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Some estimates for resolvent operators under the discretization by finite element method

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Abstract This paper is devoted to obtain some norm estimates for the difference between the two resolvent operators under the discretization of the domain $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, by finite element method.

Keywords Parabolic equations · Finite elements method · Resolvent operator approximation

Mathematics Subject Classification 35K15 · 65M60 · 47A58

1 Introduction and statement of the main result

Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, be a bounded domain, and assume either that the boundary $\partial\Omega$ is smooth or that Ω is a polyhedral domain. Let the second order uniformly strongly elliptic operator be given by

$$L = \sum_{i,j=1}^{n} a_{ij}(\cdot)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(\cdot)\frac{\partial}{\partial x_j} + (c(\cdot) + \lambda),$$

where the coefficients $a_{ij}, b_j, c : \overline{\Omega} \to \mathbb{R}$, $a_{ij} = a_{ji}$, $i, j = 1, \ldots, n$ are smooth functions and λ be a parameter to be specified later.

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Consider the initial boundary value problem of parabolic type

$$\begin{cases} u_t = Lu, \quad t > 0, \ x \in \Omega \\ u = 0, \quad t > 0, \ x \in \partial\Omega, \\ u(x,0) = \varphi(x), \quad x \in \Omega \end{cases} \tag{1}$$

where $\varphi \in H_0^1(\Omega)$.

Let $X = L^2(\Omega)$ be a Hilbert space and consider the nonselfadjoint linear operator $A : D(A) \subset X \to X$ by

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega),$$
$$Au = -Lu, \quad \forall u \in D(A).$$

In Figueroa-López and Lozada-Cruz (2014) was shown that operator A is sectorial in X and assuming that λ is chosen such that $\operatorname{Re}\sigma(A) > 0$, we can define the fractional powers A^α and the corresponding fractional power spaces $X^\alpha := D(A^\alpha)$, $\alpha > 0$, endowed with the graph norm (see Cholewa and Dlotko 2000, Section 1.3.3). X^α is a Hilbert space with the inner product $\langle \phi, \psi \rangle_\alpha = \int_\Omega A^\alpha \phi \, A^\alpha \psi$. Then, $X^1 = D(A)$, $X^0 = L^2(\Omega)$ and $X^{1/2} = H_0^1(\Omega)$.

As usual the problem (1) can be written as an abstract evolution equation in X

$$\begin{cases} \dot{u} + Au = 0, \quad t > 0, \\ u(0) = \varphi \in X^{1/2}. \end{cases} \tag{AP}$$

Fujita–Mizutani (see Fujita and Mizutani 1976) made a operator theoretical study of the finite element method applied to the initial boundary value problems for partial differential equations of parabolic type (1).

The discretization of the problem (AP) using the finite element method has the form

$$\begin{cases} \dot{u}_h + A_h u_h = 0, \\ u_h(0) = \varphi_h \in X_h^{1/2} \end{cases} \tag{AP$_h$}$$

where $X_h^{1/2} \subset X^{1/2} \cap C(\Omega)$ is a finite dimensional space obtained from the discretization of domain Ω by the finite element method, h is the largest diameter of each subdivision of the domain and $A_h : X_h^{1/2} \to X_h^{1/2}$ is the discretization of the operator A. More details on A_h and $X_h^{1/2}$ are given in the next section.

Let $u = u(t, \varphi)$ be the solution of the problem (AP) and let $u_h = u_h(t, \varphi_h)$ be the solution of the problem (1). In this context, it is natural to ask if h approaching to zero implies u_h to approximate u. To answer this question, we need to compare the solutions u and u_h in a certain sense when h tends to zero. We can see that u and u_h are living in different spaces.

Using the integral representation(the Dunford–Taylor integral), we have

$$u(t, \varphi) = \mathrm{e}^{-tA}\varphi = \frac{1}{2\pi i} \int_{\Gamma_1} \mathrm{e}^{-zt}(z - A)^{-1}\varphi \mathrm{d}z, \tag{2}$$

where Γ_1 is the positively oriented boundary, running from $\infty \mathrm{e}^{-i\theta_1}$ to $\infty \mathrm{e}^{i\theta_1}$, of the sector $\Sigma_1 = \{z \in \mathbb{C} : |\arg(z)| < \theta_1\}$ (see Pazy 1992, Theorem 1.7.7). Also, for u_h, we have a similar integral representation. With this, we can see that to compare u and u_h, we need to compare the resolvent operators $(z - A)^{-1}$ and $(z - A_h)^{-1}$ of A and A_h, respectively, i.e., we will show that $(z - A_h)^{-1}$ converge in an appropriate way to $(z - A)^{-1}$. This is a key point to our work. With the convergence of the resolvent operators A_h^{-1} to A one can show, with a Trotter-Kato-type formula, the convergence of the linear semigroups e^{-tA_h} to e^{tA}. This analysis has been proved to be successful when addressing to the behavior of the long-time dynamics in different perturbation problems (see Arrieta et al. 2006; Carvalho et al. 2013).

Now, we are in a position to formulate our main result.

Theorem 1 *If Assumption* 1 *holds, then, there exists a positive constant C and an acute angle θ_1 such that for any $f \in X$ and $z \in \mathcal{S}_{0,\theta_1}$, we have*

$$\|(z-A)^{-1}f-(z-A_h)^{-1}P_hf\|_{X^{1/2}} \leqslant Ch\|f\|_X, \tag{3}$$

$$\|(z-A)^{-1}f-(z-A_h)^{-1}P_hf\|_X \leqslant Ch^2\|f\|_X, \tag{4}$$

$$\|(z-A)^{-1}f-(z-A_h)^{-1}P_hf\|_X \leqslant Ch|z|^{-1/2}\|f\|_X, \tag{5}$$

where $\mathcal{S}_{0,\theta_1}=\{z\in\mathbb{C}:\theta_1\leqslant|\arg(z)|\leqslant\pi, z\neq 0\}\subset\rho(A)$ and P_h is projection operator from X to $X_h^{1/2}$.

Theorem 1 was proved by Fujita–Mizutani (Fujita and Mizutani 1976, Theorem 3.1) and Fujita–Saito–Suzuki (Fujita et al. 2001) for a bounded domain $\Omega\subset\mathbb{R}^2$ with smooth boundary or Ω a convex polygon.

The main goal of this paper is to show that Theorem 1 remains valid when $\Omega\subset\mathbb{R}^n$ is a bounded domain or a polyhedral domain with $n>2$, which has $\{\mathcal{T}^h\}_{h\in(0,1]}$ a quasi-uniform family of subdivisions and a reference element $(K,\mathcal{P},\mathcal{N})$ of class C^0 with K a "star-shaped" domain with respect to some ball. More details are given in Sect. 2.

Note that the norm estimates for the resolvent established in Gil' (2012, 2013) allow us to obtain the concrete bounds for the resolvent of A, when it is nonselfadjoint.

This paper is organized as follows. In Sect. 2, we introduce some notations and provide the discretization of the domain Ω needed to solve our problem. In Sect. 3, we obtain the rate of convergence of the resolvents and proceed with the proof of our main result.

2 Notations and preliminaries

In this section, we introduce some notations and we give the discretization of the domain $\Omega\subset\mathbb{R}^n$ with $n\geqslant 2$.

Since A is sectorial, we can associate it with a sesquilinear form $\sigma: X^{1/2}\times X^{1/2}\to\mathbb{C}$ given by

$$\sigma(u,v)=\langle Au,v\rangle_X,\quad u\in D(A),\ v\in X^{1/2}, \tag{6}$$

$$|\sigma(u,v)| \leqslant c_1\|u\|_{X^{1/2}}\|v\|_{X^{1/2}},\quad u,v\in X^{1/2} \tag{7}$$

$$\operatorname{Re}\sigma(u,u) \geqslant c_2\|u\|^2_{X^{1/2}}-\delta\|u\|^2_X,\quad u\in X^{1/2}, \tag{8}$$

where the constants c_1, c_2 are positives and $\delta<\infty$(see Brenner and Scott 1996, Section 5.6). In particular if $\delta=0$, we have

$$\operatorname{Re}\sigma(u,u)\geqslant c_2\|u\|^2_{X^{1/2}}. \tag{9}$$

Also, there are positive constants θ_1 and M_1 with $\theta_1<\pi/2$ such that

$$\mathcal{S}_{0,\theta_1}=\{z\in\mathbb{C}:\theta_1\leqslant|\arg(z)|\leqslant\pi, z\neq 0\}\subset\rho(A)\quad\text{and}$$

$$\|(z-A)^{-1}\|_{\mathscr{L}(X)}\leqslant\frac{M_1}{|z|},\quad\forall z\in\mathcal{S}_{0,\theta_1}. \tag{10}$$

Furthermore, a solution of the initial value problem (AP) is given by $u(t,\varphi)=\mathrm{e}^{-tA}\varphi$, $t\geqslant 0$, where $\{\mathrm{e}^{-tA}:t\geqslant 0\}$ is the analytic semigroup in $X^{1/2}$ generated by $-A$ (see Cholewa and Dlotko 2000, p. 35; Henry 1981, p. 21).

Now, we describe the discretization of the domain $\Omega \subset \mathbb{R}^n$ with $n \geqslant 2$, using the finite element method. Thus, we will be able to discretize the problem (AP) and study the limitations given in Fujita and Mizutani (1976) using the theory made in Brenner and Scott (1996) for its generalization to $n > 2$.

For a better understanding of the work and for convenience of the reader, we recall some definitions and results of Brenner and Scott (1996, Chapters 3 and 4).

Definition 1 Let

(i) $K \subseteq \mathbb{R}^n$ be a bounded closed set with nonempty interior and piecewise smooth boundary (the element domain),

(ii) $\mathcal{P}$ be a finite-dimensional space of functions on K (the space of shape functions) and

(iii) $\mathcal{N} = \{N_1, N_2, \ldots, N_k\}$ be a basis for $\mathcal{P}'$ (the set of nodal variables).

The triple $(K, \mathcal{P}, \mathcal{N})$ is called a finite element.

Definition 2 Given a finite element $(K, \mathcal{P}, \mathcal{N})$, let the set $\{\phi_i : 1 \leqslant i \leqslant k\} \subseteq \mathcal{P}$ be the basis dual to $\mathcal{N}$. If υ is a function for which all $N_i \in \mathcal{N}$, $i = 1, \ldots, k$, are defined, then we define the local interpolant by

$$\mathcal{I}_K \upsilon := \sum_{i=1}^{k} N_i(\upsilon)\phi_i.$$

Proposition 1 *Let $\mathcal{I}_K$ be the local interpolant, then*

(i) *$\mathcal{I}_K$ is linear.*

(ii) *$N_i(\mathcal{I}_K \upsilon) = N_i(\upsilon)$, $i = 1, \ldots, k$.*

(iii) *$\mathcal{I}_K(\upsilon) = \upsilon$ for $\upsilon \in \mathcal{P}$. In particular, $\mathcal{I}_K$ is idempotent, i.e., $\mathcal{I}_K^2 = \mathcal{I}_K$.*

Proof See Propositions 3.3.4, 3.3.5 and 3.3.7 in Brenner and Scott (1996). □

Definition 3 A subdivision of the domain $\Omega \subset \mathbb{R}^n$ is a finite collection of element domains $\{K_i\}_{i \in \mathbb{N}}$ such that $\operatorname{int} K_i \cap \operatorname{int} K_j = \emptyset$ if $i \neq j$ and $\bigcup_{i \in \mathbb{N}} K_i = \overline{\Omega}$.

Definition 4 Let $(K, \mathcal{P}, \mathcal{N})$ be a finite element with $K \subset \Omega \subset \mathbb{R}^n$ and let $F : \mathbb{R}^n \to \mathbb{R}^n$ be an affine map given by $F(x) = \mathbf{A}x + \mathbf{b}$ where $\mathbf{A}$ is a nonsingular matrix and $\mathbf{b}$ is a nonzero vector in $\mathbb{R}^n$. The finite element $(\widehat{K}, \widehat{\mathcal{P}}, \widehat{\mathcal{N}})$ is affine equivalent to $(K, \mathcal{P}, \mathcal{N})$ if $F(K) = \widehat{K}$, $F^*\widehat{\mathcal{P}} = \mathcal{P}$ and $F_*\mathcal{N} = \widehat{\mathcal{N}}$, where F^* is the "pull-back" of F defined by $F^*(\hat{f}) := \hat{f} \circ F$, and F_* is the "push-forward" of F defined by $(F_* N)(\hat{f}) := N(F^*(\hat{f})) = N(\hat{f} \circ F)$.

Definition 5 The domain Ω is star-shaped with respect to ball B if, for all $x \in \Omega$, the closed convex hull of $\{x\} \cup B$ is a subset of Ω. The domain Ω is star-shaped with respect to some ball if there is a ball B such that Ω is star-shaped with respect to ball B.

For example, any convex domain $\Omega \subset \mathbb{R}^n$ is star-shaped with respect to each ball $B \subset \Omega$.

Lemma 1 *If a bounded domain $\Omega \subset \mathbb{R}^n$ is start-shaped with respect to a ball $B \subset \Omega$, then it satisfies the cone condition and has a Lipschitz boundary with the parameters depending only on* $\operatorname{diam}(B)$, $\operatorname{diam}(\Omega)$ *and n.*

Proof See Lemmas 3.2.3 and 4.3.5 in Burenkov (1998). □

Definition 6 Let $\Omega \subset \mathbb{R}^n$ be a given domain and let $\{\mathcal{T}^h\}_{h \in (0,1]}$ be a family of subdivisions such that

$$\max\{\operatorname{diam}(T) : T \in \mathcal{T}^h\} \leqslant h \operatorname{diam}(\Omega).$$

(i) The family $\{\mathcal{T}^h\}_{h\in(0,1]}$ is said to be quasi-uniform if there exists $\rho > 0$ such that

$$\min\{\operatorname{diam}(B_T) : T \in \mathcal{T}^h\} \geqslant \rho\, h \operatorname{diam}(\Omega), \quad \forall h \in (0,1],$$

where B_T is the largest ball contained in T such that T is star-shaped with respect to B_T.

(ii) The family $\{\mathcal{T}^h\}_{h\in(0,1]}$ is said to be non-degenerate or regular if there exists $\rho > 0$ such that for all $T \in \mathcal{T}^h$ and for all $h \in (0,1]$,

$$\operatorname{diam}(B_T) \geqslant \rho \operatorname{diam}(T). \tag{11}$$

Remark 1 If a family is quasi-uniform, then it is non-degenerate, but not conversely.

Definition 7 A reference element $(K, \mathcal{P}, \mathcal{N})$ is said to be a C^r element if r is the largest non-negative integer for which the finite element space satisfies

$$V^h := \mathcal{I}^h(C^l(\overline{\Omega})) \subseteq C^r(\Omega) \cap W^{r+1,\infty}(\Omega), \tag{12}$$

where $\mathcal{I}^h : C^l(\overline{\Omega}) \to L^1(\Omega)$ is the global interpolation operator defined by

$$\mathcal{I}^h v\big|_T := \mathcal{I}_T^h v, \quad \text{for } T \in \mathcal{T}^h,\ h \in (0,1],$$

and $\mathcal{I}_T^h$ is the interpolation operator for the affine-equivalent element $(T, \mathcal{P}_T, \mathcal{N}_T)$.

Let $\mathcal{P}_k$ be the set of polynomials in n variables of degree less than or equal to k with $\dim(\mathcal{P}_k) = \binom{n+k}{k}$.

Theorem 2 *Let $\{\mathcal{T}^h\}_{h\in(0,1]}$ be a non-degenerate family of subdivisions of a polyhedral domain Ω in $\mathbb{R}^n$, $n \in \mathbb{N}$. Let $(K, \mathcal{P}, \mathcal{N})$ be a reference element satisfying*

(i) *K is star-shaped with respect to some ball,*

(ii) *$\mathcal{P}_{m-1} \subseteq \mathcal{P} \subseteq W^{m,\infty}(K)$,*

(iii) *$\mathcal{N} \subseteq (C^l(\overline{K}))'$,*

(iv) *$p \in [1,\infty]$ and either $m - l - n/p > 0$ when $p > 1$ or $m - l - n \geqslant 0$ when $p = 1$.*

For all $T \in \mathcal{T}^h$, $h \in (0,1]$, let $(T, \mathcal{P}_T, \mathcal{N}_T)$ be the affine-equivalent element. Then, there exist a positive constant C depending on the reference element, n, m, p and the number ρ in (11) such that for $0 \leqslant s \leqslant m$,

$$\left(\sum_{T\in\mathcal{T}^h} \|v - \mathcal{I}_T^h v\|_{W^{s,p}(T)}^p\right)^{1/p} \leqslant Ch^{m-s}|v|_{W^{m,p}(\Omega)}, \quad \forall v \in W^{m,p}(\Omega). \tag{13}$$

Moreover, in the case $p = \infty$, for $0 \leqslant s \leqslant l$,

$$\max_{T\in\mathcal{T}^h} \|v - \mathcal{I}_T^h v\|_{W^{s,\infty}(T)} \leqslant Ch^{m-s-n/p}|v|_{W^{m,p}(\Omega)}, \quad \forall v \in W^{m,p}(\Omega). \tag{14}$$

Proof See Theorem 4.4.20 in Brenner and Scott (1996). □

Theorem 3 (Inverse Estimate) *Let $\{\mathcal{T}^h\}_{h\in(0,1]}$ be a quasi-uniform family of subdivisions of a polyhedral domain $\Omega \subseteq \mathbb{R}^n$. Let $(K, \mathcal{P}, \mathcal{N})$ be a reference finite element of the class C^r such that $\mathcal{P} \subseteq W^{l,p}(K) \cap W^{m,q}(K)$ where $1 \leqslant p, q < \infty$ and $0 \leqslant m \leqslant l$. For $T \in \mathcal{T}^h$, let $(T, \mathcal{P}_T, \mathcal{N}_T)$ be the affine-equivalent element, and $V^h = \{v : v$ is mensurable and $v_{|T} \in \mathcal{P}_T, \forall T \in \mathcal{P}_T\}$. Then, there exist a positive constant $C = C(l, p, q, \rho)$ such that*

$$\left[\sum_{T\in\mathcal{T}^h} \|v\|_{W^{l,p}(T)}^p\right]^{1/p} \leqslant Ch^{m-l+\min\left(0,\frac{n}{p}-\frac{n}{q}\right)} \left[\sum_{T\in\mathcal{T}^h} \|v\|_{W^{m,q}(T)}^q\right]^{1/q}, \quad \forall v \in V^h. \tag{15}$$

Proof See Theorem 4.5.11 in Brenner and Scott (1996). □

Remark 2 In the event that the elements in the previous results form C^r elements for some $r \geqslant 0$, then for $0 \leqslant s \leqslant r+1$, we have

$$\sum_{T \in \mathcal{T}^h} \|\upsilon - \mathcal{I}_T^h \upsilon\|_{W^{s,p}(T)}^p = \|\upsilon - \mathcal{I}^h \upsilon\|_{W^{s,p}(\Omega)}^p, \tag{16}$$

$$\max_{T \in \mathcal{T}^h} \|\upsilon - \mathcal{I}_T^h \upsilon\|_{W^{s,\infty}(T)} = \|\upsilon - \mathcal{I}^h \upsilon\|_{W^{s,\infty}(\Omega)} \tag{17}$$

and

$$\left[\sum_{T \in \mathcal{T}^h} \|\upsilon\|_{W^{l,\varrho}(T)}^{\varrho}\right]^{1/\varrho} = \|\upsilon\|_{W^{s,\varrho}(\Omega)}, \quad \text{for } \varrho = p, q. \tag{18}$$

Substituting these expressions in the left-hand side leads to estimates of the form

$$\|\upsilon - \mathcal{I}^h \upsilon\|_{W^{s,p}(\Omega)} \leqslant C h^{m-s} |\upsilon|_{W^{m,p}(\Omega)}, \tag{19}$$

for all $\upsilon \in W^{m,p}(\Omega)$ and $0 \leqslant s \leqslant \min\{m, r+1\}$ and

$$\|\upsilon - \mathcal{I}^h \upsilon\|_{W^{s,\infty}(\Omega)} \leqslant C h^{m-s-n/p} |\upsilon|_{W^{m,p}(\Omega)}, \tag{20}$$

for all $\upsilon \in W^{m,p}(\Omega)$ and $0 \leqslant s \leqslant \min\{l, r+1\}$ and

$$\|v\|_{W^{l,p}(\Omega)} \leqslant C h^{m-l+\min\left(0,\frac{n}{p}-\frac{n}{q}\right)} \|v\|_{W^{m,q}(\Omega)}, \tag{21}$$

for all $v \in V^h$ and $0 \leqslant s \leqslant \min\{m, r+1\}$.

Now, we are able to define the discretization of the domain Ω, which will allow us to obtain the proof of our main result (Theorem 1).

We assume the following hypothesis about the domain Ω.

Assumption 1 Let $\Omega \subset \mathbb{R}^n$ be a polyhedral domain with $n \geqslant 2$ which has $\{\mathcal{T}^h\}_{h \in (0,1]}$ a quasi-uniform family of subdivisions with positive constant ρ and let $(K, \mathcal{P}, \mathcal{N})$ be a reference element of class C^0 satisfying

(i) K is star-shaped with respect to some ball,
(ii) $\mathcal{P}_1 \subseteq \mathcal{P} \subseteq W^{2,\infty}(K)$ and
(iii) $\mathcal{N} \subseteq (C^{\ell}(\overline{K}))'$, $\ell \in \mathbb{Z}_0^+$.

With this assumption and (12), we can define the finite element space

$$X_h^{1/2} := \{\mathcal{I}^h \upsilon : \upsilon \in C^{\ell}(\overline{\Omega}), \upsilon\big|_{\partial\Omega} = 0\} \subset X^{1/2} \cap C(\Omega),$$

which has finite dimension and $\mathcal{I}^h$ is the global interpolation operator.

We can see that in Assumption 1, we still do not know the value of ℓ. This value, ℓ, comes from the inequality $m - \ell - n/p > 0$ with $p \in [1, \infty]$ for $\upsilon \in W^{m,p}(\Omega)$ from Theorem 2. In our case, $m = 2 = p$ and $n \geqslant 2$, then $\ell < 1$. Thus, $\ell = 0$.

From Assumption 1, Remark 2 and Theorems 2 and 3, there exist positive constants C and $\hat{C}$ such that

$$\|v - \mathcal{I}^h v\|_{L^2(\Omega)} \leqslant C h^2 |v|_{W^{2,2}(\Omega)}, \ \forall v \in X^1, \tag{22}$$

$$\|v - \mathcal{I}^h v\|_{W^{1,2}(\Omega)} \leqslant C h |v|_{W^{2,2}(\Omega)}, \ \forall v \in X^1, \tag{23}$$

$$\|v\|_{W^{1,2}(\Omega)} \leqslant \hat{C} h^{-1} \|v\|_{L^2(\Omega)}, \ \forall v \in X_h^{1/2}. \tag{24}$$

In this framework, $A_h : X_h^{1/2} \to X_h^{1/2}$ given by

$$\langle A_h \phi_h, \psi_h \rangle_X = \sigma(\phi_h, \psi_h), \quad \phi_h, \psi_h \in X_h^{1/2}$$

is the finite element approximation of A. Thus, the discretization of problem (AP) can be written as (1).

Definition 8 The orthogonal projection $P_h : X \to X_h^{1/2}$ with relation to the inner product of X is defined by

$$\langle P_h g, v \rangle_X = \langle g, v \rangle_X, \quad \forall v \in X_h^{1/2}. \tag{25}$$

The following result follows immediately from the above definition.

Lemma 2 *The orthogonal projection P_h satisfies*

$$\|P_h f\|_X \leqslant \|f\|_X, \quad \forall h \in (0, 1].$$

Lemma 3 *If (7) and (9) hold, then there is a positive constant C such that*

$$\|P_h v - v\|_{X^s} \leqslant C h^{2-2s} |v|_{H^2(\Omega)}, \quad \forall v \in X \text{ and } s = 0, 1/2. \tag{26}$$

Proof First, we see the case $s = 0$. Given $v \in X^1$, we obtain

$$\|P_h v - v\|_X^2 \leqslant |\langle P_h v - v, P_h v - v \rangle_X| \leqslant \|\chi_h - v\|_X \|P_h v - v\|_X,$$

where $\chi_h = P_h v \in X_h^{1/2}$. Then, $\|P_h v - v\|_X \leqslant \min_{\chi_h \in X_h^{1/2}} \|\chi_h - v\|_X$. Thus, using (22) and the embedding of X^1 in X the result follows.

Now, let us consider the case $s = 1/2$. Recall, the definition of the projection operator $P_h : X \to X^{1/2}$ which $A_h u_h = P_h v$ for some $v \in X$. Thus, from (7) and (9), we have

$$\begin{aligned} c_2 \|P_h v - v\|_{X^{1/2}}^2 &\leqslant \operatorname{Re} \sigma(P_h v - v, P_h v - v) = \operatorname{Re} \sigma(P_h v - v, v_h - v) \\ &\leqslant c_1 \|P_h v - v\|_{X^{1/2}} \|v_h - v\|_{X^{1/2}}, \end{aligned}$$

where $v_h := A_h u_h$. Then, $\|P_h v - v\|_{X^{1/2}} \leqslant \min_{v_h \in X_h^{1/2}} \dfrac{c_1}{c_2} \|v_h - v\|_{X^{1/2}}$. Thus, using (23) we obtain

$$\|P_h v - v\|_{X^{1/2}} \leqslant \frac{c_1}{c_2} \|\mathcal{I}^h v - v\|_{X^{1/2}} \leqslant \frac{c_1 C}{c_2} h |v|_{H^2(\Omega)}.$$

Since X^1 is dense in X, hence the result follows. □

Definition 9 The map $\tilde{P}_h : X^{1/2} \to X_h^{1/2}$ defined by

$$\sigma(\tilde{P}_h v, \chi) = \sigma(v, \chi), \quad \forall \chi \in X_h^{1/2}$$

is called the elliptic projection in $X^{1/2}$.

Lemma 4 *If (7) and (9) hold, then*

(i) *The map $\tilde{P}_h$ is well defined, i.e., for all $v \in X^{1/2}$, there is a unique $v_h \in X_h^{1/2}$ such that*

$$\sigma(v - v_h, \chi) = 0, \quad \forall \chi \in X_h^{1/2}. \tag{27}$$

(ii) *There is a constant $C > 0$ such that*

$$\|\tilde{P}_h v\|_{X^{1/2}} \leqslant C \|v\|_{X^{1/2}}, \quad \forall v \in X^{1/2}. \tag{28}$$

Proof (i) Let $v \in X^{1/2}$ and define

$$F(\chi) := \sigma(v, \chi), \quad F \in (X_h^{1/2})^*. \tag{29}$$

Since σ is a continuous and coercive bilinear form in $X^{1/2}$ and $X_h^{1/2} \subset X^{1/2}$, follows that σ is a continuous and coercive bilinear form in $X_h^{1/2}$. By Lax–Milgram (see Brenner and Scott 1996, Theorem 2.7.7), there exists a unique $v_h \in X_h^{1/2}$ such that

$$\sigma(v_h, \chi) = F(\chi), \quad \forall \chi \in X_h^{1/2}. \tag{30}$$

From (29) and (30) follows the result.

(ii) For $v \in X^{1/2}$, we obtain

$$c_2 \|\tilde{P}_h v\|_{X^{1/2}}^2 \leqslant |\sigma(\tilde{P}_h v, \tilde{P}_h v)| = |\sigma(v, \tilde{P}_h v)| \leqslant c_1 \|v\|_{X^{1/2}} \|\tilde{P}_h v\|_{X^{1/2}}.$$

Therefore, $\|\tilde{P}_h v\|_{X^{1/2}} \leqslant C\|v\|_{X^{1/2}}$, where $C = c_1/c_2$. □

Theorem 4 *If Assumption 1 holds, then there exists a constant $C > 0$ such that*

$$\|u - \tilde{P}_h u\|_X \leqslant Ch\|u\|_{X^{1/2}}, \quad u \in X^{1/2}, \tag{31}$$

$$\|u - \tilde{P}_h u\|_{X^{1/2}} \leqslant Ch\|u\|_{X^1}, \quad u \in X^1, \tag{32}$$

$$\|u - \tilde{P}_h u\|_X \leqslant Ch^2\|u\|_{X^1}, \quad u \in X^1. \tag{33}$$

Proof We denote by $\tilde{u}^h := \tilde{P}_h u$, $\hat{u}^h := \mathcal{I}^h u$ and $e := u - \tilde{u}^h$. From (27), we have

$$\sigma(e, \varphi_h) = 0, \quad \forall \varphi_h \in X_h^{1/2}. \tag{34}$$

Hence, putting $\varphi_h = \tilde{u}^h - \hat{u}^h$, using (9), (34), (7) and (23), we obtain

$$\|e\|_{X^{1/2}}^2 \leqslant \frac{1}{c_2}|\sigma(e, e + \varphi_h)| \leqslant \frac{c_1}{c_2}\|e\|_{X^{1/2}}\|u - \hat{u}^h\|_{X^{1/2}} \leqslant Ch\|e\|_{X^{1/2}}\|u\|_{H^2(\Omega)},$$

which implies (32). We recall that

$$\|e\|_X = \sup_{0 \neq \phi \in X} \frac{|(e, \phi)_X|}{\|\phi\|_X}. \tag{35}$$

From theory of elliptic operators (see Friedman 2008), we know that given $\phi \in X$, there is a $\psi \in X^{1/2}$, such that

$$\operatorname{Re}\sigma(v, \psi) = (v, \phi)_X, \quad \forall v \in X^{1/2}. \tag{36}$$

Moreover, $\psi \in H^2(\Omega)$ and

$$\|\psi\|_{H^2(\Omega)} \leqslant C\|\phi\|_X. \tag{37}$$

Now, substituting (36) and (37) in (35), we have

$$\|e\|_X \leqslant C \sup_{0 \neq \psi \in X^1} \frac{|\operatorname{Re}\sigma(e, \psi)|}{\|\psi\|_{H^2(\Omega)}}. \tag{38}$$

But, again using (34), we obtain

$$\operatorname{Re}\sigma(e, \psi) = \operatorname{Re}\sigma(e, \psi) - \operatorname{Re}\sigma(e, \psi_h) = \operatorname{Re}\sigma(e, \psi - \psi_h), \quad \forall \psi_h \in X_h^{1/2}. \tag{39}$$

From continuity of σ and (39), it follows

$$|\operatorname{Re}\sigma(e, \psi)| = |\operatorname{Re}\sigma(e, \psi - \psi_h)| \leqslant c_1\|e\|_{X^{1/2}}\|\psi - \psi_h\|_{X^{1/2}}. \tag{40}$$

Choosing ψ_h satisfying the inequality (32), we obtain

$$\|\psi - \psi_h\|_{X^{1/2}} \leqslant \tilde{C}h\|\psi\|_{H^2(\Omega)}. \tag{41}$$

Then, from (38), (40) and (41), we have

$$\|e\|_X \leqslant Cc_1 \sup_{0\neq\psi\in X^1} \frac{\|e\|_{X^{1/2}}\tilde{C}h\|\psi\|_{H^2(\Omega)}}{\|\psi\|_{H^2(\Omega)}}.$$

Thus,

$$\|e\|_X \leqslant \bar{C}h\|e\|_{X^{1/2}}. \tag{42}$$

Since $\|e\|_{X^{1/2}} \leqslant \|u\|_{X^{1/2}}$ by the definition of $\tilde{u}^h$ and (42), we obtain (31). Finally, replacing (32) in (42) it follows (33). □

Lemma 5 *If* $\{\mathcal{T}_h\}_{h\in(0,1]}$ *is quasi-uniform family of subdivisions, then there exists a positive constant* C *such that*

$$\|P_h v\|_{X^{1/2}} \leqslant C\|v\|_{X^{1/2}}, \quad \forall\, h \in (0,1],\ v \in X^{1/2}. \tag{43}$$

Proof Since $\|P_h v\|^2_{X^{1/2}} \leqslant \|P_h v\|^2_X + \|\nabla(P_h - I)v\|^2_X + \|\nabla v\|^2_X$, we only need to show that there exists $\bar{C} > 0$ such that

$$\|\nabla(P_h - I)v\|_X \leqslant \bar{C}\|v\|_{X^{1/2}}, \quad \forall v \in X^{1/2}.$$

In fact, denote $P_h|_{X_h^{1/2}} = I_h$, $\chi_h := \tilde{P}_h v \in X_h^{1/2}$. Now, from (24) and (31), we obtain

$$\begin{aligned}\|\nabla(P_h - I)v\|_X &= \|\nabla(P_h - I_h)(v - \chi_h)\|_X \leqslant \|\nabla P_h(v - \chi_h)\|_X + \|\nabla(v - \chi_h)\|_X\\ &\leqslant \hat{C}h^{-1}\|P_h(v - \chi_h)\|_X + \|v\|_{X^{1/2}} + \|\chi_h\|_{X^{1/2}}\\ &\leqslant \hat{C}h^{-1}\|v - \chi_h\|_X + 2\|v\|_{X^{1/2}} \leqslant (\hat{C} + 2)\|v\|_{X^{1/2}},\end{aligned}$$

for all $v \in X^{1/2}$. Therefore, from Lemma 2 the result follows. □

3 Proof of the main result

In this section, we present the proof of our main result. For this, first we will establish some results that will be needed.

Lemma 6 *If* (6), (7) *and* (9) *hold, then there exists a positive constant* δ_1 *such that*

$$|z|\|\varphi\|^2_X + \|\varphi\|^2_{X^{1/2}} \leqslant \delta_1\big|z\,\|\varphi\|^2_X - \sigma(\varphi,\varphi)\big|, \quad \forall \varphi \in X^{1/2}, \quad \forall z \in \mathcal{S}_{0,\theta_1}. \tag{44}$$

Proof See Lemma 3.3 in Fujita and Mizutani (1976). □

Lemma 7 *Under the hypotheses of Lemma 6. Given* $z \in \mathcal{S}_{0,\theta_1}$, *there is a constant* $\delta_1 > 0$ *such that*

$$|z|\|\varphi_h\|^2_X + \|\varphi_h\|^2_{X^{1/2}} \leqslant \delta_1|\langle(z - A_h)\varphi_h, \varphi_h\rangle_X|, \quad \forall \varphi_h \in X_h^{1/2}. \tag{45}$$

Furthermore, $\mathcal{S}_{0,\theta_1} \subset \rho(A_h)$ *and the following inequalities are valid*

$$\|(z - A_h)^{-1} f_h\|_X \leqslant \delta_1\|f_h\|_X/|z|, \tag{46}$$

$$\|(z - A_h)^{-1} f_h\|_{X^{1/2}} \leqslant \delta_1\|f_h\|_X/|z|^{1/2}, \tag{47}$$

$$\|A_h(z - A_h)^{-1} f_h\|_X \leqslant (1 + \delta_1)\|f_h\|_X, \tag{48}$$

for all $f_h \in X_h^{1/2}$ *and* $z \in \mathcal{S}_{0,\theta_1}$.

Proof See Corollary 3.4 in Fujita and Mizutani (1976). □

Remark 3 Given $a \in X$ and $z \in \mathcal{S}_{0,\theta_1}$, then

$$\|A^{\alpha}(z-A)^{-1}a\|_X \leqslant C_{\alpha}|z|^{-1+\alpha}\|a\|_X, \tag{49}$$

where $C_{\alpha} = C(1+\delta_1)^{\alpha}\delta_1^{1-\alpha}$, for all $z \in \mathcal{S}_{0,\theta_1}$, $0 \leqslant \alpha \leqslant 1$. Moreover, (49) is valid when we replace A by A_h.

Proof of Theorem 1

Let $z \in \mathcal{S}_{0,\theta_1}$ and $f \in X$. Define $w = (z-A)^{-1}f$ and $w_h = (z-A_h)^{-1}P_h f$, where $w \in D(A)$ and $w_h \in D(A_h)$. From (25), w and w_h satisfy

$$z\langle w, \varphi\rangle_X - \sigma(w,\varphi) = \langle f, \varphi\rangle_X, \ \forall \varphi \in X^{1/2} \ \text{ and}$$

$$z\langle w_h, \varphi_h\rangle_X - \sigma(w_h,\varphi_h) = \langle f, \varphi_h\rangle_X, \ \forall \varphi_h \in X_h^{1/2},$$

, respectively. If we call $e_h(z) = w(z) - w_h(z)$ and we use the two previous equalities, we obtain

$$z\langle e_h, \varphi_h\rangle_X - \sigma(e_h,\varphi_h) = 0, \quad \forall \varphi_h \in X_h^{1/2}. \tag{50}$$

Thus, given $\varphi_h \in X_h^{1/2}$ along with (44) and (50), we have

$$\begin{aligned}|z|\|e_h\|_X^2 + \|e_h\|_{X^{1/2}}^2 &\leqslant \delta_1|\, z\|e_h\|_X^2 - \sigma(e_h,e_h)| \\ &= \delta_1|z\langle e_h, e_h+\varphi_h\rangle_X - \sigma(e_h, e_h+\varphi_h)|.\end{aligned}$$

Putting $\varphi_h = w_h - \tilde{w}^h$ and using (31) and (32), we obtain

$$\begin{aligned}|z|\|e_h\|_X^2 + \|e_h\|_{X^{1/2}}^2 &\leqslant \delta_1|z\langle e_h, w-\tilde{w}^h\rangle_X - \sigma(e_h, w-\tilde{w}^h)| \\ &\leqslant \delta_1(|z|\|e_h\|_X\|w-\tilde{w}^h\|_X + c_1\|e_h\|_{X^{1/2}}\|w-\tilde{w}^h\|_{X^{1/2}}) \\ &\leqslant \delta_1 Ch|z|\|e_h\|_X\|w\|_{X^{1/2}} + \delta_1 c_1 Ch\|e_h\|_{X^{1/2}}\|w\|_{H^2(\Omega)} \\ &\leqslant C_3 h(|z|\|e_h\|_X\|w\|_{X^{1/2}} + \|e_h\|_{X^{1/2}}\|w\|_{H^2(\Omega)}),\end{aligned} \tag{51}$$

where $C_3 := \max\{C\delta_1, C\delta_1 c_1\}$. On the other hand, from (9) and (6), we have that $\ker(A) = 0$. Thus, using the inequality (1.3.26) in Zheng (2004, p. 14) and (10), we get

$$\|w\|_{H^2(\Omega)} \leqslant C_4\|Aw\|_X \leqslant C_5\|f\|_X, \tag{52}$$

where $C_5 := \max\{C_4, C_4\delta_1\}$. Using (49) with $\alpha = 1/2$ and (52) in (51), we have

$$\begin{aligned}|z|\|e_h\|_X^2 + \|e_h\|_{X^{1/2}}^2 &\leqslant C_3 h(\delta_1|z|^{1/2}\|e_h\|_X\|f\|_X + C_5\|e_h\|_{X^{1/2}}\|f\|_X) \\ &\leqslant C_6 h\|f\|_X(|z|^{1/2}\|e_h\|_X + \|e_h\|_{X^{1/2}}),\end{aligned} \tag{53}$$

where $C_6 := \max\{C_3\delta_1, C_3C_5\}$. Thus, from (53), we obtain

$$\begin{aligned}|z|\|e_h\|_X^2 + \|e_h\|_{X^{1/2}}^2 &\leqslant C_6 h\|f\|_X(|z|^{1/2}\|e_h+\varphi_h\|_X + \|e_h+\varphi_h\|_{X^{1/2}}) \\ &\leqslant C_6 h\|f\|_X(|z|^{1/2}\|w-\tilde{w}^h\|_X + \|w-\tilde{w}^h\|_{X^{1/2}}) \\ &\leqslant C_6 h\|f\|_X(|z|^{1/2}Ch\|w\|_{X^{1/2}} + Ch\|w\|_{H^2(\Omega)}) \\ &\leqslant C_6 Ch^2\|f\|_X(|z|^{1/2}\|w\|_{X^{1/2}} + \|w\|_{H^2(\Omega)}) \\ &\leqslant C_6 Ch^2\|f\|_X(\delta_1\|f\|_X + C_5\|f\|_X) \leqslant C_7 h^2\|f\|_X^2,\end{aligned} \tag{54}$$

where $C_7 := \max\{C_6C\delta_1, C_6CC_5\}$. From (54), it follows (3) and (5).

Finally, we will show (4). Let A^* be the adjoint operator of A. A^* is a accretive operator associated with the sesquilinear form $\sigma^*(u, v) = \overline{\sigma(v, u)}$ (see Henry 1981, p.203). Since $D(A^*) = D(A)$ for $g \in X$ and $z \in \mathcal{S}_{0,\theta_1}$, we define

$$v = (z - A^*)^{-1} g \quad \text{and} \quad v_h = (z - A_h^*)^{-1} P_h g.$$

Applying (3) and (5), and substituting A and A_h by A^* and A_h^*, respectively. We have

$$\begin{aligned} |(e_h, g)| &= |z(e_h, v) - \sigma(e_h, v)| = |z(e_h, v - v_h) - \sigma(e_h, v - v_h)| \\ &\leqslant C(|z| \|e_h\|_X \|v - v_h\|_X + \|e_h\|_{X^{1/2}} \|v - v_h\|_{X^{1/2}}) \\ &\leqslant Ch^2(|z|^{1/2} \|f\|_X \|v\|_{X^{1/2}} + \|f\|_X \|v\|_{H^2(\Omega)}) \\ &\leqslant Ch^2 \|f\|_X \|g\|_X. \end{aligned} \tag{55}$$

Using (35) and (55), we obtain (4). □

The following remarks follow from Theorem 1.

Remark 4 We can see that (3) and (4) hold for $z = 0$.

Remark 5 From (49), the inequality of (46) and of Lemma 2, we obtain

$$\begin{aligned} \|e_h(z)\|_X &\leqslant \|w\|_X + \|w_h\|_X \leqslant \delta_1 \|f\|_X |z|^{-1} + \delta_1 \|P_h f\|_X |z|^{-1} \\ &\leqslant C_8 \|f\|_X |z|^{-1}. \end{aligned} \tag{56}$$

Using (4) and (56), given $\alpha \in [0, 1]$, we have

$$\|e_h(z)\|_X \leqslant C \|e_h(z)\|_X^{\alpha} \|e_h(z)\|_X^{1-\alpha} \leqslant C_\alpha h^{2\alpha} |z|^{\alpha - 1} \|f\|_X. \tag{57}$$

Remark 6 From Theorem 1, Lemma 3 and (52), we get

$$\begin{aligned} &\|(z - A_h)^{-1} P_h f - P_h (z - A)^{-1} f\|_{X^s} \leqslant \|e_h(z)\|_{X^s} + \|P_h w - w\|_{X^s} \\ &\quad \leqslant Ch^{2-2s} \|f\|_X + Ch^{2-2s} \|w\|_{H^2(\Omega)} \leqslant Ch^{2-2s} \|f\|_X, \end{aligned} \tag{58}$$

for all $z \in \mathcal{S}_{0,\theta_1}$, $f \in X$ and $s = 0, 1/2$.

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