\nu{}_\mu, \qquad h^a{}_\mu \, h_b{}^\mu = \delta^a{}_b \tag{A.9}$$

so that the equation (A.8) has the conversely

$$g_{\mu\nu} = \eta_{ab} \, h^a{}_\mu \, h^b{}_\nu. \tag{A.10}$$

We shall also be using the notation

$$h = det(h^a{}_\mu) = \sqrt{-g}, \tag{A.11}$$

where g is the determinant of the spacetime metric.

Appendix B

Connections and Lorentz Transformations

A spin connection or Lorentz connection A_μ is a 1-form assuming values in the Lie algebra of Lorentz group,

$$A_\mu = \frac{1}{2} A^{ab}{}_\mu S_{ab} \tag{B.1}$$

with S_{ab} a given representation of the Lorentz generators.

A general linear connection $\Gamma^\rho{}_{\nu\mu}$ is related to the corresponding spin connection $A^a{}_{b\mu}$ by

$$\Gamma^\rho{}_{\nu\mu} = h_a{}^\rho \, \partial_\mu h^a{}_\nu + h_a{}^\rho \, h^b{}_\nu \, A^a{}_{b\mu} \equiv h_a{}^\rho \, \mathcal{D}_\mu h^a{}_\nu. \tag{B.2}$$

The inverse relation is, consequently

$$A^a{}_{b\mu} = h^a{}_\nu \, \partial_\mu h_b{}^\nu + h^a{}_\nu \, h_b{}^\rho \, \Gamma^\nu{}_{\rho\mu} \equiv h^a{}_\nu \, \nabla_\mu h_b{}^\nu, \tag{B.3}$$

where ∇_μ is the usual covariant derivative in the connection $\Gamma^\nu{}_{\rho\mu}$, which, acts on spacetime indices only. Whereas the Fock Ivanenko derivative $\mathcal{D}_\mu$ acts only on the tangent space indices and can be defined for all fields, tensorial and spinorial, the covariant derivative ∇_μ can be defined for tensorial fields only.

The connection of TG representing only inertial effects is given by

$$A^b{}_{c\mu} = \Lambda^b{}_d \, \partial_\mu \Lambda_c{}^d. \tag{B.4}$$

The curvature of the teleparallel connection (B.4) vanishes identically but the torsion is non-vanishing

$$T^a{}_{\nu\mu} = \partial_\nu h^a{}_\mu - \partial_\mu h^a{}_\nu + A^a{}_{e\nu} \, h^e{}_\mu - A^a{}_{e\mu} \, h^e{}_\nu \neq 0. \tag{B.5}$$

This connection can be considered a kind of dual of the GR connection, which is a connection with vanishing torsion $\bar{T}^a{}_{\nu\mu} = 0$, but $\bar{R}^a{}_{b\nu\mu} \neq 0$.

There exist actually a six-fold infinity of tetrad fields, each one relating **g** to the Lorentz metric η. Suppose another tetrad (h'_a) such that

$$g_{\mu\nu} = \eta_{cd}\, h'^c{}_\mu\, h'^d{}_\nu, \tag{B.6}$$

contracting both sides with $h_a{}^\mu\, h_b{}^\nu$, we arrive at

$$\eta_{ab} = \eta_{cd}\, \left(h'^c{}_\mu\, h_a{}^\mu\right) \left(h'^d{}_\nu\, h_b{}^\nu\right) \tag{B.7}$$

This equation says that the matrix with entries

$$\Lambda^a{}_b = h'^a{}_\mu\, h_b{}^\mu, \tag{B.8}$$

which gives the transformation

$$h'^a{}_\mu = \Lambda^a{}_b\, h^b{}_\mu, \tag{B.9}$$

satisfies

$$\eta_{ab}\, \Lambda^c{}_a\, \Lambda^d{}_b = \eta_{ab}. \tag{B.10}$$

This is just the condition that a matrix Λ must satisfy in order to belong to the Lorentz group (vector representation). Under a local Lorentz transformation $\Lambda^a{}_b(x)$, the tetrad changes according (B.9), whereas the spin connection undergoes the transformation

$$A'^a{}_{b\mu} = \Lambda^a{}_c(x)\, A^c{}_{d\mu}\, \Lambda_b{}^d(x) + \Lambda^a{}_c\, \partial_\mu \Lambda_b{}^c(x). \tag{B.11}$$

In the same way, it is easy to verify that $T^a{}_{\nu\mu}$ and $R^a{}_{b\nu\mu}$ transform covariantly

$$T'^a{}_{\nu\mu} = \Lambda^a{}_b(x)\, T^b{}_{\nu\mu}, \tag{B.12}$$

and

$$R'^a{}_{b\nu\mu} = \Lambda^a{}_c(x)\, \Lambda_b{}^d(x)\, R^c{}_{d\nu\mu}. \tag{B.13}$$

This means that (as well as $g_{\mu\nu}$) spacetime-indexed quantities $\Gamma^\rho{}_{\nu\mu}$, $T^\lambda{}_{\mu\nu}$ and $R^\rho{}_{\lambda\nu\mu}$ are invariant under a local Lorentz transformation.

Appendix C

Autonomous system of scalar-field dark energy models

A dynamical system which plays an important role in cosmology belongs to the class of so-called autonomous systems (see Ref. [16]). Let us consider the following coupled differential equations for n variables $x^1(t)$,...,$x^n(t)$

$$\dot{x}^1 = f_1(x^1, .., x^n, t), ..., \dot{x}^n = f_n(x^1, ..., x^n, t), \tag{C.1}$$

where f_1,...,f_n are the functions in terms of x^1,...,x^n and t. The system (C.1) is said to be autonomous if the functions f_1,...f_n do not contain explicit time-dependent terms. The dynamics of the autonomous systems can be analyzed by studying the stability around the fixed points (or critical points).

C.0.1 Fixed or critical points

A point $(x_c^1, ..., x_c^n)$ is said to be a fixed point or a critical point of the autonomous system if

$$(f_1, ..., f_n)|_{x_c^1,...,x_c^n} = 0. \tag{C.2}$$

A critical point $(x_c^1, ..., x_c^n)$ is called an attractor when it satisfies the condition

$$(x^1(t), ..., x^n(t)) \to (x_c^1, ..., x_c^n), \quad \text{for} \quad t \to \infty. \tag{C.3}$$

C.0.2 Stability around the fixed points

We can find whether the system approaches one of the critical points or not by studying the stability around the fixed points. Let us consider small perturbations δx^1,...,

δx^n around the critical point $(x_c^1, ..., x_c^n)$, i.e.

$$x^1 = x_c^1 + \delta x^1, ..., x^n = x_c^n + \delta x^n. \tag{C.4}$$

Then, substituting into Eqs. (C.1) leads to the first-order differential equations:

$$\frac{d}{dN}\begin{pmatrix} \delta x^1 \\ \vdots \\ \delta x^n \end{pmatrix} = \mathcal{M}\begin{pmatrix} \delta x^1 \\ \vdots \\ \delta x^n \end{pmatrix}, \tag{C.5}$$

where $N = \ln a$ is the number of e-foldings which is convenient to use for the dynamics of dark energy. The matrix $\mathcal{M}$ depends upon x_c^1,...,x_c^n, and is given by

$$\mathcal{M} = \begin{pmatrix} \frac{df_1}{dx^1}|_{(x_c^1,...,x_c^n)} & \cdots & \frac{df_1}{dx^n}|_{(x_c^1,...,x_c^n)} \\ \vdots & \cdots & \vdots \\ \frac{df_n}{dx^1}|_{(x_c^1,...,x_c^n)} & \cdots & \frac{df_n}{dx^n}|_{(x_c^1,...,x_c^n)} \end{pmatrix}. \tag{C.6}$$

This possesses n eigenvalues μ_1,...,μ_n. The general solution for the evolution of linear perturbations can be written as

$$\begin{aligned} \delta x^1 &= C_1^1\, e^{\mu_1 N} + ... + C_n^1\, e^{\mu_n N}, \\ &\vdots \\ \delta x^n &= C_1^n\, e^{\mu_1 N} + ... + C_n^n\, e^{\mu_n N}, \end{aligned} \tag{C.7}$$

where C_j^i with $i, j = 1, ..., n$ are integration constants. Thus, the stability around the fixed points depends upon the nature of the eigenvalues. One generally uses the following classification:

- (i) Stable node: $\mu_1 < 0$,...,$\mu_n < 0$.
- (ii) Unstable node: $\mu_1 > 0$,...,$\mu_n > 0$.
- (iii) Saddle point: $\mu_i > 0$ and $\mu_j < 0$ for $i, j = 1, ..., n$.
- (iv) Stable spiral: The determinant of the matrix $\mathcal{M}$ is negative and the real parts of μ_1,...,μ_n are negative.

A fixed point is an attractor in the cases (i) and (iv), but it is not so in the cases (ii) and (iii).

Appendix D

Field Equations and Gravitational Spin Tensor

The field equations of TG are given by

$$G_a{}^\nu \equiv h^{-1}\,\partial_\mu\,(h\,S_a{}^{\nu\mu}) + h_a{}^\lambda\,T^\rho{}_{\mu\lambda}\,S_\rho{}^{\nu\mu} + \frac{1}{4}\,h_a{}^\nu\,T - h_c{}^\rho\,A^c{}_{a\sigma}\,S_\rho{}^{\nu\sigma} = \frac{\kappa^2}{2}\,\Theta_a{}^\nu, \tag{D.1}$$

where

$$\Theta_a{}^\nu = -\frac{1}{h}\,\frac{\delta S_m}{\delta h^a{}_\nu} \tag{D.2}$$

and $\Theta_\rho{}^\nu \equiv h^a{}_\rho\,\Theta_a{}^\nu$ stands for the symmetric matter energy-momentum tensor.

In equation (D.1) can be identified the gravitational energy momentum tensor

$$j_a{}^\nu \equiv -\frac{1}{h}\,\frac{\delta S_G}{\delta h^a{}_\nu} = \frac{1}{\kappa^2}\left(-h_a{}^\lambda\,T^\rho{}_{\mu\lambda}\,S_\rho{}^{\nu\mu} - \frac{1}{4}\,h_a{}^\nu\,T + h_c{}^\rho\,A^c{}_{a\sigma}\,S_\rho{}^{\nu\sigma}\right). \tag{D.3}$$

Also, the field equation (D.1) can be written in a purely space-time form

$$\partial_\mu(h\,S_{(\lambda\gamma)}{}^\mu) - \kappa^2\,h\,\tilde{j}_{\lambda\gamma} = \frac{\kappa^2}{2}\,h\,\Theta_{\lambda\gamma}. \tag{D.4}$$

where $(\lambda\gamma)$ denotes symmetric part ($[\lambda\gamma]$ anti-symmetric part) and

$$\tilde{j}_{\lambda\gamma} = j_{\lambda\gamma} - \frac{1}{\kappa^2\,h}\,\left[\partial_\mu(h\,S_{[\lambda\gamma]}{}^\mu) + h\,h^a{}_\lambda\,g_{\nu\gamma}\,S_{\tau\eta}{}^\mu\,\partial_\mu(h_a{}^\tau\,g^{\nu\eta})\right] \tag{D.5}$$

with $j_{\lambda\gamma} = h^a{}_\lambda\,g_{\gamma\nu}\,j_a{}^\nu$. Using (total covariant derivative of tetrad field)

$$\partial_\mu h^b{}_\gamma = \Gamma^\rho{}_{\gamma\mu}\,h^b{}_\rho - A^b{}_{c\mu}\,h^c{}_\gamma \tag{D.6}$$

and since

$$\partial_\mu h_a{}^\tau = -h_a{}^\gamma\, h_b{}^\tau\, \partial_\mu h^b{}_\gamma \tag{D.7}$$

thus

$$\partial_\mu h_a{}^\tau = -\Gamma^\tau{}_{a\mu} + A^\tau{}_{a\mu}. \tag{D.8}$$

Using (D.8) it is calculated the last term in (D.5)

$$\begin{aligned} h^a{}_\lambda\, g_{\nu\gamma}\, S_{\tau\eta}{}^\mu\, \partial_\mu(h_a{}^\tau\, g^{\eta\nu}) = &-S_{\tau\gamma}{}^\mu\, \Gamma^\tau{}_{\lambda\mu} - S_\lambda{}^{\tau\mu}\, \Gamma_{\gamma\tau\mu} - S_{\lambda\tau}{}^\mu\, \Gamma^\tau{}_{\gamma\mu} \\ &+ S_{\tau\gamma}{}^\mu\, A^\tau{}_{\lambda\mu} + S_\lambda{}^{\tau\mu}\, A_{\gamma\tau\mu} + S_{\lambda\tau}{}^\mu\, A^\tau{}_{\gamma\mu}. \end{aligned} \tag{D.9}$$

Substituting (D.3) and (D.9) in (D.5) it is find that

$$\kappa^2\, \tilde{j}_{\lambda\gamma} = -\frac{1}{4}\, g_{\lambda\gamma}\, T + S_{\tau\gamma}{}^\mu\, \Gamma^\tau{}_{\mu\lambda} + S_\lambda{}^{\tau\mu}\, \Gamma_{\gamma\tau\mu} + S_{\lambda\tau}{}^\mu\, \Gamma^\tau{}_{\gamma\mu} - \frac{\kappa^2}{h}\, \partial_\mu \sigma^\mu{}_{\lambda\gamma}. \tag{D.10}$$

where we use the fact of that $A_{ab\mu}$ is a Lorentz connection, that is, $A_{ab\mu} = -A_{ba\mu}$. The connection-terms transforms the gauge current $j_a{}^\rho$ into the pseudotensor $\tilde{j}_{\lambda\gamma}$. The tensor $\sigma^\mu_{\lambda\gamma}$ in (D.10) it is the spin tensor of the gravitational field

$$\sigma^\mu{}_{\lambda\gamma} \equiv -\frac{\partial \mathcal{L}_G}{\partial\partial_\mu h^a{}_\sigma}\, \frac{\delta h^a{}_\sigma}{\delta \epsilon^{\lambda\gamma}} = \frac{h}{\kappa^2}\, S_{[\lambda\gamma]}{}^\mu = -\frac{h}{4\,\kappa^2}\left(T^\mu_{\lambda\gamma} + \delta^\mu_{\ \lambda}\, T^\tau_{\ \gamma\tau} - \delta^\mu_{\ \gamma}\, T^\tau_{\ \lambda\tau}\right). \tag{D.11}$$

The anti-symmetric part of (D.10) is given by

$$\begin{aligned} \kappa^2\, \tilde{j}_{[\lambda\gamma]} = &-\frac{1}{4}\, \Gamma^\epsilon{}_{\phi\lambda}\, \Gamma^\phi{}_{\gamma\epsilon} + \frac{1}{4}\, \Gamma^\epsilon{}_{\gamma\epsilon}\, \Gamma^\phi{}_{\phi\lambda} - \frac{1}{4}\, \Gamma^\epsilon{}_{\gamma\lambda}\, \Gamma^\phi{}_{\phi\epsilon} + \frac{1}{4}\, \Gamma^\epsilon{}_{\gamma\lambda}\, \Gamma^\phi{}_{\epsilon\phi} \\ &+ \frac{1}{4}\, \Gamma^\epsilon{}_{\phi\gamma}\, \Gamma^\phi{}_{\lambda\epsilon} - \frac{1}{4}\, \Gamma^\epsilon{}_{\lambda\epsilon}\, \Gamma^\phi{}_{\phi\gamma} + \frac{1}{4}\, \Gamma^\epsilon{}_{\lambda\gamma}\, \Gamma^\phi{}_{\phi\epsilon} - \frac{1}{4}\, \Gamma^\epsilon{}_{\lambda\gamma}\, \Gamma^\phi{}_{\epsilon\phi} - \frac{\kappa^2}{h}\, \partial_\epsilon \sigma^\epsilon_{\lambda\gamma}. \end{aligned} \tag{D.12}$$

As we will see below, the above anti-symmetric part is exactly a contraction of the teleparallel Bianchi identity and thus is identically equal to zero. The Bianchi identity of TG is given by

$$D_\nu T^a{}_{\rho\mu} + D_\mu T^a{}_{\nu\rho} + D_\rho T^a{}_{\mu\nu} = 0, \tag{D.13}$$

where $D_\nu T^a{}_{\rho\mu} = \partial_\nu T^a{}_{\rho\mu} + A^a{}_{b\nu}\, T^b{}_{\rho\mu}$ is Fock-Ivanenko derivative. The Bianchi identity can be rewritten as

$$D_\nu T^\sigma{}_{\rho\mu} + D_\mu T^\sigma{}_{\nu\rho} + D_\rho T^\sigma{}_{\mu\nu} + \Gamma^\sigma{}_{\gamma\nu}\, T^\gamma{}_{\rho\mu} + \Gamma^\sigma{}_{\gamma\mu}\, T^\gamma{}_{\nu\rho} + \Gamma^\sigma{}_{\gamma\rho}\, T^\gamma{}_{\mu\nu} = 0, \tag{D.14}$$

and contracting $\sigma = \nu$ it is seen that

$$D_\nu\big(T^\nu{}_{\rho\mu} + \delta^\nu{}_\rho\, T^\gamma_{\mu\gamma} - \delta^\nu{}_\mu\, T^\gamma_{\rho\gamma}\big) + \Gamma^\nu{}_{\gamma\nu}\, T^\gamma_{\rho\mu} + \Gamma^\nu{}_{\gamma\mu}\, T^\gamma_{\nu\rho} + \Gamma^\nu{}_{\gamma\rho}\, T^\gamma_{\mu\nu} = 0. \tag{D.15}$$

Using (D.11) the equation (D.15) can be rewritten as

$$\frac{\kappa^2}{h}\,\partial_\nu \sigma^\nu{}_{\rho\mu} - \frac{\kappa^2}{h}\,\partial_\nu h\, \sigma^\nu{}_{\rho\mu} - \frac{1}{4}\,\Gamma^\nu{}_{\gamma\nu}\, T^\gamma_{\rho\mu} - \frac{1}{4}\,\Gamma^\nu{}_{\gamma\mu}\, T^\gamma_{\nu\rho} - \frac{1}{4}\,\Gamma^\nu{}_{\gamma\rho}\, T^\gamma_{\mu\nu} = 0. \tag{D.16}$$

Using again equation (D.11) together with the identity

$$\partial_\nu h = h\,\left(\Gamma^\gamma_{\nu\gamma} - K^\gamma_{\nu\gamma}\right), \tag{D.17}$$

the equation (D.16) can be rewritten as

$$\begin{aligned} &-\frac{1}{4}\,\Gamma^\epsilon{}_{\phi\lambda}\,\Gamma^\phi{}_{\gamma\epsilon} + \frac{1}{4}\,\Gamma^\epsilon{}_{\gamma\epsilon}\,\Gamma^\phi{}_{\phi\lambda} - \frac{1}{4}\,\Gamma^\epsilon{}_{\gamma\lambda}\,\Gamma^\phi{}_{\phi\epsilon} + \frac{1}{4}\,\Gamma^\epsilon{}_{\gamma\lambda}\,\Gamma^\phi{}_{\epsilon\phi} \\ &\quad + \frac{1}{4}\,\Gamma^\epsilon{}_{\phi\gamma}\,\Gamma^\phi{}_{\lambda\epsilon} - \frac{1}{4}\,\Gamma^\epsilon{}_{\lambda\epsilon}\,\Gamma^\phi{}_{\phi\gamma} + \frac{1}{4}\,\Gamma^\epsilon{}_{\lambda\gamma}\,\Gamma^\phi{}_{\phi\epsilon} - \frac{1}{4}\,\Gamma^\epsilon{}_{\lambda\gamma}\,\Gamma^\phi{}_{\epsilon\phi} - \frac{\kappa^2}{h}\,\partial_\epsilon\sigma^\epsilon_{\lambda\gamma} = 0, \end{aligned} \tag{D.18}$$

which is the right side of equation (D.12). Therefore, it is see that the pseudotensor $\tilde{j}_{\lambda\gamma}$ is symmetric and the gravitational field equation is also symmetric. Finally, the gravitational field equation of TG (D.1) can also be rewritten in the purely spacetime form

$$\partial_\mu(h\, S_{\lambda\gamma}{}^\mu) - \kappa^2\, h\, t_{\lambda\gamma} = \frac{\kappa^2}{2}\, h\, \Theta_{\lambda\gamma}, \tag{D.19}$$

where by using (D.10)

$$\kappa^2\, t_{\lambda\gamma} = -\frac{1}{4}\, g_{\lambda\gamma}\, T + S_{\tau\gamma}{}^\mu\, \Gamma^\tau{}_{\mu\lambda} + S_\lambda{}^{\tau\mu}\, \Gamma_{\gamma\tau\mu} + S_{\lambda\tau}{}^\mu\, \Gamma^\tau{}_{\gamma\mu}, \tag{D.20}$$

is the canonical energy momentum pseudotensor which has anti-symmetric part

$$t_{[\lambda\gamma]} = \frac{1}{h}\,\partial_\mu \sigma^\mu{}_{\lambda\gamma}. \tag{D.21}$$

The anti-symmetric part of the canonical energy-momentum pseudotensor measures the breaking of pure-spin conservation. The anti-symmetric part of the canonical energy-momentum pseudotensor cancels the anti-symmetric part of the first term in the field equation (D.19).

Appendix E

The Conservation Law of the Source Energy-Momentum Tensor

Following [1], under an arbitrary variation $\delta_a{}^\mu$ of the tetrad field, the action variation is written in the form

$$\delta S_m = \int d^4x\, \Theta^a{}_\mu\, \delta h_a{}^\mu\, h \tag{E.1}$$

where

$$\Theta_a{}^\rho = -\frac{1}{h}\frac{\delta \mathcal{S}_m}{\delta h^a{}_\rho} = -\frac{1}{h}\left(\frac{\partial \mathcal{L}_m}{\partial h^a{}_\rho} - \partial_\lambda \frac{\partial \mathcal{L}_m}{\partial \partial_\lambda h^a{}_\rho}\right), \tag{E.2}$$

is the matter energy-momentum tensor. The invariance of the action under local Lorentz transfomation requires that $\Theta_{\mu\nu} \equiv h^a{}_\mu\, g_{\nu\rho}\, \Theta_a{}^\rho$ must be symmetric.

Now, let us consider a general transfomation of the spacetime coordinates,

$$x'^\rho = x^\rho + \epsilon^\rho. \tag{E.3}$$

Under such transfomation, the tetrad changes according to

$$\delta h_a{}^\mu \equiv h_a^{'}{}^\mu(x) - h_a{}^\mu(x) = h_a{}^\rho\, \partial_\rho \xi^\mu - \xi^\rho\, \partial_\rho h_a{}^\mu. \tag{E.4}$$

Substituting in (E.2), we obtain

$$\delta \mathcal{S}_m = \int d^4x\, \Theta^a{}_\mu\, \left[h_a{}^\rho\, \partial_\rho \xi^\mu - \xi^\rho\, \partial_\rho h_a{}^\mu\right]\, h \tag{E.5}$$

or equivalently

$$\delta \mathcal{S}_m = \int d^4x\, \left[\Theta^\rho{}_c\, \partial_\rho \xi^c + \xi^c\, \Theta^\rho{}_\mu\, \partial_\rho h^c{}_\mu - \xi^\rho\, \partial_\rho h_a{}^\mu\right]\, h \tag{E.6}$$

Substituting the identity

$$\partial_\rho h_a{}^\mu = A^b{}_{a\rho}\, h_b{}^\mu - \Gamma^\mu{}_{\lambda\rho}\, h_a{}^\lambda, \tag{E.7}$$

and making use of the symmetry of the energy-momentum tensor, the action variation assumes the form

$$\delta\mathcal{S}_m = \int d^4x\, \Theta_c{}^\rho \left[\partial_\rho \xi^c + \left(A^c{}_{b\rho} - K^c{}_{b\rho}\right)\, \xi^b\right]\, h. \tag{E.8}$$

Integrating the first term by parts, neglecting the surface term, the invariance of the action yields and the arbitrariness of ξ^c yields

$$\mathcal{D}_\mu\left(h\,\Theta_a{}^\mu\right) \equiv \partial_\mu\left(h\,\Theta_a{}^\mu\right) - \left(A^b{}_{a\mu} - K^b{}_{a\mu}\right)\left(h\,\Theta_b{}^\mu\right) = 0. \tag{E.9}$$

Making use of the identity

$$\partial_\rho h = h\,\bar{\Gamma}^\nu{}_{\nu\rho} \equiv h\,\left(\Gamma^\nu{}_{\rho\nu} - K^\nu{}_{\rho\nu}\right), \tag{E.10}$$

the above conservation law (E.9) becomes

$$\partial_\mu \Theta_a{}^\mu + \left(\Gamma^\mu{}_{\rho\mu} - K^\mu{}_{\rho\mu}\right)\, \Theta_a{}^\rho - \left(A^b{}_{a\mu} - K^b{}_{a\mu}\right)\, \Theta_b{}^\mu = 0. \tag{E.11}$$

In a purely spacetime form, it reads

$$\nabla_\mu \Theta_\lambda{}^\nu \equiv \partial_\mu \Theta_\lambda{}^\mu + \left(\Gamma^\mu{}_{\rho\mu} - K^\mu{}_{\rho\mu}\right)\, \Theta_\lambda^\rho - \left(\Gamma^\rho{}_{\lambda\mu} - K^\rho{}_{\lambda\mu}\right)\, \Theta_\rho{}^\mu = 0. \tag{E.12}$$

This is the conservation law of the source energy-momentum tensor in TG. Due to the relation

$$\Gamma^\rho_{\mu\nu} = \bar{\Gamma}^\rho_{\mu\nu} + K^\rho_{\mu\nu}, \tag{E.13}$$

it coincides with the corresponding conservation law of GR

$$\bar{\nabla}\Theta_\lambda{}^\mu \equiv \partial_\mu \Theta_\lambda{}^\mu + \bar{\Gamma}^\mu{}_{\rho\mu}\, \Theta_\lambda{}^\rho - \bar{\Gamma}^\rho{}_{\lambda\mu}\, \Theta_\rho{}^\mu = 0. \tag{E.14}$$

Appendix F

Conformal Equivalent of Teleparallel Dark Energy

Let us consider a conformal transformation of the metric, defined as

$$\hat{g}_{\mu\nu} = \Omega^2(x)\, g_{\mu\nu}, \qquad \hat{g}^{\mu\nu} = \Omega^{-2}\, g^{\mu\nu}, \tag{F.1}$$

where $\Omega(x)$ is a smooth, non-vanishing function of the space-time point, lying in the range $0 < \Omega < \infty$ [53] (See Refs. [56–58] for conformal transformations in TG and $f(T)$ theories). Also, we will use a caret to indicate quantities in the transformed frame.

From equation (F.1) we have that [57]

$$\hat{h}^a{}_\mu = \Omega\, h^a{}_\mu, \quad \hat{h}_a{}^\mu = \Omega^{-1}\, h_a{}^\mu, \qquad h^a{}_\mu = \Omega^{-1}\, \hat{h}^a{}_\mu, \quad h_a{}^\mu = \Omega\, \hat{h}_a{}^\mu, \tag{F.2}$$

with $\hat{h} = \Omega^4\, h$. By using (F.2) and the definition of torsion tensor (1.8) we find that this is transformed as

$$\hat{T}^\rho_{\mu\nu} = T^\rho_{\mu\nu} + \Omega^{-1}\, \left(\delta^\rho_\nu\, \partial_\mu\Omega - \delta^\rho_\mu\, \partial_\nu\Omega\right). \tag{F.3}$$

On the other hand, the teleparallel lagrangian (1.7) can be rewritten as

$$T = \frac{1}{4}\, T_\rho{}^{\mu\nu}\, T^\rho_{\mu\nu} + \frac{1}{2}\, T^{\rho\mu}{}_\nu\, T^\nu_{\mu\rho} - \mathcal{V}_\mu\, \mathcal{V}^\mu. \tag{F.4}$$

Using (F.1) in (F.4) it is seen the torsion scalar is transformed as

$$\hat{T} = \Omega^{-2}\, T - 6\,\Omega^{-4}\, \partial_\mu\Omega\, \partial^\mu\Omega + 4\,\Omega^{-3}\, \partial_\mu\Omega\, \mathcal{V}^\mu, \tag{F.5}$$

with the vector torsion transformed as

$$\hat{\mathcal{V}}_\mu = \mathcal{V}_\mu - 3\,\Omega^{-1}\,\partial_\mu\Omega, \tag{F.6}$$

whereas the inverse transformation for (F.5) is given by

$$T = \Omega^2\,\hat{T} - 6\,\hat{\partial}_\mu\Omega\,\hat{\partial}^\mu\Omega - 4\,\Omega\,\hat{\partial}_\mu\Omega\,\hat{\mathcal{V}}^\mu. \tag{F.7}$$

Since x^μ is unaffected by the conformal transformation thus $\partial_\mu = \hat{\partial}_\mu$.

In what follows we consider the general scalar torsion action

$$S = \int d^4x\,h\,\left[F(\phi)\,T + \frac{1}{2}\,\zeta(\phi)\,\partial_\mu\phi\,\partial^\mu\phi - V(\phi)\right] + S_m(\psi_i, h^a{}_\mu). \tag{F.8}$$

in analogy with the general action in GR which includes a wide variety of gravity models such as Brans-Dicke theories, nonminimally coupled scalar fields and dilaton gravity [16]. The teleparallel dark energy model corresponds to $F(\phi) = \frac{1}{\kappa^2} + f(\phi)$ and $\zeta(\phi) = 1$. By using (F.2) and (F.7) in (F.8) we find

$$\hat{S} = \int d^4x\,\hat{h}\,\left[\frac{\hat{T}}{\kappa^2} + \frac{1}{2}\,\hat{\partial}_\mu\hat{\phi}\,\hat{\partial}^\mu\hat{\phi} - U(\hat{\phi}) + B(\hat{\phi})\,\hat{\partial}_\mu\hat{\phi}\,\hat{\mathcal{V}}^\mu\right] + S_m(\hat{\phi}, \hat{\psi}_m, \hat{h}^a{}_\mu) \tag{F.9}$$

where it is considered $F(\phi) = \Omega^2\,\kappa^{-2}$ and it is introduced a new scalar field $\hat{\phi}$ in order to make the kinetic term canonical

$$\hat{\phi} \equiv \int d\phi\,\sqrt{\frac{\zeta}{\kappa^4\,F^2} - 3\,\left(\frac{F_{,\phi}}{\kappa\,F}\right)^2}. \tag{F.10}$$

Also we defined

$$U(\hat{\phi}) \equiv V(\phi)\,\Omega^{-4}, \qquad B(\hat{\phi}) \equiv -\frac{2\,F_{,\phi}}{\kappa^2\,F\,\sqrt{\frac{\zeta}{\kappa^4\,F^2} - 3\,\left(\frac{F'}{\kappa\,F}\right)^2}}. \tag{F.11}$$

So, the scalar torsion theory (F.8) is not dynamically equivalent to the teleparallel action plus a scalar field via conformal transformation [57]. It is equivalent to a non-minimally coupled scalar field (through the four derivative) with explicit coupling to matter. The nonminimal coupling term in (F.9), which cannot be removed by conformal transformation, is natural in TG since the scalar field as a particle of spin zero interacts with torsion through the four derivative [1].

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