UNIVERSIDADE ESTADUAL PAULISTA – UNESP
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ELLIPTIC SYSTEMS IN POPULATION DYNAMICS:
A STUDY WITH NONLOCAL DIFFUSION COEFFICIENTS

São José do Rio Preto
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Tese apresentada, em regime de co-tutela, à Universidade Estadual Paulista (UNESP), Instituto de Biociências, Letras e Ciências Exatas, São José do Rio Preto e à Universidad de Sevilla (US), para obtenção do título de Doutorado em Matemática (UNESP) e em Doctorado en Matemáticas (US).

Área de Concentração: Análise Aplicada

Orientador: Prof. Dr. Marcos Tadeu de Oliveira Pimenta (UNESP)

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Aos meus pais, Antonio e Regina.
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"Every second is time to change everything forever."
Charlie Chaplin

RESUMO

Este trabalho investiga a existência e unicidade de estados de coexistência em sistemas elípticos não-locais que modelam interações entre duas espécies, com difusão dependente da população de outra espécie ou da própria. O problema geral analisado é representado por:

$$\begin{cases} -m\left(\int_\Omega u, \int_\Omega v\right)\Delta u = f(x,u,v) & \text{em } \Omega, \\ -n\left(\int_\Omega u, \int_\Omega v\right)\Delta v = g(x,u,v) & \text{em } \Omega, \\ u = v = 0 & \text{sobre } \partial\Omega, \end{cases} \tag{P}$$

onde Ω é um domínio regular limitado em $\mathbb{R}^N$, com $N \geq 1$, $m, n : \mathbb{R}^2_+ \to [0, +\infty)$ são funções contínuas não-lineares e $f, g : \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ funções contínuas.

No primeiro modelo, analisamos a interação entre uma bactéria e um nutriente, com difusão não-linear. Usamos Bifurcação Local e Global e o Teorema da Função Implícita para determinar condições de existência e unicidade de soluções positivas. No segundo modelo, estudamos sistemas de Lotka-Volterra com difusão cruzada não-local, modelando interações de competição, predador-presa e simbiose. Investigamos a estabilidade de soluções semi-triviais e a coexistência. No terceiro modelo, abordamos a competição de Lotka-Volterra com difusão não-local, onde a difusão depende da população de cada espécie. Garantimos a coexistência com o princípio da exclusão competitiva.

Os resultados destacam a importância da difusão não-local na modelagem de interações biológicas e na dinâmica de coexistência.

PALAVRAS-CHAVE: dinâmica de populações; sistemas elípticos não-locais; bifurcação local e global; índice de ponto fixo; método de sub-supersolução.

RESUMEN

Este trabajo investiga la existencia y unicidad de estados de coexistencia en sistemas elípticos no locales que modelan interacciones entre dos especies, con difusión dependiente de la población de otra especie o de la misma. El problema general estudiado está representado por:

$$\begin{cases} -m\left(\int_\Omega u, \int_\Omega v\right)\Delta u = f(x,u,v) & \text{en } \Omega, \\ -n\left(\int_\Omega u, \int_\Omega v\right)\Delta v = g(x,u,v) & \text{en } \Omega, \\ u = v = 0 & \text{sobre } \partial\Omega, \end{cases} \tag{P}$$

donde Ω es un dominio regular acotado en $\mathbb{R}^N$, con $N \geq 1$, $m, n : \mathbb{R}^2_+ \to [0, +\infty)$ son funciones no lineales continuas, y $f, g : \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ son funciones continuas.
En el primer modelo, analizamos la interacción entre una bacteria y un nutriente con difusión no lineal. Utilizamos bifurcación local y global y el teorema de la función implícita para determinar las condiciones de existencia y unicidad de soluciones positivas. En el segundo modelo, estudiamos sistemas de Lotka-Volterra con difusión cruzada no local, modelando interacciones de competencia, depredador-presa y simbiosis. Investigamos la estabilidad de soluciones semi-triviales y la coexistencia. En el tercer modelo, abordamos la competencia de Lotka-Volterra con difusión no local, donde la difusión depende de la población de cada especie. Garantizamos la coexistencia con el principio de exclusión competitiva.

Los resultados destacan la importancia de la difusión no local en la modelización de interacciones biológicas y en la dinámica de la coexistencia.

PALABRAS CLAVE: dinámica de poblaciones; sistemas elípticos no-locales; bifurcación local y global; índice de punto fijo; método de sub-supersoluciones.

ABSTRACT

This work investigates the existence and uniqueness of coexistence states in non-local elliptic systems that model interactions between two species, with diffusion dependent on the population of another species or on the same species. The general problem studied is represented by:

$$\begin{cases} -m\left(\int_\Omega u, \int_\Omega v\right)\Delta u = f(x,u,v) & \text{in } \Omega, \\ -n\left(\int_\Omega u, \int_\Omega v\right)\Delta v = g(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where Ω is a regular bounded domain in $\mathbb{R}^N$, with $N \geq 1$, $m, n : \mathbb{R}^2_+ \to [0, +\infty)$ are continuous nonlinear functions, and $f, g : \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions.

In the first model, we analyze the interaction between a bacterium and a nutrient with nonlinear diffusion. We use Local and Global Bifurcation and the Implicit Function Theorem to determine conditions for the existence and uniqueness of positive solutions. In the second model, we study Lotka-Volterra systems with non-local cross-diffusion, modeling competition, predator-prey and symbiosis interactions. We investigate the stability of semi-trivial solutions and coexistence. In the third model, we address Lotka-Volterra competition with non-local diffusion, where diffusion depends on the population of each species. We guarantee coexistence using the competitive exclusion principle.

The results highlight the importance of non-local diffusion in modeling biological interactions and coexistence dynamics.

KEYWORDS: population dynamics; non-local elliptic systems; local y global bifurcation; fixed-point index; sub-supersolution method.

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Introduction

In the study of *Population Dynamics*, the focus lies on understanding the complex interactions among one or more populations of living organisms over time and across domains. This field combines mathematical modeling, theoretical analysis, and empirical observations to investigate how populations change in size, structure, and distribution. Such changes are driven by various biological, environmental, and ecological factors, including birth and death rates, migration patterns, competition for resources, and predation. For example, in marine ecosystems, elevated mortality rates among orcas, a predator, can disrupt the balance of the food chain, leading to an overpopulation of their prey, such as fish and other marine species.

For this study, it is essential to understand some fundamental concepts, starting with population density, which is defined as the number of living organisms within a population per unit of area. In other words, it represents the size of the population in relation to a specific spatial domain.

In addition to population density, several other factors play a pivotal role in shaping population dynamics. These include birth and death rates, which dictate the growth or decline of populations; migration patterns, encompassing both immigration and emigration, which influence the redistribution of populations across regions; and interactions between species, such as competition, predation, and cooperation, which drive ecological balance. External influences like environmental changes, availability of resources, and human interventions further add layers of complexity to these studies.

In general, birth and immigration rates contribute to an increase in population density by adding new individuals to the population within a given area. Conversely, death and emigration rates lead to a decrease in population density by reducing the number of individuals present. These opposing processes form the foundation of population dynamics, creating a delicate balance that determines whether a population grows, shrinks, or remains stable over time.

In this context, *Mathematical Modeling* aims to describe and predict the behavior of one or more populations of living organisms, chemical substances, or viruses within a specific location. By employing mathematical tools and frameworks, these models seek to explain the growth or decline of population density through a detailed analysis of the factors involved. These factors can be classified as internal, such as interactions between species, including competition, predation, and cooperation, or external, such as environmental changes, resource availability, and anthropogenic influences.

Mathematical models provide a systematic way to capture the complexity of population dynamics, enabling researchers to simulate various scenarios and identify critical thresholds or tipping points. This approach not only enhances our understanding of population behavior but also supports practical applications, such as conservation strategies, resource management, and policy-making in the context of ecological sustainability.

In 2003, J. D. Murray (see [54]) published a book with an extensive study of biological problems modeled mathematically, among which we highlight two. The first, proposed by Malthus in 1798, is a classical formulation of continuous population growth used to describe the dynamics of a single population. This model, which assumes that population growth, is described by the following differential equation:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = bN - dN = (b-d)N, \tag{1}$$

where $N(t)$ represents the population density as a function of time, b is the birth rate, and d is the death rate. The solution to this equation, given by:

$$N(t) = N_0 e^{(b-d)t},$$

where N_0 is the initial population, indicates exponential growth or decline of the population depending on b and d, as follows:

- The population grows exponentially when $b > d$;
- The population declines exponentially when $b < d$; and
- The population remains constant over time when $b = d$.

This model is considered simplistic for describing real populations since it does not account for environmental limitations or external interactions, such as resource competition or predation. However, its importance lies in introducing the fundamental principles of population dynamics and inspiring the development of more realistic models.

The second, proposed by P. F. Verhulst in 1836, is a more realistic generalization of population growth compared to Malthus exponential model. It incorporates environmental resource limitations and intraspecific competition, being described by the following differential equation:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN\left(1 - \frac{N}{K}\right), \tag{2}$$

where $N(t)$ represents the population density as a function of time, r is the intrinsic growth rate of the population, and K is the carrying capacity of the environment, representing the maximum number of individuals that the environment can sustain.

This model predicts that population growth is initially rapid but slows as the population approaches the carrying capacity K. The explicit solution to this equation is given by:

$$N(t) = \frac{N_0 K e^{rt}}{K + N_0(e^{rt} - 1)}.$$

Also known as the *Logistic Model*, it is widely used to describe natural populations subject to environmental constraints and to study density-dependent regulatory mechanisms. Despite its limitations, the model provides an essential foundation for understanding more complex biological systems and is frequently used as a starting point for incorporating additional factors, such as predation, migration, and interspecies interactions.

In 2004, R. S. Cantrell and C. Cosner (see [12]) published a comprehensive book that presents a unified perspective of spatial ecology through *Reaction-Diffusion Model*, analyzing how biological interactions and movement processes manifest in heterogeneous environments. The work spans from classical nonspatial models—such as the Lotka-Volterra systems that describe basic population dynamics—to more sophisticated mathematical formulations of spatially explicit models. This theoretical framework not only bridges ecological scales but also provides powerful tools to study patterns of species coexistence and segregation.

Among these problems, we highlight the *Logistic Problem with a local diffusion term*, given by:

$$\begin{cases} -\alpha\Delta w = \gamma w - w^2 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where Ω is a bounded regular domain in $\mathbb{R}^N$, $N \geq 1$, $\gamma \in \mathbb{R}$, $\alpha > 0$ and $\Delta w := \sum_{k=1}^{N} \frac{\partial^2 w}{\partial x_k^2}$ denotes the **Laplacian Operator**.

This problem arises from the analysis of the steady-state behavior of the following parabolic problem:

$$\begin{cases} \dfrac{\partial w}{\partial t} - \alpha\Delta w = \gamma w - w^2 & \text{in } \Omega \times (0, +\infty), \\ w = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ w(x,0) = w_0(x) & \text{in } \Omega, \end{cases}$$

where $w = w(x,t)$ represents, for example, the density of a population over time, and $w_0(x)$ is the initial condition, typically assumed to be nonnegative and nontrivial. For a detailed study of this model and its dynamical properties, see, for instance, [12].

Problem (3), which is a generalization of the Equation (2), is a semilinear elliptic problem known for combining a diffusion term, represented by the Laplacian operator $-\alpha\Delta w$, and the reaction term with logistic growth, $\gamma w - w^2$. The parameter α serves as a diffusion coefficient, directly influencing the spatial dynamics of the solutions. Higher values of α enhance the diffusive effect, leading to a more uniform spatial distribution of the variable w throughout the domain.

Moreover, it is frequently used to model biological phenomena, such as population dynamics with logistic growth, subject to limitations imposed by finite resources. The domain Ω represents the region where the solution is studied, and the boundary condition $\partial\Omega$ imposes that the population density is zero at the boundaries of the domain, simulating a confined environment. Moreover, it is important as it allows for detailed analysis regarding the existence, uniqueness, and stability of solutions, as well as critical behaviors such as bifurcations and transitions between steady states. It serves as a basis for more complex investigations involving nonlocal terms, multi-species interactions, or heterogeneous environments.

In 1989, J. Furter and M. Grinfeld (see [40]) examined several biological models involving a single species, in which they included and deepened the importance of nonlocal interactions, such as competition for shared resources.

The nonlocal term, addressed in the text, plays a central role in extending the traditional modeling based on reaction-diffusion equations, which typically consider only local interactions. Its introduction makes it possible to represent situations where interactions between populations are not restricted to immediate proximity, such as in resource competition. This approach adds realism to the models, making it possible to explore more complex spatial patterns.

Moreover, the nonlocal term enables the study of novel dynamics, such as the formation of stable patterns in contexts where this would be unlikely without these effects. These characteristics make the models more robust and relevant for practical applications, particularly in ecology and biology, where large-scale phenomena are essential to describe the organization and evolution of populations. The text further emphasizes the importance of studying the conditions that ensure the stability of these patterns, highlighting the critical role of nonlocal interactions in understanding more complex natural systems.

The inclusion of a nonlocal term in the diffusion coefficient deserves special attention. In general, the diffusion rate is modeled by:

$$\vec{v} = -\alpha \nabla w,$$

and this modeling choice results in a partial differential equation that takes the following general form:

$$\begin{cases} -\operatorname{div}(\alpha \nabla w) = f(x, w) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

The term $f(x, w)$ plays a fundamental role in describing the local dynamics of the quantity w. Depending on the context, this term can model various biological, chemical, or physical processes. For instance, in population dynamics, $f(x, w)$ may describe the growth and interaction of a population at a given location $x \in \Omega$, incorporating effects such as spatially varying birth and death rates, as well as intraspecific and interspecific competitive interactions.

Moreover, depending on the nature of the diffusion coefficient α, we can identify three distinct modeling frameworks, each capturing different ecological assumptions and leading to qualitatively different mathematical structures:

(a) When α is a positive constant, we obtain the classical Laplace operator:

$$-\operatorname{div}(\alpha \nabla w) = -\alpha \Delta w.$$

In this case, α represents the diffusion rate, which remains constant throughout the entire habitat Ω of the species w.

(b) When α depends on the population density w, a nonlinearity arises in the diffusive term:

$$-\operatorname{div}(\alpha \nabla w) = -\operatorname{div}(\alpha(w) \nabla w).$$

In this case, the diffusion coefficient $\alpha(w)$ varies with the population density w. This model captures situations where the dispersal rate of individuals changes according to local density:

  - If α is increasing: It indicates that diffusion is more intense in high-density regions, suggesting that individuals tend to spread out more when competition for resources is stronger.
  - If α is decreasing: It implies reduced diffusion in densely populated areas, possibly due to social behavior or the benefits of aggregation.

(c) In contexts where pointwise measurements are impractical, an alternative approach is to consider nonlocal dependencies. For example, replacing α by a function of the spatial average of the population over a region $B(x, r)$, that is:

$$\alpha \left(\int_{B(x,r)} w(y) \right).$$

This leads to equations with nonlocal diffusion, such as:

$$-\text{div}\,(\alpha \nabla w) = -\text{div} \left[\alpha \left(\int_{\Omega} K(x, y) w(y) \right) \nabla w \right].$$

In this case, the diffusion coefficient is influenced by a weighted average of the population density around the point x, determined by the **Kernel Function** $K(x, y)$. The kernel $K : \Omega \times \Omega \to \mathbb{R}$ is a nonnegative, measurable function that describes the influence of the point y on the location x. Typically, $K(x, y)$ is chosen to decay as the distance between x and y increases, reflecting the idea that individuals at nearby locations have a stronger impact on the movement at point x than those farther away. This nonlocal term captures long-range interactions, where the movement at location x is affected by the population density at surrounding points y, thus extending the classical diffusion model to incorporate spatial memory or nonlocal perception effects. Such formulations have been studied, for example, in [55], where diffusion depends on the range of interactions, and in [3, 29], which explore intermediate local-nonlocal elliptic problems.

Later, M. Chipot and collaborators investigated the role of nonlocal terms in elliptic problems arising in population dynamics. Among the nonlocal terms considered, one particularly notable example involves dependence on the total population within a subdomain, that is:

$$\alpha = \alpha \left(\int_{\Omega'} w(x)\, \mathrm{d}x \right),$$

where $\Omega' \subseteq \Omega$ denotes a subdomain, which may represent, for example, a specific area of interest where the population density is being monitored or where the environmental feedback is concentrated.

This formulation is especially relevant for modeling ecological phenomena such as:

- Aggregation-driven dispersal: If α decreases with population density, it describes species that tend to avoid overcrowded areas.
- Population attraction: If α increases with density, it models species that are drawn to regions of high population concentration.

In 1992, M. Chipot and J. F. Rodrigues (see [17]) studied a class of elliptic problems with nonlocal terms, addressing both theoretical and applied aspects. These problems are characterized by the global dependence of the solution, which distinguishes them from traditional local problems. More specifically, the authors considered the following problem:

$$\begin{cases} -a\left(\int_\Omega u\right)\Delta u + \lambda u = f & \text{in } \Omega, \\ \partial_n u + \gamma\left(\int_{\Omega'} u\right) = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where $\lambda > 0$, f represents the supply of beings by external sources, a a positive factor that depends on the total population, γ a positive factor that represents the influence of the population in Ω', and $\partial_n u$ denotes the **normal derivative** of u on the boundary $\partial\Omega$, that is, the derivative of u in the direction of the outward unit normal vector to the boundary.

Problem (4) describes the behavior of a bacteria with density u within a container Ω, considering population dispersion, an external source f, and a mortality rate λ, which is proportional to the density u. The population dispersion is represented by a term that depends on the gradient of the density ∇u, with a factor a that varies according to the total population within Ω, that is, $a\left(\int_\Omega u\right)$. Additionally, the population is expelled from the container through a total flux at the boundary $\partial\Omega$, which depends both on the total population within Ω and on a dominant group within a subdomain Ω'. The complete problem is then described by two equations, where the first equation describes the population dynamics inside the container, and the second specifies the boundary flux conditions.

The nonlocal term included in this problem is essential for modeling the global interaction within the system, where the population dynamics at a given point depend not only on local conditions but also on global characteristics, such as the total population within the container and the presence of a dominant group in a subdomain.

In 1999, M. Chipot and B. Lovat (see [16]) present a problem to illustrate the study of nonlinear diffusion in the nonlocal case. This problem aims to model population density, where the diffusion rate depends on the integral over a certain region of the domain. More specifically, the authors consider the following problem:

$$\begin{cases} u_t - a\left(l(u(\cdot,t))\right)\Delta u = f & \text{in } \Omega \times (0,T), \\ u(\cdot,t) \in V & \text{for } t \in (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{5}$$

where V a subspace of $H^1(\Omega)$ that accounts for the boundary conditions of problem, u_0 the initial condition, $T > 0$ some fixed time, and $f = f(x)$ a source term, which may represent, for example, the population growth rate. Moreover, the continuous linear form l is defined as the integral of the product of L^2 with a function $g \in L^2(\Omega)$, that is:

$$l(u) = l_g(u) = \int_{\Omega'} g(x)u(x),$$

where Ω' is some subdomain of Ω.

Problem (5) models population dynamics through a nonlocal diffusion process. The term Δu describes standard dispersal from high to low density regions, while f accounts for births, deaths, and migration. The key feature is the density-dependent coefficient $a(l(u))$ which represents a weighted average population in subregion Ω'. This formulation captures faster diffusion in crowded areas (when a increasing), and attraction to populated zones (when a decreasing).

Biologically, this model can be applied to the study of bacteria moving in search of nutrients in a homogeneous medium, animals migrating between habitats based on global perception of density, or even human populations responding to social and economic stimuli. Thus, the model not only incorporates the local and global interactions of the population but also allows us to explore how these processes influence the spatial and temporal distribution of population density, with significant implications for ecology, population biology, and natural resource management.

In the course of this work, three methods will be crucial: the *Sub-Supersolution Method*, *Bifurcation Theory*, and *Fixed Point Index Theory*. We will present a brief overview of these techniques, highlighting key results from the literature and emphasizing their applicability. These methods will be systematically employed throughout the work to establish existence and uniqueness for the nonlocal elliptic systems under study.

The first method, the Sub-SuperSolution technique, is used to prove the existence of solutions to differential equations, especially when these solutions are difficult to obtain directly due to nonlinearity or the complexity of boundary conditions. This method is particularly useful for studying elliptic equations involving nonlinear terms, such as equations with reaction terms, diffusion terms, and other nonlinear interactions.

The central idea of this method is to use auxiliary functions, called subsolution and supersolution, to establish lower and upper bounds for the solutions of problem. These bounds help identify a solution that lies between these two values and, in many cases, can be used to guarantee the existence of a solution.

One of the first authors to work with this method was G. Scorza Dragoni in 1931 (see [31]), who studied the existence of an ordered pair of solutions to a differential inequality in order to determine the existence of a solution to a boundary value problem for a nonlinear second-order differential equation. Later, many authors refined this method in the context of elliptic problems (see [43]) and presented an extensive study of the method in the context of elliptic and parabolic equations (see [56]).

In 2016, Y. Baoqiang and M. Tianfu (see [7]) investigate existence and multiplicity results for positive solutions to an important class of nonlocal problems given by:

$$\begin{cases} -a\left(\int_\Omega |u|^\gamma \,\mathrm{d}x\right)\Delta u = f_\lambda(x,u) & \text{in } \Omega, \\ u > 0 & \text{on } \partial\Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

where $\gamma \in (0,+\infty)$, $a : [0,+\infty) \to (0,+\infty)$ is a continuous function with $\inf_{t\in[0,+\infty)} a(t) > 0$, and $f_\lambda : \Omega \times \mathbb{R} \to \mathbb{R}$ is a nonlinearity that may depend both on the spatial position $x \in \Omega$ and on the solution u, possibly involving a control parameter λ.

The authors emphasize that such problems naturally arise in various contexts within Applied Mathematics, particularly in ecological models describing the spatial distribution of populations subject to non-standard dispersal mechanisms. A notable special case occurs when $\gamma = 2$, in which the problem reduces to a generalized version of the well-known Carrier Equation, originally studied in the context of nonlinear vibration problems in solid mechanics.

The most prominent mathematical feature of this problem lies in the nonlocal nature of the main differential operator, represented by $a\left(\int_\Omega |u|^\gamma\right)\Delta u$. This nonlocality arises from the dependence of the diffusion coefficient on the L^γ-norm of the solution over the entire domain Ω. In particular, for $\gamma > 1$, densely populated regions exert greater influence on the diffusion process. Conversely, for $0 < \gamma < 1$, or for γ sufficiently large or small, the influence of sparsely populated areas is amplified, enhancing the role of low-density regions in shaping the dynamics.

To demonstrate the existence of at least one classical solution for Problem (6), the authors employed the Sub-Supersolution Method and ensured this existence when, for $\alpha, \beta \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ such that:

(a) α is a subsolution and β is a supersolution of Problem (6), in the sense that they satisfy the differential inequality associated with the problem, with α lying below and β above the nonlinearity, and both vanish on the boundary (see the definition in Section 1.4);

(b) $\alpha(x) \leq \beta(x)$ for all $x \in \Omega$; and

(c) $|f_\lambda(x,u)| \leq h(x)$ for all $\alpha(x) \leq u \leq \beta(x)$, with $h \in L^p(\Omega)$ and $p > N$.

In the final part of the study, the authors showcase the versatility of this result by applying it to several classes of nonlinearities $f_\lambda(x,u)$.

Sub-Supersolution Method faces significant limitations when applied to problems involving nonlocal terms. This is because, in nonlocal problems, the operators depend on integrals or global averages of the solution, which hinders the use of the method traditionally employed to construct solutions via monotone iteration. Furthermore, the lack of properties such as the classical maximum principle prevents the direct application of this method.

The second method, which will be the most used throughout this work, is Bifurcation Theory. This method can be divided into two cases: local and global. Broadly speaking, this theory deals with the transition between different types of solutions as one or more parameters of the problem are varied. This is particularly important in nonlinear problems, where parameter changes can lead to the emergence of new solutions, changes in the number of existing solutions, or even the loss of stable solutions. In the context of biological models, bifurcation theory is especially relevant for studying positive solutions, which correspond to meaningful population densities or concentrations. Understanding how positive solutions arise and evolve as parameters vary allows us to analyze critical biological phenomena such as species coexistence, extinction thresholds, and pattern formation in ecosystems.

In 1971, M. G. Crandall and P. H. Rabinowitz (see [23]) introduced a detailed investigation into Bifurcation Theory, extending it to a general context with the primary goal of determining the structure of the zero set of an equation $G(w) = 0$ in the neighborhood of a point on a known curve of zeros. Focusing on bifurcations at simple eigenvalues, a topic widely studied in the literature, the authors consolidated existing results and expanded the applicability of the theory.

The main result of the paper, Theorem 1, is frequently employed in the context of *Local Bifurcation* and is widely known as the *Crandall-Rabinowitz Theorem*. This result establishes precise conditions for classifying a point on a zero curve as a bifurcation point. Based on the properties of the derivatives of G, the theorem ensures that, in a neighborhood of this point, the set of solutions forms two continuous curves that intersect uniquely at the bifurcation point. Although there were earlier results, the main result of this paper has become the most applicable in the context of Local Bifurcation due to its generality.

Also in 1971, P. H. Rabinowitz (see [58]) extended the results previously published with M. G. Crandall by addressing the global behavior of the solution curve obtained through the Crandall-Rabinowitz Theorem. Focusing on the existence of a continuum, that is, a closed and connected sets of solutions, the author demonstrated that this continuum extends globally from an eigenvalue of odd multiplicity. This result indicates that bifurcation from eigenvalues with odd multiplicity is not merely a local phenomenon but has a global nature.

The main result of this article, Theorem 1.3, is frequently employed in the context of *Global Bifurcation* and is widely known as the Rabinowitz Theorem. This result established the existence of a global continuum of solutions for nonlinear eigenvalue problems in Banach spaces, originating from an eigenvalue of odd multiplicity. Specifically, it asserts that if an eigenvalue μ has odd multiplicity, there exists a maximal subcontinuum, denoted by $\mathcal{C}_\mu$, within the solution set, such that $(\mu, 0) \in \mathcal{C}_\mu$. Moreover, this continuum satisfies one of the following two alternatives:

- $\mathcal{C}_\mu$ is an unbounded continuum in $\mathbb{R} \times \mathcal{E}$, meaning it is not contained in any bounded subset of $\mathbb{R} \times \mathcal{E}$; or
- $\mathcal{C}_\mu$ intersects $(\lambda, 0)$, where λ is another eigenvalue distinct from μ.

This result, as previously mentioned, extends the theory of Local Bifurcation by demonstrating that bifurcation from eigenvalues of odd multiplicity is a global phenomenon. The proof of Theorem 1.3 employs the Leray-Schauder degree and the topological analysis of the solution set.

In 2018, T. S. Figueiredo-Sousa, C. Rodrigo-Morales and A. Suárez (see [39]) studied a logistic equation with a nonlocal diffusion coefficient, modeling the dynamics of a population within a bounded domain. Through bifurcation methods (local and global) and fixed-point arguments, they determined conditions for the existence of positive solutions and analyzed the global behavior of the solutions depending on the nonlocal diffusion function and parameters λ. The model proposed by the authors is as follows:

$$\begin{cases} -a\left(\int_\Omega q(x)u^p\right)\Delta u = \lambda u - b(x)u^2 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7}$$

where $p > 0$, $\lambda \in \mathbb{R}$, $a \in C(\mathbb{R})$ a positive functions, $b \in C^1(\overline{\Omega})$ a nonnegative and nontrivial function, and $q(x)$ a bounded, nonnegative and nontrivial function in Ω.

In Problem (7), the Local Bifurcation Method was employed to investigate the existence and behavior of positive solutions. The central idea is to explore the bifurcation of positive solutions from the trivial solution $u \equiv 0$ when the parameter λ reaches certain critical values. Initially, the nonlocal logistic equation is reformulated into a suitable form for bifurcation analysis, using compact and well-defined operators within the relevant Banach space. This Problem is expressed as an equation of the form:

$$\mathcal{F}(\lambda, u) = a\left(\int_\Omega q(x)u^p\right)\Delta u + \lambda u - b(x)u^2.$$

The analysis of this operator reveals that bifurcation occurs at $(a(0)\lambda_1, 0)$. Here, λ_1 denotes the **principal eigenvalue** of the Dirichlet Laplacian, that is, the unique real number λ for which there exists a nontrivial function $\varphi \in H_0^1(\Omega)$, $\varphi > 0$ in Ω, satisfying:

$$\begin{cases} -\Delta u = \lambda\varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

To ensure the existence of a curve of nontrivial solutions emanating from these critical values, the authors apply the Crandall-Rabinowitz Theorem. This requires verifying the theorem conditions: that the linearized operator has a one-dimensional kernel and satisfies the transversality condition. Furthermore, the authors investigated the direction of bifurcation (supercritical or subcritical) by analyzing the sign of certain terms in the Taylor expansion of the nonlinear operator. They concluded that the bifurcation direction depends on the parameter p and the derivative of the diffusion coefficient $a'(0)$.

There is a wide range of problems where Bifurcation Theory can be employed to prove the existence of solutions. For further applications, we recommend consulting articles [5, 34, 37, 38] and their references.

The last method we will use in this work is the Fixed Point Index Theory, which is widely employed to study the existence and multiplicity of solutions to nonlinear differential equations, as well as to identify the behavior of solutions as system parameters vary.

An important aspect of this method is its connection to Bifurcation Theory: the change in the fixed point index can indicate the presence of a bifurcation. Specifically, the change in the index of an operator around a solution can sign the transition from a trivial solution to a nontrivial solution, or even the existence of multiple solutions from a single parameter value.

Although this method had been previously used, for instance in [58], we will focus on two works. The first, published in 1976 by H. Amann (see [4]). In his research, he mainly studied results in cones with nonempty interior and demonstrated the existence of solutions for nonlinear equations under certain conditions. However, Amann conditions required that the cone have a nonempty interior, limiting the applicability of his results to specific domains.

Later, in 1983, E. N. Dancer (see [25]), generalized this method and applied it to more general those ones, including cones with empty interior. This was a significant breakthrough because many domains used in differential equations and bifurcation problems have cones with empty interior. By applying fixed point index theory to positive cones, Dancer was able to extend Amann's results to these more general situations, allowing for the analysis of bifurcations in broader contexts, such as elliptic problems in irregular domains and with mixed boundary conditions.

In 2020, B. Yan and C. An (see [61]) investigate the existence of sign-changing solutions for a class of nonlocal elliptic problems posed on an annulus, specifically addressing the following problem:

$$\begin{cases} -a\left(\int_\Omega |u|^\gamma\right)\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{8}$$

where $\Omega = \{x \in \mathbb{R}^N;\ 0 < r_1 < |x| < r_2\}$ is an annulus, $\gamma \in (0, +\infty)$, $a \in C\left([0, +\infty), (0, +\infty)\right)$, and $f \in C\left((0, +\infty), (0, +\infty)\right)$.

To establish the existence of sign-changing solutions for Problem A, the authors employ Fixed Point Index Theory. To this end, they transform the original problem on an annulus into a nonlocal boundary value problem for ordinary differential equations, in such a way that the solutions of Problem A correspond to the fixed points of a certain operator. Thus, finding solutions to the original problem reduces to identifying fixed points of the operator T.

An important mathematical model in the context of population dynamics is the *Lotka-Volterra* system. Developed by A. J. Lotka in 1925 (see [51]) and V. Volterra in 1926 (see [60]), these systems consist of differential equations modeling the evolution of two or more interacting species, such as predators and prey, competitors, or cooperative populations.

The Lotka-Volterra system generalizes the Logistic Equation, which models the growth of a single species considering environmental resource limitations. While the logistic equation includes a growth term limited by the environment's carrying capacity, Lotka-Volterra systems extend this by incorporating interactions between multiple populations, such as predation, competition, and cooperation. This generalization enables the study of more complex and realistic scenarios in ecology and other fields.

There are various ways to express the Lotka-Volterra system; however, for the purposes of this work, we will focus on the following stationary system:

$$\begin{cases} -\Delta u = \lambda u - u^2 - cuv & \text{in } \Omega, \\ -\Delta v = \mu v - v^2 - duv & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{9}$$

where $c, d, \lambda, \mu \in \mathbb{R}$.

From the population dynamics perspective, Problem (9) models the behavior of two species, u and v, inhabiting the habitat Ω. Since we are considering homogeneous Dirichlet boundary conditions, the habitat is entirely surrounded by inhospitable region, $\partial\Omega$. The terms $-\Delta u$ and $-\Delta v$ describe the spatial movement of the species. In the reactions functions, λ and μ stand for the intrinsic growth rate of each species, and c and d describe the growth limitations on the other population. This can model *competition*, *prey-predator* or *cooperation* interactions depending on the signs of the constants:

- Competition when c and d are positive;
- Prey-Predator when c is positive and d negative; and
- Cooperation when c and d are negative.

The first results on local and global bifurcation for Lotka-Volterra systems date back to the 1970s and 1980s. Among these works, we highlight the one by E. N. Dancer (see [24]), which demonstrates the existence of a global continuum for semilinear elliptic boundary value problems emanating from the trivial solution.

Classical references on Local and Global Bifurcation in Lotka-Volterra systems include, in addition to those already mentioned, the works of J. Blat and K. J. Brown (see [8, 9]), which address the existence of global branches of coexistence solutions in Competition and Predator-Prey cases. Also noteworthy are the contributions of R. S. Cantrell and C. Cosner [11], who analyze the steady-state problem with diffusion, and P. Korman and A. Leung [46], who investigate the existence and uniqueness of positive steady states in the Volterra-Lotka ecological model with diffusion.

In 1994, J. López-Gómez (see [47]) applied a Global Bifurcation theorem, originally developed by the authors themselves, to analyze the existence of coexistence states in Lotka-Volterra reaction-diffusion systems involving two species. Their focus was on proving the existence of global continuum of coexistence solutions for these systems, employing an optimized version of Rabinowitz Theorem.

The main result of this article, *Theorem 4.1*, is frequently employed in the context of Global Bifurcation for systems. It establishes that, starting from a nondegenerate positive solution $(\theta_\lambda, 0)$ of problem associated with a semi-trivial solution, there exists a continuum $\mathfrak{C}^+$ of coexistence states for the system. Basically, in Problem (9), the trivial solution $(0,0)$ always exists, along with the semi-trivial solutions of the form $(u,0)$ and $(0,v)$. The semi-trivial solution $(\theta_\lambda, 0)$ exists if, and only if, $\lambda > \lambda_1$. Similarly, the semi-trivial solution $(0, \theta_\mu)$ exists if, and only if, $\mu > \lambda_1$.

The main bifurcation result for this problem states that, fixing $\lambda > \lambda_1$ and taking $\mu \in \mathbb{R}$ as the bifurcation parameter, there exists a critical value given by

$$\mu = F(\lambda) := \sigma_1 \left[-\Delta + d\theta_\lambda\right],$$

from which a continuum of positive solutions $\mathfrak{C}^+$ bifurcates, where $\sigma_1 [-d\Delta + a]$, with $d > 0$, denotes the **principal eigenvalue** of the weighted eigenvalue problem:

$$\begin{cases} -d\Delta\varphi + a(x)\varphi = \lambda\varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega, \end{cases}$$

with $a(x)$ representing the corresponding weight function depending on the context (see [48]). Additionally, the result asserts that this continuum satisfies at least one of the following properties:

- $\mathfrak{C}^+$ is unbounded in the space of $\mathbb{R} \times X$; with X a suitable Banach space where the solutions live, indicating that the set of solutions can extend indefinitely
- $\mathfrak{C}^+$ intersects the curve of semi-trivial solutions at the point $(\mu_\infty, 0, \theta_{\mu_\infty})$, where θ_{μ_∞} is a positive solution of another related problem; or
- $\mathfrak{C}^+$ intersects the curve of semi-trivial solutions at the point $(\lambda_\infty, \theta_{\lambda_\infty}, 0)$, where θ_{λ_∞} is another positive solution of the original problem; or
- $\mathfrak{C}^+$ connects to the trivial state $(0,0)$ at another point.

In fact, it can be proved that this continuum satisfies one of the following alternatives:

(1) $\mathfrak{C}^+$ is unbounded in $\mathbb{R} \times X$; or

(2) $\mathfrak{C}^+$ intersects the curve of semi-trivial solutions at the point $(\mu_\infty, 0, \theta_{\mu_\infty})$, where $\theta_{\mu_\infty} > 0$ is a positive solution and

$$\lambda = G(\mu_\infty) := \sigma_1 \left[-\Delta + c\theta_{\mu_\infty}\right].$$

In the case of Competition $(c, d > 0)$, it can be shown that $\mathfrak{C}^+$ is bounded. Therefore, only alternative (2) can occur. Moreover, there exists a coexistence state of Problem (9) when

$$\mu \in (F(\lambda), \mu_\infty) \cup (\mu_\infty, F(\lambda))$$

On the other hand, in the cases of Predator-Prey ($d < 0 < c$) and Symbiosis ($c, d < 0$ with $cd < 1$), one can guarantee that alternative (2) does not occur, and therefore $\mathfrak{C}^+$ is unbounded. In these scenarios, for all $\mu > F(\lambda)$, there exists a positive coexistence solution.

Below we illustrate the bifurcation diagrams corresponding to the Competition, Predator-Prey, and Symbiotic cases:

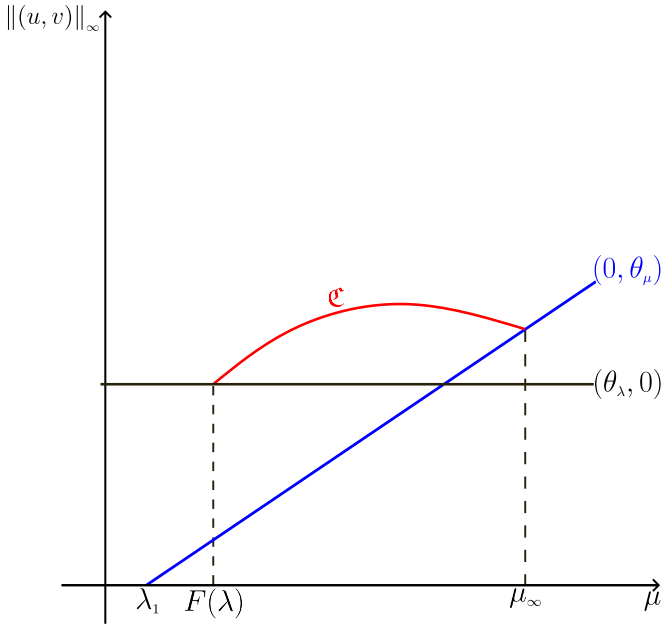


Figure 1: Behavior of the Continuum of semi-trivial solution $\mathfrak{C}$ in the Competition case.
Source: Prepared by the author.

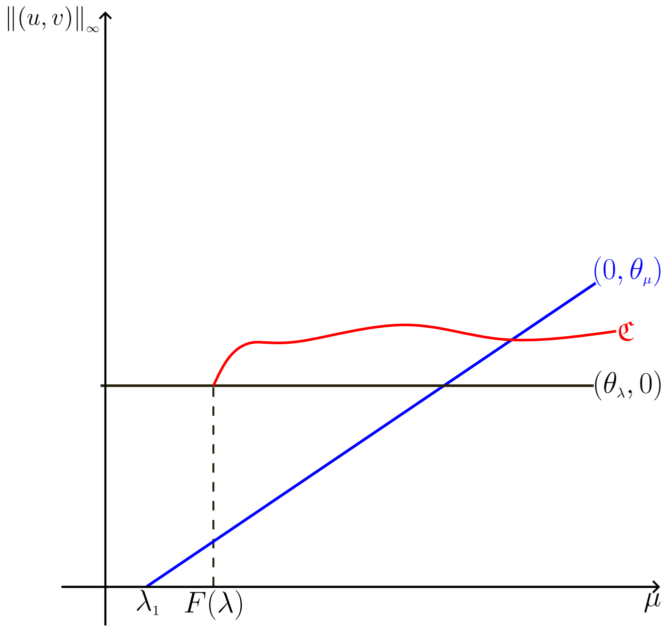


Figure 2: Behavior of the Continuum of semi-trivial solution $\mathfrak{C}$ in the Predator-Prey case.
Source: Prepared by the author.

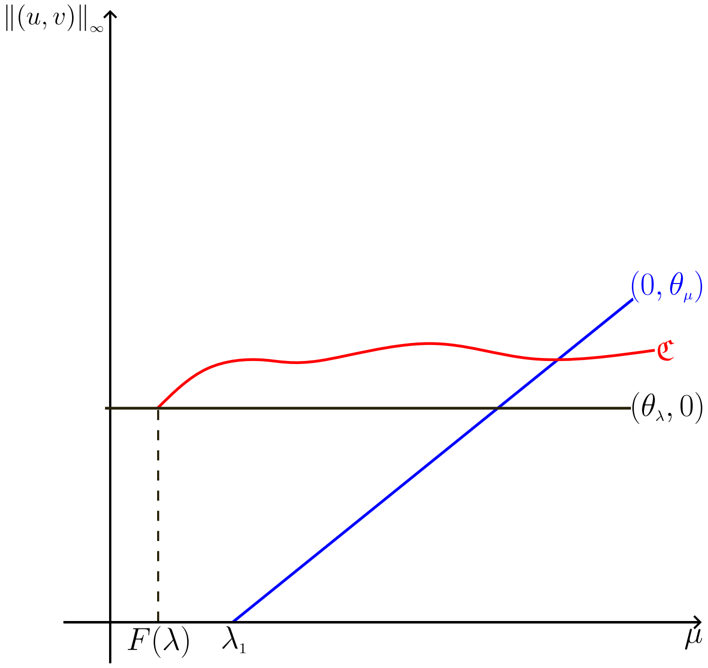


Figure 3: Behavior of the Continuum of semi-trivial solution $\mathfrak{C}$ in the Symbiotic cases.
Source: Prepared by the author.

In all cases, the region where both species can coexist is determined by the parameters for which positive solutions exist simultaneously. This region is delimited by the region:

$$\mathcal{R}_1 := \{(\lambda,\mu);\ (\mu - F(\lambda))(\lambda - G(\mu)) > 0\}.$$

This coexistence region, defined by the inequality above, is illustrated in Figure 3.4, where we highlight the set of parameters (λ,μ) for which coexistence states are possible.

In this work, we study systems of elliptic equations with nonlocal diffusion, which arise as models in Population Dynamics. These systems have garnered significant attention in the literature due to their ability to capture complex interactions between populations and the effects of nonlocal terms on spatial dispersion. Motivated by applications in Ecology, we consider systems where each variable represents the density of a specific population in a heterogeneous environment. Our primary focus is on investigating the existence of simultaneous positive solutions, known as coexistence states, which represent scenarios where both populations can persist stably within the system.

Moreover, we aim to analyze the conditions under which these solutions emerge, as well as the impact of ecological and mathematical parameters on the outcomes. Whenever possible, we interpret the obtained results in an ecological context, establishing a direct connection between theoretical aspects and phenomena observed in nature.

The main objective of this work is to investigate sufficient conditions for the existence and uniqueness of coexistence states in coupled nonlocal elliptic systems. More precisely, we consider problems with the following general structure:

$$\begin{cases} -m\left(\int_\Omega u, \int_\Omega v\right)\Delta u = f(x,u,v) & \text{in } \Omega, \\ -n\left(\int_\Omega u, \int_\Omega v\right)\Delta v = g(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $m, n : \mathbb{R}^2_+ \to [0, +\infty)$ are continuous nonlinear functions and $f, g : \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions.

Our approach will explore three distinct applications of this general problem, each treated with a different analytical method, namely: the Sub-Supersolution Method, Local and Global Bifurcation Theory, and Fixed Point Index Theory. In developing the results for these applications, we draw inspiration from techniques already established in the literature for nonlocal problems involving a single equation. In this way, we extend such theories to the more complex setting of nonlocal systems involving two coupled equations.

We have organized this work into five chapters, each designed to progressively develop the theoretical framework and main results of the study. The structure is as follows:

In **Chapter 1**, we provide a review of the main mathematical concepts necessary for the reader to properly understand the subsequent chapters. To this end, we cover topics such as the basics of Partial Differential Equations, Functional Analysis, and Nonlinear Analysis. Next, we discuss key results related to the Maximum Principle and review the Eigenvalue Problem, presenting its fundamental properties. We also explore the literature on the Method of Sub-Supersolution, Bifurcation Methods - Local and Global — and Fixed Point Index Theory. Finally, we highlight well-studied results related to the Logistic Problem, establishing a solid foundation for understanding problems addressed in the following chapters. Additionally, we present in this chapter the results concerning the logistic equation with both local and nonlocal diffusion, specifically Theorem 7.

In **Chapter 2**, we will present the results obtained in [13], where we study the existence of coexistence states for the following nonlocal elliptic system:

$$\begin{cases} -m\left(\int_\Omega v\right)\Delta u = \lambda u - u^2 + cuv & \text{in } \Omega, \\ -\Delta v + \sigma v = \rho u & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{10}$$

where $m : \mathbb{R} \to [0, +\infty)$ a continuous function, $c, \lambda \in \mathbb{R}$, $\rho \geq 0$, and $\sigma > 0$.

A particular case of Problem (10) was introduced in [14] to model the behavior of a bacterium, with density v, located in a habitat Ω, where u represents the nutrient. Specifically, in [14], the first equation of (10) is:

$$-m\left(\int_\Omega v\right)\Delta u = f(x), \text{ in } \Omega$$

where f is a constant rate of the nutrient.

In our model, the nutrient is represented by another living organism that grows according to a logistic law, $\lambda u - u^2$, where λ is the growth rate. This nutrient interacts with the bacteria at a rate c, which can be either competitive or cooperative, depending on the sign of c: negative in the first case and positive in the second. In the specific case where $c = 0$, no interaction occurs. Additionally, the bacteria have a constant death rate σ and a source rate ρ, which depends solely on the nutrients.

It is important to note that the main innovation in this model is that the nutrient's diffusion depends nonlocally and nonlinearly on the bacteria population. For related works involving nonlocal diffusivity terms, see also [33], [44], and [59].

Regarding the results of this chapter, we can summarize the findings on coexistence states through the following result:

Theorem 1. The following assertions hold:

(a) Assume that $c = 0$. Problem (10) possesses at least one coexistence state when $\lambda > m(0)\lambda_1$. Moreover, the coexistence state is unique when m is increasing.

(b) Assume that $c < 0$. Problem (10) possesses at least one coexistence state when $\lambda > m(0)\lambda_1$ and does not have a coexistence state when $\lambda \leq \lambda_1 \min_{s\geq 0} m(s)$.

(c) Assume that $c > 0$. Problem (10) possesses at least one coexistence state when $\lambda > m(0)\lambda_1$ and one of the following conditions holds: c is small, ρ is small, σ is large, or

$$\lim_{s\to\infty} \frac{m(s)}{s} = \infty. \tag{11}$$

Moreover, it does not possess a coexistence state when $\lambda \leq \lambda_0$, for some $\lambda_0 \in \mathbb{R}$.

Although the existence results in all cases are quite similar, their derivations differ. In particular, we will use the bifurcation technique, and the main difference across the cases lies in the method used to obtain the a priori bounds. It is important to highlight that in the cooperative case, we have obtained the a priori bounds in two distinct ways. In the first case, we primarily use arguments based on the maximum principle, while in the second case, we argue by contradiction and make use of the fact that the diffusion coefficient grows very rapidly.

In all cases, we prove the existence of an unbounded continuum $\mathfrak{C} \subset \mathbb{R}\times C^1_0(\overline{\Omega})\times C^1_0(\overline{\Omega})$ of positive solutions for Problem (10). Specifically, we prove:

Theorem 2. Assume that $m(0) > 0$. From the trivial solution $(u, v) = (0, 0)$ emanates an unbounded continuum $\mathfrak{C} \subset \mathbb{R} \times C^1_0(\overline{\Omega}) \times C^1_0(\overline{\Omega})$ of positive solutions of Problem (10) at $\lambda = m(0)\lambda_1$. Moreover, if one of the following conditions holds: $c \leq 0$, $c \geq 0$ and c is small, $c > 0$ and ρ is small, $c > 0$ and σ is large, or $c > 0$ and m satisfies (11), then

$$(m(0)\lambda_1, \infty) \subset \mathrm{Proj}_{\mathbb{R}}(\mathfrak{C}) \subset (\lambda_0, \infty),$$

for some $\lambda_0 \leq 0$, where $\mathrm{Proj}_{\mathbb{R}}(\lambda, u, v) = \lambda$ for $(\lambda, u, v) \in \mathfrak{C}$. As a consequence, there exists at least a positive solution for $\lambda > m(0)\lambda_1$.

We also study the local bifurcation, including the bifurcation direction. This direction depends on the relative size of the coefficients of (10) and $m'(0)$. Specifically:

Theorem 3. Assume that $m(0) > 0$. Then, the bifurcation direction from the trivial solution $(u, v) = (0, 0)$ at $\lambda = m(0)\lambda_1$ is:

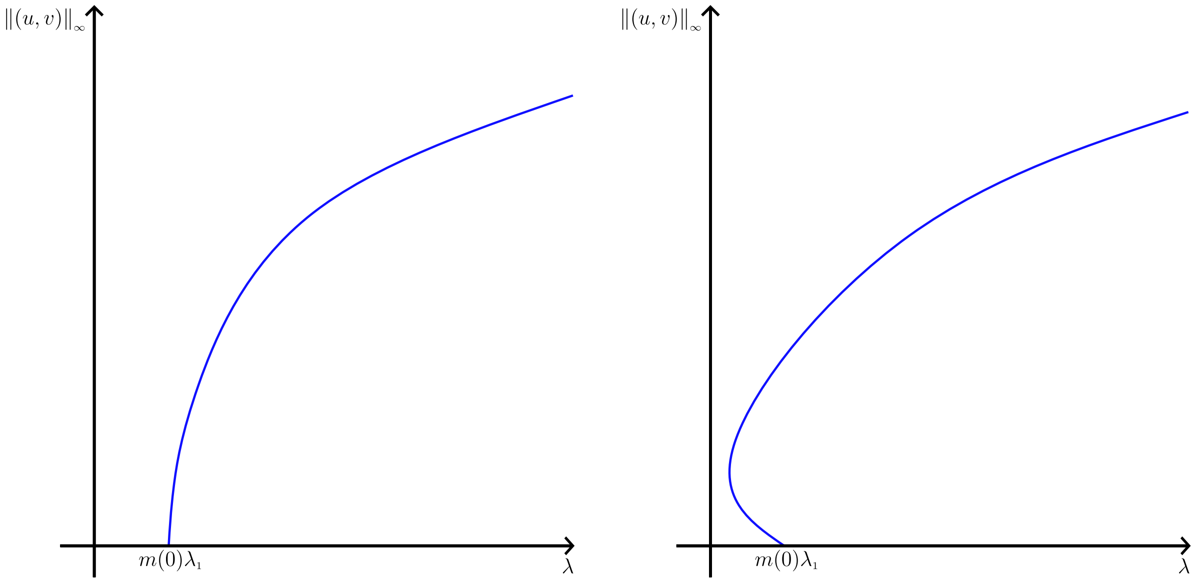
(a) Supercritical when

$$m'(0) > \frac{(c\rho - \lambda_1 - \sigma)\,\|\varphi_1\|_3^3}{\lambda_1\rho\,\|\varphi_1\|_1}.$$

(b) Subcritical when

$$m'(0) < \frac{(c\rho - \lambda_1 - \sigma)\,\|\varphi_1\|_3^3}{\lambda_1\rho\,\|\varphi_1\|_1}.$$

In the Figure 2.1 we have illustrated two possible bifurcation diagrams.



(a) Supercritical bifurcation (b) Subcritical bifurcation

Figure 4: Bifurcation diagrams of Problem (10)
Source: Prepared by the author.

Moreover, we prove a uniqueness result. We show that when m is increasing, there exists a unique coexistence state of Problem (10) for $c \in (-c_0, c_0)$ for some $c_0 > 0$. Observe that this uniqueness is optimal in the following sense: if m is increasing and c is large, the bifurcation direction is subcritical, and the multiplicity of positive solutions occurs for $\lambda \in (m(0)\lambda_1 - \delta, m(0)\lambda_1)$ for some $\delta > 0$ small.

Finally, we analyze the case $m(0) = 0$. In this case, we cannot apply the bifurcation method directly, but we can use a compactness argument to show the existence of a positive solution to Problem (10) for all $\lambda > 0$.

In **Chapter 3**, we will present the results obtained in [22], where we study the existence of coexistence states for the following nonlocal system with cross-diffusion:

$$\begin{cases} -m\left(\int_\Omega v\right)\Delta u = \lambda u - u^2 - cuv & \text{in } \Omega, \\ -n\left(\int_\Omega u\right)\Delta v = \mu v - v^2 - duv & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{12}$$

where $c, d, \lambda, \mu \in \mathbb{R}$, and $m, n : \mathbb{R} \to [0, \infty)$ are continuous functions.

From the perspective of population dynamics, Problem (12) describes the behavior of two species, u and v, inhabiting the habitat Ω. Since we consider homogeneous Dirichlet boundary conditions, the habitat is entirely surrounded by an inhospitable region.

In the reaction functions, we have used classical Lotka-Volterra type reaction terms, where λ and μ represent the intrinsic growth rates of each species, and c and d describe the interaction rates between the species. This can model interactions of competition, predator-prey, or cooperation, depending on the signs of the constants c and d.

In population dynamics, the nonlocal term was included in [40] in the reaction term to account for the interaction between species, stating that, in the ecological context of species interactions, there is no real justification for assuming that they occur locally.

On the other hand, in recent years, this kind of nonlocal term has also been incorporated into the diffusion term, considering expressions of the form:

$$-a\left(\int_\Omega u\right)\Delta u$$

to model the fact that the velocity of migration depends on the total population in the domain, see for instance [17]. Here, a is an increasing function if the species tends to leave crowded zones, while if we are dealing with species attracted by the growing population in Ω, we will assume a to decrease.

This idea can be extended to systems, see for instance [14, 33, 44] and their references. That is, the diffusion of one species u, respectively v, depends on the other species in the form

$$-m\left(\int_\Omega v\right)\Delta u,$$

respectively

$$-n\left(\int_\Omega u\right)\Delta v.$$

Let us explain the first case. In this case, if the total population of the species v increases, then u can act in two different ways:

- The species u tends to leave the crowded region when m is increasing; and
- The species u is attracted by v when m is decreasing.

The second case is analogous. This diffusion behavior is combined with and complemented by the interaction between species. Assume that u and v compete. In this case, if m increases, species u tends to escape its competitor v, while if m decreases, species u does not leave crowded zones; that is, despite the competition, u remains in a densely populated area, even if it is occupied by its competitor. However, if u and v cooperate, when m increases, species u leaves the crowded region. In this case, the cooperation with v is not being fully exploited. Conversely, if m decreases, species u benefits from its cooperation with v.

Finally, in the predator-prey scenario where u is a prey and v is a predator, when m increases, the prey u escapes from v. On the other hand, if m decreases, the prey does not leave the area occupied by the predators.

One of the main objectives of this chapter is to study the region of existence (as a function of the parameters λ and μ) of positive solutions, depending on the behavior of the functions m and n, as well as the type of interaction between the species.

In relation to the results of this chapter, we show that the semi-trivial solutions of Problem (12), denoted by:

$$(u,v)=(\theta_\lambda,0) \qquad \text{and} \qquad (u,v)=(0,\theta_\mu)$$

which are, respectively, solutions of the following problems:

$$\begin{cases} -m\,(0)\,\Delta u=\lambda u-u^2 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} -n(0)\,\Delta v = \mu v - v^2 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

exist if, and only if, $\lambda > m(0)\lambda_1$ and $\mu > n(0)\lambda_1$, respectively.

We define the maps $F : [m(0)\lambda_1, \infty) \to \mathbb{R}$ and $G : [n(0)\lambda_1, \infty) \to \mathbb{R}$ given by:

$$F(\lambda) := \sigma_1 \left[-n\left(\int_\Omega \theta_\lambda \right) \Delta + d\theta_\lambda \right] \quad \text{and} \quad G(\mu) := \sigma_1 \left[-m\left(\int_\Omega \theta_\mu \right) \Delta + c\theta_\mu \right],$$

where $\sigma_1[-d\Delta + a]$, with $d > 0$. First, we provide some results characterizing the local stability of semi-trivial solutions:

Theorem 4. The following assertions hold:

(a) Assume that $\lambda > m(0)\lambda_1$. The semi-trivial solution $(\theta_\lambda, 0)$ is linearly asymptotically stable, respectively unstable, when $\mu < F(\lambda)$, respectively $\mu > F(\lambda)$.

(b) Assume that $\mu > n(0)\lambda_1$. The semi-trivial solution $(0, \theta_\mu)$ is linearly asymptotically stable, respectively unstable, when $\lambda < G(\mu)$, respectively $\lambda > G(\mu)$.

Using the Theory of Fixed Point Index with respect to cones in Banach spaces, originally developed by [4] and later extended and refined by [25], we are able to establish important existence results for positive solutions. These foundational tools allow us to rigorously analyze nonlinear operators in ordered function spaces, leading to the following conclusions:

Theorem 5. The following assertions hold:

(a) Assume that $\lambda > m(0)\lambda_1$ and $\mu > n(0)\lambda_1$. Problem (12) possessat least one coexistence state when

$$(\lambda - G(\mu))(\mu - F(\lambda)) > 0. \tag{13}$$

(b) Assume that $\lambda > m(0)\lambda_1$ and $\mu \le n(0)\lambda_1$. Problem (12) possessat least one coexistence state when $\mu > F(\lambda)$.

(c) Assume that $\lambda \le m(0)\lambda_1$ and $\mu > n(0)\lambda_1$. Problem (12) possessat least one coexistence state when $\lambda > G(\mu)$.

Subsequently, we will conduct a detailed study of the coexistence region defined by (13), which plays a central role in understanding the structure of solutions. To this end, we will closely examine the behavior of the functions $F(\lambda)$ and $G(\mu)$, which are fundamental in characterizing the boundaries of this region. Under the assumption that the functions m and n are differentiable, it follows that both F and G inherit differentiability. Our analysis will focus on their properties and asymptotic behavior, particularly near the lower bounds $\lambda = m(0)\lambda_1$ and $\mu = n(0)\lambda_1$, as well as in the limit as λ and μ tend to $+\infty$. This investigation will provide insights into the geometry and structure of the coexistence region.

Theorem 6. The following assertions hold:

(a) Assume that $d > 0$. If $n(s) \geq ks^{-\alpha}$ for all $s \geq s_0$, for some $s_0 > 0$, where $0 < \alpha < 1$ and $k > 0$, then

$$\lim_{\lambda \to +\infty} F(\lambda) = +\infty.$$

(b) Assume that $d < 0$. If $n(s) \leq cs^{\alpha}$ for all $s \geq s_0$, for some $s_0 > 0$, where $0 < \alpha < 1$ and $c > 0$, then

$$\lim_{\lambda \to +\infty} F(\lambda) = -\infty.$$

(c) Assume that $d < 0$. If $n(s) \geq cs^{\alpha}$ for all $s \geq s_0$, for some $s_0 > 0$, where $\alpha > 1$ and $c > 0$, then

$$\lim_{\lambda \to +\infty} F(\lambda) = +\infty.$$

By symmetry, the same results apply to the function G.

Hence, we can construct the coexistence regions of Problem (12) in the $\lambda - \mu$ plane for all cases, depending on the conditions satisfied by m and n. See the figures below.

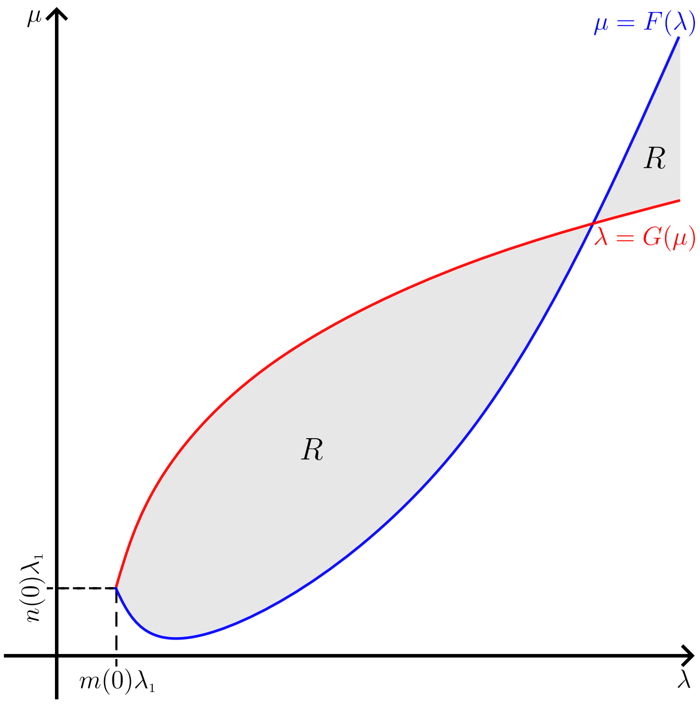
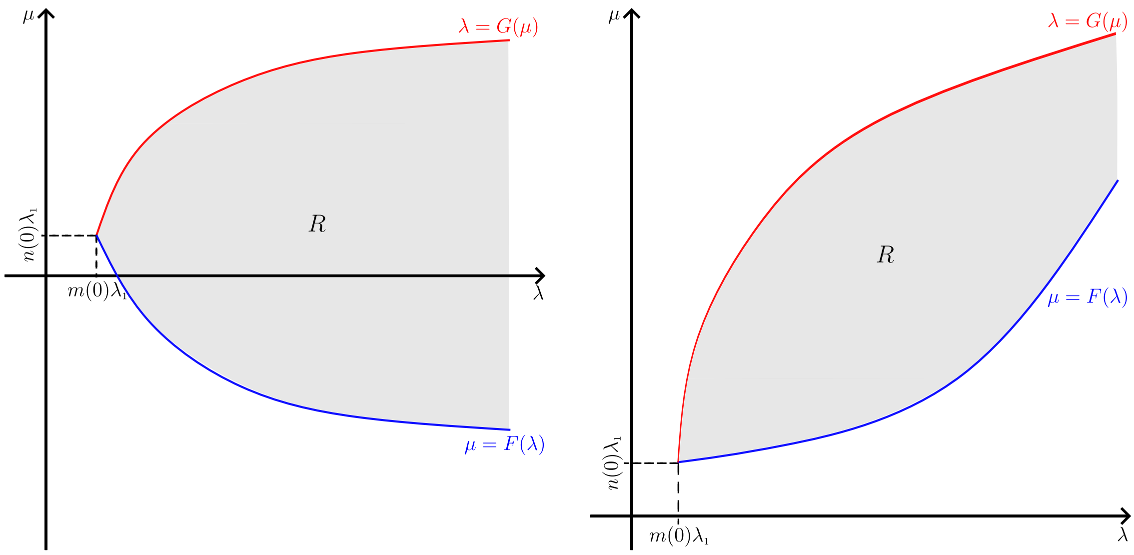


Figure 5: The coexistence region of Problem (12) in the Competition Case. In this case, we assume that m and n verify the Assertion (a) of Theorem 6, which ensures certain properties of the semi-trivial solutions. Additionally, we assume that $n'(0) < 0$, implying that $F'(m(0)\lambda_1) < 0$. Under these conditions, it follows that a coexistence state may occur even for values of μ satisfying $\mu < n(0)\lambda_1$.

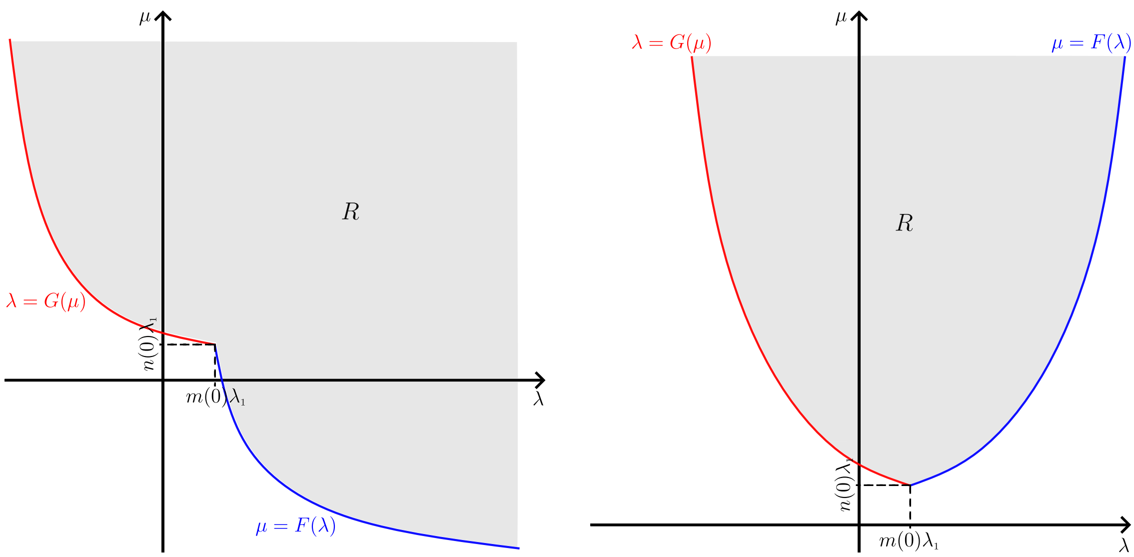
Source: Prepared by the author.



(a) In this case, we assume that m verifies the Assertion (a) of Theorem 6 and n the Assertion (b), according to the hypotheses established in the framework of the analysis.

(b) In this case, we assume that m verify the Assertion (a) of Theorem 6 and n Assertion (c), under the conditions considered throughout the current study.

Figure 6: Coexistence region of Problem (12) in the Prey-Predator Case, corresponding to the scenario analyzed under specific assumptions for the interaction coefficients.
Source: Prepared by the author.



(a) In this case, we assume that m and n verifies the Assertion (b) of Theorem 6.

(b) In this case, we assume that m verifies the Assertion (a) of Theorem 6 and n Assertion (c).

Figure 7: Coexistence region of Problem (12) in the Symbiosis Case.
Source: Prepared by the author.

Finally, we compare our model with the linear model and observe the following differences:

- Competition: In the linear diffusion case, there is no coexistence state if $\lambda \leq \lambda_1$ or $\mu \leq \lambda_1$. In other words, it is a necessary condition for coexistence that both semi-trivial solutions exist, that is, for two competing species to coexist, each must survive in the absence of competition. However, surprisingly in our case, both competing species can coexist even when one semi-trivial solution does not exist. This occurs when the growth rate of species v, μ is small, and the velocity of diffusion of this species is decreasing ($n'(0)$ negative) with respect to the population of the other species u. See Figure 5.

- Prey-predator: In the linear diffusion case, for a fixed predator growth rate μ, there exists at least one coexistence state if the prey growth rate λ is large. This significantly changes in our case when n is large. Indeed, for fixed μ, there is no coexistence state for large λ. This happens because as λ increases, the prey u grows sufficiently, and therefore so does $n(\int_\Omega u)$. As a result, the predator leaves the area populated by the prey and does not benefit from this situation.

- Cooperation: In the linear diffusion case, for a fixed growth rate μ of species v, both species coexist for large values of the growth rate λ of the other species. This is true even for negative values of μ. However, in our case, when n grows strongly, if λ increases, the species u also increases, and the high diffusion rate of v, $n(\int_\Omega u)$, drives species v to extinction because, again, v does not take advantage of the cooperation.

Regarding this comparison, we can conclude the following in relation to the model proposed by Problem (12):

- In the competition case, the coexistence region is expanded when the densities of the species are small and neither species leaves the region of competition. In this scenario, it is in their interest to remain in contact with the other species in order to survive.

- In the prey-predator case, the region of coexistence decreases when the predator abandons areas of high population density. As the prey population increases, the predator's strategy of avoiding these areas prevents coexistence.

- In the cooperation case, a similar situation arises: if a species leaves high-population zones, it no longer benefits from the cooperation, leading to the eventual extinction of both species.

In **Chapter 4**, we will present the results obtained in [21], where we study the existence of coexistence states for the following system with nonlocal diffusion:

$$\begin{cases} -m\left(\int_\Omega u\right)\Delta u = \lambda u - u^2 - cuv & \text{in } \Omega, \\ -n\left(\int_\Omega v\right)\Delta v = \mu v - v^2 - duv & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{14}$$

where $c, d, \lambda, \mu \in \mathbb{R}$, and $m, n : \mathbb{R} \to [0, \infty)$ are continuous functions.

From the perspective of Population Dynamics, (14) models the behavior of two species with population densities $u(x)$ and $v(x)$, inhabiting a habitat Ω. Since we consider homogeneous Dirichlet boundary conditions, the habitat is entirely surrounded by an inhospitable region.

Unlike Problem (12), in this problem, the diffusion terms are not cross-diffusive, that is, the diffusion of the species u, respectively v, depend on their own populations as follows

$$-m\left(\int_\Omega u\right)\Delta u,$$

respectively

$$-n\left(\int_\Omega v\right)\Delta v.$$

This type of diffusion was extensively studied in 2018 by T. S. Figueiredo-Sousa, C. Rodrigo-Morales, and A. Suárez (see [39]). Let's analyze the first expression, the second follows similarly.

- If m is an increasing function, the species tends to leave crowded zones.
- If m is a decreasing function, the species is attracted to areas with a growing population.

As in Chapter 4, our main objective in this chapter is to study the region of existence (as a function of the parameters λ and μ) of positive solutions, depending on the behavior of the functions m and n, as well as the type of interaction between the species.

Regarding the results of this chapter, we will first study the semi-trivial solutions of Problem (14), which we will denote by

$$(u,v)=(\theta_{[\lambda,m(\cdot)]},0) \qquad \text{and} \qquad (u,v)=(0,\theta_{[\mu,n(\cdot)]})$$

which are, respectively, solutions of the following problems:

$$\begin{cases} -m\left(\int_\Omega u\right)\Delta u=\lambda u-u^2 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} -n\left(\int_\Omega v\right)\Delta v=\mu v-v^2 & \text{in } \Omega,\\ v=0 & \text{on } \partial\Omega. \end{cases}$$

Note that, when one species is absent, the other one follows the following logistic equation with a nonlocal diffusivity coefficient:

$$\begin{cases} -g\left(\int_\Omega w\right)\Delta w=\gamma w-w^2 & \text{in } \Omega,\\ w=0 & \text{on } \partial\Omega, \end{cases} \tag{15}$$

where $\gamma\in\mathbb{R}$ and $g:\mathbb{R}\longrightarrow(0,+\infty)$ is a continuous function. This problem has been studied in [39] in a more general situation. In this thesis, we complete the results of [39], and we prove the following result:

Theorem 7. The following assertions hold:

(a) If $\gamma \leq \lambda_1 g_L$, where $g_L := \inf_{s\in\mathbb{R}} g(s)$, Problem (15) does not possess positive solutions.

(b) If $g(0) > 0$, there exists at least a positive solution of Problem (15) when

$$\gamma > g(0)\lambda_1,$$

Moreover, if g is increasing, there exists a unique positive solution of Problem (15) if, and only if, $\gamma > g(0)\lambda_1$.

(c) Let w_γ a positive solution of Problem (15). Then,

$$\int_\Omega w_\gamma \,\mathrm{d}x \to \infty \quad \text{and} \quad w_\gamma(x) \to \infty, \quad \text{as} \quad \gamma \to \infty.$$

We define the maps $F : [m(0)\lambda_1, \infty) \to \mathbb{R}$ and $G : [n(0)\lambda_1, \infty) \to \mathbb{R}$ given, respectively, by:

$$F(\lambda) := \sigma_1 \left[-n(0)\Delta + d\theta_{[\lambda,m(\cdot)]}\right] \qquad \text{and} \qquad G(\mu) := \sigma_1 \left[-m(0)\Delta + c\theta_{[\mu,n(\cdot)]}\right].$$

It is important to emphasize that, despite the notation, F and G are not proper maps in the strict functional sense, since for each value of λ or μ, the corresponding $\theta_{[\lambda,m(\cdot)]}$ or $\theta_{[\mu,n(\cdot)]}$ is not necessarily unique. This non-uniqueness stems from the fact that the auxiliary problems that define $\theta_{[\lambda,m(\cdot)]}$ and $\theta_{[\mu,n(\cdot)]}$ may admit more than one positive solution. This subtlety is one of the particular features of the nonlocal structure of the system under consideration and must be kept in mind throughout the analysis.

Since they are already widely studied in the literature, we will not address the results of these maps in this chapter, but we emphasize that they are well-defined and continuous. Furthermore, the following result holds:

Theorem 8. Assume that m and n are increasing. The following assumptions hold:

(a) The semi-trivial solutions $\theta_{[\lambda,m(\cdot)]}$ and $\theta_{[\mu,n(\cdot)]}$ are unique if, and only if, $\lambda > m(0)\lambda_1$ and $\mu > n(0)\lambda_1$, respectively.

(b) The maps F and G are increasing.

(c) $\lim_{\lambda\to+\infty} F(\lambda) = +\infty$ and $\lim_{\mu\to+\infty} G(\mu) = +\infty$.

Using the Sub-Supersolution Method, we will prove the following coexistence state existence result:

Theorem 9. Assume that $cd < 1$. If the conditions

$$\lambda > m_M\lambda_1 + c\mu \qquad \text{and} \qquad \mu > n_M\lambda_1 + d\lambda \tag{16}$$

are satisfied, then Problem (14) possessat least a coexistence state

In Figure 8, we represent the coexistence region R, defined by:

$$R := \{(\lambda, \mu) \in \mathbb{R}^2; (\lambda, \mu) \text{ verifies } (16)\};$$

in gray, which is ensured by the previous theorem, that is, if $(\lambda, \mu) \in R$, then Problem (14) possessat least a coexistence state. The region R is

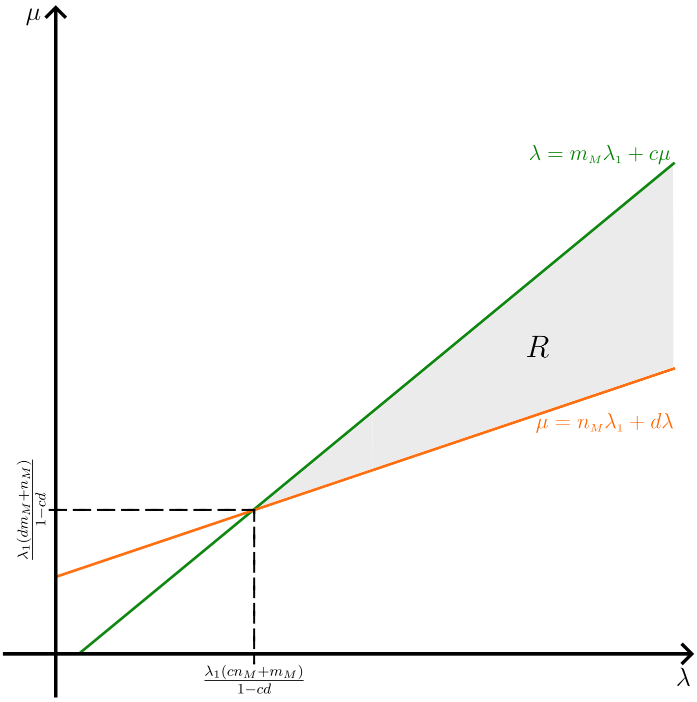


Figure 8: The coexistence region of Problem (14) defined by Condition (16).
Source: Prepared by the author.

Applying the results of Local Bifurcation, where we will strongly rely on apply *Theorem 1.7* of [23], we will present a bifurcation point for one of the semi-trivial solutions of Problem (14).

At this point, it is essential to highlight two major difficulties specific to our setting. First, the semi-trivial solutions of Problem (14) are not necessarily unique, which complicates the direct application of classical bifurcation results. Second, even when a semi-trivial solution is available, we cannot guarantee that it is non-degenerate. For this reason, we must explicitly impose condition (17).

Theorem 10. Assume that $\lambda > 0$. Suppose there exists a positive solution $\theta_{[\lambda,m(\cdot)]}$ of Problem (15), with $g \equiv m$ such that

$$1 + \frac{m'\left(\int_\Omega \theta_{[\lambda,m(\cdot)]}\right)}{m\left(\int_\Omega \theta_{[\lambda,m(\cdot)]}\right)} \int_\Omega (\lambda\theta_{[\lambda,m(\cdot)]} - \theta^2_{[\lambda,m(\cdot)]})e_\lambda \neq 0. \tag{17}$$

Then,

$$(\mu, u, v) = (F(\lambda), \theta_{[\lambda,m(\cdot)]}, 0)$$

is a bifurcation point from the semi-trivial solution $(\mu, \theta_{[\lambda,m(\cdot)]}, 0)$.

In the previous result, e_λ denotes the unique positive solution of the following auxiliary problem, whose existence is guaranteed by the Maximum Principle:

$$\begin{cases} -m\left(\int_\Omega \theta_{[\lambda,m(\cdot)]}\right)\Delta e_\lambda + (2\theta_{[\lambda,m(\cdot)]} - \lambda)e_\lambda = 1 & \text{in } \Omega, \\ e_\lambda = 0 & \text{on } \partial\Omega. \end{cases}$$

Furthermore, we will present a result addressing the bifurcation direction from the semi-trivial solution $(\theta_{[\lambda,m(\cdot)]}, 0)$ arising from the previous theorem.

Using the bifurcation point determined through Local Bifurcation, we will employ Global Bifurcation to ensure the existence of a continuum of positive solutions for Problem (14). More precisely, we will show that:

Theorem 11. Assume that m is increasing. The following assumptions hold:

(a) At $\mu = F(\lambda)$ emanates from $(\theta_{[\lambda,m(\cdot)]}, 0)$ a continuum $\mathfrak{C}$ of positive solutions of Problem (14).

(b) There exists a sequence $(\mu_n, u_n, v_n) \in \mathfrak{C}$ such that $(\mu_n, u_n, v_n) \to (\mu^*, 0, v^*_\mu)$, where v^*_μ is a positive solution of Problem (15) and μ^* satisfies the following condition

$$\lambda = G(\mu^*). \tag{18}$$

In this case, there exists a coexistence state of Problem (14) when

$$\mu \in (\min\{F(\lambda), \mu^*\}, \max\{F(\lambda), \mu^*\}). \tag{19}$$

Furthermore, assuming that n is increasing, there exists a unique semi-trivial solution $(0, \theta_{[\mu,n(\cdot)]})$ and a unique value μ^* that verifies the Condition (18).

In the following figures, we will represent several cases of the Continuum $\mathfrak{C}$ based on the behavior of the functions m and n, as well as the semi-trivial solutions.

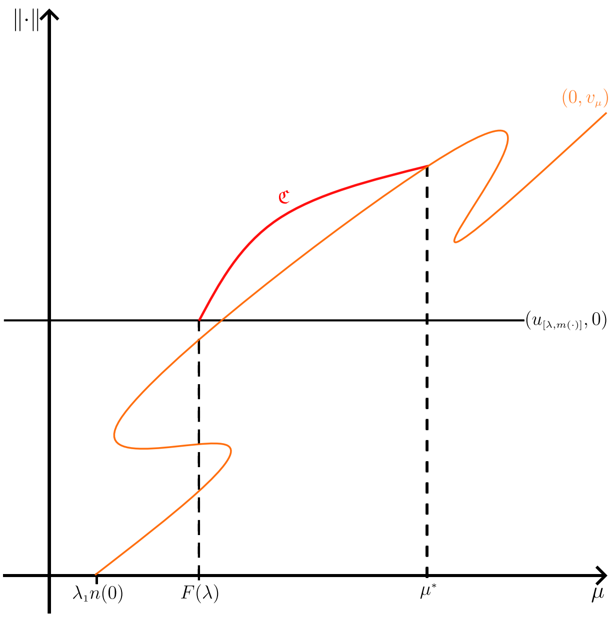


Figure 9: Behavior of the Continuum of semi-trivial solution $\mathfrak{C}_{(\mu,\theta,0)}$ when m is increasing.
Source: Prepared by the author.

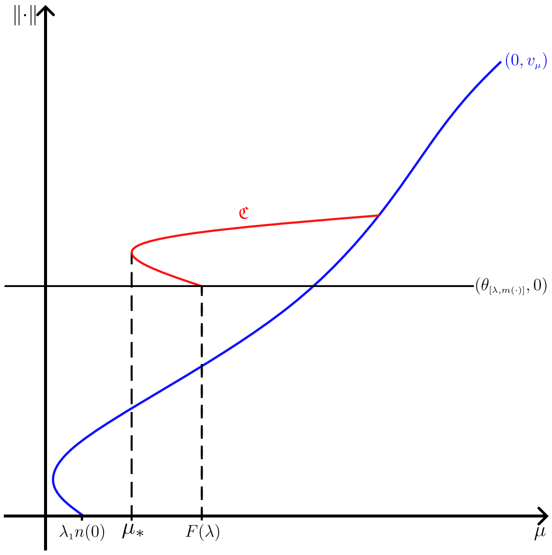


Figure 10: Behavior of the Continuum of semi-trivial solution $\mathfrak{C}_{(\mu,\theta,0)}$ when m is increasing, $\lambda \approx m(0)\lambda_1$ and $n'(0) < 0$.

Source: Prepared by the author.

A significant result obtained was the proof of the existence of coexistence states under the assumption that the functions m and n are increasing. More precisely, we established the following result:

Theorem 12. Assume that m and n are increasing. The following assumptions hold:

(a) Problem (14) does not possess coexistence states when $\lambda \leq m(0)\lambda_1$ or $\mu \leq n(0)\lambda_1$.

(b) Problem (14) possessat least a coexistence state when

$$(\mu - F(\lambda))\,(\lambda - G(\mu)) > 0.$$

In the following figure, we present the bifurcation diagram in the case where both m and n are increasing.

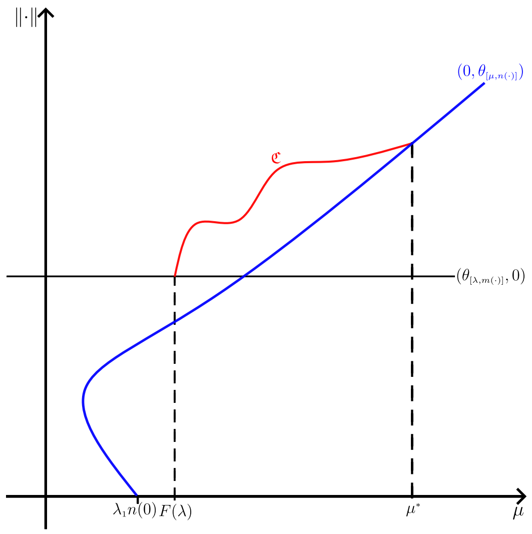


Figure 11: Bifurcation diagram when m and n both increase. In this case, there exists a unique semi-trivial solution $(u_\lambda, 0)$ and $(0, v_\mu)$.

Source: Prepared by the author.

To conclude the chapter, we will demonstrate an important relationship between the existence result obtained through the Sub-Supersolution Method and the Global Bifurcation Theory. More precisely, we will show that:

Theorem 13. If the pair $(\lambda, \mu) \in \mathbb{R}^2$ satisfies Condition (16), then this pair also satisfies Condition (19).

In the Figure 12, we represent the curves generated by conditions (16) and (19). These curves define the coexistence regions, which correspond to the areas enclosed by the curves.

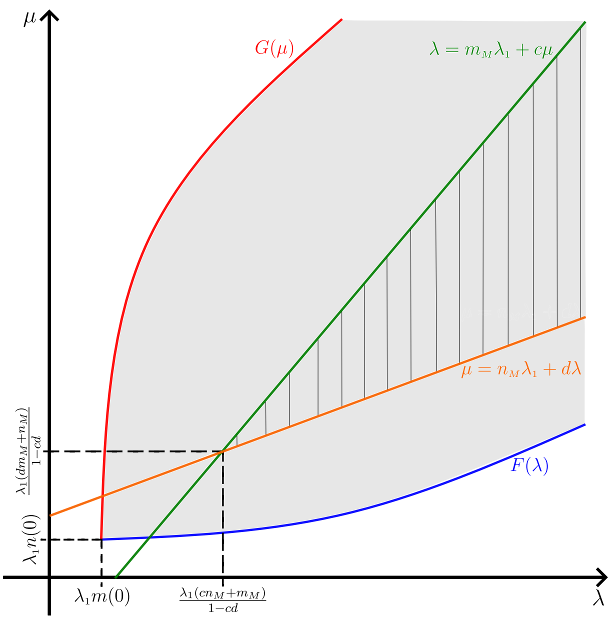


Figure 12: The coexistence region of Problem (14) defined by conditions (16) and (19). In this case, the region defined by Condition (19) is larger than the one defined by (16).

Source: Prepared by the author.

It is pertinent to highlight some relevant observations that complement and reinforce the aspects we will address in this chapter.

- It is quite surprising that the competitive exclusion principle holds regardless of the behavior of the functions m and n. Indeed, while it is well known that this principle is satisfied in the case of linear diffusion, that is, when $m \equiv n \equiv 1$, it remains valid for any choice of the functions m and n.
- Condition (16) is obtained using the Sub-Supersolution Method and does not require any monotonicity assumption on either n or m. However, it is necessary that $cd < 1$.
- Unlike the linear diffusion case, the bifurcation method cannot, in general, be applied for two main reasons. First, the semi-trivial solution is not unique. Second, it is not, in general, nondegenerate. However, under a specific condition (see (17)), it is possible to establish the bifurcation of a continuum of positive solutions. In particular, Condition (17) is satisfied if m is increasing.
- In the linear diffusion case, the continuum of positive solutions bifurcates from one semi-trivial solution and connects to the other. However, in the nonlocal diffusion case, the behavior of the continuum is not predetermined — multiple scenarios may occur, and the structure of the solution set can be significantly more complex.
- When the bifurcation method can be applied, the results obtained are more robust than those achieved through the Sub-Supersolution Method, as they provide a broader description of the set of solutions and, in particular, expand the region of coexistence.
- The inclusion of nonlocal terms leads to a more intricate behavior of the system, resulting, for instance, in the multiplicity of positive solutions under conditions where the corresponding system with linear diffusion admits a unique coexistence state.

In **Chapter 5**, we present the general conclusions drawn from the study of the nonlocal elliptic systems analyzed throughout the previous chapters. We discuss the main results concerning the existence, uniqueness, and structure of positive solutions, with particular emphasis on coexistence states in population dynamics models. Furthermore, we highlight the methodological contributions employed, such as Bifurcation Theory, Sub-Supersolution Method, and the use of Fixed Point Index Theory in Positives Cones, as well as the challenges introduced by the presence of nonlocal terms in the diffusion coefficients. This chapter also underscores the differences between nonlocal models and their local counterparts, revealing new and ecologically relevant phenomena.

We conclude this introduction by highlighting some open questions throughout this work. Some of them are difficulties that we couldn't resolve within the scope of the research conducted, while others represent areas of interest that, although not addressed here, will serve as a guide for future investigations.

1. **Applicability of the bifurcation method when $m(0) = 0$ or $n(0) = 0$:** In the case where $m(0) = 0$ or $n(0) = 0$, possibly both, the bifurcation method cannot be applied directly, as the associated linearized operator may lose key properties such as coercivity or invertibility. This scenario requires the use of compactness

arguments or alternative variational techniques to prove the existence of solutions. It highlights an inherent limitation of the bifurcation method in degenerate cases and calls for more robust analytical tools.

2. **Uniqueness of the coexistence state in systems with nonlocal diffusion:** Establishing the uniqueness of coexistence states in systems involving nonlocal diffusion terms is a major open problem. The global nature of these terms introduces strong coupling and nonlinearity, making it difficult to derive sufficient conditions for uniqueness. This issue is particularly relevant in applications, where uniqueness is often associated with predictability and stability of the model.

3. **Stability of solutions:** Analyzing the stability of the obtained solutions is crucial for understanding the long-term dynamics of the modeled populations. Determining whether a solution is stable or unstable under small perturbations allows for predictions about species persistence or extinction. However, the interplay between the coupling and the nonlocal terms significantly complicates the spectral and dynamical analysis required to assess stability.

4. **Behavior in the cooperative case under strong cooperation:** In cooperative systems, where species benefit mutually from each other's presence, it remains unclear how strong cooperation affects the existence, multiplicity, and stability of solutions. Intense cooperation may lead to new bifurcation phenomena or even loss of stability. A better understanding of this scenario is essential to model ecological systems with high levels of mutualistic interaction accurately.

5. **Eigenvalue problems in systems with nonlocal terms:** The inclusion of nonlocal terms in the diffusion operator adds a layer of complexity to the associated eigenvalue problems. Such operators are typically nonlinear and lack compactness, making it challenging to characterize the spectrum. A thorough spectral analysis is fundamental for understanding bifurcation structure, stability thresholds, and the qualitative behavior of solutions in these systems. Moreover, through this analysis, we guarantee uniqueness, as pointed out in item 2.

1 Preliminaries

1.1 Basic Concepts and Definitions

In this section, we will present some concepts and definitions that will be used throughout this work, which are essential for understanding the more complex topics introduced later. For further details on the material covered here, we recommend foundational texts on Linear and Nonlinear Algebra, Functional Analysis, Topology, and Analysis (see [4, 6, 10, 27, 45]).

Throughout this work, we will often consider a specific subset of $\mathbb{R}^N$ that possesses certain specific properties, namely:

Definition 1.1. We say that Ω is a **bounded regular domain** in $\mathbb{R}^N$, with $N \geq 1$, when Ω is an open and connected subset of $\mathbb{R}^N$ whose boundary, $\partial\Omega$, is sufficiently regular.

The following definition introduces a category of operators in Banach spaces that are fundamental in Functional Analysis, especially in Differential Equations, due to their stability and connection with topological and spectral properties.

Definition 1.2. Let E, F be Banach spaces and $T : E \to F$ a continuous linear operator. We say that T is a **Fredholm operator** when:

(a) $\dim[\mathrm{Ker}(T)] < \infty$; and

(b) $\mathrm{codim}[\mathrm{Rg}(T)] < \infty$.

For operators that satisfy this definition, we can introduce the concept of an **index**, defined as follows:

$$\mathrm{ind}(T) := \dim[\mathrm{Ker}(T)] - \mathrm{codim}[\mathrm{Rg}(T)].$$

We will denote by $\mathrm{Fred}_0(E, F)$ the set of all Fredholm operators with index zero.

In the next result, we present an important property of Fredholm operators, which will be utilized in future applications.

Theorem 1.3 (Fredholm Alternative)**.** Let E be a Banach space, $T : E \to E$ a compact operator and T^* the adjoint operator associated to T. The following assumptions hold:

(a) $\mathrm{Ker}[I - T]$ and $\mathrm{Ker}[I - T^*]$ have finite dimension. Moreover

$$\dim[\mathrm{Ker}(I - T)] = \dim[\mathrm{Ker}(I - T^*)].$$

(b) $\text{Rg}[I-T]$ and $\text{Rg}[I-T^*]$ are closed. Moreover

$$\text{Rg}[I-T] = \text{Ker}[I-T^*]^\perp \qquad \text{and} \qquad \text{Rg}[I-T^*] = \text{Ker}[I-T]^\perp.$$

(c) $\text{Ker}[I-T] = \{0\}$ if, and only if, $\text{Rg}[I-T] = E$.

(d) $\dim[\text{Ker}(I-T)] = \dim[\text{Ker}(I-T^*)]$.

Proof. See *Theorem 6.6* of [10]. □

Let E be a Banach space, $I : E \to E$ the identity operator in E and $K : E \to E$ a compact linear operator. Note that, $I-K$, called **compact perturbation of the identity**, is a Fredholm operator of index zero. Indeed, by Assumption (a) of Theorem 1.3,

$$\dim[\text{Ker}(I-K)] = \dim[\text{Ker}(I-K^*)] < \infty$$

and, by Assumption (b),

$$\dim[\text{Ker}(I-K)] = \dim[\text{Rg}(I-K)^\perp].$$

Consequently

$$\dim[\text{Rg}(I-K)^\perp] < \infty.$$

Since $\text{Rg}(I-K)$ is closed, by *Proposition 11.13* of [10],

$$\dim[\text{Rg}(I-K)^\perp] = \text{codim}[\text{Rg}(I-K)].$$

Therefore

$$\text{ind}(I-K) = 0.$$

To introduce the next concept, which will be used throughout this text, we first need to recall some fundamental ideas from Linear Algebra.

Definition 1.4. Let V be a vector real space and $\prec$ a ordering in V. We say that $\prec$ is a **linear ordering** when the following properties are satisfied:

(a) Given $x, y \in V$, for all $z \in V$, $x \prec y$ implies that $x + z \prec y + z$; and

(b) Given $x, y \in V$, for all $\alpha \in \mathbb{R}_+$, $x \prec y$ implies that $\alpha x \prec \alpha y$.

We will refer to the space V with a linear ordering as an **ordered vector space** (OVS) and denote it by $(V, \prec)$, or simply V, when the ordering is clear.

Definition 1.5. Let V be a ordered vector space. We say that the set

$$P_V := \{v \in V;\ 0 \prec v\}$$

is a **positive cone** of V associated with the ordering $\prec$ when:

(a) $P_V + P_V \subset P_V$.

(b) $\mathbb{R}_+ P_V \subset P_V$.

(c) $P_V \cap (-P_V) = \{0\}$.

Recall that a Banach space E is a vector space. We can say that E is a **ordered Banach space** when its positive cone ψ^* is closed (in the topology induced by the norm of E) and denoted it by (E, P_E) or simply E when it is clear that E has a positive cone and what it is.

Definition 1.6. Let E, F be ordered Banach spaces and $T : E \to F$ a linear transformation. We say that:

(a) T is **positive** when $T(P_E) \subset P_F$.

(b) T is **strictly positive** when $T(P_E \setminus \{0\}) \subset P_F \setminus \{0\}$.

(c) T is **strongly positive** when $T(P_E \setminus \{0\}) \subset \mathrm{Int}(P_F)$, where $\mathrm{Int}(P_F) \neq \emptyset$.

On some occasions, in addition to requiring that the cone of a given space be positive, we will also require it to satisfy another property, which we define below.

Definition 1.7. Let E be a ordered Banach space. We say that P_E is **normal** when there exists $\delta > 0$ such that $0 \prec u \prec v$ implies that $\|u\|_E \leq \delta \|v\|_E$.

With what we saw so far, we have sufficient conditions to state the next result, which is often used to demonstrate the existence of a principal eigenvalue for certain problems.

Theorem 1.8 (Krein-Rutman)**.** Let E be a ordered Banach space with $\mathrm{Int}(P_E) \neq \emptyset$ and $T : E \to E$ a strongly positive compact endomorphism. The following assumptions hold:

(a) The **spectral radius** of T, defined by

$$\mathrm{r}(T) := \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}},$$

is positive and any other eigenvalue λ of T satisfies $|\lambda| \leq \mathrm{r}(T)$.

(b) $\mathrm{r}(T)$ is a simple eigenvalue of T having a positive eigenvector and there is no other eigenvalue with a positive eigenvector.

(c) For each $v \in P_E$, with $v \neq 0_E$, the equation

$$\lambda u - T(u) = v$$

has exactly one positive solution when $\lambda > \mathrm{r}(T)$ and has no positive solution when $\lambda \leq \mathrm{r}(T)$.

Proof. See *Theorem 19.2* and *Theorem 19.3* of [27] and *Theorem 3.2* of [10]. □

We will discuss a concept that, roughly speaking, is used to measure how many times a continuous function wraps around an N-sphere with respect to a set. In the context of differential equations, this concept can be employed to prove the existence of solutions.

Theorem 1.9. Define the following set:

$$\Sigma := \left\{(f, \Omega, y);\ f \in C^1(\overline{\Omega}, \mathbb{R}^N) \text{ and } y \notin f\right\}.$$

There exists a unique application

$$\begin{aligned} d :\quad \Sigma \quad &\longrightarrow \mathbb{Z} \\ (f, \Omega, y) &\longmapsto d(f, \Omega, y) \end{aligned}$$

that satisfies the following properties:

(B_1) (Normalization) $d(f, \Omega, y) = 1$ for every constant function f.

(B_2) (Additivity) $d(f, \Omega, y) = d(f, \Omega_1, y) + d(f, \Omega_2, y)$ when Ω_1, Ω_2 are disjoint open sets of Ω and $y \notin f(\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2))$.

(B_3) (Homotopy Invariance) $d(h(t, \cdot), \Omega, y(t)) = k$ when $h : [0,1] \times \overline{\Omega} \longrightarrow \mathbb{R}^N$ is a continuous compact application, $y : [0,1] \longrightarrow \mathbb{R}^N$ is continuous and $y(t) \notin h(t, \partial\Omega)$, for all $t \in [0,1]$.

Proof. See *Chapter 1* of [27]. □

The application presented in the result above is called the **Brouwer degree** of f relative to Ω and y.

Proposition 1.10. The Brouwer degree satisfies the following properties:

(B_4) If $d(f, \Omega, y) \neq 0$ then the equation $f(x) = y$ has at least one solution in Ω.

(B_5) If $(f, \Omega, y) \in \Sigma$ there exists $r > 0$ such that, for all $g \in C(\overline{\Omega}, \mathbb{R}^N)$,

$$\|f - g\|_C < r$$

when $(g, \Omega, y) \in \Sigma$ and $d(f, \Omega, y) = d(g, \Omega, y)$.

(B_6) If y and z are in the same connected component of $\mathbb{R}^N \setminus f\partial\Omega$ then

$$d(f, \Omega, y) = d(f, \Omega, z).$$

(B_7) If $(f, \Omega, y), (g, \Omega, y) \in \Sigma$ are such that $f(x) = g(x)$, for all $x \in \partial\Omega$, then

$$d(f, \Omega, y) = d(g, \Omega, y).$$

(B_8) If $(f, \Omega, y) \in \Sigma$ and Ω_1 is a open set in Ω such that $y \notin f(\overline{\Omega} \setminus \Omega_1)$ then

$$d(f, \Omega, y) = d(f, \Omega_1, y).$$

(B_9) If $(f, \Omega, y) \in \Sigma$ then

$$d(f, \Omega, y) = d(f - y, \Omega, 0).$$

Proof. See *Chapter 1* of [27]. □

Through the concept of Brouwer degree, it is possible to define the index for functions in finite-dimensional spaces. Let $f : \overline{\Omega} \to \mathbb{R}^N$ be a continuous function and $y \notin f(\partial\Omega)$. Given x_0 as an isolated solution of $f(x) = y$, there exists $r > 0$ such that x_0 is the only solution of this equation in $\overline{\mathrm{B}(x_0, r)}$. Consequently, by Property (B_8), for every $\varepsilon > 0$ such that $0 < \varepsilon < r$,

$$d(f, \mathrm{B}(x_0, r), y) = d(f, \mathrm{B}(x_0, \varepsilon), y),$$

where $y \notin f(\overline{\mathrm{B}(x_0, r)} \setminus \mathrm{B}(x_0, \varepsilon))$ because there is no solution of $f(x) = y$. Thus, for every $\varepsilon > 0$ sufficiently small,

$$d(f, \mathrm{B}(x_0, \varepsilon), y) = k.$$

Definition 1.11. Let $f : \Omega \to \mathbb{R}^N$ be a continuous function, $y \notin f(\partial\Omega)$ and x_0 a isolated solution of the equation $f(x) = y$. We say that

$$i(f, x_0, y) := \lim_{\varepsilon \to 0} d(f, \mathrm{B}(x_0, \varepsilon), y)$$

is the **index** of f at x_0 relative to the point y.

Although not addressed here, the concept of index in finite-dimension spaces possess several results, which can be found in [4, 27, 45].

The concept we just defined has the limitation of not being directly extendable to infinite-dimensional spaces. To overcome this, we will introduce a new concept, formulated in a manner very similar to what we have seen, which is specifically designed to define the index in infinite-dimensional spaces.

Theorem 1.12. Let E be a real Banach space, Ω a bounded open set in E, $y \in E$ and define the following set:

$$\Gamma := \left\{(I - T, \Omega, y);\ T : \overline{\Omega} \to E \text{ compact and } y \notin (I - T)(\partial\Omega)\right\}.$$

There exists a unique application

$$\begin{aligned} D : \quad \Gamma \quad &\longrightarrow \mathbb{Z} \\ (I - T, \Omega, y) &\longmapsto D(I - T, \Omega, y) \end{aligned}$$

that satisfies the following properties:

(S_1) (Normalization) $D(I, \Omega, y) = 1$ for every $y \in \Omega$.

(S_2) (Additivity) $D(I - T, \Omega, y) = D(I - T, \Omega_1, y) + D(I - T, \Omega_2, y)$ when Ω_1, Ω_2 are disjoint open sets of Ω and $y \notin (I - T)(\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2))$.

(S_3) (Homotopy Invariance) $D(I - h(t, \cdot), \Omega, y(t))$ it is well-defined and does not depend of $t \in [0, 1]$, where $h : [0, 1] \times \overline{\Omega} \to \mathbb{R}^N$ is a continuous compact application, $y : [0, 1] \to E$ is continuous and $y(t) \notin (I - h(t, \cdot))(t, \partial\Omega)$, for all $t \in [0, 1]$.

Proof. See *Chapter 3* of [45]. □

The application defined in the result above is a extension of the application presented in Theorem 1.12 and is called **Leray-Schauder degree** of $I - T$ relative to Ω and y.

Proposition 1.13. The Leray-Shauder degree satisfies the following properties:

(S_4) If $D(I - T, \Omega, y) \neq 0$ then the equation $x - T(x) = y$ has at least one solution in E.

(S_5) If $(I - T, \Omega, y) \in \Gamma$ there exists $r > 0$ such that, for all $S : \overline{\Omega} \to E$ compact with $\|T - S\| < r$, $(I - S, \Omega, y) \in \Gamma$ and

$$D(I - T, \Omega, y) = D(I - S, \Omega, y).$$

(S_6) If y and z are in the same connected component of $E \setminus (I - T)(\partial\Omega)$ then

$$D(I - T, \Omega, y) = D(I - T, \Omega, z).$$

(S_7) If $(I-T,\Omega,y),(I-S,\Omega,y)\in\Gamma$ are such that $T(x)=S(x)$, for all $x\in\partial\Omega$, then

$$D(I-T,\Omega,y)=D(I-S,\Omega,y).$$

(S_8) If $(I-T,\Omega,y)\in\Gamma$ and Ω_1 is a open set in Ω such that $y\notin(I-T)(\overline{\Omega}\setminus\Omega_1)$ then

$$D(I-T,\Omega,y)=D(I-T,\Omega_1,y).$$

(S_9) If $(I-T,\Omega,y)\in\Gamma$ then

$$D(I-T,\Omega,y)=D(I-T-y,\Omega,0).$$

Proof. See *Section 3* of [45]. □

Let $T:\overline{\Omega}\to E$ be a compact operator $y\notin(I-T)(\partial\Omega)$. Given x_0 as an isolated solution of the equation $x-T(x)=y$ such that, for a certain $r>0$, it is the only solution in $\overline{\mathrm{B}(x_0,r)}$. Consequently, by Property (S_8), for all $R>0$ with $0<R<r$,

$$D(I-T,\mathrm{B}(x_0,r),y)=D(I-T,\mathrm{B}(x_0,R),y).$$

Definition 1.14. Let $T:\overline{\Omega}\to E$ be a compact operator, $y\in(I-T)(\partial\Omega)$ and x_0 a isolated solution of the equation $x-T(x)=y$. We say that

$$i(I-T,x_0,y):=\lim_{r\to0}D(I-T,\mathrm{B}(x_0,r),y)$$

is the **index** of T at x_0 relative to the point y.

The following results will provide methods to determine the index of certain differentiable operators.

Proposition 1.15. Let Ω be an open subset in E and $T:\Omega\to E$ a continuous compact operator such that one is not an eigenvalue of T for some $x_0\in\Omega$. Then x_0 is a solution of the equation $x-T(x)=y$, where $y=x_0-T(x_0)$, and

$$i(I-T,x_0,y)=(-1)^{\chi},$$

where χ is the sum of the algebraic multiplicities of the eigenvalues of $T'(x_0)$ contained in the interval $(0,1)$.

Proof. See *Lemma 3.19* of [6]. □

When $y=0$, we denote the index of T at x_0 relative to the point 0 simply as

$$i(I-T,x_0)=\lim_{r\to0}D(I-T,\mathrm{B}(x_0,r),0).$$

Proposition 1.16. Let Ω be an open subset in E and $T:\overline{\Omega}\to E$ a continuous compact operator such that one is not an eigenvalue of T for some $x_0\in\Omega$. Define

$$F(x)=x-T(x)$$

and

$$F(x_0)=p.$$

Then x_0 is an isolated solution of F and

$$i(F,x_0)=(-1)^{\chi}$$

where χ is the sum of the algebraic multiplicities of the eigenvalues of $T'(x_0)$ contained in the interval $(0,1)$.

Proof. See *Theorem 3.20* of [6]. □

To conclude this section, we will present the concept and results that ensure the existence of the fixed-point index for compact mappings. This tool is fundamental in Nonlinear Analysis, particularly in the study of differential equations and bifurcation theory, as it provides a way to quantify the behavior of fixed points and their stability.

Definition 1.17. Let E be a Banach space and X a nonempty subset of E. We say that X is a **retract** of E when there exist a continuous mapping $r : E \to X$, called **retraction**, such that $r(x) = x$ for all $x \in X$.

Although the definition above presents the concept only for Banach spaces, the notion of a retract can be generalized to any topological space.

Theorem 1.18. Let E be a Banach space and X a retract of E. Define the following set

$$\Pi := \left\{ (K, U); \begin{array}{l} U \text{ is a oper subset of } X \text{ and } K : \overline{U} \to X \text{ is a compact} \\ \text{operator that has no fixed points on } \partial U \end{array} \right\}$$

There exists a unique application

$$\begin{aligned} i_X : \quad \Pi \quad &\longrightarrow \mathbb{Z} \\ (K, U) &\longmapsto i_X(K, U) \end{aligned}$$

that satisfies the following properties:

(I_1) (Normalization) $i_X(I, U) = 1$;

(I_2) (Additivity) $i_X(K, U) = i_X(K, U_1) + i_X(K, U_2)$ when U_1, U_2 are disjoint open sets of U such that K has no fixed point on $\overline{U} \setminus (U_1 \cup U_2)$;

(I_3) (Homotopy Invariance) $i_X(h(\lambda, \cdot), U)$ it is well-defined and does not depend of $\lambda \in \Lambda$, where $\Lambda \subset \mathbb{R}$ is a compact, $h : \Lambda \times \overline{U} \to X$ is a compact operator such that $h(\lambda, x) \neq x$ for all $(\lambda, x) \in \Lambda \times \partial U$; and

(I_4) (Permanence) $i_X(K, U) = i_Y(K, U \cap Y)$, where Y is a retract of X and $f(\overline{U}) \subset Y$, with $i_Y(K, U \cap Y) := i_Y(K|_{\overline{U \cap Y}}, U \cap Y)$.

The application presented in the result above is called **fixed point index** of K over U with respect to X.

Proposition 1.19. The fixed point index satisfies the following properties:

(I_5) (Excision) If V is an open set in U such that K has no fixed point on $\overline{U} \setminus V$, then $i_X(K, U) = i_X(K, V)$.

(I_6) (Existence of Solution) If $i_X(K, U) \neq 0$, then K has at least one fixed point in U.

1.2 The Maximum Principle

In this section, we will present some results regarding the Maximum Principle, a fundamental tool in the theory of Elliptic Equations. In summary, this principle establishes that, under certain conditions, the maximum value of a solution occurs at the boundary of the domain. Throughout this work, we will examine its importance in developing comparison techniques and a priori estimates, which ensure control over the solutions in various classes of problems. In this section, we will explore the main formulations of the Maximum Principle for the Laplace operator, for more details, we recommend references [41, 49, 57].

Let Ω a bounded regular domain in $\mathbb{R}^N$, with $N \geq 1$. Throughout this section, we will consider the following second-order operator:

$$\mathcal{L} := -\sum_{i,j}^{N} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{N} b_j(x)\frac{\partial}{\partial x_j} + c(x), \tag{1.1}$$

where $x \in \Omega$, $a_{ij} \in C(\overline{\Omega})$, with $a_{ij} = a_{ji}$, and $b_j, c \in L^\infty(\Omega)$, for all $i, j \in \{1, 2, \ldots, N\}$. Additionally, we will impose that the operator $\mathcal{L}$ satisfies the following definition:

Definition 1.20. We say that the operator $\mathcal{L}$, defined in (1.1), is **uniformly elliptic** when there exists a constant $C > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq C\,|\xi|^2$$

for all $x \in \overline{\Omega}$ and $\xi = (\xi_1, \xi_2, \ldots, \xi_N) \in \mathbb{R}^N$.

Next, we will examine several formulations of this principle, starting with the "most basic" formulation.

Theorem 1.21 (Weak Maximum Principle)**.** The following assumptions hold:

(a) Assume that $c \equiv 0$. Suppose that $w \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy

$$\mathcal{L}w \leq 0.$$

Then, the maximum of w in Ω is achieved on $\partial\Omega$, that is, $\max_{\overline{\Omega}} w = \max_{\partial\Omega} w$.

(b) Assume that $c \equiv 0$. Suppose that $w \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy

$$\mathcal{L}w \geq 0.$$

Then, the minimum of w in Ω is achieved on $\partial\Omega$, that is, $\min_{\overline{\Omega}} w = \min_{\partial\Omega} w$.

(c) Assume that $c \geq 0$. Suppose that $w \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy

$$\mathcal{L}w \leq 0.$$

Then $\sup_{\overline{\Omega}} w \leq \sup_{\partial\Omega} w^+$.

(d) Assume that $c \geq 0$. Suppose that $w \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy

$$\mathcal{L}w \geq 0.$$

Then $\inf_{\overline{\Omega}} w \geq \inf_{\partial\Omega} w^-$.

(e) Assume that $c \geq 0$. Suppose that $w \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy

$$\mathcal{L}w = 0.$$

Then $\sup_{\overline{\Omega}} |w| = \sup_{\partial\Omega} |w|$.

Proof. See *Theorem 1.1.3* of [48]. □

The next version addresses the more general case when $\mathcal{L}w \leq 0$ throughout the domain Ω.

Theorem 1.22 (Maximum Principle of Hopf)**.** The following assumptions hold:

(a) Assume that $c \equiv 0$. Suppose that $w \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy

$$\mathcal{L}w \leq 0.$$

If w achieves its maximum in Ω, then w is constant.

(b) Assume that $c \equiv 0$. Suppose that $w \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy

$$\mathcal{L}w \geq 0.$$

If w achieves its minimum in Ω, then w is constant.

(c) Assume that $c \geq 0$. Suppose that $w \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy

$$\mathcal{L}w \leq 0.$$

If w achieves a non-negative maximum in the interior of Ω, then w is constant.

(d) Assume that $c \geq 0$. Suppose that $w \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy

$$\mathcal{L}w \geq 0.$$

If w achieves a non-positive minimum in the interior of Ω, then w is constant.

Proof. See *Theorem 1.2.1* of [48]. □

To conclude this section, we present the definition that allows us to verify when a uniformly elliptic operator satisfies the Strong Maximum Principle.

Definition 1.23. We say that the operator $\mathcal{L}$, defined in (1.1), satisfies the Strong Maximum Principle when, given $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ such that

$$\begin{cases} \mathcal{L}w \geq 0 & \text{in } \Omega, \\ w \geq 0 & \text{on } \partial\Omega, \end{cases}$$

implies that

(a) $w(x) > 0$ for all $x \in \Omega$.

(b) $\dfrac{\partial w}{\partial \eta}(x_0) < 0$ for all $x_0 \in \partial\Omega$ such that $w(x_0) = 0$.

For the next result, Ω must satisfy the uniform interior sphere condition.

Theorem 1.24 (Strong Maximum Principle)**.** Suppose $c \geq 0$. Then, $\mathcal{L}$ satisfies the Strong Maximum Principle.

Proof. See *Theorem 1.5.1* of [48]. □

In what follows, we will consider a particular case for the uniformly elliptic operator $\mathcal{L}$. Specifically, we will consider the Laplacian operator given by:

$$\mathcal{L}w = -\alpha(x)\Delta w + \beta(x)w,$$

where α and β which will be detailed below.

1.3 Eigenvalue Problems

In this section, we will present the main concepts and results regarding the Eigenvalue Problem. For a more comprehensive study than what we present here, we recommend [15, 41, 49].

Throughout this section, consider the following Eigenvalue Problem involving the Laplace operator:

$$\begin{cases} -\alpha(x)\Delta w + \beta(x)w = \gamma w & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\alpha, \beta \in L^{\infty}(\Omega)$, with $\alpha \geq \alpha_0$, for some constant $\alpha_0 > 0$.

This type of problem is fundamental in the study of Elliptic Equations, as the eigenvalues and eigenfunctions associated with the Laplace operator in bounded domains with Dirichlet boundary conditions provide key insights into solution behavior, namely: existence and uniqueness of solutions; solution stability; asymptotic behavior; solution regularity; and multiplicity of solutions.

The analysis of eigenvalues and eigenfunctions is typically conducted through variational techniques and the application of the Maximum Principle to ensure solution properties.

We will now present definitions and results concerning the Eigenvalue Problem (1.2) that we will frequently use throughout this work.

Definition 1.25. We say that $\gamma \in \mathbb{R}$ is an **eigenvalue** of Eigenvalue Problem (1.2) in Ω, and denoted by

$$\sigma_j^{\Omega}\left[-\alpha(x)\Delta + \beta(x)\right],$$

when the problem has a classic solution. Moreover, we say that $w \neq 0$ is an **eigenfunction** of Eigenvalue Problem (1.2) associated to γ.

We now highlight a particular eigenvalue of special interest in the study of elliptic problems. This concept plays a central role in understanding the behavior of solutions, especially in relation to the sign of the associated eigenfunctions and the structure of the spectrum.

Definition 1.26. We say that $\gamma \in \mathbb{R}$ is a **principal eigenvalue** in Ω when Eigenvalue Problem (1.2) admits a solution that does not change sign, which will be denoted by

$$\sigma_1^{\Omega}\left[-\alpha(x)\Delta + \beta(x)\right].$$

If there is no risk of confusion, we will omit the index Ω to denote the principal eigenvalue in Ω.

The next result guarantees the existence and uniqueness of an eigenvalue for Problem A, as well as some of its properties.

Theorem 1.27. Let $E = W^{2,p}(\Omega)$. There exists a unique principal eigenvalue of Eigenvalue Problem (1.2) and a unique normalized principal eigenfunction in the interior of the cone of $C_0^1(\Omega)$, denoted by $\varphi_1^{[\alpha(\cdot),\beta(\cdot)]}$. Moreover, the following properties hold:

(a) $\sigma_1\left[-\alpha(x)\Delta + \beta(x)\right]$ is **algebraically simple**, that is, all eigenfunctions are multiples of $\varphi_1^{[\alpha(\cdot),\beta(\cdot)]}$.

(b) $\sigma_1\left[-\alpha(x)\Delta + \beta(x)\right]$ is **dominant**, that is, any other eigenvalue satisfies $\Re\left(\sigma_j\left[-\alpha(x)\Delta + \beta(x)\right]\right) > \sigma_1\left[-\alpha(x)\Delta + \beta(x)\right]$.

Proof. See *Chapter 7* of [49]. □

When there is no risk of confusion, we will omit the superscript of the principal eigenfunction and simply denote it by φ_1. The next result provides monotonicity properties, which will serve as important tools throughout this work.

Proposition 1.28. The following assumptions hold:

(a) The application $L^\infty(\Omega) \ni \alpha \longmapsto \sigma_1\left[-\alpha(x)\Delta + \beta(x)\right]$ is continuous in $\mathbb{R}$.

(b) The application $L^\infty(\Omega) \ni \beta \longmapsto \sigma_1\left[-\alpha(x)\Delta + \beta(x)\right]$ is continuous and increasing in $\mathbb{R}$.

(c) The principal eigenvalue $\sigma_1\left[-\alpha(x)\Delta + \beta(x)\right]$ is decreasing with respect to Ω, that is, for $\Omega_1 \subset \Omega_2 \subset \Omega$, with $\Omega_1 \neq \Omega_2$,

$$\sigma_1^{\Omega_2}\left[-\alpha(x)\Delta + \beta(x)\right] < \sigma_1^{\Omega_1}\left[-\alpha(x)\Delta + \beta(x)\right].$$

Proof. See *Chapter 7* of [49]. □

Besides being decreasing with respect to the domain, the principal eigenvalue also varies continuously with respect to domain inclusion. For a detailed discussion, see [49].

We will now examine two particular cases of Eigenvalue Problem (1.2), which have been extensively studied in the literature.

Case $\alpha \equiv d > 0$

In this first case, we consider a perturbation introduced by a positive constant in the diffusion term, which is described by the following equation:

$$\begin{cases} -d\Delta w + \beta(x)w = \gamma w & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

In this equation we have the presence of two crucial parameters. The first, a positive constant multiplying the diffusion term, adjusts the speed of diffusion, allowing the problem to model systems with different propagation behaviors of w. The second, the reaction term, causes the behavior of w to depend on the behavior of $\beta(x)$ in Ω.

The variational characterization of Problem (1.3) is made possible by the fact that the differential operator involved, given by $-d\Delta + \beta(x)$, is self-adjoint. This property ensures that the operator's spectrum is real and that the eigenvalues can be obtained through variational principles, such as the minimization of associated energy functionals.

The next result presents findings that we will frequently use throughout this work regarding eigenvalues, eigenfunctions, and their properties.

Theorem 1.29. The Eigenvalue Problem (1.3) has a principal eigenvalue, denoted by $\sigma_1[-d\Delta + \beta(x)]$, which is the only simple eigenvalue and admits an positive eigenfunction, denoted by $\varphi_1^{[d,\beta(\cdot)]}$. Moreover, this eigenvalue has the following variational characterization:

$$\sigma_1[-d\Delta + \beta(x)] = \inf_{\substack{v \in H_0^1(\Omega) \\ \|v\|_2^2 = 1}} \left\{ \int_\Omega \left(d\,|\nabla v|^2 + \beta(x)v^2 \right) \right\}.$$

Proof. See for instance [41] and *Teorema 1.4*, *Proposiciones 1.5 and 1.7* of [36]. □

Note that, an immediate consequence of the previous results and the properties seen in Proposition 1.28 is that for $\beta(x) = \beta_0 > 0$,

$$\sigma_1[-d\Delta + \beta_0] = \sigma_1[-d\Delta] + \beta_0 = d\lambda_1 + \beta_0 > 0.$$

Case $\alpha \equiv 1$ and $\beta \equiv 0$

This case is the classic linear eigenvalue problem without perturbation, which is described by the following equation:

$$\begin{cases} -\Delta w = \gamma w & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

This type of problem is common in various areas of Physics and Mathematics, where the Laplace operator models phenomena such as heat diffusion, vibrations of structures, or wave propagation, and the parameter γ is related to the natural frequencies or modes of vibration of the system.

The following result, which is a particular case of Theorem 1.29, presents key findings concerning eigenvalues, eigenfunctions, and their properties that will be frequently used throughout this work.

Corollary 1.30. The Eigenvalue Problem (1.4) has a principal eigenvalue, denoted by λ_1, which is the only simple eigenvalue and admits an positive eigenfunction, denoted by φ_1, such that $\|\varphi_1\|_2 = 1$. Moreover, this eigenvalue has the following variational characterization:

$$\lambda_1 = \inf_{\substack{v \in H_0^1(\Omega) \\ \|v\|_2^2 = 1}} \left\{ \int_\Omega |\nabla v|^2 \right\}.$$

Proof. See *Theorem 1.13* of [6]. □

The following result establishes a fundamental equivalence between the sign of the principal eigenvalue and various formulations of the Maximum Principle. This connection links the spectral properties of the operator to the behavior of the solutions.

Theorem 1.31. The following assumptions are equivalent:

(a) $\sigma_1 [\mathcal{L}] > 0$.

(b) $\mathcal{L}$ has a strictly positive supersolution $w \in W^{2,p}(\Omega)$, with $p > N$, that is, a function w such that

$$w(x) > 0 \quad \text{for all } x \in \Omega, \quad \text{and} \quad \mathcal{L}w \geq 0 \quad \text{in } \Omega,$$

with boundary condition $w = 0$ on $\partial\Omega$.

(c) $\mathcal{L}$ satisfies the Strong Maximum Principle.

(d) $\mathcal{L}$ satisfies the Maximum Principle.

Proof. See *Theorem 7.5.2* of [48] and [49]. □

Note that, Theorem 1.31 strengthens Theorem 1.24 by establishing the Strong Maximum Principle without requiring the condition $c \geq 0$.

1.4 Sub-Supersolution Method

In this section, we will introduce some concepts and results of the Sub-Supersolution Method for equations and systems. This method is a widely used technique in the analysis of Elliptic Problems to establish the existence of solutions and to obtain estimates on their behavior. The central idea of the method is to construct two functions that "bound" the solution of the problem from above and below, creating a range within which the solution must lie. For more details about this method, we recommend [6, 15].

In what follows, we will consider the following problem involving a uniformly elliptic operator, which plays a central role in the study of linear elliptic equations:

$$\begin{cases} \mathcal{L}w = f(x, w) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $f : \overline{\Omega} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a **Caratheodory function**, that is, measurable in $x \in \overline{\Omega}$ and continuous in $w \in \mathbb{R}$. We will start by presenting the main concept of this method.

Definition 1.32. We say that $\underline{w} \in C^2(\Omega) \cap C^0(\overline{\Omega})$ it is a **subsolution** to Problem (1.5) when

(a) $\mathcal{L}\underline{w} \leq f(x, \underline{w})$ in Ω.

(b) $\underline{w} \leq 0$ on $\partial\Omega$.

Furthermore, we say that $\overline{w} \in C^2(\Omega) \cap C^0(\overline{\Omega})$ it is a **supersolution** to Problem (1.5) when

(c) $\mathcal{L}\overline{w} \geq f(x, \overline{w})$ in Ω.

(d) $\overline{w} \geq 0$ on $\partial\Omega$.

Using the previous definition, given $\underline{w}, \overline{w} \in C^2(\Omega) \cap C^0(\overline{\Omega})$ as a subsolution and supersolution, respectively, we say that the pair $(\underline{w}, \overline{w})$ is a **sub-supersolution** to Problem (1.5) when:

$$\underline{w}(x) \leq \overline{w}(x)$$

for all $x \in \Omega$.

The first result of this section guarantees the existence of classical solutions to Problem (1.5) between the sub-supersolution pair.

Theorem 1.33. Assume that $f \in C^1(\overline{\Omega} \times \mathbb{R})$. Suppose that $\underline{w}, \overline{w} \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a sub-supersolution to Problem (1.5). Then, there exists $w_*, w^* \in C^2(\overline{\Omega})$, called, respectively, **minimal solution** and **maximal solution**, not necessarily different, solutions to Problem (1.5). Furthermore

(a) $w_* \leq w^*$.

(b) If $w \in C^2(\overline{\Omega})$ is a solution to Problem (1.5) such that $\underline{w} \leq w \leq \overline{w}$, then

$$w_* \leq w \leq w^*.$$

Proof. See *Theorem 11.1* of [15]. □

Throughout this thesis, we will repeatedly employ the Sub-Supersolution Method for systems involving nonlocal terms. It is known (see [2]) that the Maximum Principle generally does not hold when an operator includes nonlocal components. Nevertheless, the method can still be applied in such cases, provided that an appropriate definition of sub-supersolutions is adopted. To illustrate this approach, consider the following problem:

$$\begin{cases} -\Delta w = f(x, w, B(w)) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where $B : L^\infty(\Omega) \to \mathbb{R}$ is a continuous operator and $f : \Omega \times \mathbb{R}^2 \mapsto \mathbb{R}$ a continuous function. In this context, the concept of sub-supersolutions is adapted and formalized as follows:

Definition 1.34. We say that the pair $(\underline{w}, \overline{w})$, with $\underline{w}, \overline{w} \in H^1(\Omega) \cap L^\infty(\Omega)$, is a **pair of sub-supersolution** of Problem (1.6) when

(a) $\underline{w} \leq \overline{w}$ in Ω and $\underline{w} \leq 0 \leq \overline{w}$ on $\partial\Omega$.

(a) The pair $(\underline{w}, \overline{w})$ satisfies, for all $w \in [\underline{w}, \overline{w}]$, in the weak sense

$$-\Delta\underline{w} - f(x, \underline{w}, B(w)) \leq 0 \leq -\Delta\overline{w} - f(x, \overline{w}, B(w)).$$

In this context, the existence of a solution can be ensured by the following result:

Theorem 1.35. Assume that there exists a pair of sub-supersolution of Problem (1.6) in the sense of Definition 1.34. Then, there exists a solution $w \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that $w \in [\overline{w}, \underline{w}]$.

Proof. See *Theorem 3.2* of [20]. □

Now, we will present the definition of sub-supersolution for the following Elliptic System involving two equations:

$$\begin{cases} \mathcal{L}_1 u = f(x, u, v) & \text{in } \Omega, \\ \mathcal{L}_2 v = g(x, u, v) & \text{in } \Omega, \\ \quad u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

where $f, g : \overline{\Omega} \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ are continuous functions. Although in the problem above we have the same operator in both lines, it is possible to consider a different uniformly elliptic operator in one of the lines.

Definition 1.36. We say that the pairs $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$, with $\underline{u}, \underline{v}, \overline{u}, \overline{v} \in H^2(\Omega) \cap L^\infty(\Omega)$, are a sub-supersolution to Problem (1.7) when:

(a) $\underline{u} \leq \overline{u}$ and $\underline{v} \leq \overline{v}$ in Ω, and $\underline{u} \leq 0 \leq \overline{u}$ and $\underline{v} \leq 0 \leq \overline{v}$ on $\partial\Omega$.

(b) The pair $(\underline{u}, \overline{u})$ satisfies, for all $v \in [\underline{v}, \overline{v}]$,

$$\mathcal{L}_1\underline{u} - f(x, \underline{u}, v) \leq 0 \leq \mathcal{L}_1\overline{u} - f(x, \overline{u}, v).$$

(c) The pair $(\underline{v}, \overline{v})$ satisfies, for all $u \in [\underline{u}, \overline{u}]$,

$$\mathcal{L}_2\underline{v} - g(x, u, \underline{v}) \leq 0 \leq \mathcal{L}_2\overline{v} - g(x, u, \overline{v}).$$

In this context of the Sub-Supersolution Method for elliptic systems, we have the following existence result:

Theorem 1.37. Assume that there exists a pair of sub-supersolution of Problem (1.7) in the sense of Definition 1.36. Then, there exists a solution $(u, v) \in \left(H_0^1(\Omega) \cap L^\infty(\Omega)\right)^2$ of Problem (1.7) such that

$$\underline{u} \leq u \leq \overline{u} \quad \text{and} \quad \underline{v} \leq v \leq \overline{v}.$$

Proof. See Theorem 1.4 of [42]. □

To conclude this section, we will apply the same reasoning previously developed in the context of Problem (1.6), now adapted to the study of elliptic systems. This transition requires additional considerations, particularly due to the interactions between the systems equations, but the core idea remains valid. For a more detailed and comprehensive exposition of this approach, we recommend consulting [35]. To illustrate this approach, consider the following problem:

$$\begin{cases} -\Delta u = g_1(x,u,v,B_1(u),B_2(v)) & \text{in } \Omega, \\ -\Delta v = g_2(x,u,v,B_1(u),B_2(v)) & \text{in } \Omega, \\ \quad u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

where $g_1, g_2 : \Omega \times \mathbb{R}^4 \to \mathbb{R}$ is a continuous function and $B_1, B_2 : L^\infty(\Omega) \to \mathbb{R}$. In this setting, the notion of sub-supersolution is redefined and precisely stated as follows:

Definition 1.38. We say that the pair $(\underline{u}, \overline{u})$ and $(\underline{v}, \overline{v})$, with $\underline{u}, \overline{u}, \underline{v}, \overline{v} \in H^1(\Omega) \cap L^\infty(\Omega)$ is a **pair of sub-supersolution** of Problem (1.8) when

(a) $\underline{u} \le \overline{u}$ and $\underline{v} \le \overline{v}$ in Ω, and $\underline{u} \le 0 \le \overline{u}$ and $\underline{v} \le 0 \le \overline{v}$ on $\partial\Omega$.

(b) The pair $(\underline{u}, \overline{u})$ satisfies, for all $(u,v) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$, in the weak sense

$$-\Delta\underline{u} - g_1(x,\underline{u},v,B_1(u),B_2(v)) \le 0 \le -\Delta\overline{u} - g_2(x,\overline{u},v,B_1(u),B_2(v)).$$

(c) The pair $(\underline{v}, \overline{v})$ satisfies, for all $(u,v) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$, in the weak sense

$$-\Delta\underline{v} - g_2(x,u,\underline{v},B_1(u),B_2(v)) \le 0 \le -\Delta\overline{v} - g_2(x,u,\overline{v},B_1(u),B_2(v)).$$

In this context, the existence of a solution for elliptic system can be ensured by the following result:

Theorem 1.39. Assume that there exists a pair of sub-supersolution of Problem (1.8) in the sense of Definition 1.38. Then, there exists a solution $(u,v) \in \left(H_0^1(\Omega) \cap L^\infty(\Omega)\right)^2$ of Problem (1.8) such that $u \in [\underline{u}, \overline{u}]$ and $v \in [\underline{v}, \overline{v}]$.

Proof. See *Theorem 2.2* of [35]. □

1.5 Bifurcations Methods

In this section, we present the main concepts of the Bifurcation Method applied to nonlinear Elliptic Problems, stating fundamental results and discussing their conditions of applicability. This method is widely used to study the behavior of solutions to nonlinear problems, particularly when system parameters change. For more details of this method, we recommend [6, 27, 41, 48].

As a motivation for studying this method, this method allows us to obtain the existence of solutions for elliptic problems of the form

$$\begin{cases} -\Delta w = \gamma w + f(x,w) & \text{in } \Omega, \\ \quad w = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.9}$$

where $f : \overline{\Omega} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a nonlinear function satisfying some conditions and $\gamma \in \mathbb{R}$ acts as the **bifurcation parameter**. The main idea behind this method is as follows: given a known solution for a specific value of γ, for example $w \equiv 0$, it aims to:

(BL) Determine the values of γ at which new solutions "bifurcate" and, if possible, identify the direction of this "bifurcation".

(BG) Understand the global behavior of these new set of solutions.

More precisely, we will say that Item (BL) addresses *Local Bifurcation*, while Item (BG) deals with *Global Bifurcation.* These topics will be explored in the following sections.

Consider E, F as Banach spaces and I as interval in $\mathbb{R}$. A natural way to visualize the solutions of Problem (1.9) is as the zeros of a nonlinear operator over these spaces, as follows:

$$\begin{aligned} \mathcal{F} : \mathbb{R} \times E &\longrightarrow F \\ (\gamma, w) &\longmapsto \mathcal{F}(\gamma, w) = -\Delta w - \gamma w - f(x, w). \end{aligned}$$

By defining the operator $\mathcal{F}$, a function w is a solution to Problem (1.9) if, and only if, $\mathcal{F}(\gamma, w) = 0_F$. This provides the basic framework to apply classical results on Local Bifurcation, which we will explore in Section 1.5.

Another way to view the solutions of Problem (1.9) utilizes the fact that the operator $-\Delta$ is invertible. This allows us to rewrite the equation of the problem as:

$$0 = w - \gamma \left(-\Delta\right)^{-1} u - \left(-\Delta\right)^{-1} f(x, w),$$

where $(-\Delta)^{-1}$ is the solution operator for the Laplacian under Dirichlet boundary conditions. By appropriately choosing spaces so that $(-\Delta)^{-1}$ is compact, the above equation can be seen as the sum of a compact perturbation of the identity,

$$L(\gamma) = I - \gamma \left(-\Delta\right)^{-1},$$

and a compact term involving w,

$$h(\gamma, w) = -\left(-\Delta\right)^{-1} f(x, w),$$

such that $h(\gamma, w) = 0(\|w\|_E)$ as $w \to 0_E$. Consequently, the solutions of Problem (1.9) can be obtained by defining the operator:

$$\begin{aligned} \mathcal{G} : \mathbb{R} \times E &\longrightarrow E \\ (\gamma, w) &\longmapsto \mathcal{G}(\gamma, w) = L(\gamma) w + h(\gamma, w). \end{aligned}$$

With this definition, w is a solution to Problem (1.9) if, and only if, $\mathcal{G}(\gamma, w) = 0_E$. This formulation also forms the basis for applying classical results on Local Bifurcation, which we will examine in Section 1.5.

Local Bifurcation

Let E, F Banach spaces and $\mathcal{F} : \mathbb{R} \times E \to F$. We aim to study problems of the form (1.9) through the following equation:

$$\mathcal{F}(\gamma, w) = 0_F. \tag{1.10}$$

To study the operator above, it is necessary to assume that the following hypotheses are satisfied:

(H$_1$) $\mathcal{F} \in C^2(\mathbb{R} \times E; F)$, and

(H$_2$) $\mathcal{F}(\gamma, 0_E) = 0_F$, for all $\gamma \in \mathbb{R}$.

Observe that, by Hypothesis (H$_1$), the function $w \equiv 0_E$ is a solution to Equation (1.10) for all $\gamma \in \mathbb{R}$, which we will call **trivial solution**. Thus, we are led to seek non-trivial solutions, which can be expressed as elements of the set

$$\mathcal{S} := \{(\gamma, w) \in \mathbb{R} \times E;\ w \not\equiv 0_E \text{ and } \mathcal{F}(\gamma, w) = 0_F\}.$$

We will call the set $\mathcal{S}$ as the **set of non-trivial solutions**. Thus, it becomes natural to search for points along the $\mathbb{R} \times \{0_E\}$ axis where we can ensure the "emergence" a new family of non-trivial solutions for Equation (1.10). These points are the central theme of the next definition.

Definition 1.40. We say that $\gamma^* \in \mathbb{R}$ is a **bifurcation point** of $\mathcal{F}$ arising from the trivial solution when there exist a sequence $(\gamma_n, w_n) \in \mathbb{R} \times E$, with $w_n \not\equiv 0_E$ and $\mathcal{F}(\gamma_n, w_n) = 0_F$ for all $n \in \mathbb{N}$, such that $(\gamma_n, w_n) \to (\gamma^*, 0_E)$.

In Figure 1.1, we illustrate the concept of a bifurcation point in three distinct cases.

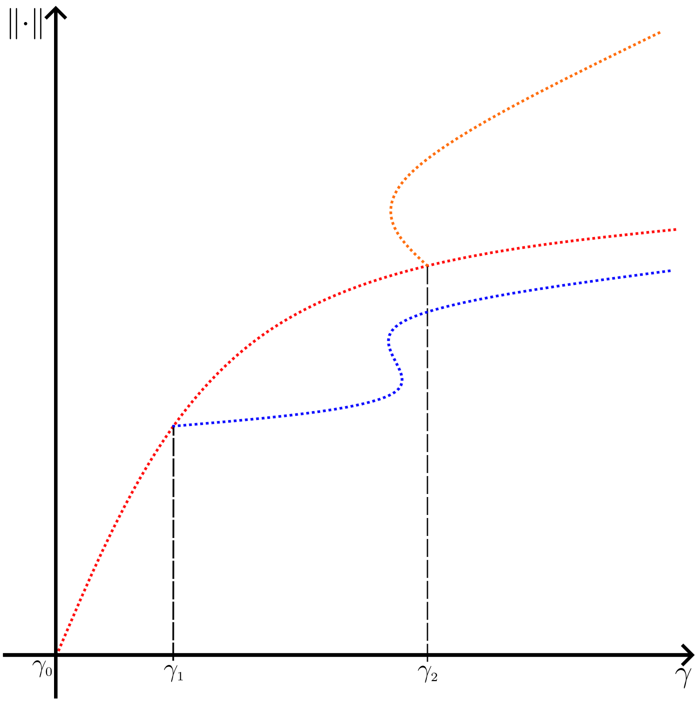


Figure 1.1: Bifurcation Diagram. In this case, γ_0 bifurcates from the trivial solution, whereas γ_1 and γ_2 bifurcate from non-trivial solutions.

Source: Prepared by the author.

Say that γ^* is a bifurcation point means that $(\gamma^*, 0_E) \in \overline{\mathcal{S}}$. Furthermore, a necessary but not sufficient condition for $(\gamma^*, 0_E)$ to be a bifurcation point is that $\mathcal{F}_w(\gamma^*, 0_E)$ is non-invertible, the following proposition guarantees this observation.

Proposition 1.41. Let $\gamma^* \in \mathbb{R}$ be a bifurcation point. Then $\mathcal{F}_w(\gamma^*, 0_E)$ is non-invertible. In particular, if $\mathcal{F}(\gamma, w) = \gamma w - T(w)$ for some $T : E \to F$, then any bifurcation point of $\mathcal{F}$ belongs to the spectrum of $T'(0_E)$.

Proof. See *Proposition 2.2* of [49]. □

The following result, due to M. G. Crandall and P. H. Rabinowitz (see [58]), provides sufficient conditions for the existence of a bifurcation point arising from the trivial solution, from which a unique curve of non-trivial solutions emanates in a neighborhood of the bifurcation point. Furthermore, the result offers a parametrization of this solution curve.

Theorem 1.42 (Crandall-Rabinowitz)**.** Assume that hypotheses (H_1)-(H_2) are satisfied and denote

$$\mathcal{L}(\gamma) := \mathcal{F}_w(\gamma, 0_E).$$

If Z is the topological complement of $\text{Ker}\,[\mathcal{L}(\gamma^*)]$ on E, that is,

$$E = \text{Ker}\,[\mathcal{L}(\gamma^*)] \oplus Z,$$

and $\mathcal{F}$ verifies the following conditions:

(CR_1) $\text{Ker}\,[\mathcal{L}(\gamma^*)] = \text{Span}\{\varphi^*\}$ for some $\varphi^* \in E$, with $\varphi^* \not\equiv 0_E$;

(CR_2) $\mathcal{L}'(\gamma^*)\varphi^* \notin \text{Rg}\,[\mathcal{L}(\gamma^*)]$; and

(CR_3) $\text{codim}\,[\text{Rg}\,(\mathcal{L}(\gamma^*))] = 1$.

Then γ^* is a bifurcation point of $\mathcal{F}$ and the set of nontrivial solution of $\mathcal{F} \equiv 0_F$ in a neighborhood of $(\gamma^*, 0)$ is a unique Cartesian curve of class C^1 with parametric representation in Z, that is, there are $\varepsilon > 0$, $\rho > 0$ and applications

$$\begin{aligned} \gamma :&(-\varepsilon, \varepsilon) \longrightarrow \mathbb{R} \\ &s \longmapsto \gamma(s) \end{aligned} \qquad\qquad \begin{aligned} \varphi :&(-\varepsilon, \varepsilon) \longrightarrow Z \\ &s \longmapsto \varphi(s), \end{aligned}$$

with $\lambda(0) = \gamma^*$, $\psi(0) = 0$ and

(a) The family (γ, w), with $\gamma = \gamma(s)$ and $w = s\,(\varphi^* + \varphi(s))$, is a curve of solutions (nontrivial if $s \neq 0$) for the equation (1.10) that bifurcate from $(\gamma^*, 0)$.

(b) If $(\gamma^*, w) \in B_\rho(\gamma^*, 0)$ is any nontrivial solution of $\mathcal{F}(\gamma^*, w) = 0_F$, then there exists $0 < |s| < \varepsilon$ such that

$$(\gamma, w) = (\gamma(s), s\,(\varphi^* + \varphi(s)))\,.$$

Furthermore, if $\mathcal{F} \in C^k(\mathbb{R} \times E; F)$, then $\gamma(s)$ and $w(s)$ are of class C^{k-1}.

Proof. See *Theorem 2.2.1* of [48]. □

In the results above, Item (a) ensures the existence of a smooth curve of nontrivial solutions emerging from the bifurcation point, whereas Item (b) establishes the local uniqueness of such a curve in a neighborhood of the bifurcation. Furthermore, Conditions (CR_1) and (CR_2) guarantee that γ^* is a **simple eigenvalue** of the linearized operator $\mathcal{F}$, a crucial requirement for the application of the Crandall-Rabinowitz Theorem.

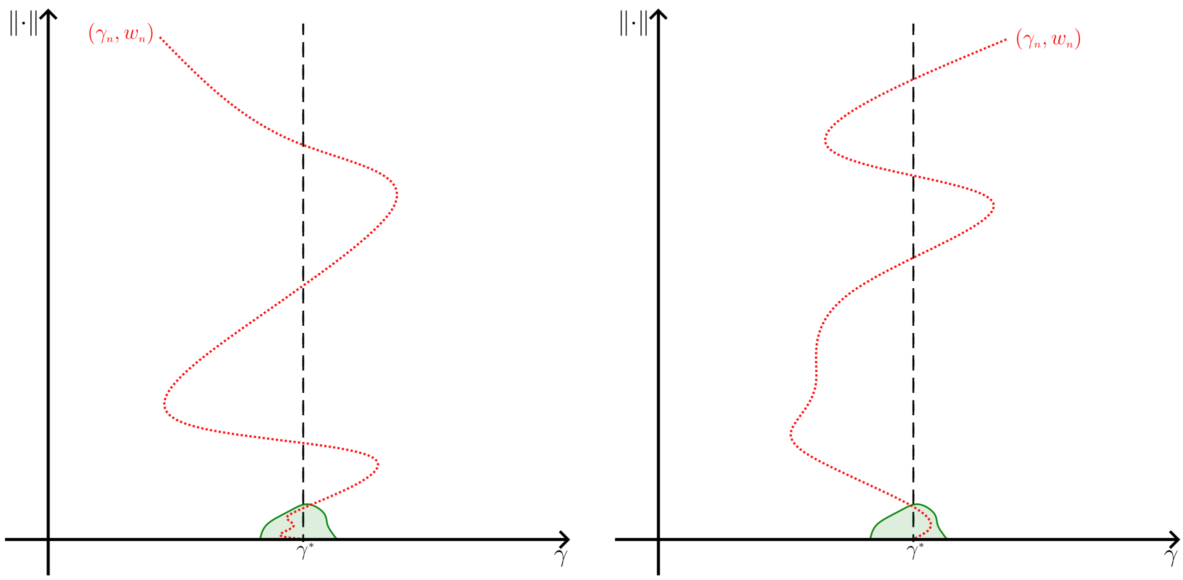
In addition to the results obtained from the previous theorem, another important concept concerns the behavior of nontrivial solutions with respect to the parameter γ near the bifurcation point $(\gamma^*, 0)$, more precisely:

Definition 1.43. Let $\gamma^* \in \mathbb{R}$ be a bifurcation point of $\mathcal{F}$ from the trivial solution. We say that γ^* has a:

(a) **subcritical direction** when there exists a neighborhood V of $(\gamma^*, 0)$ in $\mathbb{R} \times E$ such that every nontrivial solution $(\gamma, w) \in V$ is such that $\gamma < \gamma^*$.

(b) **supercritical direction** when there exists a neighborhood V of $(\gamma^*, 0)$ in $\mathbb{R} \times E$ such that every nontrivial solution $(\gamma, w) \in V$ is such that $\gamma > \gamma^*$.

The figure below presents a graphical interpretation of the bifurcation direction. It illustrates that the direction of bifurcation is determined precisely by those values of γ that belong to the projection of the set V onto the real line $\mathbb{R}$, highlighting the connection between the bifurcation parameter and the structure of the solution set.



(a) Supercritical Bifurcation. (b) Subcritical Bifurcation

Figure 1.2: Geometric interpretation of the Definition 1.43.
Source: Prepared by the author.

Global Bifurcation

We will now present results that improve the Cradall-Rabinowitz Theorem, providing insights into the global behavior of the solution set emanating from a bifurcation point. To this end, we will assume that $E = F$ and that the operator $\mathcal{F}$ can be expressed as follows:

$$\mathcal{F}(\gamma, w) = w - T_\gamma(w), \tag{1.11}$$

where $T_\gamma : E \to E$ is given by $T_\gamma(w) = \gamma L(\gamma) + h(\gamma, w)$, satisfying the following hypothesis:

(H_3) $L : E \to E$ is linear and compact;

(H_4) $h : \mathbb{R} \times E \to E$ is compact; and

(H_5) $h(\gamma, w) = o(\|w\|_E)$ in $w \equiv 0_E$ uniformly on compact intervals of $\mathbb{R}$, that is,

$$\lim_{w \to 0} \frac{h(\gamma, w)}{\|w\|_E} = 0$$

uniformly on compact intervals of $\mathbb{R}$.

In the results concerning Global Bifurcation, we aim to study a specific category of sets with relevant properties. The following definition introduces this set and its characteristics.

Definition 1.44. We say that $\mathfrak{C} \subseteq \mathcal{S}$ is a **continuum** when it is closed and connected in the topology of $\mathbb{R} \times E$. Furthermore, we say that this set is **maximal** when it is not a proper subset of any other closed connected subset of $\mathcal{S}$.

In the next result, due to P. H. Rabinowitz (see [58]), provides the global behavior of the curve of nontrivial solutions of $\mathcal{F}(\gamma, w) = 0_E$ that bifurcates from $(\gamma^*, 0_E)$.

Theorem 1.45. Assume that hypotheses (H_3)–(H_5) are satisfied and. Let $\gamma^* \in \mathbb{R} \setminus \{0\}$ be such that $1/\gamma^*$ is an eigenvalue of L with odd multiplicity. Then, from $(\gamma^*, 0_E)$ emanates a maximal continuum $\mathfrak{C}$ of $\mathcal{S}$ that satisfies at least one of the following assumptions:

(a) $\mathfrak{C}$ is unbounded; or

(b) There exists a sequence (γ_n, w_n) in $\mathfrak{C}$ such that $\gamma_n \to \overline{\gamma} \neq \gamma^*$ in $\mathbb{R}$ and $w_n \to 0_E$ in E, where $1/\overline{\gamma}$ is eigenvalue of L.

Proof. See *Theorem 1.3* of [58]. □

The previous result can be interpreted differently, removing the necessity for $1/\gamma^*$ to be an eigenvalue of L with odd multiplicity and instead imposing that the index of $I - T_\gamma$ changes when γ crosses γ^*. In other words:

Theorem 1.46. Assume that hypotheses (H_3)–(H_5) are satisfied and. Let $\gamma^* \in \mathbb{R} \setminus \{0\}$ be such that $i_E(I - T_\gamma, 0_E)$ changes when γ crosses γ^*. Then, from $(\gamma^*, 0_E)$ emanates a maximal continuum $\mathfrak{C}$ of $\mathcal{S}$ that satisfies at least one of the following assumptions:

(G_1) $\mathfrak{C}$ is unbounded; or

(G_2) There exists a sequence (γ_n, w_n) in $\mathfrak{C}$ such that $\gamma_n \to \overline{\gamma} \neq \gamma^*$ in $\mathbb{R}$ and $w_n \to 0_E$ in E, where $1/\overline{\gamma}$ is eigenvalue of L.

Proof. See *Theorem 1.3* of [58]. □

The figure below shows the geometric interpretation of Theorem 1.45. The red curve represents Item (a), while the blue curve represents Item (b).

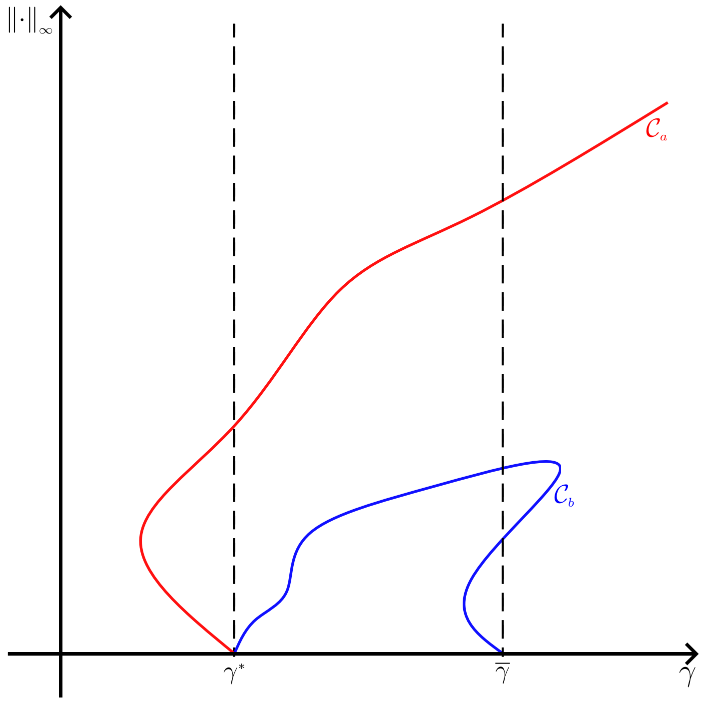


Figure 1.3: Geometric interpretation of the Theorem 1.46.
Source: Prepared by the author.

There is an alternative way to express Item (b) of the Theorems 1.45 and 1.46, which is more common in the literature, namely:

(G_2) $\mathfrak{C}$ reaches $w \not\equiv 0_E$ at another point $(\overline{\gamma}, 0_E) \not\equiv (\gamma^*, 0_E)$, where $1/\overline{\gamma}$ is eigenvalue of L.

The next result, due to J. López-Gómez (see [48]), provides a curve of solutions with better proprieties than those presented in the previous result, namely, positive solutions in the interior of the cone.

Theorem 1.47. Assume that the following conditions hold:

(i) E is an ordered Banach space and has a non-empty interior;

(ii) L is a strongly positive operator with a simple eigenvalue associated with a positive eigenfunction; and

(iii) Any solution $w \in P_E$ of $\mathcal{F}(\gamma, w) = 0_E$ is strictly positive.

Then, from $(\gamma^*, 0_E)$ emanates a unbounded continuum $\mathfrak{C}^+ \subset \text{Int}(P_E)$.

Proof. See *Theorem 6.5.5* of [48]. □

Bifurcation Methods for Elliptic Systems

To conclude this section, we will present a study of Local and Global Bifurcation for problems involving elliptic systems with two equations. This category of problem will be the focus of the upcoming chapters of this work.

We will consider, in a generic form, the following problem involving an elliptic system:

$$\begin{cases} -m(x)\Delta u = \lambda u + a(x,u)u + c(x,u,v)uv & \text{in } \Omega, \\ -n(x)\Delta v = \mu v + b(x,v)v + d(x,u,v)uv & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.12}$$

where $\lambda, \mu \in \mathbb{R}$ and the functions in the system satisfying the following hypotheses:

(H_{mn}) $m, n : \mathbb{R} \to (0, \infty)$ are continuous functions such that $m(0)$ and $n(0)$ are positive an non-zero;

(H_{ab}) $a, b : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ are continuous functions in $\overline{\Omega}$ and of class C^1 in $\mathbb{R}$ such that

$$a(x,0) = b(x,0) = 0$$

for all $x \in \overline{\Omega}$; and

(H_{cd}) $c, d : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions in $\overline{\Omega}$ and of class C^1 in $\mathbb{R}^2$.

Systems with the same structure as Problem (1.12) admit three types of non-negative solutions, namely:

(S_1) Trivial solution, that is, $(0,0) \in E^2$, with E is an ordered Banach space and has a non-empty interior;

(S_2) **Semi-trivial solution**, that is, $(u,0) \in E^2$ and $(0,v) \in E^2$, where $u \not\equiv 0$ and $v \not\equiv 0$ are non-negative solution of

$$\begin{cases} -m(x)\Delta u = \lambda u + a(x,u)u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.13}$$

and

$$\begin{cases} -n(x)\Delta v = \mu v + b(x,v)v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.14}$$

respectively; and

(S_3) **Coexistence states**, that is, $(u,v) \in E^2$ with both components non-negative and non-trivial. In fact, thanks to the Strong Maximum Principle any (u,v) non-negative and non-trivial solution of Problem (1.12) satisfies that $u, v \in \text{Int}(P_E)$.

In our analysis, we concentrate on solutions of type (S_3), which are especially relevant in population dynamics, where the central goal is to determine the conditions that allow for the coexistence of both species.

The definition below is fundamental for analyzing bifurcation points that arise from semi-trivial, rather than trivial, solutions.

Definition 1.48. Consider the following problem:

$$\begin{cases} -g(x)\Delta w = \gamma w + f(x,w)w & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.15}$$

Let $\theta_{[\gamma,g,f]}$ be a non-negative solution of Problem (1.15). We say that $\theta_{[\gamma,g,f]}$ is a **non-degenerate solution** of (1.15) when $w \equiv 0$ is the unique strongly solution of the linearized equation at $\theta_{[\gamma,g,f]}$, which is given by

$$\begin{cases} -g(x)\Delta w = \gamma w + \left[f_w(x,\theta_{[\gamma,g,f]})\theta_{[\gamma,g,f]} + f(x,\theta_{[\gamma,g,f]})\right] w & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.16}$$

We will now seek to generalize Theorem 1.42, which addresses Local Bifurcation, to systems with the same structure as Problem (1.12).

Let E, F Banach spaces and $\mathcal{F} : \mathbb{R} \times E^2 \to F^2$. We aim to study systems of the form (1.12) through the following equation:

$$\mathcal{F}(\lambda, u, v) = 0_{F\times F}. \tag{1.17}$$

The hypotheses (H_1) and (H_2) discussed earlier generalize naturally as follows:

(H_1') $\mathcal{F} \in C^2(\mathbb{R} \times E^2; F^2)$; and

(H_2') $\mathcal{F}(\lambda, \theta_{[\gamma,g,f]}, 0) = 0_{F\times F}$, for all $\lambda \in \mathbb{R}$.

The next result provides the global behavior of the curve os nontrivial solutions of $\mathcal{F}(\lambda, 0_E, 0_E) = 0_{F\times F}$ that bifurcates from one of the semi-trivial solutions.

Theorem 1.49. Let E be an ordered Banach space with a normal positive cone that has a non-empty interior, $\theta_{[\lambda,m,a]} \in \operatorname{Int}(P_E)$ a non-degenerate solution of (1.13) and

$$\mu_\lambda := \sigma_1\left[-n(x)\Delta - d(x,\theta_{[\lambda,m,a]}, 0_E)\theta_{[\lambda,m,a]}\right].$$

From $(\mu_\lambda, \theta_{[\lambda,m,a]}, 0_E)$ emanates a continuum $\mathfrak{C} \subset \mathbb{R} \times \operatorname{Int}(P_E) \times \operatorname{Int}(P_E)$ of coexistence states of Problem (1.12) that satisfies at least one of the following assumptions:

(G_1') $\mathfrak{C}$ is unbounded in $\mathbb{R} \times E^2$;

(G_2') There exist a sequence $(\mu_n, u_n, v_n) \in \mathfrak{C}$ such that $\mu_n \to \overline{\mu}$ in $\mathbb{R}$ and $(u_n, v_n) \to (0_E, 0_E)$ in E^2, where $\overline{\mu} \neq \mu_\lambda$;

(G_3') There exist a sequence $(\mu_n, u_n, v_n) \in \mathfrak{C}$ such that $\mu_n \to \overline{\mu}$ in $\mathbb{R}$ and $(u_n, v_n) \to (\overline{u}, 0_E)$ in E^2, with $\overline{\mu} \neq \mu_\lambda$ and $\overline{u} > 0_E$; or

(G_4') There exist a sequence $(\mu_n, u_n, v_n) \in \mathfrak{C}$ such that $\mu_n \to \overline{\mu}$ in $\mathbb{R}$ and $(u_n, v_n) \to (0_E, \overline{v})$ in E^2, with $\overline{\mu} \neq \mu_\lambda$ and $\overline{v} > 0_E$.

Proof. See *Theorem 4.1* of [47]. □

Theorem 1.49 above ensures that the continuum emanating from the semi-trivial solution $(\mu_\lambda, \theta_{[\lambda,m,a]}, 0_E)$ is unbounded, goes to the origin $(0_E, 0_E)$, return to the curve of semi-trivial solutions in the form $(\overline{u}, 0_E)$; or goes to another curve os semi-trivial solutions in the the form $(0_E, \overline{v})$. Visually:

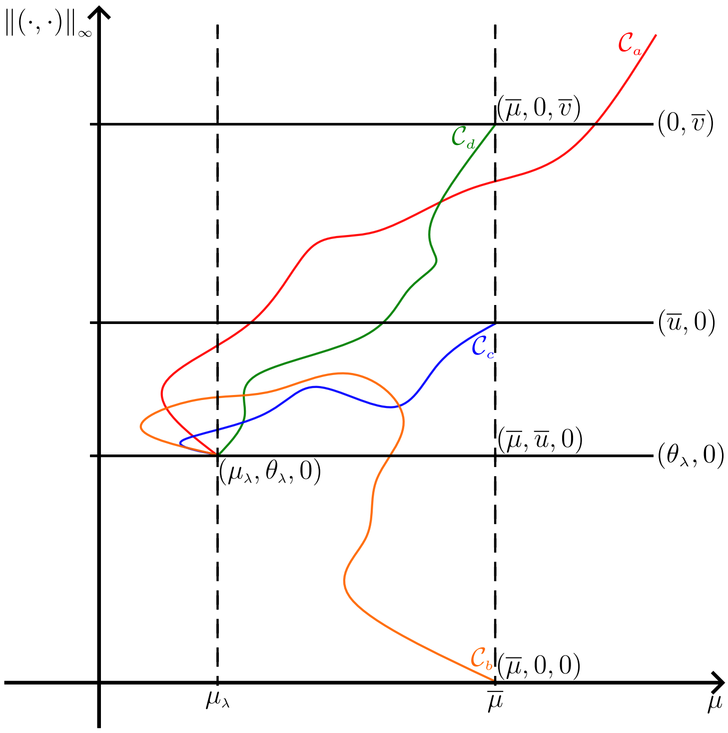


Figure 1.4: Geometric interpretation of the Theorem 1.49.
Source: Prepared by the author.

There is an alternative way to express items (G_2')–(G_4') of the previous theorem, which is more common in the literature, namely:

(G_2') $\lambda = \sigma_1[-m(x)\Delta]$ and $(\sigma_1[-n(x)\lambda_1], 0_E, 0_E) \in \overline{\mathfrak{C}}$;

(G_3') There exist a positive solution $\overline{\theta}_{[\lambda,m,a]}$ of Problem (1.13), with $\overline{\theta}_{[\lambda,m,a]} \not\equiv \theta_{[\lambda,m,a]}$, such that $\left(\sigma_1\left[-n(x)\Delta - d(x,\overline{\theta}_{[\lambda,m,a]},0_E)\overline{\theta}_{[\lambda,m,a]}\right], \overline{\theta}_{[\lambda,m,a]}, 0_E\right) \in \overline{\mathfrak{C}}$; and

(G_4') There exist $\overline{\mu} \in \mathbb{R}$ and a positive solution $\overline{\theta}_{[\mu,n,b]}$ of Problem (1.14) for $\mu = \overline{\mu}$ such that $(\overline{\mu}, 0_E, \overline{\theta}_{[\mu,n,b]}) \in \overline{\mathfrak{C}}$ and $\lambda = \sigma_1\left[-m(x)\Delta - c(x,0_E,\overline{\theta}_{[\mu,n,b]})\overline{\theta}_{[\mu,n,b]}\right]$.

Observation 1.50. In the case where Problem (1.13) has a unique positive solution, assumption (c) of Theorem 1.49 cannot occur. Similarly, in the case of $\lambda \neq \sigma_1[-m(x)\Delta]$, assumption (d) cannot occur.

Observation 1.51. In fact, the Bifurcation Theorem we have stated applies to a system like Problem (1.12). However, the result is actually more general. For instance, the term $c(x,u,v)$ can be replaced by a more general function $F(x,u,v)$, provided certain regularity conditions are satisfied. Specifically, we assume that

$$F(x,u,0) = F(x,0,v) = 0$$

for all (x,u,v), and that

$$F(x, w + \theta_{[\gamma,g,f]}, v) = o(\|w\| + \|v\|) \quad \text{as } \|w\| + \|v\| \to 0.$$

These conditions ensure that F is a sufficiently small nonlinear term in a neighborhood of the semi-trivial state, thereby allowing bifurcation techniques to remain valid in this more general setting.

1.6 Fixed Point Index Theory with Respect to the Positive Cone

In this section, we provide an introductory study presenting the main results of the Fixed Point Theory in Positive Cones. This theory extends the classical fixed point theory to deal with the existence of solutions in Ordered Banach Spaces. Within the scope of Elliptic Problems, it aids in identifying positive solutions in bounded domains. In many cases, the cone is selected as the set of non-negative or strictly positive functions within the domain.

In this context, we will employ an approach based on the concept of the fixed-point index to count the number of solutions in regions of the cone. This index provides significant topological insights into the multiplicity of solutions. For further details on this theory, we recommend reading references [4, 25], and for applications, reference [28, 52].

To apply the previous results to systems, we will need to use certain tools, and for this purpose, we will assume that X is a subspace of the Banach space E, $U \subset X$ an open set in X and E. Consider $x_0 \in U$ as an isolated fixed point of a compact operator $T : \overline{U} \to X$, which has no fixed points on ∂U. Note that there exists $\rho_0 \in \mathbb{R}$ such that $\overline{\mathrm{B}_E(x_0 + \rho, 1)} \subset U$ for every $\rho \in [0, \rho_0]$. Furthermore, assuming without loss of generality that x_0 is the only fixed point of T in $\mathrm{B}_E(x_0 + \rho, 1)$, it follows that, by the Excision property, Proposition 1.19, the integer

$$i_X(T, x_0) := i_X(T, \mathrm{B}_E(x_0 + \rho, 1))$$

is well-defined. This integer, referred to as **Local Index** of T at x_0, does not depend of $\rho \in [0, \rho_0]$. Consequently, we can state the following result, which involves the derivative of the operator T.

Proposition 1.52. Assume that X is an ordered Banach space and $T : \overline{B}_\rho \to P_X$, with $\overline{B}_\rho := \mathrm{B}_E(\rho, 1)$, is a compact operator such that $T(0) = 0$. Suppose that T has a right derivative at zero, $T'_+(0)$, such that 1 is not an eigenvalue associated with a positive eigenfunction. Then, there exists a constant $s_0 \in (0, \rho]$ such that, for any $\sigma \in (0, s_0]$, the following assumptions hold:

(a) If $T'_+(0)$ does not have a positive eigenfunction for any eigenvalue greater than 1, then $i_X(T, \overline{B}_\sigma) = 1$.

(b) If $T'_+(0)$ has a positive eigenfunction for any eigenvalue greater than 1, then $i_X(T, \overline{B}_\sigma) = 0$.

Proof. See *Lemma 13.1* of [4]. □

1.7 Analysis of a Logistic Problem

In this section, we will study the Logistic Problem with a nonlocal diffusion term. We will address the existence of a solution and its main properties. We will see that this problem generalizes two others that will be frequently used throughout this work. For more details regarding the problems discussed in this section, we recommend the references [12, 30, 39, 50].

Logistic Problem with a Perturbed Diffusion Term

The first case of the Logistic Problem features the particularity of including a constant that multiplies the diffusion term, which can be interpreted as a perturbation. Specifically:

$$\begin{cases} -\alpha\Delta w = \gamma w - w^2 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{PL$_1$}$$

where $\alpha > 0$ and $\gamma \in \mathbb{R}$.

The reaction term, expressed as $\gamma w - w^2$, describes the local population dynamics, where γ represents the growth rate, and the saturation term $-w^2$ accounts for factors like mortality, resource limitation, or internal competition as the population density rises.

The constant α represents the diffusion rate, or the speed at which the population spreads across the habitat Ω. There are two cases for this constant: $\alpha > 1$ and $0 < \alpha < 1$. In the first case, the higher the value, the more intense the diffusion, meaning the population moves more quickly within the habitat. In the second case, the population moves more slowly.

The first result of this section ensures the existence and uniqueness of a positive solution of Problem (PL$_1$), as well as some properties of this solution.

Theorem 1.53. Problem (PL$_1$) admits a unique positive solution if, and only if,

$$\gamma > \alpha\lambda_1.$$

In this case, we will denoted this solution by $\theta_{[\gamma,\alpha]}$. Moreover, the following assertions hold:

(a) $\dfrac{\gamma - \alpha\lambda_1}{\|\varphi_1\|_\infty}\varphi_1 \le \theta_{[\gamma,\alpha]} \le \gamma.$

(b) $\theta_{[\gamma,\alpha]} = \dfrac{\gamma - \alpha\lambda_1}{\displaystyle\int_\Omega \varphi_1^3}\varphi_1 + G_{\gamma,\alpha}(x)$, where

$$\lim_{\gamma\to\alpha\lambda_1} \frac{G_{\gamma,\alpha}(x)}{\gamma - \alpha\lambda_1} = 0 \quad \text{in} \quad C^2(\overline{\Omega}).$$

(c) $\sigma_1\left[-\alpha\Delta + 2\theta_{[\gamma,\alpha]} - \gamma\right] > 0.$

(d) The application $[\alpha\lambda_1, +\infty) \ni \gamma \longmapsto \theta_{[\gamma,\alpha]} \in C_0^2(\overline{\Omega})$ is increasing and derivable.

Logistic Problem with a Nonlocal Diffusion Term

Consider the following Logistic Problem with a nonlocal diffusion term:

$$\begin{cases} -g\left(\int_\Omega w\right)\Delta w = \gamma w - w^2 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{PL$_2$}$$

where $\gamma \in \mathbb{R}$ and $g : \mathbb{R} \to [0, +\infty)$ is a continuous function.

The biological model proposed by Problem (PL$_2$) describes the dynamics of a population, represented by the density $w(x)$, that moves spatially within a habitat Ω. The equation incorporates two key aspects: **Nonlocal Diffusion** and a **Reaction Term**.

The Nonlocal Diffusion, given by the expression $-g\left(\int_\Omega w\right)\Delta w$, describes the population dispersion in the habitat Ω. The diffusion rate depends on the total population in the domain and are two possible scenarios for the diffusion coefficient function, which significantly alters the way the population spreads through space.

When g is increasing, it indicates that as the integral of the population density, $\int_\Omega w$, rises, the diffusion coefficient also increases. This means that a larger overall population in the habitat leads to greater dispersion, where crowding causes individuals to spread out more quickly across the area. Conversely, if g is decreasing, then as the population density increases, the diffusion coefficient decreases. In this case, a larger total population results in slower dispersion, suggesting a scenario where individuals move less or tend to cluster together as population density rises, leading to reduced movement or aggregation.

The next result ensures the existence and uniqueness of a positive solution for Problem (PL_2), as well as some properties of this solution, which can be seen as a generalization of Theorem 1.53.

Theorem 1.54. Problem (PL_2) admits a positive solution when

$$\gamma > g(0)\lambda_1.$$

In this case, we will denoted this solution by $\theta_{[\gamma,g(\cdot)]}$. Moreover, this solution is unique when g is increasing. Finally, the following assertions hold:

(a) $\dfrac{\gamma - g\left(\int_\Omega \theta_{[\gamma,g(\cdot)]}\right)\lambda_1}{\|\varphi_1\|_\infty}\varphi_1 \leq \theta_{[\gamma,g(\cdot)]} \leq \gamma.$

(b) $\theta_{[\gamma,g(\cdot)]} = \dfrac{\gamma - g\left(\int_\Omega \theta_{[\gamma,g(\cdot)]}\right)\lambda_1}{\int_\Omega \varphi_1^3}\varphi_1 + G_{\gamma,g(\cdot)}(x)$, where

$$\lim_{\gamma - g\left(\int_\Omega \theta_{[\gamma,g(\cdot)]}\right)\lambda_1 \to 0} \frac{G_{\gamma,g(\cdot)}(x)}{\gamma - g\left(\int_\Omega \theta_{[\gamma,g(\cdot)]}\right)\lambda_1} = 0 \quad \text{in} \quad C^2(\overline{\Omega}).$$

(c) $\sigma_1\left[-g\left(\int_\Omega \theta_{[\gamma,g(\cdot)]}\right)\Delta + 2\theta_{[\gamma,g(\cdot)]} - \gamma\right] > 0.$

(d) The application $[g(0)\lambda_1, +\infty) \ni \gamma \longmapsto \theta_{[\gamma,g(\cdot)]} \in C_0^2(\Omega)$ is increasing and derivable.

Proof. The existence and uniqueness follow by *Theorem 5* of [39]. We will now examine the validity of each property.
<u>Assertion (a)</u>: This is a consequence of Item (a) of Theorem 1.53.
<u>Assertion (b)</u>: See *Lemma 4.3* of [30].
<u>Assertion (c)</u>: This is a consequence of Item (c) of Theorem 1.53.
<u>Assertion (d)</u>: The increasing and derivable of the function follows from the way $\theta_{[\gamma,g(\cdot)]}$ was defined in (PL_2). □

Now, we analyze the structure of the positive solutions of Problem (PL_2) concerning the nonlocal term and the behavior of γ. First, we examine some properties of the positive solutions of the problem.

Proposition 1.55. Let $\theta_{[\gamma,g(\cdot)]}$ be a positive solution of Problem (PL_2). The following assumptions hold:

(a) $\dfrac{\theta_{[\gamma,g(\cdot)]}}{\rho} = \theta_{[\gamma/\rho,1]}$, where $\rho = g\left(\int_\Omega \theta_{[\gamma,g(\cdot)]}\right)$.

(b) $\gamma > g\left(\int_\Omega \theta_{[\gamma,g(\cdot)]}\right)\lambda_1$.

Proof. Assumption (a): Since $\theta_{[\gamma,g(\cdot)]}$ is a solution to (PL_2), follows that

$$-g\left(\int_\Omega \theta_{[\gamma,g(\cdot)]}\right)\gamma\theta_{[\gamma,g(\cdot)]} = \lambda\theta_{[\gamma,g(\cdot)]} - \theta^2_{[\gamma,g(\cdot)]}.$$

Note that, dividing the previous expression by ρ^2,

$$-\Delta\left(\frac{\theta_{[\gamma,g(\cdot)]}}{\rho}\right) = \frac{\gamma}{\rho}\left(\frac{\theta_{[\gamma,g(\cdot)]}}{\rho}\right) - \left(\frac{\theta_{[\gamma,g(\cdot)]}}{\rho}\right)^2.$$

Thus, the previous equation admits a positive solution, which we will denoted by $\theta_{[\gamma/\rho,1]}$, if, and only if,

$$\frac{\gamma}{\rho} > \lambda_1.$$

Consequently, $\dfrac{\theta_{[\gamma,g(\cdot)]}}{\rho} = \theta_{[\gamma/\rho,1]}$.
Assumption (b): From the monotonicity of the principal eigenvalue, follow that

$$\begin{aligned}\gamma &= \sigma_1\left[-g\left(\int_\Omega \theta_{[\gamma,g(\cdot)]}\right)\Delta + d\theta_{[\gamma,g(\cdot)]}\right]\\ &> \sigma_1\left[-g\left(\int_\Omega \theta_{[\gamma,g(\cdot)]}\right)\Delta\right]\\ &> g\left(\int_\Omega \theta_{[\gamma,g(\cdot)]}\right)\lambda_1.\end{aligned}$$

This concludes the proof. □

In the next result, we will study the behavior of the positive solutions of Problem (PL_2) when $\gamma \to +\infty$. It is worth emphasizing that this result is presented in [21].

Proposition 1.56. Let $\theta_{[\gamma,g(\cdot)]}$ be a positive solution of Problem (PL_2). Then

$$\int_\Omega \theta_{[\gamma,g(\cdot)]} \to +\infty \tag{1.18}$$

as $\gamma \to +\infty$. Moreover, for $x \in \Omega$,

$$\theta_{[\gamma,g(\cdot)]}(x) \to +\infty \tag{1.19}$$

as $\gamma \to +\infty$.

Proof. Note that, by Assertion (a) of Theorem 1.54, we obtain

$$\left(\gamma - g\left(\int_\Omega \theta_{[\gamma,g(\cdot)]}\right)\lambda_1\right)\varphi_1 \leq \theta_{[\gamma,g(\cdot)]}, \tag{1.20}$$

where we assume that $\|\varphi_1\|_\infty = 1$. Consequently, integrating the previous expression,

$$\int_\Omega \theta_{[\gamma,g(\cdot)]} + g\left(\int_\Omega \theta_{[\gamma,g(\cdot)]}\right)\lambda_1 \geq \gamma.$$

Thus, taking $\gamma \to +\infty$, we deduce (1.18).
Now, we prove (1.19). For this purpose, we need to analyze two cases, namely:

(1) $g(\infty) := \lim_{s\to\infty} g(s) < \infty$, and

(2) $g(\infty) = \infty$.

For the case (1), we have that

$$g(s) \leq g_M := \sup_{x\in\mathbb{R}} g(x) < \infty,$$

for all $s \in \mathbb{R}$. By (1.20), we obtain that

$$\theta_{[\gamma,g(\cdot)]} \geq \left(\gamma - g\left(\int_\Omega \theta_{[\gamma,g(\cdot)]}\right)\lambda_1\right)\varphi_1 \geq (\gamma - g_M\lambda_1)\varphi_1.$$

Thus, taking $\gamma \to +\infty$, we deduce (1.19) in this case.
For the case (2), by (1.18), we have that

$$g\left(\int_\Omega \theta_{[\gamma,g(\cdot)]}\right) \to \infty \tag{1.21}$$

as $\gamma \to \infty$. Denote

$$\rho = g\left(\int_\Omega \theta_{[\gamma,g(\cdot)]}\right). \tag{1.22}$$

Dividing the first equation of (PL$_2$) by ρ^2, we get that

$$\theta_{[\gamma,g(\cdot)]} = \rho\theta_{[\gamma/\rho,1]}. \tag{1.23}$$

Note that, if $\dfrac{\gamma}{\rho} \geq b > \lambda_1$, for some b, by Assertion (a) of Theorem 1.54, we obtain

$$\theta_{[\gamma/\rho,1]} \geq \theta_{[b,1]}$$

and, consequently,

$$\theta_{[\gamma,g(\cdot)]} = \rho\theta_{[\gamma/\rho,1]} \geq \rho\theta_{[b,1]}.$$

Thus, we deduce (1.19) in this case from (1.21) and (1.22). Assume that there exists a sequence such that

$$\frac{\gamma_n}{\rho_n} \to \lambda_1.$$

By Assertion (b) of Theorem 1.54 follows that

$$\theta_{[\gamma_n/\rho_n,1]} = \left(\frac{\gamma_n}{\rho_n} - \lambda_1\right) \frac{\varphi_1}{\int_\Omega \varphi_1^3} + G_n(x) \tag{1.24}$$

where

$$\lim_{\gamma_n/\rho_n \to \lambda_1} \frac{G_n(x)}{\frac{\gamma_n}{\rho_n} - \lambda_1} = 0 \tag{1.25}$$

in $C^2(\overline{\Omega})$. Consequently, by (1.22),

$$\begin{aligned} \theta_{[\gamma_n,g(\cdot)]} &= \rho_n \left[\left(\frac{\gamma_n}{\rho_n} - \lambda_1\right) \frac{\varphi_1}{\int_\Omega \varphi_1^3} + G_n(x)\right] \\ &= (\gamma_n - \rho_n\lambda_1) \frac{\varphi_1}{\int_\Omega \varphi_1^3} + \rho_n G_n(x) \\ &= (\gamma_n - \rho_n\lambda_1) \left[\frac{\varphi_1}{\int_\Omega \varphi_1^3} + \frac{G_n(x)}{\left(\frac{\gamma_n}{\rho_n} - \lambda_1\right)}\right]. \end{aligned} \tag{1.26}$$

Integrating the previous expression over Ω, it follows that

$$\int_\Omega \theta_{[\gamma_n,g(\cdot)]} = (\gamma_n - \rho_n\lambda_1) \left[\frac{1}{\int_\Omega \varphi_1^3} \int_\Omega \varphi_1 + \frac{\int_\Omega G_n(x)}{\left(\frac{\gamma_n}{\rho_n} - \lambda_1\right)}\right].$$

Using (1.24) and (1.18), we get that

$$(\gamma_n - \rho_n\lambda_1) \to +\infty \tag{1.27}$$

Thus, come back to (1.26), we deduce (1.19). $\square$

Observation 1.57. Observe that, the previous result reveals a remarkable feature: the solution $\theta_{[\gamma,g(\cdot)]}$ diverges to infinity, regardless of the specific behavior of the function g. This particularity indicates that the influence of g on the problem is limited regarding the behavior of $\theta_{[\gamma,g(\cdot)]}$. In other words, even if g possesses specific properties, such as growth or decay, it cannot prevent $\theta_{[\gamma,g(\cdot)]}$ from diverging to infinity.

2 Population Dynamics Between Bacteria and a Living Nutrient

In this chapter, we will study the existence and uniqueness of coexistence states for the following nonlocal elliptic system:

$$\begin{cases} -m\left(\int_\Omega v\right)\Delta u = \lambda u - u^2 + cuv & \text{in } \Omega, \\ -\Delta v + \sigma v = \rho u & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{P$_1$}$$

where Ω is a bounded regular domain in $\mathbb{R}^N$, where $N \geq 1$, $m : \mathbb{R} \to [0, +\infty)$ a continuous function, $c, \lambda \in \mathbb{R}$, $\rho \geq 0$ and $\sigma > 0$.

Throughout this chapter and the following ones, we will frequently employ the following auxiliary problem in various arguments:

$$\begin{cases} -\Delta e + \sigma e = 1 & \text{in } \Omega, \\ e = 0 & \text{on } \partial\Omega, \end{cases} \tag{P$_e$}$$

which admits a unique positive solution, denoted by e_σ.

First, we will analyze the semi-trivial solutions and show that, under the given hypotheses, Problem (P$_1$) does not admit a semi-trivial solution. In the case $\rho = 0$, the bacteria do not receive nutrients and, consequently, the bacteria die ($v \equiv 0$), while the nutrients follow the classical logistic equation:

$$\begin{cases} -m(0)\Delta u = \lambda u - u^2 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

In this case, assuming that $m(0) > 0$, it is well-known that Problem (2.1) admits a positive solution if, and only if, $\lambda > m(0)\lambda_1$.

From now on, we will assume $\rho > 0$ to ensure that neither of the populations goes extinct. In this case, the coexistence state results will depend solely on the sign of the constant c.

In his work [14], M. Chipot examines a particular case of Problem (P$_1$) to model the behavior of a bacteria, with density v, living in a habitat Ω, which feeds on a nutrient u. Specifically:

$$-a\left(\int_\Omega v\right)\Delta u = f(x), \text{ in } \Omega,$$

where f is a constant rate of the nutrient.

In our model, (P_1), the nutrient is another living organism that grows following the logistic equation:

$$f(\lambda, u, v) = \lambda u - u^2 + cuv,$$

where λu denotes the growth rate of the nutrient, $-u^2$ the term responsible for limiting growth, and cuv the interaction between the nutrient and the bacteria. This interaction is competitive when c is negative and cooperative when is positive. A special case is when $c = 0$, meaning there is no interaction.

We will divide this chapter as follows: In Section 2.1, we will explore *a priori* bounds and nonexistence results for coexistence states of Problem (P_1). Sections 2.2 and 2.3 will focus on reformulating the Problem (P_1) into a suitable nonlinear equation for applying local and global bifurcation methods. Section 2.4 will analyze the conditions under which the uniqueness of coexistence states of Problem (P_1) occurs. Finally, in Section 2.5, we will address the case where $m(0) = 0$.

2.1 A Priori Bounds and Non-existence Results of Coexistence States

In this section, we will investigate *a priori* estimates and non-existence results for coexistence states of Problem (P_1). To address this, it will be necessary to analyze the case where the sign of c is negative and positive separately.

Proposition 2.1. Assume that $c \leq 0$. The following assertions hold:

(a) If $\lambda < m_L\lambda_1$, with $m_L := \min_{x\in\overline{\Omega}} m(x)$, then Problem ($P_1$) does not possess a coexistence states.

(b) If (u, v) is a coexistence states of Problem (P_1) then

$$u \leq \lambda \qquad \text{and} \qquad v \leq \rho\lambda e_\sigma,$$

where e_σ is defined in (P_e).

Proof. Assertion (a): Let (u, v) be a coexistence states of (P_1). Since u is a positive solution, it follows from Definition 1.26 that

$$\lambda = \sigma_1\left[-m\left(\int_\Omega v\right)\Delta + u + cv\right].$$

Since $c \geq 0$ and $m(x) \geq m_L$ for all $x \in \mathbb{R}_+$, we obtain, from Proposition 1.28, that

$$\lambda > \sigma_1\left[-m_L\Delta\right] = m_L\lambda_1.$$

Assertion (b): Let x_M be a element of Ω such that $u_M = \max_{x\in\overline{\Omega}} u(x) = u(x_M)$. Since $-\Delta u(x_M) \geq 0$, by the Maximum Principle (see Section 1.2),

$$-m\left(\int_\Omega v\right)\Delta u(x_M) = \lambda u_M - u_M^2 + cu_M v(x_M) \geq 0$$

and, rearranging the terms to isolate u_M,

$$u_M \leq \lambda + cv(x_M). \tag{2.2}$$

Since $c \leq 0$, we obtain

$$u \leq \lambda.$$

Using the previous expression in the second equation of (P_1), it follows that

$$-\Delta v + \sigma v = \rho u \leq \rho\lambda.$$

Consider $w = \rho\lambda e_\sigma$. Multiplying Problem (P_e) by $\rho\lambda$, we obtain

$$\begin{cases} -\Delta w + \sigma w = \rho\lambda & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Note that, the previous problem admits v as a subsolution, and consequently,

$$v \leq \rho\lambda e_\sigma.$$

□

The next result, in addition to providing a priori estimates for the maximum values attained by the functions u and v, gives an upper bound on the infinity norm for coexistence states when the function m satisfies certain properties.

Proposition 2.2. Assume that $c > 0$. The following assertions hold:

(a) Suppose that

$$c\rho \left\| e_\sigma \right\|_\infty < 1. \tag{H_1}$$

Then

$$u_M \leq \frac{\lambda}{1 - c\rho \left\| e_\sigma \right\|_\infty} \qquad \text{and} \qquad v_M \leq \frac{\rho\lambda \left\| e_\sigma \right\|_\infty}{1 - c\rho \left\| e_\sigma \right\|_\infty}$$

(b) Suppose that

$$\lim_{s\to\infty} \frac{m(s)}{s} = \infty. \tag{H_2}$$

If $\lambda \in \Lambda \subset \mathbb{R}$, with Λ a compact set, there exists $k > 0$ such that

$$\left\| (u, v) \right\|_\infty \leq k.$$

Furthermore, there exists $\lambda_0 \in \mathbb{R}$ such that Problem (P_1) does not possess a coexistence state for $\lambda \leq \lambda_0$.

Proof. Assertion (a): Analogously to what was done in Assertion (b) of Proposition 2.1, we obtain

$$v \leq \rho u_M e_\sigma$$

and, consequently,

$$v_M \leq \rho u_M \|e_\sigma\|_\infty . \tag{2.3}$$

Thus, using the expression (2.2) in (2.3), we obtain

$$v_M \leq \frac{\rho\lambda \|e_\sigma\|_\infty}{1 - c\rho \|e_\sigma\|_\infty}, \tag{2.4}$$

and the expression (2.3) in (2.2), we obtain

$$u_M \leq \frac{\lambda}{1 - c\rho \|e_\sigma\|_\infty}. \tag{2.5}$$

Assertion (b): Let y_M be a element of Ω such that $v_M = \max_{x \in \overline{\Omega}} v(x) = v(y_M)$. Note that, $-\Delta v(y_M) \geq 0$ and, consequently,

$$-\Delta v(y_M) + \sigma v_M = \rho u(y_M) \leq \rho u_M$$

and, consequently,

$$\sigma v_M \leq \rho u_M.$$

Using the expression (2.2) in the previous expression, we obtain

$$\sigma v_M \leq \rho u_M \leq \rho\lambda + c\rho v_M.$$

Rearranging the terms in the previous expression, it follows that

$$\frac{\sigma}{\rho} \leq \frac{u_M}{v_M} \leq \frac{\lambda}{v_M} + c, \tag{2.6}$$

which we can deduce that

$$\frac{1}{c}\left(1 - \frac{\lambda}{u_M}\right) \leq \frac{v_M}{u_M} \leq \frac{\rho}{\sigma}. \tag{2.7}$$

Suppose there exists a sequence (λ_n) in $\mathbb{R}$ and a sequence (u_n, v_n) of positive solutions of (P_1) in $C^0(\overline{\Omega}) \times C^0(\overline{\Omega})$ such that $\lambda_n \to \overline{\lambda}$, with $\overline{\lambda} < \infty$ and $\|u_n\|_\infty + \|v_n\|_\infty \to \infty$. Note that, by inequality (2.6), $\|v_n\|_\infty \to \infty$ when $\|u_n\|_\infty \to \infty$. Furthermore, by inequality (2.7), $\|u_n\|_\infty \to \infty$ when $\|v_n\|_\infty \to \infty$. Consequently

$$\|u_n\|_\infty \to \infty \qquad \text{and} \qquad \|v_n\|_\infty \to \infty. \tag{2.8}$$

Consider the sequences (z_n) and (w_n) in $C^0(\overline{\Omega})$ such that

$$z_n := \frac{v_n}{\|v_n\|_\infty} \qquad \text{and} \qquad w_n := \frac{u_n}{\|u_n\|_\infty}. \tag{2.9}$$

Note that

$$\|z_n\|_\infty \geq \frac{1}{c}\left(1 - \frac{\lambda}{\|u_n\|_\infty}\right) \qquad \text{and} \qquad \|w_n\|_\infty = 1. \tag{2.10}$$

Using the expression (2.9) in the second equation of (P_1), we obtain

$$\begin{cases} -\Delta z_n + \sigma z_n = \rho z_n & \text{in } \Omega, \\ z_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Thus, by the *Agmon-Douglis-Nirenber Theorem* (see [1]), it follows that

$$\|z_n\|_{W^{2,p}} \leq K \|w_n\|_p \leq K,$$

where $K > 0$. Consequently, there exists $z_0 \in C^1(\overline{\Omega})$, with $z_0 \geq 0$, such that

$$z_n \to z_0 \text{ in } C^1(\overline{\Omega}).$$

By the expressions of (2.10), $z_n \not\equiv 0$ in Ω. Furthermore, taking λ_n, u_n and v_n in the first equation of (P_1), we obtain

$$-m\left(\int_\Omega v_n\right) \Delta u_n = \lambda_n u_n - u_n^2 + c u_n v_n$$

and, dividing the previous equation by $\|u_n\|_\infty^2$,

$$-\frac{m\left(\int_\Omega v_n\right)}{\|u_n\|_\infty} \Delta w_n = \frac{\lambda_n}{\|u_n\|_\infty} w_n - w_n^2 + c w_n z_n. \tag{2.11}$$

On the left-hand side of the equation, it follows that

$$\begin{aligned} \frac{m\left(\int_\Omega v_n\right)}{\|u_n\|_\infty} &= \frac{m\left(\|u_n\|_\infty \int_\Omega \frac{v_n}{\|u_n\|_\infty}\right)}{\|u_n\|_\infty} \\ &= \frac{m\left(\|u_n\|_\infty \int_\Omega z_n\right)}{\|u_n\|_\infty} \\ &= \frac{m(s_n)}{s_n} \int_\Omega z_n, \end{aligned}$$

where $s_n = \|u_n\|_\infty \int_\Omega z_n$. Since $z_0 \not\equiv 0$, it follows that $s_n \to \infty$ and, consequently,

$$-\frac{m(s_n)}{s_n} \int_\Omega z_n \Delta w_n = \frac{\lambda_n}{\|u_n\|_\infty} w_n - w_n^2 + c w_n z_n.$$

Since the right-hand side of the equation is bounded in $L^p(\Omega)$, from the equation above,

$$\|w_n\|_{W^{2,p}} \leq \frac{C \left\| \frac{\lambda_n}{\|u_n\|_\infty} w_n - w_n^2 + c w_n z_n \right\|_p}{\frac{m(s_n)}{s_n} \int_\Omega z_n} \to 0$$

and, consequently, by Hypothesis (H$_2$), $w_n \to 0$ in $C^1(\overline{\Omega})$. A contradiction due to the fact that $\|w_n\|_\infty = 1$. Finally, (P$_1$) does not possess a coexistence state when $\lambda \le \lambda_0$. Assume by contradiction that there exists a coexistence state (λ_n, u_n, v_n) such that $\lambda_n \to -\infty$. Since $\lambda_n = \sigma_1\left[-m\left(\int_\Omega v_n\right)\Delta + u_n - cv_n\right]$, it follows that

$$\|v_n\|_\infty \to \infty.$$

Indeed, suppose by contradiction that $\|v_n\|_\infty \le K$. Thus

$$\lambda_n \ge m_L\lambda_1 - cK,$$

a contradiction. Consequently, $\|v_n\|_\infty \to \infty$ implies that $\|u_n\|_\infty \to \infty$. Note that, by (2.3),

$$1 \le c\frac{\|v_n\|_\infty}{\|u_n\|_\infty} - \frac{\lambda_n}{\|u_n\|_\infty} \le c\frac{\rho}{\sigma} - \frac{\lambda_n}{\|u_n\|_\infty}$$

and, consequently,

$$1 - c\frac{\rho}{\sigma} \le -\frac{\lambda_n}{\|u_n\|_\infty} \le 0. \tag{2.12}$$

Using the same reasoning as before, we can conclude that

$$\|w_n\|_{W^{2,p}} \le \frac{K\left\|\dfrac{\lambda_n}{\|u_n\|_\infty}w_n - w_n^2 + cw_nz_n\right\|_p}{\dfrac{m(s_n)}{s_n}\displaystyle\int_\Omega z_n}$$

and, by (2.12),

$$\|w_n\|_{W^{2,p}} \to 0,$$

concluding the demonstration. □

To conclude this section, we present a result that highlights an important property of the mapping $[0,\infty) \ni \sigma \longmapsto e_\sigma \in C(\overline{\Omega})$.

Lemma 2.3. The mapping $[0,\infty) \ni \sigma \longmapsto e_\sigma \in C(\overline{\Omega})$ is decreasing and

$$e_\sigma \to 0$$

uniformly in $\overline{\Omega}$ when $\sigma \to \infty$.

Proof. Consider $\sigma_1, \sigma_2 \in [0,\infty)$, with $\sigma_1 < \sigma_2$, and e_1 and e_2 distinct positive solutions solution of (P$_e$). Note that

$$\begin{cases} -\Delta(e_2 - e_1) + \sigma_2(e_2 - e_1) = (\sigma_1 - \sigma_2)e_1 \le 0 & \text{in } \Omega, \\ e_2 - e_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

Then, by the Maximum Principle, it follows that

$$e_2 - e_1 \le 0$$

and, consequently, $e_2 < e_1$ in Ω. On the other hand, for $x_\sigma \in \Omega$ such that $e_\sigma(x_\sigma) = \max_{x\in\Omega} e_\sigma(x)$, we have

$$\sigma e_\sigma(x_\sigma) \leq -\Delta e_\sigma(x_\sigma) + \sigma e_\sigma(x_\sigma) = 1$$

and, consequently,

$$\|e_\sigma\|_\infty \leq \frac{1}{\sigma},$$

as expected. □

2.2 Local Bifurcation Analysis

In this section, we will rewrite (P_1) to utilize the results of local bifurcation in order to obtain a curve of non-trivial solutions in a neighborhood of the trivial solution in the Cooperative case. To achieve this, we will employ the theory developed in Section 1.5, with particular emphasis on Theorem 1.42.

We will consider the spaces $E := C_0^2(\overline{\Omega})$, $F := C(\overline{\Omega})$ and the operator $\mathcal{F} : \mathbb{R}\times E^2 \to F^2$ given by:

$$\mathcal{F}(\lambda, u, v) = \begin{bmatrix} -m\left(\int_\Omega v\right)\Delta u - \lambda u + u^2 - cvu \\ -\Delta v + \sigma v - \rho u \end{bmatrix}.$$

Note that $\mathcal{F}$ is a continuous and differentiable operator. By the construction of $\mathcal{F}$, $(u, v) \in E^2$ is a nonnegative solution of (P_1) if, and only if,

$$\mathcal{F}(\lambda, u, v) = 0_{F\times F} \tag{2.13}$$

for all $\lambda \in \mathbb{R}$. In particular

$$\mathcal{F}(\lambda, 0, 0) = 0_{F\times F} \tag{2.14}$$

for all $\lambda \in \mathbb{R}$. Furthermore, the derivative of $\mathcal{F}$ in $(\lambda, 0, 0)$ is given by

$$\mathcal{L}(\lambda) := \mathcal{F}_{(u,v)}(\lambda, 0, 0)$$

where, for $(\xi, \eta) \in E^2$,

$$\mathcal{L}(\lambda)(\xi, \eta) = \begin{bmatrix} -m(0)\Delta\xi - \lambda\xi \\ -\Delta\eta + \sigma\eta - \rho\xi \end{bmatrix}.$$

The next result will ensure the existence and uniqueness of a curve of non-trivial solutions emanating from $0_{E\times E} := (0, 0)$ and will identify a specific bifurcation point.

Theorem 2.4. Assume that $m(0) > 0$. There exists $\varepsilon > 0$ and applications of class C^1

$$\lambda : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R} \qquad \varphi : (-\varepsilon, \varepsilon) \longrightarrow Z \qquad \psi : (-\varepsilon, \varepsilon) \longrightarrow Z,$$

where Z is a topological complementary of $\text{Ker}\,[\mathcal{F}_{(u,v)}(m(0)\lambda_1, 0, 0)]$ in E^2, such that

$$\begin{cases} \lambda(s) = m(0)\lambda_1 + \rho(s) \\ u(s) = s(\varphi_1 + \varphi(s)) \\ v(s) = s(\psi_1 + \psi(s)) \end{cases}, \tag{2.15}$$

where $\psi_1 = \dfrac{\rho}{\lambda_1 + \sigma}\varphi_1$ is a positive function, which will be detailed below, $\varphi(0) = \psi(0) = 0$ and $\lambda(0) = m(0)\lambda_1$. Furthermore, the coexistence states of Problem (P_1) in a neighborhood of $(m(0)\lambda_1, 0, 0)$ are given by the triple $(\lambda(s), u(s), v(s))$, for each $s \in (-\varepsilon, \varepsilon)$.

Proof. Consider λ as the main bifurcation parameter. We will apply Theorem 1.42 and to do this, it is necessary to verify that $\mathcal{F}$ satisfies the conditions (CR_4)–(CR_6).
Condition (CR_4): We need to determine λ^* such that $\dim\,[\text{Ker}\,(\mathcal{L}(\lambda^*))] = 1$. Note that, $(\xi, \eta) \in \text{Ker}\,(\mathcal{L}(\lambda^*))$ if, and only if, (ξ, η) is a solution of the following problem:

$$\begin{cases} -m(0)\Delta\xi - \lambda^*\xi = 0 & \text{in } \Omega, \\ -\Delta\eta + \sigma\eta - \rho\xi = 0 & \text{in } \Omega, \\ \xi = \eta = 0 & \text{on } \partial\Omega. \end{cases}$$

From the first equation, by Theorem 1.53, we deduce that $\lambda^* := m(0)\lambda_1$. Thus, $\xi \in \text{Span}\,\{\varphi_1\}$. For the second equation, substituting η with φ_1, we obtain the following linear equation:

$$\begin{cases} -\Delta\eta + \sigma\eta = \rho\varphi_1 & \text{in } \Omega, \\ \eta = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.16}$$

The Problem (2.16) admits a unique positive solution, which we will denote by ψ_1. Indeed, observe that it is enough to take

$$\psi_1 := K\varphi_1, \tag{2.17}$$

where $K = \dfrac{\rho}{\lambda_1 + \sigma}$. Thus

$$\text{Ker}\,[\mathcal{L}\,(m(0)\lambda_1)] = \text{Span}\,\{(\varphi_1, K\varphi_1)\}\,.$$

Condition (CR_5): We claim that

$$\text{Rg}\,[\mathcal{L}(m(0)\lambda_1)] = \text{Rg}\,(-\Delta - \lambda_1) \times C_0^1(\overline{\Omega}). \tag{2.18}$$

Indeed, given $(\varphi, \psi) \in \text{Rg}\,[\mathcal{L}(m(0)\lambda_1)]$, there exists $(u, v) \in E$ such that

$$\begin{cases} -m(0)\Delta u - m(0)\lambda_1 u = \varphi & \text{in } \Omega, \\ -\Delta v + \sigma v - \rho u = \psi & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.19}$$

Multiplying the first equation of the system above by φ_1, integrating over Ω, and using the concept of eigenvalue, we obtain:

$$\int_\Omega \varphi_1\varphi = 0. \tag{2.20}$$

Furthermore, $\varphi \in \mathrm{Rg}\,(-\Delta - \lambda_1)$, that is, φ satisfies (2.20). Since $\sigma_1[-\Delta + \sigma] > 0$, it is clear that, for any $\psi \in C_0^1(\overline{\Omega})$, there exists $v \in C_0^2(\overline{\Omega})$ solution of the second equation of (2.19). Thus

$$\mathrm{codim}\,[\mathrm{Rg}\,(\mathcal{L}(m(0)\lambda_1))] = 1.$$

Condition (CR$_6$): Note that, we can write $\mathcal{L}(m(0)\lambda_1)$ as follows

$$\mathcal{L}(m(0)\lambda_1) = \begin{bmatrix} \mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{bmatrix} - \begin{bmatrix} -m(0)\Delta - \lambda + \mathrm{Id} & 0 \\ -\rho & -\Delta + \sigma + \mathrm{Id} \end{bmatrix}.$$

Furthermore

$$\mathcal{L}'(m(0)\lambda_1)\,(\varphi_1, \psi_1) = (-\varphi_1, 0)\,.$$

Then, if $\mathcal{L}'(m(0)\lambda_1)\,(\varphi_1, \psi_1) \in \mathrm{Rg}\,[\mathcal{L}(m(0)\lambda_1)]$, by (2.20),

$$\int_\Omega \varphi_1^2 = 0,$$

which is a contradiction. Thus

$$\mathcal{L}'(m(0)\lambda_1) \notin \mathrm{Rg}\,[\mathcal{L}(m(0)\lambda_1)] \tag{2.21}$$

Therefore, there exists $\varepsilon > 0$ and class C^1 applications such that (2.15) holds. □

In the next result, we will analyze the bifurcation direction from the trivial solution.

Theorem 2.5. Assume that $m(0) > 0$. Then, the bifurcation direction from the trivial solution $(u, v) = (0, 0)$ in $\lambda = m(0)\lambda_1$ is:

(a) Supercritical, when

$$m'(0) > -\frac{(\lambda_1 + \sigma - c\rho)\displaystyle\int_\Omega \varphi_1^3}{\lambda_1\rho\displaystyle\int_\Omega \varphi_1}.$$

(b) Subcritical, when

$$m'(0) < -\frac{(\lambda_1 + \sigma - c\rho)\displaystyle\int_\Omega \varphi_1^3}{\lambda_1\rho\displaystyle\int_\Omega \varphi_1}.$$

Proof. Using the expressions from (2.15) in the first equation of (P$_1$), we obtain

$$\begin{aligned} -m\left(\int_\Omega s\,(\psi_1 + \psi(s))\right)\Delta\,(s\,(\varphi_1 + \varphi(s))) = {} & (\lambda_1 a(0) + \rho(s))\,(s\,(\varphi_1 + \varphi(s))) \\ & - (s\,(\varphi_1 + \varphi(s)))^2 \\ & + c\,(s\,(\varphi_1 + \varphi(s)))\,(s\,(\psi_1 + \psi(s)))\,. \end{aligned}$$

Consider the Taylor expansion of $m(s)$, that is, $m(s) = m(0) + sm'(0) + o(s)$, and $\rho(s) = s\rho_1(s) + o(s)$. Note that

$$
\begin{aligned}
-\left(m(0) + sm'(0)\int_\Omega (\psi_1 + \psi(s)) + o(s)\right)\Delta(\varphi_1 + \varphi(s))\, s = \\
&= (\lambda_1 m(0) + s\rho_1(s) + o(s))(\varphi_1 + \varphi(s))\, s - (\varphi_1 + \varphi(s))^2 s^2 \\
&\quad + c(\varphi_1 + \varphi(s))(\psi_1 + \psi(s))\, s^2
\end{aligned}
$$

and, consequently,

$$
\begin{aligned}
-m(0)\Delta(\varphi_1 + \varphi(s))\, s - s^2 m'(0)\Delta(\varphi_1 + \varphi(s))\int_\Omega (\psi_1 + \psi(s)) - o(s)\Delta(\varphi_1 + \varphi(s))\, s = \\
= \lambda_1 m(0)(\varphi_1 + \varphi(s))\, s + \rho_1(s)(\varphi_1 + \varphi(s))\, s^2 + o(s)(\varphi_1 + \varphi(s))\, s - s^2(\varphi_1 + \varphi(s))^2 \\
+ c\varphi_1(\psi_1 + \psi(s))\, s^2 + c\varphi(s)(\psi_1 + \psi(s))\, s^2
\end{aligned}
$$

Equating the terms involving s, it follows that

$$
-m(0)\Delta(\varphi_1 + \varphi(s)) - o(s)\Delta(\varphi_1 + \varphi(s)) = \lambda_1 m(0)(\varphi_1 + \varphi(s)) + o(s)(\varphi_1 + \varphi(s))
$$

and, consequently,

$$
-m(0)\Delta\varphi_1 = m(0)\lambda_1\varphi_1.
$$

Equating the terms involving s^2, it follows that

$$
\begin{aligned}
-m'(0)\Delta(\varphi_1 + \varphi(s))\int_\Omega (\psi_1 + \psi(s)) = &\rho_1(s)(\varphi_1 + \varphi(s)) - (\varphi_1 + \varphi(s))^2 + c\varphi_1(\psi_1 + \psi(s)) \\
&+ c\varphi(s)(\psi_1 + \psi(s)).
\end{aligned}
$$

Multiplying the previous expression by φ_1, integrating over Ω, using Green's Identity (see [10]), it follows that

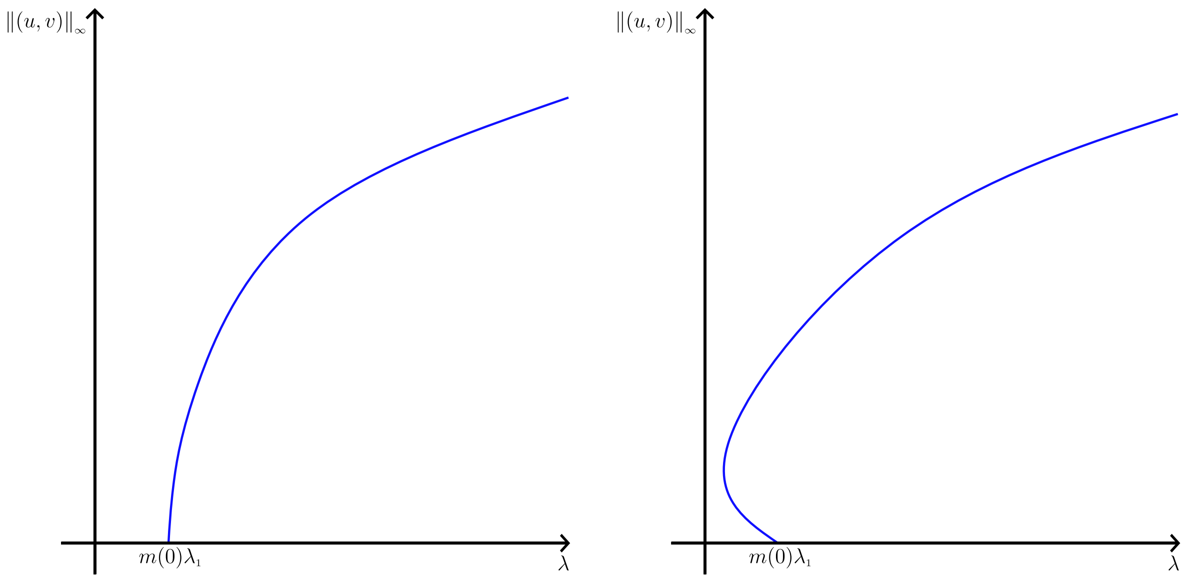
$$
\begin{aligned}
m'(0)\lambda_1\int_\Omega (\psi_1 + \psi(s))\int_\Omega \varphi_1(\varphi_1 + \varphi(s)) = &\rho_1(s)\int_\Omega \varphi_1(\varphi_1 + \varphi(s)) - \int_\Omega \varphi_1(\varphi_1 + \varphi(s))^2 \\
&+ c\int_\Omega \varphi_1^2(\psi_1 + \psi(s)) + c\int_\Omega \varphi_1\varphi(s)(\psi_1 + \psi(s)).
\end{aligned}
$$

Note that, as shown in (2.16), ψ_1 is the unique solution of the above equation, and consequently, using (2.17), we obtain

$$
\begin{aligned}
\lim_{s\to 0}\rho_1(s) &= \frac{m'(0)\lambda_1 K\int_\Omega \varphi_1\int_\Omega \varphi_1^2 + \int_\Omega \varphi_1^3 - cK\int_\Omega \varphi_1^3}{\int_\Omega \varphi_1^2} \\
&= m'(0)\lambda_1 K\int_\Omega \varphi_1 + (1 - cK)\int_\Omega \varphi_1^3.
\end{aligned}
$$

□

The following figure provides a geometric interpretation of the bifurcation direction of the curve of non-trivial solutions ensured by Theorem 2.4, which emanates from the trivial solution $(0, 0)$.



(a) Supercritical bifurcation. (b) Subcritical bifurcation.

Figure 2.1: Bifurcation Diagram of Problem (P_1).
Source: Prepared by the author.

2.3 Global Bifurcation Analysis

In this section, similarly to what was done in the previous section, we will rewrite (P_1) and apply the Global Bifurcation to obtain a bounded continuum $\mathfrak{C}$ of coexistence states for (P_1) in the Cooperative case. To achieve this, we will employ the theory developed in Section 1.5, with particular emphasis on Theorem 1.49.

We will consider the Ordered Banach Space $E := C_0^1(\overline{\Omega})$ and its positive cone, that is,

$$P_E := \{u \in E;\ u(x) \geq 0 \text{ for all } x \in \Omega\}.$$

In the next result, we will prove the existence of a bounded continuum of coexistence states $\mathfrak{C}$ of Problem (P_1), that is, the existence of a maximal connected closed subset in the set of coexistence states of (P_1).

Theorem 2.6. Assume that $m(0) > 0$. Then, from the point

$$(\lambda, u, v) = (m(0)\lambda_1, 0, 0)$$

bifurcates a unbounded continuum $\mathfrak{C} \subset \mathbb{R} \times \mathrm{Int}(P_E) \times \mathrm{Int}(P_E)$ of coexistence states of Problem (P_1).

Proof. We proved in Theorem 2.4 that $m(0)\lambda_1$ is a bifurcation point. On the other hand, it is possible to rewrite the equations of (P_1) in such a way that we obtain an operator such that

$$\mathcal{G}(\lambda, u, v) = 0_{E \times E},$$

where $\mathcal{G} : \mathbb{R} \times E^2 \mapsto E^2$ is defined by

$$\mathcal{G}(\lambda, u, v) = \begin{bmatrix} u \\ v \end{bmatrix} - L \begin{bmatrix} \dfrac{\lambda u}{m(0)} \\ \rho u \end{bmatrix} + N(\lambda, u, v).$$

In the previous expression

$$L = \begin{bmatrix} (-\Delta)^{-1} & 0 \\ 0 & (-\Delta + \sigma)^{-1} \end{bmatrix}$$

and

$$N(\lambda, u, v) = L \begin{bmatrix} \lambda u \left(\dfrac{1}{m\left(\int_\Omega v\right)} - \dfrac{1}{m(0)} \right) + \dfrac{cuv - u^2}{m\left(\int_\Omega v\right)} \\ 0 \end{bmatrix}.$$

Note that, the way we defined N,

$$\frac{\|N(\lambda, u, v)\|_{E\times E}}{\|(u, v)\|_{E\times E}} \to 0,$$

when $\|(u, v)\|_{E\times E} \to 0$. Now, we can apply Theorem 1.49 (see Observation 1.51), and following the steps of *Theorem 1.1* of [19] (see also *Theorem 4.1* of [47] and *Theorem 6.4.3* of [48]) and conclude that there exists a continuum $\mathfrak{C}$ of coexistence states for (P_1) emanating from $(m(0)\lambda_1, 0, 0)$ and satisfies at least one of the following alternatives:

(G_1') $\mathfrak{C}$ is unbounded in $\mathbb{R} \times E^2$;

(G_2') There exists $(\lambda_n, u_n, v_n) \in \overline{\mathfrak{C}}$ such that $\lambda_n \to \overline{\lambda}$ and $(u_n, v_n) \to (0, 0)$ in E^2, with $\overline{\lambda} \neq m(0)\lambda_1$;

(G_3') There exists $(\lambda_n, u_n, v_n) \in \overline{\mathfrak{C}}$ such that $\lambda_n \to \overline{\lambda}$ and $(u_n, v_n) \to (\overline{u}, 0)$ in E^2, with $\overline{\lambda} \neq m(0)\lambda_1$ and $\overline{u} > 0$; or

(G_4') There exists $(\lambda_n, u_n, v_n) \in \overline{\mathfrak{C}}$ such that $\lambda_n \to \overline{\lambda}$ and $(u_n, v_n) \to (0, \overline{v})$ in E^2, with $\overline{\lambda} \neq m(0)\lambda_1$ and $\overline{v} > 0$.

We go check the validity of each alternative and prove that only (G_1') holds.
Condition (G_2') is not possible: Suppose there exists $(\lambda_n, u_n, v_n) \in \overline{\mathfrak{C}}$ such that $\lambda_n \to \overline{\lambda}$ and $(u_n, v_n) \to (0, 0)$ in E^2, with $\overline{\lambda} \neq m(0)\lambda_1$. Observe, by Definition 1.26, that

$$\lambda_n = \sigma_1 \left[-m\left(\int_\Omega v_n\right)\Delta + u_n - cv_n \right].$$

Since $(u_n, v_n) \to (0, 0)$ in E^2, we conclude that

$$\lambda_n \to m(0)\lambda_1,$$

a contradiction.

Condition (G_3') is not possible: Suppose there exists $(\lambda_n, u_n, v_n) \in \overline{\mathfrak{C}}$ such that $\lambda_n \to \overline{\lambda}$ and $(u_n, v_n) \to (\overline{u}, 0)$ in E^2, with $\overline{\lambda} \neq m(0)\lambda_1$ and $\overline{u} > 0$. Since v_n is bounded in $C^{2,\gamma}(\overline{\Omega})$, $\gamma \in (0,1)$, it is easy to deduce that $v_n \to v^* \geq 0$ solution of

$$-\Delta v^* + \sigma v^* = \rho \overline{u} > 0,$$

a contradiction because $v^* = 0$.
Condition (G_4') is not possible: Suppose there exists $(\lambda_n, u_n, v_n) \in \overline{\mathfrak{C}}$ such that $\lambda_n \to \overline{\lambda}$ and $(u_n, v_n) \to (0, \overline{v})$ in E^2, with $\overline{\lambda} \neq m(0)\lambda_1$ and $\overline{v} > 0$. If we denote $y_M \in \Omega$ such that $v(y_M) = v_M = \max_{x\in\overline{\Omega}} v(x)$, we get

$$-\Delta v(y_M) + \sigma v_M = \rho u(y_M) \leq \rho \|u\|_\infty .$$

Consequently

$$\sigma \|v_n\|_\infty \leq \rho \|u_n\|_\infty .$$

Thus, taking limit,

$$\sigma \|\overline{v}\|_\infty \leq 0,$$

which is a contradiction.
Therefore, from the point $(m(0)\lambda_1, 0, 0)$ bifurcates an unbounded continuum $\mathfrak{C} \subset \mathbb{R} \times \text{Int}(P_E) \times \text{Int}(P_E)$ of coexistence states of (P_1). □

The following result will present some consequences of the existence of the bounded continuum.

Corollary 2.7. The following statements hold:

(a) If $c \leq 0$ then $(m(0)\lambda_1, \infty) \subset \text{Proj}_{\mathbb{R}}(\mathfrak{C}) \subset (m_L\lambda_1, \infty)$.

(b) If $c > 0$ and (H_1) is satisfied then $(m(0)\lambda_1, \infty) \subset \text{Proj}_{\mathbb{R}}(P_E) \subset (0, \infty)$.

(c) If $c > 0$ and (H_2) is satisfied, then there exists $\lambda_0 \leq 0$ such that $(m(0)\lambda_1, \infty) \subset \text{Proj}_{\mathbb{R}}(P_E) \subset (\lambda_0, \infty)$.

Proof. To prove these statements, we need only use the estimates obtained in Section 2.2 along with the properties of the continuum established in the previous result.
Statement (a): From Proposition 2.1, we obtain the existence of a priori bound in L^∞ and, by elliptic regularity, in $C^1(\overline{\Omega})$. Furthermore, from the same proposition, we obtain the nonexistence of coexistence states when $\lambda \leq m_L\lambda_1$.
Statement (b): From Proposition 2.2, we obtain the existence of a priori bound and nonexistence for $\lambda \leq 0$.
Statement (c): From Proposition 2.2, we obtain the existence of a priori bound and nonexistence for $\lambda \leq \lambda_0$. □

2.4 Uniqueness of Coexistence State

In this section, we will demonstrate the conditions under which the coexistence state of Problem (P_1) in the cooperative case occurs uniquely.

Theorem 2.8. Assume that m is increasing. Then, there exists $c_0 > 0$ such that Problem (P_1) possesses at most one coexistence state for $c \in (-c_0, c_0)$.

Proof. Firstly, we will study the case where $c = 0$. Note that, in this case, by multiplying the second equation of (P_1) by e_σ, integrating over Ω, and using Green's Identity, we obtain:

$$\int_\Omega v = \rho \int_\Omega e_\sigma u.$$

Then, when $c = 0$, (P_1) is equivalent to the problem

$$\begin{cases} -m\left(\rho \int_\Omega e_\sigma u\right) \Delta u = \lambda u - u^2 & \text{in } \Omega, \\ -\Delta v + \sigma v = \rho u & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.22}$$

Since m is increasing, from Theorem 1.54 (see also *Theorem 5* in [39]), there is a unique positive solution, which we will denote by u_0, of the problem

$$\begin{cases} -m\left(\rho \int_\Omega e_\sigma u\right) \Delta u = \lambda u - u^2 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Furthermore, the existence of a positive solution, which we will denote by v_0, for the following problem is evident

$$\begin{cases} -\Delta v + \sigma v = \rho u_0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Finally, since

$$\sigma_1\left[-m\left(\rho \int_\Omega e_\sigma u_0\right) \Delta + u_0 - \lambda\right] = 0$$

it follows, by Theorem 1.27, that

$$\sigma_1\left[-m\left(\rho \int_\Omega e_\sigma u_0\right) \Delta + 2u_0 - \lambda\right] > 0. \tag{2.23}$$

Now, we will study the case where $c \neq 0$. In this case, we will apply the *Implicit Function Theorem* and, for this purpose, we will consider the operator $\mathcal{G} : \mathbb{R} \times E^2 \to F^2$, where $E = C_0^2(\overline{\Omega})$ and $F = C(\overline{\Omega})$, given by:

$$\mathcal{G}(c, u, v) = \begin{bmatrix} -m\left(\int_\Omega v\right) \Delta u - \lambda u + u^2 - cuv \\ -\Delta v + \sigma v - \rho u \end{bmatrix}.$$

Note that, $\mathcal{G}$ is derivable and $\mathcal{G}(0, u_0, v_0) = 0_{F\times F}$. Its derivative at $(0, u_0, v_0)$ is given by

$$\mathcal{G}_{(u,v)}(0, u_0, v_0)(\xi, \eta) = \begin{bmatrix} -m\left(\int_\Omega v_0\right)\Delta\xi + M(x)\int_\Omega \eta + N(x)\xi \\ -\Delta\eta + \sigma\eta - \rho\xi \end{bmatrix},$$

where

$$M(x) = \frac{m'\left(\int_\Omega v_0\right)}{m\left(\int_\Omega v_0\right)}(\lambda u_0 - u_0^2) \qquad \text{and} \qquad N(x) = 2u_0 - \lambda.$$

We claim that $\mathcal{G}_{(u,v)}(0, u_0, v_0)$ is an isomorphism from E^2 to F^2. Indeed, given $(f, g) \in F^2$, it is necessary to show the existence and uniqueness of a solution for

$$\mathcal{G}_{(u,v)}(0, u_0, v_0)(\xi, \eta) = \begin{bmatrix} f \\ g \end{bmatrix},$$

that is, for the problem

$$\begin{cases} -m\left(\int_\Omega v_0\right)\Delta\xi + M(x)\int_\Omega \eta + N(x)\xi = f & \text{in } \Omega, \\ -\Delta\eta + \sigma\eta - \rho\xi = g & \text{in } \Omega, \\ \eta = \xi = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.24}$$

Note that, (2.24) is a linear system with a nonlocal term in η. It is possible to transform this system into another one by multiplying the second equation by e_σ, obtaining

$$\int_\Omega \eta = \rho\int_\Omega e_\sigma\xi + \int_\Omega e_\sigma g.$$

Then, equation (2.24) becomes

$$\begin{cases} -m\left(\rho\int_\Omega e_\sigma u_0\right)\Delta\xi + N(x)\xi + \rho M(x)\int_\Omega e_\sigma\xi = h & \text{in } \Omega, \\ -\Delta\eta + \sigma\eta - \rho\xi = g & \text{in } \Omega, \\ \eta = \xi = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.25}$$

where $h(x) = f(x) - M(x)\int_\Omega e_\sigma g$. From expression (2.23), we obtain:

$$\sigma_1\left[-m\left(\rho\int_\Omega e_\sigma u_0\right)\Delta + N(x)\right] > 0. \tag{2.26}$$

Hence, there exists a unique positive solution, which we denote by H, of the following linear problem:

$$\begin{cases} -m\left(\rho\int_\Omega e_\sigma u_0\right)\Delta H + N(x)H = e_\sigma & \text{in } \Omega, \\ H = 0, & \text{on } \partial\Omega. \end{cases}$$

Multiplying the first equation of the previous problem by ξ and integrating over Ω, we obtain:

$$\int_\Omega Hh - \rho\int_\Omega HM\int_\Omega e_\sigma\xi = \int_\Omega e_\sigma\xi$$

and, consequently,

$$\int_\Omega e_\sigma \xi = \frac{\displaystyle\int_\Omega Hh}{1 + \rho \displaystyle\int_\Omega HM},$$

where $M \geq 0$ in Ω. Thus, the first equation of (2.25) is equivalent to

$$\begin{cases} -m\left(\rho \displaystyle\int_\Omega e_\sigma u_0\right) \Delta\xi + N(x)\xi = h - \rho M(x) \dfrac{\displaystyle\int_\Omega Hf}{1 + \rho \displaystyle\int_\Omega HM} & \text{in } \Omega, \\ \xi = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.27}$$

Finally, from (2.26), there exists a unique solution ξ of (2.27). Since $\sigma_1[-\Delta + \sigma] > 0$ there exists a unique solution η. □

2.5 Coexistence State for the Case $m(0) = 0$

In this section, we will examine the particular case where $m(0) = 0$. In this scenario, the results from the previous sections cannot be directly applied. Therefore, we need to consider the following auxiliary problem:

$$\begin{cases} -m_\varepsilon\left(\displaystyle\int_\Omega v\right) \Delta u = \lambda u - u^2 + cuv & \text{in } \Omega, \\ -\Delta v + \sigma v = \rho u & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{NP$_1$}$$

where

$$m_\varepsilon(s) := m(s) + \varepsilon, \quad \text{for } \varepsilon > 0.$$

Theorem 2.9. Assume that $m(0) = 0$, $c \leq 0$ (or $c > 0$) and that (H_1) (or (H_2)) is satisfied. Then, for $\lambda > 0$, (P_1) possess at least one coexistence state.

Proof. Note that, for each $\varepsilon > 0$, by Theorem 2.6 there exists a bounded continuum $\mathfrak{C}_\varepsilon \subset \mathbb{R} \times \mathrm{Int}(P_E) \times \mathrm{Int}(P_E)$ of coexistence states of (NP_1) that bifurcates from $(\lambda, u, v) = (\lambda_\varepsilon^*, 0, 0)$, where

$$\lambda_\varepsilon^* = m_\varepsilon(0)\lambda_1 = \varepsilon\lambda_1.$$

Furthermore, from Corollary 2.7,

$$(\varepsilon\lambda_1, \infty) \subset \mathrm{Proj}_{\mathbb{R}}(\mathfrak{C}_\varepsilon).$$

For fixed $\lambda > 0$, there exists $\varepsilon_0 > 0$ such that there exists at least one coexistence state $(u_\varepsilon, v_\varepsilon)$ of (P_1), for $\varepsilon < \varepsilon_0$. Note that, for $c \leq 0$,

$$-m\left(\int_\Omega v_\varepsilon\right) \Delta u_\varepsilon \geq u_\varepsilon(\lambda + (cv_\varepsilon)_L - u_\varepsilon)$$

and, for $c > 0$,

$$-m\left(\int_\Omega v_\varepsilon\right)\Delta u_\varepsilon \geq u_\varepsilon(\lambda - u_\varepsilon).$$

In both cases, u_ε is a super-solution of the following problem involving the logistic equation

$$\begin{cases} -d\Delta w = w(\mu - w) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

with $d = m\left(\int_\Omega v_\varepsilon\right)$. Remember that

$$\frac{\mu - d\lambda_1}{\|\varphi_1\|_\infty}\varphi_1 \leq w.$$

Thus

$$\frac{\lambda + (cv_\varepsilon)_L - m\left(\int_\Omega v_\varepsilon\right)\lambda_1}{\|\varphi_1\|_\infty}\varphi_1 \leq u_\varepsilon. \tag{2.28}$$

We will show that, for some constant independent of ε,

$$\|u_\varepsilon\|_\infty \leq C. \tag{2.29}$$

Suppose, first, that $c \leq 0$. In this case, from Proposition 2.1, we obtain

$$\|u_\varepsilon\|_\infty \leq \lambda,$$

where λ does not depend on ε. Now, for the case $c > 0$ we need to analyze when (H_1) or (H_2) is satisfied. For (H_1) and Proposition 2.2, we obtain

$$\|u_\varepsilon\|_\infty \leq C,$$

where $C > 0$ does not depend of ε. For (H_2), we can follow what was proven in Proposition 2.2 and conclude the same. Therefore, in both cases, (2.29) occurs. Finally, since u_ε is bounded in $L^\infty(\overline{\Omega})$, it follows that v_ε is bounded in $W^{2,p}(\Omega)$, for any $p > 1$. Consequently,

$$v_\varepsilon \to v^* \geq 0 \text{ in } C^1(\overline{\Omega}).$$

Suppose that $v^* \equiv 0$. From the equation (2.28)

$$u_\varepsilon \geq \frac{\lambda}{2\|\varphi_1\|_\infty}\varphi_1.$$

Then

$$-\Delta v_\varepsilon + \sigma v_\varepsilon = \rho u_\varepsilon \geq C\varphi_1$$

implying that $v^*(x) > 0$, for all $x \in \Omega$. Consequently,

$$m\left(\int_\Omega v_\varepsilon\right) \to m\left(\int_\Omega v^*\right) > 0.$$

Thus, u_ε is bounded in $W^{2,p}(\Omega)$ and, consequently,

$$u_\varepsilon \to u^* > 0 \text{ in } C^1(\overline{\Omega}).$$

Therefore, (u^*, v^*) is a coexistence state of (P_1). □

3 Modeling Population Dynamics in Lotka-Volterra Systems with Nonlocal Cross-Diffusivity Terms

In this chapter, we will study the existence of coexistence states for the following nonlocal elliptic system:

$$\begin{cases} -m\left(\int_\Omega v\right)\Delta u = \lambda u - u^2 - cuv & \text{in } \Omega, \\ -n\left(\int_\Omega u\right)\Delta v = \mu v - v^2 - duv & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{P$_2$}$$

where Ω is a bounded regular domain in $\mathbb{R}^N$, where $N \geq 1$, $c, d, \lambda, \mu \in \mathbb{R}$, and $m, n : \mathbb{R} \to [0, \infty)$ are continuous and differentiable functions.

Regarding the solution, Problem (P$_2$) admits three types of non-negative strong solutions, namely:

(S$_1$) Trivial solution $(0, 0)$.

(S$_2$) Semi-trivial solutions $(u, 0)$ and $(0, v)$, where $u \not\equiv 0$ and $v \not\equiv 0$ are positive solutions of

$$\begin{cases} -m\,(0)\,\Delta u = \lambda u - u^2 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

and

$$\begin{cases} -n\,(0)\,\Delta v = \mu v - v^2 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

respectively.

(S$_3$) Coexistence states (u, v), with both components non-negative and non-trivial. In fact, thanks to the Strong Maximum Principle any (u, v) non-negative and non-trivial solution of Problem (P$_2$) satisfies that $u, v \in \text{Int}(P_E)$.

From a population dynamics perspective, Problem (P$_2$) models the interaction between two populations, u and v, inhabiting the same habitat, Ω. In this context, $u(x)$ and $v(x)$

represent the population densities at point $x \in \Omega$. Since we are considering homogeneous Dirichlet boundary conditions, the habitat is surrounded by an uninhabitable region, $\partial\Omega$.

For the reaction functions, we adopt the classical Lotka-Volterra type reaction terms, where λ and μ denote the growth rates of each population, $-u^2$ and $-v^2$ represent the limiting effects caused by overcrowding within each population, and cuv and duv describe the growth limitations of each population by the presence of the other. Moreover, the signs of the constants c and d determine the type of interaction between the species: competition when $c, d > 0$, predator-prey when $c > 0 > d$, and cooperation when $c, d < 0$.

The terms $-m\left(\int_\Omega v\right)\Delta u$ and $-n\left(\int_\Omega u\right)\Delta v$ are referred to as diffusion terms and describe the spatial movement of each population. In this context, the expressions $m\left(\int_\Omega v\right)$ and $n\left(\int_\Omega u\right)$ indicating Cross-Diffusion, meaning that the diffusion rate of one species depends on the total population of the other.

We have divided this chapter as follows: In Section 3.1, a priori estimates related to the Problem (P_2) and results on the non-existence of coexistence states are established. In Section 3.2, we will examine the stability changes in semi-trivial solutions. Section 3.3, we will employ the Fixed Point Index Theory with respect to the Positive Cone to determine the existence of coexistence states. Section 3.4 is dedicated to studying the coexistence region obtained in the previous section. Finally, in Section 3.5, we will compare the results obtained for our problem with the Local Diffusion Lotka-Volterra System.

3.1 A Priori Bounds and Non-Existence Results of Coexistence States

In this section, we will investigate *a priori* estimates and non-existence results for coexistence states of Problem (P_2). To this end, we will analyze each type of population interaction separately.

We will consider the Ordered Banach Space $E := C_0^1(\overline{\Omega})$ and P_E its positive cone. We will begin by studying the Competition case.

Proposition 3.1. Assume that $c, d > 0$. If $(u, v) \in E^2$ is a coexistence state of Problem (P_2), then

$$u \leq \lambda \tag{3.3}$$

and

$$v \leq \mu. \tag{3.4}$$

Proof. Consider $(u, v) \in E^2$ a coexistence state of (P_2). Let x_M be a element of Ω such that $u_M = u(x_M)$. Since $-\Delta u(x_M) \geq 0$, by Maximum Principle (see Section 1.2),

$$-m\left(\int_\Omega v\right)\Delta u(x_M) = \lambda u_M - u_M^2 - c u_M v(x_M) \geq 0.$$

Dividing the previous expression by u_M and isolating λ, we obtain

$$\lambda \geq u_M + cv(x_M).$$

Since $c > 0$, it follows that

$$\lambda \geq u_{M} \geq u(x).$$

Similarly, it is possible to prove Expression (3.4). $\square$

We will now consider the Prey-Predator case.

Proposition 3.2. Assume that $d < 0 < c$. If $(u, v) \in E^2$ is a coexistence state of Problem (P_2), then

$$u \leq \lambda$$

and

$$v \leq \mu - d\lambda. \tag{3.5}$$

Proof. Consider $(u, v) \in E^2$ a coexistence state of (P_2). Since $c > 0$, by a similar reasoning to the used in Proposition 3.1, we obtain

$$u \leq \lambda.$$

Using this fact on the second equation of (P_2), it follows that

$$-n\left(\int_{\Omega} u\right)\Delta v \leq \mu v - v^2 - d\lambda v.$$

Let y_{M} be a element of Ω such that $v_{M} = v(y_{M})$. Since $-\Delta v(y_{M}) \geq 0$, by Maximum Principle,

$$0 \leq -n\left(\int_{\Omega} u\right)\Delta v_{M} \leq \mu v_{M} - v_{M}^2 - d\lambda v_{M}.$$

Dividing the previous expression by v_{M} and isolating this term, we obtain

$$v(x) \leq v_{M} \leq \mu - d\lambda.$$

This concludes the proof. $\square$

Finally, we will study the Cooperation case, that is, the case in which $c, d < 0$.

Proposition 3.3. Assume that $c, d < 0$ with $cd < 1$. If $(u, v) \in E^2$ is a coexistence state of Problem (P_2), then

$$u \leq \frac{\lambda - c\mu}{1 - cd} \tag{3.6}$$

and

$$v \leq \frac{\mu - d\lambda}{1 - cd}. \tag{3.7}$$

Proof. Consider $(u, v) \in E^2$ a coexistence state of (P_2). Let $x_M \in \Omega$ such that $u(x_M) = \max_{x\in\overline{\Omega}} u(x) = u(x_M)$. Since $-\Delta u(x_M) \geq 0$ and x_M is the maximum point,

$$0 \leq -m\left(\int_\Omega v\right)\Delta u(x_M) = \lambda u_M - u_M^2 - cu_M v(x_M).$$

Dividing the previous expression by u_M, isolating this term and using the fact that $v(x_M) \geq v(x)$ for all $x \in \Omega$, we obtain

$$u_M \leq \lambda - cv(x_M) \leq \lambda - cv_M. \tag{3.8}$$

By a similar reasoning,

$$v_M \leq \mu - du(y_M) \leq \mu - du_M. \tag{3.9}$$

Using the expression (3.9) in (3.8) and isolating u_M, it follows that

$$u(x) \leq u_M \leq \frac{\lambda - c\mu}{1 - cd}.$$

By a similar reasoning,

$$v(x) \leq v_M \leq \frac{\mu - d\lambda}{1 - cd}.$$

□

As a consequence of the bounds obtained above, we can conclude the following non-existence result for coexistence states.

Corollary 3.4. The following assertions hold:

(a) For the Competition case $(c, d > 0)$, Problem (P_2) does not possess coexistence state when $\lambda \leq 0$ or $\mu \leq 0$.

(b) For the Prey-Predator case $(d < 0 < c)$, Problem (P_2) does not possess coexistence state when $\lambda \leq 0$ or $\mu - d\lambda \leq 0$.

(c) For the Cooperative case $(c, d < 0)$ with $cd < 1$, Problem (P_2) does not possess coexistence state when $\lambda - c\mu \leq 0$ or $\mu - d\lambda \leq 0$.

3.2 Curves of Change of Stability

In this section, we will investigate the stability of the semi-trivial solutions of Problem (P_2) in the Competition case. The focus will be on analyzing the behavior of these solutions in response to small perturbations to determine whether they are linearly asymptotically stable or linearly asymptotically unstable.

Consider the real-valued maps $F : [m(0)\lambda_1, \infty) \to \mathbb{R}$ and $G : [n(0)\lambda_1, \infty) \to \mathbb{R}$, given explicitly by

$$F(\lambda) := \sigma_1\left[-n\left(\int_\Omega \theta_{[\lambda,m(0)]}\right)\Delta + c\theta_{[\lambda,m(0)]}\right] \tag{3.10}$$

and

$$G(\mu) := \sigma_1 \left[-m \left(\int_\Omega \theta_{[\mu,n(0)]} \right) \Delta + d\theta_{[\mu,n(0)]} \right]. \tag{3.11}$$

The following result characterizes the linear stability of the semi-trivial solution $(\theta_{[\lambda,m(0)]}, 0)$.

Proposition 3.5. The following assertions hold:

(a) If $\mu < F(\lambda)$, then the semi-trivial solution $(\theta_{[\lambda,m(0)]}, 0)$ is linearly asymptotically stable.

(b) If $\mu > F(\lambda)$, then the semi-trivial solution $(\theta_{[\lambda,m(0)]}, 0)$ is unstable.

Proof. Note that, the linearity stability of semi-trivial solution $(\theta_{[\lambda,m(0)]}, 0)$ is given by the sign of the real parts of the eigenvalues of the linearization of (P_2) at $(\theta_{[\lambda,m(0)]}, 0)$, that is, by the real parts of the τ's for which the following linear problem possess a solution $(\xi, \eta) \in E^2$, with $(\xi, \eta) \not\equiv 0_{E\times E}$,

$$\begin{cases} -m(0)\Delta\xi + 2\theta_{[\lambda,m(0)]}\xi - \lambda\xi + c\theta_{[\lambda,m(0)]}\eta - m'(0)\Delta\theta_{[\lambda,m(0)]} \int_\Omega \eta = \tau\xi & \text{in } \Omega, \\ -n \left(\int_\Omega \theta_{[\lambda,m(0)]} \right) \Delta\eta + d\theta_{[\lambda,m(0)]}\eta - \mu\eta = \tau\eta & \text{in } \Omega, \\ \xi = \eta = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.12}$$

<u>Assertion (a)</u>: Assume that $\mu < F(\lambda)$. We need to prove that the first eigenvalue of (3.12) is positive. To do this, we need to analyze two cases: $\eta \not\equiv 0$ and $\eta \equiv 0$. Suppose that $\eta \equiv 0$ in Ω. Note that, in this case, (3.12) becomes

$$\begin{cases} -m(0)\Delta\xi + 2\theta_{[\lambda,m(0)]}\xi - \lambda\xi = \tau\xi & \text{in } \Omega, \\ \xi = 0 & \text{on } \partial\Omega, \end{cases}$$

and, consequently,

$$\tau = \sigma_j \left[-m(0)\Delta + 2\theta_{[\lambda,m(0)]} - \lambda \right]$$

for some $j \geq 1$. Thus, by Proposition 1.28,

$$\tau \geq \sigma_1 \left[-m(0)\Delta + 2\theta_{[\lambda,m(0)]} - \lambda \right] > 0.$$

Now, suppose that $\eta \not\equiv 0$ in Ω. Note that, from second equation of (3.12),

$$\tau = \sigma_j \left[-n \left(\int_\Omega \theta_{[\lambda,m(0)]} \right) \Delta + d\theta_{[\lambda,m(0)]} \right] - \mu$$

for some $j \geq 1$. Since $\mu \leq F(\lambda)$, it follows that

$$\tau \geq F(\lambda) - \mu > 0.$$

Thus $\tau > 0$, that is, $(\theta_{[\lambda,m(0)]}, 0)$ is linearly asymptotically stable.
<u>Assertion (b)</u>: Assume that $\mu > F(\lambda)$. In this case

$$\overline{\tau} := \sigma_1 \left[-n \left(\int_\Omega \theta_{[\lambda,m(0)]} \right) \Delta + d\theta_{[\lambda,m(0)]} - \mu \right] < 0$$

is an eigenvalue corresponding to a positive eigenvalue $\overline{\psi}$ of the second equation of (3.12). Note that, the first equation of (3.12) can be rewritten as

$$\begin{cases} -m(0)\Delta\xi + 2\theta_{[\lambda,m(0)]}\xi - \lambda\xi - \overline{\tau}\xi = f(\overline{\psi}) & \text{in } \Omega, \\ \xi = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.13}$$

where $f(\overline{\psi}) = m'(0)\Delta\theta_{[\lambda,m(0)]}\int_\Omega \overline{\psi} - c\theta_{[\lambda,m(0)]}\overline{\psi}$. Since $\overline{\tau} < 0$, by Proposition 1.28,

$$\sigma_1\left[-m(0)\Delta + 2\theta_{[\lambda,m(0)]} - \lambda - \overline{\tau}\right] > 0.$$

Consequently, (3.13) admits a unique solution given by

$$\xi = \left(-m(0)\Delta + 2\theta_{[\lambda,m(0)]} - \lambda - \overline{\tau}\right)^{-1}(f(\overline{\psi})).$$

Thus, $\tau = \overline{\tau} < 0$ is an eigenvalue of (3.12), that is, $(\theta_{[\lambda,m(0)]}, 0)$ is unstable. □

Similarly, we can characterizes the linear stability of the semi-trivial solution $(0, \theta_{[\mu,n(0)]})$ as follows:

Proposition 3.6. The following assertions hold:

(a) If $\lambda < G(\mu)$, then the semi-trivial solution $(0, \theta_{[\mu,n(0)]})$ is linearly asymptotically stable.

(b) If $\lambda > G(\mu)$, then the semi-trivial solution $(0, \theta_{[\mu,n(0)]})$ is unstable.

Although we have only considered the case of Competition in the previous results, for the other cases, Predator-Prey and Cooperation, the conclusion remains the same, and the proof follows in an analogous manner.

3.3 Fixed Point Index Analysis with Respect to the Positive Cone

In this section, we will study the existence of coexistence states for Problem (P_2), in the Competition case, using the Fixed Point Theory with Respect to the Positive Cone. In Section 1.6, we conducted a thorough study of this theory, presenting the main concepts and results relevant for a clear understanding of the current section. For further details, we recommend reading the articles [28] and [52].

Given any fixed $t \in [0, 1]$, we will consider the following continuous family of problems arising from Problem (P_2):

$$\begin{cases} -m\left(t\int_\Omega v\right)\Delta u = \lambda u - u^2 - tcuv & \text{in } \Omega, \\ -n\left(t\int_\Omega u\right)\Delta v = \mu v - v^2 - tduv & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.14}$$

Let $(u_0, v_0) \in E^2$ be a coexistence state of Problem (3.14). Note that, by Propositions 3.1, 3.2, and 3.3, for any coexistence state (u, v), its components are bounded by terms presented in Problem (P_2). Consequently, there exist constants $C_1, C_2 > 0$ independent of $t \in [0,1]$ such that:

$$\|u_0\|_\infty \leq C_1 \qquad \text{and} \qquad \|v_0\|_\infty \leq C_2. \tag{3.15}$$

Taking $p > N$ such that $W^{2,p}(\Omega) \hookrightarrow E$, and using the first inequality of (3.15) in the first equation of Problem (3.14), by the *Agmon-Douglas-Nirenberg Theorem* (see [1]), there exists $C > 0$ independent of u_0 such that:

$$\|u_0\|_E \leq C \|u_0\|_{W^{2,p}} \leq C \frac{\|\lambda u_0 - u_0^2 - tcu_0v_0\|_{L^p}}{m\left(t\int_\Omega v_0\right)} \leq C_1,$$

with $C_1 > 0$ independent of t. Similarly, we obtain:

$$\|v_0\|_E \leq C_2,$$

with $C_2 > 0$ also independent of t.

We will now define two fundamental sets for applying the Fixed Point Theory with Respect to the Positive Cone in the study of the existence of coexistence states for Problem (P_2). These sets ensure that the elements we use have bounded norms in E. Define, for each $k \in \{0,1\}$,

$$N_k := \{w \in E;\ \|w\|_E < C_k + 1\}$$

and

$$N := N_1 \times N_2.$$

Given $M > 0$ large enough, it is possible to rewrite the first equation of Problem (3.14) as follows:

$$\begin{aligned}
-m\left(t\int_\Omega v\right)\Delta u = \lambda u - u^2 - tcuv \Rightarrow -\Delta u &= \frac{1}{m\left(t\int_\Omega v\right)}\left(\lambda u - u^2 - tcuv\right)\\
\Rightarrow -\Delta u + Mu &= Mu + \frac{1}{m\left(t\int_\Omega v\right)}\left(\lambda u - u^2 - tcuv\right)\\
\Rightarrow (-\Delta + M)\,u &= Mu + \frac{1}{m\left(t\int_\Omega v\right)}\left(\lambda u - u^2 - tcuv\right)\\
\Rightarrow u &= L\left[Mu + \frac{1}{m\left(t\int_\Omega v\right)}\left(\lambda u - u^2 - tcuv\right)\right],
\end{aligned}$$

where $L := (-\Delta + M)^{-1}$ under homogeneous Dirichlet conditions, which is possible because $M > 0$. Consequently,

$$M + \frac{1}{m\left(t\int_\Omega v\right)}(\lambda - u - tcv) > 0 \tag{3.16}$$

for all $(t, u, v) \in [0, 1] \times N$. Similarly, we obtain

$$M + \frac{1}{n\left(t \int_\Omega u\right)} (\mu - v - tdu) > 0. \tag{3.17}$$

Finally, through the previous considerations, we can define the operator $H : [0, 1] \times N \to E^2$ as follows:

$$H(t, u, v) := \begin{pmatrix} L\left[Mu + \dfrac{1}{m\left(t \int_\Omega v\right)} (\lambda u - u^2 - tcuv)\right] \\ L\left[Mv + \dfrac{1}{n\left(t \int_\Omega u\right)} (\mu v - v^2 - tduv)\right] \end{pmatrix}. \tag{3.18}$$

It is clear that, given the way the operator H was constructed, the fixed points of $H(t, \cdot, \cdot)$ correspond to the coexistence states of Problem (P_2).

In the next result, we will demonstrate several properties of the operator H, which will be crucial for the remaining results in this section.

Lemma 3.7. The operator $H : [0, 1] \times N \to E^2$ is well-defined, compact positive differentiable operator and is an admissible homotopy.

Proof. Note that, from expressions (3.16) and (3.17), H is well-defined, compact positive differentiable operator for a choice of $M > 0$ large enough. Moreover, due to the way the set N was defined, given $(u_0, v_0) \in \partial N$, it follows that

$$H(t, u_0, v_0) \not\equiv (u_0, v_0)$$

for all $t \in [0, 1]$. Consequently, H is an admissible homotopy. □

In the next result, we will address the index value with respect to the positive cone P_E of the operator $H(1, \cdot, \cdot)$ relative to the set N.

Theorem 3.8. Assume that $\lambda, \mu \in \mathbb{R}$. Then

$$i_{P_E^2}(H(1, \cdot, \cdot); N) = 1.$$

Proof. Observe that, from the properties of index and homotopy (see Section 1.1),

$$\begin{aligned} i_{P_E^2}(H(1, \cdot, \cdot); N) &= i_{P_E^2}(H(0, \cdot, \cdot); N) = i_{P_E^2}(H(0, \cdot, \cdot); N_1 \times N_2) \\ &= i_{P_E}(H(0, \cdot, \cdot); N_1) \cdot i_{P_E}(H(0, \cdot, \cdot); N_2) \\ &= \prod_{k=1}^{2} i_{P_E}(H_k; N_k) \end{aligned} \tag{3.19}$$

where

$$H_1(u) = L\left(Mu + \frac{\lambda u - u^2}{m(0)}\right) \qquad \text{and} \qquad H_2(v) = L\left(Mv + \frac{\mu v - v^2}{n(0)}\right).$$

We show now that, for each $k = \{1,2\}$, $i_{P_E}(H_k; N_k) = 1$ and, by expression (3.19), the result will be proved. Define the following homotopies

$$\begin{aligned} G_1 : [0,1] \times N_1 &\longrightarrow X \\ (t,u) \quad &\longmapsto G_1(t,u) = L\left(Mu + t\frac{\lambda u - u^2}{m(0)}\right) \end{aligned}$$

and

$$\begin{aligned} G_2 : [0,1] \times N_2 &\longrightarrow X \\ (t,v) \quad &\longmapsto G_2(t,v) = L\left(Mv + t\frac{\mu v - v^2}{n(0)}\right). \end{aligned}$$

These homotopies are defined such that $H_1(u) = G_1(1,u)$ and $H_2(v) = G_2(1,v)$. Then, by Property (B_3) of Theorem 1.9,

$$\begin{aligned} i_{P_E}(H_k(\cdot); N_k) &= i_{P_E}(G_k(1,\cdot); N_k) \\ &= i_{P_E}(G_k(0,\cdot); N_k) \\ &= i_{P_E}(G_k(0,\cdot); 0) \end{aligned}$$

for $k \in \{1,2\}$, where the last equality in the previous expression follows from the fact that 0 is the only solution of $G_k(0,u) = u$ in N_k. We claim that

$$\operatorname{spr}[G_k(0,\cdot)] < 1, \tag{3.20}$$

for $k \in \{1,2\}$. Consider the case $k = 1$, the case $k = 2$ follows in a similar way. Take $r \in \mathbb{R}$, $r \not\equiv 0$, such that

$$G_1'(0,u) = G_1(0,u) = ru,$$

with $u \in N_1$, $u \in P_E$, $u \not\equiv 0$. Then

$$\begin{cases} -\Delta u = M\left(\dfrac{1}{r} - 1\right)u & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $u \in P_E$ follows that $u \not\equiv 0$. Consequently

$$\lambda_1 = M\left(\frac{1}{r} - 1\right) > 0,$$

whence $r < 1$. Since $\operatorname{spr}[G_k(0,u)] < 1$, for each $k \in \{1,2\}$, $G_k(0,u)$ does not admits a positive eigenfunction associated to an eigenvalue greater than one and, by *Lemma 13.1* of [4], we get that

$$i_{P_E}(G_k(0,\cdot); N_k) = 1$$

for $k \in \{1,2\}$. Therefore,

$$i_{P_E^2}(H(1,\cdot,\cdot); N) = 1.$$

□

Now, we will show that $(0,0)$ is an isolated solution and determine the index value of the operator $H(1,0,0)$.

Theorem 3.9. Assume that $\lambda > m(0)\lambda_1$ or $\mu > n(0)\lambda_1$. Then $(0,0)$ is an isolated solution of $H(1,\cdot,\cdot)$ such that

$$i_{P_E^2}\left(H(1,\cdot,\cdot);(0,0)\right) = 0.$$

Proof. Observe that, the derivative of $H(1,\cdot,\cdot)$ with respect to (u,v) at $(0,0)$ is given by

$$H_{(u,v)}(1,0,0)(u,v) = \begin{pmatrix} L\left[Mu + \dfrac{\lambda u}{m(0)}\right] \\ L\left[Mv + \dfrac{\mu v}{n(0)}\right] \end{pmatrix}.$$

Since $\lambda > m(0)\lambda_1$ or $\mu > n(0)\lambda_1$, the operator $I - H_{(u,v)}(1,0,0)$ is invertible on P_E^2, that is, 1 is not an eigenvalue to a positive eigenfunction of $H_{(u,v)}(1,0,0)$. Note that, $H_{(u,v)}(1,0,0)$ admits an eigenvalue greater than one associated to a positive eigenfunction. Indeed, we want determite $r \in \mathbb{R}$, with $r \neq 0$, such that

$$H_{(u,v)}(1,0,0)(\varphi_1, 0) = r(\varphi_1, 0).$$

This is equivalent to

$$\lambda_1 = \frac{1}{r}\left(M + \frac{\lambda}{m(0)}\right) - M.$$

Since $\lambda > m(0)\lambda_1$, it follows that

$$r = \frac{M + \frac{\lambda}{m(0)}}{\lambda_1 + M} > \frac{M + \lambda_1}{M + \lambda_1} = 1.$$

Similarly, in the case $\mu > n(0)\lambda_1$ we can prove that $H_{(u,v)}(1,0,0)(0,\varphi_1) = \overline{r}(0,\varphi_1)$ for some $\overline{r} > 1$. Therefore, $H_{(u,v)}(1,0,0)$ admits an eigenvalue greater than one associated to a positive eigenfunction and, by *Lemma 13.1* of [4],

$$i_{P_E^2}\left(H(1,\cdot,\cdot);(0,0)\right) = 0.$$

□

The next step is to study the index of the semi-trivial solutions of Problem (P_2). To do this, we will need to introduce some notations and results from [25], which can also be found in [28, 52].

Define the spaces $X := E \times E$, $Y := P_E \times P_E$ and denote the semi-trivial solutions, for which we intend to determine the index, by $y_1 = (\theta_{[\lambda,m(0)]}, 0)$ and $y_2 = (0, \theta_{[\mu,n(0)]})$. We will also consider the following sets

$$W_y := \{x \in X;\ y + tx \in Y \text{ for some } t > 0\}$$

and

$$S_y := \{x \in W_y;\ -x \in \overline{W_y}\}.$$

It is easy to see that the sets defined above, in relation to points y_1 and y_2, are such that

$$W_{y_1} = E \times P_E, \qquad W_{y_1} = P_E \times E, \qquad S_{y_1} = E \times \{0\} \qquad \text{and} \qquad S_{y_2} = \{0\} \times E.$$

Furthermore, we will denote by $M_{y_1} = \{0\} \times E$ and $M_{y_2} = E \times \{0\}$ the complement of the sets S_{y_1} and S_{y_2}, respectively, and define the following continuous projections of the space X onto these sets as

$$\begin{aligned} P_{y_1} : \quad X &\longrightarrow \quad M_{y_1} \\ (u,v) &\longmapsto P_{y_1}(u,v) = (0,v) \end{aligned}$$

and

$$\begin{aligned} P_{y_2} : \quad X &\longrightarrow \quad M_{y_2} \\ (u,v) &\longmapsto P_{y_2}(u,v) = (u,0). \end{aligned}$$

The following result, developed by E. N. Dancer in 1983 (see *Lemma 2* of [25]), is widely used to determine the index of semi-trivial solutions.

Lemma 3.10. The following assertions hold:

(a) If $I - H_x(1,y)$ is an invertible operator on X and the spectral radius of $P_y H_x(1,y)|_{M_y}$ is greater than one, then $i_Y(H(1,\cdot);y) = 0$.

(b) If $I - H_x(1,y)$ is an invertible operator on X and the spectral radius of $P_y H_x(1,y)|_{M_y}$ is smaller than one, then $i_Y(H(1,\cdot);y) = (-1)^{\chi}$ where χ is the sum of the multiplicities of the eigenvalues of $H_x(1,y)$ greater than one.

(c) If $I - H_x(1,y)$ is an invertible in W_y intead of X and there exists $w \in W_y$ such that the equation $(I - H_x(1,y))(z) = w$ has no solution $z \in W_y$, then $i_Y(H(1,\cdot);y) = 0$.

In the next result, we will study the index of the semi-trivial solution $(\theta_{[\lambda,m(0)]}, 0)$ with respect to the parameter μ.

Theorem 3.11. Assume that $\lambda > m(0)\lambda_1$. Then

(a) $i_Y(H(1,\cdot,\cdot);(\theta_{[\lambda,m(0)]},0)) = 0$, when $\mu > F(\lambda)$.

(b) $i_Y(H(1,\cdot,\cdot);(\theta_{[\lambda,m(0)]},0)) = 1$, when $\mu < F(\lambda)$.

Proof. Observe that, the derivative of $H(1,\cdot,\cdot)$ with respect to (u,v) at $(\theta_{[\lambda,m(0)]},0)$ is given by

$$H_{(u,v)}(1,\theta_{[\lambda,m(0)]},0)(u,v) = \begin{pmatrix} L\left[Mu + A(u,v)\right] \\ L\left[Mv + B(v)\right] \end{pmatrix}$$

where the functions $A(u,v)$ and $B(v)$ in the operator above are defined by

$$A(u,v) = \frac{\lambda u - 2\theta_{[\lambda,m(0)]}u}{m(0)} - \frac{m(0)c\theta_{[\lambda,m(0)]}v + \left(\lambda\theta_{[\lambda,m(0)]} - \theta^2_{[\lambda,m(0)]}\right)m'(0)\int_\Omega v}{(m(0))^2},$$

and

$$B(v) = \frac{\mu v - d\theta_{[\lambda,m(0)]}v}{n\left(\int_\Omega \theta_{[\lambda,m(0)]}\right)}.$$

Assumption (a): To apply Item (c) of Lemma 3.10, we must first demonstrate that $I - H_{(u,v)}(1, \theta_{[\lambda,m(0)]}, 0)$ is invertible in W_{y_1}. Consider $(\xi, \eta) \in W_{y_1} = E \times P_E$ such that $H_{(u,v)}(1, \theta_{[\lambda,m(0)]}, 0)(\xi, \eta) = (\xi, \eta)$, that is,

$$L\left[M\xi + \frac{\lambda\xi - 2\theta_{[\lambda,m(0)]}\xi}{m(0)} - \frac{m(0)c\theta_{[\lambda,m(0)]}\eta + \left(\lambda\theta_{[\lambda,m(0)]} - \theta^2_{[\lambda,m(0)]}\right)m'(0)\int_\Omega \eta}{m^2(0)}\right] = \xi \quad (3.21)$$

and

$$L\left[M\eta + \frac{\mu\eta - d\theta_{[\lambda,m(0)]}\eta}{n\left(\int_\Omega \theta_{[\lambda,m(0)]}\right)}\right] = \eta. \quad (3.22)$$

From Equation (3.22), we derive the following problem

$$\begin{cases} -n\left(\int_\Omega \theta_{[\lambda,m(0)]}\right)\Delta\eta + d\theta_{[\lambda,m(0)]}\eta = \mu\eta & \text{in } \Omega, \\ \eta = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $(\xi, \eta) \in W_{y_1}$, we obtain that $\eta \in P_E$ and, consequently, $\mu = F(\lambda)$ when $\eta \not\equiv 0$, a contradiction. Since $\mu > F(\lambda)$, it follows that $\eta \equiv 0$. Using this fact in Equation (3.21), we derive the following problem

$$\begin{cases} -m(0)\Delta\xi + 2\theta_{[\lambda,m(0)]}\xi = \lambda\xi & \text{in } \Omega, \\ \xi = 0 & \text{on } \partial\Omega. \end{cases}$$

By Assertion (c) of Theorem 1.53, it follows that $\xi \equiv 0$. Then, $I - H_{(u,v)}(1, \theta_{[\lambda,m(0)]}, 0)$ is invertible in W_{y_1}. In particular, $(0, 0)$ is the only solution of $H_{(u,v)}(1, \theta_{[\lambda,m(0)]}, 0)(u, v) = (0, 0)$. Finally, let us prove that there exists $(u_0, v_0) \in W_{y_1}$ such that

$$\left(I - H_{(u,v)}(1, \theta_{[\lambda,m(0)]}, 0)\right)(z) = w$$

has no solution $z \in W_{y_1}$. We will suppose, by contradiction, there exits $(u_0, v_0) \in W_{y_1}$ such that $\eta \in P_E$, with $\eta \not\equiv 0$ and $L(\eta) \in P_E$. Note that, $\xi \in E$ satisfying

$$\left(I - H_{(u,v)}(1, \theta_{[\lambda,m(0)]}, 0)\right)(u_0, v_0) = (\xi, L(\eta)).$$

Since $\eta \in P_E$, it follows that

$$\Delta v_0 - \frac{\mu v_0 - d\theta_{[\lambda,m(0)]}v_0}{n\left(\int_\Omega \theta_{[\lambda,m(0)]}\right)} = \eta > 0$$

and, consequently,

$$-n\left(\int_\Omega \theta_{[\lambda,m(0)]}\right)\Delta v_0 + d\theta_{[\lambda,m(0)]}v_0 - \mu v_0 = f(\eta),$$

where $f(\eta) = n\left(\int_\Omega \theta_{[\lambda,m(0)]}\right)\eta > 0$. Thus, since $f(\eta) > 0$ we get that

$$\sigma_1\left[-n\left(\int_\Omega \theta_{[\lambda,m(0)]}\right)\Delta + d\theta_{[\lambda,m(0)]} - \mu\right] > 0,$$

or equivalently $F(\lambda) > \mu$, a contradiction. Therefore, by Item (c) of Lemma 3.10, the result follows.
Assumption (b): To apply Item (b) of Lemma (3.10), we must first demonstrate that $I - H_{(u,v)}(1, \theta_{[\lambda,m(0)]}, 0)$ is invertible in X. Consider $(\xi, \eta) \in X$ such that equations (3.21) and (3.22) hold. From Equation (3.22), we derive the following problem

$$\begin{cases} -\Delta\eta = \dfrac{\mu\eta - d\theta_{[\lambda,m(0)]}\eta}{n\left(\int_\Omega \theta_{[\lambda,m(0)]}\right)} & \text{in } \Omega, \\ \eta = 0 & \text{on } \partial\Omega. \end{cases}$$

In the case where $\eta \not\equiv 0$, we can deduce that

$$\mu = \sigma_j\left[-n\left(\int_\Omega \theta_{[\lambda,m(0)]}\right)\Delta + d\theta_{[\lambda,m(0)]}\eta\right] \geq F(\lambda).$$

Since $\mu < F(\lambda)$, it follows that $\eta \equiv 0$. Similarly, by applying the same reasoning as in the assumption above, we can conclude that $\xi \equiv 0$. Now, we need to show that the spectral radius of $P_{y_1}H_{(u,v)}(1, \theta_{[\lambda,m(0)]}, 0)|_{M_{y_1}}$ is smaller than one and $\chi = 0$. To do this, consider $T : E \to E$ given by

$$T(v) := P_{y_1}H_{(u,v)}(1, \theta_{[\lambda,m(0)]}, 0)(0, v).$$

Consider $\varphi > 0$ an eigenfunction associated to λ_1. We want to determine $r \in \mathbb{R}$ such that

$$T(\varphi) = r\varphi.$$

Observe that, the expression above is equivalent to

$$L\left[M\varphi + \frac{\mu\varphi - d\theta_{[\lambda,m(0)]}\varphi}{n\left(\int_\Omega \theta_{[\lambda,m(0)]}\right)}\right] = r\varphi \Leftrightarrow -\Delta\varphi + \frac{1}{r}\left(\frac{d\theta_{[\lambda,m(0)]} - \mu}{n\left(\int_\Omega \theta_{[\lambda,m(0)]}\right)}\right)\varphi = \left(\frac{M}{r} - M\right)\varphi$$

and, consequently,

$$\sigma_1\left[-\Delta + \frac{1}{r}\left(\frac{d\theta_{[\lambda,m(0)]} - \mu}{n\left(\int_\Omega \theta_{[\lambda,m(0)]}\right)} - M\right) + M\right] = 0.$$

Consider the application $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(s) := \sigma_1\left[-\Delta + \frac{1}{s}\left(\frac{d\theta_{[\lambda,m(0)]} - \mu}{n\left(\int_\Omega \theta_{[\lambda,m(0)]}\right)} - M\right) + M\right].$$

Note that, by the choose of M, this application is increasing in s. Since $\mu < F(\lambda)$, it follows that

$$f(1) = \sigma_1 \left[-\Delta + \frac{d\theta_{[\lambda,m(0)]} - \mu}{n\left(\int_\Omega \theta_{[\lambda,m(0)]}\right)} \right] > \sigma_1 \left[-\Delta + \frac{d\theta_{[\lambda,m(0)]} - F(\lambda)}{n\left(\int_\Omega \theta_{[\lambda,m(0)]}\right)} \right] > 0.$$

Since $f(r) = 0$, we conclude that $r < 1$, and consequently,

$$\operatorname{spr}\left[P_{y_1} H_{(u,v)}(1, \theta_{[\lambda,m(0)]}, 0)|_{M_{y_1}} \right] < 1.$$

Finally, let us show that $\chi = 0$. To do this, consider α an eigenvalue of $H_{(u,v)}(1, \theta_{[\lambda,m(0)]}, 0)$ with corresponding eigenfunction (φ, ψ), that is,

$$L \left[M\varphi + \frac{\lambda\varphi - 2\theta_{[\lambda,m(0)]}\varphi}{m(0)} - \frac{m(0)c\theta_{[\lambda,m(0)]}\psi + (\lambda\theta_{[\lambda,m(0)]} - \theta^2_{[\lambda,m(0)]}) \int_\Omega \psi}{m^2(0)} \right] = \alpha\varphi$$

and

$$L \left[M\psi + \frac{\mu\psi - d\theta_{[\lambda,m(0)]}\psi}{n\left(\int_\Omega \theta_{[\lambda,m(0)]}\right)} \right] = \alpha\psi.$$

We need to analyze two cases, namely: $\psi \not\equiv 0$ and $\psi \equiv 0$. For the first case, α is an eigenvalue of

$$L \left[M + \frac{\mu - d\theta_{[\lambda,m(0)]}}{n\left(\int_\Omega \theta_{[\lambda,m(0)]}\right)} \right].$$

Since $\operatorname{spr}\left[P_{y_1} H_{(u,v)}(1, \theta_{[\lambda,m(0)]}, 0)|_{M_{y_1}} \right] < 1$, it follows that α must necessarily be less than one, which implies that $\chi = 0$. For the second case, we have $\varphi \not\equiv 0$ and α is an eigenvalue of

$$L \left[M + \frac{\lambda - 2\theta_{[\lambda,m(0)]}}{m(0)} \right].$$

Using the same reasoning as before, but with $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(s) := \sigma_1 \left[-\Delta + \frac{1}{s}\left(\frac{2\theta_{[\lambda,m(0)]} - \lambda}{m(0)} - M \right) + M \right],$$

by Assertion (b) of Theorem 1.53, it follows that $\alpha < 1$, which implies that $\chi = 0$. Therefore, by Item (b) of Lemma 3.10, the assumption follows. □

Similarly, we can state the result for the study of the index of the semi-trivial solution $(0, \theta_{[\mu,n(0)]})$ with respect to the parameter λ, whose proof follows by symmetry.

Theorem 3.12. Assume that $\mu > n(0)\lambda_1$. Then

(a) $i_Y(H(1,\cdot,\cdot); (0, \theta_{[\mu,n(0)]})) = 0$, when $\lambda > G(\mu)$.

(b) $i_Y(H(1,\cdot,\cdot);(0,\theta_{[\mu,n(0)]})) = 1$, when $\lambda < G(\mu)$.

The next result provides conditions for the coexistence states to Problem (P_2) by analyzing the possible cases for λ and μ. This result is an immediate consequence of the study of the indices of the semi-trivial solutions that were conducted in the two previous theorems.

Theorem 3.13. The following assertions hold:

(a) Assume that $\lambda > m(0)\lambda_1$ and $\mu > n(0)\lambda_1$. Then, Problem (P_2) possesses at least one coexistence states when

$$(\lambda - G(\mu))(\mu - F(\lambda)) > 0. \tag{3.23}$$

(b) Assume that $\lambda > m(0)\lambda_1$ and $\mu \leq n(0)\lambda_1$. Then, Problem (P_2) possesses at least one coexistence state when $\mu > F(\lambda)$.

(c) Assume that $\lambda \leq m(0)\lambda_1$ and $\mu > n(0)\lambda_1$. Then, Problem (P_2) possesses at least one coexistence state when $\lambda > G(\mu)$.

Proof. We will prove only assumptions (a) and (b), assumption (c) follows by symmetry of the previous one.
Assertion (a): Observe that, there exist two semi-trivial solutions, $(\theta_{[\lambda,m(0)]}, 0)$ and $(0, \theta_{[\mu,n(0)]})$. By the concept of the sum of fixed-point indices, it follows that

$$i_Y(H_{(u,v)}(1,\cdot,\cdot);(\theta_{[\lambda,m(0)]},0)) + i_Y(H_{(u,v)}(1,\cdot,\cdot);(0,\theta_{[\mu,n(0)]})) = \begin{cases} 0, & \lambda > G(\mu) \text{ and } \mu > F(\lambda), \\ 2, & \lambda < G(\mu) \text{ and } \mu < F(\lambda). \end{cases}$$

Since the total index $i_{P_E^2}(H(1,\cdot,\cdot);N) = 1$ and the local index of the trivial solution $i_{P_E^2}(H(1,\cdot,\cdot);(0,0)) = 0$, it follows the existence of at least one coexistence states when (3.23) is satisfied.
Assertion (b): Observe that, the unique semi-trivial solution is $(\theta_{[\lambda,m(0)]}, 0)$ and $i_Y(H(1,\cdot,\cdot);(\theta_{[\lambda,m(0)]},0)) = 0$. Using the same reasoning as in the previous assumption, by summing indices, we conclude the result. □

3.4 Study of the Coexistence States Region

From the previous section, Theorem 3.13 provides us a region of coexistence in the plane $\lambda - \mu$. Note that, $\theta_{[\mu,n(0)]} \equiv 0$ when $\mu = n(0)\lambda_1$, and by extending this statement, $\theta_{[\mu,n(0)]} \equiv 0$ when $\mu \leq n(0)\lambda_1$. Thus,

$$G(\mu) = m(0)\lambda_1, \quad \text{when } \mu \leq n(0)\lambda_1.$$

Similarly, we can assert the same for the semi-trivial solution $\theta_{[\lambda,m(0)]}$, namely:

$$F(\lambda) = n(0)\lambda_1, \quad \text{when } \lambda \leq m(0)\lambda_1.$$

Thus, from Theorem 3.13, we can conclude the existence of a region of coexistence given by

$$R := \left\{(\lambda,\mu) \in \mathbb{R}^2;\ (\lambda - G(\mu))(\mu - F(\lambda)) > 0\right\}. \tag{3.24}$$

To gain information into the region R, we need to examine the maps F and G, which were defined in (3.10) and (3.11), respectively. Since both maps are similar, we will study only F.

In the next result, we will demonstrate some properties of the map F that are independent of the sign of the constant d.

Proposition 3.14. The following properties hold:

(a) The map F is a derivable function.

(b) $F(m(0)\lambda_1) = n(0)\lambda_1$.

(c) $\displaystyle\lim_{\lambda\to m(0)\lambda_1} F'(\lambda) = \frac{c}{\int_\Omega \varphi_1^2} + \frac{\lambda_1 n'(0)}{\int_\Omega \varphi_1^3}\int_\Omega \varphi_1.$

Proof. Property (a): The statement follows directly from the fact that the application $\lambda \longmapsto \theta_{[\lambda,m(0)]}$ is differentiable.
Property (b): Recall that $\theta_{[\lambda,m(0)]}$ is a positive solution of Problem (PL_1) with $\alpha = m(0)$. Note that, for $\lambda = m(0)\lambda_1$, it follows that $\theta_{[\lambda,m(0)]} \equiv 0$, and consequently,

$$F(m(0)\lambda_1) = \sigma_1\left[-n(0)\Delta\right] = n(0)\lambda_1.$$

Property (c): Consider $\varphi_{F(\lambda)} > 0$ an eigenfunction associated to the principal eigenvalue of $F(\lambda)$, that is

$$\begin{cases} -n\left(\int_\Omega \theta_{[\lambda,m(0)]}\right)\Delta\varphi_{F(\lambda)} + c\theta_{[\lambda,m(0)]}\varphi_{F(\lambda)} = F(\lambda)\varphi_{F(\lambda)} & \text{in } \Omega, \\ \varphi_{F(\lambda)} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.25}$$

Observe that, the derivative of the previous expression with respect to λ is given by

$$\begin{aligned} -n\left(\int_\Omega \theta_{[\lambda,m(0)]}\right)\Delta\varphi'_{F(\lambda)} + c\theta_{[\lambda,m(0)]}\varphi'_{F(\lambda)} - F(\lambda)\varphi'_{F(\lambda)} = &\left[n'\left(\int_\Omega \theta_{[\lambda,m(0)]}\right)\int_\Omega \theta'_{[\lambda,m(0)]}\right]\Delta\varphi_{F(\lambda)} \\ &+ F'(\lambda)\varphi_{F(\lambda)} - c\theta'_{[\lambda,m(0)]}\varphi_{F(\lambda)}. \end{aligned}$$

Multiplying the previous expression by φ_λ, integrating over Ω and taking the *Green Identity*, it follows that

$$F'(\lambda)\int_\Omega \varphi^2_{F(\lambda)} = -\left[n'\left(\int_\Omega \theta_{[\lambda,m(0)]}\right)\int_\Omega \theta'_{[\lambda,m(0)]}\right]\int_\Omega \varphi_{F(\lambda)}\Delta\varphi_{F(\lambda)} + c\int_\Omega \theta'_{[\lambda,m(0)]}\varphi^2_{F(\lambda)}$$

and, substituting the value of $\Delta\varphi_{F(\lambda)}$,

$$\begin{aligned} F'(\lambda)\int_\Omega \varphi^2_{F(\lambda)} = &\frac{n'\left(\int_\Omega \theta_{[\lambda,m(0)]}\right)}{n\left(\int_\Omega \theta_{[\lambda,m(0)]}\right)}\int_\Omega \theta'_{[\lambda,m(0)]}\int_\Omega \varphi_{F(\lambda)}\left[F(\lambda)\varphi_{F(\lambda)} - c\theta_{[\lambda,m(0)]}\varphi_{F(\lambda)}\right] \\ &+ c\int_\Omega \theta'_{[\lambda,m(0)]}\varphi^2_{F(\lambda)}. \end{aligned}$$

Since $\theta_{[m(0)\lambda_1,m(0)]} \equiv 0$, by Assertion (b) of Theorem 1.53, it follows that $\theta'_{[m(0)\lambda_1,m(0)]}$. Thus

$$\lim_{\lambda\to m(0)\lambda_1} F'(\lambda) = \frac{c}{\int_\Omega \varphi_1^2} + \frac{\lambda_1 n'(0)}{\int_\Omega \varphi_1^3}\int_\Omega \varphi_1.$$

Completing the proof of the result □

Through the previous result, we can state that the sign of $F'(m(0)\lambda_1)$ depends on both the sign and the size of d and $n'(0)$.

In the following results, we will study the behavior of the map $F(\lambda)$ at infinity, that is, when λ tends to $+\infty$ or $-\infty$. Note that, the presence of the nonlocal term $n\left(\int_\Omega \theta_{[\lambda,m(0)]}\right)$ breaks up the monotonicity of the map $F(\lambda)$. For this reason, we will need to separate the analysis into two cases, namely: $d>0$ and $d<0$.

Proposition 3.15. Assume that $d>0$. Suppose that the function n satisfies the following hypothesis:

$$n(s)\geq k_1 s^{-\alpha},\ \text{for all } s\geq s_0,\ \text{where } 0<\alpha<1,\ k_1>0 \text{ and } s_0 \text{ large enough.} \qquad (\mathrm{H}_+)$$

Then

$$\lim_{\lambda\to+\infty} F(\lambda)=+\infty.$$

Proof. We need to analyze two cases, namely: $n(s)\geq n_L>0$ and $n(s)\to 0$ as $s\to+\infty$. For the first case, observe that

$$F(\lambda)\geq \sigma_1\left[-n_L\Delta+d\theta_{[\lambda,m(0)]}\right]\to+\infty.$$

For the second case, multiplying the first line of Equation (3.25) by $m(0)\theta_{[\lambda,m(0)]}$ and the first line of Problem (PL_1), with $d=m(0)$ and $\gamma=\lambda$, by $n\left(\int_\Omega \theta_{[\lambda,m(0)]}\right)\varphi_\lambda$, then integrating both resulting expressions over Ω and substituting them, we obtain

$$\left(F(\lambda)m(0)-\lambda n\left(\int_\Omega \theta_{[\lambda,m(0)]}\right)\right)\int_\Omega \theta_{[\lambda,m(0)]}\varphi_\lambda=\left(dm(0)-n\left(\int_\Omega \theta_{[\lambda,m(0)]}\right)\right)\int_\Omega \theta^2_{[\lambda,m(0)]}\varphi_\lambda.$$

Since $n\left(\int_\Omega \theta_{[\lambda,m(0)]}\right)\to 0$ as $\lambda\to+\infty$, it follows that

$$F(\lambda)m(0)\geq \lambda n\left(\int_\Omega \theta_{[\lambda,m(0)]}\right).$$

Since $\theta_{[\lambda,m(0)]}\leq\lambda$, by hypothesis ($\mathrm{H}_+$), it follows that

$$F(\lambda)m(0)\geq k_1\lambda\left(\int_\Omega \theta_{[\lambda,m(0)]}\right)^{-\alpha}\geq k_1\lambda\left(\int_\Omega \lambda\right)^{-\alpha}=k_1\lambda^{1-\alpha}\left|\Omega\right|^{-\alpha}.$$

Therefore, $F(\lambda)\to+\infty$ as $\lambda\to+\infty$. □

The above result shows that $F(\lambda)\to+\infty$ when n does not decrease very quickly to zero as $\lambda\to+\infty$.

Proposition 3.16. Assume that $d<0$. The following assertions hold:

(a) Suppose that the function n satisfies the following hypothesis:

$$n(s)\leq k_2 s^{\alpha},\ \text{for all } s\geq s_0,\ \text{where } 0<\alpha<1,\ k_2>0 \text{ and } s_0 \text{ large enough.} \qquad (\mathrm{h}_-)$$

Then

$$\lim_{\lambda\to+\infty} F(\lambda)=-\infty.$$

(b) Suppose that the function n satisfies the following hypothesis:

$$n(s) \geq k_3 s^\alpha, \text{ for all } s \geq s_0, \text{ where } \alpha > 1, \; k_3 > 0 \text{ and } s_0 \text{ large enough.} \tag{H$_-$}$$

Then

$$\lim_{\lambda \to +\infty} F(\lambda) = +\infty.$$

Proof. Observe that, by Assertion (a) of Theorem 1.53, on one hand

$$F(\lambda) \geq \sigma_1 \left[-n\left(\int_\Omega \theta_{[\lambda, m(0)]} \right) \Delta + d\lambda \right] = n\left(\int_\Omega \theta_{[\lambda, m(0)]} \right) \lambda_1 + d\lambda.$$

and, on the other hand,

$$\begin{aligned} F(\lambda) &\leq \sigma_1 \left[-n\left(\int_\Omega \theta_{[\lambda, m(0)]} \right) \Delta + d\frac{\lambda - m(0)\lambda_1}{\|\varphi_1\|_\infty} \varphi_1 \right] \\ &\leq \sigma_1^B \left[-n\left(\int_\Omega \theta_{[\lambda, m(0)]} \right) \Delta + \frac{d}{2}(\lambda - m(0)\lambda_1) \right]. \end{aligned}$$

In the last expression of the inequality above, we take $\|\varphi_1\|_\infty = 1$ and B as a subdomain of Ω such that $\frac{1}{2} \leq \varphi_1 \leq 1$. Consequently,

$$n\left(\int_\Omega \theta_{[\lambda, m(0)]} \right) \lambda_1 + d\lambda \leq F(\lambda) \leq \sigma_1^B \left[-n\left(\int_\Omega \theta_{[\lambda, m(0)]} \right) \Delta + \frac{d}{2}(\lambda - m(0)\lambda_1) \right].$$

<u>Assertion (a)</u>: Note that

$$\begin{aligned} n\left(\int_\Omega \theta_{[\lambda, m(0)]} \right) &\leq k_2 \left(\int_\Omega \theta_{[\lambda, m(0)]} \right)^\alpha \\ &\leq k_2 \lambda^\alpha |\Omega|^\alpha. \end{aligned}$$

Consequently

$$\sigma_1^B \left[-n\left(\int_\Omega \theta_{[\lambda, m(0)]} \right) \Delta + \frac{d}{2}(\lambda - m(0)\lambda_1) \right] \leq k_2 \lambda_1 \lambda^\alpha |\Omega|^\alpha + \frac{d}{2}(\lambda - m(0)\lambda_1),$$

which goes to $-\infty$ as $\lambda \to +\infty$. Thus $F(\lambda) \to -\infty$ as $\lambda \to +\infty$.
<u>Assertion (b)</u>: By Assertion (a) of Theorem 1.53

$$\begin{aligned} n\left(\int_\Omega \theta_{[\lambda, m(0)]} \right) &\geq k_3 \left(\int_\Omega \theta_{[\lambda, m(0)]} \right)^\alpha \\ &\geq k_3 (\lambda - m(0)\lambda_1)^\alpha \int_\Omega \varphi_1^\alpha. \end{aligned}$$

Consequently

$$\begin{aligned} n\left(\int_\Omega \theta_{[\lambda, m(0)]} \right) \lambda_1 + d\lambda &\geq k_3 \lambda_1 (\lambda - m(0)\lambda_1)^\alpha \int_\Omega \lambda d \\ &\geq k_3 \lambda_1 (\lambda - m(0)\lambda_1)^\alpha d\lambda \, |\Omega|, \end{aligned}$$

which goes to $+\infty$ as $\lambda \to +\infty$. Thus $F(\lambda) \to +\infty$ as $\lambda \to +\infty$. □

The above result shows that the behavior of the map F at infinity differs depending on the behavior of the function n.

By symmetry, we can obtain the following results, which addresses the behavior of the map G at infinity.

Proposition 3.17. Assume that $c > 0$ and that the function m satisfies hypothesis (H_+). Then

$$\lim_{\mu \to +\infty} G(\mu) = +\infty.$$

Proposition 3.18. Assume that $c < 0$

(a) Suppose that the function m satisfies hypothesis (h_-). Then

$$\lim_{\mu \to +\infty} G(\mu) = -\infty.$$

(b) Suppose that the function m satisfies hypothesis (H_-). Then

$$\lim_{\mu \to +\infty} G(\mu) = +\infty.$$

In the following figures, we will highlight some examples of the coexistence region R according to the assumptions considered for the maps F and G. In all of them, the gray area represents the coexistence region, the blue line indicates the image of the map F, and the red line indicates the image of the map G.

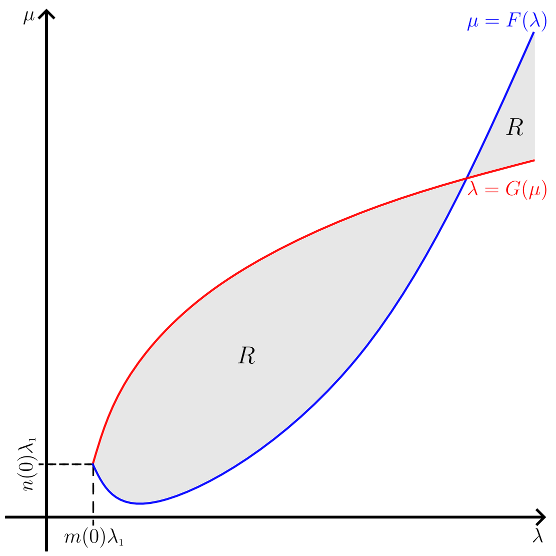
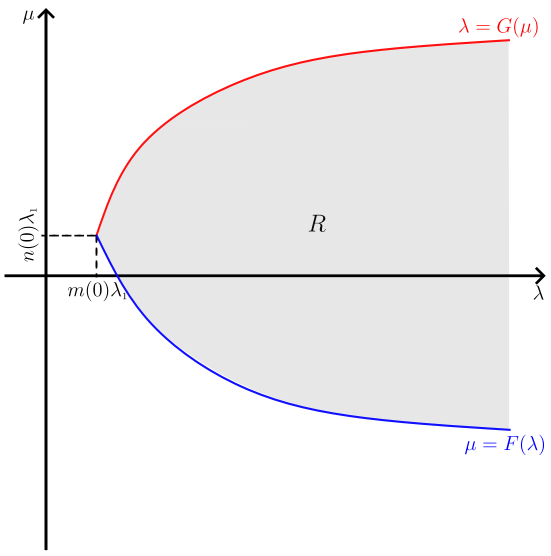


Figure 3.1: The coexistence region of Problem (P_2) in the Competition Case. In this case, we assume that m and n verifies the Condition (H_+). Additionally, it is assumed that $n'(0) < 0$ such that $F'(m(0)\lambda_1) < 0$. Under these conditions, it is notable that a coexistence state exists for $\mu < n(0)\lambda_1$.

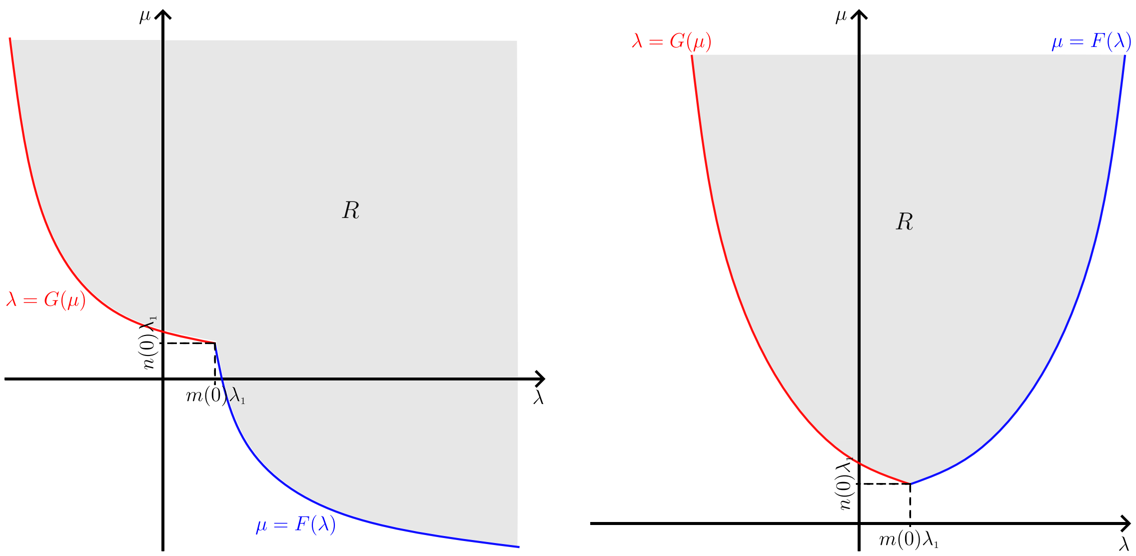
Source: Prepared by the author.



(a) In this case, we assume that m verifies the Condition (H_+) and n the Condition (h_-).

(b) In this case, we assume that m verifies the Condition (H_+) and n Condition (H_-).

Figure 3.2: Coexistence region of Problem (P_2) in the Prey-Predator Case.
Source: Prepared by the author.



(a) In this case, we assume that m and n verifies the Condition (h_-).

(b) In this case, we assume that m verifies the Condition (H_+) and n Condition (H_-).

Figure 3.3: Coexistence region of Problem (P_2) in the Symbiosis Case.
Source: Prepared by the author.

3.5 Comparison of the Proposed Model and the Local Lotka-Volterra System

In this section, we will compare the results obtained in the previous sections with the classical Lotka-Volterra model with local diffusion. To simplify the analysis, we will consider the case $m \equiv 1$. In this way, we will compare our model:

$$\begin{cases} -\Delta u = \lambda u - u^2 - cuv & \text{in } \Omega, \\ -n\left(\int_\Omega u\right)\Delta v = \mu v - v^2 - duv & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{NP$_2$}$$

with the classical model of the local diffusion Lotka-Volterra system:

$$\begin{cases} -\Delta u = \lambda u - u^2 - cuv & \text{in } \Omega, \\ -\Delta v = \mu v - v^2 - duv & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.26}$$

To perform this comparison, we will consider the coexistence region of problems (NP$_2$) and (3.26), that is, the region R defined by Expression (3.24) and the region defined by the following expression:

$$R_1 := \{(\lambda,\mu) \in \mathbb{R}^2;\ (\lambda - G_1(\mu))(\mu - F_1(\lambda)) > 0\}$$

where

$$F_1(\lambda) := \sigma_1[-\Delta + d\theta_\lambda] \qquad \text{and} \qquad G_1(\mu) := \sigma_1[-\Delta + c\theta_\mu],$$

θ_λ and θ_μ denote the positive solutions of Logistic Problem with $\gamma = \lambda$ and $\gamma = \mu$, respectively.

Several works address results on Problem (3.26) and region R_1. We specifically recommend [32, 52] for the Competition case, [26, 53] in the Prey-Predator case, and [30] in the Cooperation case.

The following results summarizes the main properties of the mappings F_1 and G_1 used to determine the behavior of region R_1.

Proposition 3.19. Assume that $d > 0$, respectively $d < 0$. The following assertions hold:

(a) The application $[\lambda_1, +\infty) \ni \lambda \longmapsto F_1(\lambda) \in \mathbb{R}$ is increasing, respectively decreasing.

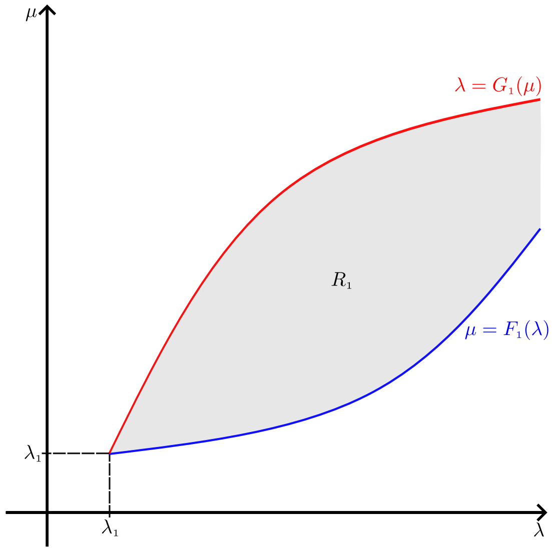
(b) $\lim_{\lambda \to +\infty} F_1(\lambda) = +\infty$, respectively $\lim_{\lambda \to +\infty} F_1(\lambda) = -\infty$.

Proposition 3.20. Assume that $c > 0$, respectively $c < 0$. The following assertions hold:

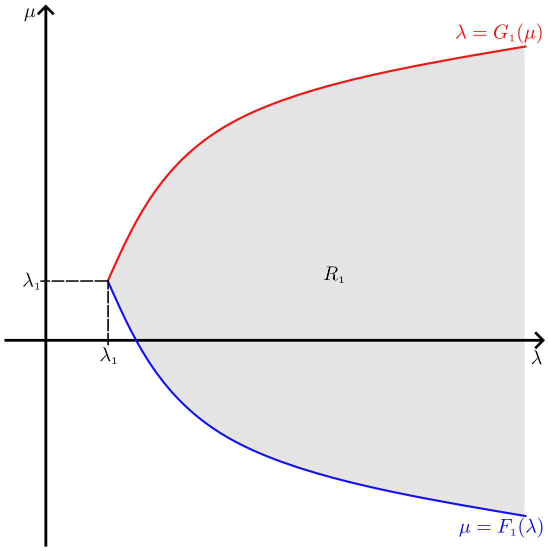
(a) The application $[\lambda_1, +\infty) \ni \mu \longmapsto G_1(\mu) \in \mathbb{R}$ is increasing, respectively decreasing.

(b) $\lim_{\mu \to +\infty} G_1(\mu) = +\infty$, respectively $\lim_{\mu \to +\infty} G_1(\mu) = -\infty$.

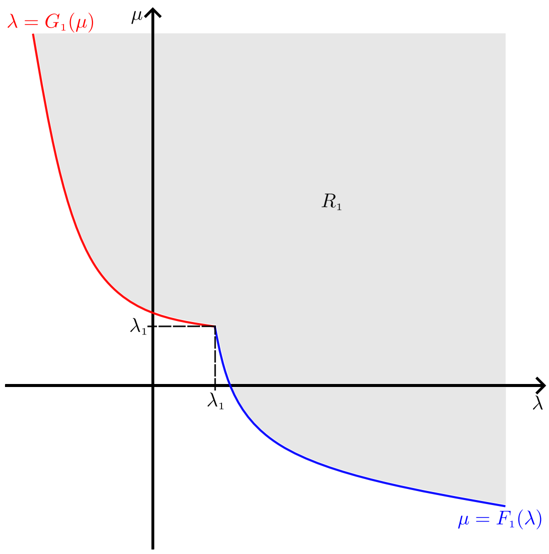
The following figures illustrate the behavior of region R_1 in three cases: Competition, Prey-Predator and Cooperation.



(a) Competition case.



(b) Prey-Predator case.



(c) Cooperation case

Figure 3.4: The coexistence region of (3.26).

Source: Prepared by the author.

Competition Case

We will begin by studying the Competition case, that is, the case in which $c, d > 0$.

In this case, the regions R and R_1 are quite similar (see Figure 3.1 and Figure 3.4-(a)). The primary difference is that, in the Problem (NP_2), positive solutions can exist for $\mu < n(0)\lambda_1$, whereas in the Problem (3.26), this is not possible. The first result of this section guarantees that Problem (NP_2) also satisfies the *Principle of Competitive Exclusion*.

Proposition 3.21. Assume that the function n satisfies the hypothesis (H_+). For fixed μ, the Problem (NP_2) does not possess coexistence states for λ large enough.

Proof. Suppose, by contradiction, that there exists at least one coexistence state (u, v) for the Problem (NP_2) for λ large enough. Since (u, v) is a coexistence state, we have that

$$\mu = \sigma_1\left[-n\left(\int_\Omega u\right)\Delta + v + du\right]. \tag{3.27}$$

By Proposition 3.1 and the fact that the function n satisfies hypothesis (H_+), it follows that

$$\begin{aligned} n\left(\int_\Omega u\right) &\geq k_1\left(\int_\Omega u\right)^\alpha \\ &\geq k_1\left(\int_\Omega \lambda\right)^\alpha \\ &\geq k_1\,|\Omega|^{-\alpha}\,\lambda^{-\alpha}. \end{aligned} \tag{3.28}$$

On the other hand, once again by Proposition 3.1, we get

$$-\Delta u \geq \lambda u - u^2 - c\mu u$$

and, consequently,

$$u \geq \theta_{\lambda - c\mu}, \tag{3.29}$$

where $\theta_{\lambda-c\mu}$ denotes the positive solution of Logistic Problem with $\gamma = \lambda - c\mu$. Hence, Using equations (3.28) and (3.29) in (3.27), we obtain

$$\mu \geq \sigma_1\left[-k_1\,|\Omega|^{-\alpha}\,\lambda^{-\alpha} + v + d\theta_{\lambda-c\mu}\right] =: R(\lambda) \tag{3.30}$$

Denote by $\varphi_{R(\lambda)}$ an positive eigenfunction associated to $R(\lambda)$. Note that, multiplying the equation that satisfies $\varphi_{R(\lambda)}$ by $\theta_{\lambda-c\mu}$, we obtain

$$R(\lambda)\varphi_{R(\lambda)}\theta_{\lambda-c\mu} = -(k_1\,|\Omega|^{-\alpha}\,\lambda^{-\alpha})\Delta\varphi_{R(\lambda)}\theta_{\lambda-c\mu} + d\varphi_{R(\lambda)}\theta^2_{\lambda-c\mu} \tag{3.31}$$

Furthermore, multiplying the equation of Logistic Problem, with $\gamma = \lambda - c\mu$, by $k_1\,|\Omega|^{-\alpha}\,\lambda^{-\alpha}\varphi_{R(\lambda)}$, we get

$$-(k_1\,|\Omega|^{-\alpha}\,\lambda^{-\alpha})\Delta\theta_{\lambda-c\mu}\varphi_{R(\lambda)} = (k_1\,|\Omega|^{-\alpha}\,\lambda^{-\alpha})(\lambda - c\mu)\theta_{\lambda-c\mu}\varphi_{R(\lambda)} - (k_1\,|\Omega|^{-\alpha}\,\lambda^{-\alpha})\theta^2_{\lambda-c\mu}\varphi_{R(\lambda)}. \tag{3.32}$$

Substituting Equation (3.32) in (3.31), integrating over Ω, and rearranging the terms, we obtain

$$\left[R(\lambda) - k_1\,|\Omega|^{-\alpha}\,\lambda^{-\alpha}(\alpha - c\mu)\right]\int_\Omega \theta_{\lambda-c\mu}\varphi_{R(\lambda)} = (b - k_1\,|\Omega|^{-\alpha}\,\lambda^{-\alpha})\int_\Omega \theta^2_{\lambda-c\mu}\varphi_{R(\lambda)}$$

and, consequently,

$$R(\lambda) \geq k_1\,|\Omega|^{-\alpha}\,\lambda^{-\alpha}(\alpha - c\mu).$$

Since $\alpha < 1$, it follows that $R(\lambda) \to +\infty$ as $\lambda \to +\infty$, a contraction with (3.30). Therefore, the Problem (NP_2) does not possess coexistence states for λ large enough. □

By symmetry, we can prove the next result, obtained by following the same steps as the previous proof, when λ is fixed.

Proposition 3.22. For fixed λ, the Problem (NP_2) does not possess coexistence states for μ large enough.

Proof. Suppose, by contradiction, that there exists at least one coexistence state (u, v) for the Problem (NP_2) for μ is large enough. Since (u, v) is a coexistence state, we have that

$$\lambda = \sigma_1 [-\Delta + u + cv] .$$

By Proposition 3.1, it follows that

$$-n\left(\int_\Omega u\right) \Delta v \geq \mu v - v^2 - d\lambda v$$

and, consequently,

$$v \geq \left[\mu - c\lambda - \lambda_1 n\left(\int_\Omega u\right)\right] \varphi_1.$$

Thus

$$\lambda \geq \sigma_1 \left[-\Delta + c\left[\mu - c\lambda - \lambda_1 n\left(\int_\Omega u\right)\right] \varphi_1\right] \to +\infty$$

as $\mu \to +\infty$, a contraction. Therefore, the Problem (NP_2) does not possess coexistence states for μ large enough. □

Prey-Predator Case

We will now consider the Prey-Predator case, that is, the case in which $c > 0 > d$.

In this case, the regions R and R_1 are quite similar when n satisfies the Hypothesis (h_-) (see Figure 3.2-(a) and Figure 3.4-(b)). However, when n verifies the Hypothesis (H_-), a significant difference emerges between the two cases. In Problem (3.26), even with a negative growth rate of the predator, μ, both species coexist for large values of λ (see Figure 3.2-(b)). In contrast, Problem (NP_2), for fixed μ, the species do not coexist for large values of λ (see Figure 3.2-(a)). Indeed, we can establish the following result:

Proposition 3.23. Assume that $1 + cd > 0$ and the function n satisfies the hypothesis (H_-). For fixed μ, the Problem (NP_2) does not possess coexistence states for λ large enough.

Proof. Suppose, by contradiction, that there exists at least one coexistence state (u, v) for the Problem (NP_2) for λ is large enough. Since (u, v) is a coexistence state, we have that

$$\mu = \sigma_1 \left[-n\left(\int_\Omega u\right) \Delta + v - du\right]. \tag{3.33}$$

By Proposition 3.2, it follows that

$$-\Delta u \geq u(\lambda(1 - cd) - u - c\mu)$$

and, by Assertion (a) of Theorem (1.53),

$$u \geq (\lambda(1+cd) - \lambda_1 - c\mu)\varphi_1.$$

Using the fact that the function n satisfies hypothesis (H_-), it follows that

$$\begin{aligned} n\left(\int_\Omega u\right) &\geq k_3 \left(\int_\Omega u\right)^\alpha \\ &\geq k_3(\lambda(1+cd) - \lambda_1 - c\mu)^\alpha \int_\Omega \varphi_1^\alpha. \end{aligned} \tag{3.34}$$

Substituting de expression (3.34) in (3.33) and using Proposition 3.2, we obtain

$$\mu \geq k_3\lambda_1(\lambda(1-cd) - \lambda_1 - c\mu)^\alpha \int_\Omega \varphi_1^\alpha - d\lambda \to -\infty$$

as $\lambda \to +\infty$, a contraction. Therefore, the Problem (NP_2) does not possess coexistence states for λ large enough. □

Cooperation Case

Finally, we will study the Cooperation case, that is, the case in which $c, d < 0$.

In this case, the regions R and R_1 are quite similar when n satisfies the Hypothesis (h_-), but they differ significantly when n satisfies the Hypothesis (H_-) (see Figure 3.4 and Figure 3.4-(b)). Once again, for small values of μ, even negative, in the Problem (3.26) (see Figure 3.4-(c), the species coexist for large values of λ due to cooperation. However, this does not happen in the Problem (NP_2) when n satisfies the Hypothesis (H_-), as illustrated in Figure 3.4-(b). Here, the species v leaves the densely populated regions, failing to benefit from cooperation. Indeed, using a reasoning similar to that in Proposition 3.21, we can establish:

Proposition 3.24. Assume that $cd < 1$ and the function n satisfies the hypothesis (H_-). For fixed μ, the Problem (NP_2) does not possess coexistence states for λ large enough.

4 Modeling Population Dynamics in Lotka-Volterra Systems with Nonlocal Coefficient Diffusion

In this chapter, we will study the existence of coexistence states for the following nonlocal elliptic system:

$$\begin{cases} -m\left(\int_\Omega u\right)\Delta u = \lambda u - u^2 - cuv & \text{in } \Omega, \\ -n\left(\int_\Omega v\right)\Delta v = \mu v - v^2 - duv & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{P$_3$}$$

where Ω is a bounded regular domain in $\mathbb{R}^N$, where $N \geq 1$, $c, d, \lambda, \mu \in \mathbb{R}$, and $m, n : \mathbb{R} \to [0, \infty)$ are continuous functions.

Regarding the solution, Problem (P$_3$) admits three types of non-negative strong solutions, namely:

(S$_1$) Trivial solution $(0, 0)$.

(S$_2$) Semi-trivial solutions $(u, 0)$ and $(0, v)$, where $u \not\equiv 0$ and $v \not\equiv 0$ are positive solutions of

$$\begin{cases} -m\left(\int_\Omega u\right)\Delta u = \lambda u - u^2 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

and

$$\begin{cases} -n\left(\int_\Omega v\right)\Delta v = \mu v - v^2 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.2}$$

respectively.

(S$_3$) Coexistence states (u, v), with both components non-negative and non-trivial. In fact, thanks to the Strong Maximum Principle any (u, v) non-negative and non-trivial solution of Problem (P$_3$) satisfies that $u, v \in \text{Int}(P_E)$.

Note that, for Problem (P$_2$), the semi-trivial equations (3.1) and (3.2) admit unique positive solutions. In contrast, Problem (P$_3$), semi-trivial (4.1) and (4.2) include a nonlocal diffusion term depending on the solution itself, in general, in this case the uniqueness of solution is not guaranteed.

4.1 A Priori Bounds and Non-Existence Results of Coexistence States

In this section, we will investigate *a priori* estimates and non-existence results for coexistence states of Problem (P_3). To this end, we will analyze each type of population interaction separately.

We will consider the Ordered Banach Space $E := C_0^1(\overline{\Omega})$ and P_E its positive cone. The following result provides a priori bounds in the case of competition, with the proof proceeding similarly to that of Proposition 3.1.

Proposition 4.1. Assume that $c, d > 0$. If $(u, v) \in E^2$ is a coexistence state of Problem (P_3), then

$$u \leq \lambda \tag{4.3}$$

and

$$v \leq \mu. \tag{4.4}$$

As a consequence of these bound, we obtain the following nonexistence result for coexistence states.

Corollary 4.2. Assume that $c, d > 0$. Then Problem (P_3) does not possess coexistence state when $\lambda \leq 0$ or $\mu \leq 0$.

To conclude this section, we investigate the existence of a solution under the condition of a sufficiently large parameter λ and $\mu > 0$ fixed.

Proposition 4.3. Fix $\mu > 0$. Then, there is no positive solution to Problem (P_3) for sufficiently large values of λ. Analogously, for $\lambda > 0$ fixed, there is no positive solution to Problem (P_3) for sufficiently large values of μ.

Proof. Let us fix $\mu > 0$ and suppose, by contradiction, that there exists at least one coexistence state (u, v) for Problem (P_3) for sufficiently large values of λ. Since (u, v) is a coexistence state, we have that

$$\mu = \sigma_1 \left[-n \left(\int_\Omega v \right) \Delta + v + du \right].$$

Note that, by Proposition 1.28 and Proposition 4.1, we have

$$\mu \geq \sigma_1 \left[-n_\mu \Delta + du \right], \tag{4.5}$$

where $n_\mu := \min_{0 \leq v \leq \mu} n \left(\int_\Omega v \right)$. On the other hand, once again by Proposition 4.1,

$$-m \left(\int_\Omega u \right) \Delta \geq u \left(\lambda - c\mu - u. \right).$$

Hence u is a supersolution of Problem (PL_1), with $\alpha = m \left(\int_\Omega u \right)$ and $\gamma = \lambda - c\mu$, we have that

$$u \geq \theta_{[\gamma,\alpha]}$$

and, consequently, by Assertion (a) of Theorem 1.53, we get

$$u \geq (\gamma - \alpha\lambda_1)\varphi_1 = \left[\lambda - c\mu - \lambda_1 m\left(\int_\Omega u\right)\right]\varphi_1. \tag{4.6}$$

Integrating the previous expression over Ω and rearranging the terms, we obtain

$$\int_\Omega u + \lambda_1 m\left(\int_\Omega u\right)\|\varphi_1\|_1 \geq (\lambda - c\mu)\|\varphi_1\|_1$$

and, consequently,

$$\lim_{\lambda\to+\infty}\int_\Omega u = +\infty. \tag{4.7}$$

We need to analyze two cases based on the value attained by the maximum of the function m, namely: $m_M < +\infty$ and $m_M = +\infty$. For the first case, by Expression (4.6),

$$\begin{aligned} u(x) &\geq \left[\lambda - c\mu - \lambda_1 m\left(\int_\Omega u\right)\right]\varphi_1(x) \\ &\geq [\lambda - c\mu - m_M]\,\varphi_1(x). \end{aligned}$$

Note that, using Proposition 1.28 on Expression (4.1), we get

$$\mu \geq \sigma_1\left[-n_\mu\Delta + d\left(\lambda - c\mu - m_M\right)\varphi_1\right]$$

and, by the reasoning presented in [49], we obtain

$$\lim_{\lambda\to+\infty}\sigma_1\left[-n_\mu\Delta + d\left(\lambda - c\mu - m_M\right)\varphi_1\right] = +\infty,$$

a contradiction. For the second case, by Expression (4.1),

$$\lim_{\lambda\to+\infty} m\left(\int_\Omega u\right) = +\infty. \tag{4.8}$$

On the other hand, dividing the first equation of Problem (P_3) by $\left[m\left(\int_\Omega u\right)\right]^2$ and applying Proposition 4.1,

$$\begin{aligned} -\Delta\left(\frac{u}{m\left(\int_\Omega u\right)}\right) &= \frac{u}{m\left(\int_\Omega u\right)}\left[\frac{\lambda - cv}{m\left(\int_\Omega u\right)} - \frac{u}{m\left(\int_\Omega u\right)}\right] \\ &\geq \frac{u}{m\left(\int_\Omega u\right)}\left[\frac{\lambda - c\mu}{m\left(\int_\Omega u\right)} - \frac{u}{m\left(\int_\Omega u\right)}\right] \end{aligned}$$

and, consequently, $\dfrac{u}{m\left(\int_\Omega u\right)}$ is a supersolution of Problem (PL_1) with $\alpha = 1$ and $\gamma = \dfrac{\lambda - c\mu}{m\left(\int_\Omega u\right)}$. Thus

$$u \geq m\left(\int_\Omega u\right)\theta_{[\gamma,1]}. \tag{4.9}$$

Let us analyze the previous expression in the following two cases:

1. Assume that $\gamma \geq b \geq \lambda_1$, for some positive constant b. Thus, by Expression (4.9) and Assertion (a) of Theorem 1.53, we get

$$u \geq m\left(\int_\Omega u\right)\theta_{[\gamma,1]} \geq m\left(\int_\Omega u\right)\theta_{[b,1]}$$

Combining the Expression and Expression , by the reasoning presented in [49], we obtain

$$\mu \geq \sigma_1\left[-n_\mu\Delta + dm\left(\int_\Omega u\right)\theta_{[b,1]}\right] \to +\infty \quad \text{as} \quad \lambda \to +\infty,$$

a contradiction.

2. Assume that there exists a sequence (λ_n) such that $\lambda_n \to +\infty$ and

$$\gamma_n = \frac{\lambda_n - c\mu}{m\left(\int_\Omega u_n\right)} \to \lambda_1. \tag{4.10}$$

Then, by Expression (4.9) and Theorem 1.53, we get

$$u_n \geq m\left(\int_\Omega u_n\right)\left[(\gamma_n - \lambda_1)\frac{\varphi_1}{\int_\Omega \varphi_1^3} + G_n(x)\right],$$

where $G_n(x) := o(\gamma_n - \lambda_1)$ in $C^2(\overline{\Omega})$. Note that

$$\begin{aligned}
-\Delta\left(\frac{u_n}{m\left(\int_\Omega u_n\right)}\right) &= \frac{u_n}{m\left(\int_\Omega u_n\right)}\left[\frac{\lambda_n - cv_n}{m\left(\int_\Omega u_n\right)} - \frac{u_n}{m\left(\int_\Omega u_n\right)}\right] \\
&\leq \frac{u_n}{m\left(\int_\Omega u_n\right)}\left[\frac{\lambda_n}{m\left(\int_\Omega u_n\right)} - \frac{u_n}{m\left(\int_\Omega u_n\right)}\right]
\end{aligned}$$

and, consequently,

$$u_n \leq m\left(\int_\Omega u_n\right)\theta_{[\tilde{\gamma}_n,1]},$$

where $\theta_{[\tilde{\gamma}_n,1]} = \dfrac{\lambda_n}{m\left(\int_\Omega u_n\right)}$. It is straightforward to see that

$$\tilde{\gamma}_n = \gamma_n + \frac{c\mu}{m\left(\int_\Omega u_n\right)}$$

and, by Expression (4.10), we have

$$\tilde{\gamma}_n \to \lambda_1.$$

Thus

$$u_n \le m\left(\int_\Omega u_n\right)\left[(\tilde{\gamma}_n - \lambda_1)\frac{\varphi_1}{\displaystyle\int_\Omega \varphi_1^3} + \tilde{a}_n(x)\right],$$

where $\tilde{a}_n(x) := o(\tilde{\gamma}_n - \lambda_1)$. Integrating the previous expression over Ω, we obtain

$$\int_\Omega u_n \le m\left(\int_\Omega u_n\right)(\tilde{\gamma}_n - \lambda_1)\frac{\displaystyle\int_\Omega \varphi_1}{\displaystyle\int_\Omega \varphi_1^3} + m\left(\int_\Omega u_n\right)\int_\Omega \tilde{a}_n(x). \tag{4.11}$$

We claim that

$$\lim_{\lambda_n\to+\infty} m\left(\int_\Omega u_n\right)(\tilde{\gamma}_n - \lambda_1) = +\infty. \tag{4.12}$$

Indeed, assume by contradiction that

$$m\left(\int_\Omega u_n\right)(\tilde{\gamma}_n - \lambda_1) \le C \quad \text{for some} \quad C > 0. \tag{4.13}$$

Note that

$$\lim_{\lambda_n\to+\infty} m\left(\int_\Omega u_n\right)\int_\Omega \tilde{a}_n(x) = \lim_{\lambda_n\to+\infty} m\left(\int_\Omega u_n\right)(\tilde{\gamma}_n - \lambda_1)\frac{\displaystyle\int_\Omega \tilde{a}_n(x)}{\tilde{\gamma}_n - \lambda_1} = 0$$

and, consequently,

$$\int_\Omega u_n \le C,$$

a contradiction with Expression (4.1). This proves the Expression (4.12). Thus

$$\lim_{\lambda_n\to+\infty} m\left(\int_\Omega u_n\right)(\gamma_n - \lambda_1) = +\infty.$$

Using the expression of γ_n, we obtain

$$\lim_{\lambda_n\to+\infty}\left[\lambda_n - c\mu - \lambda_1 m\left(\int_\Omega u_n\right)\right] = +\infty$$

Then

$$\mu \ge \lim_{\lambda_n\to+\infty}\sigma_1\left[-n_\mu\Delta + d\left(\lambda_n - c\mu - \lambda_1 m\left(\int_\Omega u_n\right)\right)\varphi_1\right] = +\infty.$$

Therefore, we get a contradiction and we complete the proof. □

4.2 Sub-Supersolution Analysis

In this section, we demonstrate the existence of a coexistence state for Problem (P_3) using the Sub-Supersolution Method. Here, we will consider only the Competition case, as the others follow analogously to what we will demonstrate.

We will begin this section by presenting the result that ensures the existence of a coexistence state for Problem (P_3). To achieve this, we will strongly rely on Definition 1.38.

Theorem 4.4. Assume that $cd < 1$. If the conditions

$$\lambda > m_M\lambda_1 + c\mu \qquad \text{and} \qquad \mu > n_M\lambda_1 + d\lambda \tag{4.14}$$

are satisfied, then Problem (P_3) possesses at least a coexistence state

Proof. We claim that the pairs $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$, given by

$$(\underline{u}, \underline{v}) = (\varepsilon_1\varphi_1, \varepsilon_2\varphi_1) \qquad \text{and} \qquad (\overline{u}, \overline{v}) = (\lambda, \mu),$$

are sub-supersolution of (P_3), where $\varepsilon_1, \varepsilon_2 > 0$ are small enough. Indeed, we must verify that items (a)-(c) of Definition 1.36 are satisfied. Note that Item (a) is straightforward, so we will focus on verifying (b) and (c). Observe that they are sub-supersolution of (P_3), provided that:

$$m\left(\int_\Omega u\right)\lambda_1 \leq \lambda - \varepsilon_1\varphi_1 - c\mu, \quad \text{for all } u \in [\underline{u}, \overline{u}]$$

and

$$n\left(\int_\Omega v\right)\lambda_1 \leq \mu - \varepsilon_2\varphi_1 - d\lambda, \quad \text{for all } v \in [\underline{v}, \overline{v}].$$

Since (λ, μ) satisfies the inequalities of (4.14), it is possible to choose ε_1 and ε_2 small enough so that the conditions (b) and (c) are satisfied. Thus, $(\varepsilon_1\varphi_1, \varepsilon_2\varphi_1)$ and (λ, μ) are sub-supersolutions of (P_3). Therefore, there exists at least one coexistence state (u^*, v^*). □

In the figure below, we illustrate the curves that delineate the region of coexistence provided by Condition (4.14). The gray region represents the set of coexistence states for Problem (P_3), that is

$$R := \left\{(\lambda, \mu) \in \mathbb{R}^2;\ (\lambda, \mu) \text{ verifies the Condition (4.14)}\right\}.$$

This region is determined by the Sub-Supersolution Method. Note that, when $cd < 1$, $R \neq \emptyset$

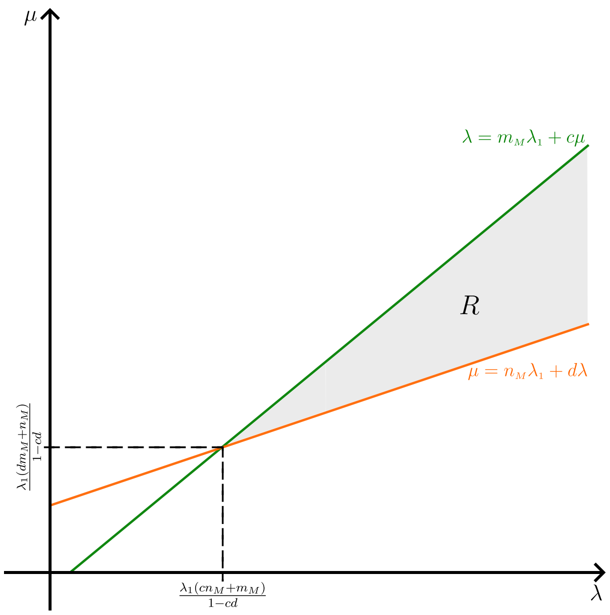


Figure 4.1: Region defined by Condition (4.14). If $(\lambda, \mu) \in R$, then Problem (P_3) possesses at least a coexistence state.

Source: Prepared by the author.

4.3 Local Bifurcation Analysis

In this section, we will apply the Crandall-Rabinowitz Theorem to ensure the existence of a local curve of non-trivial solutions for Problem (P_3) for the Competition case. Unlike what was done in Section 2.2 of Chapter 2, here we will consider a semi-trivial solution as the bifurcation point. Moreover, differently from the classical model of the local diffusion Lotka-Volterra system (see Problem (3.26)), Problem (P_3) does not generally admit a unique semi-trivial solution. Consequently, we demonstrate bifurcation from any semi-trivial solution. To achieve this, it is necessary to include conditions that guarantee such a bifurcation.

Fix $\lambda \in \mathbb{R}$ such that there exists a positive solution $\theta_{[\lambda,m(\cdot)]}$ of Problem (4.1). We then study the bifurcation from the point $(\theta_{[\lambda,m(\cdot)]}, 0)$ with μ as bifurcation parameter. We will consider the spaces $E := C_0^2(\overline{\Omega})$, $F := C(\overline{\Omega})$ and the operator $\mathcal{F} : \mathbb{R} \times E^2 \to F^2$ given by:

$$\mathcal{F}(\mu, u, v) = \begin{bmatrix} -m\left(\int_\Omega u\right) \Delta u - \lambda u + u^2 + cuv \\ -n\left(\int_\Omega v\right) \Delta v - \mu v + v^2 + duv \end{bmatrix}. \tag{4.15}$$

It is clear that $\mathcal{F} \in C^2(\mathbb{R} \times E, F)$ and $(u, v) \in E^2$ is a non-negative strong solution of Problem (P_3) if, and only if,

$$\mathcal{F}(\mu, u, v) = 0_{E \times E} \tag{4.16}$$

for all $\mu \in \mathbb{R}$. In particular

$$\mathcal{F}(\mu, \theta_{[\lambda,m(\cdot)]}, 0) = 0_{E \times E} \tag{4.17}$$

for all $\mu \in \mathbb{R}$, where $\theta_{[\lambda,m(\cdot)]}$ is a positive solution of Problem (4.1). Furthermore, the derivative of $\mathcal{F}$ in $(\mu, \theta_{[\lambda,m(\cdot)]}, 0)$ is given by

$$\mathcal{L}(\mu) := \mathcal{F}_{(u,v)}(\mu, \theta_{[\lambda,m(\cdot)]}, 0),$$

where, for $(\xi, \eta) \in E^2$,

$$\mathcal{L}(\mu)(\xi, \eta) = \begin{bmatrix} A(\xi, \eta) \\ -n(0)\Delta\eta + d\theta_{[\lambda,m(\cdot)]}\eta - \mu\eta \end{bmatrix},$$

with

$$\begin{aligned} A(\xi, \eta) = &-\left[m'\left(\int_\Omega \theta_{[\lambda,m(\cdot)]}\right) \int_\Omega \xi\right] \Delta\theta_{[\lambda,m(\cdot)]} - m\left(\int_\Omega \theta_{[\lambda,m(\cdot)]}\right) \Delta\xi \\ &- \lambda\xi + 2\theta_{[\lambda,m(\cdot)]}\xi + c\theta_{[\lambda,m(\cdot)]}\eta. \end{aligned} \tag{4.18}$$

Before presenting the main result of this section, we will need to study the following nonlocal problem:

$$\begin{cases} -d\Delta w + \alpha(x)w + \beta(x)\int_\Omega w = f(x) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.19}$$

where $d > 0$, $\alpha \in L^{\infty}(\Omega)$ such that $\sigma_1[-d\Delta + \alpha] > 0$ and $\beta \in L^{\infty}(\Omega)$. Since $\sigma_1[-d\Delta + \alpha] > 0$, there exists a unique positive solution, denoted by $e \in C^2(\Omega)$, of the equation

$$\begin{cases} -d\Delta e + \alpha(x)e = 1 & \text{in } \Omega, \\ e = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.20}$$

The next result guarantees that Problem (4.19) admits a unique solution under certain conditions on the coefficients of problem.

Theorem 4.5. Assume that $\sigma_1[-d\Delta + \alpha] > 0$ and

$$1 + \int_{\Omega} \beta(x)e(x) \neq 0, \tag{4.21}$$

where e is the unique positive of Problem (4.20). For each $f \in L^2(\Omega)$, Problem (4.19) admits a unique solution $w \in W^{2,2}(\Omega)$. Moreover, the following assertions hold:

(a) If $f \in L^p(\Omega)$, where $p > 1$, then $w \in W^{2,p}(\Omega)$.

(b) If $\alpha, \beta, f \in C^{0,a}(\overline{\Omega})$, where $a \in (0,1)$, then $w \in C^{2,a}(\overline{\Omega})$.

Proof. We will use some ideas of *Lemma 3.1* of [18]. Since $\sigma_1[-d\Delta + \alpha] > 0$, there exists $w_0 \in W^{2,2}(\Omega)$, unique solution of the following problem

$$\begin{cases} -d\Delta w_0 + \alpha(x)w_0 = f(x) - \dfrac{\int_{\Omega} f(x)e}{1 + \int_{\Omega} \beta(x)e}\beta(x) & \text{in } \Omega, \\ w_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.22}$$

We will show that w_0 is the unique solution of (4.20). Note that proving this statement is equivalent to show that

$$\int_{\Omega} w_0 = \frac{\int_{\Omega} f(x)e}{1 + \int_{\Omega} \beta(x)e}.$$

Multiplying the first equation of (4.22) by e and integrating over Ω, it follows that

$$\int_{\Omega} w_0 = \int_{\Omega} e(x)f - \frac{\int_{\Omega} f(x)e}{1 + \int_{\Omega} \beta(x)e} \int_{\Omega} \beta(x)e = \frac{\int_{\Omega} f(x)e}{1 + \int_{\Omega} \beta(x)e}.$$

Now, we show the uniqueness of solution of (4.20). Let w be a solution of (4.20). Then, multiplying (4.20) by e and integrating in Ω, we get

$$\int_{\Omega} w \left[1 + \int_{\Omega} \beta(x)e\right] = \int_{\Omega} f(x)e,$$

and, consequently,

$$\int_{\Omega} w = \frac{\int_{\Omega} f(x)e}{1 + \int_{\Omega} \beta(x)e}.$$

Hence, $w \equiv w_0$. Furthermore, Parts (a) and (b) follow as direct consequences of elliptic regularity theory. This completes the proof. □

The following result ensures the existence and uniqueness of a continuum of nontrivial solutions bifurcating from the point $(\theta_{[\lambda,m(\cdot)]}, 0)$, corresponding to a specific bifurcation value.

Theorem 4.6. Assume that $\lambda > 0$. Suppose there exists a positive solution $\theta_{[\lambda,m(\cdot)]}$ of Problem (4.1) such that

$$1 + \frac{m'\left(\int_\Omega \theta_{[\lambda,m(\cdot)]}\right)}{m\left(\int_\Omega \theta_{[\lambda,m(\cdot)]}\right)} \int_\Omega (\lambda\theta_{[\lambda,m(\cdot)]} - \theta^2_{[\lambda,m(\cdot)]})e_\lambda \neq 0. \tag{4.23}$$

Then,

$$(\mu, u, v) = (F(\lambda), \theta_{[\lambda,m(\cdot)]}, 0) \tag{4.24}$$

is a bifurcation point from the semi-trivial solution $(\mu, \theta_{[\lambda,m(\cdot)]}, 0)$. Moreover, there exists $\varepsilon > 0$ and application of class C^1

$$\mu : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R} \qquad \varphi : (-\varepsilon, \varepsilon) \longrightarrow Z \qquad \psi : (-\varepsilon, \varepsilon) \longrightarrow Z,$$

where Z is a topological complementary of $\mathrm{Ker}\,[\mathcal{F}_{(u,v)}\,(F(\lambda), \theta_{[\lambda,m(\cdot)]}, 0)]$ in E^2, such that

$$\begin{cases} \mu(s) = F(\lambda) + s\mu_1 + \rho(s) \\ u(s) = \theta_{[\lambda,m(\cdot)]} + s(\varphi^* + \varphi(s)) \\ v(s) = s(\psi^* + \psi(s)) \end{cases}, \tag{4.25}$$

where φ^* and ψ^* are functions, which will be detailed below, $\varphi(0) = \psi(0) = 0$ and $\rho(0) = 0$.

Proof. Consider μ as the main bifurcation parameter. We will apply Theorem 1.47 and to do this, it is necessary to verify that $\mathcal{F}$ satisfies the conditions (CR_4)–(CR_6).
Condition (CR_4): We need to determine μ^* such that $\dim\,[\mathrm{Ker}\,(\mathcal{L}(\mu^*))] = 1$. Note that, $(\xi, \eta) \in \mathrm{Ker}\,[\mathcal{L}(\mu^*)]$ if, and only if, (ξ, η) is solution of

$$\begin{cases} A(\xi, \eta) = 0 & \text{in } \Omega, \\ -n(0)\Delta\eta + d\theta_{[\lambda,m(\cdot)]}\eta = \mu^*\eta & \text{in } \Omega, \\ \xi = \eta = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.26}$$

From de second equation we deduce that $\mu^* := F(\lambda)$. Thus, the solutions of the second equation are given by $C\varphi^*$, where $C \in \mathbb{R}$ and $\varphi^* := \varphi_{F(\lambda)}$ is the principal eigenfunction associated to $F(\lambda)$ with $\|\varphi^*\|_\infty = 1$ and, consequently, $\eta \in \mathrm{Span}\{\varphi^*\}$. For the first equation, substituting η by $\varphi_{F(\lambda)}$, since $\theta_{[\lambda,m(\cdot)]}$ is a positive solution of (4.1), we can write $A(\xi, \eta) = 0$ as follows

$$\begin{cases} -m\left(\int_\Omega \theta_{[\lambda,m(\cdot)]}\right)\Delta\xi + (2\theta_{[\lambda,m(\cdot)]} - \lambda)\xi + \beta(x)\int_\Omega \xi = -C\theta_{[\lambda,m(\cdot)]}\varphi_{F(\lambda)} & \text{in } \Omega, \\ \xi = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.27}$$

where

$$\beta(x) := \frac{m'\left(\int_\Omega \theta_{[\lambda,m(\cdot)]}\right)}{m\left(\int_\Omega \theta_{[\lambda,m(\cdot)]}\right)}\left(\lambda\theta_{[\lambda,m(\cdot)]} - \theta^2_{[\lambda,m(\cdot)]}\right).$$

Moreover, for $\alpha = m\left(\int_\Omega \theta_{[\lambda,m(\cdot)]}\right)$ and $\gamma = 2\theta_{[\lambda,m(\cdot)]} - \lambda$, by Assertion (c) of Theorem 1.53,

$$\sigma_1\left[-\alpha\Delta + \alpha(x)\right] > 0.$$

Consequently, using the condition (4.25), by Theorem 4.5, (4.27) admits a unique positive solution in $W^{2,p}(\Omega)$, with $p > 1$, which we will denote as $\psi^* := \varphi_A$. Thus, the solution of (4.26) are given by $(C\varphi_{F(\lambda)}, \psi^*)$ and, consequently,

$$\mathrm{Ker}\left[\mathcal{L}(F(\lambda))\right] = \mathrm{Span}\{(C\varphi_{F(\lambda)}, \varphi_A)\}.$$

Condition (CR_5): We claim that

$$\mathrm{Rg}\left[\mathcal{L}(\mu^*)\right] = \left\{(u,v) \in E^2;\ \int_\Omega v\varphi_A = 0\right\}. \tag{4.28}$$

Indeed, given $(\varphi, \psi) \in \mathrm{Rg}\left[\mathcal{L}(\mu^*)\right]$, there exists $(u,v) \in E^2$ such that

$$\begin{cases} A(u,v) = \varphi, & \text{in } \Omega \\ -n(0)\Delta v + d\theta_{[\lambda,m(\cdot)]}v - F(\lambda)v = \psi, & \text{in } \Omega \\ u = v = 0, & \text{on } \partial\Omega \end{cases}. \tag{4.29}$$

Consequently, $\psi \in \mathrm{Rg}(-n(0)\Delta + d\theta_{[\lambda,m(\cdot)]} - F(\lambda))$, that is, ψ is such that

$$\int_\Omega \psi\varphi_A = 0.$$

Using the same reasoning as the previous condition, it follows that for every $\varphi \in E$, there exists $u \in E$ solution of the first equation of (4.29). Thus, the claim holds and, consequently,

$$\mathrm{codim}\left[\mathrm{Rg}\left(\mathcal{L}(F(\lambda))\right)\right] = 1$$

Condition (CR_6): Note that

$$\mathcal{L}_1(F(\lambda)) := \mathcal{L}_\mu(F(\lambda)) = \begin{bmatrix} 0 & 0 \\ 0 & -\mathrm{Id} \end{bmatrix}$$

In the case that $\mathcal{L}_1(F(\lambda))\,(\varphi_{F(\lambda)}, \varphi_A) \in \mathrm{Rg}\left[\mathcal{L}(F(\lambda))\right]$, there exists $(\xi, \eta) \in E^2$ such that

$$\mathcal{L}(F(\lambda))(\xi,\eta) = \mathcal{L}_1\left(F(\lambda)\right)\left(\varphi_{F(\lambda)}, \varphi_A\right).$$

Consequently, η satisfies

$$\begin{cases} -n(0)\Delta\eta + d\theta_{[\lambda,m(\cdot)]}\eta - F(\lambda)\eta = -\varphi_{F(\lambda)} & \text{in } \Omega, \\ \eta = 0 & \text{on } \partial\Omega. \end{cases}$$

Multiplying the previous equation by $\varphi_{F(\lambda)}$, integrating over Ω and using the fact that it is a eigenfunction associated to $F(\lambda)$, we obtain

$$\int_\Omega \varphi^2_{F(\lambda)} = 0,$$

a contradiction. Thus

$$\mathcal{L}_1\left(F(\lambda), \theta_{[\lambda,m(\cdot)]}, 0\right)\left(\varphi_{F(\lambda)}, \varphi_A\right) \notin \operatorname{Rg}\left[\mathcal{L}(F(\lambda))\right].$$

Therefore, the solutions of (P_3) in a neighborhood of $(F(\lambda), \theta_{[\lambda,m(\cdot)]}, 0)$ are given by $(\mu(s), u(s), v(s))$, for all $s \in (-\varepsilon, \varepsilon)$. □

The following observation provides an important remark regarding Condition (4.23), which establishes a condition for its validity, taking into account the positivity of the derivative of the diffusion function.

Observation 4.7. Consider the following problem

$$\begin{cases} -m\left(\int_\Omega \theta_{[\lambda,m(\cdot)]}\right)\Delta e_\lambda + (2\theta_{[\lambda,m(\cdot)]} - \lambda)e_\lambda = 1 & \text{in } \Omega, \\ e_\lambda = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.30}$$

Note that, by Assertion (c) of Theorem 1.53, Problem (4.30) admits a unique positive solution, denoted by e_λ. Moreover, $\lambda \geq \theta_{[\lambda,m(\cdot)]}$ in Ω and, consequently,

$$\lambda\theta_{[\lambda,m(\cdot)]} - \theta^2_{[\lambda,m(\cdot)]} \geq 0.$$

On the other hand, by the Maximum Principle, the solution of Problem (4.30) is positive. Thus

$$\int_\Omega (\lambda\theta_{[\lambda,m(\cdot)]} - \theta^2_{[\lambda,m(\cdot)]})e_\lambda \geq 0$$

and, consequently, the Condition (4.23) holds when the derivative of m is positive.

In the next result, we will analyze the bifurcation direction from the semi-trivial solution $(\theta_{[\lambda,m(\cdot)]}, 0)$.

Theorem 4.8. The following assertions hold:

(a) $\mu_1 = \dfrac{n'(0)\int_\Omega \varphi_A \int_\Omega |\nabla\varphi_A|^2 + \int_\Omega \varphi_A^3 + d\int_\Omega \varphi_A^2\varphi_{F(\lambda)}}{\int_\Omega \varphi_A^2}$, where φ_A is an eigenfunction associated with Equation (4.18).

(b) $\operatorname{sign}(\mu_1) = \operatorname{sign}\left(n'(0)\lambda_1 \int_\Omega \varphi_1 \int_\Omega \varphi_1^2 + (1 - cd)\int_\Omega \varphi_1^3\right)$, where $\lambda \approx m(0)\lambda_1$.

Moreover, the bifurcation direction from the semi-trivial solution $(\theta_{[\lambda,m(\cdot)]}, 0)$ in $F(\lambda)$ is given by

(a) Supercritical, when $n'(0) > 0$ and cd is sufficiently small.

(b) Subcritical, when $n'(0) < 0$ is sufficiently small and cd is sufficiently small.

Proof. Note that, using the expressions of (4.25) in the second equation of (P_3),

$$\begin{aligned}-n\left(\int_\Omega s(\varphi_A+\psi(s))\right)s\Delta(\varphi_A+\psi(s)) =& s(F(\lambda)+s\mu_1+o(s))(\varphi_A+\psi(s))\\ &- s^2(\varphi_A+\psi(s))^2 - d\theta_{[\lambda,m(\cdot)]}s(\varphi_A+\psi(s))\\ &- ds^2(\varphi_{F(\lambda)}+\varphi(s))(\varphi_A+\psi(s))\end{aligned}$$

Dividing the previous expression by s, taking the Taylor Expansion for the function $n\left(\int_\Omega v(s)\right)$ and substituting it, we obtain

$$\begin{aligned}-\left(n(0)+n'(0)s\int_\Omega\varphi_A+o(s)\right)\Delta(\varphi_A+\psi(s)) =& (F(\lambda)+s\mu_1+o(s))(\varphi_A+\psi(s))\\ &- s(\varphi_A+\psi(s))^2 - d\theta_{[\lambda,m(\cdot)]}(\varphi_A+\psi(s))\\ &- ds(\varphi_{F(\lambda)}+\varphi(s))(\varphi_A+\psi(s)).\end{aligned}$$

Taking the terms independent of s, we get

$$-n(0)\Delta\varphi_A = F(\lambda)\varphi_A - d\varphi_A\theta_{[\lambda,m(\cdot)]}.$$

Moreover, taking the terms of first order in s, it follows that

$$\begin{aligned}-n(0)\Delta\psi(s) - n'(0)\Delta(\varphi_A+\psi(s))\int_\Omega\varphi_A =& F(\lambda)\psi(s)+\mu_1(\varphi_A+\psi(s))\\ &- (\varphi_A+\psi(s))^2 - d\theta_{[\lambda,m(\cdot)]}\psi(s)\\ &- d(\varphi_{F(\lambda)}+\varphi(s))(\varphi_A+\psi(s))\end{aligned}$$

and, consequently,

$$\begin{aligned}-n(0)\Delta\psi(s) - F(\lambda)\psi(s) + d\theta_{[\lambda,m(\cdot)]}\psi(s) =& n'(0)\Delta(\varphi_A+\psi(s))\int_\Omega\varphi_A\\ &+ \mu_1(\varphi_A+\psi(s)) - (\varphi_A+\psi(s))^2\\ &- d(\varphi_{F(\lambda)}+\varphi(s))(\varphi_A+\psi(s))\end{aligned}$$

Multiplying by φ_A and integrating, we get

$$\begin{aligned}n'(0)\int_\Omega\varphi_A\int_\Omega\Delta(\varphi_A+\psi(s))\varphi_A =& \mu_1\int_\Omega(\varphi_A+\psi(s))\varphi_A - \int_\Omega(\varphi_A+\psi(s))^2\varphi_A\\ &- d\int_\Omega(\varphi_{F(\lambda)}+\varphi(s))(\varphi_A+\psi(s))\varphi_A.\end{aligned}$$

Taking $s\to 0$, we obtain

$$n'(0)\int_\Omega\varphi_A\int_\Omega\varphi_A\Delta\varphi_A + \mu_1\int_\Omega\varphi_A^2 - \int_\Omega\varphi_A^3 - d\int_\Omega\varphi_{F(\lambda)}\varphi_A^2 = 0,$$

and, consequently,

$$\mu_1\int_\Omega\varphi_A^2 = \frac{n'(0)}{n(0)}\int_\Omega\varphi_A\int_\Omega\left(F(\lambda)-d\theta_{[\lambda,m(\cdot)]}\right)\varphi_A^2 + \int_\Omega\varphi_A^3 + d\int_\Omega\varphi_A^2\varphi_{F(\lambda)}$$

Moreover, $\varphi_{F(\lambda)}$ is the unique solution of

$$-m\left(\int_\Omega\theta_{[\lambda,m(\cdot)]}\right)\Delta\varphi_{F(\lambda)} + (2\theta_{[\lambda,m(\cdot)]}-\lambda)\varphi_{F(\lambda)} + \beta(x)\int_\Omega\varphi_{F(\lambda)} = -c\theta_{[\lambda,m(\cdot)]}\varphi_A,$$

where

$$\beta(x) = \frac{m'\left(\int_\Omega \theta_{[\lambda,m(\cdot)]}\right)}{m\left(\int_\Omega \theta_{[\lambda,m(\cdot)]}\right)}\left(\lambda\theta_{[\lambda,m(\cdot)]} - \theta^2_{[\lambda,m(\cdot)]}\right).$$

Thus, $\varphi_{F(\lambda)} = -c\eta_1$, where η_1 verifies

$$-m\left(\int_\Omega \theta_{[\lambda,m(\cdot)]}\right)\Delta\eta_1 + (2\theta_{[\lambda,m(\cdot)]} - \lambda)\eta_1 + \beta(x)\int_\Omega \eta_1 = -\theta_{[\lambda,m(\cdot)]}\varphi_A.$$

Consequently

$$\mu_1 \int_\Omega \varphi_A^2 = \frac{n'(0)}{n(0)}\int_\Omega \varphi_A \int_\Omega \left(F(\lambda) - d\theta_{[\lambda,m(\cdot)]}\right)\varphi_A^2 + \int_\Omega \varphi_A^3 - cd\int_\Omega \varphi_A^2\eta_1. \tag{4.31}$$

Note that, by Assertion (b) of Theorem 1.53,

$$\theta_{[\lambda,m(\cdot)]} \cong (\lambda - m(0)\lambda_1)\frac{\varphi_1}{\int_\Omega \varphi_1^3},$$

and, consequently, we can express the application F as

$$F(\lambda) = \sigma_1\left[-n(0) + d\theta_{[\lambda,m(\cdot)]}\right] \cong n(0)\lambda_1 + d(\lambda - m(0)\lambda_1)\frac{\varphi_1}{\int_\Omega \varphi_1^3}.$$

Since $\varphi_{F(\lambda)}$ is the principal eigenfunction associated to $F(\lambda)$ it follows that $\varphi_{F(\lambda)} = \varphi_{F(\lambda)}(\lambda)$ admits a expansion of the form

$$\varphi_{F(\lambda)}(\lambda) \cong \varphi_1 + (\lambda - m(0)\lambda_1)\frac{\eta}{\int_\Omega \varphi^3_{F(\lambda)}} + o(\lambda - m(0)\lambda_1).$$

Moreover

$$\eta_1 \cong \varphi_1 + o(\lambda - m(0)\lambda_1).$$

Using the four expressions obtained above in (4.31) and tanking $\lambda \to m(0)\lambda_1$, we obtain

$$\mu_1 \int_\Omega \varphi_1^2 = n'(0)\lambda_1 \int_\Omega \varphi_1 \int_\Omega \varphi_1^2 + (1 - cd)\int_\Omega \varphi_1^3.$$

Therefore, the bifurcation direction is supercritical when $n'(0) > 0$ and cd is sufficiently small, and subcritical when $n'(0) < 0$ is sufficiently small, and cd is sufficiently small. □

4.4 Global Bifurcation Analysis

In this section, similarly to what was done in Section 2.3, we will rewrite (P_3) and apply the Global Bifurcation to obtain a bounded continuum $\mathfrak{C}$ of coexistence states for (P_3) by analyzing the behavior of the functions m and n.

We will consider the Ordered Banach Space $E := C_0^1(\overline{\Omega})$ and its positive cone, that is,

$$P_E := \{u \in E;\ u(x) \geq 0 \text{ for all } x \in \Omega\}.$$

In the next result, we prove a existence of a continuum of positive solutions of Problem (P_3) assuming that $\lambda > 0$ and that there exists a positive solution for Problem (4.1) such that Condition (4.23) holds.

Theorem 4.9. Assume that $\lambda > 0$. Suppose that there exists a positive solution $(\theta_{[\lambda,m(\cdot)]}, 0)$ of Problem (4.1) such that Condition (4.23) holds. Then, from the point

$$(\mu, u, v) = (F(\lambda), \theta_{[\lambda,m(\cdot)]}, 0) \tag{4.32}$$

bifurcates a continuum $\mathfrak{C}_{(\mu,\theta,0)} \subset \mathbb{R} \times \mathrm{Int}(P_E) \times \mathrm{Int}(P_E)$ of coexistence states of Problem (P_3). Moreover, $\mathrm{Proj}_{\mathbb{R}}(\mathfrak{C}_{(\mu,\theta,0)})$ is bounded in $\mathbb{R}$, where $\mathrm{Proj}_{\mathbb{R}}(\mu, u, v) := \mu$ for $(\mu, u, v) \in \mathfrak{C}_{(\mu,\theta,0)}$, and one of the following possibilities holds:

(C_1) There exists a sequence $(\mu_n, u_n, v_n) \in \mathfrak{C}_{(\mu,\theta,0)}$ such that $(\mu_n, u_n, v_n) \to (\mu_*, u_*, 0)$, where u_* is a positive solution of Problem (4.1), with $u_* \not\equiv \theta_{[\lambda,m(\cdot)]}$, and

$$\mu_* = \sigma_1 \left[-n(0)\Delta + du_*\right].$$

(C_2) There exists a sequence $(\mu_n, u_n, v_n) \in \mathfrak{C}_{(\mu,\theta,0)}$ such that $(\mu_n, u_n, v_n) \to (\mu^*, 0, v^*_\mu)$, where v^*_μ is a positive solution of Problem (4.2) and

$$\lambda = G(\mu^*) = \sigma_1 \left[-m(0)\Delta + cv^*_\mu\right].$$

(C_3) There exists a sequence $(\mu_n, u_n, v_n) \in \mathfrak{C}_{(\mu,\theta,0)}$ such that $(\mu_n, u_n, v_n) \to (n(0)\lambda_1, 0, 0)$ and $\lambda = m(0)\lambda_1$.

Proof. Consider the operator $\mathcal{G} : \mathbb{R} \times E^2 \to E^2$ defined by

$$\mathcal{G}(\mu, u, v) = \begin{pmatrix} u - L\left[Mu + \dfrac{1}{m\left(\int_\Omega u\right)}(\lambda u - u^2 - cuv)\right] \\ \\ v - L\left[Mv + \dfrac{1}{n\left(\int_\Omega v\right)}(\mu v - v^2 - duv)\right] \end{pmatrix}, \tag{4.33}$$

where $M > 0$ is a constant and $L := (-\Delta + M)^{-1}$ under homogeneous Dirichlet conditions. It is clear that (u, v) is a non-negative solution of (P_3) if, and only if, $\mathcal{G}(\mu, u, v) = 0_{E\times E}$. Moreover, $\mathcal{G}(\mu, \theta_{[\lambda,m(\cdot)]}, 0) = 0_{E\times E}$ for all $\mu \in \mathbb{R}$, and, by Theorem 4.6, from semi-trivial solution $(\theta_{[\lambda,m(\cdot)]}, 0)$ bifurcates a branch os positive solution at $\mu = F(\lambda)$. Now, we can apply Theorem 1.49, and following the steps of *Theorem 1.1* of [19] (see also *Theorem 4.1* of [47] and *Theorem 6.4.3* of [48]) aand conclude that there exists a continuum $\mathfrak{C}_{(\mu,\theta,0)}$ of coexistence states for (P_3) emanating from $(F(\lambda), \theta_{[\lambda,m(\cdot)]}, 0)$ and satisfies at least one of the following alternatives:

(G'_1) $\mathfrak{C}_{(\mu,\theta,0)}$ is unbounded in $\mathbb{R} \times E^2$;

(G'_2) There exist a sequence $(\mu_n, u_n, v_n) \in \mathfrak{C}_{(\mu,\theta,0)}$ such that $\mu_n \to \overline{\mu}$ in $\mathbb{R}$ and $(u_n, v_n) \to (0_E, 0_E)$ in E^2, with $\overline{\mu} \neq F(\lambda)$;

(G'_3) There exist a sequence $(\mu_n, u_n, v_n) \in \mathfrak{C}$ such that $\mu_n \to \overline{\mu}$ in $\mathbb{R}$ and $(u_n, v_n) \to (\overline{u}, 0_E)$ in E^2, where $\overline{u}$ is a positive solution of (4.1) with $\overline{u} \not\equiv \theta_{[\lambda,m(\cdot)]}$; or

(G'_4) There exist a sequence $(\mu_n, u_n, v_n) \in \mathfrak{C}$ such that $\mu_n \to \overline{\mu}$ in $\mathbb{R}$ and $(u_n, v_n) \to (0_E, \overline{v})$ in E^2, where $\overline{v}$ is a positive solution of (4.2).

We go check the validity of each alternative.
Alternative (G'_1) is not possible: Note that, by Proposition 4.1, the positive solution of (P_3) are bounded in $L^\infty(\Omega)$ and, by Elliptic Regularity, are bounded in E. On the other hand, by Assertion (a) of Corollary 4.2 and Proposition 4.3, $\mathrm{Proj}_{\mathbb{R}}(\mathfrak{C}_{(\mu,\theta,0)})$ is bounded in $\mathbb{R}$.
Alternative (G'_2) is possible: Suppose there exist a sequence $(\mu_n, u_n, v_n) \in \mathfrak{C}_{(\mu,\theta,0)}$ such that $\mu_n \to \overline{\mu}$ in $\mathbb{R}$ and $(u_n, v_n) \to (0_E, 0_E)$ in E^2, with $\overline{\mu} \neq F(\lambda)$. Note that

$$\mu_n = \sigma_1 \left[-n \left(\int_\Omega v_n \right) \Delta + v_n + du_n \right]. \tag{4.34}$$

Since $(u_n, v_n) \to (0_E, 0_E)$ in E^2, follows that

$$\mu_n \to n(0)\lambda_1.$$

Analogously, observe that

$$\lambda = \sigma_1 \left[-m \left(\int_\Omega u_n \right) \Delta + u_n + cv_n \right]. \tag{4.35}$$

Since $(u_n, v_n) \to (0_E, 0_E)$ in E^2, follows that

$$\lambda = m(0)\lambda_1,$$

Thus, Possibility (C_3) occurs.
Alternative (G'_3) is possible: Suppose there exist a sequence $(\mu_n, u_n, v_n) \in \mathfrak{C}$ such that $\mu_n \to \overline{\mu}$ in $\mathbb{R}$ and $(u_n, v_n) \to (\overline{u}, 0_E)$ in E^2, where $\overline{u}$ is a positive solution of (4.1) with $\overline{u} \not\equiv \theta_{[\lambda,m(\cdot)]}$. Note that, from $(\theta_{[\lambda,m(\cdot)]}, 0)$ the unique bifurcation point is $\mu = F(\lambda)$ and, consequently, $\overline{u} \not\equiv \theta_{[\lambda,m(\cdot)]}$. Since $(u_n, v_n) \to (0_E, 0_E)$ in E^2, taking the limit in (4.34), follows that

$$\mu_n \to \sigma_1 \left[-n(0)\Delta + d\overline{u} \right].$$

Thus, Possibility (C_1) occurs.
Alternative (G'_4) is possible: Suppose there exist a sequence $(\mu_n, u_n, v_n) \in \mathfrak{C}$ such that $\mu_n \to \overline{\mu}$ in $\mathbb{R}$ and $(u_n, v_n) \to (0_E, \overline{v})$ in E^2, where $\overline{v}$ is a positive solution of (4.2). Since $\mu_n \to \overline{\mu}$ and $(u_n, v_n) \to (0_E, \overline{v})$, taking the limit in (4.34), follows that

$$\overline{\mu} = \sigma_1 \left[-n \left(\int_\Omega \overline{v} \right) \Delta + \overline{v} \right].$$

Analogously, taking the limit in (4.35),

$$\lambda = \sigma_1 \left[-m(0)\Delta + d\overline{v} \right].$$

Thus, Possibility (C_2) occurs. □

In the figure below, we sketch the possible behaviors of the continuum $\mathfrak{C}_{(\mu,\theta,0)}$ of positive solutions to Problem (P_3). The diagram represents a scenario in which two semi-trivial solutions of the form $(u, 0)$ coexist, denoted by $(u_\lambda, 0)$ and $(u_{[\lambda,m(\cdot)]}, 0)$. Additionally, there exists a continuum of semi-trivial solutions of the form $(0, v_\mu)$ for $\mu > n(0)\lambda_1$. The continuum $\mathfrak{C}_{(\mu,\theta,0)}$ bifurcates from the semitrivial solution $(u_\lambda, 0)$ at the bifurcation point $\mu = F(\lambda)$. Depending on the possibility of Theorem 4.9, the continuum may follow one of three distinct scenarios

- In the possibility (C_1), $\mathfrak{C}_{(\mu,\theta,0)}$ behaves like $\mathcal{C}_1$, connecting the two semitrivial solutions $(u_\lambda, 0)$ and $(u_{[\lambda,m(\cdot)]}, 0)$ at $\mu = \mu_*$, with at least one coexistence state for $\mu \in (F(\lambda), \mu_*)$.

- In the possibility (C_2), $\mathfrak{C}_{(\mu,\theta,0)}$ behaves like $\mathcal{C}_2$, connecting $(u_\lambda, 0)$ to the semitrivial solution $(0, v_\mu)$ at $\mu = \mu^*$, and there exists at least one coexistence state for $\mu \in (F(\lambda), \mu^*)$.

- In the possibility (C_3), $\mathfrak{C}_{(\mu,\theta,0)}$ behaves like $\mathcal{C}_3$, connecting $(u_\lambda, 0)$ to the trivial solution $(0, 0)$ at $\mu = n(0)\lambda_1$. In this scenario, a coexistence state exists for $\mu \in (n(0)\lambda_1, F(\lambda))$.

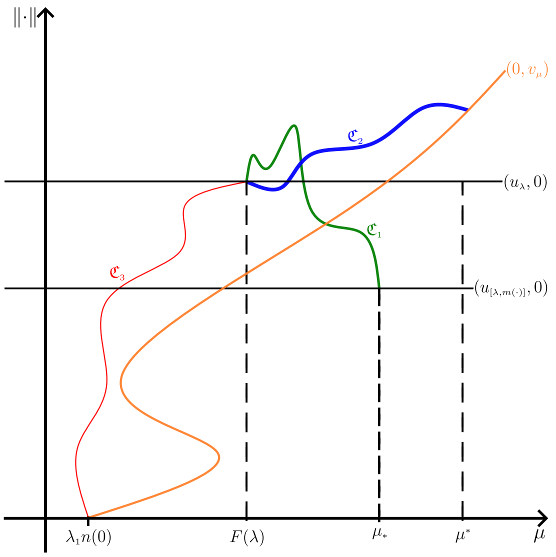


Figure 4.2: Possible bifurcation diagrams in the case of two semitrivial solutions $(u, 0)$ and one semtrivial solution $(0, v)$.

Source: Prepared by the author.

In general, it is challenging to determine which of the alternatives in Theorem 4.9 occurs. However, when m is increasing, we can confidently assert that at least one of these possibilities will always occur.

Corollary 4.10. Assume that $\lambda > m(0)\lambda_1$ and m increasing. Then, from the point

$$(\mu, u, v) = (F(\lambda), \theta_{[\lambda,m(\cdot)]}, 0)$$

bifurcates a continuum $\mathfrak{C}_{(\mu,\theta,0)} \subset \mathbb{R} \times \mathrm{Int}(P_E) \times \mathrm{Int}(P_E)$ of coexistence sates of Problem (P_3). Moreover, there exists a sequence $(\mu_n, u_n, v_n) \in \mathfrak{C}_{(\mu,\theta,0)}$ such that $(\mu_n, u_n, v_n) \to (\mu^*, 0, v^*_\mu)$, where v^*_μ is a positive solution of Problem (4.2) and

$$\lambda = G(\mu^*).$$

As consequence, Problem (P_3) possesses at least a coexistence state when

$$\mu \in (\min\{F(\lambda), \mu^*(\lambda)\}, \max\{F(\lambda), \mu^*(\lambda)\}) .$$

Proof. Note that, by Observation 4.7, the Condition (4.23) holds. In this case, there exists a unique positive solution $\theta_{[\lambda,m(\cdot)]}$ of Problem (4.1) and, consequently, $\lambda > m(0)\lambda_1$. Thus possibilities (C_1) and (C_3) do not occur. □

In the figures below, we illustrate the case where m is increasing, which ensures the existence of a unique semitrivial solution of the form $(u, 0)$. However, since we do not assume that n is increasing, there may exist multiple semitrivial solutions of the form $(0, v)$ for a given $\mu > 0$. In Figure 4.3, we represent a continuum of semitrivial solutions $(0, v_\mu)$ and sketch the associated bifurcation diagram. In this scenario, the global continuum $\mathfrak{C}_{(\mu,\theta,0)}$ connects to the semitrivial branch $(0, v_\mu)$.

It is important to note that, in this case, the set $G(\mu)$ is not a curve, not even a function. As a consequence, we cannot guarantee the existence of coexistence states for every pair (λ, μ) in the region bounded by the curve $\mu = F(\lambda)$ and the set $\lambda = G(\mu)$. In Figure 4.4, we display a possible coexistence region, highlighted in gray. In contrast, the existence of coexistence states cannot be ensured in the white region.

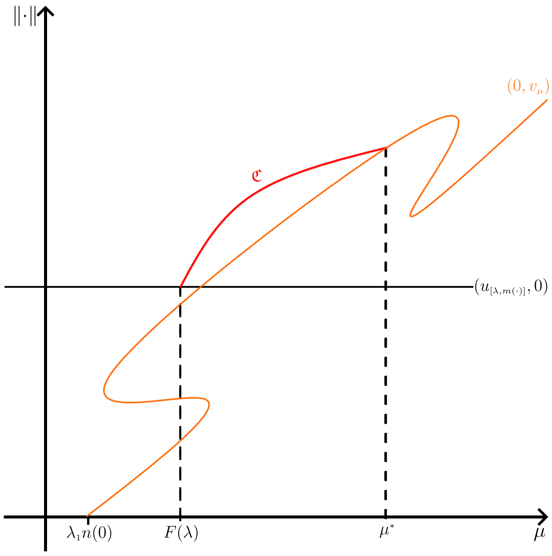


Figure 4.3: Behaviour of the global continuum $\mathfrak{C}_{(\mu,\theta,0)}$.

Source: Prepared by the author.

In the figure below, we illustrate a possible coexistence region in the case m increases.

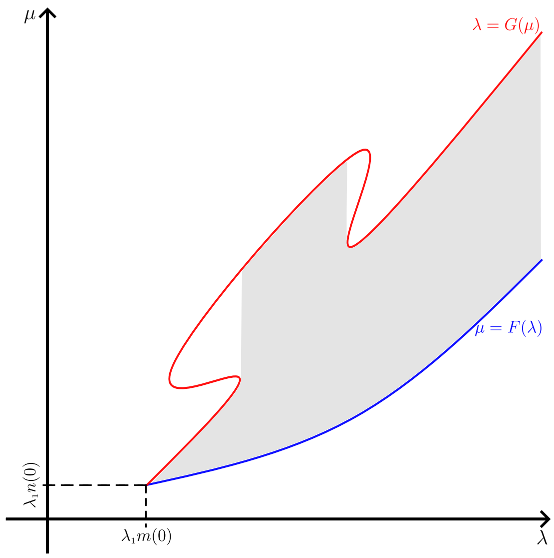


Figure 4.4: A possible coexistence region.

Source: Prepared by the author.

In the figure below, we have represented the case m increasing and $n'(0) < 0$. In this case, the bifurcation direction is subcritical and then there exist at least two coexistence states for $\mu \in (\mu_*, F(\lambda))$.

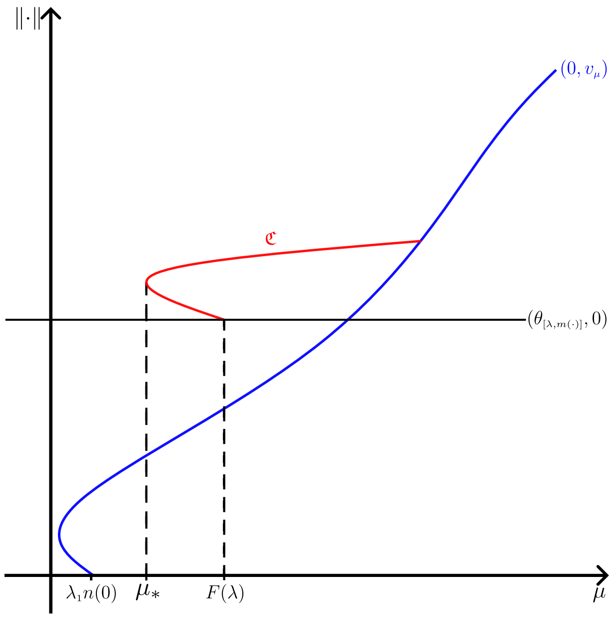


Figure 4.5: Bifurcation diagram for $\lambda \approx m(0)\lambda_1$, $cd < 1$ and $n'(0) < 0$. In this case, the bifurcation is subcritical.

Source: Prepared by the author.

For the case where m and n are increasing, there exist unique semi-trivial solutions $\theta_{[\lambda,m(\cdot)]}$ and $\theta_{[\mu,n(\cdot)]}$ for $\lambda > m(0)\lambda_1$ and $\mu > n(0)\lambda_1$, respectively. Moreover, the maps $F(\lambda)$ and $G(\mu)$ are continuous and increasing. In the next result, we will address what can be established regarding this scenario.

Theorem 4.11. Assume that m and n are increasing. The following assertions holds:

(a) Problem (P_3) does not possesses coexistence states when $\lambda \le m(0)\lambda_1$ or $\mu \le n(0)\lambda_1$.

(b) Problem (P_3) possesses at least a coexistence states when

$$(\mu - F(\lambda))(\lambda - G(\mu)) > 0. \tag{4.36}$$

Proof. Assertion (a): Consider $(u,v) \in E^2$ a coexistence state of (P_3). Since m is increasing, follows that

$$\lambda = \sigma_1 \left[-m\left(\int_\Omega u\right)\Delta + u + cv \right] > \sigma_1[-m(0)\Delta] = m(0)\lambda_1.$$

Analogously, it is possible to show that $\mu > n(0)\lambda_1$.
Assertion (b): Since m and n are increasing, there exists a unique semi-trivial solution $(\theta_{[\lambda,m(\cdot)]}, 0)$ and $(0, \theta_{[\mu,n(\cdot)]})$. Note that, by Corollary 4.10, there exists at least a coexistence states when

$$\mu \in (\min\{F(\lambda), \mu^*(\lambda)\}, \max\{F(\lambda), \mu^*(\lambda)\}).$$

Suppose, without loss of generality, that $F(\lambda) < \mu^*(\lambda)$. Thus, there exists at least a coexistence state when

$$F(\lambda) < \mu < \mu^*(\lambda).$$

Since G is increasing, follows that

$$G(\mu) < G(\mu^*(\lambda)) = \lambda.$$

Hence, $\lambda > G(\mu)$ and $\mu > F(\lambda)$. Analogously, $\lambda < G(\mu)$ and $\mu < F(\lambda)$ when $\mu^*(\lambda) < F(\lambda)$. □

In the figure bellow, we have represented the bifurcation diagram for the case m and n increase. In this case, there exists a unique semi-trivial solution $(u,0)$ and $(0,v)$.

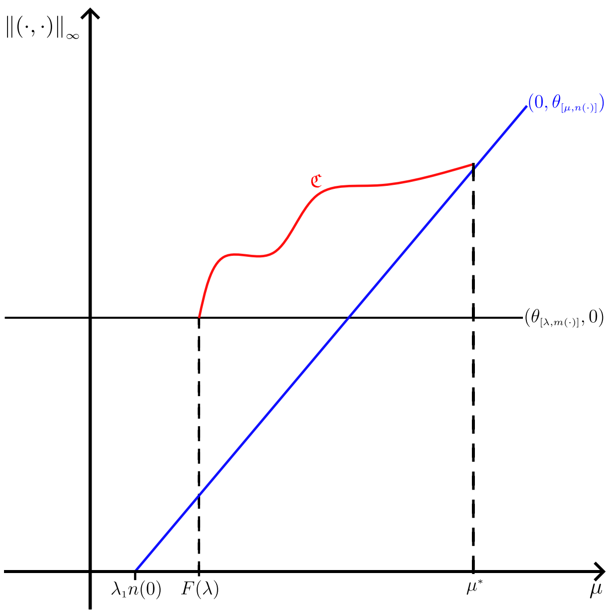


Figure 4.6: Behavior of the Continuum for m and n increase.
Source: Prepared by the author.

To conclude this section, we will show that by assuming m and n are increasing, we can improve the result obtained in Theorem (4.4), which did not impose any conditions on the functions m and n. To achieve this improvement, we will once again employ the Sub-Supersolution Method.

Consider (λ, μ) such that Condition (4.14) is satisfied. Note that $\mu > F(\lambda)$ e $\lambda > G(\mu)$. Indeed, on the one hand,

$$F(\lambda) < \sigma_1 [-n_M \Delta + d\lambda] = n_M \lambda_1 + d\lambda,$$

and on the other hand,

$$G(\mu) < \sigma_1 [-m_M \Delta + c\mu] = m_M \lambda_1 + c\mu,$$

The following figure illustrates a comparative analysis of the coexistence regions defined by conditions (4.14) and (4.36). The results highlight the differences in the spatial extent of these regions, providing insights into their respective boundaries and implications.

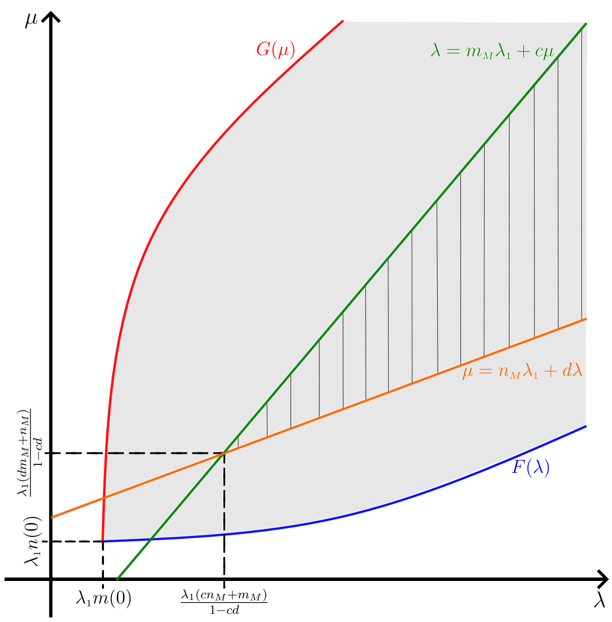


Figure 4.7: Comparison of the coexistence regions defined by (4.14) and (4.36).
Source: Prepared by the author.

5 Conclusions / Conclusiones / Conclusões

Conclusions

In this work, we explore the existence and uniqueness of coexistence states for various classes of nonlocal elliptic systems arising in the context of Population Dynamics and biological interactions. These systems model the behavior of species whose diffusion rates depend on the total population of one or both species in a nonlocal and nonlinear manner. The inclusion of nonlocal terms in the diffusion coefficients introduces significant technical challenges and leads to a richer solution structure compared to classical local diffusion models.

In conclusion, this work advances the understanding of nonlocal elliptic systems in Population Dynamics, providing new tools and techniques to analyze the existence and uniqueness of coexistence states. The results obtained here pave the way for a broader exploration of nonlocal models in ecology and other fields where nonlocal interactions play a crucial role.

The first part of this work was dedicated to studying a nonlocal elliptic system that models the interaction between a bacterium and a nutrient, where the nutrient's diffusion depends on the bacterial population. Using bifurcation methods and the Implicit Function Theorem, we established conditions for the existence and uniqueness of positive solutions. Specifically, we proved that the system admits at least one coexistence state under certain conditions on the growth rates and interaction parameters. Furthermore, we showed that the uniqueness of the coexistence state is guaranteed when the diffusion coefficient is an increasing function.

In the second part, we extended our analysis to Lotka-Volterra systems with nonlocal cross-diffusion terms, covering competition, predator-prey, and symbiosis interactions. The nonlocal terms in the diffusion coefficients, which depend on the total population of the other species, significantly alter the structure of coexistence states compared to classical models. Using the fixed-point index in cones, we characterized the stability of semi-trivial solutions and established conditions for the existence of coexistence states. Notably, we found that, in the case of nonlocal competition, coexistence can occur even when one of the semi-trivial solutions does not exist, a result that contrasts with the classical case of local diffusion.

The third part of this work focused on a Lotka-Volterra competition system with nonlocal diffusion, where the diffusion coefficients depend on the total population of the species themselves. This model captures scenarios where species tend to leave overcrowded areas or are attracted to regions with higher population density, depending on the behavior of the diffusion functions. We proved that the competitive exclusion principle holds regardless of the behavior of the diffusion functions, a surprising result given the complexity introduced by the nonlocal terms. Additionally, we established conditions for the existence of coexistence states using sub-supersolution methods and bifurcation techniques.

The results obtained in this work contribute to the understanding of nonlocal elliptic systems and their applications in Population Dynamics. The inclusion of nonlocal terms in the diffusion coefficients introduces new phenomena and challenges that require innovative mathematical techniques to address. Our findings highlight the importance of considering nonlocal interactions in ecological models, as they can lead to more realistic and complex dynamics compared to traditional local models.

Conclusiones

En este trabajo, exploramos la existencia y unicidad de estados de coexistencia para diversas clases de sistemas elípticos no-locales que surgen en el contexto de la Dinámica de Poblaciones y las interacciones biológicas. Estos sistemas modelan el comportamiento de especies cuyas tasas de difusión dependen de la población total de una o ambas especies de manera no-local y no-lineal. La inclusión de términos no-locales en los coeficientes de difusión introduce desafíos técnicos significativos y conduce a una estructura de soluciones más rica en comparación con los modelos clásicos de difusión local.

En conclusión, este trabajo avanza en la comprensión de los sistemas elípticos no-locales en la Dinámica de Poblaciones, proporcionando nuevas herramientas y técnicas para analizar la existencia y unicidad de los estados de coexistencia. Los resultados obtenidos aquí abren camino para una exploración más amplia de modelos no-locales en ecología y en otros campos donde las interacciones no-locales juegan un papel crucial.

La primera parte de este trabajo se dedicó al estudio de un sistema elíptico no-local que modela la interacción entre una bacteria y un nutriente, donde la difusión del nutriente depende de la población de la bacteria. Utilizando métodos de bifurcación y el Teorema de la Función Implícita, establecimos condiciones para la existencia y unicidad de soluciones positivas. Específicamente, demostramos que el sistema admite al menos un estado de coexistencia bajo ciertas condiciones sobre las tasas de crecimiento y los parámetros de interacción. Además, mostramos que la unicidad del estado de coexistencia está garantizada cuando el coeficiente de difusión es una función creciente.

En la segunda parte, ampliamos nuestro análisis a sistemas de Lotka-Volterra con términos de difusión cruzada no-local, abarcando interacciones de competencia, presa-depredador y simbiosis. Los términos no-locales en los coeficientes de difusión, que dependen de la población total de la otra especie, alteran significativamente la estructura de los estados de coexistencia en comparación con los modelos clásicos. Utilizando el índice de punto fijo en conos, caracterizamos la estabilidad de las soluciones semi-triviales y establecimos condiciones para la existencia de estados de coexistencia. Es notable que encontramos que, en el caso de competencia no-local, la coexistencia puede ocurrir incluso cuando una de las soluciones semi-triviales no existe, un resultado que contrasta con el caso clásico de difusión local.

La tercera parte de este trabajo se centró en un sistema de competencia de Lotka-Volterra con difusión no-local, donde los coeficientes de difusión dependen de la población total de las propias especies. Este modelo captura escenarios en los que las especies tienden a abandonar áreas superpobladas o son atraídas a regiones con mayor densidad poblacional, dependiendo del comportamiento de las funciones de difusión. Demostramos que el principio de exclusión competitiva se mantiene independientemente del comportamiento de las funciones de difusión, un resultado sorprendente dada la complejidad introducida por los términos no-locales. Además, establecimos condiciones para la existencia de estados de coexistencia utilizando métodos de sub-soluciones y super-soluciones y técnicas de bifurcación.

Los resultados obtenidos en este trabajo contribuyen a la comprensión de los sistemas elípticos no-locales y sus aplicaciones en la Dinámica de Poblaciones. La inclusión de términos no-locales en los coeficientes de difusión introduce nuevos fenómenos y desafíos que requieren técnicas matemáticas innovadoras para abordarlos. Nuestros hallazgos destacan la importancia de considerar interacciones no-locales en los modelos ecológicos, ya que pueden conducir a dinámicas más realistas y complejas en comparación con los modelos locales tradicionales.

Conclusões

Neste trabalho, exploramos a existência e unicidade de estados de coexistência para diversas classes de sistemas elípticos não-locais, que surgem no contexto da Dinâmica de Populações e interações biológicas. Esses sistemas modelam o comportamento de espécies cujas taxas de difusão dependem da população total de uma ou ambas as espécies de maneira não-local e não-linear. A inclusão de termos não-locais nos coeficientes de difusão introduz desafios técnicos significativos e leva a uma estrutura de soluções mais rica em comparação com os modelos clássicos de difusão local.

Em conclusão, este trabalho avançou na compreensão de sistemas elípticos não-locais na Dinâmica de Populações, fornecendo novas ferramentas e técnicas para analisar a existência e unicidade de estados de coexistência. Os resultados aqui obtidos abrem caminho para uma exploração mais ampla de modelos não-locais na ecologia e em outros campos onde interações não-locais desempenham um papel crucial.

A primeira parte deste trabalho foi dedicada ao estudo de um sistema elíptico não-local que modela a interação entre uma bactéria e um nutriente, onde a difusão do nutriente depende da população da bactéria. Usando métodos de bifurcação e o Teorema da Função Implícita, estabelecemos condições para a existência e unicidade de soluções positivas. Especificamente, provamos que o sistema admite pelo menos um estado de coexistência sob certas condições sobre as taxas de crescimento e os parâmetros de interação. Além disso, mostramos que a unicidade do estado de coexistência é garantida quando o coeficiente de difusão é uma função crescente.

Na segunda parte, estendemos nossa análise para sistemas de Lotka-Volterra com termos de difusividade cruzada não-local, abrangendo interações de competição, presa-predador e simbiose. Os termos não-locais nos coeficientes de difusão, que dependem da população total da outra espécie, alteram significativamente a estrutura dos estados de coexistência em comparação com os modelos clássicos. Utilizando o índice de ponto fixo em cones, caracterizamos a estabilidade de soluções semi-triviais e estabelecemos condições para a existência de estados de coexistência. Notavelmente, descobrimos que, no caso de competição não-local, a coexistência pode ocorrer mesmo quando uma das soluções semi-triviais não existe, um resultado que contrasta com o caso clássico de difusão local.

A terceira parte deste trabalho focou em um sistema de competição de Lotka-Volterra com difusão não-local, onde os coeficientes de difusão dependem da população total das próprias espécies. Este modelo captura cenários em que as espécies tendem a deixar áreas superlotadas ou são atraídas para regiões com maior densidade populacional, dependendo do comportamento das funções de difusão. Provamos que o princípio da exclusão competitiva se mantém independentemente do comportamento das funções de difusão, um resultado surpreendente dada a complexidade introduzida pelos termos não-locais. Além disso, estabelecemos condições para a existência de estados de coexistência utilizando métodos de sub-super soluções e técnicas de bifurcação.

Os resultados obtidos neste trabalho contribuem para a compreensão de sistemas elípticos não-locais e suas aplicações na Dinâmica de Populações. A inclusão de termos não-locais nos coeficientes de difusão introduz novos fenômenos e desafios que exigem técnicas matemáticas inovadoras para serem abordados. Nossos achados destacam a importância de considerar interações não-locais em modelos ecológicos, pois elas podem levar a dinâmicas mais realistas e complexas em comparação com os modelos locais tradicionais.

References

[1] S. Agmon, A. Douglis, and L. Nirenberg. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I. *Communications on Pure and Applied Mathematics*, 12(4):623–727, 1959.

[2] W. Allegretto and A. Barabanova. Positivity of solutions of elliptic equations with nonlocal terms. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, 126(3):643–663, 1996.

[3] C. O. Alves, F. J. S. A. Corrêa, and M. Chipot. On a class of intermediate local-nonlocal elliptic problems. *Topological Methods in Nonlinear Analysis*, 49:497–509, 2017.

[4] H. Amann. Fixed Point Equations and Nonlinear Eigenvalue Problems in Ordered Banach Spaces. *SIAM Review*, 18(4):620–709, 1976.

[5] A. Ambrosetti and D. Arcoya. Positive solutions of elliptic Kirchhoff equations. *Advanced Nonlinear Studies*, 17(1):3–15, 2017.

[6] A. Ambrosetti and A. Malchiodi. *Nonlinear Analysis and Semilinear Elliptic Problems*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2007.

[7] Y. Baoqiang and M. Tianfu. The existence and multiplicity of positive solutions for a class of nonlocal elliptic problems. *Bound. Value Probl.*, pages Paper No. 165, 35, 2016.

[8] J. Blat and K. J. Brown. Bifurcation of steady-state solutions in predator-prey and competition systems. *Proceedings of the Royal Society of Edinburgh. Section A. Mathematics*, 97:21–34, 1984.

[9] J. Blat and K. J. Brown. Global bifurcation of positive solutions in some systems of elliptic equations. *SIAM Journal on Mathematical Analysis*, 17(6):1339–1353, 1986.

[10] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Springer, New York, 2011.

[11] R. S. Cantrell and C. Cosner. On the steady-state problem for the Volterra-Lotka competition model with diffusion. *Houston Journal of Mathematics*, 13:337–352, 1987.

[12] R. S. Cantrell and C. Cosner. *Spatial Ecology via Reaction-Diffusion Equations*. John Wiley & Sons, 2004.

[13] Y. C. Carranza, M. A. V. Costa, C. Morales-Rodrigo, and A. Suárez. Nonlocal diffusion elliptic system modelling the behaviour of a bacteria and a living nutrient. *Discrete and Continuous Dynamical Systems. Series B. A Journal Bridging Mathematics and Sciences*, 30(1):82–98, 2025.

[14] M. Chipot. Remarks on some class of nonlocal elliptic problems. In *Recent advances on elliptic and parabolic issues*, pages 79–102. 2006.

[15] M. Chipot. *Elliptic equations: an introductory course*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Verlag, Basel, 2009.

[16] M. Chipot and B. Lovat. On the asymptotic behaviour of some nonlocal problems. *Positivity. An International Journal devoted to the Theory and Applications of Positivity in Analysis*, 3(1):65–81, 1999.

[17] M. Chipot and J. F. Rodrigues. On a class of nonlocal nonlinear elliptic problems. *RAIRO Modélisation Mathématique et Analyse Numérique*, 26(3):447–467, 1992.

[18] W. Cintra, M. Molina-Becerra, and A. Suárez. The Lotka-Volterra models with nonlocal reaction terms. *Communications on Pure and Applied Analysis*, 21(11):3865–3886, 2022.

[19] W. Cintra, C. Morales-Rodrigo, and A. Suárez. Unilateral global bifurcation for a class of quasilinear elliptic systems and applications. *Journal of Differential Equations*, 267:619–657, 2019.

[20] F.J. Corrêa, M. Delgado, and A. Suárez. Some nonlinear heterogeneous problems with nonlocal reaction term. *Advances in Differential Equations*, 16(7/8):623–641, 2011.

[21] M. A. V. Costa, C. Morales-Rodrigo, and A. Suárez. Lotka-Voterra competition model with nonlocal coefficient diffusion. *Journal of Differential Equations*, 439, 2025.

[22] M. A. V. Costa, C. Morales-Rodrigo, and A. Suárez. The Lotka-Volterra models with nonlocal cross-diffusivity terms. *Journal of Mathematical Analysis and Applications*, 547(1):129316, 2025.

[23] M. G. Crandall and P. H. Rabinowitz. Bifurcation from simple eigenvalues. *Journal of Functional Analysis*, 8(2):321–340, 1971.

[24] E. N. Dancer. Global Solution Branches for Positiue Mappings. *Archive for Rational Mechanics and Analysis*, 52:181–192, 1973.

[25] E. N. Dancer. On the indices of fixed points of mappings in cones and applications. *Journal of Mathematical Analysis and Applications*, 91(1):131–151, 1983.

[26] E. N. Dancer. On positive solutions of some pairs of differential equations II. *Journal of Differential Equations*, 60(2):236–258, 1985.

[27] K. Deimling. *Nonlinear functional analysis*. Springer-Verlag, Berlin, 1985.

[28] M. Delgado, I. B. M. Duarte, and A. Suárez. Nonlocal elliptic system arising from the growth of cancer stem cells. *Discrete and Continuous Dynamical Systems. Series B*, 23(4):1767–1795, 2018.

[29] M. Delgado, J. Santos Junior, and A. Suárez. An intermediate local-nonlocal eigenvalue elliptic problem. *Communications in Contemporary Mathematics*, 24:Paper No. 2050076, 21 pp., 2022.

[30] M. Delgado, J. López-Gómez, and A. Suárez. On the Symbiotic Lotka–Volterra Model with Diffusion and Transport Effects. *Journal of Differential Equations*, 160(1):175–262, 2000.

[31] G. Scorza Dragoni. II problema dei valori ai limiti studiato in grande per le equazioni differenziali del secondo ordine. *Mathematische Annalen*, 105(1):133–143, 1931.

[32] J.C. Eilbeck, J.E. Furter, and J. López-Gómez. Coexistence in the Competition Model with Diffusion. *Journal of Differential Equations*, 107(1):96–139, 1994.

[33] J. Ferreira and H. B. Oliveira. Parabolic reaction-diffusion systems with nonlocal coupled diffusivity terms. *Discrete and Continuous Dynamical Systems*, 37(5):2431–2453, 2017.

[34] G. M. Figueiredo, C. Morales-Rodrigo, J. R. Santos Junior, and A. Suárez. Study of a nonlinear Kirchhoff equation with non-homogeneous material. *Journal of Mathematical Analysis and Applications*, 416(2):597–608, 2014.

[35] G. M. Figueiredo and A. Suárez. *The sub-supersolution method for Kirchhoff systems: applications*, volume 86, pages 217–227. Springer International Publishing, 2015.

[36] T. S. Figueiredo-Sousa. Existencia de soluciones positivas para ecuaciones elípticas con coeficiente de difusión no-local. Master's thesis, Universidad de Sevilla, Sevilla, España, 2015.

[37] T. S. Figueiredo-Sousa, C. Morales-Rodrigo, and A. Suárez. A non-local non-autonomous diffusion problem: linear and sublinear cases. *Z. Angew. Math. Phys.*, 68(5):108, 20, 2017.

[38] T. S. Figueiredo-Sousa, C. Morales-Rodrigo, and A. Suárez. Some superlinear problems with nonlocal diffusion coefficient. *Journal of Mathematical Analysis and Applications*, 482(1):123519, 25, 2020.

[39] T. S. Figueiredo-Sousa, C. Rodrigo-Morales, and A. Suárez. The influence of a metasolution on the behaviour of the logistic equation with nonlocal diffusion coefficient. *Calculus of Variations and Partial Differential Equations*, 57(4):Paper No. 100, 26, 2018.

[40] J. Furter and M. Grinfeld. Local vs. nonlocal interactions in population dynamics. *Journal of Mathematical Biology*, 27(1):65–80, 1989.

[41] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*, volume 224. Springer-Verlag, Berlin, second edition, 1983.

[42] J. Hernández. *Qualitative methods for nonlinear diffusion equations.* Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1986.

[43] P. Hess. On the solvability of nonlinear elliptic boundary value problems. *Indiana University Mathematics Journal*, 25(5):461–466, 1976.

[44] D. Hilhorst and J-F. Rodrigues. On a nonlocal diffusion equation with discontinuous reaction. *Advances in Differential Equations*, 5(4-6):657 – 680, 2000.

[45] S. Kesavan. *Nonlinear Functional Analysis: A First Course.* Hindustan Book Agency, 2004.

[46] P. Korman and A. Leung. On the existence and uniqueness of positive steady-states in the Volterra-Lotka ecological model with diffusion. *Applied Analysis*, 26:145–160, 1987.

[47] J. López-Gómez. Nonlinear eigenvalues and global bifurcation application to the search of positive solutions for general Lotka-Volterra reaction diffusion systems with two species. *Differential Integral Equations*, 7(5-6):1427–1452, 1994.

[48] J. López-Gómez. *Spectral theory and nonlinear functional analysis.* Chapman & Hall/CRC, Boca Raton, FL, 2001.

[49] J. López-Gómez. *Linear second order elliptic operators.* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2013.

[50] J. López-Gómez. *Metasolutions of parabolic equations in population dynamics.* CRC Press, Boca Raton, FL, 2016.

[51] A. J. Lotka. *Elements of Mathematical Biology.* Dover Publications, Inc., New York, 1959.

[52] J. López-Gómez and J. C. Sabina de Lis. Coexistence States and Global Attractivity for Some Convective Diffusive Competing Species Models. *Transactions of the American Mathematical Society*, 347(10):3797–3833, 1995.

[53] J. López-Gómez and R. Pardo. Existence and uniqueness of coexistence states for the predator-prey model with diffusion: the scalar case. *Differential and Integral Equations. An International Journal for Theory and Applications*, 6(5):1025–1031, 1993.

[54] J. D. Murray. *Mathematical biology. II*, volume 18 of *Interdisciplinary Applied Mathematics.* Springer-Verlag, New York, third edition, 2003. Spatial models and biomedical applications.

[55] A. A. Ovono and A. Rougirel. Elliptic equations with diffusion parameterized by the range of nonlocal interactions. *Annali di Matematica Pura ed Applicata*, 189:163–183, 2010.

[56] C. V. Pao. *Nonlinear parabolic and elliptic equations.* Plenum Press, New York, 1992.

[57] M. H. Protter and H. F. Weinberger. *Maximum Principles in Differential Equations.* Prentice Hall, New Jersey, 1967.

[58] P. H. Rabinowitz. Some global results for nonlinear eigenvalue problems. *Journal of Functional Analysis*, 7(3):487–513, 1971.

[59] N. H. Tuan, T. Caraballo, P. T. K. Van, and V. V Au. On a terminal value problem for parabolic reaction-diffusion systems with nonlocal coupled diffusivity terms. *Communications In Nonlinear Science And Numerical Simulation*, 108:1427–1452, 2022.

[60] V. Volterra. *Variazioni e fluttuazioni del numero d'individui in specie animali conviventi*. Memorie della Classe di Scienze Fisiche, Matematiche e Naturali. Societá anonima tipografica "Leonardo da Vinci", 1927.

[61] B. Yan and C. An. The sign-changing solutions for a class of nonlocal elliptic problem in an annulus. *Topological Methods in Nonlinear Analysis*, 55(1):1–18, 2020.