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de Sitter-Invariant Approach to Cosmology

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Abstract

A crucial problem of spacetime physics is the lack of a special relativity that could describe the Planck scale kinematics. The problem stems from the existence of an invariant length at the Planck scale, dubbed Planck length. Since the Poincaré-invariant Einstein's special relativity does not admit an invariant length, it cannot describe the Planck scale kinematics. A possible solution to this puzzle is to replace the Poincaré-invariant Einstein's special relativity with a de Sitter-invariant special relativity. Considering that the latter admits a finite invariant length while preserving the Lorentz symmetry, it meets the quantum requirements of the Planck scale physics and provides a consistent description of the Planck scale kinematics. Under such replacement, general relativity changes to the de Sitter-invariant general relativity, which differs substantially from ordinary general relativity. In this thesis, the associated de Sitter-invariant Friedmann equations are obtained, and a discussion of the main consequences for cosmology is presented.

Key words: de Sitter invariant special relativity; de Sitter modified gravitational theory; dark energy problem; cosmology.

Knowledge area: Exact and Earth Science.

Resumo

Um problema crucial da física do espaço-tempo é a ausência de uma relatividade especial que possa descrever a cinemática na escala de Planck. Esse problema decorre da existência de um comprimento invariante na escala de Planck, conhecido como comprimento de Planck. Como a cinemática da relatividade especial de Einstein é governada pelo grupo de Poincaré, ela não admite um comprimento invariante e, por isso, ela é incapaz de descrever a cinemática na escala de Planck. Uma possível solução para esse problema é substituir a relatividade especial invariante sob Poincaré por uma relatividade especial invariante sob de Sitter. Como esta última admite um comprimento invariante, ao mesmo tempo em que preserva a simetria de Lorentz, ela atende aos requisitos quânticos da física da escala de Planck e, por esta razão, fornece uma descrição consistente da cinemática na escala de Planck. Quando a substituição acima é efetuada, a relatividade geral muda para a relatividade geral invariante sob de Sitter, que difere substancialmente da relatividade geral ordinária. Nesta tese, as equações de Friedmann invariantes por de Sitter são obtidas e uma discussão das principais consequências para a cosmologia é apresentada.

Palavras Chaves: Relatividade especial de de Sitter; Teoria gravitacional modificada por de Sitter; problema da energia escura; cosmologia.

Área de conhecimento: Ciências Exatas e da Terra.

Notation

The notation that will be used in this thesis is the following:

- Capital latin indices $A, B, C...$ run over five space-time coordinates $(0, 1, 2, 3, 4)$.
- Latin indices $i, j, k...$ run over three spatial coordinates $(1, 2, 3)$.
- Greek indices μ, ν, ρ, σ run over space-time coordinates $(0, 1, 2, 3, 4)$.
- The Ricci curvature defined as
  $R_{\mu\nu} = \partial_\nu \Gamma^\lambda_{\ \mu\lambda} - \partial_\lambda \Gamma^\lambda_{\ \mu\nu} + \Gamma^\lambda_{\ \mu\sigma}\Gamma^\sigma_{\ \nu\lambda} - \Gamma^\lambda_{\ \mu\nu}\Gamma^\sigma_{\ \lambda\sigma}$.

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Introduction

The rationale for a de Sitter special relativity

In the 17th century, the Newtonian theory gained a high degree of trustworthiness because it could adequately explain classical mechanics and gravitation. Also, this theory went hand in hand with the first kinematic group in physics history known as the Galileo group, a transformation group 10-dimensional, formed by: three rotations, three spatial translations, one time-translation, and three Galilean boost transformations, this group of transformations left invariant the physics laws, based on absolute time and space.

With the advent of Maxwell's equations, a remarkable conclusion was found: the existence of electromagnetic waves that travel at the speed of light. This discovery brought a theoretical conflict between Maxwell's theory and Galilei's group.

To solve such a conflict, a new kinematic group was necessary, one where the speed of light c takes part in the local kinematics, thus allowing the passage from Galilean relativity to Einstein's special relativity. The latter includes the principle of invariant light speed, which states that light always propagates in empty space with a definite velocity c and is independent of the state of motion of the emitting body. Einstein's special relativity changed the fundamental definitions of space and time intervals and abandoned the idea of absolute time.

Local symmetry of Einstein's special relativity is ruled by the Poincaré group, which contains two subgroups: the Lorentz group $\mathcal{L}$, which is in charge of preserving the symmetries associated with rotations and boosts, and the translation group $\mathcal{T}$, which is a four-dimensional Abelian group that is in charge of enforcing the isotropy around any other point in a homogeneous space. Mathematically, Poincaré group is defined as $\mathcal{P} = \mathcal{L} \oslash \mathcal{T}$, the semi-direct product of Lorentz and the translation group $\mathcal{T}$.

It is well-known that the Poincaré group offers a good understanding of elementary particle physics because all particles can be classified through the irreducible representation of the Poincaré group. For that reason, the Poincaré group represents an exact symmetry of Nature, and Einstein's special relativity represents the kinematic group consistent with all the velocity ranges, from low velocities up to the speed of light c. However, Einstein's special relativity faces experimental and theoretical problems. On the experimental side, the problem comes from the propagation of very high-energy photons. More specifically, from ultra-high-energy extragalactic gamma-ray flares, which seem to travel slower than lower-energy ones. These experiments established that, if the photons' speed does deviate from the speed of light, it is by a factor of less than 1.7×10^{-15} [1, 2].

On the theoretical side, at the Planck scale [3], the problem is that quantum mechanics predicts the existence of a Lorentz invariant length-scale, known as Planck length, which is interpreted as the shortest physical distance that exists in nature. Such a length clashes with Einstein's special relativity because the Lorentz group does not allow the existence of such an invariant parameter. Based on the above problems, it is perfectly valid to think that the Lorentz symmetry must be broken at the Planck scale, or another relativistic theory must exist to describe the kinematics at this energy scale.

Alternative models have been proposed to solve such the conflict. Amelino-Camelia proposed an interesting model after being continued by João Magueijo and Lee Smolin, called *doubly special relativity*. This model was constructed by introducing in the dispersion relation of the special relativity, scale-suppressed terms of higher-order in the momentum to allow an invariant length at the Planck scale. The importance of these terms is determined by a parameter κ , which essentially modifies the kinematic group of special relativity from Poincaré to a κ-deformed Poincairé group [4, 5]

$$E^2 - p^2c^2 - m^2c^4 + f(E, p, m; l_p) = 0. \tag{1}$$

Far away from the Planck scale, these terms are suppressed, and Lorentz symmetry is recovered. As a result, we obtain special relativity back.

A possible weakness of this model is that the Lorentz symmetry is explicitly violated. The problem is that a violation of the Lorentz symmetry is easy to accept. Since it is deeply related to causality [6], any violation of the first implies a violation of the second. The point is that causality is one of the most fundamental principles of physics. Even if it is violated preponderantly at the Planck scale, we cannot be sure Nature is prepared to afford it.

A different solution to the same problem is to assume that, instead of being governed by the Poincaré, the spacetime kinematics is governed by the de Sitter group. Thus, if the underlying spacetime is de Sitter, the local kinematics will be ruled by the de Sitter group, which amounts to replacing ordinary special relativity with de Sitter-invariant special relativity [7, 8]. It is important to mention that this approach is not as new as it is believed to be. The first ideas about de Sitter's special relativity are due to L. Fantappié, who in 1952 introduced what he called *Projective Relativity*, a theory that G. Arcidiacono further developed. Readers interested in historical details can consult Ref. [9]. Since the Lorentz group is usually blamed for the non-existence of an invariant length, and considering that it is also a subgroup of de Sitter, how is it possible that in a de Sitter-invariant kinematics the existence of an invariant length turns out to be possible?

To understand this point, let us first recall that Lorentz transformations can be performed only in *homogeneous spacetimes*. In addition to Minkowski, therefore, they can be performed in de Sitter[1] and anti-de Sitter spaces, which are the unique homogeneous spacetimes in $(1+3)$-dimensions [10]. As a maximally symmetric space, the de Sitter spacetime has constant sectional curvature

$$R = 12\, l^{-2}\,, \tag{2}$$

where l is a length parameter, also known as pseudo-radius.

By definition, Lorentz transformations do not change the curvature of the homogeneous spacetime in which they are performed. Since the scalar curvature is given by (2), *Lorentz transformations are found to leave the length parameter l invariant* [11]. Although somewhat hidden in Minkowski spacetime, because what is invariant in this case is an infinite length, in de Sitter spacetime, whose pseudo-radius is finite, such a property becomes transparent. Therefore,

> *Lorentz transformations do leave invariant a very particular length parameter: that defining the scalar curvature of the homogeneous spacetime in which they are performed.*

If the Planck length l_P is to be invariant under Lorentz transformations, it must then represent the pseudo-radius of spacetime at the Planck scale, which will be a

[1]Hereafter, our interest will be restricted to the de Sitter case because it has a positive cosmological constant and presents appropriate properties to deal with cosmology and quantum gravity.

de Sitter space with a Planck cosmological term

$$\Lambda_P = 3/l_P^2 \simeq \pm 1.2 \times 10^{70}\,\mathrm{m}^{-2}. \tag{3}$$

In the de Sitter invariant special relativity, therefore, the existence of an invariant length parameter at the Planck scale does not clash with the Lorentz invariance, which remains a symmetry at all scales. Consequently, causality is always preserved, even at the Planck scale.

Instead of Lorentz, translation invariance is broken down. In this theory, physics turns out to be invariant under the so-called *de Sitter translations*, which in stereographic coordinates are given by a combination of translations and proper conformal transformations [12]. We can then say that in the same way Einstein's special relativity may be thought of as a generalization of Galilei relativity for velocities near the speed of light, the de Sitter invariant special relativity may be thought of as a generalization of Einstein's special relativity for high energies. Nevertheless, this can describe physics at all energy scales.

General Purposes

Cosmological observations indicate that the universe expansion is accelerating [13, 14, 15]. The most straightforward and favored approach to explain this acceleration is to suppose the existence of a cosmological constant Λ. This is the case, for example, of the Λ-CDM model, where the cosmological constant Λ represents the dark energy. Despite the Λ-CDM being successful in meeting most of the cosmological data, it faces troubles with a growing number new observational data [16, 17, 18].

On the other hand, as quotient spaces, Minkowski and de Sitter are fundamental spacetimes in the sense that they can be algebraically obtained without the need for Einstein's equations. As fundamental spaces, they represent a background for constructing physical theories. For example, general relativity can be constructed on any of them. In either case, the dynamics will be the same: only the local kinematics will differ.

If the background space is Minkowski, the local kinematics will be governed by the Poincaré-invariant Einstein's special relativity. In this case, the solutions of Einstein's equation will be spacetimes that reduce locally to Minkowski. If the background space is de Sitter, the local kinematics will be governed by the de

Sitter-invariant special relativity. In this case, the solutions to the corresponding gravitational field equation, will be spacetimes that reduce locally to de Sitter. The corresponding gravitational theory is the de *Sitter-modified general relativity*. By considering this theory, the primary purpose of this thesis is to derive the de Sitter-modified Friedmann equations and study some implications for cosmology.

The structure of the thesis is as follows.

- Chapter 1 will provide the theoretical framework for the de Sitter space and group.
- Chapter 2 will review how the de Sitter-modified general relativity is obtained.
- Chapter 3 will be devoted to studying the equation of motion and conservation laws of the cosmological fluid, assumed to be a perfect fluid.
- Chapter 4 will be devoted to obtaining the de Sitter-modified Friedmann equations and discussing some physical consequences.
- Chapter 5 will present the conclusions and discuss future perspectives.

Part I

Theoretical Framework

Chapter 1

Basic Notions: de Sitter space, group, and all that

At the Planck length, the spacetime short-distance structure is ruled by a Lorentz invariant length scale, known as *Planck length*. Since Einstein's special relativity does not admit such an invariant length, it cannot describe the Planck scale kinematics. A solution to this inconsistency is to replace Einstein's special relativity with de Sitter-invariant special relativity, which means the de Sitter group will now rule spacetime's local kinematics. Note that this replacement does not change the kinematic group dimension.

The de Sitter invariant special relativity naturally incorporates two invariant quantities, the speed of light and the Lorentz invariant length. An important aspect of this theory is that it lives between two limit cases: the first corresponds to large values of de Sitter length parameter l, which translates into tiny values of the cosmological constant Λ. The second corresponds to small values of the de Sitter parameter l, which translates into large values of cosmological constant Λ.

Considering the above context, we aim to review in this chapter the theoretical framework of de Sitter invariant special relativity. It is structured in three sections. Section 1.1 outlines the main characteristics of maximally symmetric spaces, emphasizing the de Sitter spacetime. Section 1.2 describes the structure of the de Sitter group and its principal properties. Afterward, we will discuss the possible contractions of the de Sitter group employing the **Inönü-Wigner procedure**. Section 1.3 describes the de Sitter Killing vectors.

1.1 de Sitter spacetime

Spacetimes with constant sectional curvature are maximally symmetric because they can host the maximum number of Killing vectors. Their curvature tensor is entirely specified by the scalar curvature R, which is constant throughout. The simplest maximally symmetric spacetime is Minkowski M, with vanishing scalar curvature, $R = 0$. Apart from Minkowski, there are two other maximally symmetric four-dimensional spacetimes: de Sitter, with a positive scalar curvature, and anti-de Sitter space, with negative scalar curvature. In this thesis, our interest will be restricted to the de Sitter spacetime and group.

The de Sitter spacetime, denoted dS, is a maximally symmetric spacetime that can be seen as hypersurface embedded in a $(1+4)$-dimensional pseudo-Euclidean space, whose points in Cartesian coordinates χ^A $(A, B = 0, ..., 4.)$ satisfy the relation [19, 20]

$$\eta_{AB}\chi^A\chi^B = -l^2 \tag{1.1}$$

where $\eta_{AB} = (+1, -1, -1, -1, -1)$ and l stands for the de Sitter length parameter (or pseudo radius). In terms of four-dimensional coordinates, it assumes the form

$$\eta_{\mu\nu}\chi^\mu\chi^\nu - (\chi^4)^2 = -l^2. \tag{1.2}$$

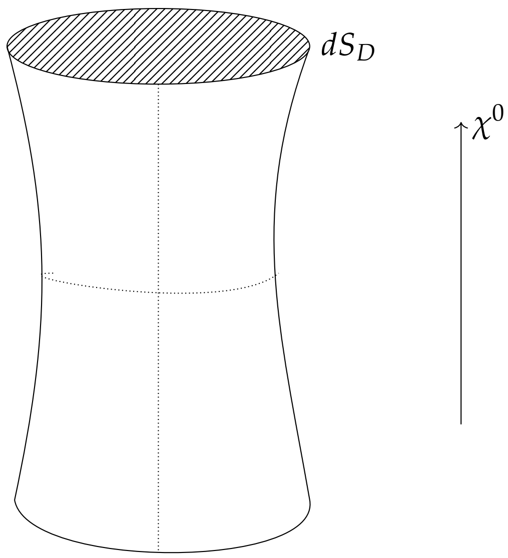

Figure 1.1: *Graphical representation of de Sitter hyperbolid.*

1.1.1 Stereographic coordinates

The de Sitter space possesses several coordinate systems [21] each one with relevant attributes. This thesis will pay attention to stereographic coordinates $\{x^\mu\}$ because they represent a natural generalization of Minkowski Cartesian coordinates. In fact, for a vanishing cosmological term, they reduce to the Cartesian coordinates of Minkowski spacetime. The stereographic coordinates are obtained through a stereographic projection from the de Sitter hypersurface into a target Minkowski spacetime.

The stereographic projection present the form [22]

$$\chi^\mu = \Omega x^\mu \,, \qquad \text{and} \qquad \chi^4 = -l\Omega(1+\sigma^2/4l^2) \tag{1.3}$$

where

$$\Omega = (1-\sigma^2/4l^2)^{-1} \,, \tag{1.4}$$

with σ^2 the Lorentz invariant quadratic form

$$\sigma^2 = \eta_{\mu\nu}x^\mu x^\nu. \tag{1.5}$$

In terms of the coordinates χ^A, the de Sitter metric is written as

$$ds^2 = \eta_{AB}d\chi^A d\chi^B \,. \tag{1.6}$$

In terms of the stereographic coordinates x^μ, it reads

$$d\hat{s}^2 = \hat{g}_{\mu\nu}dx^\mu dx^\nu \tag{1.7}$$

where the de Sitter metric components are

$$\hat{g}_{\mu\nu} = \Omega^2\eta_{\mu\nu} \,, \qquad \hat{g}^{\mu\nu} = \Omega^{-2}\eta^{\mu\nu}. \tag{1.8}$$

The Levi-Civita connection of the metric (1.8) is given by

$$\hat{\Gamma}^\lambda{}_{\mu\nu} = \frac{\Omega}{2l^2}\left(\delta^\lambda_\mu\,\eta_{\nu\alpha}\,x^\alpha + \delta^\lambda_\nu\,\eta_{\mu\alpha}\,x^\alpha - \eta_{\mu\nu}\,x^\lambda\right). \tag{1.9}$$

The corresponding Riemann tensor, defined as

$$R^\mu{}_{\nu\rho\sigma} = \partial_\sigma\Gamma^\mu{}_{\nu\rho} - \partial_\rho\Gamma^\mu{}_{\nu\sigma} + \Gamma^\mu{}_{\alpha\sigma}\Gamma^\alpha{}_{\nu\rho} - \Gamma^\mu{}_{\alpha\rho}\Gamma^\alpha{}_{\nu\sigma}, \tag{1.10}$$

assumes the form

$$\hat{R}^{\mu}{}_{\nu\rho\sigma} = \frac{\Omega^2}{l^2}\left(\delta^{\mu}_{\rho}\eta_{\nu\sigma} - \delta^{\mu}_{\sigma}\eta_{\nu\rho}\right). \tag{1.11}$$

The Ricci tensor and the scalar curvature are, consequently

$$\hat{R}_{\mu\nu} = \frac{3\Omega^2}{l^2}\eta_{\mu\nu}\,, \qquad \text{and} \qquad \hat{R} = \frac{12}{l^2}\,. \tag{1.12}$$

1.2 de Sitter group and algebra

Minkowski spacetime, with vanishing curvature, is the simplest maximally symmetric spacetime. Its kinematic group is the Poincaré group $\mathcal{P} = \mathcal{L} \oslash \mathcal{T}$, the semi-direct product between Lorentz $\mathcal{L}$ and the translation group $\mathcal{T}$. Mathematically, Minkowski is defined as a quotient space [23]

$$M = \mathcal{P}/\mathcal{L}\,. \tag{1.13}$$

The Lorentz subgroup is responsible for the isotropy around a given point of M. The translation symmetry enforces this isotropy around all the other points. Homogeneity implies that all points of Minkowski are equivalent under spacetime translations, whose generators are

$$P_\rho = \delta^{\alpha}_{\rho}\partial_\alpha\,, \tag{1.14}$$

with δ^{α}_{ρ} the Killing vector of translations. Since the translations Killing vectors are Kronecker delta's, the translation generators are usually written in the form $P_\rho = \partial_\rho$.

The kinematic group of the de Sitter spacetime dS is the de Sitter group $SO(1,4)$. This spacetime is defined as the quotient space of the de Sitter group under the Lorentz group [23]

$$dS = SO(1,4)/\mathcal{L}. \tag{1.15}$$

Similarly to Minkowski, the Lorentz group is responsible for the isotropy around a given point of dS. However, de Sitter's homogeneity is entirely different.

1.2.1 Generators of the de Sitter group

In Cartesian coordinate χ^A, the generators of the infinitesimal de Sitter transformations are written in the form

$$L_{AB} = \eta_{AC}\chi^C \frac{\partial}{\partial \chi^B} - \eta_{BC}\chi^C \frac{\partial}{\partial \chi^A}. \tag{1.16}$$

These generators satisfy the commutation relations

$$\left[L_{AB}, L_{CD}\right] = \eta_{BC}L_{AD} + \eta_{AD}L_{BC} - \eta_{BD}L_{AC} - \eta_{AC}L_{BD} \ . \tag{1.17}$$

In terms of stereographic coordinates $\{x^\mu\}$, the de Sitter generators are given by

$$L_{\mu\nu} = \eta_{\mu\rho}x^\rho\partial_\nu - \eta_{\nu\rho}x^\rho\partial_\mu \ , \tag{1.18}$$

and

$$L_{4\mu} = l\, P_\mu - K_\mu \ , \tag{1.19}$$

where

$$P_\mu = \partial_\mu \qquad \text{and} \qquad K_\mu = (1/4l^2)(2\eta_{\mu\nu}x^\nu x^\rho - \sigma^2\delta^\rho_\mu)\partial_\rho \tag{1.20}$$

are the generators of ordinary translations and proper conformal transformation, respectively. Generators $L_{\mu\nu}$ refer to the Lorentz subgroup, whereas the $L_{4\mu}$ define the transitivity[1] on the homogeneous space. These generators satisfy the commutation relations

$$\begin{aligned}
\left[L_{\mu\nu}, L_{\sigma\rho}\right] &= \eta_{\nu\sigma}L_{\mu\rho} + \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\nu\rho}L_{\mu\sigma} \ , \\
\left[L_{4\mu}, L_{\nu\rho}\right] &= \eta_{\mu\nu}L_{4\rho} - \eta_{\mu\rho}L_{4\nu} \ , \\
\left[L_{4\mu}, L_{4\nu}\right] &= L_{\mu\nu} \ .
\end{aligned} \tag{1.21}$$

We can see from Eq. (1.19) that the de Sitter spacetime is transitive under a combination of translation and proper conformal transformation, usually called *de Sitter translations*. It is important to remark that the de Sitter spacetime naturally includes the proper conformal transformations as part of spacetime kinematics, while Minkowski is transitive under ordinary translation only.

1.2.2 Inönü-Wigner contractions of the de Sitter group

In this subsection, we will discuss a powerful method known as Inönü-Wigner

[1] Transitivity is a property that specifies how one moves from one point to another in a given spacetime. For instance, any two points of Minkowski spacetime are connected by the spacetime translation (1.14), while any two points of the de Sitter spacetime are connected by the combination (1.19) of translation and proper conformal transformation.

contraction of the de Sitter group. Our starting point is to consider a semisimple group, the Inönü-Wigner contraction yields a non-semisimple group with the same dimension of the original group. The most common example is the contraction limit of the speed of light going to infinity $c \to \infty$, which leads the Poincaré group to the non-relativistic Galileo group. Another example is the contraction of the de Sitter group to the Newton-Hooke group, this latter is a non-relativistic group in presence of a non-vanishing cosmological constant [24].

$$\begin{tikzcd}
\mathrm{CP} \arrow[d] & \mathrm{dS} \arrow[l, "0 \leftarrow l"'] \arrow[r, "l \to \infty"] \arrow[d, "c \to \infty"] & \mathrm{P} \arrow[d] \\
\mathrm{CG} & \mathrm{N\text{-}H} \arrow[l] \arrow[r] & \mathrm{G}
\end{tikzcd}$$

Figure 1.2: *Hierarchy between groups. dS : de Sitter group, P : Poincaré group, CP : Conformal Poincaré group, N-H : Newton-Hooke group, G : Galileo group, and CG : Conformal Galileo group*

In what follows we will consider the limit of a large value of de Sitter length parameter $l \to \infty$, which translates into a vanishing cosmological term $\Lambda \to 0$. With the aim of studying said limit, it is convenient to rewrite the de Sitter generators according to

$$\Pi_\mu \equiv \frac{1}{l} L_{4\mu} = P_\mu - K_\mu \ , \tag{1.22}$$

whereas the generators $L_{\mu\nu}$ keep their original form. In terms of these generators, the commutation relations take the form

$$\begin{aligned}
\left[L_{\mu\nu}, L_{\sigma\rho}\right] &= \eta_{\nu\sigma} L_{\mu\rho} + \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\mu\sigma} L_{\nu\rho} - \eta_{\nu\rho} L_{\mu\sigma} \ , \\
\left[\Pi_\mu, L_{\nu\rho}\right] &= \eta_{\mu\nu} \Pi_\rho - \eta_{\mu\rho} \Pi_\nu \ , \\
\left[\Pi_\mu, \Pi_\nu\right] &= l^{-2} L_{\mu\nu} .
\end{aligned} \tag{1.23}$$

From the last commutation relation we can see that de Sitter "translation" are not really translation, but rotations. By taking the formal limit $l \to \infty$, we see that from Eq. (1.22) de Sitter generators Π_μ reduces to generators of the ordinary translations. Mathematically, it writes

$$\Pi_\mu \to P_\mu \,. \tag{1.24}$$

Likewise, the commutation relations (1.23) reduces to

$$\begin{aligned}
\left[L_{\mu\nu}, L_{\sigma\rho}\right] &= \eta_{\nu\sigma}L_{\mu\rho} + \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\nu\rho}L_{\mu\sigma} \\
\left[P_\mu, L_{\nu\rho}\right] &= \eta_{\mu\nu}P_\rho - \eta_{\mu\rho}P_\nu \\
\left[P_\mu, P_\nu\right] &= 0.
\end{aligned} \tag{1.25}$$

which are the commutation relations of the Poincaré group. This means that, under the formal limit, the de Sitter group $SO(1,4)$ contracts to the Poincaré group $\mathcal{P}$. Concomitantly, the de Sitter space $dS = SO(1,4)/\mathcal{L}$ contracts to the Minkowski space $M = \mathcal{P}/\mathcal{L}$, and the curvature tensor vanishes

$$\hat{R}^\mu{}_{\rho\nu\sigma} \to 0, \qquad \hat{R}_{\nu\sigma} \to 0, \qquad \hat{R} \to 0\,. \tag{1.26}$$

1.3 de Sitter transformation and Killing vectors

The de Sitter spacetime is maximally symmetric, in the sense that this can host the highest possible number of Killing vectors. In terms of the five-dimensional space coordinates, an infinitesimal de Sitter transformation is written as

$$\delta\chi^C = \frac{1}{2}\epsilon^{AB}L_{AB}\chi^C \tag{1.27}$$

where $\epsilon^{CD} = -\epsilon^{DC}$ are the transformation parameter, and

$$L_{AB} = \eta_{AC}\chi^C\partial_B - \eta_{BC}\chi^C\partial_A \tag{1.28}$$

are the de Sitter generators. In terms of stereographic coordinates it can be written as

$$\delta x^\mu \equiv \delta_L x^\mu + \delta_\Pi x^\mu = \frac{1}{2}\epsilon^{\rho\sigma}L_{\rho\sigma}x^\mu + \epsilon^{4\rho}L_{4\rho}x^\mu \,, \tag{1.29}$$

where

$$L_{\rho\sigma} = \eta_{\rho\lambda}x^\lambda P_\sigma - \eta_{\sigma\lambda}x^\lambda P_\rho \tag{1.30}$$

are the Lorentz generators, and $L_{4\rho}$ are the generators that defines the transitivity on the de Sitter spacetime. Identifying

$$\epsilon^{4\rho} = \frac{1}{l}\epsilon^{\rho} \quad \text{and} \quad L_{4\rho} = l\Pi_{\rho}, \tag{1.31}$$

we get

$$\delta x^{\mu} \equiv \delta_L x^{\mu} + \delta_{\Pi} x^{\mu} = \frac{1}{2}\epsilon^{\nu\sigma} L_{\nu\sigma} x^{\mu} + \epsilon^{\nu}\Pi_{\nu} x^{\mu} \tag{1.32}$$

where

$$\Pi_{\nu} = P_{\nu} - K_{\nu} \tag{1.33}$$

are the de Sitter "translation" generators. Substituting the generators (1.30) and (1.33) in the transformation (1.32), the Lorentz transformation takes the form

$$\delta_L x^{\mu} = \frac{1}{2}\xi^{\mu}_{\nu\sigma}\epsilon^{\nu\sigma} \tag{1.34}$$

where

$$\xi^{\mu}_{\nu\sigma} = (\eta_{\nu\lambda}\delta^{\mu}_{\sigma} - \eta_{\sigma\lambda}\delta^{\mu}_{\nu}) \tag{1.35}$$

are the Killing vectors of the Lorentz group. The de Sitter "translations" are written as

$$\delta_{\Pi} x^{\mu} = \xi^{\mu}_{\nu} x^{\nu} \tag{1.36}$$

where

$$\xi^{\mu}_{\nu} = \delta^{\mu}_{\nu} - \vartheta^{\mu}_{\nu} \tag{1.37}$$

where δ^{μ}_{ν} are the Killing vectors of the ordinary translations, and

$$\vartheta^{\mu}_{\nu} = \frac{1}{4l^2}\left(2\eta_{\nu\rho}x^{\rho}x^{\mu} - \sigma^2\delta^{\mu}_{\nu}\right) \tag{1.38}$$

are the Killing vectors of the proper conformal transformation . In the contraction limit $l \to \infty$, they reduce to the transformation of the Poincaré group

$$\delta_L x^{\mu} = \frac{1}{2}\xi^{\mu}_{\nu\sigma}\epsilon^{\nu\sigma} \qquad \text{and} \qquad \delta_P x^{\mu} = \delta^{\mu}_{\nu}\epsilon^{\nu}. \tag{1.39}$$

Of course, the ten Killing vectors $\xi^{(\nu\sigma)}_{\mu}$ and $\xi^{(\rho)}_{\mu}$ of the de Sitter spacetime satisfy the Killing equations

$$-\nabla_{\rho}\xi^{(\nu\sigma)}_{\mu} - \nabla_{\mu}\xi^{(\nu\sigma)}_{\rho} = 0\,, \tag{1.40}$$

and

$$- \nabla_\rho \xi^{(\nu)}_\mu - \nabla_\mu \xi^{(\nu)}_\rho = 0 \; . \tag{1.41}$$

It is worth remarking that Eq. (1.36) will produce changes in the notion of diffeomorphism. As a consequence, we will be forced to re-define the conserved current through Noether's theorem, which will be explored in the next section.

Chapter 2

de Sitter-modified General relativity

Local symmetries play an important role in the construction of any physics theory. For example, general relativity, a geometric theory of gravitation, is constructed taking into account the local symmetry is ruled by the Poincaré group, and its underlying spacetime is the Minkowski spacetime. Indeed, this makes lucid in the strong equivalence principle, which states:

In the presence of a gravitational field, it is always possible to find a local coordinate system in which the laws of physics reduce to those of special relativity.

In simple words, one can always find a local region where the local inertial effects are compensated by the effects of gravity, such a region is well represented by the Minkowski space, and the local kinematic is ruled by the Poincaré group. Based on these assumptions, general relativity in locally-Minkowski provides a set of equations, a conservation law

$$\nabla_\nu \left(\delta^\mu_\alpha T^{\alpha\mu}\right) \equiv \nabla_\nu T^{\mu\nu} = 0 \, , \tag{2.1}$$

with $T^{\alpha\mu}$ the symmetric energy-momentum tensor and δ^μ_α the Killing vectors of translation-which are the transformation defining Minkowski's transitivity. Further, the field gravitational equations present the following form

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = \frac{8\pi G}{c^4} T^{\mu\nu} \, . \tag{2.2}$$

We can see that a characteristic feature of Eq. (2.2) is that the source for gravitation is the entire energy-momentum tensor. Also, the second Bianchi identity implies that

$$\nabla_\nu (R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R) = 0 \; . \tag{2.3}$$

It should be remarked that all solutions to Einstein's equations are spacetimes that reduce locally to Minkowski.

With this short introduction, the main purpose of this chapter is to re-define general relativity, such that, now, this will be constructed on de Sitter space, which means we will replace the Poincaré invariant special relativity with de Sitter-invariant special relativity. This chapter is organized into five sections. Section 2.1 is devoted to re-define the definition of stress-energy and the conservation law, then a discussion of its implications is presented. Section 2.2 is devoted to presenting the de Sitter-modified Einstein equations and the principal properties. Section 2.3 provides an alternative form to express the associate field equations in terms of stereographic coordinates. Section 2.4 focuses on the physical explanation of the source of the cosmological term Λ. Section 2.5 studies the matter content of the universe modeled as a perfect fluid.

2.1 Conservation law of the source fields

Let us consider the action integral of the general matter field

$$\mathcal{S}_m = \frac{1}{c} \int \mathcal{L}_m \sqrt{-g} d^4 x \; , \tag{2.4}$$

with $\mathcal{L}$ the Lagrangian density. As local transformations, diffeomorphisms can ascertain the local homogeneity properties of spacetime. For example, in the locally Minkowski spacetimes, such a diffeomorphism is defined by

$$\delta_P x^\mu = \delta^\mu_\alpha \epsilon^\alpha(x) \; , \tag{2.5}$$

with δ^μ_α the Killing vectors of translation — which are the transformations that define Minkowski's transitivity. The corresponding variation of the metric tensor is given by

$$\delta_P g_{\mu\nu} = -\delta^\alpha_\mu \nabla_\nu \epsilon^\alpha(x) - \delta^\alpha_\nu \nabla_\mu \epsilon^\alpha(x) \; . \tag{2.6}$$

Under this transformations, the variation of $\mathcal{S}_m$ is written as

$$\delta \mathcal{S}_m = \frac{1}{2c} \int T^{\alpha\nu} \delta_P g_{\mu\nu} \sqrt{-g} d^4 x \; , \tag{2.7}$$

where

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_m}{\delta g_{\mu\nu}} \tag{2.8}$$

is the symmetric energy-momentum tensor. Using the symmetry of the energy-momentum tensor $T^{\mu\nu}$, the variation (2.6) can be re-expressed in the form

$$\delta \mathcal{S}_m = \frac{1}{2c} \int (\delta^\mu_\alpha T^{\alpha\nu}) \delta g_{\mu\nu} \sqrt{-g} d^4 x \ , \tag{2.9}$$

where $\delta g_{\mu\nu}$ is the generic metric transformation

$$\delta g_{\mu\nu} = -\nabla_\nu \epsilon_\mu(x) - \nabla_\mu \epsilon_\nu(x) \ . \tag{2.10}$$

By generic we mean a transformation not related to any coordinate transformation, which is the relevant transformation in variational calculus. Using Eq. (2.10), the invariance of $\mathcal{S}_m$ yields the covariant conservation law

$$\nabla_\nu \left(\delta^\mu_\alpha T^{\alpha\nu} \right) \equiv \nabla_\nu T^{\mu\nu} = 0 \ . \tag{2.11}$$

Note that the conserved current is the projection of the energy-momentum tensor along the Killing vectors of ordinary translations — which in this case is the energy-momentum tensor itself. It should be remarked that the presence of these translational Killings vectors in the conserved current leaves it clear that the conservation law (2.11) is a legitimate consequence of Noether's theorem, in the sense that the local properties of spacetime were adequately taken into account.

On the other hand, in the case of the de Sitter-modified general relativity, where the local kinematics is ruled by the de Sitter group, and all solutions to the de Sitter-modified Einstein equations are spacetimes that reduce locally to de Sitter. In this case, a diffeomorphism has the form [25]

$$\delta_\Pi x^\mu = \xi^\mu_\alpha \epsilon^\alpha(x) \ , \tag{2.12}$$

with ξ^μ_α the Killing vectors of the Sitter "translation" — the transformations that define de Sitter's transitivity. The corresponding change of the metric tensor is

$$\delta_\Pi g_{\mu\nu} = -\xi^\alpha_\mu \nabla_\nu \epsilon_\alpha(x) - \xi^\alpha_\nu \nabla_\mu \epsilon_\alpha(x) \ . \tag{2.13}$$

Variation of $\mathcal{S}_m$ with respect to the diffeomorphism (2.12) is now written as

$$\delta \mathcal{S}_m = \frac{1}{2c} \int T^{\mu\nu} \delta_{\Pi} g_{\mu\nu} \sqrt{-g}\, d^4x \,, \tag{2.14}$$

with $T^{\mu\nu}$ the symmetric energy-momentum tensor (2.8). Using the symmetry of $T^{\mu\nu}$, variation (2.14) assumes the form

$$\delta \mathcal{S}_m = \frac{1}{2c} \int (\xi^{\mu}_{\alpha} T^{\alpha\nu}) \delta g_{\mu\nu} \sqrt{-g}\, d^4x \,, \tag{2.15}$$

with $\delta g_{\mu\nu}$ the generic metric transformation (2.10). The invariance $\delta S_m = 0$ of the action yields to the covariant conservation law

$$\nabla_{\nu}(\xi^{\mu}_{\alpha} T^{\alpha\nu}) \equiv \nabla_{\nu} \Pi^{\mu\nu} = 0 \,. \tag{2.16}$$

We see from Eq. (2.16) that the conserved current is the projection of the energy-momentum tensor along the Killing vectors of the de Sitter's translations. The presence of these vectors clearly remark the fact that the conserved current depends on the local properties of the underlying spacetime. Note that, since the current $\Pi^{\mu\nu} = \xi^{\mu}_{\alpha} T^{\alpha\nu}$ appears contracted with $\delta g_{\mu\nu}$ in Eq. (2.15), only its symmetric part contributes to the covariant conservation law, and consequently to the gravitational field equations. Hereafter, it is understood that $\Pi^{\mu\nu}$ denotes the symmetric part of the current.

In stereographic coordinates, $\Pi^{\mu\nu}$ can be decomposed in the form

$$\Pi^{\mu\nu} = T^{\mu\nu} - K^{\mu\nu} \,, \tag{2.17}$$

where $K_{\mu\nu}$ is the proper conformal current, whose explicit form is

$$K_{\mu\nu} = \frac{1}{4l^2} (2\eta_{\mu\rho} x^{\rho} x^{\alpha} - \sigma^2 \delta^{\alpha}_{\mu}) T_{\alpha\nu} \,, \tag{2.18}$$

with $\sigma^2 = \eta_{\mu\nu} x^{\mu} x^{\nu}$ the Lorentz invariant quadratic form. Notice that Eq. (2.17) presents important additional freedom, which allows ordinary energy-momentum $T^{\mu\nu}$ to transform into proper conformal current $K^{\mu\nu}$ and vice versa while keeping the total current covariantly conserved. In general, neither $T^{\mu\nu}$ nor $K^{\mu\nu}$ is conserved separately. Their conservation laws have the form

$$\nabla_{\nu} T^{\mu\nu} \sim T^{\rho}{}_{\rho}\, x^{\mu} \qquad \text{and} \qquad \nabla_{\nu} K^{\mu\nu} \sim T^{\rho}{}_{\rho}\, x^{\mu} \,. \tag{2.19}$$

When the physical system has a traceless energy-momentum tensor, it is conformal invariant and, consequently, the proper conformal current is covariantly conserved:

$$\nabla_\nu K^{\mu\nu} = 0. \tag{2.20}$$

Of course, the energy-momentum is also covariantly conserved:

$$\nabla_\nu T^{\mu\nu} = 0. \tag{2.21}$$

This is the case, for example, of the electromagnetic field.

2.2 de Sitter-modified Einstein equations

As we discussed in the previous chapter, the de Sitter spacetime represents a different background for the construction of physical theories, standing with an equal footing with Minkowski spacetime. General relativity, for instance, can be constructed on any one of them. Of course, in either case, gravitation will have the same dynamics, but their fundamental difference will reside in the local kinematics. In this way, by constructing general relativity on a de Sitter space, the spacetime not will be described by a Riemannian structure anymore but will be described by a more general structure known as Cartan geometry [26]. In simple words, Cartan geometry generalizes the local geometry of Riemann, which means that for each point p of the manifold M the tangent space will now be characterized by an osculating de Sitter space. This modification on the geometry has a deep impact on the strong equivalence principle, which passes from now, to state that

In the presence of a gravitational field, it is always possible to find a local coordinate system in which the laws of physics reduce to those of de Sitter-invariant special relativity.

It implies that one observer can choose a frame where the local inertial effects are compensated by the effects of gravity; in such a region, the spacetime will be represented by a de Sitter space and the local symmetry will be governed by the de Sitter group. Since the dynamics of the gravitational field does not change due to the replacement of the local kinematics, it is valid to assume that Lagrangian density for gravitation must maintain its same form

$$S_g = -\frac{c^3}{16\pi G}\int \mathcal{R}\sqrt{-g}d^4x\ , \tag{2.22}$$

where $\mathcal{R}$ is the scalar curvature. The curvature tensor $\mathcal{R}^{\rho}{}_{\nu\sigma\beta}$ represents now both the general relativity *dynamic curvature*, whose source is the energy momentum tensor, and the *kinematic curvature* implied by the underlying de Sitter spacetime, and these aspects will be explored in more detail the next section. Considering an arbitrary variation of the metric tensor $\delta g_{\mu\nu}$, the gravitational action S_g changes according to

$$\delta S_g = \frac{c^3}{16\pi G}\int (\mathcal{R}^{\mu\nu} - \frac{1}{2}g^{\mu\nu}\mathcal{R})\delta g_{\mu\nu}\sqrt{-g}d^4x\ . \tag{2.23}$$

The invariance of the total action integral

$$\delta\mathcal{S}_g + \delta\mathcal{S}_m = 0\ , \tag{2.24}$$

with $\delta\mathcal{S}_m$ given by Eq. (2.15), yields the de Sitter-modified Einstein equations [27]

$$\mathcal{R}^{\mu\nu} - \frac{1}{2}g^{\mu\nu}\mathcal{R} = \frac{8\pi G}{c^4}\Pi^{\mu\nu}\ . \tag{2.25}$$

This is the equation that replaces ordinary Einstein equations when the Poincaré invariant special relativity is replaced with the de Sitter-invariant special relativity. It is important to mention that all solutions of de Sitter-modified Einstein equations are spacetimes that reduce locally-de Sitter spacetimes.

Another relevant aspect of this approach is that the background de Sitter curvature is included in the curvature tensor $\mathcal{R}^{\mu}{}_{\nu\sigma\rho}$, that is, the cosmological term Λ does not appear explicitly in the field equation (2.25). As a consequence, from the second Bianchi identity

$$\nabla_\nu \mathcal{G}^{\mu\nu} \equiv \nabla_\mu\big(\mathcal{R}^{\mu\nu} - \frac{1}{2}g^{\mu\nu}\mathcal{R}\big) = 0\ , \tag{2.26}$$

the cosmological term is no longer restricted to be constant. Finally, in the formal limit when $l \to \infty$, the underlying de Sitter spacetime contracts to Minkowski spacetime, then the de Sitter-modified Einstein equations reduces to the ordinary Einstein equation [28]

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = \frac{8\pi G}{c^4}T^{\mu\nu}\ . \tag{2.27}$$

2.3 Field equations in stereographic coordinates

In the first chapter was mentioned the fact that stereographic coordinates present interesting characteristics, for example, these coordinates allow us to split the gravitational field equations into two parts, which facilitates to study its contributions.

In sterographic coordinates, the Killing vectors of the *de Sitter translation* are given by Eq. (1.37). Substituting in the covariantly conserved current $\Pi^{\mu\nu} = \xi^{\mu}_{\alpha} T^{\alpha\nu}$, it decompose into

$$\Pi^{\mu\nu} = T^{\mu\nu} - K^{\mu\nu}, \tag{2.28}$$

where

$$T^{\mu\nu} = \delta^{\mu}_{\alpha} T^{\alpha\nu} \quad \text{and} \quad K^{\mu\nu} = \vartheta^{\mu}_{\alpha} T^{\alpha\nu} \ , \tag{2.29}$$

are, respectively, the energy-momentum tensor and the proper conformal current [29]. Analogously, the Einstein tensor $\mathcal{G}^{\mu\nu} = \xi^{\mu}_{\alpha} G^{\alpha\nu}$ decomposes into [1]

$$\mathcal{G}^{\mu\nu} = G^{\mu\nu} - \hat{G}^{\mu\nu} \ , \tag{2.30}$$

where

$$G^{\mu\nu} \equiv \delta^{\mu}_{\alpha} G^{\alpha\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \ , \tag{2.31}$$

is general relativity's Einstein tensor and

$$\hat{G}^{\mu\nu} \equiv \vartheta^{\mu}_{\alpha} G^{\alpha\nu} = \hat{R}^{\mu\nu} - \frac{1}{2} g^{\mu\nu} \hat{R} \ , \tag{2.32}$$

is the Einstein tensor of the background de Sitter spacetime. In stereographic coordinates, therefore, the de Sitter-modified Einstein equation (2.25) assumes the form

$$(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R) - (\hat{R}^{\mu\nu} - \frac{1}{2} g^{\mu\nu} \hat{R}) = -\frac{8\pi G}{c^4} \left(T^{\mu\nu} - K^{\mu\nu}\right) . \tag{2.33}$$

2.4 The source of the cosmological term Λ

According to the above equation, any source field gives rise to both an energy-momentum and a proper conformal current. The energy-momentum tensor $T^{\mu\nu}$ keeps its role as the source of general relativity's dynamics curvature, whereas the proper conformal current $K^{\mu\nu}$ shows up as the source of the kinematic curvature

[1] This decomposition is only valid in stereographic coordinates.

of the local de Sitter spacetimes [30]. In fact, taking the trace of the field equations (2.33) and identifying

$$\hat{G}^{\mu}{}_{\mu} = -\hat{R} \equiv \Lambda, \tag{2.34}$$

we obtain

$$R - \Lambda = \frac{8\pi G}{c^4}\left(T^{\mu}_{\mu} - K^{\mu}_{\mu}\right) . \tag{2.35}$$

To get a separate expression for Λ, we consider the case of the electromagnetic field, whose energy-momentum tensor is traceless, $T^{\mu}_{\mu} = 0$, and consequently, the gravitational scalar curvature vanishes $R = 0$. In this case, the field equation (2.35) reduces to

$$\Lambda = \frac{8\pi G}{c^4} K^{\mu}_{\mu} . \tag{2.36}$$

In contrast to ordinary special relativity, the de Sitter-invariant special relativity creates a non-gravitational underlying homogeneous spacetime, whose sectional curvature Λ is given by Eq. (2.36). Since Λ shows up as a kinematic effect, this equation is not dynamic but essentially algebraic. Therefore, the gravitational field equation (2.35) has a dynamical part, whose source is the energy-momentum current, and an algebraic part, whose source is the proper conformal current.

2.5 Perfect fluid universe

We assume now that the matter content of the universe can be modeled by a perfect fluid, whose energy-momentum tensor in co-moving coordinate is written as

$$T^{\mu}_{\nu} = (\epsilon_m + p_m) u^{\mu} u_{\nu} - p_m \delta^{\mu}_{\nu} , \tag{2.37}$$

where ϵ_m and p_m are the matter energy density and pressure, respectively. Its trace is

$$T^{\mu}_{\mu} \equiv \delta^{\mu}_{\alpha} T^{\alpha}_{\mu} = \epsilon_m - 3p_m . \tag{2.38}$$

On the other hand, the trace of the proper conformal current is

$$K^{\mu}_{\mu} = \vartheta^{\mu}_{\alpha} T^{\alpha}_{\mu} \equiv \frac{1}{4l^2}\left(2\eta_{\alpha\rho} x^{\rho} x^{\mu} - \sigma^2 \delta^{\mu}_{\alpha}\right) T^{\alpha}_{\mu} . \tag{2.39}$$

At large scale our universe is assumed to be isotropic and homogeneous. Such an homogeneity implies that the energy density and pressure should be position-independent. Hence, they can only be time-dependent. Moreover, since all points of a homogeneous space are equivalents, without losing generality, we will make the computation at any point of the homogeneous space. For sake of simplicity, we perform the computation at $x^i = 0$. In this case, the trace of the proper conformal current takes the form

$$K^\mu_{\ \mu} = \gamma(t)(\epsilon_m + p_m) \; , \tag{2.40}$$

where

$$\gamma(t) \equiv \vartheta^0_0 = \frac{c^2 t^2}{4l^2} \; , \tag{2.41}$$

is a time-dependent dimensionless parameter, given by the time component of the proper conformal Killing vectors. Considering that the trace of the proper conformal current is the source of Λ, we identify

$$\epsilon_\Lambda \equiv \gamma(t)\epsilon_m \quad \text{and} \quad p_\Lambda \equiv \gamma(t) p_m \; . \tag{2.42}$$

In this case, the trace of the proper conformal current takes the form

$$K^\mu_{\ \mu} = \epsilon_\Lambda + 3p_\Lambda \; . \tag{2.43}$$

Using the traces (2.38) and (2.43), the field equation (2.35) takes the form

$$R - \Lambda = \frac{8\pi G}{c^4}\Big[(\epsilon_m - 3p_m) - (\epsilon_\Lambda + 3p_\Lambda)\Big] \; . \tag{2.44}$$

As can be seen from Eq. (2.42), the energy densities ϵ_Λ and ϵ_m differ by the same coefficient as p_Λ differs from p_m. Consequently, p_m and p_Λ satisfy the same equation of state

$$p_m = w\epsilon_m \qquad \text{and} \qquad p_\Lambda = w\epsilon_\Lambda \; , \tag{2.45}$$

with w a numerical constant. This is expected because, as we have already seen, in locally-de Sitter, any source field gives rise to both an energy-momentum and a proper conformal current. Comparing the trace (2.38) of the energy-momentum current with the trace (2.43) of the proper conformal current, we see that, even though p_m and p_Λ satisfy the same equation state, they naturally appear in the field equation (2.44) with opposite signs. This difference uniquely resides in the nature of sources: whereas the source of gravitation is the energy-momentum current of ordinary matter, the source of Λ is the proper conformal current of the ordinary matter. In this theory, the source of the dark energy does not need to be a mysterious fluid that satisfies an exotic equation of state ($w = -1$) [31].

It is important to highlight that in de Sitter-modified general relativity the notion of dark energy is different from the usual Λ-CDM model. In the latter case, it is known the cosmological term Λ is constant, as a consequence, the dark energy ϵ_Λ is uniformly distributed across space, even in an expanding universe, and is independent of the matter-energy density ϵ_m distribution. However, in the de Sitter-modified general relativity, the spatial distribution of both energy densities coincide, this can be lucidly seen from the relation (2.42), ϵ_Λ is non-vanishing only in the space regions where ϵ_m is non-vanishing. This peculiar characteristic is uniquely present in de Sitter-modified general relativity. It is important to mention that evolving dark energy could be an alternative for a solution to the cosmological constant problem and the coincidence problem [32].

Part II

de Sitter-Invariant Approach to Cosmology

Chapter 3

Source conservation in the FLWR geometry

When considering the large-scale structure of the universe, observations indicate that on average galaxies are spread uniformly throughout the universe at any given time. This means that if we consider a portion of the universe that is large compared to the distance between typical nearest galaxies, then the number of galaxies in the portion is roughly the same as the number in another portion with the same volume at any given time.

This proviso *at any given time* about the uniform distribution of galaxies is important because the universe is in a dynamic state, and so the number of galaxies in any given volume changes with time. Also, the distribution of galaxies appears to be isotropic about us, that is, it is the same, on average, in all directions from us. Now, if we assume that we do not occupy a special position amongst the galaxies, and we continue observing that the distribution of galaxies is isotropic about every

Figure 3.1: *One observer sits on the circle of the fig.(a), (b), and (c), will view that on sufficiently large scales the distributions of galaxies in the universe look similar.*

galaxy, then it necessarily implies that galaxies are spread uniformly throughout the universe (see Fig. 3.1). Therefore, we can say that the universe appears to be homogeneous and isotropic as far as we can detect. These properties lead us to establish a fundamental principle of the universe called *the Cosmological Principle* [33], a large portion of modern cosmology models are built on it, which states that all positions in the universe are essentially equivalents. The cosmological principle allows us to consider that the geometry of the universe can be split into spatial sections along the cosmic time, and every section of constant time must be a homogeneous and isotropic three-dimensional space of constant curvature.

On the other hand, all solutions of de Sitter-modified gravitational field equations are spacetimes that reduce locally to de Sitter space. Therefore, to construct our cosmological model, the metric that should be viable to describe the universe's geometry as a whole must include these two concepts.

Considering the above comments, the principal goal of this chapter is to present the first original part of this thesis project, which is organized into two sections. Sec 3.1 outlines the construction of the FLRW metric valid in locally-de Sitter spaces. Sec 3.2 explores the equation of motion for a cosmological fluid in the corresponding FLRW metric and the corresponding conservation law.

3.1 The FLRW metric

The cosmological principle states that the space section of the universe is assumed to be isotropic and homogeneous at sufficiently large scales. This assumption places tight constraints on the geometry of spacetime. In fact, there are only three possibilities: the universe space section can be Euclidean, spherical, or hyperbolic. In the case of locally-de Sitter spacetimes, the space section of the universe must be a (hyperbolic) de Sitter 3-space. Such space can be described by embedding it into a four-dimensional pseudo-Euclidean space coordinate (x^0, x^i), inclusion whose points set is determined by the condition

$$\delta_{ij}x^i x^j - (x^0)^2 = -a^2, \tag{3.1}$$

where a is an arbitrary constant. The corresponding line element has the form

$$ds^2 = \delta_{ij}dx^i dx^j - (dx^0)^2. \tag{3.2}$$

Re-scaling the coordinates according to

$$x^0 \to ax^0 \quad \text{and} \quad x^i \to ax^i, \tag{3.3}$$

and using the identity (3.1) to eliminate x^0, the line elements assumes the form

$$dl^2 = a^2\gamma_{ij}dx^i dx^j, \tag{3.4}$$

with

$$\gamma_{ij} = \delta_{ij} - \frac{x_i x_j}{1 + (x_k x^k)} \tag{3.5}$$

The line element coincides with the one obtained by choosing the curvature parameter $k = -1$ in the standard approach [34].

The FLRW metric is obtained by including, the line element dl^2 into the space-time line element, with the parameter assumed to be an arbitrary function of the cosmic time,

$$ds^2 = c^2dt^2 - a^2\gamma_{ij}dx^i dx^j \tag{3.6}$$

with $a = a(t)$ the cosmic scale factor. Let us compute the non-vanishing components of the Levi-Civita

$$\begin{aligned} \Gamma^0_{ij} &= (a\dot{a}/c)\gamma_{ij}\,, \quad \Gamma^i{}_{0j} = (\dot{a}/ac)\delta^i_j\,, \\ \Gamma^i_{jk} &= \frac{1}{2}\gamma^{il}\left(\partial_j\gamma_{kl} + \partial_k\gamma_{jl} - \partial_l\gamma_{jk}\right) \equiv -x^i\delta_{jk}, \end{aligned} \tag{3.7}$$

here a dot represents the derivative with respect to the cosmic time t.

3.2 Noether continuity equations

In the previous chapter, taking into account two assumptions: solutions valid in locally-de Sitter and the space sections of the universe are isotropic and homogeneous on a large scale, we have found the FLRW metric. Now, we can obtain a deeper insight into the behavior of matter in a FLRW universe by plugging it into its corresponding conservation law. For simplicity, the matter's content of the universe can be modeled by a fluid perfect

$$T^{\mu\nu} = (\epsilon + p)u^{\mu\nu} - pg^{\mu\nu}\,. \tag{3.8}$$

Recalling that $\Pi^{\mu\nu}$ denotes the symmetric part of the tensor, the zero-component of the covariant conservation law (2.16) reads

$$\nabla_\mu \Pi^\mu_{\ 0} \equiv \partial_\mu \Pi^\mu_{\ 0} + \Gamma^\mu_{\ \rho\mu} \Pi^\rho_{\ 0} - \Gamma^\rho_{\ 0\mu} \Pi^\mu_{\ \rho} = 0 \,. \tag{3.9}$$

Separating the time and space components and noting that $\Pi^j_{\ 0} = \Pi_j^{\ 0}$ vanish due to the homogeneity of the universe space section, we obtain

$$\partial_0 \Pi^0_{\ 0} + \Gamma^j_{\ 0j} \Pi^0_{\ 0} - \Gamma^i_{\ 0j} \Pi^j_{\ i} = 0 \,. \tag{3.10}$$

Substituting the connections (3.7) computed at $x^i = 0$, we get

$$\partial_0 \Pi^0_{\ 0} + 3\frac{\dot{a}}{ac} \Pi^0_{\ 0} - \frac{\dot{a}}{ac} \Pi^j_{\ j} = 0 \,. \tag{3.11}$$

Using Eqs. (2.17) and (2.18), the components of the source current, computed at $x^i = 0$ are found to be

$$\Pi^0_{\ 0} = \epsilon_m \left(1 - \frac{c^2 t^2}{4l^2}\right) \quad \text{and} \quad \Pi^i_{\ j} = -3p_m \left(1 + \frac{c^2 t^2}{4l^2}\right) \,. \tag{3.12}$$

Using the identifications (2.42), the conservation law (3.11) can be rewritten in the form

$$\frac{d}{dt}(\epsilon_m - \epsilon_\Lambda) + 3\frac{\dot{a}}{a}(\epsilon_m - \epsilon_\Lambda) + 3\frac{\dot{a}}{a}(p_m + p_\Lambda) = 0 \,, \tag{3.13}$$

Assuming that the energy densities depend on the cosmic time t through the scale factor $a = a(t)$, and using the equations of state (2.45), the conservation law (3.13) becomes

$$\frac{d}{da}\left[a^3(\epsilon_m - \epsilon_\Lambda)\right] + 3a^2 w(\epsilon_m + \epsilon_\Lambda) = 0 \,. \tag{3.14}$$

Its solution is

$$\epsilon_m - \epsilon_\Lambda \propto a^{-3-3w} + a^{-3+3w} \,. \tag{3.15}$$

In the contraction limit $\Lambda \to 0$, the continuity equation (3.15) reduces to the usual expression of locally-Minkowski spacetimes

$$\frac{d}{da}(a^3 \epsilon_m) + 3a^2 w \epsilon_m = 0 \,, \tag{3.16}$$

whose solution is [35]

$$\epsilon_m \propto a^{-3-3w} \,. \tag{3.17}$$

Chapter 4

de Sitter-invariant approach to Cosmology

Up to now we have explored the universe's geometry and how its matter content can evolve through along the cosmic time. To study the universe's dynamics, we need to solve the de Sitter-modified Einstein equations in order to obtain the corresponding Friedmann equations [36].

4.1 The de Sitter-modified Friedmann equations

First, we need to compute the Ricci tensor and scalar. As already mentioned, the space section of the universe is isotropic and homogeous. This means that all points of this space are equivalent. Consequently, the computations can be made at any point. For the sake of simplicity, it is usual to choose the point $x^i = 0$ [33]. In this case, the components of the Levi-Civita connection are found to be

$$\Gamma^0_{ij} = (a\dot{a}/c)\delta_{ij}, \quad \Gamma^i_{0j} = \frac{\dot{a}}{ac}\delta^i_j, \quad \Gamma^i_{jk} = 0, \tag{4.1}$$

and its derivative

$$\partial_l \Gamma^i_{jk} = \delta^i_l \delta_{jk}. \tag{4.2}$$

Using these Levi-Civita connections, we compute the non-vanishing components of the Ricci tensor. The component $\mathcal{R}_{00}$ is

$$\begin{aligned}\mathcal{R}_{00} &= \partial_0 \Gamma^{\lambda}{}_{0\lambda} - \partial_\lambda \Gamma^{\lambda}{}_{00} + \Gamma^{\lambda}{}_{0\rho}\Gamma^{\rho}{}_{0\lambda} - \Gamma^{\lambda}{}_{\lambda\rho}\Gamma^{\rho}{}_{00} \\ &= \partial_0 \Gamma^{i}{}_{0i} + \Gamma^{i}{}_{0j}\Gamma^{j}{}_{0i} \\ &= \partial_0\Big(\frac{3\dot{a}}{ac}\Big) + \big(\frac{3\dot{a}}{ac}\big)^2 \\ &= 3\frac{\ddot{a}}{ac^2}.\end{aligned} \tag{4.3}$$

The component $\mathcal{R}_{ij}$ has the form

$$\mathcal{R}_{ij} = \partial_j \Gamma^{\lambda}{}_{i\lambda} - \partial_\lambda \Gamma^{\lambda}{}_{ij} + \Gamma^{\lambda}{}_{i\rho}\Gamma^{\rho}{}_{j\lambda} - \Gamma^{\lambda}{}_{\lambda\rho}\Gamma^{\rho}{}_{ij}\,. \tag{4.4}$$

For sake of simplicity, we label the first two terms with the letter (A) and the other two terms with the letter (B). We then obtain

$$\begin{aligned}A &= \partial_j \Gamma^{\lambda}{}_{i\lambda} - \partial_\lambda \Gamma^{\lambda}{}_{ij} \\ &= \partial_j \Gamma^{k}{}_{ik} - \partial_0 \Gamma^{0}{}_{ij} - \partial_k \Gamma^{k}{}_{ij} \\ &= -\delta^k_j \delta_{ik} - \partial_0\big(a\dot{a}/c\big)\delta_{ij} - \delta^k_k \delta_{ij} \\ &= -\frac{1}{c^2}(\dot{a}^2 + a\ddot{a} - 2c^2)\delta_{ij}\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}B &= \Gamma^{\lambda}{}_{i\rho}\Gamma^{\rho}{}_{j\lambda} - \Gamma^{\lambda}{}_{\lambda\rho}\Gamma^{\rho}{}_{ij} \\ &= \Gamma^{0}{}_{il}\Gamma^{l}{}_{j0} + \Gamma^{k}{}_{i0}\Gamma^{0}{}_{jk} - \Gamma^{k}{}_{k0}\Gamma^{0}{}_{ij} \\ &= (a\dot{a}/c)\delta_{il}\frac{\dot{a}}{ac}\delta^l_j + \frac{\dot{a}}{ac}\delta^k_i(a\dot{a}/c)\delta_{jk} - 3\frac{\dot{a}}{ac}(a\dot{a}/c)\delta_{ij} \\ &= -\frac{\dot{a}^2}{c^2}.\end{aligned} \tag{4.6}$$

Hence, we obtain

$$\mathcal{R}_{ij} = -\frac{1}{c^2}(\ddot{a}a + 2\dot{a}^2 - 2c^2), \tag{4.7}$$

and the corresponding scalar curvature is

$$\begin{aligned}\mathcal{R} &= g^{00}\mathcal{R}_{00} + g^{ij}\mathcal{R}_{ij} \\ &= \frac{6}{c^2}\Big(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} - \frac{c^2}{a^2}\Big).\end{aligned} \tag{4.8}$$

Using these tensors in the de Sitter-modified Einstein equation (2.25), we obtain the de Sitter-modified Friedmann equations

$$H^2 = \frac{8\pi G}{3c^2}(\epsilon_m - \epsilon_\Lambda) + \frac{c^2}{a^2} \tag{4.9}$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}\Big[(\epsilon_m - \epsilon_\Lambda) + 3(p_m + p_\Lambda)\Big] \tag{4.10}$$

with $H = \dot{a}/a$ being the Hubble parameter. Similarly to the ordinary Friedmann equations, taking the time derivative of the equation (4.9), and using (4.10) to eliminate $\ddot{a}$, one obtains the conservation law (3.13) .

One may wonder why ϵ_m and ϵ_Λ have opposite signs. The reason comes from their different physical effects: whereas the matter's energy density represents attractive effects, dark energy density represents a repulsive effect. Since the Friedmann equations are dynamic equations, it is natural that ϵ_m and ϵ_Λ enter the equations with opposite signs. The difference $\epsilon_m - \epsilon_\Lambda$ is the fundamental variable for describing the universe dynamics: the balance between them determines how the universe evolves along the cosmic time. Although the energy densities ϵ_m and ϵ_Λ can transform into each other, they do it in such a way that the difference $\epsilon_m - \epsilon_\Lambda$ satisfies the Noether continuity equation (3.15). One should mention that the minus sign of ϵ_Λ is determined by the de Sitter group. For the anti-de Sitter group, the sign would be positive, leading to an unstable universe.

Considering the critical energy density $3H^2c^2/8\pi G = \epsilon_c$, the Friedmann equation (4.9) can be written in the form

$$1 = \Omega_m - \Omega_\Lambda + \frac{c^2}{H^2a^2}, \tag{4.11}$$

where

$$\Omega_m \equiv \frac{\epsilon_m}{\epsilon_c} = \frac{8\pi G}{3H^2c^2}\epsilon_m\,, \quad \text{and} \quad \Omega_\Lambda \equiv \frac{\epsilon_\Lambda}{\epsilon_c} = \frac{8\pi G}{3H^2c^2}\epsilon_\Lambda \tag{4.12}$$

are, respectively, the matter and the dark-energy density parameters. An important feature of the de Sitter-modified Friedmann equations is that the energy densities ϵ_m and ϵ_Λ are not independent, a property inherited from the dependence of the proper conformal current $K^{\mu\nu}$ on the energy-momentum tensor $T^{\mu\nu}$, as evinced by Eq. (2.29). This dependence establishes the constraint

$$\epsilon_\Lambda = \gamma(t)\epsilon_m \tag{4.13}$$

with $\gamma(t)$ given by the Eq. (2.42). Of course, the density parameter Ω_m and Ω_Λ are also related through[1]

$$\Omega_\Lambda = \gamma(t)\Omega_m \tag{4.14}$$

the cosmological data estimates [37] that

$$\Omega_\Lambda \approx 0.69\,, \quad \text{and} \quad \Omega_m \approx 0.31 \tag{4.15}$$

the present-day value of $\gamma(t)$ is

$$\gamma(t_0) \approx 2.2. \tag{4.16}$$

Similarly, the present-day value of the last term on the right-hand side of Eq. (4.11) which represents effects coming from the universe's space topology, is found to be

$$\frac{c^2}{H^2a^2} \approx 1.38. \tag{4.17}$$

Unlike the density parameters Ω_m and Ω_Λ which represent the natural constituent of the universe, the last term on the right-hand side of Eq. (4.11) does not represent a constituent of the universe. Even though this term contributes to the universe dynamics, as far as the universe's inventory is concerned, only Ω_m and Ω_Λ must be taken into account. Since it is irrelevant for the universe's inventory whether their effects are attractive or repulsive, they must satisfy the algebraic equation

$$\Omega_m + \Omega_\Lambda = 1 \tag{4.18}$$

4.2 The deceleration parameter q

According to Sandage [38], the knowledge of two fundamental cosmological parameters provides a deep understanding of the behavior of our universe. The first one is the Hubble parameter H which determines the expansion rate of the universe while the second parameter is the deceleration parameter q which is a dimensionless measure of the cosmic acceleration of the expansion of the universe.

[1] In the Λ-CDM, the matter and dark energy density parameters are related by $\Omega_\Lambda/\Omega_m \propto a^3$. Since this ratio changes rapidly as the universe expands, the fact that they are of the same order today is known as the *cosmological coincidence problem*. The situation drastically changes within de Sitter-invariant approach to cosmology, where the ratio of these quantities is smooth because this is not determined by the scale factor, but the cosmic time.

These parameters can be naturally defined making use of Taylor series for the scale factor $a(t)$ in a vicinity of the present time t_0

$$a(t) = a(t_0) + \dot{a}(t_0)(t - t_0) + \frac{1}{2}\ddot{a}(t_0)(t - t_0)^2 + ... \tag{4.19}$$

or alternatively

$$\frac{a(t)}{a(t_0)} = 1 + H_0(t - t_0) - \frac{1}{2}q_0 H_0^2 (t - t_0)^2 + ... \tag{4.20}$$

here we have defined

$$q_0 = -\frac{\ddot{a}(t_0)}{a(t_0)}\frac{1}{H_0^2} = -\frac{a(t_0)\ddot{a}(t_0)}{\dot{a}^2(t_0)}, \qquad \text{and} \qquad H_0 = \frac{\dot{a}(t_0)}{a(t_0)}, \tag{4.21}$$

the deceleration parameter and the Hubble parameter at the present time, respectively. In general,

$$q(t) = -\frac{a(t)\ddot{a}(t)}{\dot{a}^2(t)}, \qquad \text{and} \qquad H(t) = \frac{\dot{a}(t)}{a(t)}. \tag{4.22}$$

Now let us discuss some implications. It is well-known that in the ordinary case of locally-Minkowski spacetimes with a cosmological constant Λ, the deceleration parameter has the form [39]

$$q_M = \frac{1}{2}(1 + 3w)\Omega_i - \Omega_\Lambda \; . \tag{4.23}$$

On the contrary, using Eq. (4.9) and Eq. (4.10) the deceleration parameter in locally-de Sitter spacetimes is found to be

$$q_{dS} = \frac{1}{2}\left[\Omega_i(1 + 3w) - \Omega_\Lambda(1 - 3w)\right] \; . \tag{4.24}$$

The present-day universe's content can be approximately described by dust ($w = 0$), and using the values Eq. (4.15), the deceleration parameters are found to be

$$q_M \equiv \frac{1}{2}\Omega_m - \Omega_\Lambda \simeq -0.54 \; , \tag{4.25}$$

and

$$q_{dS} \equiv \frac{1}{2}(\Omega_m - \Omega_\Lambda) \simeq -0.19 \; , \tag{4.26}$$

It is worth mentioning that, although q_M and q_{dS} are formally and numerically similar since Λ is constant in the expression for q_M and a function of the cosmic time in the expression for q_{dS}, the physical outcome of the two cases differ substantially.

4.3 The adiabaticity condition

The initial conditions for a Bing Bang universe are characterized by low entropy [40]. Likewise, in an expanding universe is expected that the entropy of the matter content is increasing [41]

$$dS_m \geq 0\ . \tag{4.27}$$

In the Λ-CDM model, the dark energy stays constant despite the expansion and does not present a mechanism to convert it into particles or radiation. The entropy of the matter content is not increasing or decreasing with the expansion but

$$dS_m = 0\ , \tag{4.28}$$

the last equation offers us a limited understanding of the concept of entropy of the universe. In the Sitter-modified general relativity, by construction, all solutions to the de Sitter-modified Einstein's equation are space-times that reduce locally to de Sitter. Consequently, any source field gives rise to both an energy-momentum and a proper conformal current. There will be a translational energy density $\epsilon_m \equiv T^0{}_0$, which corresponds to the usual matter energy-density, and a (proper) conformal energy density. In terms of energies $E_m = V\epsilon_m$ and $E_\Lambda = V\epsilon_\Lambda$, with $V \sim a^3$ as the volume. Thus, the continuity equation (3.14) is written as

$$d(E_m - E_\Lambda) = -(p_m + p_\Lambda)dV. \tag{4.29}$$

Since there are two different notions of energy, there will exist two corresponding notions of entropy [42], denoted S_m and S_Λ. In locally-de Sitter spacetimes, therefore, the first law of the thermodynamics applied to the universe reads

$$d(E_m - E_\Lambda) = Td(S_m + S_\Lambda) - (p_m + p_\Lambda)dV. \tag{4.30}$$

Comparing (4.29) with (4.30), we see that

$$d(S_m + S_\Lambda) = 0 \tag{4.31}$$

This expression ensures that the de Sitter-modified expansion is adiabatic. Even though the total entropy $(S_m + S_\Lambda)$ remains constant, S_m and S_Λ are allowed to transform into each other along with the cosmic evolution. Such additional freedom, which is not present in locally-Minkowski spacetimes, may have significant consequences for the universe's evolution. In particular, it could eventually unveil a strategy to circumvent the restrictions imposed by the second law of thermodynamics, opening the possibility of the existence of cyclic cosmology [43].

4.4 Understanding the cosmological term Λ

Differently from general relativity, the de Sitter-modified general relativity creates, besides the dynamic curvature representing the gravitational field, a non-gravitational homogeneous spacetime, whose sectional curvature represents the cosmological term Λ. The source of the cosmological term is the trace of the proper conformal current of ordinary matter

$$\Lambda = \frac{8\pi G}{c^4} K^\mu{}_\mu, \tag{4.32}$$

with the trace given by

$$K^\mu{}_\mu \equiv \vartheta^\mu{}_\alpha T^\alpha{}_\mu = (1/4l^2)(2\eta_{\alpha\rho} x^\rho x^\mu - \sigma^2 \delta^\mu_\alpha)\, T^\alpha{}_\mu. \tag{4.33}$$

As an example, let us consider the case of the universe as a whole. Assuming that the universe's matter content is preponderantly made up of dust, the trace of the proper conformal current computed at $x^i = 0$ is given by

$$K^\mu{}_\mu = \gamma(t)\, \varepsilon_m, \tag{4.34}$$

where $\gamma(t) = c^2 t^2/4l^2$. Equation (4.32) can then be rewritten in the form

$$\Lambda = \frac{8\pi G}{c^4}\, \gamma(t)\, \varepsilon_m. \tag{4.35}$$

Since $\gamma(t) = \varepsilon_\Lambda/\varepsilon_m$ is currently of order unity, $\gamma(t_0) \sim 1$, we obtain

$$\Lambda \simeq \frac{8\pi G}{c^4}\, \varepsilon_m. \tag{4.36}$$

Assuming that the universe's energy density ε_m is of the order of the critical energy density

$$\varepsilon_m \simeq \varepsilon_c = \frac{3H_0^2c^2}{8\pi G}, \tag{4.37}$$

Eq. (4.36) becomes

$$\Lambda \simeq \frac{3H_0^2}{c^2}. \tag{4.38}$$

Using the value $H_0 \simeq 67.4$ Km/s/Mpc, the cosmological term is found to be

$$\Lambda \simeq 10^{-52}\ \mathrm{m}^{-2}, \tag{4.39}$$

which is of the order of magnitude of the present-day observed value. We see from these considerations that the de Sitter-modified general relativity not only explains the origin of Λ but correctly determines its order of magnitude.

4.5 Spacetime Kinematics and conformal transformations

Local (or proper) conformal symmetry is a broken symmetry of nature, which is expected to play a significant role in the Planck scale [44]. There are convincing arguments that, at the Planck scale, it must be an exact symmetry of nature. However, Poincaré-invariant Einstein's special relativity does not include local conformal transformation in the spacetime kinematics. Consequently, it is unclear how it could become relevant at the Planck scale. For this reason, local conformal transformation is sometimes considered the missing component of the spacetime physics [45].

The great virtue of de Sitter invariant special relativity is that it naturally includes the proper conformal transformation in the spacetime kinematics. Such inclusion comes into play because the de Sitter group is obtained from Poincaré's by replacing ordinary translations with a combination of translation and local conformal transformations, usually known as de Sitter "translations". Observe that such replacement does not change the dimension of the spacetime's local kinematics as both Poincaré and de Sitter are ten-dimensional groups. The unique effect of such replacement is to change the local transitivity of spacetime from ordinary translation to the de Sitter "translations". As a consequence of this replacement, all solutions to the de Sitter-modified Einstein's equation will be spacetimes that reduce locally to de Sitter.

Chapter 5

Conclusions and Future Perspectives

Cosmological observations in the last few decades have shown that the universe's expansion is accelerating [13, 14, 15]. The problem is that Einstein's equation does not have a solution for a universe with accelerated expansion. Therefore, to describe the present-day universe, it is necessary to change general relativity or incorporate external elements to Einstein's equation, generically called *dark energy*, which would drive the observed accelerated expansion. For example, in the ΛCDM model, one incorporates a cosmological constant Λ into Einstein's equation.

There are two approaches to performing this incorporation. The first is to suppose the existence of a (perfect) fluid permeating the whole universe, whose energy-momentum current shows up as the source of the dark energy. However, to produce the necessary repulsive effect, the fluid must satisfy a nonphysical equation of state, never seen in any natural fluid. The second approach is to repeat Einstein and add a positive cosmological constant to the left-hand side of Einstein's equation.

On the other hand, although constructed to be consistent with quantum mechanics, the de Sitter-modified general relativity comes out deeply different from ordinary general relativity. For example, in the de Sitter-modified general relativity, the cosmological term Λ is incorporated into the spacetime's local kinematics, governed now by the de Sitter group. This means that the cosmological term is a constitutive part of the theory and does not need to be added by hand. Furthermore, Λ is no longer constrained to be constant, being allowed to change along with the cosmic time. Also, the source of the cosmological term is not the energy-momentum current of an exotic fluid but the *proper conformal current of ordinary matter*. Consequently, the de Sitter-modified general relativity can determine the origin and value of Λ. For these reasons, we can say that the de Sitter-invariant Friedmann equations provide an appealing alternative approach to cosmology.

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