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Thermodynamic properties of QCD matter and multiplicity fluctuations

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Thermodynamic properties of QCD matter and multiplicity fluctuations

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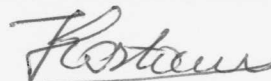
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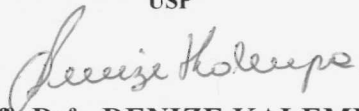
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This thesis is dedicated to my parents.

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“There is only one good, knowledge, and one evil, ignorance.”
(Socrates)

RESUMO

Uma característica vital da cromodinâmica quântica (QCD) está relacionada à simetria quiral. Isso é particularmente intrigante devido ao papel crítico da simetria quiral não abeliana dos spinores de Lorentz na física teórica moderna. Muitos esforços teóricos foram dedicados à sua quebra espontânea no vácuo, bem como a restauração da mesma no ambiente extremamente quente ou denso. Além disso, quarks e glúons tornam-se os graus de liberdade relevantes por meio da transição de desconfinamento do estado dos hádrons. O significado desta última está intimamente ligado às implicações da equação de Callan-Symanzik e à teoria do grupo renormalizado. No entanto, em princípio, ambas as transições acima podem ser descritas pela QCD. Os estudos da QCD na rede demonstraram que a transição do sistema é um cruzamento suave com a densidade bariônica nula e a massa de quarks estranhos grandes. No potencial químico finito, por outro lado, uma variedade de modelos prevê a ocorrência de uma transição de fase de primeira ordem entre a fase hadrônica e o plasma de quarks e glúons (QGP). Esses resultados indicam que um ponto crítico (CEP) pode estar localizado em algum lugar no diagrama de fases da QCD no qual a linha de transições de fase de primeira ordem termina. Espera-se que a transição seja de segunda ordem neste caso. De fato, entre outros objetivos estabelecidos, o programa Beam Energy Scan (BES) em andamento no Relativistic Heavy Ion Collider (RHIC) é impulsionado pela busca do CEP.

Nesta tese, exploramos alguns tópicos relacionados às propriedades termodinâmicas da matéria QCD no contexto de colisões de íons pesados. Esses estudos são, até certo ponto, motivados a alcançar uma melhor compreensão do diagrama de fases, enquanto fazem parte do esforço de busca pelo CEP. Em particular, investigamos a consistência termodinâmica do modelo de quasipartículas no potencial químico bariônico nulo e não-nulo, bem como as flutuações da razão de partículas e de multiplicidade. Entende-se que uma fase desconfiada de matéria QCD foi atingida no colisor de íons pesados ultra-relativista. Embora o modelo de sacola MIT seja simplificado demais e incapaz de descrever a equação de estado (EoS) de QGP obtida pelas simulações da QCD na rede, os cálculos de QCD perturbativa (pQCD) são confiáveis apenas a temperaturas extremamente altas. Nesse contexto, o modelo de quase-partícula é uma ferramenta valiosa para acomodar tanto a noção de graus efetivos de liberdade quanto os dados da QCD na rede. Como uma abordagem efetiva, o modelo é capaz de descrever as propriedades termodinâmicas do QGP em uma ampla faixa de temperatura T e potencial químico bariônico μ . No entanto, como apontado por Gorenstein e Yang, a consistência termodinâmica apresenta um problema, pois coloca uma restrição não trivial à EoS em questão. Sua receita foi proposta originalmente para o potencial químico bariônico nulo, enquanto vários autores generalizaram a abordagem para o caso com a densidade bariônica finita. Nestas, revisamos a questão relativa à autoconsistência termodinâmica do modelo de quase-partículas, com potencial químico bariônico finito, adaptado aos cálculos da QCD na rede. Aqui, investigamos a possibilidade da massa efetiva de quase-partículas também ser uma função de seu momento, k , além da temperatura T e do potencial químico μ . Verifica-se que a consistência termodinâmica pode ser expressa em termos de uma equação integro-diferencial em termos de k , T e μ . Discutimos ainda duas soluções especiais, ambas podem ser vistas como condição suficiente para a consistência termodinâmica, enquanto expressas em termos de uma equação diferencial parcial. O primeiro caso mostra-se equivalente aos discutidos anteriormente por Peshier *et al.* O segundo,

obtido por meio de uma suposição *ad hoc*, é uma solução intrinsecamente diferente em que a massa das partículas é dependente do momento. Essas equações podem ser resolvidas usando a condição de contorno determinada pelos dados da QCD na rede com potencial químico nulo. Pelos cálculos numéricos, mostramos que ambas as soluções podem reproduzir razoavelmente os resultados recentes da QCD na rede das Colaborações Wuppertal-Budapest e HotQCD, e em particular os relativos à densidade bariônica finita.

O programa BES realizado no RHIC é dedicado a explorar o diagrama de fases da matéria nuclear que interage fortemente. Em termos de colisões Au+Au com energias relativamente baixas, estão sendo realizadas medidas precisas para a região da matéria QCD com a sua densidade bariônica alta. Intuitivamente, deve-se procurar quantidades sensíveis à física subjacente enquanto acessíveis experimentalmente. Os cumulantes de ordens mais elevantes das cargas conservadas e as suas combinações, como as taxas de cumulantes, nessa conta, tornam-se observáveis focados. Essas quantidades cumprem os requisitos, pois carregam informações vitais no meio primordial criado nas colisões. Além disso, argumentou-se que eles são sensíveis à estrutura de fases da questão QCD e, em particular, ao paradeiro do CEP. Nesse sentido, recentemente, as flutuações da multiplicidade chamaram muita atenção como um dos principais observáveis. Nesta tese, estudamos os aspectos não críticos das flutuações da multiplicidade em colisões de íons pesados, empregando um modelo hidrodinâmico. A abordagem atual é aprimorada principalmente nas abordagens existentes do modelo HRG, que levam em consideração flutuações térmicas, correção de volume finito e decaimento de ressonâncias. Nosso modelo é focado nos aspectos da expansão hidrodinâmica do sistema e nas flutuações de estado inicial evento a evento (EbE). Calculamos as flutuações da razão de partículas de K/π , K/p , e p/π usando o código hidrodinâmico SPheRIO com ou sem as condições iniciais (IC) flutuantes EbE. Além disso, também são avaliadas as flutuações de multiplicidade de prótons, kaon líquido e carga líquida. Os resultados obtidos são então comparados com os dos modelos HRG, UrQMD e os dados experimentais da colaboração STAR. Em geral, relativo aos dados existentes, os resultados obtidos pelo SPheRIO são razoáveis em comparação com as demais abordagens. Em particular, observa-se que as IC EbE podem causar um efeito considerável, que pode não apenas ultrapassar as flutuações térmicas, mas também superestimar os dados. Por sua vez, isso implica potencialmente um requisito mais rigoroso para o gerador de eventos em relação às flutuações EbE.

PALAVRAS-CHAVE: equação de estado. transição de fase QCD. ponto final crítico. flutuação.

ABSTRACT

One vital characteristic of the quantum chromodynamics (QCD) is regarding the chiral symmetry. This is particularly intriguing owing to the critical role of non-abelian gauge symmetry of Lorentz spinors in modern theoretical physics. Many theoretical efforts have been devoted concerning its spontaneously breaking in the vacuum, as well as the restoration at the extremely hot or dense environment. Furthermore, quarks and gluons become the relevant degrees of freedom through the deconfinement transition from the hadron state of matter. The significance of the latter is closely connected to the implications of the Callan-Symanzik equation and the theory of the renormalized group. Nonetheless, in principle, both of the above transitions can be described by the QCD. Lattice QCD studies demonstrated that the transition of the system is a smooth crossover at vanishing baryon density and large strange quark mass. At finite chemical potential, on the other hand, a variety of models predict the occurrence of a first-order transition between the hadronic phase and quark-gluon plasma (QGP). These results indicate that a critical endpoint (CEP) might be located somewhere on the QCD phase diagram at which the line of first-order phase transitions terminates. The transition is expected to be of second-order at this point. As a matter of fact, among other established goals, the ongoing Beam Energy Scan (BES) program at the Relativistic Heavy Ion Collider (RHIC) is driven by the search for the CEP.

In this thesis, we explore a few topics regarding the thermodynamic properties of QCD matter in the context of heavy-ion collisions. These studies are, to a certain extent, motivated to achieve a better understanding of the phase diagram, while being a part of the endeavor to search for the CEP. In particular, we investigate the thermodynamic consistency of the quasi-particle model at both vanishing and non-vanishing baryon chemical potential, as well as the particle ratio and multiplicity fluctuations. It is understood that a deconfined phase of QCD matter has been attained at the ultra-relativistic heavy-ion collider. While the MIT bag model is oversimplified and unable to describe the equation of state (EoS) of QGP from the lattice QCD simulations, perturbative QCD (pQCD) calculations are reliable only at extremely high temperature. In this context, the quasi-particle model is a valuable tool to accommodate both the notion of effective degrees of freedom and the lattice QCD data. As an effective approach, the model is capable of describing the thermodynamical properties of QGP over a wide range of temperature T and baryon chemical potential μ . However, as pointed out by Gorenstein and Yang, the thermodynamic consistency poses an issue as it places a nontrivial restriction to the EoS in question. Their proposed recipe was originally for vanishing baryon chemical potential, while several authors have generalized the approach to the case of finite baryon density. In this thesis, we revisit the matter regarding the thermodynamical self-consistency of quasi-particle model at finite baryon chemical potential adapted to lattice QCD calculations. Here, we investigate the possibility where the effective quasi-particle mass is also a function of its momentum, k , in addition to temperature T and chemical potential μ . It is found that the thermodynamic consistency can be expressed in terms of an integro-differential equation concerning k , T , and μ . We further discuss two special solutions, both can be viewed as sufficient condition for the thermodynamical consistency, while expressed in terms of a partial differential equation. The first case is shown to be equivalent to those previously discussed by Peshier *et al.* The second one, obtained through an *ad hoc* assumption, is an intrinsically different

solution where the particle mass is momentum dependent. These equations can be solved by using boundary condition determined by the lattice QCD data at vanishing baryon chemical potential. By numerical calculations, we show that both solutions can reasonably reproduce the recent lattice QCD results of the Wuppertal-Budapest and HotQCD Collaborations, and in particular, those concerning finite baryon density.

The BES program carried out at RHIC is dedicated to exploring the phase diagram of the strongly interacting nuclear matter. In terms of Au+Au collisions at relatively low energies, precise measurements are being carried out for the high baryon density region of the QCD matter. Intuitively, one shall look for quantities that are sensitive to the underlying physics while accessible experimentally. The higher cumulants of conserved charges and combinations of them, such as cumulant ratios, on that account, become such focused observables. These quantities fulfill the requirement as they carry vital information on the primordial medium created in the collisions. Moreover, it has been argued that they are sensitive to the phase structure of the QCD matter, and in particular, the whereabouts of the CEP. In this regard, recently, multiplicity fluctuations have drawn much attention as one of the key observables. In this thesis, we studied the noncritical aspects of the multiplicity fluctuations in heavy-ion collisions by employing a hydrodynamic model. The present approach is mainly improved upon existing HRG model approaches, which take into consideration thermal fluctuations, finite volume correction, and resonance decay. Our model is focused on the aspects of the hydrodynamic expansion of the system and the event-by-event (EbE) initial state fluctuations. We calculated the particle ratio fluctuations of K/π , K/p , and p/π by using the hydrodynamic code SPHeRIO with or without the EbE fluctuating initial conditions (IC). Besides, the net-proton, net-kaon, and net-charged multiplicity fluctuations are also evaluated. The obtained results are then compared to those of the HRG, UrQMD models, as well as the experimental data from the STAR Collaboration. Overall, regarding the existing data, the results obtained by SPHeRIO are reasonable in comparison with those by using different approaches. In particular, it is observed that the EbE IC may cause a sizable effect, which may not only overwhelm the thermal fluctuations but also overestimate the data. This, in turn, potentially implies a more stringent requirement for the event generator regarding EbE fluctuations.

KEYWORDS: equation of state. QCD phase transition. critical end point. fluctuation.

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LIST OF ABBREVIATIONS AND ACRONYMS

BES	Beam Energy Scan
BNL	Brookhaven National Laboratory
CEP	critical endpoint
CERN	the European Organization for Nuclear Research
CGC	the color glass condensate
EoS	equation of state
EoM	equation of motion
GCE	the grand canonical ensemble
HRG	hadron resonance gas
HTL	hard thermal loop
LHC	Large Hadron Collider
pQCD	perturbative QCD
QCD	quantum chromodynamics
QFT	quantum field theory
QGP	the quark gluon plasma
RHIC	Relativistic Heavy Ion Collider
SM	standard model
SPH	Smoothed Particle Hydrodynamics
SSM	standard statistical mechanics

LIST OF SYMBOLS

g_s	QCD coupling constant
Λ	QCD scale
$g_{\mu\nu}$	metric tensor
τ	proper time
τ_0	initial time
η	space time rapidity
u^μ	four velocity
$T^{\mu\nu}$	energy-momentum tensor
J_i^μ	conserved current
$f(x, p)$	distribution function
T_{fo}	freeze-out temperature
T_c	critical temperature
μ_c	critical chemical potential
ε	energy density
p	pressure
s	entropy density
c_s^2	the squared speed of sound
n_B	baryon number density
n_S	strangeness number density

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1 INTRODUCTION

Throughout human history, we have always been attracted to questions regarding the most fundamental building blocks of the Universe. Today, these questions have been transformed into the study of elementary particles and their interactions, strongly associated with the development of modern physics. It is generally understood that there are four types of fundamental interactions in nature, namely, strong, electromagnetic, weak, and gravitational. The Standard Model (SM) provides a consistent scheme to accommodate three out of the four interactions, where relevant fermions interact through the exchange of bosons. The fermions consist of three generations of quarks and leptons. To be specific, there are six quarks in three generations, namely, up (u), down (d), strange (s), charm (c), bottom (b), and top quark (t) and the antiquarks. Also, there are six leptons in the three generations, namely, electron (e), muon (μ), tau (τ), the corresponding neutrinos, as well as their antiparticles. The bosons, on the other hand, are composed of gluons (g), photons (γ), $W^\pm$ and Z^0 bosons, as well as the Higgs bosons. The latter was found recently by ATLAS (AAD et al., 2012) and CMS (CHATRCHYAN et al., 2012) at Large Hadron Collider (LHC) in 2012. Graviton, the quantum of gravity, apart from the issue concerning renormalizability, is also a boson which mediates the gravitational force. On the other hand, quantum field theory (QFT) serves as a theoretical framework that encompasses the other known interactions. In particular, the electromagnetic and weak interactions were unified by Glashow (GLASHOW, 1961) in 1961. Thereafter, Weinberg (WEINBERG, 1967) and Salam (SALAM, 1968) introduced the Higgs mechanism into Glashow's theory, which gives rise to the origin of the mass for Fermions in particle physics. Besides, the strong interaction is described by the theory of quantum chromodynamics (QCD). Figure 1 illustrates all the elementary particles in SM.

As a theory for describing the strong interaction between quarks and gluons, QCD is a SU(3) gauge theory in terms of the quark and gluon degrees of freedom (PESKIN; SCHROEDER, 2006). There are two main characteristic properties, namely, color confinement and asymptotic freedom. We will briefly review the QCD theory below in Sec. 2. Due to the color confinement, quarks can only move inside the volume of the hadron at standard temperature and density. However, Lattice QCD calculations indicated a phase transition from the hadronic to quark-gluon plasma (QGP) state for extremely high temperature or baryon density. For the latter, the quarks and gluons are free to move over a large volume compared with the typical size of hadrons.

It is also understood that the QGP can be produced in relativistic heavy-ion collisions. However, due to the color confinement, individual quarks or gluons still can not be directly observable in the final state particles. Moreover, the QGP phase only exists for a very short period, as the system rapidly expands and cools down. The process continues as hadronization takes place until the condition for freeze-out is satisfied. Therefore, one can only obtain valuable information about the QGP from the observed hadrons spectrum. In other words, the existence of QGP can only be inferred from the experimentally accessible quantities. These experimental observables, which indicate the presence of QGP, are of great theoretical interest, will be discussed in the following section.

Standard Model of Elementary Particles

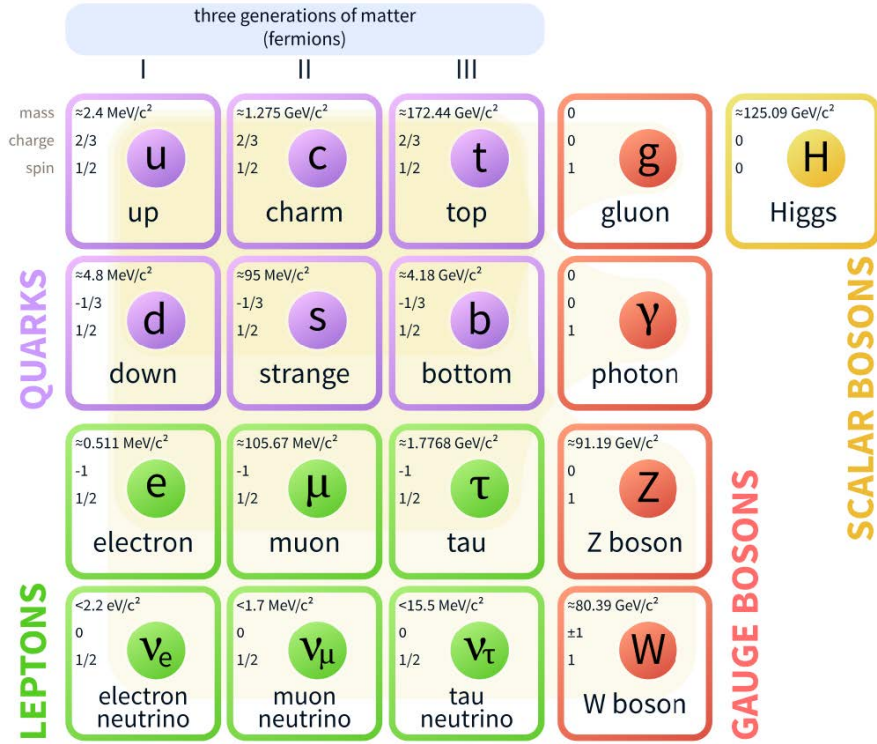


Figure 1 – The schematic diagram of the element particles in SM

1.1 SIGNALS OF QGP

There are, in literature, several relevant signatures of QGP which has aroused much attention. One of those signatures is strangeness enhancement (RAFELSKI, 1982). This enhancement means strange quarks will produce in more abundance than non-strange anti-quarks. To be more specific, in equilibrating QGP, the number density of strange quark is

$$n_s = g_s \frac{T m_s^2}{\pi^2} K_2 \left(\frac{m_s}{T} \right) \quad (1.1)$$

and the number density of massless non-strange anti-quark $\bar{q}$ is

$$n_{\bar{q}} = g_{\bar{q}} \frac{T^3}{\pi^2} e^{-\mu_q/T}, \quad (1.2)$$

where $g_{\bar{q}} = g_s = 6$ is the degeneracy of the anti-quark and/or s quark, q denotes u or d quark and $\mu_q = \mu_B/3$ is the quark chemical potential. Then the ratio of strange quarks to non-strange quarks is

$$\frac{n_s}{n_{\bar{q}}} = \frac{1}{2} \left(\frac{m_s}{T} \right)^2 K_2 \left(\frac{m_s}{T} \right) e^{-\mu_q/T}. \quad (1.3)$$

It is easy to derive that this ratio is greater than one for $\mu_q > 0$. In other words, the number of strange quark will be produced more than the one of non-strange quark. Therefore the production of strange hadron will enhance after hadronization. There are some experiments that support this enhancement

of strange particles, particularly strange and multistrange antibaryons (BARI et al., 1995; KINSON; ABATZIS; ANDERSEN, 1995).

Another signature is J/ψ suppression (MATSUI; SATZ, 1986). This idea is also simple. As Matsui and Satz (MATSUI; SATZ, 1986) suggested, the production of J/ψ will be suppressed if the QGP is produced in the collisions. Such a suppression was indeed discovered by the experiment (GAVIN; VOGT, 1996). Besides, there are also some other experimental evidences in support of the signatures of QGP, like the dilepton pairs signature (SHURYAK, 1978a) discovered by (AGAKICHIEV et al., 1995) and Jet quenching phenomena (WANG; GYULASSY, 1992) which is discovered by (ADAMS et al., 2003; ADLER et al., 2003). Finally, in 2005, Relativistic Heavy Ion Collider (RHIC) announced that QGP had been discovered (GYULASSY, 2004; GYULASSY; MCLERRAN, 2005; JACOBS et al., 2007).

1.2 THE EQUATION OF STATE OF THE QGP PHASE

In order to investigate the QGP's properties and formation, we should firstly derive its equation of state (EoS). Due to the asymptotic freedom of QCD at high energy, QGP can be treated as a weakly coupled gas when $T \sim 10^5 T_c$. Therefore, considering the free QGP is a relativistic ideal gas, its pressure, energy density and other thermodynamics can be obtained from a MIT bag model (CHODOS et al., 1974). In this model, a bag constant B , as the 'external bag pressure', is introduced.

In addition, Lattice QCD, as a non-perturbative approach can be applied to study the QCD properties at finite temperature T (BORSANYI et al., 2014; BAZAVOV et al., 2014). The EoS at vanishing chemical potential for $N_f = 2 + 1$ QCD is shown in Figure 2. However, there exists the sign problem of Fermions when the chemical potential μ is non-zero (FORCRAND, 2009). In order to evade such a sign problem, there are three main approaches: reweighting, Taylor expansion and analytic continuation from imaginary μ . But those approaches can give reliable results only when μ is small enough (FORCRAND, 2009). For now, those methods can allow us to explore the QCD phase diagram up to $\mu_B/T \simeq 2.5$ (BAZAVOV et al., 2017).

For high energies, it is understood that perturbative QCD (pQCD) at finite temperature T and chemical potential μ is suitable for computing the thermodynamics of QGP, for the asymptotic freedom of QCD. The pQCD expansion of free energy had been pushed up to $g_s^6 \ln(1/g_s)$ order (KAJANTIE et al., 2003). However such a pQCD series shows a bad convergence (KAJANTIE et al., 2003; ARNOLD; ZHAI, 1995; ZHAI; KASTENING, 1995). We can reorganize the perturbative series to improve the convergence. There are some attempts to solve this problem, such as HTL perturbation theory (ANDERSEN; BRAATEN; STRICKLAND, 1999; ANDERSEN et al., 2002), the 2-loop ϕ -derivable approximation (BLAIZOT; IANCU; REBHAN, 1999; BLAIZOT; IANCU; REBHAN, 2003; BLAIZOT; IANCU; REBHAN, 2001), principle of maximum conformality (PMC) (BU et al., 2018) and others.

To characterize the EoS of QGP over a wide range of temperature and chemical potential, we adopt a phenomenological model called quasi-particle model. This model is assumed to treat the QGP phase as a non-interacting system with massive quasi-particle. Generally, the mass of quasi-particle in this phenomenological model is dependent on the temperature. If we follow the standard

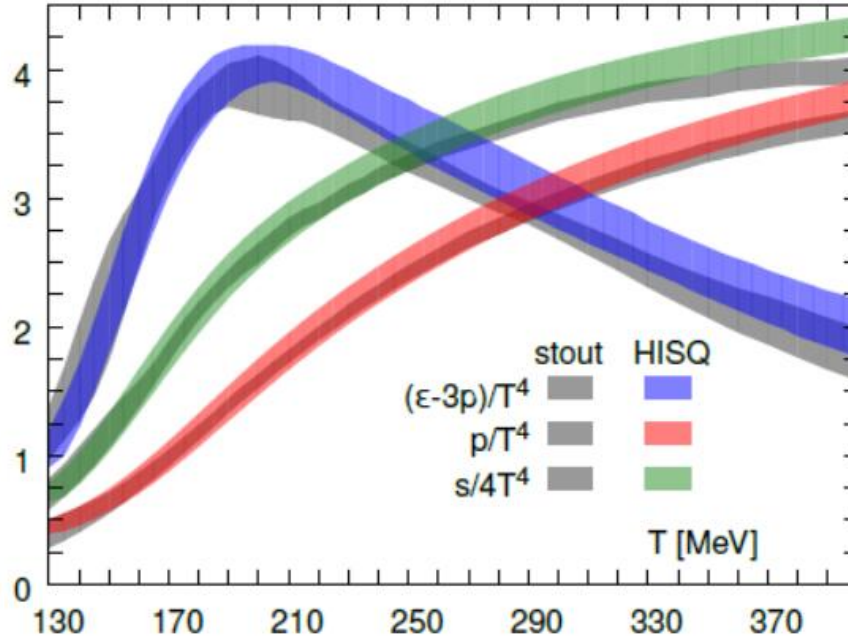


Figure 2 – The Lattice QCD results for the trace anomaly, the pressure, and the entropy density from (BAZAVOV et al., 2014)

procedure of statistical mechanics for this ideal massive gas naively, the thermodynamic inconsistency appears (GORENSTEIN; YANG, 1995). As an in-depth discuss, we will revisit the possible solutions for this inconsistency problem and propose a special solution where the quasi-particle mass m is a function of temperature, chemical potential and momentum.

1.3 QCD PHASE DIAGRAM

Figure 3 shows the QCD phase diagram. As indicated in this figure, hadronic phase (Hadron Gas) exist in low temperature T with low baryon chemical potential μ_B which will be switched to QGP phase at high T or high μ_B . However the thermodynamic properties on the boundary separating the two phases seems interesting. At non-vanishing baryon chemical potential, *i.e.* $\mu_B \neq 0$, theoretical model calculations (BARI et al., 1995; KINSON; ABATZIS; ANDERSEN, 1995; MATSUI; SATZ, 1986; GAVIN; VOGT, 1996; AGAKICHIEV et al., 1995; HALASZ et al., 1998; BERGES; RAJAGOPAL, 1999; STEPHANOV; RAJAGOPAL; SHURYAK, 1998; SCHWARZ; KLEVANSKY; PAPP, 1999; FODOR; KATZ, 2004) suggest it is a first order phase transition. On the other hands, non-perturbative approach, like Lattice QCD (FODOR; KATZ, 2002; KARSCH, 2002), indicates that there is a smooth crossover for $\mu_B = 0$ and large strange quark mass. Therefore, it implies the existence of critical endpoint (CEP) where the line of first order phase transitions terminates, and the transition is expected to be of second order at this point. However, the exact location of the CEP is also unknown due to the difficulty of lattice QCD calculations at finite μ_B (STEPHANOV, 2006; FORCRAND, 2009). Therefore the experiment to search for the CEP of QCD falls (AGGARWAL et al., 2010). So far, the existence and properties of the critical point is a long-standing intriguing topic.

It is worth mentioning that, at the CEP, the QCD phase transition is continuous rather than the first

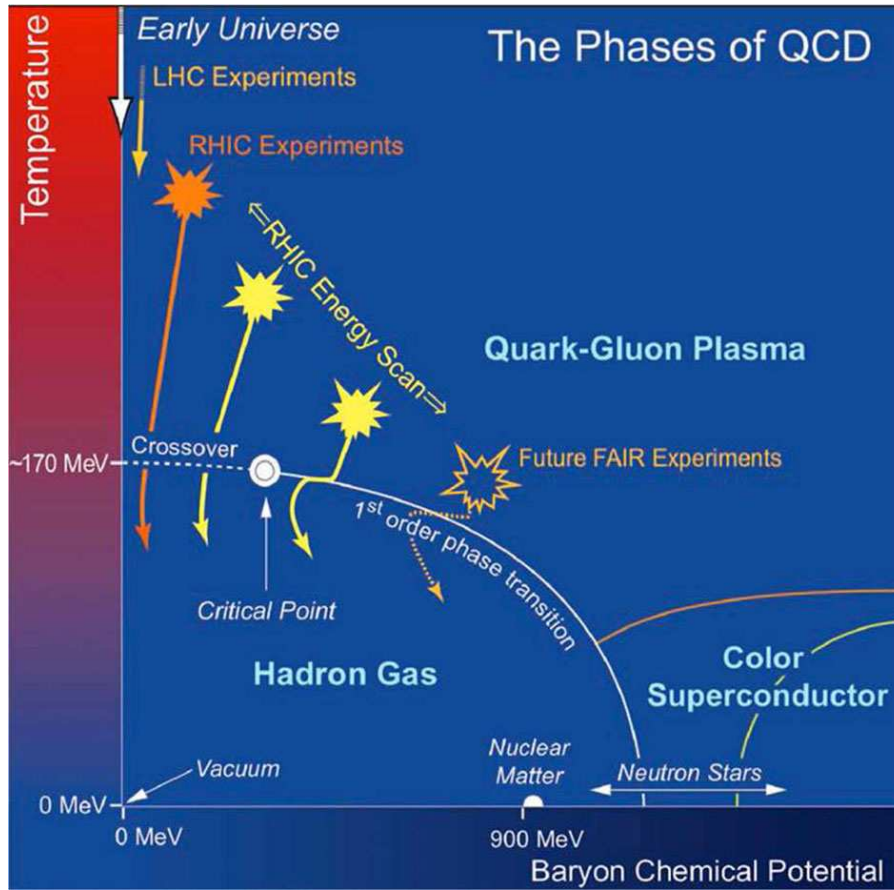


Figure 3 – Diagram of QCD phase

order, such that we should observe the divergent susceptibility for the critical opalescence phenomenon. In fact there are large fluctuations near the QCD critical temperature according to Lattice QCD calculations (KARSCH, 2007; CHENG et al., 2009). Figure 4 shows the calculated quadratic and quartic fluctuations of baryon number, electric charge, and strangeness by the Lattice QCD. Thus, in experiment, the fluctuations of baryon number, charge number can possibly be served as the signature of QCD CEP.

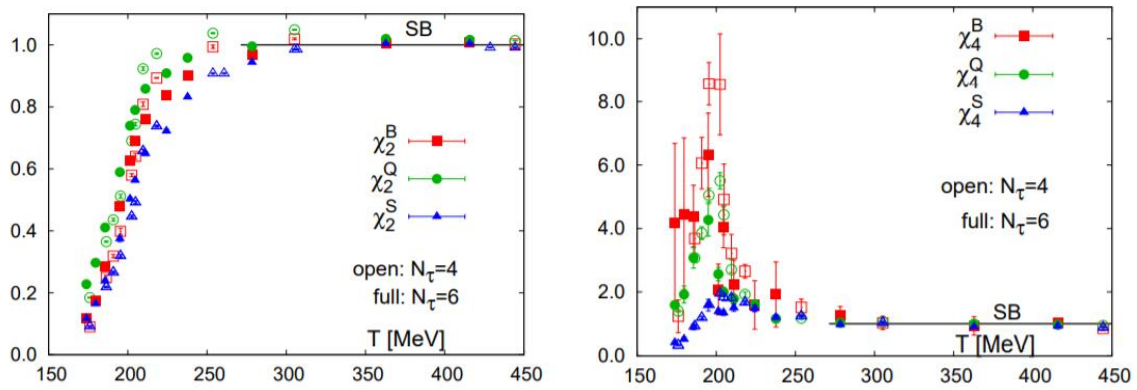


Figure 4 – The quadratic and quartic fluctuations of baryon number, electric charge and strangeness, given by (CHENG et al., 2009)

1.4 OUTLINES

In chapter 2 and 3, we mainly review some theoretical backgrounds, QCD theory and hydrodynamics. In chapter 2, The QCD theory, including its two main properties, color confinement and asymptotic freedom, is revisited. The asymptotic freedom property of QCD allows that one can apply perturbation theory to the physical processes with high momentum transfer. For the physical process of low momentum transfer, some non-perturbative theories should be adopted, such as Lattice QCD and sum rules. Another property of QCD, color confinement, indicates gluons can not be observable. Thus the only way to investigate the properties of QGP is to study the observed hadrons at the final stage of ultra-relativistic heavy ion collisions. Hydrodynamics which can describe heavy ion collision process is introduced in chapter 3.

Because the EoS is essential to the hydrodynamic analysis, so that some different models or theoretical tools to describe the thermodynamic properties of QCD matter is shown in chapter 4. In this chapter, we adopt the HRG model with excluded volume correction to obtain the EoS for hadronic matter and pQCD calculations or quasi-particle model fitting with Lattice QCD results to obtain the one for QGP. We also discussed the thermodynamic self-consistency in HRG model with excluded volume correction and as well the thermodynamic inconsistency for quasi-particle model with a temperature-dependent mass. Then, some solutions for solving this inconsistency are presented. Finally, we proposed the general solutions coming from the Maxwell equation and give a special solution with a temperature-, chemical potential- and momentum- dependent mass.

In chapter 5, we present a hybrid EoS with a phenomenological CEP at finite chemical potential. The results we obtained coincide with the Lattice data. Then we focus on the noncritical aspects of the multiplicity fluctuations in heavy-ion collisions by employing a hydrodynamic model. The calculated multiplicity fluctuations are compared to those of the hadronic resonance gas model, UrQMD model as well as to the experimental data.

The conclusions and summaries are presented in the last chapter 6

2 QUANTUM CHROMODYNAMICS

Nuclear physics, as a new branch of physics, is a subject of both profound theoretical significance and great practical significance. In 1911, Rutherford *et al* bombarded various atoms with α rays to observe the deflection of α rays which proved the nuclear structure of atoms. Thus the planetary model of atomic structure was established, which laid a foundation for the study of atomic structure. Soon afterwards, we made a preliminary understanding of the shell structure of atoms and the motion of electrons. Based on this understanding, the quantum mechanics was established to describe the motion rules of matter in the microscopic world. In the late 1920s, the principle of accelerating charged particles was being explored. By the early 1930s, electrostatic, linear and cyclotron accelerators had taken shape, and initial nuclear reaction experiments were carried out on high-voltage multipliers. By using accelerators, stronger beams, higher energies and more kinds of beam can be obtained, which greatly expands the research work of nuclear reactions. Since then, accelerators have gradually become necessary tools for studying nuclei. Hundreds of short-lived particles, known as baryons, mesons, leptons and a variety of resonant states, have been discovered through the interaction of high-energy and ultra-high-energy ray beams with atomic nuclei. The discovery of large families of particles has taken the study of the physical world to a new stage and established a new discipline, particle physics, also called high-energy physics. In the past, through the study of macroscopic objects, one knows that there are two kinds of long range interaction, electromagnetic interaction and gravitational interaction; A closer look at the nuclei of atoms reveals that there are two short-range interactions between matter, the strong interaction and the weak interaction. It has become an important topic of particle physics to study the laws of these four interactions and their possible relations, and to explore other possible new interactions. There is no doubt that nuclear physics research will make important new contributions in this regard.

The gauge theory that describes strong interactions is called QCD (APPELQUIST; POLITZER, 1975; BARBIERI; GATTO; KÖGERLER, 1976; RÚJULA; GLASHOW, 1975). It describes the interactions between the quarks that make up the strongly interacting particles (hadrons) and the gauge fields associated with the quantum number of color. It can uniformly describe the structure of hadrons and the strong interactions between them. The strong interaction satisfies SU(3) group symmetry. The lagrangian established by non-Abelian gauge invariance is

$$\mathcal{L} = \sum_f \bar{\Psi}_f(x) (i\gamma^\mu D_\mu - m_f) \Psi_f(x) - \frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu}, \quad (2.1)$$

where $F_{\mu\nu}^a \equiv (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc} A_\mu^b A_\nu^c)$. An important feature of the Lagrangian in Eq.(2.1) is the occurrence of the self-interaction of the gauge field, which comes from the kinetic energy term $\frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu}$. There is an interaction between three gauge fields and four gauge fields in the Lagrangian. This feature is unique to non-abelian gauge fields. It is the existence of self-interaction of gauge field that makes the non-abelian gauge field theory appear a completely new world. For example, the asymptotic freedom in QCD is an inevitable result of the self-interaction of the gauge field. In quantum

field theory, the β -function is negative and has the anti-shielding property to make the effective coupling constant $\alpha_s(Q^2)$ decreases with the increase of Q^2 , that is, the property of asymptotical freedom. In this field theory, the mediators in the strong interaction are massless gluons. Besides those gluons are colored, and they produce a transfer between different colored quarks, and they transfer strong interactions. Because the gluons have color charge, the interaction between gluons can cause anti-shielding effect, which determining the asymptotic free nature of the strong interaction.

The basic degrees of freedom of QCD are quarks and gluons, which are tightly bounded inside the hadron. They cannot present free states and can only be observed indirectly by hadron experiments. In QCD theory, the three-generations quark and gluons have quark-gluon interactions. There are also three gluon and four gluon interaction vertices. It's these vertices that determine the strong interaction coupling constant g_s decreases as the energy increases and the β -function closely related to g_s is negative value. Therefore, the effective coupling constant in strong interaction satisfies

$$\alpha_s(Q^2) = \frac{4\pi}{\beta_0 \log \frac{Q^2}{\Lambda^2}} \left[1 - \frac{\beta_1}{\beta_0^2} \frac{\log \left(\log \frac{Q^2}{\Lambda^2} \right)}{\log \left(\frac{Q^2}{\Lambda^2} \right)} \right] + \dots, \quad (2.2)$$

where $\alpha_s = g_s^2/(4\pi)$, $\beta_0 = -(\frac{2}{3}N_f - 11)$ and $\beta_1 = \frac{2}{3}(153 - 19N_f)$ are the β -function value for the one-loop and two-loop approximation respectively. N_f is the flavor of quark. From the above Eq.(2.2), it can be seen that when the energy Q^2 tends to infinity, the strong interaction coupling constant $\alpha_s(Q^2)$ tends to zero, as shown in Figure 5. This means that the interaction between quarks tends to zero. Eq.(2.2) also quantitatively expressed the strongly interacting asymptotic properties of freedom. The correctness of Eq.(2.2), which reflects the asymptotic freedom, is proved by experiments. To visually reflect this feature of the coupling, the constant is also called the running coupling constant.

Perturbation theory had made a great success due to the asymptotic freedom of QCD. However, perturbation theory can only be applied to the physical processes with high momentum transfer. For the physics with low momentum transfer and hadronic structures, it is powerless. Among the six kinds of quarks in nature, the first five quarks (u, d, s, c, b) exist only within the hadronic bound states. Meanwhile, the t quark as the heaviest quark had a very short life after its creation and soon decayed into the b quark. From Eq.(2.2), one can also see that the running coupling constant increases to infinity as the energy Q^2 decreases. This qualitatively explains why quarks cannot be separated in free states within hadrons. As we known, when the distance between quarks increases, the energy Q^2 of the exchanged gluons between quarks would decrease. Therefore, the running coupling constant would increase, as the increase of the distance. In the other word, the interactions between the quarks increase as the distance between them increases, which means the quarks and gluons would be forever bounded into hadrons. Thus, such a physical phenomenon is known figuratively as "quark confinement". In the QCD framework, the physicist can only qualitatively explain the structure of quarks trapped inside hadrons, but how to quantitatively explain the structures image of quarks trapped inside hadrons is still a major problem in high energy physics. Lattice gauge theory is trying to solve the problem of quark confinement from QCD theory. Since lattice gauge theory is essentially non-perturbation theory, its theoretical method does not depend on the interaction, so scientists are trying to get all the solutions to the strong interactions, not just asymptotic ones by the solution. Asymptotic freedom

and quark confinement are two important features of QCD theory. For now, one thought the quark confinement is based on the property of physical vacuum in QCD and perturbation QCD theory is based on perturbation vacuum. While the QCD physical vacuum is completely different from the perturbed vacuum. In a physical vacuum, it's full of quarks, antiquark pairs and gluons. The continuous interaction of matter, quark-antiquark pairs with gluons results in new images of hadron structures. Thus revealing the nature of vacuum will lead to the discovery of the quark confinement puzzle. At present, the relativistic heavy ion collider in brookhaven is to reveal the properties of physical vacuum experimentally. The idea is to liberate quarks and gluons from protons and neutrons under extreme conditions, and to make the transition from the confinement phase of a quark to the deconfinement phase of a quark. Only by fully understanding asymptotic freedom and quark confinement can one say that one has a profound understanding of strong interactions.

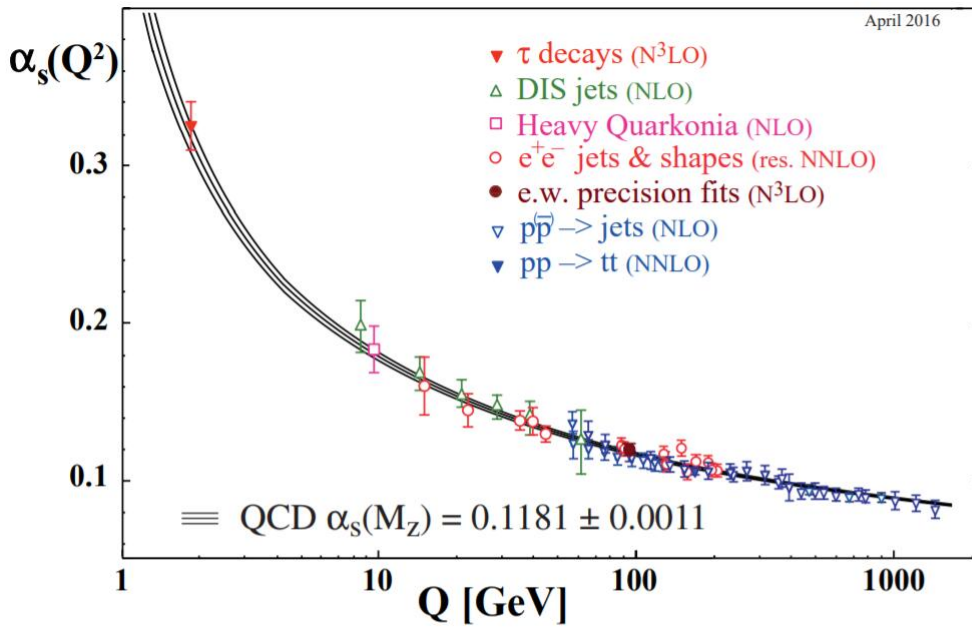


Figure 5 – The running behavior of QCD strong coupling constant from the Particle Data Group (TANABASHI et al., 2018) . The world average value and uncertainty of $\alpha_s(M_Z)$ is also given, here M_Z is the mass of Z^0 boson

As we had discussed before, although pQCD can be applied to compute the physical observables in high-energy processes successfully, it will be no longer applicable for the low-energy processes. The momentum transfer in a low-energy process is small which will lead to a relatively large α_s , as shown in Figure 5. To be more specific, the perturbation expansion of the physical observables will no longer converge, if the value of the running coupling constant is close to or even greater than 1, *i.e.*, $\alpha_s \sim 1$. Therefore, some other non-perturbative methods, such as lattice gauge theory, Schwinger-Dyson equation and sum rule techniques should be introduced to describe this process.

3 HYDRODYNAMICS

According to the color confinement of QCD theory, it is thought that hadrons are consisting of the confined quarks and gluons. Meantime, this property of QCD means the isolated quarks or gluons can not be observed. However, the hadronic matter would be transferred into a new QCD phase—QGP in extreme conditions (extremely high temperature and densities) due to Debye screening effect. It is easy to be told that QGP should exist in the core of neutron stars or at the very early universe for their extremely high temperature and densities environments. It is almost impossible to study the properties of QGP matter in the inner core of neutron stars or at the very early universe after the Big Bang. Fortunately, such extreme conditions can be formed by colliding two heavy nuclei at ultra-relativistic energy. After decades of years of efforts, more and more evidences showed that the QGP matter can be produced at the high energy collisions of heavy nucleus. In 2000, “a new state of matter” seemed to be discovered in SPS energy ($\sqrt{s} \sim 17\text{GeV}$) at the European Organization for Nuclear Research (CERN) while there is no solid evidence. Then, the RHIC at Brookhaven National Laboratory (BNL) found the unambiguous evidences and annouced the existence of QGP. Currently, the heavy ion collider is a only reasonable tool to study the properties of deconfined QGP matter on earth.

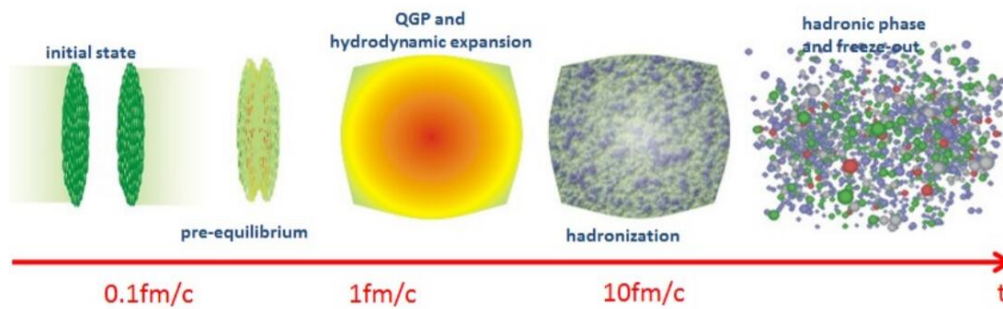


Figure 6 – Different dynamical evolution stages of heavy-ion collisions (QIN, 2015)

Figure 6 illustrates the different dynamical evolution stages of heavy-ion collisions with ultra-relativistic energy. Before the collisions, the speed of two nuclei will be almost accelerated up to the speed of light in the laboratory frame. Then the nuclear matter (extremely hot and dense matter) in a highly-excited state is produced after these two Lorentz contracted nuclei collide. Such a highly-excited nuclear matter can be called pre-equilibrium matter, because it is not in equilibrium yet. After a certain thermalisation time (about 1 fm/c), the pre-equilibrium matter will reach local thermal equilibrium. Then the equilibrated QGP matter expands and cools down. When the temperature reaches the critical temperature $T_c \simeq 150 \text{ MeV}$ (BORSANYI et al., 2010; BAZAVOV et al., 2012b; BHATTACHARYA et al., 2014), the QCD phase transition from QGP to hadronic matter occurs. It is worth mentioning that, the formed hadronic matter is consisting of stable and unstable hadrons and hadrons resonances. Then, the hadronic matter also expands while the QCD phase transition takes place. The hadrons will freeze-out since the system continues to cool and expand. Finally, the hadrons at the freeze-out surface will be detected in the detector. However, the QGP matter can not be observed directly in the detector. To study the properties of QGP matter in energetic heavy-ion collisions, one should find a reasonable

connection between the QGP produced at the beginning and the final hadrons observed in the detector and identify reliable signatures for the formation of QGP.

Here, we use the hydrodynamic model to describe heavy ion collision process (WONG, 1994; CHAUDHURI, 2014). However, hydrodynamics can not be applied to describe the initial thermalization process. Thus, in hydrodynamics, one assumes that the local thermal equilibrium is already achieved and such a state of matter can be specified by some appropriate macroscopic variables, like thermodynamical quantities and flow velocity. We also call those macroscopic variables the initial conditions (IC). Then the hydrodynamical model can simulate the relativistic heavy ion collisions from the equilibrium stage to the hadronic phase and freeze-out stage successfully (KOLB et al., 2001; KOLB; HEINZ, 2003; TEANEY, 2003; CHAUDHURI, 2008). In another word, hydrodynamics can describe the evolution of the QGP and hadronic matter with a given and appropriate IC. In the following Sec. 3.1, 3.2 and 3.3, we will introduce the equation of motion (EoM) for the evolution, the Cooper-Frye prescription for the decoupling process and the IC.

3.1 EQUATION OF MOTION

As discussed before, it is assumed that the expanding system possesses local thermal equilibrium. Hydrodynamics suggests that the expansion is collective, which can essentially be described by conservation equations, such as energy-momentum, baryon number, strangeness and isotopic spin, etc.

$$\partial_\mu T^{\mu\nu} = 0, \quad (3.1)$$

$$\partial_\mu J_i^\mu = 0, \quad (3.2)$$

where

$$T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu - pg^{\mu\nu} \quad (3.3)$$

is the energy-momentum tensor, ε , p , and u^μ are the energy density, the pressure and the four-velocity of the fluid respectively. J_i^μ is the conserved current which can be specified by some conserved number, like baryon number

$$J_B^\mu = n_B u^\mu, \quad (3.4)$$

or strangeness,

$$J_S^\mu = n_S u^\mu, \quad (3.5)$$

where n_B and n_S are the baryon number density and strangeness density.

In this thesis, we consider the fluid in Minkowski space where the metric tensor is

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (3.6)$$

Then four vectors, like the trajectory $x^\mu = (t, x, y, z)$, should transform as follows

$$x_\mu = g_{\mu\nu} x^\nu = (t, -x, -y, -z). \quad (3.7)$$

However, the (τ, x, y, η) coordinate is more suitable to solve the hydrodynamical equations rather than (t, x, y, z) , where τ is the proper time. Those two coordinates are shown in Figure 7

$$x^\mu = (t, x, y, z) \rightarrow x^m = (\tau, x, y, \eta), \quad (3.8)$$

where

$$\begin{aligned} t &= \tau \cosh \eta; & \tau &= \sqrt{t^2 - z^2}, \\ z &= \tau \sinh \eta; & \eta &= \frac{1}{2} \ln \left(\frac{t+z}{t-z} \right). \end{aligned} \quad (3.9)$$

The metric tensors in this coordinate system are $g^{mn} = \text{diag}(1, -1, -1, -1/\tau^2)$ and $g_{mn} = \text{diag}(1, -1, -1, -\tau^2)$.

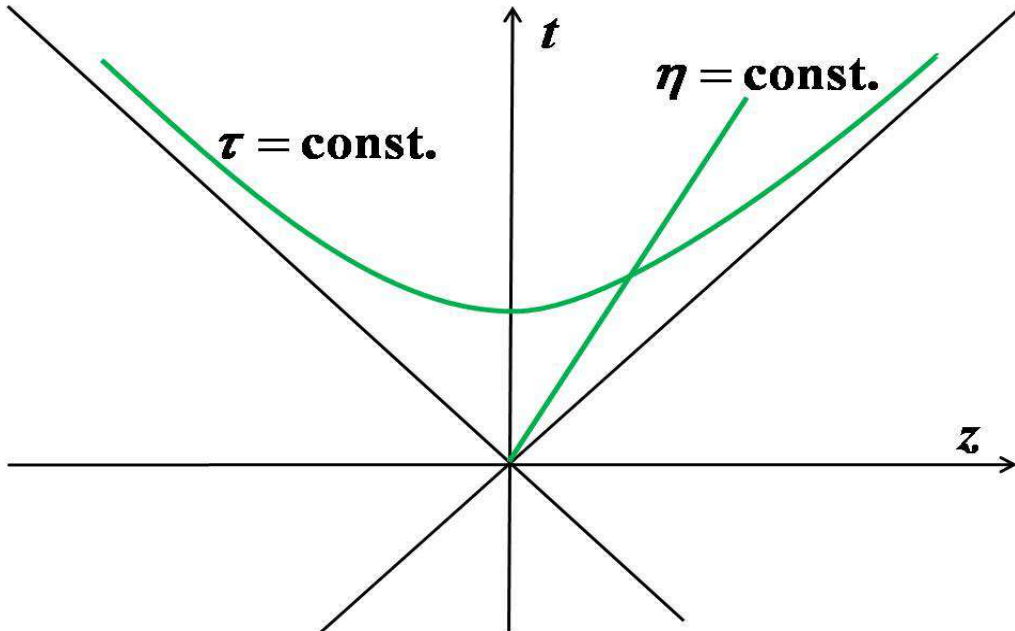


Figure 7 – A schematic diagram for (τ, x, y, η) and (t, x, y, z) coordinates taken from (SONG, 2009)

Thus, it is easy to find that the four-velocity $u^\mu = (u^0, \vec{u})$ is normalized,

$$u^\mu u_\mu = 1, \quad (3.10)$$

whose definition is $u^\mu = \frac{dx^\mu}{d\tau}$.

Following the procedure suggested by (ELZE et al., 1999; ELZE et al., 1998; HAMA; KODAMA; SOCOLOWSKI JR., 2005), we can derive the equations of relativistic hydrodynamics, *i.e.* Eqs. (3.1)-(3.5) by using the variational formulation. Firstly, for a system with the proper energy density ε , its action is

$$I = \int d^4x \{-\varepsilon\}. \quad (3.11)$$

Meantime, we also introduce other two conserved number which is the proper baryon density, n , and proper entropy density, s ,

$$\begin{aligned} \partial_\mu(nu^\mu) &= 0, \\ \partial_\mu(su^\mu) &= 0. \end{aligned} \quad (3.12)$$

Additionally, the energy density can be expressed in form of n and s ,

$$\varepsilon = \varepsilon(n, s). \quad (3.13)$$

The other thermodynamics, like pressure, temperature and baryon chemical potential can be obtained from the following thermodynamic relations,

$$\begin{aligned} p &= n \frac{\partial[\varepsilon(n, s)/n]}{\partial n}, \\ T &= \frac{\partial\varepsilon(n, s)}{\partial s}, \\ \mu &= \frac{\partial\varepsilon(n, s)}{\partial n}. \end{aligned} \quad (3.14)$$

Then, with the aid of constraints Eqs.(3.10) and (3.12), the hydrodynamical EoM for the fluid can be obtained by the Lagrangian equation with respect to n , s and u^μ . By introducing the Lagrangian multipliers, λ , ζ and w , we can merge the previous conditions, Eqs.(3.10) and (3.12) into the variational principle. Then we have

$$\delta \int d^4 \left[-\varepsilon(n, s) + \lambda \partial_\mu(nu^\mu) + \zeta \partial_\mu(su^\mu) - \frac{1}{2}w(u^\mu u_\mu - 1) \right] = 0. \quad (3.15)$$

In other word, we can obtain the hydrodynamical fluid in terms of the effective Lagrangian, which is written as

$$\begin{aligned} \mathcal{L}_{eff}^{(fluid)}(n, s, u^\mu, \lambda, \zeta, w) &= -\varepsilon(n, s) - nu^\mu \partial_\mu \lambda \\ &\quad - su^\mu \partial_\mu \zeta - \frac{w}{2}(u^\mu u_\mu - 1). \end{aligned} \quad (3.16)$$

Here all the variational variables in the brackets, *i.e.*, n , s , u^μ , λ , ζ and w are independent. Therefore the EoM for the fluid can be obtained from the Lagrange's equations

$$\partial_\mu \left(\frac{\partial \mathcal{L}(\alpha)}{\partial (\partial_\mu \alpha)} \right) - \frac{\partial \mathcal{L}(\alpha)}{\partial \alpha} = 0. \quad (3.17)$$

It is easy to derive that the variations with respect to the Lagrangian multipliers λ , ζ and w show the constraints, Eqs.(3.10) and (3.12). And the variations with respect to n, s and u^μ lead to

$$-\mu - u^\mu \partial_\mu \lambda = 0, \quad (3.18)$$

$$-T - u^\mu \partial_\mu \zeta = 0, \quad (3.19)$$

$$-n \partial_\mu \lambda - s \partial_\mu \zeta - w u_\mu = 0. \quad (3.20)$$

After solving the above equations with the constraint, Eq.(3.10), we have

$$\begin{aligned} w &= n\mu + Ts \\ &= \varepsilon + p, \end{aligned} \quad (3.21)$$

where p is the pressure and w is shown as the enthalpy density. Then we substitute back this known w into Eq.(3.20) and multiplying the both sides of Eq.(3.20) by u_ν , we get

$$w u_\mu u_\nu = -(n u_\nu)(\partial_\mu \lambda) - (s u_\nu)(\partial_\mu \zeta). \quad (3.22)$$

Thus, with the aid of Eqs.(3.18) and (3.19), it is not difficult to be derived

$$\begin{aligned} \partial^\nu (w u_\mu u_\nu) &= - \partial^\nu (n u_\nu)(\partial_\mu \lambda) - \partial^\nu (s u_\nu)(\partial_\mu \zeta) \\ &\quad - (n u_\nu)(\partial^\nu \partial_\mu \lambda) - (s u_\nu)(\partial^\nu \partial_\mu \zeta) \\ &= - (n u_\nu)(\partial^\nu \partial_\mu \lambda) - (s u_\nu)(\partial^\nu \partial_\mu \zeta), \end{aligned} \quad (3.23)$$

with

$$\begin{aligned} (n u_\nu)(\partial^\nu \partial_\mu \lambda) &= n \partial_\mu (u_\nu \partial^\nu \lambda) - n (\partial^\nu \lambda)(\partial_\mu u_\nu) \\ &= -n \partial_\mu \mu - n (\partial^\nu \lambda)(\partial_\mu u_\nu), \end{aligned} \quad (3.24)$$

and

$$(s u_\nu)(\partial^\nu \partial_\mu \zeta) = -s \partial_\mu T - s (\partial^\nu \zeta)(\partial_\mu u_\nu). \quad (3.25)$$

By using Eq.(3.20) and the Gibbs-Duhem relation,

$$dp = s dT + n d\mu, \quad (3.26)$$

and the property,

$$u^\nu (\partial_\mu u_\nu) = 0, \quad (3.27)$$

the Eq.(3.23) can be rewritten as

$$\begin{aligned} \partial^\nu (w u_\mu u_\nu) &= n \partial_\mu \mu + s \partial_\mu T + (\partial_\mu u_\nu) w u^\nu \\ &= \partial_\mu p + u^\nu (\partial_\mu u_\nu) w \\ &= \partial_\mu p, \end{aligned} \quad (3.28)$$

which means

$$\partial^\nu ((\varepsilon + p)u_\mu u_\nu - pg_{\mu\nu}) = 0, \quad (3.29)$$

or

$$\partial_\mu ((\varepsilon + p)u^\mu u^\nu - pg^{\mu\nu}) = 0. \quad (3.30)$$

The relativistic hydrodynamic equation (3.1) is revisited.

3.2 COOPER-FRYE PRESCRIPTION

In the previous section, we have shown that the EoM governs the space-time evolution of the fluid. As the fluid further expands, the temperature and density of the system become lower and lower. In the end, the decoupling of the constituent particles occurs, which means that there will be no interactions among them. Such a decoupling or freeze-out process can be described by using Cooper-Frye prescription (COOPER; FRYE, 1974). There are also some other models to describe the freeze-out process, such as continuous emission and transport model, but we don't address them in this thesis. According to Cooper-Frye prescription, the constituent particles in the fluid will suddenly stop interacting when meets the freeze-out conditions, namely temperature T arrives at the freeze-out temperature T_{fo} . Figuratively speaking, considering a three-dimensional freeze-out hypersurface σ in a four-dimensional space, the temperature on this surface is freeze-out temperature T_{fo} which can be characterized as

$$T_{fo} = T(\tau, x, y, \eta). \quad (3.31)$$

Then we can obtain the particle spectrum by counting the number of hadrons crossing σ . That is to say, the total number of particles crossing σ , N , can be expressed as

$$N = \int_\sigma d\sigma_\mu j^\mu, \quad (3.32)$$

where $d\sigma_\mu$ is an infinitesimal element who is perpendicular to σ and towards outside. j^μ is the current of particles, which is

$$j^\mu = \int \frac{d^3p}{E} p^\mu f(x, p). \quad (3.33)$$

Here E is the energy of the particle, p^μ is four-momentum and $f(x, p)$ denotes the one particle distribution function. In case of the ideal hydrodynamics, $f(x, p)$ should correspond to Lorentz-covariant equilibrium distribution function

$$f(x, p) = \frac{g_i}{2\pi^3} \frac{1}{(e^{(p^\mu u_\mu - \mu_B)/T} \pm 1)}. \quad (3.34)$$

Here the signs "+" and "-" in the denominator stand for Fermi-Dirac and Bose-Einstein distribution respectively. g_i is the degeneracy of the particle specie i . The factor $p^\mu u_\mu$ in the exponent is the energy of the particle in the local rest frame.

In differential form, we get

$$E \frac{dN}{d^3p} = \int_\sigma p^\mu f(x, p) d\sigma_\mu. \quad (3.35)$$

This formula shows the invariant distribution of particles described by the Cooper-Frye prescription (COOPER; FRYE, 1974). After considering the unstable particles decay and resonances effect, the calculated observable quantities can be compared with experimental data.

3.3 INITIAL CONDITIONS

For one-dimensional systems satisfying Bjorken scaling, as long as the density or temperature at some initial time $\tau_0 \sim 1 fm/c$ is given, the evolution of the density or temperature of the fluid can be obtained. Besides, the freeze-out temperature could be determined by fitting with the experimental data. Consequently, hydrodynamic analysis of ultra-relativistic heavy ion is actually an initial value problem.

For (3+1)-dimensional relativistic hydrodynamic evolution and freeze-out, we use the hydrodynamic code SPheRIO which is developed by the Sao Paulo-Rio de Janeiro Collaboration. In order to obtain the IC, one may use the event generator, such as HIJING (GYULASSY; RISCHKE; ZHANG, 1997), VNI (SCHLEI; STROTTMAN, 1999), URASiMA (NONAKA; HONDA; MUROYA, 2000), the NeXuS (DRESCHER et al., 2002), the string model (MAGAS; CSERNAI; STROTTMAN, 2001) and color glass condensate (CGC) (HIRANO; NARA, 2004; LUZUM; ROMATSCHKE, 2008; ROY; CHAUDHURI; MOHANTY, 2012). The NeXuS is adopted in this thesis. It is worth mentioning that the IC for hydrodynamics given by event generator is not necessary to correspond to the one of local equilibrium. That is to say, the matter produced by event generator can be not in local equilibrium, or in other word, the energy-momentum tensor $T^{\mu\nu}$ can be not diagonal. We can adopt certain procedures to convert such energy-momentum tensor to the one of local equilibrium.

Following the procedure suggested by Landau (LANDAU; LIFSHITZ, 1987), we know the following eigenvalue equation

$$T^\mu_\nu u^\nu = \varepsilon u^\mu, \quad (3.36)$$

where the four-velocity u^μ is used as a normalized time-like eigenvector and the energy density ε is the corresponding eigenvalue. Then, according to Eqs.(3.4) and (3.5), the proper baryon and strangeness density in this frame can be expressed as

$$n_B = J_B^\mu u_\mu \quad (3.37)$$

and

$$n_S = J_S^\mu u_\mu. \quad (3.38)$$

Finally, the other thermodynamic quantities can be obtained from the thermodynamic relations.

4 EQUATION OF STATE

In this chapter, we mainly introduce the HRG model for describing the hadronic phase, as well as the quasi-particle model and pQCD calculations fitted with Lattice QCD results for the QGP phase. Moreover, we also develop the QCD equation of state by the requirement of thermodynamic self-consistency.

Generally speaking, QGP only exists in a short period after heavy-ion collisions. Afterwards, the density and temperature of the system will decrease rapidly. The QGP phase will be switched to hadronic phase at low temperature and density. As discussed in Introduction, Lattice QCD can be used to study the QCD properties at finite temperature T and relatively low chemical potential. Meantime, there are some attempts to study thermal properties of the QGP such as dimensional reduction (GINSPARG, 1980; APPELQUIST; PISARSKI, 1981; NADKARNI, 1988), hard thermal loop (HTL) resummation scheme (PISARSKI, 1989; FRENKEL; TAYLOR, 1990; BLAIZOT; IANCU; REBHAN, 1999; BLAIZOT; IANCU; REBHAN, 2003; ANDERSEN; BRAATEN; STRICKLAND, 1999; IPP; REBHAN; VUORINEN, 2004; MOGLIACCI et al., 2013), Polyakov-loop model (PISARSKI, 2000; DUMITRU; PISARSKI, 2001), as well as approaches in terms of hadronic degrees of freedom (ASAKAWA; KO, 1994; LENAGHAN; RISCHKE; SCHAFFNER-BIELICH, 2000; RODER; RUPPERT; RISCHKE, 2003). However, in order to generalize the EoS at high chemical potential and a wide range of temperature, we adopt the phenomenological model, quasi-particle model or pQCD calculations with some free parameters for fitting to Lattice QCD results at vanishing chemical potential. The results show that the quasi-particle model can reproduce the Lattice QCD results quite well not only in zero chemical potential but also at low finite chemical potential.

The arrangement of this chapter is shown as follows. In Sec.4.1, we introduce some fundamental concepts of standard statistical mechanics. HRG model and its recent developments are shown in Sec.4.2. In the Sec.4.3, the EoS obtained from pQCD calculations fitted with Lattice QCD results are listed. In Sec.4.4, we review thermodynamical inconsistency which occurs in quasi-particle model at finite temperature and chemical potential, and some solutions for solving this inconsistency. The general solution with a momentum dependent mass is given in the last Sec. 4.5.

4.1 A BRIEF REVIEW OF STATISTICAL MECHANICS

Here we consider an open system in contact with a thermal and particulate reservoir, which means not only the temperature of the reservoir is hold at T but also the reservoir can exchange particles with the system. The system will achieve thermal and chemical equilibrium when its temperature arrives at T with a certain chemical potential μ . Then it is interesting to use the T and μ instead of S and N for describing the macroscopic state of the system. To be more specific, we suppose a microscopic state i of our system whose occupation numbers are

$$i \equiv \{n_1, n_2, \dots, n_j, \dots\}, \quad (4.1)$$

where n_j is the number of particles which are in the j th “single particle state”.

Thus, for the possible microscopic state i , its total energy E_i and number of particles N_i can be written as

$$E_i = \sum_j n_j \varepsilon_j, \quad (4.2)$$

$$N_i = \sum_j n_j. \quad (4.3)$$

Here, following the definition of statistical mechanics, we define the partition function for the grand canonical ensemble as,

$$Z = Z(V, \beta, \alpha) \equiv \sum_i \exp(-\beta E_i - \alpha N_i), \quad (4.4)$$

with the probability for finding a given microstate i in the grand ensemble

$$P_i = \frac{\exp(-\beta E_i - \alpha N_i)}{Z}. \quad (4.5)$$

Then the total energy and particle number of the system can be expressed as

$$\begin{aligned} \langle E \rangle &= \frac{\sum_i E_i \exp(-\alpha N_i - \beta E_i)}{\sum_i \exp(-\alpha N_i - \beta E_i)}, \\ \langle N \rangle &= \frac{\sum_i N_i \exp(-\alpha N_i - \beta E_i)}{\sum_i \exp(-\alpha N_i - \beta E_i)}, \end{aligned} \quad (4.6)$$

where α and β can be determined by comparing with the first law of thermodynamics, namely,

$$d\langle E \rangle = TdS - pdV + \mu d\langle N \rangle. \quad (4.7)$$

It is inferred that

$$\begin{aligned} \beta &= \frac{1}{k_B T}, \\ \alpha &= -\frac{\mu}{k_B T}. \end{aligned} \quad (4.8)$$

All the thermodynamical quantities can be expressed in terms of the grand partition function with respect to α and β ,

$$\langle E \rangle = -\frac{\partial \ln Z(V, \beta, \alpha)}{\partial \beta}, \quad (4.9)$$

$$\langle N \rangle = -\frac{\partial \ln Z(V, \beta, \alpha)}{\partial \alpha}, \quad (4.10)$$

$$\begin{aligned} \langle S \rangle &= -k_B \langle \ln P_i \rangle = k_B \sum_i (\beta E_i + \alpha N_i) + k_B \ln Z \\ &= \frac{1}{T} (\langle E \rangle - \mu \langle N \rangle) + k_B \ln Z. \end{aligned} \quad (4.11)$$

When we assume that the volume of the system V goes to infinity, *i.e.*, the thermodynamic limits and the general extensive thermodynamical relation

$$\langle S \rangle = \frac{1}{T} \langle E \rangle - \frac{\mu}{T} \langle N \rangle + \frac{1}{T} pV \quad (4.12)$$

is valid, then we have

$$p = \frac{\partial \ln Z(V, \beta, \alpha)}{\partial V}, \quad (4.13)$$

which is in fact the “general force” for a grand canonical ensemble (GCE). Also it is not difficult to prove the following thermodynamic identities

$$\begin{aligned} \varepsilon &= Ts + \mu n - p, \\ s &= \left(\frac{\partial p}{\partial T} \right)_\mu, \\ n &= \left(\frac{\partial p}{\partial \mu} \right)_T. \end{aligned} \quad (4.14)$$

In the following, we present the basic expressions for thermodynamics for ideal Fermi gas and ideal boson gas case. For a Fermi gas, the occupation number of particles n_j for each state j is limited to be 0 or 1 for the Pauli exclusion principle. Thus, the partition function should be rewritten as

$$\begin{aligned} Z(V, \beta, \alpha) &= \sum_{\alpha} \exp(-\beta E_i - \alpha N_i) \\ &= \sum_{n_1} \sum_{n_2} \sum_{n_3} \cdots \sum_{n_j} \cdots \exp \left[- \sum_j n_j (\beta \varepsilon_j + \alpha) \right] \\ &= \prod_j \sum_{n_j=0,1} \exp[-n_j (\beta \varepsilon_j + \alpha)] \\ &= \prod_j [1 + e^{-n_j (\beta \varepsilon_j + \alpha)}] \\ &= \exp \left\{ \sum_j \ln [1 + e^{-n_j (\beta \varepsilon_j + \alpha)}] \right\} \end{aligned} \quad (4.15)$$

For bosons, there is no restriction for the occupation numbers which means the occupation number of

particles n_j for each state j should be summed from 0 to ∞ . Therefore, the partition function should be expressed as

$$\begin{aligned}
 Z(V, \beta, \alpha) &= \sum_{\alpha} \exp(-\beta E_i - \alpha N_i) \\
 &= \sum_{n_1} \sum_{n_2} \sum_{n_3} \cdots \sum_{n_j} \cdots \exp \left[- \sum_j n_j (\beta \varepsilon_j + \alpha) \right] \\
 &= \prod_j \sum_{n_j=0}^{\infty} \exp[-n_j (\beta \varepsilon_j + \alpha)] \\
 &= \prod_j \frac{1}{1 - e^{-n_j(\beta \varepsilon_j + \alpha)}}
 \end{aligned} \tag{4.16}$$

With the assumption

$$\beta \varepsilon_i + \alpha > 0,$$

we have

$$Z(V, \beta, \alpha) = \exp \left\{ - \sum_j \ln [1 - e^{-n_j(\beta \varepsilon_j + \alpha)}] \right\}. \tag{4.17}$$

Reminding that the single particle state j for an ideal gas is taken as a plane wave state of momentum k , we may replace the sum over states j by an integral in k

$$\sum_j \rightarrow \frac{gV}{(2\pi\hbar)^3} \int dk, \tag{4.18}$$

where g is the statistical factor of the particle. For simplicity, we will adopt the natural units in this thesis, where $\hbar = c = k_B = 1$. By substituting Eq.(4.78) into Eqs.(4.15) and (4.17), the partition function can be expressed in terms of β and chemical potential μ as

$$\ln Z(V, T, \mu) = \pm \frac{gV}{(2\pi)^3} \int d^3k \ln [1 \pm \exp(\beta(\varepsilon_k - \mu))]. \tag{4.19}$$

Here, ε_k is the energy of the state with momentum k . “+” represents for bosons and “-” stands for fermions respectively. Then, the total energy of the system is

$$\begin{aligned}
 \langle E \rangle &= - \frac{\partial}{\partial \beta} \ln Z(V, T, \mu) \\
 &= \frac{gV}{(2\pi)^3} \int d^3k \frac{\varepsilon_k e^{-\beta(\varepsilon_k - \mu)}}{1 + e^{-\beta(\varepsilon_k - \mu)}} \\
 &= \frac{gV}{(2\pi)^3} \int d^3k \frac{\varepsilon_k}{e^{\beta(\varepsilon_k - \mu)} + 1},
 \end{aligned} \tag{4.20}$$

and the total number of particles of the system becomes

$$\begin{aligned}\langle N \rangle &= \frac{1}{\beta} \frac{\partial}{\partial \mu} \ln Z(V, T, \mu) \\ &= \frac{gV}{(2\pi)^3} \int d^3k \frac{1}{e^{\beta(\varepsilon_k - \mu)} + 1}.\end{aligned}\quad (4.21)$$

From the above Eqs.(4.20) and (4.21), we know that the average of occupation number for the energy level ε_k is given by

$$f(\varepsilon_k) = \frac{1}{e^{\beta(\varepsilon_k - \mu)} + 1}. \quad (4.22)$$

The pressure is given by

$$p = \frac{g}{(2\pi)^3} \frac{1}{\beta} \int d^3k \ln [1 + e^{-\beta(\varepsilon_k - \mu)}]. \quad (4.23)$$

Finally, the entropy is calculated from

$$T\langle S \rangle = \langle E \rangle - \mu\langle N \rangle + pV. \quad (4.24)$$

4.2 HADRON RESONANCE GAS MODEL

At the final stage of heavy ion collisions, lots of hadronic matter are formed and they can be decided by the HRG model. Generally, we think that the interacting hadronic matter can be well approximated by a non-interacting gas of resonance. One way is to use Virial expansion to obtain an effective interaction. For example, in Ref (VENUGOPALAN; PRAKASH, 1992), thermodynamics of low temperature hadronic matter is studied by Virial expansion together with experimental phase shifts. It is meaningful to mention that, for the HRG with vanishing chemical potential, the contributions from pion for the thermodynamics dominate at $T \lesssim 120\text{MeV}$. If temperature is bigger than 150 MeV, the heavy states will dominate the energy density due to the excitation of more and more heavier resonances. However, the particle number density of the heavy states is still small, because the number density is proportional to a suppression factor, namely $n_i \sim e^{M_i/T}$

The expressions for energy density, pressure and number density for hadronic resonance gas can be expressed as

$$\begin{aligned}\varepsilon(T, \mu) &= \sum_{i=1}^N \varepsilon_i(T, \mu_i), \\ p(T, \mu) &= \sum_{i=1}^N p_i(T, \mu_i), \\ n(T, \mu) &= \sum_{i=1}^N n_i(T, \mu_i),\end{aligned}\quad (4.25)$$

where N represents for the number of the hadrons, T is the temperature and μ is the chemical potential of the system. n_i and $\mu_i = B_i\mu_B + S_i\mu_S + Q_i\mu_Q$ are the number and chemical potential of i th hadron, where B_i , S_i and Q_i is the baryon, strangeness and charge quantum numbers, respectively.

However, those expressions are obtained by assuming that the hadrons are served as point particles. The expressions can be corrected to account for finite size of hadrons. The correction is called “excluded volume correction”. The pressure of HRG with excluded volume correction (RISCHKE et al., 1991) can be determined by the following self-consistent equations

$$\begin{aligned} p^H(T, \mu_B, \mu_S, \mu_Q) &= \sum_{i=1} p_i^{id}(T, \tilde{\mu}_i), \\ \tilde{\mu}_i &\equiv \mu_i - v_i p^H, \end{aligned} \quad (4.26)$$

besides the energy density and particle number density can be obtained with the thermodynamic consistency shown in Appendix B,

$$\begin{aligned} \varepsilon(T, \mu_B, \mu_S, \mu_Q) &= \sum_i \frac{\varepsilon_i^{id}(T, \tilde{\mu}_i)}{1 + \sum_j v_j n_j^{id}(T, \tilde{\mu}_j)}, \\ n(T, \mu_B, \mu_S, \mu_Q) &= \sum_i \frac{n_i^{id}(T, \tilde{\mu}_i)}{1 + \sum_j v_j n_j^{id}(T, \tilde{\mu}_j)}. \end{aligned} \quad (4.27)$$

In (HAMA; KODAMA; SOCOLOWSKI JR., 2005), the excluded volume $v_i = (4\pi r_0^3/3)$, with $r_0 = 0.7 fm$ for baryons and $r_0 = 0$ for mesons.

In the case of zero baryon and strangeness density, one has $\mu_B = \mu_S = 0$. However, at finite baryon density, even though the strangeness density is zero, the strangeness chemical potential does not necessarily vanish. This is because in this case the net strangeness density from baryons and their anti-particles does not vanish at zero strangeness chemical potential, namely, the net strangeness density $n_S(\mu_B(\neq 0), \mu_S = 0) - n_S(\mu_B \rightarrow \mu_B, \mu_S = 0) \neq 0$. Thus the value of strangeness chemical potential has to be determined by solving Eq.(4.26) numerically.

We note that some improved HRG model with excluded volume correction (VOVCHENKO; GORENSTEIN; STOECKER, 2017) has been proposed recently. However, since there is no significant deviation between the models in the region of our interest, the HRG model used in (HAMA; KODAMA; SOCOLOWSKI JR., 2005) is adopted for our present study.

The particle species we used in the calculations are listed in the Appendix A. Those particles and their properties are taken from the Particle Data Group (TANABASHI et al., 2018)

4.3 PQCD CALCULATIONS FOR EQUATION OF STATE OF $N_f = 3$ QCD

QGP phase exists in extremely high temperature and/or high density environment. It is worth mentioning that under the extremely high temperature, far above the critical temperature, the effective QCD coupling constant will become very small. This nature of QCD is called the asymptotic freedom (POLITZER, 1973; GROSS; WILCZEK, 1973). Thus, it allows us to use the pQCD theory to describe the EoS of QGP phase at this region. There have been a lot of theoretical physicists committed to obtain the pressure expression from the perturbative calculations in the series of the weak coupling (SHURYAK, 1978b; CHIN, 1978; KAPUSTA, 1979; TOIMELA, 1983; ARNOLD; ZHAI, 1994; ARNOLD; ZHAI, 1995; ZHAI; KASTENING, 1995; KAJANTIE et al., 2003; VUORINEN,

2003b; VUORINEN, 2003a; IPP et al., 2006). Here the expression of pressure calculated up to the $g_s^6 \ln(1/g)$ order at finite temperature and/or finite chemical potential is adopted (VUORINEN, 2003b; VUORINEN, 2003a; IPP et al., 2006)

$$P = \frac{8\pi^2}{45} T^4 \left[p_0 + p_2 \left(\frac{\alpha_s}{\pi} \right) + p_3 \left(\frac{\alpha_s}{\pi} \right)^{3/2} + p_4 \left(\frac{\alpha_s}{\pi} \right)^2 + p_5 \left(\frac{\alpha_s}{\pi} \right)^{5/2} + p_6 \left(\frac{\alpha_s}{\pi} \right)^3 \right], \quad (4.28)$$

where

$$p_0 = 1 + \frac{21}{32} N_f \left(1 + \frac{120}{7} \hat{\mu}_q^2 + \frac{240}{7} \hat{\mu}_q^4 \right), \quad (4.29)$$

$$p_2 = -\frac{15}{4} \left[1 + \frac{5N_f}{12} \left(1 + \frac{72}{5} \hat{\mu}_q^2 + \frac{144}{5} \hat{\mu}_q^4 \right) \right], \quad (4.30)$$

$$p_3 = 30 \left[1 + \frac{1}{6} (1 + 12\hat{\mu}_q^2) N_f \right]^{3/2}, \quad (4.31)$$

$$\begin{aligned} p_4 = & 237.223 + (15.963 + 124.773\hat{\mu}_q^2 - 319.849\hat{\mu}_q^4) N_f \\ & - (0.415 + 15.926\hat{\mu}_q^2 + 106.719\hat{\mu}_q^4) N_f^2 \\ & + \frac{135}{2} \left[1 + \frac{1}{6} (1 + 12\hat{\mu}_q^2) N_f \right] \log \left[\frac{\alpha_s}{\pi} \left(1 + \frac{1}{6} (1 + 12\hat{\mu}_q^2) N_f \right) \right] \\ & - \frac{165}{8} \left[1 + \frac{5}{12} \left(1 + \frac{72}{5} \hat{\mu}_q^2 + \frac{144}{5} \hat{\mu}_q^4 \right) N_f \right] \left(1 - \frac{2}{33} N_f \right) \log \hat{M}, \end{aligned} \quad (4.32)$$

$$\begin{aligned} p_5 = & - \left(1 + \frac{1 + 12\hat{\mu}_q^2}{6} N_f \right)^{1/2} [799.149 + (21.963 - 136.33\hat{\mu}_q^2 + 482.171\hat{\mu}_q^4) N_f \\ & + (1.926 + 2.0749\hat{\mu}_q^2 - 172.07\hat{\mu}_q^4) N_f^2] \\ & + \frac{495}{2} \left(1 + \frac{1 + 12\hat{\mu}_q^2}{6} N_f \right) \left(1 - \frac{2}{33} N_f \right) \log \hat{M}, \end{aligned} \quad (4.33)$$

$$\begin{aligned} p_6 = & - [659.175 + (65.888 - 341.489\hat{\mu}_q^2 + 1446.514\hat{\mu}_q^4) N_f \\ & + (7.653 + 16.225\hat{\mu}_q^2 - 516.210\hat{\mu}_q^4) N_f^2 \\ & - \frac{1485}{2} \left(1 + \frac{1 + 12\hat{\mu}_q^2}{6} N_f \right) \left(1 - \frac{2}{33} N_f \right) \log \hat{M}] \log \left[\frac{\alpha_s}{\pi} \left(1 + \frac{1 + 12\hat{\mu}_q^2}{6} N_f \right) 4\pi^2 \right] \\ & - 475.587 \log \left[\frac{\alpha_s}{\pi} 4\pi^2 C_A \right]. \end{aligned} \quad (4.34)$$

Here we consider the system of massless particles with the flavor number $N_f = 3$, color number $N_c=3$ and $C_A=3$. M is the renormalization scale, μ_q is the chemical potential of quarks which equals $\mu_q = \mu_B/3$ and the superscript *hat* denotes the division by $2\pi T$ which means $\hat{M} = M/(2\pi T)$ and $\hat{\mu}_q = \mu_q/(2\pi T)$. The three-loop coupling is adopted here (TANABASHI et al., 2018)

$$\begin{aligned} \alpha_s = & \frac{1}{b_0 t} \left[1 - \frac{b_1 \ln t}{b_0^2 t} + \frac{b_1^2 (\ln^2 t - \ln t - 1) + b_0 b_2}{b_0^4 t^2} \right. \\ & \left. - \frac{b_1^3 (\ln^3 t - \frac{5}{2} \ln^2 t - 2 \ln t + \frac{1}{2}) + 3b_0 b_1 b_2 \ln t}{b_0^6 t^3} \right], \end{aligned} \quad (4.35)$$

where $b_i = \beta_i/(4\pi)^{i+1}$ with

$$\begin{aligned}
\beta_0 &= \frac{11}{3}C_A - \frac{4}{3}T_F N_f, \\
\beta_1 &= \frac{34}{3}C_A^2 - \frac{20}{3}C_A T_F N_f - 4C_F T_F N_f, \\
\beta_2 &= \frac{2857}{54}C_A^3 - \frac{1415}{27}C_A^2 T_F N_f + \frac{158}{27}C_A T_F^2 N_f^2 \\
&\quad + \frac{44}{9}C_F T_F^2 N_f^2 - \frac{205}{9}C_F C_A T_F N_f + 2C_F^2 T_F N_f, \\
\beta_3 &= C_A C_F T_F^2 N_f^2 \left(\frac{17152}{243} + \frac{448}{9}\zeta_3 \right) + C_A C_F^2 T_F N_f \left(-\frac{4204}{27} + \frac{352}{9}\zeta_3 \right) \\
&\quad + \frac{424}{243}C_A T_F^3 N_f^3 + C_A^2 C_F T_F N_f \left(\frac{7073}{243} - \frac{656}{9}\zeta_3 \right) + C_A^2 T_F^2 N_f^2 \left(\frac{7930}{81} + \frac{224}{9}\zeta_3 \right) \\
&\quad + \frac{1232}{243}C_F T_F^3 N_f^3 + C_A^3 T_F N_f \left(-\frac{39143}{81} + \frac{136}{3}\zeta_3 \right) + C_A^4 \left(\frac{150653}{486} - \frac{44}{9}\zeta_3 \right) \\
&\quad + C_F^2 T_F^2 N_f^2 \left(\frac{1352}{27} - \frac{704}{9}\zeta_3 \right) + 46C_F^3 T_F N_f + N_f \frac{d_F^{abcd} d_A^{abcd}}{N_A} \left(\frac{512}{9} - \frac{1664}{3}\zeta_3 \right) \\
&\quad + N_f^2 \frac{d_F^{abcd} d_F^{abcd}}{N_A} \left(-\frac{704}{9} + \frac{512}{3}\zeta_3 \right) + \frac{d_A^{abcd} d_A^{abcd}}{N_A} \left(-\frac{80}{9} + \frac{704}{3}\zeta_3 \right).
\end{aligned} \tag{4.36}$$

Under the renormalization scheme $\overline{MS}$, the above the β -function can be expressed as,

$$\begin{aligned}
\beta_0 &= \frac{33 - 2N_f}{3}, \\
\beta_1 &= \frac{153 - 19N_f}{6}, \\
\beta_2 &= \frac{1}{2} \left(2857 - \frac{5033}{9}N_f + \frac{325}{27}N_f^2 \right).
\end{aligned}$$

Here $t = \ln(M^2/\Lambda_{\overline{MS}}^2)$ and $\Lambda_{\overline{MS}}$ is the QCD scale under the scheme $\overline{MS}$

There exist two main doubts when we adopt the pQCD calculation.

- The first one is the convergence of the perturbative expansions.
- The second one is the renormalization scale ambiguity.

The typical momentum flow is commonly chosen as the renormalization scale of the corresponding perturbation process. And the perturbation expansion convergence may be improved through higher and higher order calculation. However, the difficulty of the perturbation calculation is exponentially increased with the increase of the order for pQCD calculation. Even worse is that the value for each order of the series in Eq.(4.28) are observed as oscillation shown in Figure 8 and the dependence on the renormalization scale increases rather than decreases. Then this perturbative series only becomes convergent at very high temperature $T \sim 10^5 T_c$. The renormalization scale setting method PMC may be a choice to improve the convergence and give a reasonable result (BU et al., 2018).

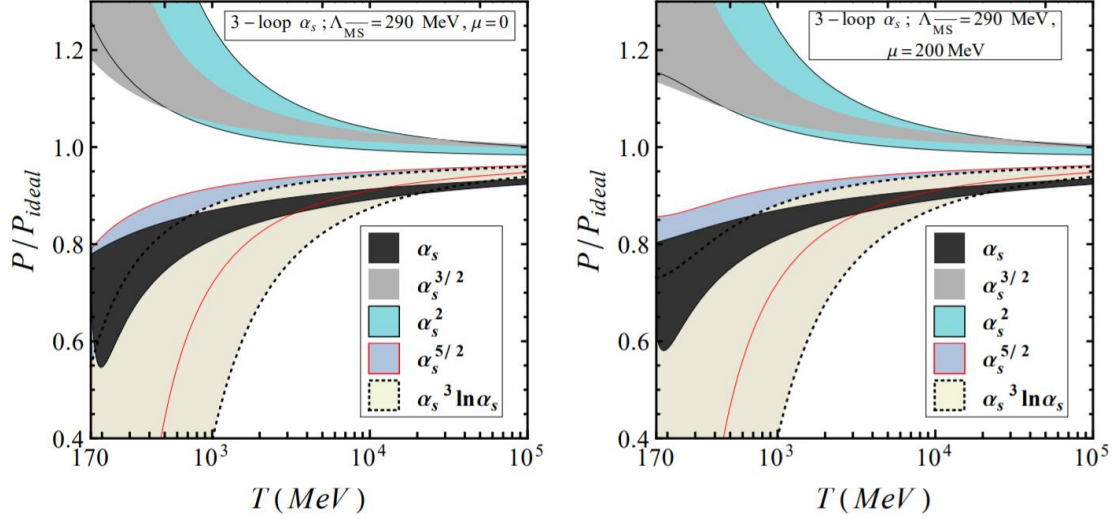


Figure 8 – (color line) The temperature dependence of the calculated ratio p/p_{ideal} from the pQCD calculation with $C_M = 0.5$ and $C_S = 0$ (HAQUE; MUSTAFA; STRICKLAND, 2013) for the different α_s order at $\mu_q = 0$ and $\mu_q = 200$ MeV. The shaded bands show the scale uncertainties by varying the renormalization scale $M \in [\sqrt{(\pi T)^2 + \mu_q^2}, 4\sqrt{(\pi T)^2 + \mu_q^2}]$.

In accordance with the general setting $M = \pi T$, we choose

$$M = C_M \sqrt{(\pi T)^2 + \mu_q^2}, \quad (4.37)$$

where C_M is a free parameter to fit with the Lattice results. Meanwhile, in order to avoid Landau pole produced under the low finite temperature and chemical potential, namely divergence of coupling constant, t should be revised as follows according to the Refs. (ALBRIGHT; KAPUSTA; YOUNG, 2014; ALBRIGHT, 2015),

$$t = \ln (C_S^2 + M^2/\Lambda_{MS}^2). \quad (4.38)$$

Here C_S is also a free parameter which is used to avoid Landau pole and fit with Lattice results.

Thus, we can present the pressure through the pQCD calculation. The other thermodynamics, like energy density, entropy density and number density can be derived through the thermodynamic relations. The corresponding results are shown in Figure 9 and Figure 10 where the results from the pQCD calculation near T_c can not reproduce the Lattice data. Thus, we adopted a phenomenological approach, called the quasi-particle model to describe the QGP phase.

4.4 QUASI-PARTICLE MODEL FOR QGP

With the increase of temperature and density, quarks and gluons are deconfined from hadrons and the hadronic phase matter is transferred into a new QCD phase called QGP. As suggested by Lattice simulations, this QCD phase transition will occur around the critical temperature $T_c \sim 150$ MeV (BORSANYI et al., 2010; BAZAVOV et al., 2012b; BHATTACHARYA et al., 2014). Under such circumstances, the perturbation theory is no longer applicable at such region, *i.e.*, around the

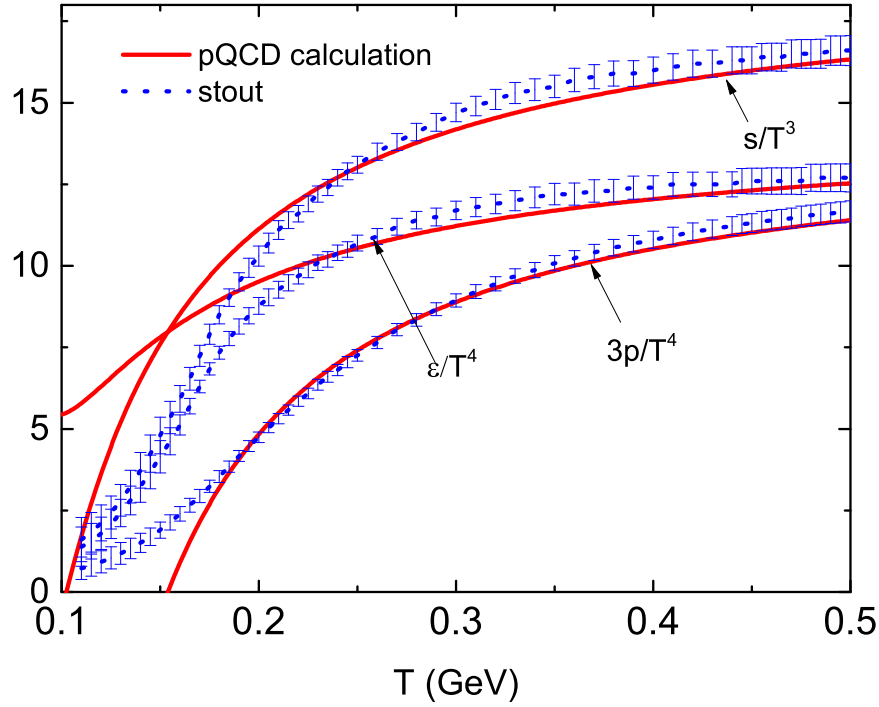


Figure 9 – (Color online) The calculated of $3p/T^4$, ε/T^4 and s/T^3 from the pQCD calculation where $C_M = 3.293$ and $C_S = 1.509$. The lattice QCD with stout action (BORSANYI et al., 2012; BORSANYI et al., 2014) at zero chemical potential is also given.

critical temperature, $T \sim T_c$, because the pQCD is used to compute the various QGP thermodynamic quantities at the extremely high temperature. To describe the thermodynamic properties of the quark-gluon plasma phenomenologically and reasonably, the quasi-particle model for QGP is proposed which is able to fit with Lattice QCD results over a wide range of temperatures: not only near the critical temperature T_c , but also at $T > 20T_c$ as in the hard thermal loop theory, or at the extremely high temperature as in pQCD theory. The main idea of the quasi particle model is that quark-gluon plasma is made up of free quarks and gluons, and characterize the interactions between quarks and gluons as the effective masses of quarks and gluons. At year 1994, Peshier *et al* (PESHER et al., 1994) applied such model for describing the QGP, and treat the effective masses of the elements of QGP as the function of temperature T without consideration of chemical potential. Taking the consideration that the standard statistical mechanics (SSM) introduces the mass of the element as a constant, we need to re-derive thermodynamics if the effective mass is treated as the function of temperature T . Gorenstein and Yang (GORENSTEIN; YANG, 1995) found that thermodynamics does not meet the thermodynamic relations by using the definitions of SSM directly. In order to solve this problem, Gorenstein and Yang introduced a Bag constant related to the temperature as the background field, which can be determined by the requirement of the thermodynamic relationship. Soon afterwards, there were many other attempts to find the solution of the problem (YIN; SU, 2008; YIN; SU, 2007; YIN; SU, 2010; BANNUR, 2007a; BANNUR, 2013; BANNUR, 2012; BANNUR, 2008; BANNUR, 2007b; GARDIM; STEFFENS, 2007; GARDIM; STEFFENS, 2009). Since Gorenstein and Yang only

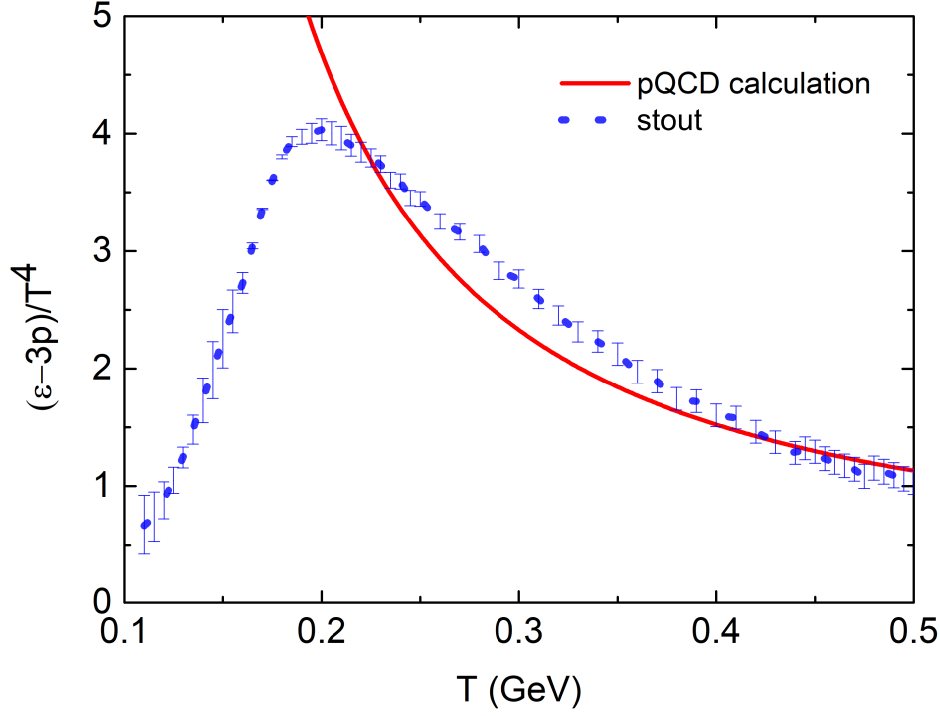


Figure 10 – (Color online) The calculated trace anomaly from the pQCD calculation where $C_M = 3.293$ and $C_S = 1.509$. The lattice QCD with stout action (BORSANYI et al., 2012; BORSANYI et al., 2014) at zero chemical potential is also given.

discussed the situation without chemical potential, Peshier tried to generalize Gorenstein and Yang' idea into finite temperature and density case (PESHER; KAMPFER; SOFF, 2000). We found that such a generalization is just a special solution for an integro-differential equation (MA et al., 2019).

4.4.1 Thermodynamic inconsistency

For convenience, we do not consider the chemical potential and choose the chemical potential as zero. According to the statistical mechanics (PATHRIA, 1996; HUANG, 1987; LANDAU; LIFSHITZ, 1980), the grand partition function for the ideal gas consisting of the particles with mass m can be characterized as

$$\ln Z(V, T) = \pm \frac{g_i V}{(2\pi)^3} \int d^3k \ln [1 \pm \exp(-\omega_k/T)], \quad (4.39)$$

where $\omega_k = [k^2 + m^2(T)]^{1/2}$ is the energy of the massive "non-interacting" quasi-particles. "+" stands for the Fermions and "-" stands for the Bosons. g_i is the degeneracy factor. According to the definition of SM, it is easy to obtain its pressure energy density and entropy density,

$$p_{id}(T, m) = \frac{T}{V} \ln Z(V, T), \quad (4.40)$$

$$\varepsilon_{id}(T, m) = -\frac{1}{V} \left. \frac{\partial \ln Z(V, T)}{\partial \beta} \right|_V, \quad (4.41)$$

$$s_{id}(T, m) = \frac{1}{V} \left(\ln Z(V, T) + \left. \frac{\partial \ln Z(V, T)}{\partial T} \right|_V \right). \quad (4.42)$$

Or

$$p_{id}(T, m) = \pm T \frac{g_i}{2\pi^2} \int_0^\infty k^2 dk \ln [1 \pm \exp(-\omega_k/T)], \quad (4.43)$$

$$\varepsilon_{id}(T, m) = \frac{g_i}{2\pi^2} \int_0^\infty k^2 dk \frac{\omega_k}{\exp(\omega_k/T) \pm 1}, \quad (4.44)$$

$$s_{id}(T, m) = \pm \frac{g_i}{2\pi^2} \int_0^\infty k^2 dk \ln [1 \pm \exp(-\omega_k/T)] + \frac{1}{T} \frac{g}{2\pi^2} \int_0^\infty k^2 dk \frac{\omega_k \exp(-\omega_k/T)}{1 \pm \exp(-\omega_k/T)}. \quad (4.45)$$

However, we can also derive the energy density ε^* from pressure with the aid of thermodynamic relation, namely,

$$\varepsilon^*(T, m) = Ts - p, \quad s = \frac{dp(T, m)}{dT}, \quad (4.46)$$

or

$$\varepsilon^*(T, m) = T \left. \frac{\partial p(T, m)}{\partial T} \right|_m - p(T, m) + T \left. \frac{\partial p(T, m)}{\partial m} \right|_T \frac{dm(T)}{dT}. \quad (4.47)$$

Substituting Eq.(4.43) into the above equation (4.47), we can obtain

$$\begin{aligned} \varepsilon^*(T, m) &= \frac{g_i}{2\pi^2} \int_0^\infty k^2 dk \frac{\omega_k}{\exp(\omega_k/T) \pm 1} + \left. \frac{\partial p(T, m)}{\partial m} \right|_T \frac{dm(T)}{dT} \\ &= \varepsilon_{id}(T, m) + \left. \frac{\partial p(T, m)}{\partial m} \right|_T \frac{dm(T)}{dT} \neq \varepsilon_{id}(T, m). \end{aligned} \quad (4.48)$$

Then the thermodynamics relation,

$$\varepsilon(T, m) = T \frac{\partial p(T, m)}{\partial T} - p(T, m), \quad (4.49)$$

is not satisfied, which means the energy density obtained from ensemble average may not be the same as that from thermodynamic relation. Turning to the situation, $m(T) = m_0 = \text{const}$, the additional term $\left. \frac{\partial p(T, m)}{\partial m} \right|_T \frac{dm(T)}{dT}$ will equal zero due to $\frac{dm(T)}{dT} = \frac{dm_0}{dT} = 0$ and these expressions would satisfy the thermodynamic relation, *i.e.*, Eq.(4.49).

4.4.2 Some attempts for solving thermodynamic inconsistency

Gorenstein and Yang reformulated the SSM to solve thermodynamic inconsistency, by introducing a temperature dependent term in the expressions for pressure and energy density (GORENSTEIN; YANG, 1995). This extra term, called as the Bag constant, was fixed by applying a constraint relation such that the thermodynamic inconsistency term was cancelled. There are some solutions, such as

the the model proposed by V. M. Bannur (BANNUR, 2007a; BANNUR, 2013; BANNUR, 2012; BANNUR, 2008; BANNUR, 2007b). The idea for this model is to start from the definition of energy density and number density in SSM, and then the other thermodynamic quantities can be derived from the thermodynamic relations. Moreover, another model suggested by F.G. Gardim and F.M. Steffens (GARDIM; STEFFENS, 2007; GARDIM; STEFFENS, 2009) tried to parameterize all of possible quasi-particle models with the requirement of thermodynamic consistency and SSM. By assuming that the Bag constant can be expressed in terms of the quasi-particle mass, they show that the different values or combinations of those parameters represent for different solutions.

4.4.2.1 Gorenstein and Yang

Based on the quasi-particle model proposed by (GORENSTEIN; YANG, 1995), we revisit the thermodynamic consistency for quasi-particle model at finite temperature and baryon chemical potential. Firstly, following their traditional definition in statistical physics (PATHRIA, 1996), we can write the the grand partition function for ideal gas in the form of Hamiltonian as follows,

$$Q_G = \langle \exp[-\alpha \hat{N} - \beta \hat{H}_{\text{id}}] \rangle, \quad (4.50)$$

where $\hat{H}_{\text{id}}$ is the Hamiltonian of ideal gas of particle

$$\hat{H}_{\text{id}} = \sum_{i=1}^d \sum_{\mathbf{k}} \omega(\mathbf{k}) a_{\mathbf{k},j}^\dagger a_{\mathbf{k},j}. \quad (4.51)$$

Here, $\omega(\mathbf{k})$ is the energy of a single particle and the index j corresponds to internal degrees of freedom for the particle, *i.e.*, the different spin and color states of the particle. However, in finite baryon chemical potential and temperature case, $\omega(\mathbf{k})$ is replaced by $\omega_k(\mathbf{k}, T, \mu)$. Then the thermodynamic inconsistency occurs due to the Hamiltonian of the system becomes temperature- and/or baryon chemical potential- dependent. To solve the thermodynamic inconsistency, Gorenstein and Yang introduced an effective temperature- and/or chemical potential- dependent Hamiltonian to guarantee the thermodynamic consistency of the quasi-particle model, namely,

$$Q_G = \langle \exp[-\alpha \hat{N} - \beta \hat{H}_{\text{eff}}] \rangle, \quad (4.52)$$

where

$$\hat{H}_{\text{eff}} = \hat{H}_{\text{id}} + E_0. \quad (4.53)$$

Here E_0 is a temperature and chemical potential dependent function associated with the bag constant B proposed by Gorenstein and Yang (GORENSTEIN; YANG, 1995). This term, a specific temperature- and chemical potential- dependent bag constant, is used to cancel out the effects of the temperature (and chemical potential) dependence of the quasi-particle mass and it can be determined by the requirement

of thermodynamic self-consistency, *i.e.*,

$$\left. \frac{\partial p(T, \mu, m)}{\partial m} \right|_{T, \mu} - \left. \frac{\partial m(T, \mu)}{\partial T} \right|_{\mu} = 0. \quad (4.54)$$

To be more specific, the expressions for energy and particle number can be formulated as ensemble average, *i.e.*, Eq.(4.6) can be rewritten in terms of the grand partition function, $\ln Q_G$, namely

$$\begin{aligned} \langle E \rangle &= -\frac{\partial \ln Q_G}{\partial \beta}, \\ \langle N \rangle &= -\frac{\partial \ln Q_G}{\partial \alpha}. \end{aligned} \quad (4.55)$$

quasi-particle ansatz assumes that one may carry out the calculations in the momentum space where the Hamiltonian is diagonal. To be more specific, one makes the following substitutions for the ideal gas part

$$\sum_j \sum_{\mathbf{k}} \rightarrow \frac{g_i V}{(2\pi)^3} \int d\mathbf{k}. \quad (4.56)$$

where g_i is the degeneracy factor. Now, thermodynamical quantities can also be expressed regarding the derivatives of the grand partition function. Then the energy density, number density, pressure and entropy density read

$$\varepsilon = \frac{\langle E \rangle}{V} = -\frac{1}{V} \frac{\partial \ln Q_G}{\partial \beta} = \varepsilon_{\text{id}} + \frac{E_0}{V} = \varepsilon_{\text{id}} + B. \quad (4.57)$$

$$n = \frac{\langle N \rangle}{V} = -\frac{1}{V} \frac{\partial \ln Q_G}{\partial \alpha} = n_{\text{id}}, \quad (4.58)$$

$$p = \frac{1}{\beta} \frac{\partial \ln Q_G}{\partial V} = \frac{1}{\beta} \frac{\ln Q_G}{V} = p_{\text{id}} - B, \quad (4.59)$$

$$s = \frac{\varepsilon + p - \mu n}{T} = \frac{\varepsilon_{\text{id}} + p_{\text{id}} - \mu n_{\text{id}}}{T} = s_{\text{id}}, \quad (4.60)$$

where

$$\begin{aligned} \varepsilon_{\text{id}} &= \frac{g_i}{2\pi^2} \int_0^\infty \frac{k^2 dk \omega^*(k, T, \mu)}{\exp[(\omega^*(k, T, \mu) - \mu)/T] \mp 1} \\ &+ \frac{g_i}{2\pi^2} \int_0^\infty \frac{k^2 dk \omega^*(k, T, -\mu)}{\exp[(\omega^*(k, T, -\mu) + \mu)/T] \mp 1}, \end{aligned} \quad (4.61)$$

$$\begin{aligned} n_{\text{id}} &= \frac{g_i}{2\pi^2} \int_0^\infty \frac{k^2 dk}{\exp[(\omega^*(k, T, \mu) - \mu)/T] \mp 1} \\ &- \frac{g_i}{2\pi^2} \int_0^\infty \frac{k^2 dk}{\exp[(\omega^*(k, T, -\mu) + \mu)/T] \mp 1}, \end{aligned} \quad (4.62)$$

$$\begin{aligned}
p_{id} &= \frac{\mp g_i}{2\pi^2} \int_0^\infty k^2 dk \ln \{1 \mp \exp[(\mu - \omega^*(k, T, \mu))/T]\} \\
&\mp \frac{g_i}{2\pi^2} \int_0^\infty k^2 dk \ln \{1 \mp \exp[(-\mu - \omega^*(k, T, -\mu))/T]\} \\
&= \frac{g_i}{12\pi^2} \int_0^\infty \frac{k^3 dk}{\exp[(\omega^*(k, T, \mu) - \mu)/T] \mp 1} \frac{\partial \omega^*(k, T, \mu)}{\partial k} \Big|_{T, \mu} \\
&+ \frac{g_i}{12\pi^2} \int_0^\infty \frac{k^3 dk}{\exp[(\omega^*(k, T, -\mu) + \mu)/T] \mp 1} \frac{\partial \omega^*(k, T, -\mu)}{\partial k} \Big|_{T, \mu}, \tag{4.63}
\end{aligned}$$

with on-shell dispersion relation

$$\omega^*(k, T, \mu) = \sqrt{m(T, \mu)^2 + k^2}, \tag{4.64}$$

where $B = \lim_{V \rightarrow \infty} \frac{E_0}{V}$ is the bag constant. Generally, one thought the effective quasi-particle mass will change with temperature and density due to its interaction with the environment. Some theoretical studies, for example, Brown-Rho scaling (BROWN; RHO, 1991), QCD sum rules (HATSUDA; KOIKE; LEE, 1993), and as well the experiment (LOLOS et al., 1998) all indicate such a phenomenon. The temperature-dependent quasi-particle mass $m(T, \mu)$ is usually used to mimic the medium effects, which means $\frac{\partial m(T, \mu)}{\partial T} \Big|_\mu$ should not always equal 0.

Then, the constraint condition suggested by Gorenstein and Yang (GORENSTEIN; YANG, 1995) can be rewritten as

$$\frac{\partial p(T, \mu, m)}{\partial T} \Big|_{\mu, m} = 0, \quad \frac{\partial p(T, \mu, m)}{\partial \mu} \Big|_{T, m} = 0. \tag{4.65}$$

Here, the contribution from the temperature dependence of quasi-particle mass has already been canceled out with the temperature dependence of E_0 . If the system has vanishing chemical potential $\mu = 0$, one has $B = B(\mu = 0, T) \equiv B(T)$ and $m = m(\mu = 0, T) \equiv m(T)$, in general, one can invert the second function to find $T = T(m)$ and express B as a function of m . Thus, the above requirement Eq.(4.65) regarding E_0 implies

$$\frac{dB}{dm} = \frac{\partial p_{id}(T, \mu, m)}{\partial m} \Big|_{T, \mu}. \tag{4.66}$$

i.e.,

$$\frac{dB}{dm} = -\frac{g_i m}{2\pi^2} \int_0^\infty \frac{k^2 dk}{\omega^*(k, T)} \frac{1}{\exp[(\omega^*(k, T))/T] \mp 1}. \tag{4.67}$$

If one already known the function $\omega^*(k, T)$, the bag constant $B(m)$ can be obtained by the following equation,

$$B(m) = B_0 - \frac{d}{2\pi^2} \int_{T_0}^T dT' m \frac{dm}{dT'} \times \int_0^\infty \frac{k^2 dk}{\omega_k} \frac{1}{\exp[\omega_k/T'] - 1}, \tag{4.68}$$

where $B_0 = B(m(T_0))$ is an arbitrary integration constant and it can be determined by comparing with Lattice QCD data.

Considering a system for SU(3) gluodynamics, one suppose the linear relation between effective mass (m_g) and temperature (T), $m_g = aT$. Then, the thermodynamics suggested by Eqs.(4.57), (4.59) and (4.60) will be reduced to

$$\varepsilon(T) = \sigma T^4 + B_0, \quad p(T) = \frac{\sigma}{3} T^4 - B_0, \quad s(T) = \frac{4\sigma}{3} T^3. \quad (4.69)$$

Here σ equals

$$\sigma = \frac{3g}{2\pi^2} \sum_{n=1}^{\infty} \left[\frac{a^2}{n^2} K_2(na) + \frac{a^3}{4n} K_1(na) \right], \quad (4.70)$$

where K_1 and K_2 are the modified Bessel functions. As found by Begun *et al* (BEGUN; GORENSTEIN; MOGILEVSKY, 2011; BEGUN; GORENSTEIN; MOGILEVSKY, 2010), no matter how we choose the integration constant B_0 , the lattice results can not be reproduced very well, shown in Figure 11

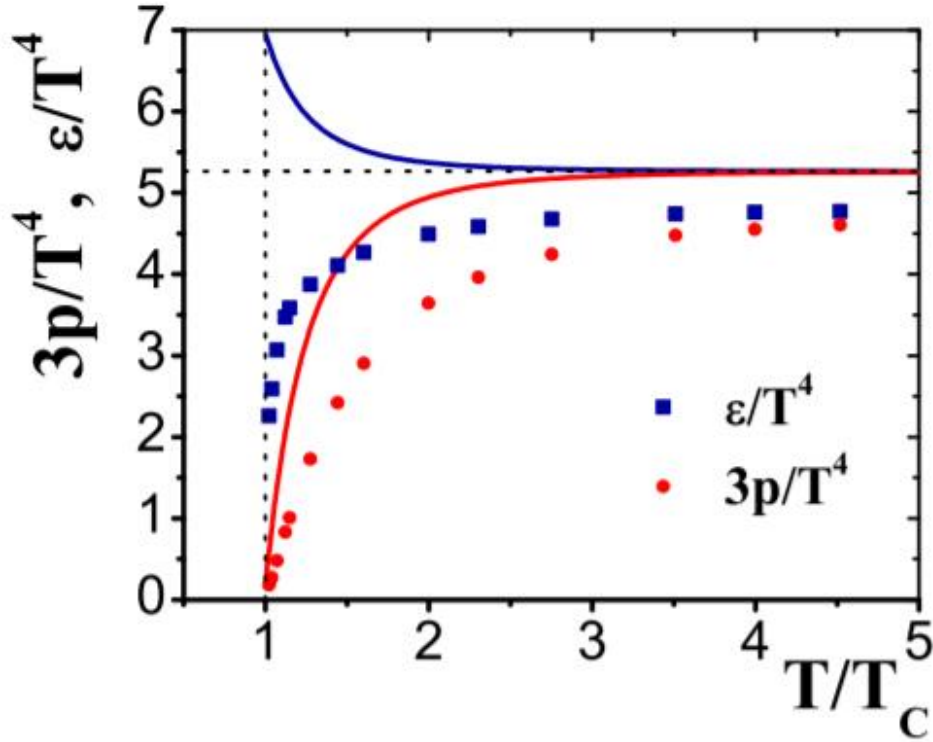


Figure 11 – The calculated energy density and pressure for SU(3) gluodynamics. The blue squares and red circles represent for the energy density and pressure of Lattice QCD respectively. The dashed horizontal line shows the Stefan-Boltzmann limits. The calculated energy density and pressure by Eq.(4.69) by setting the parameters as, $g = 16$ and $B = 1.7T_c^4$

Begun *et al* suggested that we can introduce an extra, free parameter E_1 into the Hamiltonian, *i.e.*,

$$\hat{H}_{\text{eff}} = \hat{H}_{\text{id}} + E_0 + E_1, \quad (4.71)$$

which is proportional to the temperature. Then, the thermodynamics can be expressed as,

$$\varepsilon = \frac{\langle E \rangle}{V} = -\frac{1}{V} \frac{\partial \ln Q_G}{\partial \beta} = \varepsilon_{id} + \frac{E_0}{V} + \frac{E_1}{V} + \frac{1}{V} \langle \beta \frac{\partial E_1}{\partial \beta} \rangle = \varepsilon_{id} + B, \quad (4.72)$$

$$n = \frac{\langle N \rangle}{V} = -\frac{1}{V} \frac{\partial \ln Q_G}{\partial \alpha} = n_{id}, \quad (4.73)$$

$$p = \frac{1}{\beta} \frac{\partial \ln Q_G}{\partial V} = \frac{1}{\beta} \frac{\ln Q_G}{V} = p_{id} - B - \frac{E_1}{V}, \quad (4.74)$$

$$s = \frac{\varepsilon + p - \mu n}{T} = \frac{\varepsilon_{id} + p_{id} - \mu n_{id}}{T} = s_{id} + \frac{E_1}{VT}. \quad (4.75)$$

The E_1 term is singled out from E_0 owing to its peculiar nature. We note the presence of the term regarding E_1 in the resulting expression for the pressure, but not in that for the energy density. By choosing $E_1/V = AT$ (BEGUN; GORENSTEIN; MOGILEVSKY, 2011; BEGUN; GORENSTEIN; MOGILEVSKY, 2010), We can replace Eq.(4.69), those thermodynamics can be replaced by

$$\varepsilon(T) = \sigma T^4 + B_0, \quad p(T) = \frac{\sigma}{3} T^4 - B_0 - AT, \quad s(T) = \frac{4\sigma}{3} T^3 - A. \quad (4.76)$$

Then it allows one to further adjust the value of the pressure for any given energy density (BEGUN; GORENSTEIN; MOGILEVSKY, 2011; BEGUN; GORENSTEIN; MOGILEVSKY, 2010). As shown in Figure 12, the lattice results can be reproduced quite well with an additional negative term E_1 .

We note that the resultant expressions for thermodynamic quantities, namely, Eqs.(4.72), (4.73) and (4.74), are thermodynamically as well as statistically consistent. The reasons are twofold. Firstly, the expressions for energy and particle density are in accordance with the conventional definition regarding ensemble average¹, while they can also be conveniently expressed in standard form as derivatives of the grand partition function, as emphasized by other authors (BIRO; SHANENKO; TONEEV, 2003; YIN; SU, 2008). Moreover, from the viewpoint of statistical physics, those ensemble averages are meaningful, only when one can match those quantities, term by term, to the first law of thermodynamics (PATHRIA, 1996). In this context, thermodynamical consistency is guaranteed. Subsequently, any other thermodynamical quantities can be derived from a thermodynamic potential, which summarizes all the constitutive properties of a body that thermodynamics represents. Now, it is not difficult to see that the second requirement is indeed achieved by evaluating the total derivative of $q = \ln Q_G$, to be specific, one can readily verify that

$$dq = -\langle N \rangle d\alpha - \langle E \rangle d\beta - \beta p dV, \quad (4.77)$$

where

$$\begin{aligned} \beta &= \frac{1}{T}, \\ \alpha &= -\frac{\mu}{T}, \\ q + \alpha N + \beta E &= S. \end{aligned} \quad (4.78)$$

¹ This can be seen by comparing the r.h.s. of Eq.(4.72) and Eq.(4.73) against Eq.(4.6).

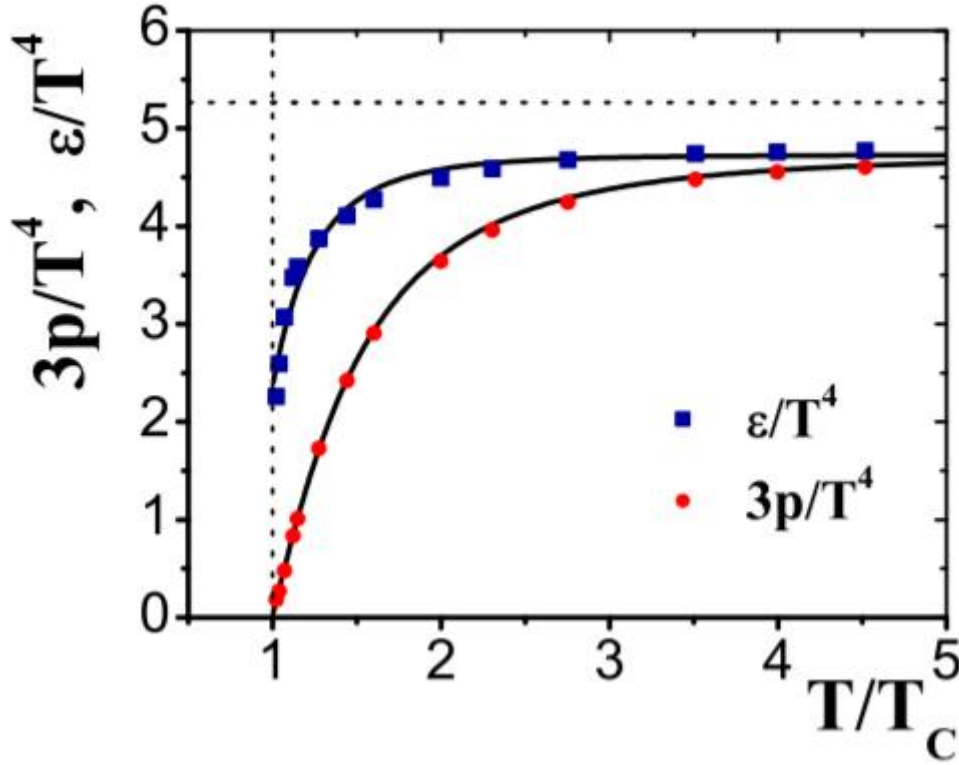


Figure 12 – The energy density and pressure for SU(3) gluodynamics. The blue squares and red circles represent for the energy density and pressure of Lattice QCD respectively. The dashed horizontal line shows the Stefan-Boltzmann limits. The calculated energy density and pressure by Eq.(4.76) by setting the parameters as, $\sigma = 4.73$, $A = 3.94T_c^3$ and $B = -2.37T_c^4$

Since the first law of thermodynamics holds, it is natural to expect that all thermodynamical quantities defined through the above procedure automatically satisfy any thermodynamical relations, such as:

$$\epsilon \equiv \frac{E}{V} = T \left. \frac{\partial p}{\partial T} \right|_{V, \mu} - p + \mu n.$$

4.4.2.2 Bannur

As we showed before, Gorenstein and Yang and/or Peshier guaranteed the thermodynamic consistency by reformulating the SSM with introducing a temperature- and/or chemical potential- dependent bag function B in the expressions for energy density and pressure, shown in Eq.(4.57) and Eq.(4.59), respectively. In order to obtain B , one should solve the constraint relation, *i.e.*, Eq. (4.54) which is used to cancel the term of thermodynamic inconsistency, $\left. \frac{\partial p(T, m)}{\partial m} \right|_T \frac{dm(T)}{dT}$. However, in the view of Bannur, the wrong choice of pressure leads to the problem of thermodynamic inconsistency. Thus, there is no need to introduce such a temperature- and/or chemical potential- dependent term instead of redefine the correct expression of pressure with the aid of thermodynamic relations. To obtain the expression of pressure, one should use the statistical mechanics's definition of ϵ and n and the thermodynamic relations.

Following the standard procedure of statistical mechanics (PATHRIA, 1996) and considering the

quasi-particle mass is a function of temperature, then the expressions for the total energy and particle numbers can be expressed in terms of grand partition function as

$$\begin{aligned}\langle E_i \rangle &= -\frac{\partial}{\partial \beta} \left\{ \ln \sum_i \exp(-\alpha N_i - \beta E_i) \right\} - \beta \frac{\partial m}{\partial \beta} \left\langle \frac{\partial E_i}{\partial m} \right\rangle, \\ \langle N_i \rangle &= -\frac{\partial}{\partial \alpha} \left\{ \ln \sum_i \exp(-\alpha N_i - \beta E_i) \right\}.\end{aligned}\quad (4.79)$$

Here we note there is an extra term involving $\left(\frac{\partial m}{\partial \beta}\right)$ in the expression for the ensemble average of energy owing to the temperature dependence of quasi-particle mass.

Thereafter, other thermodynamic quantities are subsequently identified by matching the total derivative of $q = \ln Q_G$ to the first law of thermodynamics (PATHRIA, 1996). Thus, one has

$$dq = -\langle N_i \rangle d\alpha - \langle E_i \rangle d\beta - \beta \left\langle \frac{\partial E_i}{\partial V} \right\rangle dV - \beta \frac{\partial m}{\partial \beta} \left\langle \frac{\partial E_i}{\partial m} \right\rangle d\beta. \quad (4.80)$$

Subsequently, one finds the expression for pressure with the aid of Eq.(4.78),

$$\frac{pV}{T} = \frac{(E + pV - TS) - E + TS}{T} = \frac{\mu N - E + TS}{T} = q + \int d\beta \beta \frac{\partial m}{\partial \beta} \left\langle \frac{\partial E_i}{\partial m} \right\rangle, \quad (4.81)$$

or

$$p = \frac{1}{V\beta} q + \frac{1}{V\beta} \int d\beta \beta \frac{\partial m}{\partial \beta} \left\langle \frac{\partial E_i}{\partial m} \right\rangle. \quad (4.82)$$

Then the partial derivatives $\frac{\partial(\beta p)}{\partial \beta}$ can be written as

$$\begin{aligned}-\frac{\partial(\beta p)}{\partial \beta} &= T \frac{\partial p}{\partial T} - p \\ &= -\frac{1}{V} \frac{\partial q}{\partial \beta} - \frac{1}{V} \frac{\partial \int d\beta \beta \frac{\partial m}{\partial \beta} \left\langle \frac{\partial E_i}{\partial m} \right\rangle}{\partial \beta} \\ &= -\frac{1}{V} \frac{\partial \ln Q_G}{\partial \beta} - \beta \frac{\partial m}{\partial \beta} \left\langle \frac{\partial E_i}{\partial m} \right\rangle \\ &= \frac{E}{V} \equiv \varepsilon.\end{aligned}\quad (4.83)$$

The Eq.(4.81) is in consistency with the thermodynamical relation (BANNUR, 2007a). Thus, with the acknowledge of energy density, the expression of pressure can be derived from the above thermodynamic relation,

$$\frac{p}{T} = \frac{p_0}{T_0} + \int_{T_0}^T dT \frac{\varepsilon}{T^2}, \quad (4.84)$$

where p_0 and T_0 are the pressure and temperature at some reference points.

For the 2+1 flavor QGP system, it is supposed to be consist of the non-interacting quasi-particles

carrying the quantum numbers of the gluons, the up, down as well as strange quarks. We employ the quasi-particle model proposed by Bannur (BANNUR, 2008) to reproduce the lattice QCD data (BORSANYI et al., 2014). As we have described previously, this model can be achieved by choosing the form of energy and particle number the same as formulated as grand ensemble averages in statistical mechanics and the other thermodynamics can be obtained from the thermodynamic relations.

Firstly, we need to know the explicit expression of the quasi-particle masses. With the consideration of the on-shell dispersion relation in high temperature, the single particle energy of quasi-particles ω_k can be written as

$$\omega_k^2 = k^2 + m_{g,q}^2, \quad (4.85)$$

where the following prescriptions (BANNUR, 2008; LEVAI; HEINZ, 1998) for the thermal masses of quasi-particles are adopted, *i.e.*,

$$m_g^2 = \frac{3}{2}\omega_p^2 \quad (4.86)$$

for gluons and

$$m_q^2 = (m_{q0} + m_f)^2 + m_f^2 \quad (4.87)$$

for quarks, where q stands for u , d , or s quark respectively. Here m_{q0} is the rest mass of the quarks. We take $m_{s0} = 0.150$ GeV for strange quark, and $m_{u0,d0} = m_{s0}/28.15 \approx 5.33$ MeV for up and down quarks. Due to mutual interactions among particles, the plasmon frequency ω_p and the effective mass of soft massless quark m_f are associated with the collective behavior of the system. They can be obtained by analysing the poles of the relevant propagators. By using HTL approximation (PISARSKI, 1989; FRENKEL; TAYLOR, 1990),

$$\omega_p^2 = \frac{g_s^2 T^2}{18} (2N_c + N_f), \quad (4.88)$$

$$m_f^2 = \frac{N_c^2 - 1}{2N_c} \frac{g_s^2 T^2}{8}, \quad (4.89)$$

where the number of colors $N_c = 3$, the number of flavors $N_f = 2 + 1 = 3$ and g is the coupling constant. To describe the QGP at relative low temperature region, the parameterization of model II proposed in (BANNUR, 2008) are adopted here, shown as

$$\omega_p^2 = a_g^2 g_s^2 \frac{n_g}{T} + \sum_q a_q^2 g_s^2 \frac{n_q}{T}, \quad (4.90)$$

$$m_f^2 = b_q^2 g_s^2 \frac{n_q}{T}. \quad (4.91)$$

Here n_g and n_q are number densities of gluons and quarks as follows

$$\frac{n_g}{T^3} = \frac{g_g}{2\pi^2} \int_0^\infty dx \frac{x^2}{e^{\sqrt{x^2+m_g^2/T^2}} - 1}, \quad (4.92)$$

$$\frac{n_q}{T^3} = \frac{g_q N_f^q}{2\pi^2} \int_0^\infty dx \frac{x^2}{e^{\sqrt{x^2+m_q^2/T^2}} + 1} \quad (4.93)$$

where, $x = k/T$. g_g and g_q are the degeneracy for gluons and quarks, respectively. For gluons, there are two spin and eight color states. Then the degeneracy factor g_g equals to

$$g_g = \text{spin} \times \text{color} = 2 \times 8 = 16. \quad (4.94)$$

For quarks, there are two spin and three color states, besides they can come in different flavor and there is an anti-quark for each one. For 2+1 flavor QCD, we assume that there are three light quarks (u, d, s) and u and d quarks are degenerate. Then, the degeneracy factor g_q is equal

$$\begin{aligned} g_q &= \text{spin} \times \text{color} \times \text{particle} - \text{antiparticle} \times \text{flavor} \\ &= 2 \times 3 \times 2 \times N_f^q \end{aligned} \quad (4.95)$$

where $N_f^{u,d} = 2$, $N_f^s = 1$.

As shown in Figure 5, the effective running coupling constant decreases as the momentum transfer increases due to the property of asymptotic freedom in zero-temperature QCD. However, turning to the finite temperature QCD, the characteristic momentum transfer between quanta is of the order of the temperature. Therefore, the running coupling constant g_s will fall as the temperature increases, and the explicit expression is given in the two-loop approximation (CASWELL, 1974; YNDURAIN, 2006),

$$\alpha_s(T) \equiv \frac{g_s^2}{4\pi} = \frac{6\pi}{(33 - 2N_f) \ln(T/\Lambda_T)} \left(1 - \frac{3(153 - 19N_f) \ln(2 \ln(T/\Lambda_T))}{(33 - 2N_f)^2 \ln(T/\Lambda_T)} \right). \quad (4.96)$$

Under the limitation $T \rightarrow \infty$, the coupling constant g_s goes to 0. Hence, it is not difficult to know

$$\frac{n_g}{T} \rightarrow \frac{d_g}{2\pi^2} \zeta(3) T^2$$

and

$$\frac{n_q}{T} \rightarrow \frac{d_q}{2\pi^2} \eta(3) T^2.$$

Then the coefficients a_g , a_q and b_q can be determined by demanding Eqs.(4.90)-(4.91) approach the perturbative results, Eqs.(4.88)-(4.89), at $T \rightarrow \infty$. Thus, we can obtain the quasi-particle masses self-consistency for 2+1 flavor QGP.

Following the idea of Bannur, the energy density and the number density of the 2+1 flavor QGP

Table 1 – List of parameters used in the Bannur's model for the $N_f = 2 + 1$ QCD system at vanishing chemical potential.

T_0 (GeV)	p_0/T_0^4	Λ_T (GeV)	m_{s0} (GeV)
0.175	1.08	0.135	0.15
$m_{u0,d0}$ (GeV)	a_g^2	a_q^2	b_q^2
5.33×10^{-4}	0.171	0.101	0.304

read

$$\varepsilon = \varepsilon_g + \varepsilon_u + \varepsilon_d + \varepsilon_s, \quad (4.97)$$

$$n = n_g + n_u + n_d + n_s. \quad (4.98)$$

where

$$\begin{aligned} \varepsilon_i &= \frac{g_i}{2\pi^2} \int_0^\infty dk \frac{(k^2 + m_i^2)k^2}{e^{(\sqrt{k^2 + m_i^2} - \mu_i)/T} \pm 1} + \frac{g_i}{2\pi^2} \int_0^\infty dk \frac{(k^2 + m_i^2)k^2}{e^{(\sqrt{k^2 + m_i^2} + \mu_i)/T} \pm 1} \\ &\equiv \varepsilon_i^{id}, \end{aligned} \quad (4.99)$$

$$\begin{aligned} n_i &= \frac{g_i}{2\pi^2} \int_0^\infty dk \frac{k^2}{e^{(\sqrt{k^2 + m_i^2} - \mu_i)/T} \pm 1} - \frac{g_i}{2\pi^2} \int_0^\infty dk \frac{k^2}{e^{(\sqrt{k^2 + m_i^2} + \mu_i)/T} \pm 1} \\ &\equiv n_i^{id}, \end{aligned} \quad (4.100)$$

where “−” in the denominator applies to bosons and “+” is for fermions and g_i is the degeneracy for particle specie i . For the present case of zero chemical potential, $\mu_i = 0$, the number density vanishes identically. As discussed before, the pressure can be calculated by using the thermodynamic relation, *i.e.*, Eq.(4.84). Here we choose the reference point $(T_0, p_0/T_0^4)$ as (0.175, 1.08) according to the Lattice data.

To reproduce the Lattice data (BORSANYI et al., 2012; BORSANYI et al., 2014), the parameters we used at vanishing chemical potential are listed in Table.1. In the Figure 13, we show that the quasi-particle model suggested by Banuur can reproduce the Lattice data over a quite wide range of temperature, $[0.8T_c, 3.3T_c]$.

4.4.2.3 Gardim and Steffen

To solve the inconsistency in the thermodynamics relations, Gorenstein and Yang suggested to introduce an extra term, E_0 , into the original Hamiltonian, $\hat{H}_{id} \rightarrow \hat{H}_{eff} = \hat{H}_{id} + E_0$. Here E_0 is thought as the zero point energy of the system. And the thermodynamic relations are achieved with the modifications of the usual thermodynamics functions. To generalize this idea, Gardim and Steffen (GARDIM; STEFFENS, 2007; GARDIM; STEFFENS, 2009) rewrite the thermodynamics functions, like the grand potential $\Phi = -pV$, the internal energy U , the entropy S and the average net quark number N as follows,

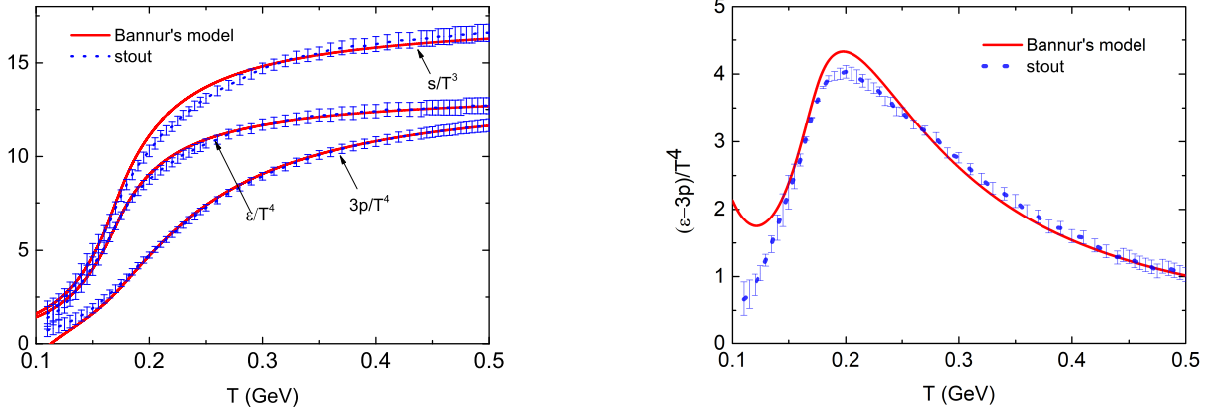


Figure 13 – (Color online) The calculated of $3p/T^4$, ε/T^4 , s/T^3 and trace anomaly from the Bannur's model in comparison with those by lattice QCD with stout action (BORSANYI et al., 2012; BORSANYI et al., 2014) at zero chemical potential.

$$\Phi(V, T, \mu, m_f^2, m_b^2) = \Phi_f(V, T, \mu, m_f^2) + \Phi_b(V, T, \mu, m_b^2) + \alpha B(V, m_f^2, m_b^2), \quad (4.101)$$

$$U(V, T, \mu, m_f^2, m_b^2) = U_f(V, T, \mu, m_f^2) + U_b(V, T, \mu, m_b^2) + \eta B(V, m_f^2, m_b^2), \quad (4.102)$$

$$S(V, T, \mu, m_f^2, m_b^2) = S_f(V, T, \mu, m_f^2) + S_b(V, T, \mu, m_b^2) - \frac{\gamma}{T} B(V, m_f^2, m_b^2), \quad (4.103)$$

$$N(V, T, \mu, m_f^2, m_b^2) = N_f(V, T, \mu, m_f^2) + N_b(V, T, \mu, m_b^2) - \frac{\lambda}{\mu} B(V, m_f^2, m_b^2), \quad (4.104)$$

where the subscripts f and b denote Fermions and Bosons. The effective mass for fermions and bosons are considered as a function of temperature T and chemical potential μ , *i.e.*, $m_f^2 = m_f^2(T, \mu)$ and $m_b^2 = m_b^2(T, \mu)$, respectively. Considering the requirement of thermodynamic consistency, given by the following relations

$$P = - \left(\frac{\partial \Phi}{\partial V} \right)_{T, \mu}, \quad S = - \left(\frac{\partial \Phi}{\partial T} \right)_{V, \mu}, \quad N = - \left(\frac{\partial \Phi}{\partial \mu} \right)_{T, V}, \quad U = \Phi + TS + \mu N, \quad (4.105)$$

one should have the constraint among the arbitrary constants α , η , γ and λ (BLUHM; KAMPFER, 2008; GARDIM; STEFFENS, 2007; GARDIM; STEFFENS, 2009),

$$\alpha = \eta + \gamma + \lambda. \quad (4.106)$$

Then, the following differential equations for solving B can be obtained by substituting the Eqs.(4.101), (4.102), (4.103) and (4.104) into Eq.(4.105),

$$\gamma B = T \alpha \frac{\partial B}{\partial T} + T \sum_{i=b,f} \left\langle \frac{\partial H_i}{\partial T} \right\rangle, \quad (4.107)$$

$$\lambda B = \mu \alpha \frac{\partial B}{\partial \mu} + \mu \sum_{i=b,f} \left\langle \frac{\partial H_i}{\partial \mu} \right\rangle. \quad (4.108)$$

Here, Gardim and Steffen (GARDIM; STEFFENS, 2007; GARDIM; STEFFENS, 2009) had assumed that the bag function B is a function with respect to V , m_f and/or m_b and its dependence on T and μ all comes from the mass term. Thus we can rewrite those two terms $\left\langle \frac{\partial H_i}{\partial T} \right\rangle$ and $\left\langle \frac{\partial H_i}{\partial \mu} \right\rangle$ as

$$\left\langle \frac{\partial H_i}{\partial T} \right\rangle = \frac{\partial m_i^2}{\partial T} \left(\frac{\partial \Phi_i}{\partial m_i^2} \right)_{T,V,\mu}, \quad (4.109)$$

and

$$\left\langle \frac{\partial H_i}{\partial \mu} \right\rangle = \frac{\partial m_i^2}{\partial \mu} \left(\frac{\partial \Phi_i}{\partial m_i^2} \right)_{T,V,\mu}, \quad (4.110)$$

respectively. Substituting the above two equations (4.109) and (4.110) into Eqs.(4.107) and 4.108) accordingly, one have

$$\gamma B = T\alpha \frac{\partial B}{\partial T} + T \sum_{i=b,f} \frac{\partial m_i^2}{\partial T} \left(\frac{\partial \Phi_i}{\partial m_i^2} \right)_{T,V,\mu}, \quad (4.111)$$

$$\lambda B = \mu\alpha \frac{\partial B}{\partial \mu} + \mu \sum_{i=b,f} \frac{\partial m_i^2}{\partial \mu} \left(\frac{\partial \Phi_i}{\partial m_i^2} \right)_{T,V,\mu}. \quad (4.112)$$

If one assumes $\alpha \neq 0$, it is easy to derive that the above two equations (4.111) and 4.112) can be summed up as one equation

$$\sum_{i=b,f} \frac{\partial}{\partial m_i^2} \left(B T^{-\frac{\gamma}{\alpha}} \mu^{-\frac{\lambda}{\alpha}} \right) = -\frac{T^{-\frac{\gamma}{\alpha}} \mu^{-\frac{\lambda}{\alpha}}}{\alpha} \sum_{i=b,f} \left(\frac{\partial \Phi_i}{\partial m_i^2} \right)_{T,V,\mu}, \quad \alpha \neq 0. \quad (4.113)$$

If $\alpha = 0$, Eqs.(4.107) and (4.108) should be rewritten as

$$\gamma B = T \sum_{i=b,f} \left\langle \frac{\partial H_i}{\partial T} \right\rangle, \quad (4.114)$$

and

$$\lambda B = \mu \sum_{i=b,f} \left\langle \frac{\partial H_i}{\partial \mu} \right\rangle. \quad (4.115)$$

However, the bag function B in Eq.(4.114) appears in the modified expression of entropy and the one in Eq.(4.115) comes from the modified expression of net particle number. Since we can not guarantee those two bag functions are the same, then the two different bag constant, B_S and B_N correspond to the entropy and net particle number are chosen, then the above equation can be expressed as,

$$\gamma B_S = T \sum_{i=b,f} \left\langle \frac{\partial H_i}{\partial T} \right\rangle, \quad \lambda B_N = \mu \sum_{i=b,f} \left\langle \frac{\partial H_i}{\partial \mu} \right\rangle. \quad (4.116)$$

Then the solution of B_S and B_N can be derived obviously,

$$B_S = \frac{T}{\gamma} \sum_{i=b,f} \left\langle \frac{\partial H_i}{\partial T} \right\rangle, \quad B_N = \frac{\mu}{\lambda} \sum_{i=b,f} \left\langle \frac{\partial H_i}{\partial \mu} \right\rangle \quad \alpha = 0 \quad (4.117)$$

From the Eqs.(4.101)-(4.104), we can easily find that

- If we follow the idea proposed by Bannur, the total energy and particle number is chosen as the same as the one from ensemble average, *i.e.*, $\lambda = \eta = 0$, $\alpha = \gamma$.
- If $\lambda = \gamma = 0$, $\alpha = \eta$, it is Gorenstein and Yang's model.

4.5 GENERAL SOLUTIONS FOR THERMODYNAMIC INCONSISTENCY AT FINITE CHEMICAL POTENTIAL

In this section, we generalize the solution inspired by Gorenstein and Yang into finite baryon chemical potential by revisiting the thermodynamical self-consistency of quasi-particle model. In the case of finite baryon chemical potential, Peshier *et al* (PESHIER; KAMPFER; SOFF, 2000) investigated the QGP system described by an effective Hamiltonian with temperature- and chemical potential-dependent mass through a Maxwell equation. However, there exists another possibility where the effective quasi-particle mass is also a function of its momentum, k , in addition to temperature T and chemical potential μ . In this case, the Maxwell equation for achieving thermodynamic consistency should be expressed to be of an integro-differential equation concerning k , T , and μ . We show that the generalization suggested by Peshier is just one special solution. Moreover, another special solution through an *ad hoc* assumption that the equation can be expressed to a differential equation by moving out of the integrals with respect to k is also given. Both of those solutions can achieve the thermodynamic consistency. With the boundary condition determined by the lattice QCD data at zero baryon chemical potential, we can obtain those two special solutions. We found that both solutions can reasonably reproduce the recent lattice QCD results of the Wuppertal-Budapest Collaboration, and in particular, those concerning finite baryon density.

For the case of finite density, the bag constant B is considered as a bivariate function $B = B(\mu, T)$. We cannot express B as a function of m . This means Eq.(4.67) is not valid here. To obtain the specific expression of B , one can write down

$$\begin{aligned} \frac{\partial B}{\partial T} = & - \frac{g_i}{2\pi^2} \int_0^\infty \frac{k^2 dk}{\omega^*(k, T, \mu)} \frac{1}{\exp[(\omega^*(k, T, \mu) - \mu)/T] \mp 1} m \frac{\partial m}{\partial T} \\ & + \frac{g_i}{2\pi^2} \int_0^\infty \frac{k^2 dk}{\omega^*(k, T, -\mu)} \frac{1}{\exp[(\omega^*(k, T, -\mu) + \mu)/T] \mp 1} m \frac{\partial m}{\partial T}, \end{aligned} \quad (4.118)$$

and

$$\begin{aligned} \frac{\partial B}{\partial \mu} = & - \frac{g_i}{2\pi^2} \int_0^\infty \frac{k^2 dk}{\omega^*(k, T, \mu)} \frac{1}{\exp[(\omega^*(k, T, \mu) - \mu)/T] \mp 1} m \frac{\partial m}{\partial \mu} \\ & + \frac{g_i}{2\pi^2} \int_0^\infty \frac{k^2 dk}{\omega^*(k, T, -\mu)} \frac{1}{\exp[(\omega^*(k, T, -\mu) + \mu)/T] \mp 1} m \frac{\partial m}{\partial \mu}. \end{aligned} \quad (4.119)$$

Then at the finite density, B has to be satisfied both Eq.(4.118) and Eq.(4.119) simultaneously, which is not equivalent to Eq.(4.65). In fact, the symmetry of second derivatives for Eq.(4.118) and

Eq.(4.119) implies the following integro-differential equation:

$$\left\langle \left\langle m \frac{\partial m}{\partial T} \right\rangle \right\rangle_- = \left\langle \left\langle m \frac{\partial m}{\partial \mu} \right\rangle \right\rangle_+, \quad (4.120)$$

where

$$\begin{aligned} \langle \langle O \rangle \rangle_- &= \int_0^\infty k^2 dk \left\{ \frac{\exp[(\omega^*(k, T, \mu) - \mu)/T]}{(\exp[(\omega^*(k, T, \mu) - \mu)/T] \mp 1)^2 T} \right. \\ &\quad \left. - \frac{\exp[(\omega^*(k, T, -\mu) + \mu)/T]}{(\exp[(\omega^*(k, T, -\mu) + \mu)/T] \mp 1)^2 T} \right\} O(k), \\ \langle \langle O \rangle \rangle_+ &= \int_0^\infty k^2 dk \left\{ \frac{\exp[(\omega^*(k, T, \mu) - \mu)/T] (\omega^*(k, T, \mu) - \mu)}{(\exp[(\omega^*(k, T, \mu) - \mu)/T] \mp 1)^2 T^2} \right. \\ &\quad \left. + \frac{\exp[(\omega^*(k, T, -\mu) + \mu)/T] (\omega^*(k, T, -\mu) + \mu)}{(\exp[(\omega^*(k, T, -\mu) + \mu)/T] \mp 1)^2 T^2} \right\} O(k). \end{aligned} \quad (4.121)$$

Generally, the quasi-particle mass is thought of a function of moment, $m = m(k, T, \mu)$, and therefore B is actually a functional of m besides a function of T and μ , and derivatives in equations such as Eq.(4.66) should be understood as functional derivatives. There will be many different solutions for solving such an integro-differential equation with different assumptions. We will discuss two different special solutions of Eq.(4.120).

4.5.1 The momentum independent solution

If the quasi-particle mass is only a function of temperature and chemical potential, not a function of momentum, *i.e.*, $m = m(T, \mu)$. Then both derivatives $\partial m / \partial T$ and $\partial m / \partial \mu$ can be moved out of the integrals with respect to k . Thus, Eq.(4.120) gives

$$\frac{\partial m}{\partial T} \langle \langle 1 \rangle \rangle_- = \frac{\partial m}{\partial \mu} \langle \langle 1 \rangle \rangle_+, \quad (4.122)$$

where

$$\begin{aligned} \langle \langle 1 \rangle \rangle_- &= \int_0^\infty k^2 dk \left\{ \frac{\exp[(\omega^*(k, T, \mu) - \mu)/T]}{(\exp[(\omega^*(k, T, \mu) - \mu)/T] \mp 1)^2 T} \right. \\ &\quad \left. - \frac{\exp[(\omega^*(k, T, -\mu) + \mu)/T]}{(\exp[(\omega^*(k, T, -\mu) + \mu)/T] \mp 1)^2 T} \right\}, \\ \langle \langle 1 \rangle \rangle_+ &= \int_0^\infty k^2 dk \left\{ \frac{\exp[(\omega^*(k, T, \mu) - \mu)/T] (\omega^*(k, T, \mu) - \mu)}{(\exp[(\omega^*(k, T, \mu) - \mu)/T] \mp 1)^2 T^2} \right. \\ &\quad \left. + \frac{\exp[(\omega^*(k, T, -\mu) + \mu)/T] (\omega^*(k, T, -\mu) + \mu)}{(\exp[(\omega^*(k, T, -\mu) + \mu)/T] \mp 1)^2 T^2} \right\}. \end{aligned} \quad (4.123)$$

Besides, the above equation can be expressed in terms of p_{id} of Eq.(4.63) as

$$\frac{\partial m}{\partial T} \frac{\partial}{\partial \mu} \left(\frac{\partial p_{id}}{\partial m} \Big|_{T, \mu} \right) = \frac{\partial m}{\partial \mu} \frac{\partial}{\partial T} \left(\frac{\partial p_{id}}{\partial m} \Big|_{T, \mu} \right). \quad (4.124)$$

It implies the Maxwell relation of the ideal gas,

$$\frac{\partial}{\partial \mu} \left(\frac{\partial p_{\text{id}}}{\partial T} \Big|_{m, \mu} \right) = \frac{\partial}{\partial T} \left(\frac{\partial p_{\text{id}}}{\partial \mu} \Big|_{m, T} \right), \quad (4.125)$$

and taking into account Eq.(4.118) and (4.119), one recovers

$$\frac{\partial s}{\partial \mu} = \frac{\partial^2 p}{\partial T \partial \mu} = \frac{\partial^2 p}{\partial \mu \partial T} = \frac{\partial n}{\partial T}. \quad (4.126)$$

This Maxwell relation was already given as Eq.(7) of Ref.(PESHIER; KAMPFER; SOFF, 2000) and it could be expressed as the flow equation for the running coupling constant α_s (PESHIER; KAMPFER; SOFF, 2000; PESHIER; KAMPFER; SOFF, 2002; BLUHM; KAMPFER; SOFF, 2005; BLUHM et al., 2007a). That is say, given a specific expression of $\alpha_s(T, \mu)$ at vanishing chemical potential $\mu = 0$, one can obtain the $\alpha_s(T, \mu \neq 0)$. Alternatively, we think the Maxwell relation can be used to determine the quasi-particle mass $m(T, \mu)$. One can also solve the Eq.(4.122) formally by using the method of characteristics. As shown in Appendix C, its solution consists of characteristic curves for given m satisfying

$$\frac{d\mu}{dT} = - \frac{\langle\langle 1 \rangle\rangle_+}{\langle\langle 1 \rangle\rangle_-}.$$

Similarly, once we know the explicit $m(T, 0)$ by fitting with Lattice data at vanishing chemical potential, $m(T, \mu)$ can be acquired by carrying out numerical integral from the $\mu = 0$ boundary onto the $T - \mu$ plane where $\mu \neq 0$.

4.5.2 A special momentum dependent solution

Many works show that the quasi-particle mass is also dependent on the momentum, in addition to the temperature T and chemical potential μ . For example, the results on the gluon (DUDAL et al., 2008; DUDAL; SORELLA; VANDERSICKEL, 2011; CUCCHIERI et al., 2012; CAPRI et al., 2017) and quark propagator (BANDYOPADHYAY et al., 2016) within the Gribov-Zwanziger framework show that the pole masses indeed are dependent on momentum and the non-perturbative approaches, Schwinger-Dyson equation also shows that both dynamic masses of gluon (AGUILAR; BINOSI; PAPAVALASSILIOU, 2016) and quark (AGUILAR; PAPAVALASSILIOU, 2011; AGUILAR et al., 2018) are functions of momentum. Besides, in the framework of quasi-particle model, there are also many authors investigate the concept of momentum dependent self-energy (BLUHM et al., 2007b; SCHULZE; BLUHM; KAMPFER, 2008; BERREHRAH et al., 2016b; BERREHRAH et al., 2016a). Thus, instead of considering the quasi-particle mass is only dependent on T and μ , we suggest that the quasi-particle mass is a function of k , T and μ . Then the integro-differential equation (4.120) may have infinite solutions. For convenience, using the assumption that the integrands on the both sides of Eq.(4.120) are the same, we can have a special solution, which means,

$$\frac{\partial m}{\partial T} \langle\langle 2 \rangle\rangle_- = \frac{\partial m}{\partial \mu} \langle\langle 2 \rangle\rangle_+, \quad (4.127)$$

where

$$\begin{aligned}
\langle\langle 2 \rangle\rangle_- &= \frac{\exp[(\omega^*(k, T, \mu) - \mu)/T]T}{(\exp[(\omega^*(k, T, \mu) - \mu)/T] \mp 1)^2} \\
&\quad - \frac{\exp[(\omega^*(k, T, -\mu) + \mu)/T]T}{(\exp[(\omega^*(k, T, -\mu) + \mu)/T] \mp 1)^2}, \\
\langle\langle 2 \rangle\rangle_+ &= \frac{\exp[(\omega^*(k, T, \mu) - \mu)/T](\omega^*(k, T, \mu) - \mu)}{(\exp[(\omega^*(k, T, \mu) - \mu)/T] \mp 1)^2} \\
&\quad + \frac{\exp[(\omega^*(k, T, \mu) - \mu)/T](\omega^*(k, T, \mu) - \mu)}{(\exp[(\omega^*(k, T, \mu) - \mu)/T] \mp 1)^2}.
\end{aligned} \tag{4.128}$$

Since ω^* is involved in the above equation, the resultant particle mass is indeed a function of k . Then again, the above equation can be solved by using the method of characteristics, and its solution consists of characteristic curves for given ω^* .

Here, we solve Eq.(4.127) numerically to obtain $m(k, T, \mu)$, and subsequently the bag constant B as well as other thermodynamical quantities. The obtained results are compared to those by using a presumed quasi-particle mass used in Sec. 4.4.2.2. If one evaluates B by integrating on the (T, μ) plane, using its partial derivatives determined by Eq.(4.118) and Eq.(4.119) after substituting an *a priori* function $m(T, \mu)$, then the results are usually path dependent.

Table 2 – A comparison of the calculated bag constant $B(T, \mu)$ by using two different integral paths. The integral for B is carried out from $(T_0 = 0.16 \text{ GeV}, \mu_0 = 0)$ to (T, μ) , where path 1 is defined by $(T_0, \mu_0) \rightarrow (T, \mu_0) \rightarrow (T, \mu)$, while path 2 is through $(T_0, \mu_0) \rightarrow (T_0, \mu) \rightarrow (T, \mu)$. The table shows the results obtained for several different values of (T, μ) .

T(GeV)	$\mu(\text{GeV})$	the present model		discrepancy	presumed form for $m(T, \mu)$		discrepancy
		path 1	path 2		path 1	path 2	
0.1	0.1	5.22×10^{-3}	5.23×10^{-3}	0.016%	1.52×10^{-2}	1.32×10^{-2}	7.11%
0.15	0.1	1.10×10^{-3}	1.10×10^{-2}	0.014%	9.27×10^{-3}	8.78×10^{-3}	2.72%
0.20	0.1	6.40×10^{-2}	6.40×10^{-2}	0.018%	1.04×10^{-1}	1.05×10^{-1}	0.43%
0.1	0.2	5.03×10^{-3}	5.04×10^{-3}	0.11%	2.15×10^{-2}	1.37×10^{-2}	22.2%
0.15	0.2	1.17×10^{-3}	1.17×10^{-2}	0.017%	1.52×10^{-2}	1.34×10^{-2}	6.35%
0.20	0.2	6.56×10^{-2}	6.56×10^{-2}	0.033%	1.05×10^{-1}	1.08×10^{-1}	1.56%
0.1	0.3	4.76×10^{-3}	4.78×10^{-3}	0.122%	3.11×10^{-2}	1.47×10^{-2}	35.9%
0.15	0.3	1.30×10^{-3}	1.30×10^{-2}	0.003%	2.43×10^{-2}	2.07×10^{-2}	8.00%
0.20	0.3	6.84×10^{-2}	6.85×10^{-2}	0.030%	1.07×10^{-1}	1.13×10^{-1}	2.97%

In Table 2, we show the comparison between the two different approaches. Here the integral for B is carried out from $(T_0 = 0.16 \text{ GeV}, \mu_0 = 0)$ to (T, μ) by using two different paths: path 1 is defined by $(T_0, \mu_0) \rightarrow (T, \mu_0) \rightarrow (T, \mu)$, while path 2 is defined by $(T_0, \mu_0) \rightarrow (T_0, \mu) \rightarrow (T, \mu)$. It is found that for the case where $m(T, \mu)$ is presumed, the discrepancy may grow out of control to some unacceptably large value. While on the other hand, the discrepancy found in the present model is consistently small, and therefore we are confident that they are purely due to numerical errors.

Apparently, the uncertainty in B also spreads to other thermodynamic quantities. Besides, a thermodynamic quantity can be evaluated by using ensemble average or by thermodynamic relations. For instance, entropy can be calculated by using the third line of Eq.(4.78), or by the relation $s = \left. \frac{\partial p}{\partial T} \right|_{V, \mu}$. So in the following, we verify the thermodynamic consistency by evaluating the entropy density and baryon density by using different methods. The results are presented in Figure 14. For the calculated

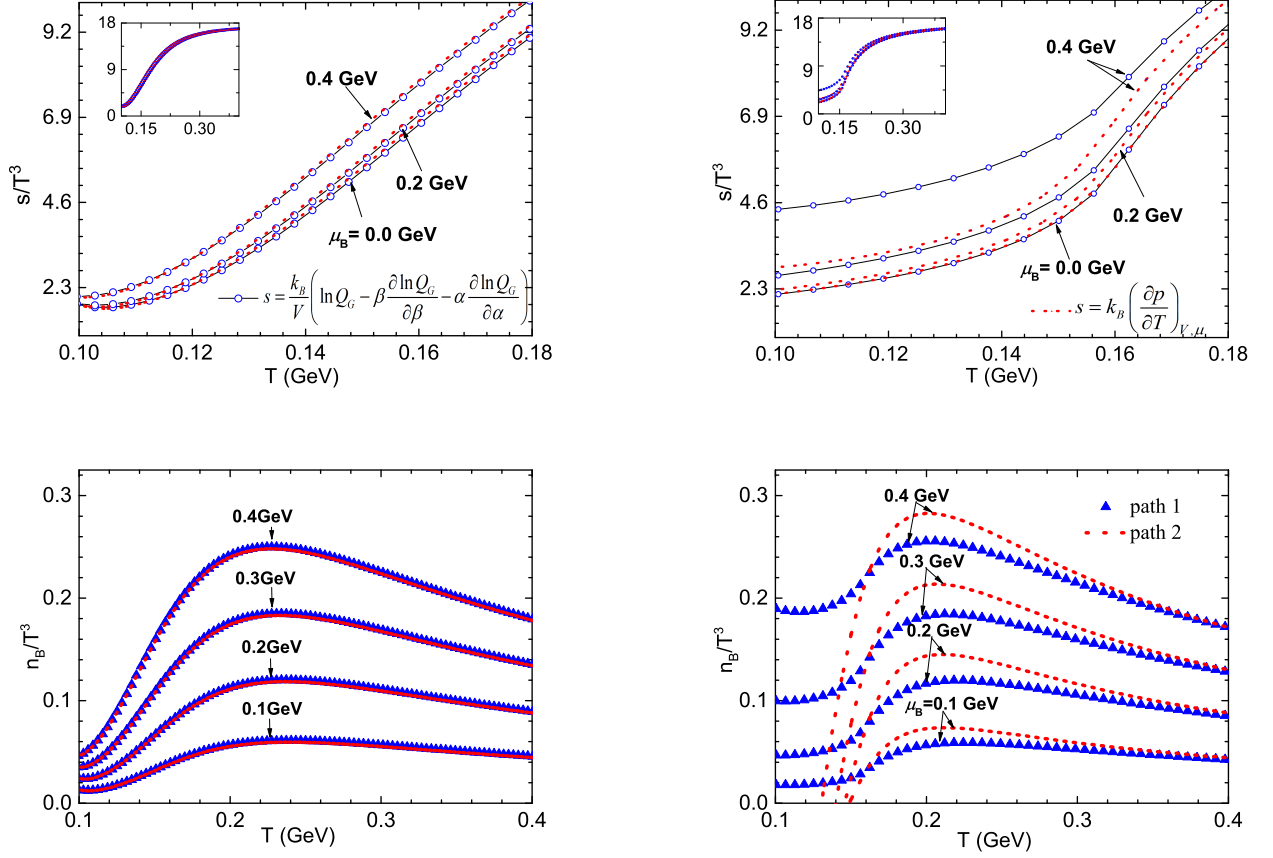


Figure 14 – (Color online) The calculated entropy density and pressure by using different expressions. The entropy density calculated by using ensemble average is presented in empty blue circles and that by using thermodynamic relation is in dotted purple curves. Here both calculations were done by choosing path 1. The baryon density is evaluated using Eq.(4.73), but by choosing different integral paths defined in the text, the results are shown in solid blue triangles and dotted red curves respectively. Left column: the present method; right column: the method with a presumed form for particle mass.

entropy density, one finds that discrepancy arises for the case where $m(T, \mu)$ is presumed. The discrepancy increases with increasing chemical potential. This behavior is expected, as at $\mu = 0$ the formula restores to that proposed by Gorenstein and Yang, $m = m(T)$, and therefore the expression is thermodynamically consistent. Also, it is observed that the discrepancy becomes smaller as temperature increases. Numerically, the discrepancy in the transition region $T \sim 0.18$ GeV is already sufficiently small so that the problem is less severe. For the baryon density, again the discrepancies become more significant at low temperature as well as at larger baryon chemical potential.

If the contributions from anti-particles are insignificant, namely, $\mu \gg 1$, the Eq.(4.127) can be further simplified to

$$\frac{\partial m}{\partial \mu} = \frac{T}{(\omega^*(k, T, \mu) - \mu)} \frac{\partial m}{\partial T}. \quad (4.129)$$

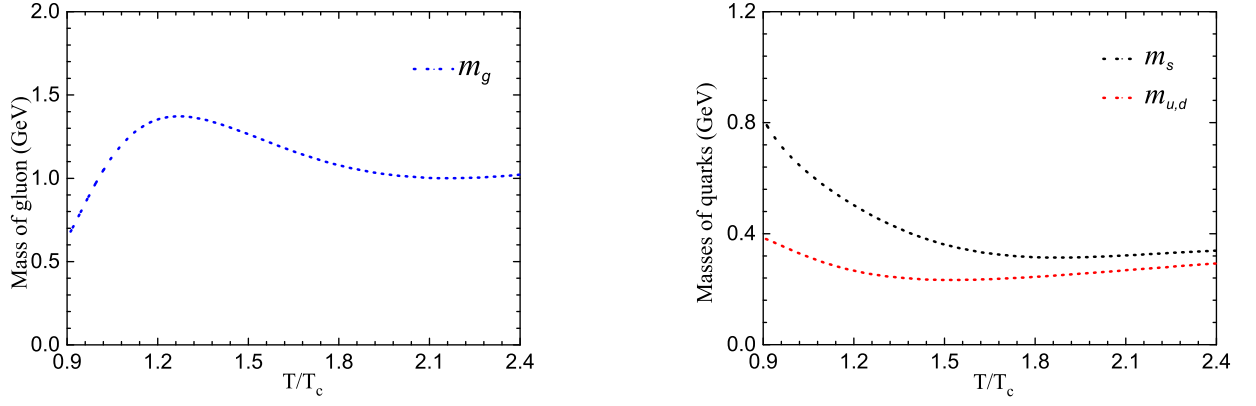


Figure 15 – (Color online) The resultant temperature dependent quasi-particle mass for gluons, light and strange quarks at zero chemical potential.

Then we can obtain an analytic solution by using the method of characteristics shown in Appendix C,

$$m = f\left(\frac{T\omega^*}{\omega^* - \mu}\right), \quad (4.130)$$

where $f(T) \equiv m(T, \mu = 0)$ is determined by the boundary condition.

4.5.3 Numerical Results

Now, we are in a position to present the numerical results and to compare them to the recent Lattice data for $N_f = 2 + 1$ flavor QCD system (BORSANYI et al., 2012; BORSANYI et al., 2014; BAZAVOV et al., 2012a; BAZAVOV et al., 2014; BAZAVOV et al., 2017). We first show the calculated thermodynamical quantities for the case of momentum dependent quasi-particle mass. Here, the free parameters are the effective masses of gluons, of light as well as strange quarks as functions of temperature at zero chemical potential, and a constant related to E_1 . Once they are determined, one may evaluate all thermodynamical quantities such as energy density, pressure, and entropy density at zero as well as finite baryon density. In addition, we also calculate the trace anomaly, sound velocity, and the particle number susceptibility defined as

$$\chi_2^{ab} = \frac{T}{V} \frac{1}{T^2} \frac{\partial^2 \ln Q_G(T, \mu_u, \mu_d, \mu_s)}{\partial \mu_a \partial \mu_b} \Big|_{\mu_a = \mu_b = 0}. \quad (4.131)$$

In particular, the relevant quantities χ_2^B and χ_2^L in the present model (BORSANYI et al., 2012) read:

$$\chi_2^B = \frac{1}{9} [\chi_2^u + \chi_2^d + \chi_2^s + 2\chi_{11}^{us} + 2\chi_{11}^{ds} + 2\chi_{11}^{ud}] = \frac{1}{9} [2\chi_2^u + \chi_2^s], \quad (4.132)$$

and

$$\chi_2^L = \frac{1}{9} [\chi_2^u + \chi_2^d + 2\chi_{11}^{ud}] = \frac{2}{9} \chi_2^u. \quad (4.133)$$

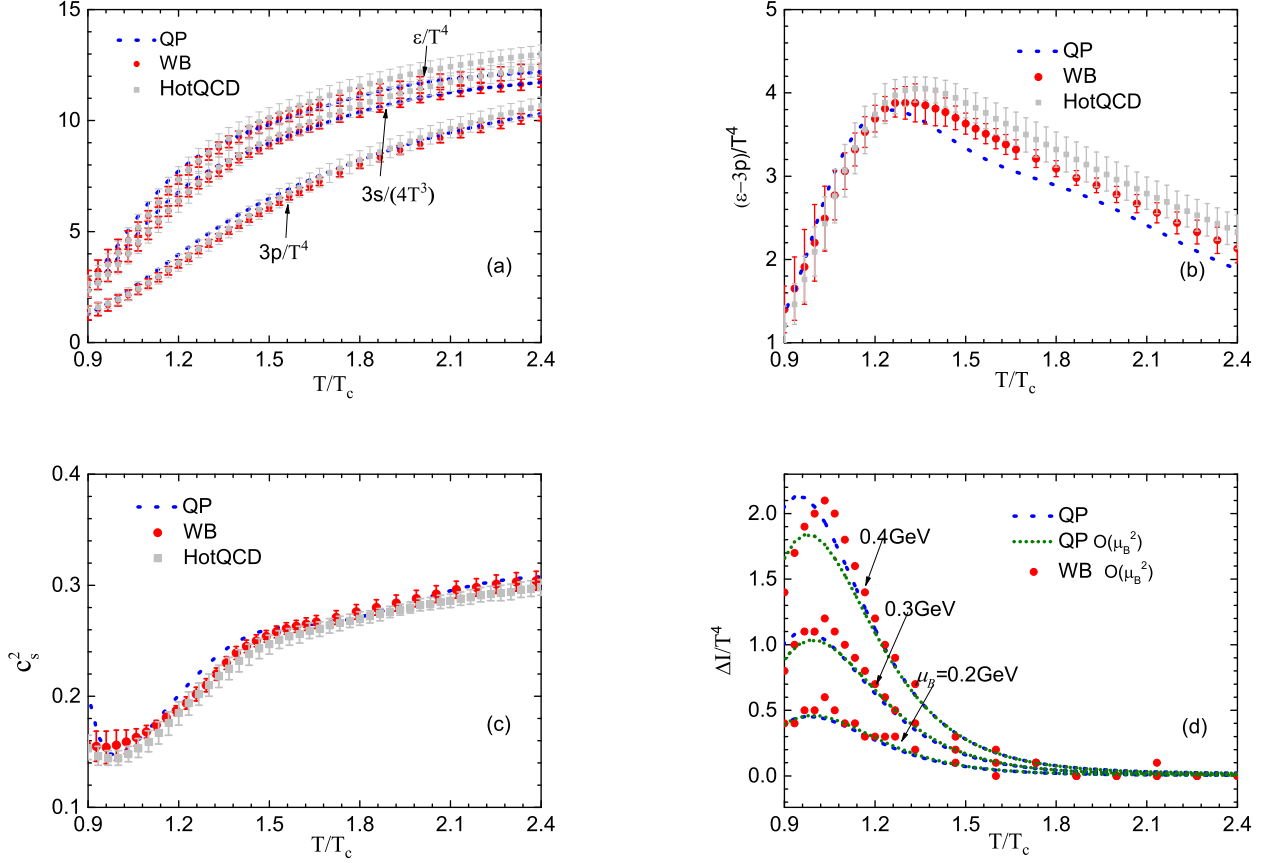


Figure 16 – (Color online) The calculated thermodynamical quantities for both vanishing and finite baryon chemical potential. The thermodynamical quantities obtained by the present model is shown in dotted blue curves. The calculated results truncated in terms of $\frac{\mu}{T}$ up to second are shown in dotted green curves. They are compared with those of lattice QCD calculations the Wuppertal-Budapest (BORSANYI et al., 2012; BORSANYI et al., 2014) and HotQCD (BAZAVOV et al., 2012a; BAZAVOV et al., 2014; BAZAVOV et al., 2017) Collaborations, indicated by filled red circles and grey squares (with error bars when it applies) respectively. The first row shows the results of entropy density, energy density, pressure (on the left) and trace anomaly (on the right) at zero baryon chemical potential. The left plot on the second row shows the speed of sound and the right plot gives trace anomaly for different values of chemical potential.

The x-axis of the plots is chosen to be T/T_c , where the value for the transition temperature $T_c = 0.15\text{GeV}$ is taken (BORSANYI et al., 2014; BAZAVOV et al., 2014; BORSANYI et al., 2010; AOKI et al., 2006b). Then all these results are compared to those obtained by the lattice QCD calculations by Budapest-Wuppertal (BORSANYI et al., 2012; BORSANYI et al., 2014) as well as HotQCD collaborations (BAZAVOV et al., 2012a; BAZAVOV et al., 2014; BAZAVOV et al., 2017).

The parameters of the present approach are determined as follows. Firstly, the lattice data (BORSANYI et al., 2014) on particle susceptibility of light χ_2^L quarks is used to determine the quasi-particle mass of light quarks at vanishing chemical potential. Subsequently, the quasi-particle mass of strange quark as a function of temperature is determined by the particle susceptibility regarding baryon chemical potential χ_2^B . Then the gluon mass is used to fit the energy density for $n_B = 0$. Also, to compare the results between Eq.(4.122) and Eq.(4.127), we assume that the quasi-particle mass is momentum independent at $\mu = 0$, so that both equations are solved by using the same boundary condition. Finally, E_1 is tuned to further improve the pressure as a function of temperature at zero baryon density. The resultant particle masses at $\mu_q = 0$ are shown in Figure 15, the constant of integration for the bag constant is taken to be $B(T_c, \mu = 0) = 0.12 \times T_c^4$, and the value of E_1 is found to be $\beta E_1/V = 2.305 \times 10^{-4}\text{GeV}^3$. The particle mass at finite chemical potential is subsequently evaluated according to Eq.(4.127).

We note that, in principle, it seems to be more reasonable to adjust the model parameters to the lattice data of χ_2^u and χ_2^s , instead of χ_2^L and χ_2^B . This is because the lattice results show that χ_2^B contains flavor correlations. However, since our quasi-particle model does not take into account the contributions from mixed cumulant terms, such as χ_{11}^{ud} in Eq.(4.133), it is found that in practice the proposed model calibration leads to a better fit to the existing lattice data.

The resultant thermodynamic quantities are presented in Figure 16 and Figure 17. One observes that, overall, a reasonably good agreement is achieved, especially for quark number susceptibility, besides the energy density, entropy density, and pressure. It is also worth pointing out that in our present approach, we did not introduce any renormalization for the degeneracy factor, which is adopted as an additional free parameter by some of the quasi-particle approaches. The only discrepancies are observed for the quantities associated with the first and second derivative of the grand partition function for the region where $T < T_c$. For instance, the pressure difference is related to the expansion in terms of μ/T . Therefore, the deviation becomes larger for smaller temperature. It is probably related to the peak of χ_4 at T_c (BELLWIED et al., 2015) which has not been appropriately considered in the present study. As explained above, the fit was only carried out regarding the χ_2 lattice data. Since the lattice QCD results were obtained by a Taylor expansion in terms of $\frac{\mu}{T}$, it is thus meaningful to show our results also truncated to the corresponding order when comparing to them. This is shown in the second row of Figure 17. It is noted when we evaluate the pressure difference expanded up to the order of $(\frac{\mu}{T})^2$, the calculated curve stays closer to the lattice results, as expected. This is shown by the dotted green curves in the plot on the left of the first row of Figure 17. But since the present quasi-particle model does not consider any contribution from the mixed second order derivative such as χ_{11}^{ud} , it is merely understood as a result of appropriate parameterization. For the same reason, the results on fourth-order cumulant χ_4^B presents more substantial discrepancies. Probably due to a similar reason,

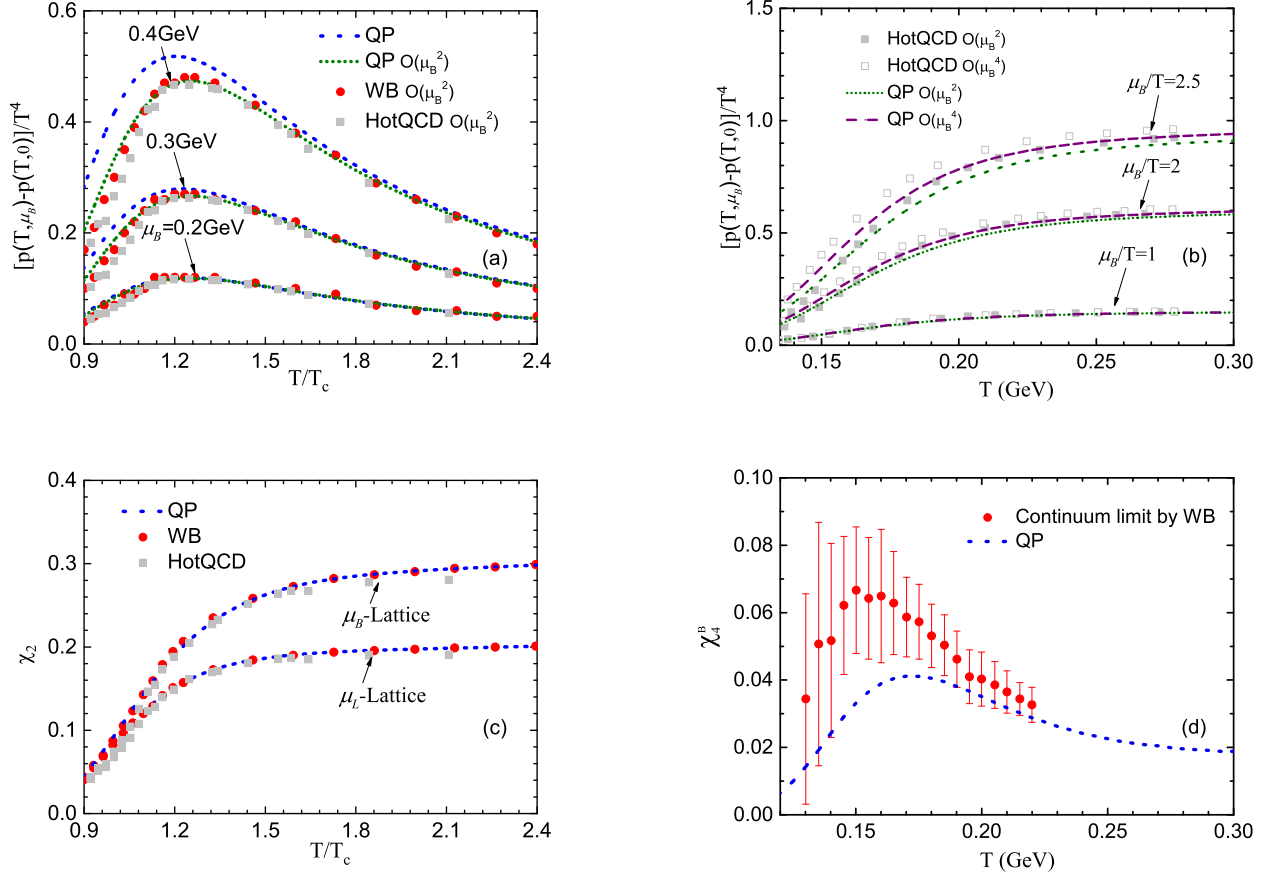


Figure 17 – (Color online) The calculated thermodynamical quantities for both vanishing and finite baryon chemical potential. The thermodynamical quantities obtained by the present model is shown in dotted blue curves. The calculated results truncated in terms of $\frac{\mu}{T}$ up to second and fourth order are shown in dotted green and dashed purple curves respectively. They are compared with those of lattice QCD calculations the Wuppertal-Budapest (BORSANYI et al., 2012; BORSANYI et al., 2014) and HotQCD (BAZAVOV et al., 2012a; BAZAVOV et al., 2014; BAZAVOV et al., 2017) Collaborations, indicated by filled red circles and grey squares (with error bars when it applies) respectively. The first row shows the difference of pressure for given μ_B (on the left) and μ_B/T (on the right) for different as a function of temperature. The calculations have been carried out by using different truncations and the results are compared against corresponding lattice data. The second row presents the second and fourth order cumulants of particle number fluctuations, χ_2 and χ_4 .

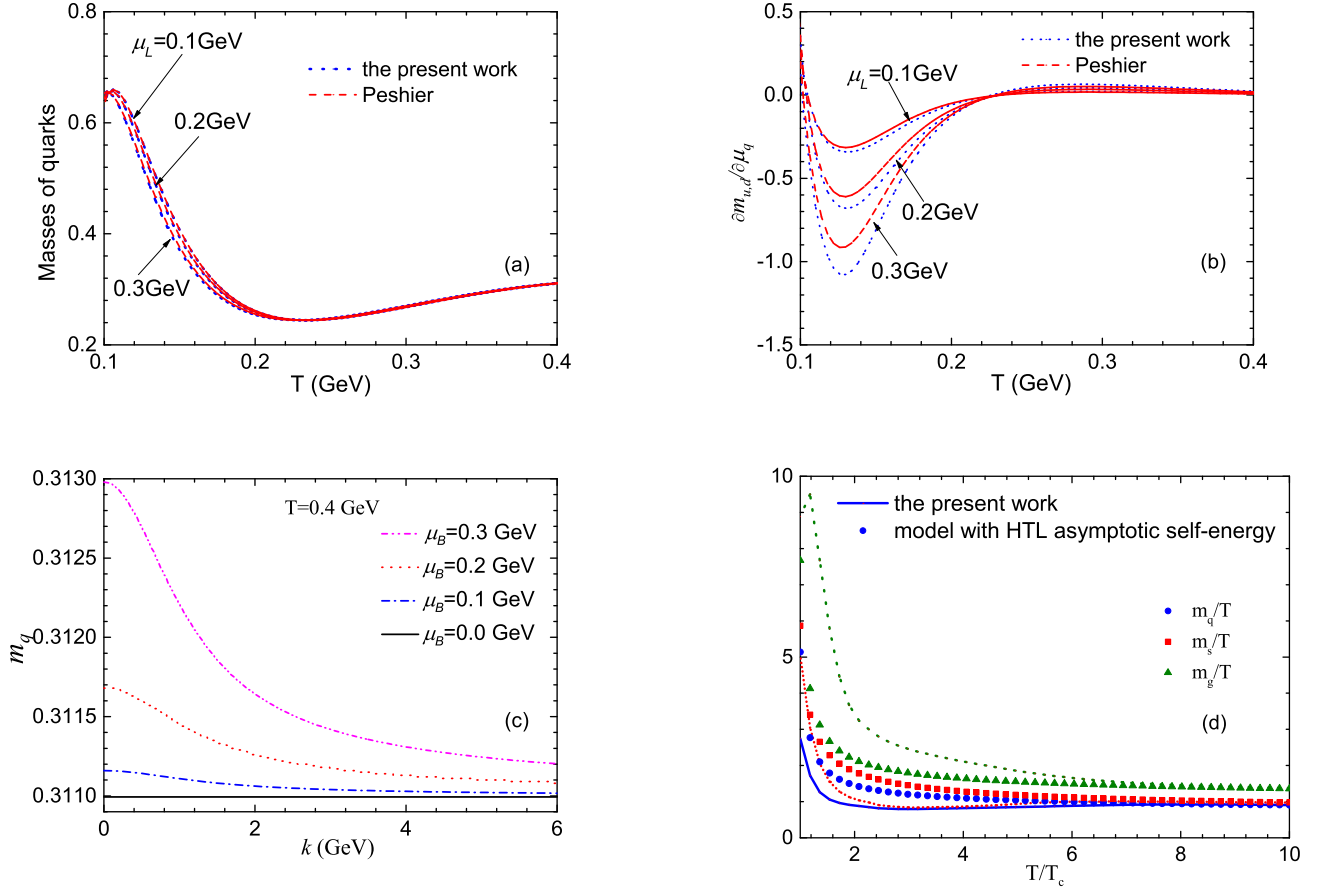


Figure 18 – (Color online) Top: the calculated quasi-particle mass of light quarks and its derivative as functions of temperature for different baryon chemical potentials, obtained by solving Eq.(4.127), in comparison with those by solving Eq.(4.122), the latter is equivalent to the approach by Peshier et al. (PESHER et al., 1994). Bottom left: the quasi-particle mass of light quarks as a function of momentum for the solution discussed in the present work. Bottom right: the calculated asymptotic behavior of quasi-particle masses, in comparison with a model (PESHER; KAMPFER; SOFF, 2002) inspired by the gauge-independent HTL calculations.

some small deviation is also found for the calculated sound velocity as a function of temperature. However, by adjusting the gluon mass in the region of temperature $T \sim T_c$, we were able to reproduce the behavior of the sound speed which increases again as the temperature reaches the sector associated with the hadronic resonance gas. From a practical viewpoint, these difference can also be amended by manually connecting the quasi-particle EoS to that of the hadronic resonance gas model.

In order to compare two different solutions discussed in the previous section, we solve Eq.(4.122) and Eq.(4.127) respectively but fitting to the given boundary condition at $n_B = 0$ defined by the lattice data. The corresponding results are shown in Figure 18, where the obtained particle masses are presented as a function of temperature. In the first case, since the mass is also a function of momentum k , the presented results are average values evaluated by using the same weight on the r.h.s. of Eq.(4.118) or Eq.(4.119). Numerically, one finds that the particles masses from two different schemes are quite close to each other. Though it seems to be a somewhat a surprising result, we understand that it could be merely owing to that both approaches are tuned to reproduce the lattice

data and the fact the numerically obtained momentum dependence of quasi-particle mass is not strong at all. The latter is observed in the plot at the bottom left of Figure 18, which presents the obtained momentum dependence of quark masses for a given temperature but with different values of chemical potential. It is observed that the quasi-particle mass decreases slightly but monotonically and converges to a given value as the momentum increases. As the chemical potential increases, the dependence becomes stronger, though the overall dependence is not significant.

Last but not least, we show that the results obtained in the present approach are consistent with the established perturbative limit. This is achieved by carrying out calculations by using the quasi-particle model proposed in Ref. (BLUHM; KAMPFER; SOFF, 2005) with the following forms for the quasi-particle masses

$$m_a^2 = m_{a0}^2 + \Pi_a, \quad (4.134)$$

where $a = g, q, s$ and the quasi-particle self-energies adopt the asymptotic forms of the gauge-independent HTL calculations (PISARSKI, 1989; BELLAC, 2011):

$$\Pi_g = \left(\left[3 + \frac{N_f}{2} \right] T^2 + \frac{3}{2\pi^2} \sum_f \mu_f^2 \right) \frac{g_s^2}{6}, \quad (4.135)$$

$$\Pi_q = 2m_{q0} \sqrt{\frac{g_s^2}{6} \left(T^2 + \frac{\mu_q^2}{\pi^2} \right)} + \frac{g_s^2}{3} \left(T^2 + \frac{\mu_q^2}{\pi^2} \right), \quad (4.136)$$

$$\Pi_s = 2m_{s0} \sqrt{\frac{g_s^2}{6} T^2} + \frac{g_s^2}{3} T^2. \quad (4.137)$$

For the high temperature region, the coupling is taken to have the form of the perturbative running coupling at two-loop order

$$g_s^2(T, \mu_q = 0) = \frac{16\pi^2}{\beta_0 \log \xi^2} \left[1 - \frac{2\beta_1}{\beta_0} \frac{\log(\log \xi^2)}{\log \xi^2} \right], \quad (4.138)$$

with

$$\xi = \lambda \frac{T - T_s}{T_c}. \quad (4.139)$$

which regulates the infrared divergence of the running coupling. For the parameters, the scale parameter and the temperature shift are chosen to be $\lambda = 1.5$ and $T_s = 0.15T_c$. This is done so that the model may adequately reproduce the recent lattice data (BORSANYI et al., 2012; BORSANYI et al., 2014) in the intermediate temperature region, while the remaining parameters are taken to be the same as used in literature (MA; QIAN, 2018). The calculated asymptotic behavior of the quasi-particle mass is shown in the bottom right of Figure 18. As described above, the quasi-particle masses at vanishing chemical potential are adjusted to reproduce the lattice data at the intermediate temperature. We first interpolate the lattice data, and then make use of the obtained expression to evaluate the particle masses for the whole temperature range. The interpolation is carried out by specifically requiring the asymptotic behavior in Eq.(4.137) is attained at the limit $T \rightarrow \infty$. It is shown that our present approach is indeed

consistent with the established perturbative limit. Owing to Eq.(4.137), at very high temperature but physically relevant finite chemical potential, the limit established above does not change at all, which is also confirmed by the numerical calculations.

5 THE QCD PHASE TRANSITION AND CROSS-OVER

As we mentioned before, the QGP could be formed in the Early Universe or in the heavy ion collision experiments on earth, like the LHC at CERN and the RHIC at BNL. This QGP phase with high temperature and density will be switched to hadronic phase when the temperature arrives at T_c . Lattice QCD suggested that such a phase transition is a “cross-over” (AOKI et al., 2006a; KARSCH, 2002) at zero chemical potential. At non-vanishing chemical potential, on the other hand, it is indicated that there is a first order phase transition by some theoretical model calculations (HALASZ et al., 1998; BERGES; RAJAGOPAL, 1999; STEPHANOV; RAJAGOPAL; SHURYAK, 1998; SCHWARZ; KLEVANSKY; PAPP, 1999; FODOR; KATZ, 2004). These results imply that the phase diagram is probably featured by a CEP where the line of first order phase transition terminates, and the transition is expected to be of second order at this point. As a matter of fact, the existence and properties of the critical point is a long-standing intriguing topic.

The arrangement of this chapter is shown as follows. In Sec.5.1, an EoS with a phenomenological CEP was proposed. As an important signature of QCD CEP, the multiplicity fluctuations as a function of beam energy are shown in Sec.5.2

5.1 A HYBRID EOS WITH QCD CRITICAL POINT

In this section, we will propose a new hybrid EoS by using the combination of EoS proposed by Hama *et al.* between quasi-particle model at high temperatures fitting to Lattice QCD and hadron resonance gas at low temperatures. In 2009, an EoS by combining HRG at low temperatures with lattice EoS at high temperatures had already been proposed by Huovinen and Petreczky (HUOVINEN; PETRECZKY, 2010) which is widely used. In fact, they used inverse polynomial fit for the lattice data at zero baryon density at high temperature region, and it is matched to the HRG model at the joining temperature T_0 by requiring that the trace anomaly as well as its first and second derivatives are continuous. However, it cannot be generalized to finite chemical potential naturally and a new lattice EoS data (BORSANYI et al., 2014) is given which are in accordance with the one from HTL calculations (BAZAVOV et al., 2014).

In our treatment, HRG with excluded volume correction is used to produce the EoS at low temperatures, while quasi-particle model fitting to the lattice data explains the EoS at high temperature. To demonstrate the phenomenological QCD critical point discovered by Lattice QCD, Hama *et al.* assumed that there will be a smoothed connection between the EoS of hadronic phase and QGP phase. Thus, we can combine those two models with the aid of the smoothed connection suggested by Hama *et al.* (HAMA et al., 2006) and give a hybrid EoS which can demonstrate the critical point. Due to that both HRG and quasi-particle model we used can be generalized to finite chemical potential without ambiguity, the new EoS with phenomenological critical point can not only explain the EoS at vanishing chemical potential but also the EoS at non-vanishing chemical potential.

5.1.1 Quasi-particle Model for 2+1 flavor QGP at finite chemical potential

Here, considering the $N_f = 2 + 1$ QCD system, the quasi-particle model for reproducing the Lattice data at high temperature (BAZAVOV et al., 2014) proposed by Bannur (BANNUR, 2008) is adopted. An advantage for this quasi-particle model is that it does not introduce a temperature and/or chemical potential dependent bag constant B . Because B should be solved from a restrictive condition (GORENSTEIN; YANG, 1995; BANNUR, 2007a) which is not easy to be adjusted to fit with Lattice data.

In Sec. 4.4.2.2, we already presented the results of EoS at vanishing chemical potential. However, for finite chemical potential, the term T/Λ_T in the coupling constant should be adjusted to (BANNUR, 2007c; SCHNEIDER, 2003)

$$\frac{T}{\Lambda_T} = \frac{T}{\Lambda_T} \sqrt{1 + (1.91/2.91)^2 \frac{\mu^2}{T^2}}. \quad (5.1)$$

Meanwhile, the plasma frequencies shown in Eq.(4.89) are replaced by (PESHER; KAMPFER; SOFF, 2000)

$$m_f^2 = \frac{g_s^2 T^2}{18} N_f \left(1 + \frac{\mu^2}{\pi^2 T^2}\right). \quad (5.2)$$

Then, the number density n_q at finite chemical potential can be calculated with the aid of the above modified plasma frequencies. Therefore, the pressure at finite chemical potential can be determined by the thermodynamic relation $n = \left(\frac{\partial p}{\partial \mu}\right)_T$ via an integral from its value at zero chemical potential

$$\Delta p = p(T, \mu) - p(T, 0) = \int_0^\mu n_q d\mu. \quad (5.3)$$

Besides, other thermodynamic quantities are obtained according to the thermodynamic relations

$$\Delta s = \partial \Delta p / \partial T, \quad (5.4)$$

$$\Delta \varepsilon = T \Delta s - \Delta p + \mu_B n_B + \mu_S n_S. \quad (5.5)$$

Within BQS ensemble for 2+1 flavor QGP, it is well known that

$$\mu_B = \mu_u + 2\mu_d, \quad (5.6)$$

$$\mu_Q = \mu_u - \mu_d, \quad (5.7)$$

$$\mu_S = \mu_d - \mu_s, \quad (5.8)$$

where B , Q and S are Baryon number, Charge number and Strangeness number, respectively.

Considering strangeness neutrality condition, *i.e.*, $\mu_s = 0$, we have $\mu_Q = 0$, $\mu_d = \mu_S$ and $\mu_u = \mu_d = \mu_B/3$ for light quarks. One sees that Eq.(5.2) can restore Eq.(4.89) at vanishing chemical potential. However, in our present study, we employ Eq.(4.91) which only approaches Eq.(4.89) as $T \rightarrow \infty$. To compensate their difference at the low-temperature region, we take $\Lambda_T = 0.130$ GeV for

finite chemical potential.

5.1.2 The smoothed connection in transition region

As described previously, the hadronic phase of QCD will be transferred to QGP phase at high temperature and/or density. As suggested by (AOKI et al., 2006a; KARSCH, 2002), such a transition is a “cross-over” instead of first-order in the region of small baryon density. This property of QCD phase transition has been studied by some Lattice QCD group (AOKI et al., 2006b; AOKI et al., 2009; BORSANYI et al., 2010; BAZAVOV et al., 2012b). Here, to implement this “cross-over” phenomenologically, we proposed a hybrid EoS by using the interpolation scheme suggested by Hama *et al* (HAMA et al., 2006) which is shown as follows.

As we all know, Gibbs condition can be used to determine the chemical potential and temperature of the two phases when the phase transition is of first order, as follows

$$(p - p^Q)(p - p^H) = 0, \quad (5.9)$$

where Q and H is the superscript stand for QGP and hadronic phase, respectively.

If the phase transition is of the cross-over, the above Gibbs condition Eq. (5.9) should be modified as (HAMA et al., 2006)

$$(p - p^Q)(p - p^H) = \delta(\mu, T), \quad (5.10)$$

where

$$\delta(\mu, T) = \delta_0(T) \exp [-(\mu/\mu_c)^4], \quad (5.11)$$

and μ_c is the critical chemical potential, which is taken to be $\mu_c = 0.3$ GeV in this thesis.

To solve Eq.(5.10), one should rewrite this equation as

$$p^2 - (p^Q + p^H)p + p^Q p^H = \delta(\mu, T). \quad (5.12)$$

The solutions of the above quadratic function can be obtained easily,

$$p = \frac{1}{2} \left[(p^Q + p^H) \pm \sqrt{(p^Q - p^H)^2 + 4\delta(\mu, T)} \right]. \quad (5.13)$$

We noticed that $\delta(\mu, T) \rightarrow 0$ when $\mu \rightarrow \infty$. Then the first order phase transition is recovered, which means $p^H = p^Q$. Therefore we can choose $p = p^H$ (negative sign) or $p = p^Q$ (positive sign). Here we take $p = p^Q$ in the following calculations. By substituting the following definition

$$\lambda \equiv \frac{1}{2} \left[1 - \frac{(p^Q - p^H)}{\sqrt{(p^Q - p^H)^2 + 4\delta(\mu, T)}} \right] \quad (5.14)$$

into Eq.(5.13), one knows

$$\begin{aligned}
p &= \frac{1}{2} \left[p^H + p^Q + \sqrt{(p^Q - p^H)^2 + 4\delta(\mu, T)} \right] \\
&= p^Q + \frac{1}{2} \left[\sqrt{(p^Q - p^H)^2 + 4\delta(\mu, T)} - (p^Q - p^H) \right] \\
&= p^Q + \frac{1}{2} \left[\sqrt{(p^Q - p^H)^2 + 4\delta(\mu, T)} - (p^Q - p^H) \right] \\
&\quad - \frac{2\delta(\mu, T)}{\sqrt{(p^Q - p^H)^2 + 4\delta(\mu, T)}} + \frac{2\delta(\mu, T)}{\sqrt{(p^Q - p^H)^2 + 4\delta(\mu, T)}} \\
&= p^Q + \frac{1}{2} \left[\frac{(p^Q - p^H)^2}{\sqrt{(p^Q - p^H)^2 + 4\delta(\mu, T)}} - (p^Q - p^H) \right] \\
&\quad + \frac{2\delta(\mu, T)}{\sqrt{(p^Q - p^H)^2 + 4\delta(\mu, T)}} \\
&= p^Q + \lambda(p^H - p^Q) + \frac{2\delta(\mu, T)}{\sqrt{(p^Q - p^H)^2 + 4\delta(\mu, T)}}.
\end{aligned} \tag{5.15}$$

That means,

$$p = \lambda p^H + (1 - \lambda) p^Q + \frac{2\delta(\mu, T)}{\sqrt{(p^Q - p^H)^2 + 4\delta(\mu, T)}}. \tag{5.16}$$

With the aid of the thermodynamic relation $n = \left(\frac{\partial p}{\partial \mu} \right)_T$, the number density can be expressed as

$$\begin{aligned}
n &= \frac{\partial(\lambda p^H)}{\partial \mu} + \frac{\partial(1 - \lambda)p^Q}{\partial \mu} + \frac{\partial}{\partial \mu} \left[\frac{2\delta(\mu, T)}{\sqrt{(p^Q - p^H)^2 + 4\delta(\mu, T)}} \right] \\
&= \lambda n^H + (1 - \lambda) n^Q + \frac{\partial \lambda}{\partial \mu} (p^H - p^Q) + \frac{\partial}{\partial \mu} \left[\frac{2\delta(\mu, T)}{\sqrt{(p^Q - p^H)^2 + 4\delta(\mu, T)}} \right].
\end{aligned} \tag{5.17}$$

It is easy to derive that

$$\begin{aligned}
\frac{\partial \lambda}{\partial \mu} &= -\frac{1}{2} \frac{(n^Q - n^H)}{\sqrt{(p^Q - p^H)^2 + 4\delta(\mu, T)}} + \frac{1}{2} \frac{(n^Q - n^H)(p^Q - p^H)^2}{((p^Q - p^H)^2 + 4\delta(\mu, T))^{3/2}} \\
&\quad - \frac{(p^Q - p^H)}{((p^Q - p^H)^2 + 4\delta(\mu, T))^{3/2}} \frac{\partial \delta(\mu, T)}{\partial \mu} \Big|_T,
\end{aligned} \tag{5.18}$$

and

$$\begin{aligned} \frac{\partial}{\partial \mu} \left[\frac{2\delta(\mu, T)}{\sqrt{(p^Q - p^H)^2 + 4\delta(\mu, T)}} \right] &= \frac{2}{\sqrt{(p^Q - p^H)^2 + 4\delta(\mu, T)}} \frac{\partial \delta(\mu, T)}{\partial \mu} \Big|_T \\ &+ \frac{2(n^Q - n^H)(p^Q - p^H)\delta(\mu, T)}{((p^Q - p^H)^2 + 4\delta(\mu, T))^{3/2}} \\ &- \frac{4\delta(\mu, T)}{((p^Q - p^H)^2 + 4\delta(\mu, T))^{3/2}} \frac{\partial \delta(\mu, T)}{\partial \mu} \Big|_T. \end{aligned} \quad (5.19)$$

Then, we have

$$\begin{aligned} n &= \lambda n^H + (1 - \lambda) n^Q + \frac{\partial \lambda}{\partial \mu} (p^H - p^Q) + \frac{\partial}{\partial \mu} \left[\frac{2\delta(\mu, T)}{\sqrt{(p^Q - p^H)^2 + 4\delta(\mu, T)}} \right] \\ &= \lambda n^H + (1 - \lambda) n^Q + \frac{1}{\sqrt{(p^Q - p^H)^2 + 4\delta(\mu, T)}} \frac{\partial \delta(\mu, T)}{\partial \mu} \Big|_T. \end{aligned} \quad (5.20)$$

Analogously,

$$s = \lambda s^H + (1 - \lambda) s^Q + \frac{1}{\sqrt{(p^Q - p^H)^2 + 4\delta(\mu, T)}} \frac{\partial \delta(\mu, T)}{\partial T} \Big|_\mu. \quad (5.21)$$

Thus, owing to the relation $\varepsilon = -p + Ts + \mu n$, the energy density reads

$$\begin{aligned} \varepsilon &= \lambda \varepsilon^H + (1 - \lambda) \varepsilon^Q \\ &+ \frac{1}{\sqrt{(p^Q - p^H)^2 + 4\delta(\mu, T)}} \left(\mu \frac{\partial \delta(\mu, T)}{\partial \mu} \Big|_T + T \frac{\partial \delta(\mu, T)}{\partial T} \Big|_\mu - 2\delta(\mu, T) \right). \end{aligned} \quad (5.22)$$

We note, when $\delta_0 = 0$, a first order phase transition is recovered. To be specific, we have $\lambda = 0, p = p^Q$ when $p^Q > p^H$ and $\lambda = 1, p = p^H$ when $p^Q < p^H$. On the other hand, when $\delta(\mu_b)ne0$, the phase transition is smoothed out by an interpolation between the two phases. In other words, instead of a sudden jump, λ varies continuously from 0 to 1 during the transition. Also, it is readily to verify that the above expression guarantees that the resulting pressure satisfies $p > p^Q$ and $p > p^H$. Though strangeness chemical potential is considered in the model, we only consider the case of strange neutrality, and therefore, strangeness chemical potential is not a free parameter.

According to Eq.(5.14), the ratio of δ to $|p^Q - p^H|$ determines whether the relevant thermodynamic quantities are dominated by one phase or more of a mixture of two phases. We note that the Gibbs condition implies $|p^Q - p^H| = 0$, and consequently, $|p^Q - p^H|$ becomes non-zero and increases once the system evolves away from the two-phase equilibrium. In particular, $p^Q - p^H$ possesses different signs on different sides of the transition point. As a result, the interpolation should work without any intervention as it is intended. However, in practice, it is found that sometimes the magnitude of $|p^Q - p^H|$ decreases again as the system moves further away from the line of the first order phase

Table 3 – List of parameters used in the present hybrid EoS at non-vanishing baryon chemical potential

δ_0 (GeV ⁸)	Λ_T (GeV)	μ_c (GeV)	c (GeV ⁻²)
5.90×10^{-10}	0.130	0.3	10^3

transition, which may potentially jeopardize the interpolation scheme. In a view to amending this issue, one defines a temperature interval so that the size of δ is suppressed on the outside of this region. This is achieved by choosing $\delta_0(T)$ to be a piecewise function as follows

- $\delta_0(T) = \delta_0 e^{-c|T-T_p|^2}, \quad T \leq T_p$
- $\delta_0(T) = \delta_0, \quad T_p < T \leq T_p + 0.02$
- $\delta_0(T) = \delta_0 e^{-c|T-T_p-0.02|^2}, \quad T > T_p + 0.02$

where $\delta_0 = 5.90 \times 10^{-10}$ GeV⁸ and $c = 10^3$. T_p stands for the temperature (in GeV) of the corresponding first order transition.

5.1.3 Numerical results

In this section, we present the numerical results of the obtained EoS by using the parameters summarized in Table 3. It is worth mentioning that the expression of running coupling constant, *i.e.*, Eq.(4.96) is not valid when $T \leq \Lambda_T$ due to the term $\ln(2 \ln(T/\Lambda_T))$. To describe the EoS of 2+1 flavor QGP at low temperature $T < \Lambda_T$, we use an extrapolation in this region. Fortunately, the contributions from the HRG dominate at the low temperature and the results are not sensitive to any particular choice of extrapolation scheme.

As in (SCHNEIDER; WEISE, 2001; IVANOV; SKOKOV; TONEEV, 2005), an overall normalization factor 1.06 is introduced to take into account the unknown correction to the effective number of degrees of freedom. For zero chemical potential, the resulting entropy density, energy density, and pressure are shown in Figure 19 in comparison with the lattice QCD results in dotted blue curves with uncertainties (BORSANYI et al., 2012; BORSANYI et al., 2014). We see that all three quantities are reasonably well reproduced.

Another physical quantity of interest is trace anomaly, which is a measure of deviation from the conformal symmetry. By lattice QCD simulations, the square of the speed of sound, $c_s^2 = \frac{\partial p}{\partial \epsilon}$, is found to be smaller than that of an ideal gas of massless particles. In particular, it is found that as T approaches the transition region, c_s^2 reaches down to a minimum and then increases again in accordance with the HRG description of the system. Since the above properties have potentially observable consequences during the hydrodynamical expansion of the system, it is, therefore, an important feature for the EoS. The calculated trace anomaly and the squared speed of sound are presented in Figure 20. We found that the trace anomaly is fairly reproduced. The main feature of the speed of sound is also obtained, though the location of the minimum is slightly shifted towards higher temperature.

For finite chemical potential, pressure differences are calculated for different chemical potentials. The results are shown in Figure 21, in comparison with the lattice QCD results by stout action (BOR-

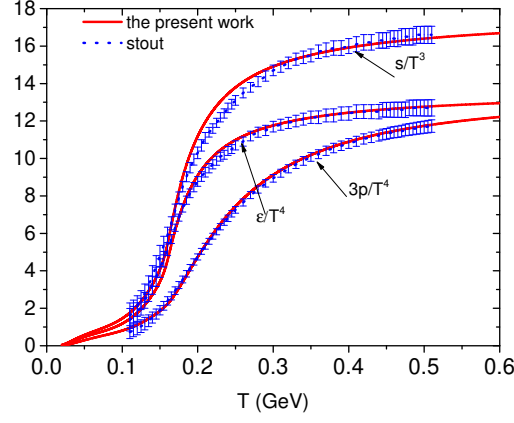


Figure 19 – (Color online) The calculated of $3p/T^4$, ε/T^4 and s/T^3 using the quasi-particle model in comparison with those by lattice QCD with stout action (BORSANYI et al., 2012; BORSANYI et al., 2014) at zero chemical potential.

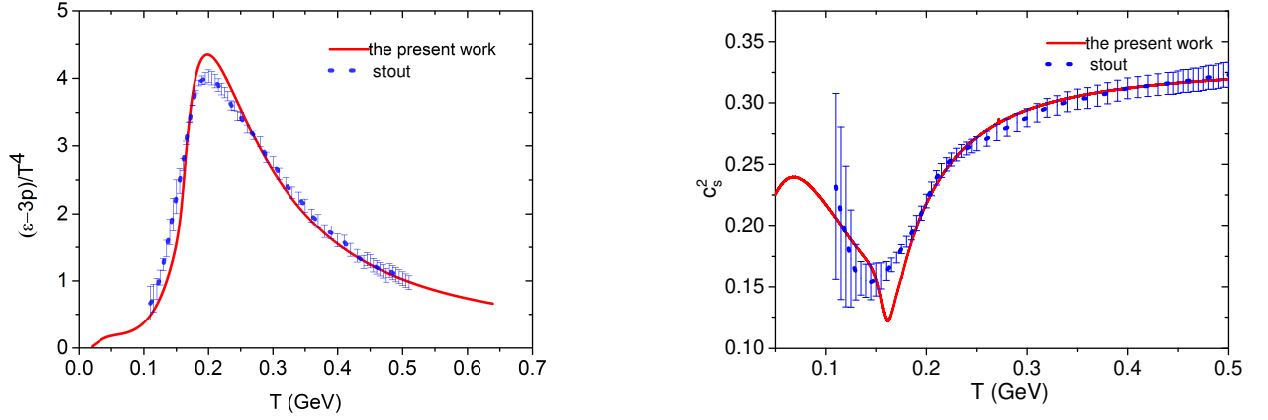


Figure 20 – (Color online) The calculated results in comparison with the lattice QCD data (BORSANYI et al., 2012; BORSANYI et al., 2014). (a) the trace anomaly as a function of temperature, (b) the speed of sound as a function of temperature.

SANYI et al., 2012). As discussed above, our choice of Λ_T ensures appropriate behavior at low temperature, while the results are insensitive to the specific value of Λ_T at high temperature, due to Eq.(4.96). The calculated results of Δp agree well with the lattice data.

Now, we compare the results of the proposed model with those obtained by first order phase transition. For $\mu_B < \mu_c$, the present interpolation scheme gives a smooth crossover in the transition region, which is distinct from that of a first order phase transition. When one goes beyond the critical chemical potential, the transition gradually approaches of that of a first order which involves discontinuous changes of thermodynamic quantities related to the first order derivative of the Gibbs thermodynamical potential. At high temperature, the quasi-particle model guarantees that the results approach those of lattice QCD calculations. This is shown in Figure 22, where we present the pressure, energy density and entropy density as functions of temperature, and pressure as functions of energy density for different chemical potentials. It is found that for $\mu_B = 0$, all physical quantities vary

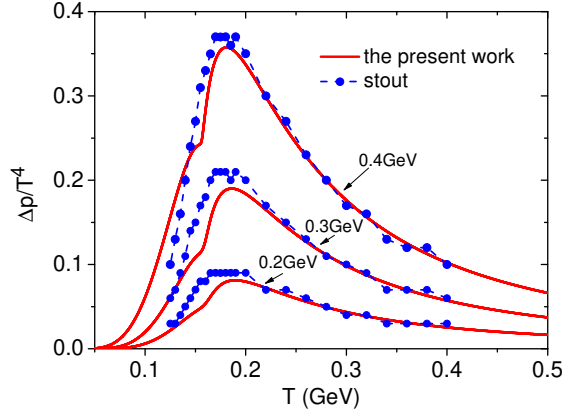


Figure 21 – The calculated $\Delta p/T^4$ as a function of temperature for different chemical potentials, in comparison with the lattice QCD results by stout action (BORSANYI et al., 2012).

smoothly during the transition for the present interpolation scheme, while the results for the first order phase transition EoS show sudden jumps in energy density and entropy density. Similar results are obtained for $\mu_B = 0.2$ GeV, which is also below the critical chemical potential μ_c . On the other hand, for $\mu_B = 0.5$ GeV, the obtained results are almost identical to those of the first order phase transition, as expected.

5.2 MULTIPLICITY FLUCTUATIONS

At the CEP, it is a second-order phase transition between hadronic and QGP phase. This leads to the divergent fluctuations of the conserved number near the CEP. Therefore, the nonmonotonic behavior may be observed in the energy dependence of the fluctuations of the conserved number (KARSCH; REDLICH, 2011; ASAKAWA; HEINZ; MULLER, 2000; KOCH, 2010).

We assume that the produced hadrons on the freeze-out hypersurface inherit the thermal properties of the hot and dense matter performed in the collisions. Therefore, one can investigate the properties of the QGP through the produced hadrons at the final stage. Phenomenologically, there is a relation between baryon chemical potential and collision energy which can be determined by fitted with experimental data (BRAUN-MUNZINGER et al., 2011). With such a relation, the calculated results from HRG model can coincide with the experimental data (FU, 2009; FU, 2012; FU, 2013; FU, 2017; KARSCH; REDLICH, 2011; GARG et al., 2013). Because the higher moments of fluctuations are more sensitive to the CEP, one expects that there is a significant difference between the results of the fluctuations from the HRG model and the ones by experiments.

On the other hand, a hydrodynamic study can be involved for the multiplicity fluctuations. The hydrodynamic evolution can be expressed in terms of the Smoothed Particle Hydrodynamics (SPH) algorithm. In our model, every elementary degree of freedom of the system, namely, a small fluid element denoted by an SPH particle, is treated as a quantum GCE. In comparison with statistical model approaches, system expansion is encoded in terms of freeze-out surface. Moreover, event-by-event (EbE) fluctuating of ICs are explicitly considered and shown to play a significant role in the resulting

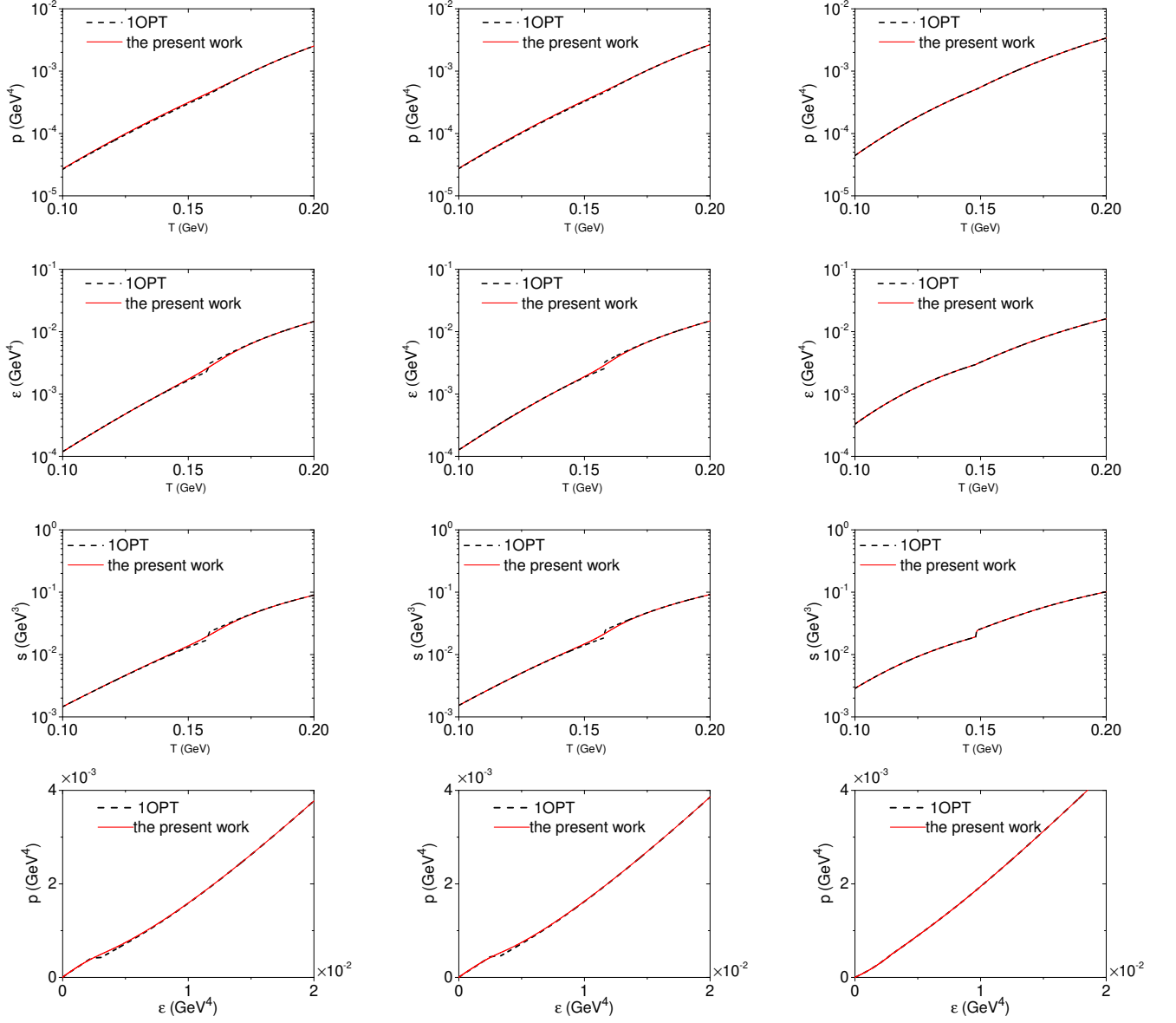


Figure 22 – (Color online) The calculated pressure, energy density and entropy density as functions of temperature, and pressure as function of energy density for different chemical potentials. The results of the present interpolation scheme is compared to those of a first order phase transition (1OPT). From left to right, the three columns are for $\mu_B = 0, 0.2$ and 0.5 GeV respectively.

quantities.

5.2.1 Thermodynamical fluctuations and resonance decay

In what follows, we adopt the notations used in Refs.(BEGUN et al., 2006; FU, 2013). For static ideal gas, the GCE average values and fluctuations of the occupation density read:

$$\langle n_{p,i} \rangle = \frac{1}{\exp \left[\sqrt{p^2 + m_i^2} - \mu_i/T \right] - \gamma_i}, \quad (5.23)$$

$$\langle \Delta n_{p,i}^2 \rangle \equiv \langle (n_{p,i} - \langle n_{p,i} \rangle)^2 \rangle \equiv v_{p,i}^2 = \langle n_{p,i} \rangle (1 + \gamma_i \langle n_{p,i} \rangle), \quad (5.24)$$

where the subscript i indicates particle species, T is the temperature, m_i, μ_i are the particle mass and chemical potential respectively, i corresponds to Bose (+1), Fermi (-1) or Boltzmann (0) statistics. The system is at chemical equilibrium, so that

$$\mu_i = Q_i \mu_Q + B_i \mu_B + S_i \mu_S. \quad (5.25)$$

Owing to the fact that the fluctuations are independent for different particle species as well as different momentum space,

$$\langle \Delta n_{p,i} \Delta n_{k,j} \rangle = v_{p,i}^2 \delta_{ij} \delta_{pk}, \quad (5.26)$$

the average number of particles of species i is given by

$$\langle N_i \rangle = \sum_p \langle n_{p,i} \rangle = \frac{g_i V}{2\pi^2} \int_0^\infty p^2 dp \langle n_{p,i} \rangle. \quad (5.27)$$

The variance σ^2 for species i reads

$$\langle (\Delta N_i)^2 \rangle = T \left(\frac{\partial N_i}{\partial \mu} \right)_T, \quad (5.28)$$

and similarly, the covariance reads

$$\langle \Delta N_i \Delta N_j \rangle = \sum_{p,k} \langle \Delta n_{p,i} \Delta n_{k,j} \rangle. \quad (5.29)$$

Besides, statistical moments of multiplicity distributions like skewness $S \propto \langle \Delta N^3 \rangle$ and kurtosis $\kappa \propto \langle \Delta N^4 \rangle$ are also of particular importance. These quantities are sensitive enough to the correlation length. Furthermore, products $\kappa \sigma^2$ and $S \sigma$ are directly related to the ratios of susceptibilities. While such higher moments can be evaluated in a similar fashion, though the calculations are somewhat

tedious. Similarly, the higher order cumulants of primary particle i reads (LANDAU; LIFSHITZ, 1980),

$$\langle (\Delta N_i)^3 \rangle = T^2 \left(\frac{\partial^2 N_i}{\partial^2 \mu} \right)_T, \quad (5.30)$$

$$\langle (\Delta N_i)^4 \rangle - 3 \langle (\Delta N_i)^2 \rangle = T^3 \left(\frac{\partial^3 N_i}{\partial^3 \mu} \right)_T. \quad (5.31)$$

Considering the vanishing covariance between different particle species and with the aid of Eq. (5.27), it is easy to be found

$$\begin{aligned} \langle \Delta N_i \Delta N_j \Delta N_k \rangle &= \sum_p \langle (\Delta n_{p,i})^3 \rangle \\ &= \sum_p 2 \langle n_{p,i} \rangle (1 + \gamma_i \langle n_{p,i} \rangle)^2 - \langle n_{p,i} \rangle (1 + \gamma_i \langle n_{p,i} \rangle) \\ &= \sum_p \langle n_{p,i} \rangle (1 + 3\gamma_i \langle n_{p,i} \rangle + 2\gamma_i^2 \langle n_{p,i} \rangle^2), \end{aligned} \quad (5.32)$$

$$\begin{aligned} \langle \Delta N_i \Delta N_j \Delta N_k \Delta N_l \rangle &= \langle (\Delta N_i \Delta N_j) (\Delta N_k \Delta N_l) \rangle - \langle (\Delta N_i \Delta N_k) (\Delta N_j \Delta N_l) \rangle - \langle (\Delta N_i \Delta N_l) (\Delta N_j \Delta N_k) \rangle \\ &= \sum_p \langle (\Delta n_{p,i})^4 \rangle - 3 \langle (\Delta n_{p,i})^2 \rangle \\ &= \sum_p 6 \langle n_{p,i} \rangle (1 + \gamma_i \langle n_{p,i} \rangle)^3 - 6 \langle n_{p,i} \rangle (1 + \gamma_i \langle n_{p,i} \rangle)^2 + \langle n_{p,i} \rangle (1 + \gamma_i \langle n_{p,i} \rangle) \\ &= \sum_p \langle n_{p,i} \rangle (1 + 7\gamma_i \langle n_{p,i} \rangle + 12\gamma_i^2 \langle n_{p,i} \rangle^2 + 6\gamma_i^3 \langle n_{p,i} \rangle^3). \end{aligned} \quad (5.33)$$

In order to consider the effect of conserved charges, one may follow Ref. (BEGUN et al., 2006; BECATTINI et al., 2005) to insert some additional factor into the phase space integral of the grand partition function. To be specific,

$$\prod_i \frac{1}{2\pi} \int_0^{2\pi} d\phi_i \exp[-iQ_i \phi_i], \quad (5.34)$$

where Q_i stands for the total charge of type i , for instance, $Q_i = Q, B, S$, etc. The integral can be evaluated by further making use of the saddle point expansion technique, and therefore approximated but analytic results can be obtained. The resulting partition function is usually referred to as “canonical” in the literature. We note that for the above prescription, the conservation is demanded for specific net charges but not for individual particle species. Otherwise, the variance of any particle species shall vanishing by definition. It was shown that, depending on specific model parameters, the effect of conserved charges can be substantial.

The resonance decay can be considered by introducing the following generating function (BEGUN

et al., 2006)

$$G \equiv \prod_R \left(\sum_r b_r^R \prod_i \lambda_i^{n_{i,r}^R} \right)^{N_R}, \quad (5.35)$$

where for a given resonance R , a specific decay channel is denoted by r with the branching ratio b_r^R . Also, $n_{i,r}^R$ indicates the number of particles i obtained through the decay channel r of the resonance in question. Here λ_i is the “external source” which will be taken to be 1 by the end of the calculations. Since in Eq.(5.35), r is a dummy variable, the resulting particle number of a specific particle species i can be mostly substituted by operation $\lambda_i \frac{\partial}{\partial \lambda_i}$. As a result

$$\overline{N_i} \equiv \sum_R \langle N_i \rangle = \lambda_i \frac{\partial}{\partial \lambda_i} G = \sum_R N_R \sum_r b_r^R n_{i,r}^R \equiv \sum_R N_R \langle n_i \rangle_R, \quad (5.36)$$

$$\begin{aligned} \overline{N_i N_j} &\equiv \sum_R \langle N_i N_j \rangle_R + \sum_{R \neq R'} \langle N_i N_j \rangle_{R,R'} = \lambda_i \frac{\partial}{\partial \lambda_i} \left(\lambda_j \frac{\partial}{\partial \lambda_j} G \right) \\ &= \sum_R [N_R(N_R - 1) \langle n_i \rangle_R \langle n_j \rangle_R + N_R \langle n_i n_j \rangle_R] + \sum_{R \neq R'} N_R N_{R'} \langle n_i \rangle_R \langle n_j \rangle_{R'}. \end{aligned} \quad (5.37)$$

Here we have used an overline “—” to indicate the resulting ensemble average value after considering all possible decay modes. Thus, $\langle \dots \rangle_R$ means the average over different decay modes regarding a given resonance. For instance, $\langle n_i n_j \rangle_R = \sum_r b_r^R n_{i,r}^R n_{j,r}^R$. Therefore, the overlined value is obtained by summing up all the contributions from different resonances. The derivations for other relevant higher moments used in this thesis can be found in the Appendix E.

According to the definition, the third and fourth order moments of multiplicity distributions via resonance decays can be found as follows,

$$\begin{aligned} \overline{N_i N_j N_k} &\equiv \sum_R \langle N_i N_j N_k \rangle_R + \sum_{R \neq R'} \langle N_i N_j N_k \rangle_{R,R'} + \sum_{R \neq R' \neq R''} \langle N_i N_j N_k \rangle_{R,R',R''} \\ &= \lambda_i \frac{\partial}{\partial \lambda_i} \left(\lambda_j \frac{\partial}{\partial \lambda_j} \left(\lambda_k \frac{\partial}{\partial \lambda_k} G \right) \right) \\ &= \sum_R [N_R \langle n_i n_j n_k \rangle_R + N_R(N_R - 1) (\langle n_i n_j \rangle_R \langle n_k \rangle_R + \langle n_i n_k \rangle_R \langle n_j \rangle_R + \langle n_k n_j \rangle_R \langle n_i \rangle_R) \\ &\quad + N_R(N_R - 1)(N_R - 2) \langle n_i \rangle_R \langle n_j \rangle_R \langle n_k \rangle_R] \\ &\quad + \sum_{R \neq R'} N_{R'} (N_R \langle n_i n_j \rangle_R + N_R(N_R - 1) \langle n_i \rangle_R \langle n_j \rangle_R) \langle n_k \rangle_{R'} \\ &\quad + \sum_{R \neq R'} N_{R'} (N_R \langle n_i n_k \rangle_R + N_R(N_R - 1) \langle n_i \rangle_R \langle n_k \rangle_R) \langle n_j \rangle_{R'} \\ &\quad + \sum_{R \neq R'} N_{R'} (N_R \langle n_j n_k \rangle_R + N_R(N_R - 1) \langle n_j \rangle_R \langle n_k \rangle_R) \langle n_i \rangle_{R'} \\ &\quad + \sum_{R \neq R' \neq R''} N_R N_{R'} N_{R''} \langle n_i \rangle_R \langle n_j \rangle_{R'} \langle n_k \rangle_{R''}, \end{aligned} \quad (5.38)$$

$$\begin{aligned}
\overline{N_i N_j N_k N_l} &\equiv \sum_R \langle N_i N_j N_k N_l \rangle_R + \sum_{R \neq R'} \langle N_i N_j N_k N_l \rangle_{R, R'} + \sum_{R \neq R' \neq R''} \langle N_i N_j N_k N_l \rangle_{R, R', R''} \\
&+ \sum_{R \neq R' \neq R'' \neq R'''} \langle N_i N_j N_k N_l \rangle_{R, R', R'', R'''} = \lambda_i \frac{\partial}{\partial \lambda_i} \left(\lambda_j \frac{\partial}{\partial \lambda_j} \left(\lambda_k \frac{\partial}{\partial \lambda_k} \left(\lambda_l \frac{\partial}{\partial \lambda_l} G \right) \right) \right) \\
&= \sum_R [N_R \langle n_i n_j n_k n_l \rangle_R + N_R(N_R - 1) (\langle n_i n_j n_k \rangle_R \langle n_l \rangle_R) \\
&+ N_R(N_R - 1) [(\langle n_i n_j n_l \rangle_R \langle n_k \rangle_R + \langle n_i n_j \rangle_R \langle n_l n_k \rangle_R) + (\langle n_i n_k n_l \rangle_R \langle n_j \rangle_R + \langle n_i n_k \rangle_R \langle n_l n_j \rangle_R)] \\
&+ N_R(N_R - 1) (\langle n_j n_k n_l \rangle_R \langle n_i \rangle_R + \langle n_j n_k \rangle_R \langle n_l n_i \rangle_R) \\
&+ N_R(N_R - 1)(N_R - 2) (\langle n_i n_j \rangle_R \langle n_k \rangle_R \langle n_l \rangle_R + \langle n_i n_k \rangle_R \langle n_j \rangle_R \langle n_l \rangle_R + \langle n_j n_k \rangle_R \langle n_i \rangle_R \langle n_l \rangle_R) \\
&+ N_R(N_R - 1)(N_R - 2) (\langle n_i n_l \rangle_R \langle n_j \rangle_R \langle n_k \rangle_R + \langle n_j n_l \rangle_R \langle n_i \rangle_R \langle n_k \rangle_R + \langle n_k n_l \rangle_R \langle n_i \rangle_R \langle n_j \rangle_R) \\
&+ N_R(N_R - 1)(N_R - 2)(N_R - 3) \langle n_i \rangle_R \langle n_j \rangle_R \langle n_k \rangle_R \langle n_l \rangle_R] \\
&+ \sum_{R \neq R'} N_{R'} [N_R \langle n_i n_j n_k \rangle_R + N_R(N_R - 1) (\langle n_i n_j \rangle_R \langle n_k \rangle_R + \langle n_i n_k \rangle_R \langle n_j \rangle_R + \langle n_j n_k \rangle_R \langle n_i \rangle_R) \\
&+ N_R(N_R - 1)(N_R - 2) \langle n_i \rangle_R \langle n_j \rangle_R \langle n_k \rangle_R] \langle n_l \rangle_{R'} \\
&+ \sum_{R \neq R'} N_{R'} [N_R \langle n_i n_j n_l \rangle_R + N_R(N_R - 1) (\langle n_i n_j \rangle_R \langle n_l \rangle_R + \langle n_i n_l \rangle_R \langle n_j \rangle_R + \langle n_j n_l \rangle_R \langle n_i \rangle_R) \\
&+ N_R(N_R - 1)(N_R - 2) \langle n_i \rangle_R \langle n_j \rangle_R \langle n_l \rangle_R] \langle n_k \rangle_{R'} \\
&+ \sum_{R \neq R'} N_{R'} [N_R \langle n_i n_k n_l \rangle_R + N_R(N_R - 1) (\langle n_i n_k \rangle_R \langle n_l \rangle_R + \langle n_i n_l \rangle_R \langle n_k \rangle_R + \langle n_k n_l \rangle_R \langle n_i \rangle_R) \\
&+ N_R(N_R - 1)(N_R - 2) \langle n_i \rangle_R \langle n_k \rangle_R \langle n_l \rangle_R] \langle n_j \rangle_{R'} \\
&+ \sum_{R \neq R'} N_{R'} [N_R \langle n_j n_k n_l \rangle_R + N_R(N_R - 1) (\langle n_j n_k \rangle_R \langle n_l \rangle_R + \langle n_j n_l \rangle_R \langle n_k \rangle_R + \langle n_k n_l \rangle_R \langle n_j \rangle_R) \\
&+ N_R(N_R - 1)(N_R - 2) \langle n_j \rangle_R \langle n_k \rangle_R \langle n_l \rangle_R] \langle n_i \rangle_{R'} \\
&+ \sum_{R \neq R'} [N_R \langle n_i n_j \rangle_R + N_R(N_R - 1) \langle n_i \rangle_R \langle n_j \rangle_R] [N_{R'} \langle n_k n_l \rangle_{R'} + N_{R'}(N_{R'} - 1) \langle n_k \rangle_{R'} \langle n_l \rangle_{R'}] \\
&+ \sum_{R \neq R'} [N_R \langle n_i n_k \rangle_R + N_R(N_R - 1) \langle n_i \rangle_R \langle n_k \rangle_R] [N_{R'} \langle n_j n_l \rangle_{R'} + N_{R'}(N_{R'} - 1) \langle n_j \rangle_{R'} \langle n_l \rangle_{R'}] \\
&+ \sum_{R \neq R'} [N_R \langle n_j n_k \rangle_R + N_R(N_R - 1) \langle n_j \rangle_R \langle n_k \rangle_R] [N_{R'} \langle n_i n_l \rangle_{R'} + N_{R'}(N_{R'} - 1) \langle n_i \rangle_{R'} \langle n_l \rangle_{R'}] \\
&+ \sum_{R \neq R' \neq R''} N_{R'} N_{R''} [N_R \langle n_i n_j \rangle_R + N_R(N_R - 1) \langle n_i \rangle_R \langle n_j \rangle_R] \langle n_k \rangle_{R'} \langle n_l \rangle_{R''} \\
&+ \sum_{R \neq R' \neq R''} N_{R'} N_{R''} [N_R \langle n_i n_k \rangle_R + N_R(N_R - 1) \langle n_i \rangle_R \langle n_k \rangle_R] \langle n_j \rangle_{R'} \langle n_l \rangle_{R''} \\
&+ \sum_{R \neq R' \neq R''} N_{R'} N_{R''} [N_R \langle n_i n_l \rangle_R + N_R(N_R - 1) \langle n_i \rangle_R \langle n_l \rangle_R] \langle n_j \rangle_{R'} \langle n_k \rangle_{R''} \\
&+ \sum_{R \neq R' \neq R''} N_{R'} N_{R''} [N_R \langle n_j n_k \rangle_R + N_R(N_R - 1) \langle n_j \rangle_R \langle n_k \rangle_R] \langle n_i \rangle_{R'} \langle n_l \rangle_{R''} \\
&+ \sum_{R \neq R' \neq R''} N_{R'} N_{R''} [N_R \langle n_j n_l \rangle_R + N_R(N_R - 1) \langle n_j \rangle_R \langle n_l \rangle_R] \langle n_i \rangle_{R'} \langle n_k \rangle_{R''} \\
&+ \sum_{R \neq R' \neq R''} N_{R'} N_{R''} [N_R \langle n_k n_l \rangle_R + N_R(N_R - 1) \langle n_k \rangle_R \langle n_l \rangle_R] \langle n_i \rangle_{R'} \langle n_j \rangle_{R''} \\
&+ \sum_{R \neq R' \neq R'' \neq R'''} N_R N_{R'} N_{R''} N_{R'''} \langle n_i \rangle_R \langle n_j \rangle_{R'} \langle n_k \rangle_{R''} \langle n_l \rangle_{R'''} .
\end{aligned} \tag{5.39}$$

When one evaluates the variance and covariance which involves more than one particle, it is noted that the contribution may come from a variety of possible decay processes. For instance, two decayed particle might originate from the same resonance, two distinct resonances of the same type, and two different resonance. But all these possibilities are automatically taken care of as referred from the last line of Eq.(5.37).

Subsequently, one proceeds to evaluate experimental observables. One quantity frequently cited by other authors is the scaled variance. For a given initial resonance distribution, it is found to be

$$\omega_R^{i*} \equiv \frac{\langle N_i^2 \rangle_R - \langle N_i \rangle_R^2}{\langle N_i \rangle_R} = \frac{\langle n_i^2 \rangle_R - \langle n_i \rangle_R^2}{\langle n_i \rangle_R} = \frac{\sum_r b_r^R (n_{i,r}^R)^2 - (\sum_r b_r^R n_{i,r}^R)^2}{\sum_r b_r^R n_{i,r}^R}. \quad (5.40)$$

The resulting expression is obtained as one takes into consideration all different resonances, which reads

$$\overline{\omega_R^{i*}} = \frac{\overline{N_i^2} - \overline{N_i}^2}{\overline{N_i}} = \frac{\sum_R N_R \langle n_i^2 \rangle_R - \sum_R N_R \langle n_i \rangle_R^2}{\sum_R N_R \langle n_i \rangle_R}. \quad (5.41)$$

In practice, resonance yields N_R also fluctuate, and the resultant scaled variance reads

$$\omega^{i*} \equiv \frac{\langle \overline{N_i^2} \rangle_T - \langle \overline{N_i} \rangle_T^2}{\langle \overline{N_i} \rangle_T} = \overline{\omega_R^{i*}} + \sum_R \langle n_i \rangle_R \omega_R, \quad (5.42)$$

where

$$\omega_R \equiv \frac{\langle N_R^2 \rangle_T - \langle N_R \rangle_T^2}{\langle N_R \rangle_T} \quad (5.43)$$

is the scaled variance of the resonance.

If the system is static and in thermal and chemical equilibrium, the thermal fluctuations used in the above expressions are those discussed above in Eqs.(5.28) and (5.29).

To include the effect of particle decay, according to (BEGUN et al., 2006), it satisfies

$$\langle N_i \rangle = \langle N_i^* \rangle + \sum_R \langle N_R \rangle \langle n_i \rangle_R, \quad (5.44)$$

and

$$\langle \Delta N_i \Delta N_j \rangle = \langle \Delta N_i^* \Delta N_j^* \rangle + \sum_R [\langle N_R^2 \rangle \langle n_i \rangle_R \langle n_j \rangle_R + \langle N_R \rangle \langle \Delta n_i \Delta n_j \rangle_R], \quad (5.45)$$

where

$$\begin{aligned} \langle n_i \rangle_R &= \sum_r b_r^R n_{i,r}^R, \\ \langle \Delta n_i \Delta n_j \rangle_R &= \sum_r b_r^R \Delta n_{i,r}^R \Delta n_{j,r}^R = \langle n_{i,r}^R n_{j,r}^R \rangle_R - \langle n_i \rangle_R \langle n_j \rangle_R, \end{aligned} \quad (5.46)$$

with $\sum_r b_r^R = 1$.

By definition, the third and fourth order moments of multiplicity distribution can be written as

$$\begin{aligned}\langle \Delta N_i \Delta N_j \Delta N_k \rangle &= \langle N_i N_j N_k \rangle - \langle N_i N_j \rangle \langle N_k \rangle - \langle N_i N_k \rangle \langle N_j \rangle \\ &\quad - \langle N_k N_j \rangle \langle N_i \rangle + 2 \langle N_i \rangle \langle N_j \rangle \langle N_k \rangle,\end{aligned}\quad (5.47)$$

$$\begin{aligned}\langle \Delta N_i \Delta N_j \Delta N_k \Delta N_l \rangle &= \langle N_i N_j N_k N_l \rangle - \langle N_i N_j N_k \rangle \langle N_l \rangle - \langle N_i N_k N_l \rangle \langle N_j \rangle \\ &\quad - \langle N_i N_j N_l \rangle \langle N_k \rangle - \langle N_j N_k N_l \rangle \langle N_i \rangle \\ &\quad + \langle N_i N_j \rangle \langle N_l \rangle \langle N_k \rangle + \langle N_i N_k \rangle \langle N_l \rangle \langle N_j \rangle + \langle N_i N_l \rangle \langle N_j \rangle \langle N_k \rangle \\ &\quad + \langle N_j N_k \rangle \langle N_i \rangle \langle N_l \rangle + \langle N_j N_l \rangle \langle N_i \rangle \langle N_k \rangle + \langle N_k N_l \rangle \langle N_i \rangle \langle N_j \rangle \\ &\quad - 3 \langle N_i \rangle \langle N_j \rangle \langle N_k \rangle \langle N_l \rangle,\end{aligned}\quad (5.48)$$

where the subscripts i, j, k , and l represent for the particle species.

Therefore, the three and four particle correlations after resonance decays can be obtained by using the expressions listed in Appendix E,

$$\begin{aligned}\langle \Delta N_i \Delta N_j \Delta N_k \rangle &= \langle \Delta N_i^* \Delta N_j^* \Delta N_k^* \rangle + \sum_R \langle N_R \rangle \langle \Delta n_i \Delta n_j \Delta n_k \rangle_R \\ &\quad + \langle (\Delta N_R)^2 \rangle (\langle \Delta n_i \Delta n_j \rangle_R \langle n_k \rangle_R + \langle \Delta n_i \Delta n_k \rangle_R \langle n_j \rangle_R + \langle \Delta n_k \Delta n_j \rangle_R \langle n_i \rangle_R) \\ &\quad + \langle (\Delta N_R)^3 \rangle \langle n_i \rangle_R \langle n_j \rangle_R \langle n_k \rangle_R,\end{aligned}\quad (5.49)$$

and

$$\begin{aligned}C_4(N_i, N_j, N_k, N_l) &= C_4(N_i^*, N_j^*, N_k^*, N_l^*) + \sum_R \langle N_R \rangle \langle \Delta n_i \Delta n_j \Delta n_k \Delta n_l \rangle_R \\ &\quad - \langle N_R \rangle (\langle \Delta n_i \Delta n_j \rangle_R \langle \Delta n_k \Delta n_l \rangle_R + \langle \Delta n_i \Delta n_k \rangle_R \langle \Delta n_j \Delta n_l \rangle_R) \\ &\quad - \langle N_R \rangle (\langle \Delta n_i \Delta n_l \rangle_R \langle \Delta n_k \Delta n_j \rangle_R) \\ &\quad + \langle (\Delta N_R)^2 \rangle (\langle \Delta n_i \Delta n_j \Delta n_k \rangle_R \langle n_l \rangle_R + \langle \Delta n_i \Delta n_j \Delta n_l \rangle_R \langle n_k \rangle_R) \\ &\quad + \langle (\Delta N_R)^2 \rangle (\langle \Delta n_i \Delta n_k \Delta n_l \rangle_R \langle n_j \rangle_R + \langle \Delta n_k \Delta n_j \Delta n_l \rangle_R \langle n_i \rangle_R) \\ &\quad + \langle (\Delta N_R)^2 \rangle (\langle \Delta n_i \Delta n_j \rangle_R \langle \Delta n_k \Delta n_l \rangle_R + \langle \Delta n_i \Delta n_k \rangle_R \langle \Delta n_j \Delta n_l \rangle_R) \\ &\quad + \langle (\Delta N_R)^2 \rangle (\langle \Delta n_i \Delta n_l \rangle_R \langle \Delta n_k \Delta n_j \rangle_R) \\ &\quad + \langle (\Delta N_R)^3 \rangle (\langle \Delta n_i \Delta n_j \rangle_R \langle n_k \rangle_R \langle n_l \rangle_R + \langle \Delta n_i \Delta n_k \rangle_R \langle n_j \rangle_R \langle n_l \rangle_R) \\ &\quad + \langle (\Delta N_R)^3 \rangle (\langle \Delta n_i \Delta n_l \rangle_R \langle n_j \rangle_R \langle n_k \rangle_R + \langle \Delta n_j \Delta n_k \rangle_R \langle n_i \rangle_R \langle n_l \rangle_R) \\ &\quad + \langle (\Delta N_R)^3 \rangle (\langle \Delta n_j \Delta n_l \rangle_R \langle n_i \rangle_R \langle n_k \rangle_R + \langle \Delta n_k \Delta n_l \rangle_R \langle n_i \rangle_R \langle n_j \rangle_R) \\ &\quad + \langle (\Delta N_R)^4 \rangle - 3 \langle (\Delta N_R)^2 \rangle^2 \langle n_i \rangle_R \langle n_j \rangle_R \langle n_k \rangle_R \langle n_l \rangle_R,\end{aligned}\quad (5.50)$$

where the term C_4 on both sides of the equality is defined to be

$$\begin{aligned}C_4(N_i, N_j, N_k, N_l) &= \langle \Delta N_i \Delta N_j \Delta N_k \Delta N_l \rangle - \langle \Delta N_i \Delta N_j \rangle \langle \Delta N_k \Delta N_l \rangle \\ &\quad - \langle \Delta N_i \Delta N_l \rangle \langle \Delta N_j \Delta N_k \rangle - \langle \Delta N_i \Delta N_k \rangle \langle \Delta N_j \Delta N_l \rangle.\end{aligned}\quad (5.51)$$

5.2.2 A hydrodynamic approach

In this section, we elaborate an approach which incorporates the effect of hydrodynamical evolution, as well as EbE fluctuating IC, on multiplicity fluctuations. To take into consideration the temporal expansion into our framework, we employ SPheRIO (HAMA; KODAMA; SOCOLOWSKI JR., 2005), a hydrodynamic code based on SPH algorithm. The latter parameterize the system in terms of discrete Lagrangian coordinates, known as SPH particles. In the entropy-based scheme, the SPH particles are attached to particular conserved quantities, associated with small fluid elements. In term of the SPH particle degree of freedom, the EoM can be derived by using the variational principle. We neglect in the present work any dissipative effects and assume Cooper-Frye sudden freeze-out take place at constant temperatures. The latter, when transformed into the local rest frame, provides the baseline to evaluate the thermal fluctuations at the moment of hadronization. We do not introduce any additional free parameter into the model as the existing ones have been determined as to appropriately reproduce the experimental data regarding the particle spectra (QIAN et al., 2007; QIAN et al., 2014; ANDRADE et al., 2008; QIAN et al., 2013; DUDEK et al., 2014; QIAN et al., 2014; CASTILHO et al., 2017; CASTILHO et al., 2018; CASTILHO; QIAN, 2018).

On the freeze-out surface, every small fluid element, denoted by an SPH particle, is treated as a GCE. One might proceed further to take into account conserved charges, as discussed in the previous section. Unfortunately, the latter is highly nontrivial, owing to precisely the same difficulties to explicitly incorporate charge conservation at hadronization in most hydrodynamical models. To be specific, this is because the momentum space integral involving a conserved charge is then to be carried out on all individual freeze-out surface elements resolved numerically. In this context, one does not possess the access to an analytic form of the partition function, whose derivatives give rise to the multiplicity fluctuations. Therefore, as a first approximation, we will ignore the condition of charge conservation in our present approach.

In the case of system created through hydrodynamic expansion, the system is at local equilibrium, but the volume in Eq.(5.27) is substituted by an integral on a time-like 3-surface. Also the particle number flux depends on the momentum, the part of integral in momentum space possesses a different form. To be specific, the average number of particles of species i is replaced by the covariant form

$$\frac{Ed^3N}{dp^3} = \frac{d^3N_i}{2\pi p_T dp_T dy} = \int_{\sigma} d\sigma_{\mu} p^{\mu} \langle n_i(u, p, x) \rangle, \quad (5.52)$$

where one makes use of the dynamical variables such as rapidity y and transverse momentum p_T . As mentioned before, the volume is replaced by an integral carried out on σ_{μ} , which is a 3-surface element. For the latter, if the time-component is only non-vanishing one, contracting with p^{μ} brings it back to Eq.(5), as

$$\langle n_i(u, p, x) \rangle = \frac{1}{\exp[(u(x) \cdot p - \mu_i)/T] - \gamma_i}, \quad (5.53)$$

is the local occupation density in the co-moving frame. When expressed in terms of SPH degree of

freedom (HAMA; KODAMA; SOCOLOWSKI JR., 2005), one finds,

$$E \frac{d^3 N_i}{dp^3} = \sum_j \frac{\nu_j n_{j\mu} p^\mu}{s_j |n_{j\rho} u_j^\rho|} \theta(n_{j\delta} p^\delta) \langle n_i(n_{j\nu} p^\nu, x) \rangle, \quad (5.54)$$

where the sum is carried out for SPH particles. Therefore,

$$\langle N_i \rangle = \int p_\perp dp_\perp dy d\phi \sum_j \frac{\nu_j n_{j\mu} p^\mu}{s_j |n_{j\rho} u_j^\rho|} \theta(n_{j\delta} p^\delta) \langle n_i(n_{j\nu} p^\nu, x) \rangle, \quad (5.55)$$

where different fluid elements are viewed as statistical independent, and treated as individual GCE. The variance can still be calculated via (LANDAU; LIFSHITZ, 1980)

$$\langle \Delta n_i^2 \rangle = T \frac{\partial \langle n_i \rangle}{\partial \mu_i}. \quad (5.56)$$

Therefore, one finds

$$\langle (\Delta N_i)^2 \rangle = \int p_\perp dp_\perp dy d\phi \sum_j \frac{\nu_j n_{j\mu} p^\mu}{s_j |n_{j\rho} u_j^\rho|} \theta(n_{j\delta} p^\delta) v_i^2(n_{j\nu} p^\nu, x). \quad (5.57)$$

Turning to the representation of SPH method, the previous expressions can be rewritten as

$$\langle \Delta N_i \Delta N_j \Delta N_k \rangle = \int p_\perp dp_\perp dy d\phi \sum_j \frac{\nu_j n_{j\mu} p^\mu}{s_j |n_{j\rho} u_j^\rho|} \theta(u_{j\delta} p^\delta) v_i^3(u_{j\nu} p^\nu, x), \quad (5.58)$$

$$\langle \Delta N_i \Delta N_j \Delta N_k \Delta N_l \rangle = \int p_\perp dp_\perp dy d\phi \sum_j \frac{\nu_j n_{j\mu} p^\mu}{s_j |n_{j\rho} u_j^\rho|} \theta(u_{j\delta} p^\delta) v_i^4(u_{j\nu} p^\nu, x), \quad (5.59)$$

where

$$\begin{aligned} v_i^3(u_{j\nu} p^\nu, x) &= n_i(u_{j\nu} p^\nu, x) (1 + 3\gamma_i n_i(u_{j\nu} p^\nu, x) + 2\gamma_i^2 n_i^2(u_{j\nu} p^\nu, x)), \\ v_i^4(u_{j\nu} p^\nu, x) &= n_i(u_{j\nu} p^\nu, x) (1 + 7\gamma_i n_i(u_{j\nu} p^\nu, x) + 12\gamma_i^2 n_i^2(u_{j\nu} p^\nu, x) + 6\gamma_i^3 n_i^3(u_{j\nu} p^\nu, x)) \end{aligned} \quad (5.60)$$

As one further considers EbE fluctuating ICs, the above quantities are further modified to include the fluctuations between different events. For a total of n events, $E \equiv \{E_1, E_2, \dots, E_n\}$, one has

$$\begin{aligned} \langle (\Delta N_i)(\Delta N_j) \rangle_E &= \frac{1}{n} [\langle (\Delta N_i)(\Delta N_j) \rangle_{E_1} + \langle (\Delta N_i)(\Delta N_j) \rangle_{E_2} + \dots + \langle (\Delta N_i)(\Delta N_j) \rangle_{E_n}] \\ &+ \frac{1}{n} [(\langle N_i \rangle_{E_1} - \langle N_i \rangle_E)(\langle N_j \rangle_{E_1} - \langle N_j \rangle_E) + (\langle N_i \rangle_{E_2} - \langle N_i \rangle_E)(\langle N_j \rangle_{E_2} - \langle N_j \rangle_E) \\ &+ \dots + (\langle N_i \rangle_{E_n} - \langle N_i \rangle_E)(\langle N_j \rangle_{E_n} - \langle N_j \rangle_E)], \end{aligned} \quad (5.61)$$

where $\langle \dots \rangle_{E_k}$ indicates the ensemble average discussed above for a given event k , where $\langle \dots \rangle_E$ stands for the event average among different events.

5.2.3 Numerical Results

Firstly, we show the p_T and rapidity distribution from the hydrodynamic simulation code SPheRIO with or without the resonance decay. To perform the resonance decay mode of the hadrons, we wrote a code which is based on THERMUS (WHEATON; CLEYMANS, 2009; WHEATON; CLEYMANS; HAUER, 2011). From Figure 23 and Figure 24, it can be told that the results obtained by the hydrodynamic code SPheRIO with the resonance decay are consistent with the ones from PHOBOS (BACK et al., 2004).

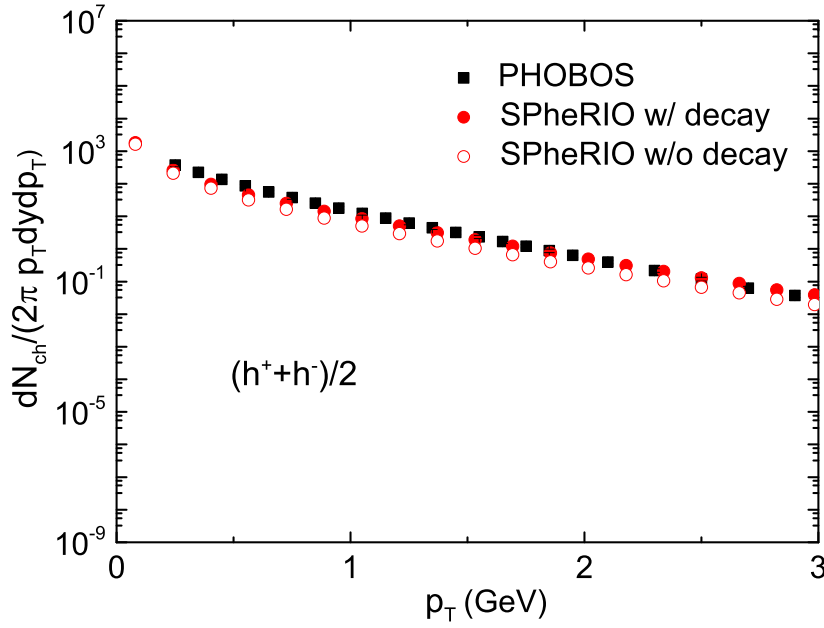


Figure 23 – The p_T spectra of the identified particles from SPheRIO with and without the resonance decay and PHOBOS (BACK et al., 2004). Here the pseudorapidity interval is takes as $0.2 < y < 1.4$.

5.2.3.1 Particle ratio p/π , K/π and K/p fluctuations

In accordance with the existing data of the Beam Energy Scan (BES) program (ALT et al., 2009; TARNOWSKY, 2012; THÄDER, 2016), the SPheRIO code has been utilized to carry out simulations of Au+Au collisions for different centrality windows at different energies. The IC are generated by using NeXuS (DRESCHER et al., 2002; DRESCHER et al., 2001). The results presented below are from simulations carried out for 989 events for 0 - 3.5% Pb+Pb collisions at 8.8 GeV, as well as 757, 455, and 533 events for 0 - 5% Au+Au collisions at 19.6, 62.4, and 200 GeV respectively. Besides EbE fluctuating, calculations are also done by using averaged ICs.

In Figure 25, we first show the calculated dynamical fluctuations of particle ratios p/π , K/π , and K/p at different energies. For instance, the quantity $\nu_{dyn,p/\pi}$ measures the deviation in the ratios of

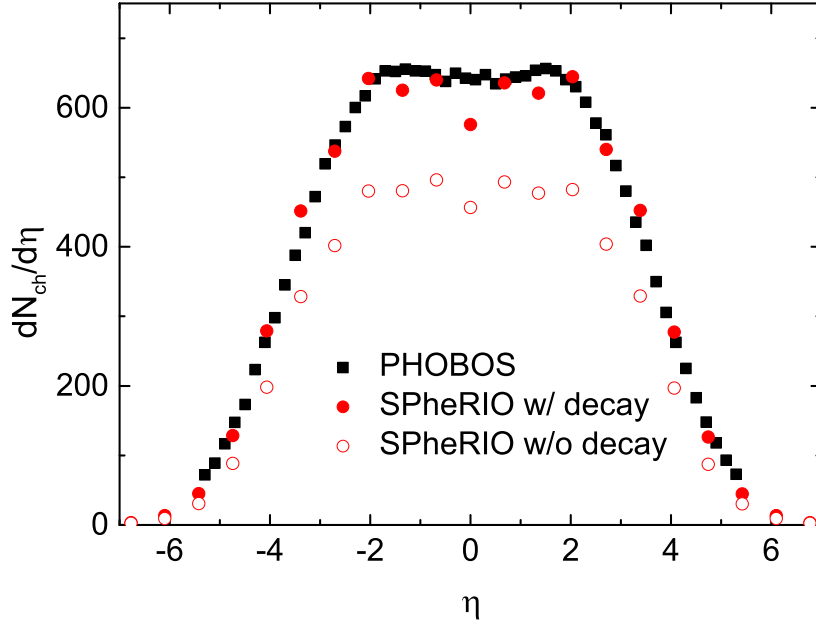


Figure 24 – The rapidity distribution of the identified particles from SPheRIO with and without the resonance decay and PHOBOS (BACK et al., 2004). Here the p_T interval is taken as $0 < p_T$ (GeV) < 5 .

p/π with respect to those of an ideal statistical Poissonian distribution. It is defined as,

$$\nu_{dyn,p/\pi} = \frac{\langle N_p(N_p - 1) \rangle}{\langle N_p \rangle^2} + \frac{\langle N_\pi(N_\pi - 1) \rangle}{\langle N_\pi \rangle^2} - 2 \frac{\langle N_p N_\pi \rangle}{\langle N_p \rangle \langle N_\pi \rangle}. \quad (5.62)$$

The results of hydrodynamic simulations by SPheRIO, and those of the Ultrarelativistic Quantum Molecular Dynamics (UrQMD) model (BASS et al., 1998; BLEICHER et al., 1999) as well as HRG models are presented. Here, the UrQMD model¹ is a microscopic model which can be used to simulate the relativistic heavy ion collisions, such as $p + p$, $p + \text{Au}$ and $\text{Au} + \text{Au}$ (BASS et al., 1998; BLEICHER et al., 1999). In the case of SPheRIO, the calculations have been carried out both with EbE fluctuating IC turned on and off. The model calculations are then compared with the data from the NA49 (ALT et al., 2009) and STAR (TARNOWSKY, 2012) Collaborations. It is found that the results which consider an expanding medium give moderate corrections in comparison to those regarding a static system. The resulting dynamical fluctuations from SPheRIO are slightly more significant than those from HRG models for p/π , smaller than those for K/π , while comparable in the case of K/p .

The effect due to the inclusion of hydrodynamic evolution is found to be sensitive on particle species. On the other hand, the order for the ratios of different isospin states, such as p/π , $p^{+(-)}/\pi^{+(-)}$, and $p^{-(+)} / \pi^{+(-)}$ are, by and large, correctly reproduced. In most cases, the effect of hydrodynamical evolution is found to be a monotonical function of energy. However, an exception is observed for p/π at 8.8 GeV, where the calculated values are smaller than those of the HRG model and the data, different

¹ The details can be found on the website <https://urqmd.org/>

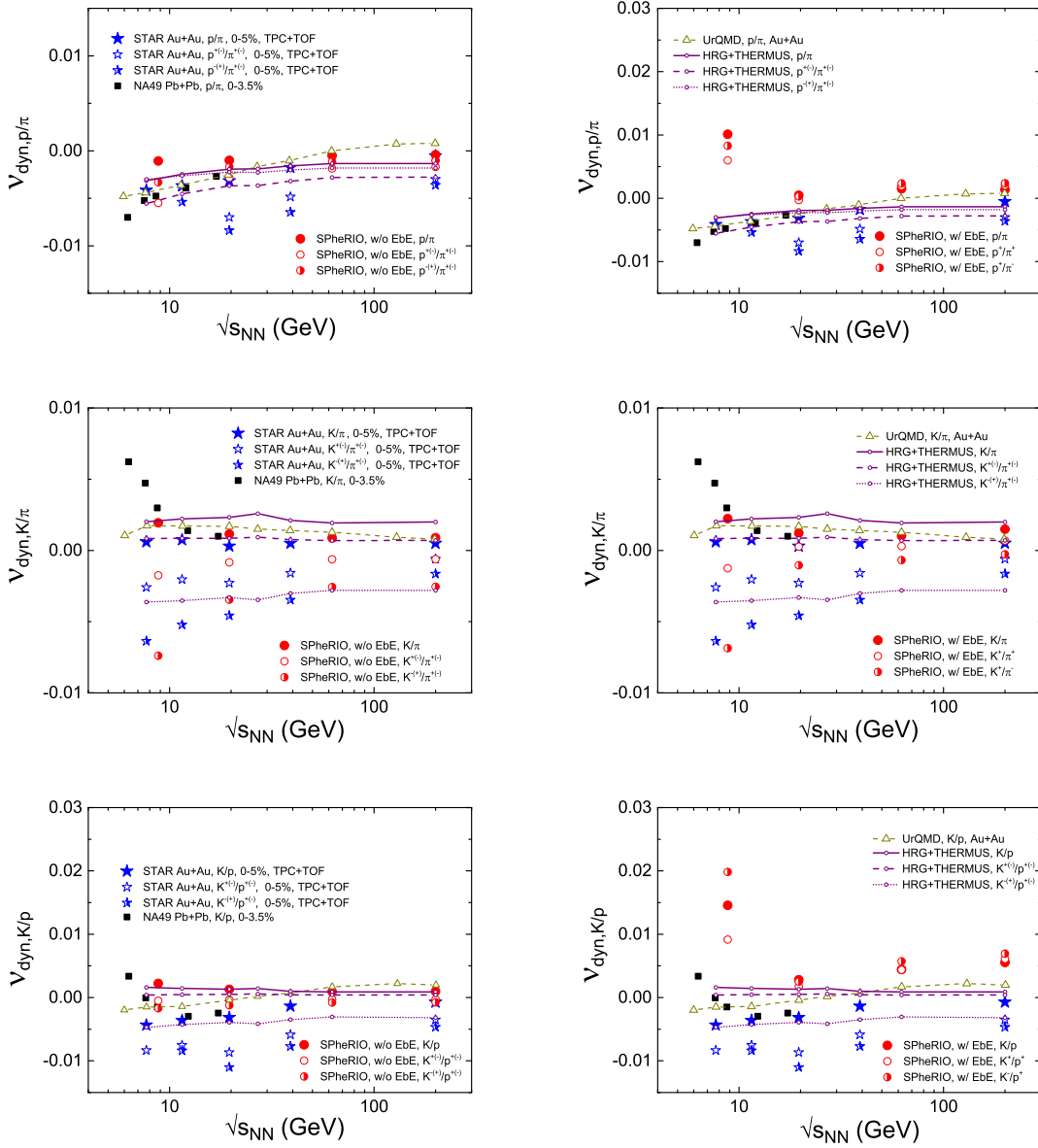


Figure 25 – The dynamical fluctuations of particle ratio of p/π , K/π and K/p at the different energies. The results are shown from the STAR collaboration (blue solid star, blue open star and blue half-filled star) (TARNOWSKY, 2012), the NA49 collaboration of the most central Pb+Pb collisions (black solid square) (ALT et al., 2009), UrQMD model calculations (dark yellow open triangle with dashed line) (BASS et al., 1998; BLEICHER et al., 1999) and the HRG model calculations (purple open circle with the straight, dashed and dotted line). Also shown are the results by hydrodynamical simulation program SPheRIO with(without) the consideration of EbE fluctuation at the right(left) column (red solid circle, red open circle and red half-filled circle). “w/” and “w/o” represent for “with” and “without” respectively.

from those for other incident energies. This might be related to the degree of collective motion of the entire system predicted by the hydrodynamical model, as it might overestimate the collectivity at lower energy.

When the EbE fluctuating are switched on, one finds that the calculated dynamical fluctuations are augmented. This result is readily to be believed. By definition, EbE ICs introduce additional

Table 4 – Average number of efficiency uncorrected identified particles used in the analysis of ν_{dyn} which are measured by STAR for 0 -5 % centrality

$\sqrt{S_{NN}}(GeV)$	$\langle K^\pm \rangle$	$\langle \pi^\pm \rangle$	$\langle p^\pm \rangle$
200	34 ± 2	366 ± 18	23 ± 1
62.4	35 ± 2	345 ± 17	25 ± 1
39	31 ± 2	317 ± 16	23 ± 1
27	24 ± 1	282 ± 14	22 ± 1
19.6	25 ± 1	272 ± 14	26 ± 1
11.5	21 ± 1	209 ± 11	32 ± 2
7.7	16 ± 1	161 ± 8	39 ± 2

fluctuations, on top of the thermal ones, in terms of particle multiplicities and their ratios. Subsequently, the calculated dynamical fluctuations predominantly overestimate the experimental data. In particular, for the case of lower energy Pb+Pb, the obtained dynamical fluctuations are found to be significantly over the data and HRG model calculations. This is mostly attributed to the significant event by event fluctuations found in the ICs generated by NeXuS. Overall, regarding the experimental data, the results obtained from SPheRIO are reasonable when compared against those from different approaches. The identified particle numbers used in the calculated ν_{dyn} are shown in Table.4.

5.2.3.2 Net-proton, net-kaon and net-charge multiplicity fluctuations

In the RHIC BES data, skewness S and kurtosis κ are two quantities closely related to the measurements, and their definitions are closely related to the cumulants. To be more specific, the following ratios are frequently being used

$$\begin{aligned}
 \frac{\sigma}{M} &= \frac{\langle (\Delta N)^2 \rangle}{\langle N \rangle}, \\
 S\sigma &= \frac{\langle (\Delta N)^3 \rangle}{\langle (\Delta N)^2 \rangle}, \\
 \kappa\sigma^2 &= \frac{\langle (\Delta N)^4 \rangle - 3 \langle (\Delta N)^2 \rangle^2}{\langle (\Delta N)^2 \rangle}.
 \end{aligned} \tag{5.63}$$

The reason for the ratio combinations is that the above three quantities are identical to unit in the case of ideal Poissonian distribution.

In practice, measurements are carried out for the net-particle multiplicity distribution regarding the above cumulant ratios. For instance, for net-proton multiplicity distribution, one finds

$$\begin{aligned}
 \frac{\sigma_{p-\bar{p}}}{M_{p-\bar{p}}} &= \frac{\langle (\Delta N_{p-\bar{p}})^2 \rangle}{\langle N_{p-\bar{p}} \rangle}, \\
 S_{p-\bar{p}}\sigma_{p-\bar{p}} &= \frac{\langle (\Delta N_{p-\bar{p}})^3 \rangle}{\langle (\Delta N_{p-\bar{p}})^2 \rangle}, \\
 \kappa_{p-\bar{p}}\sigma_{p-\bar{p}}^2 &= \frac{\langle (\Delta N_{p-\bar{p}})^4 \rangle - 3 \langle (\Delta N_{p-\bar{p}})^2 \rangle^2}{\langle (\Delta N_{p-\bar{p}})^2 \rangle},
 \end{aligned} \tag{5.64}$$

where

$$\langle N_{p-\bar{p}} \rangle = \langle N_p \rangle - \langle N_{\bar{p}} \rangle, \quad (5.65)$$

$$\langle (\Delta N_{p-\bar{p}})^2 \rangle = \langle (\Delta N_p)^2 \rangle + \langle (\Delta N_{\bar{p}})^2 \rangle - 2\langle \Delta N_p \Delta N_{\bar{p}} \rangle, \quad (5.66)$$

$$\langle (\Delta N_{p-\bar{p}})^3 \rangle = \langle (\Delta N_p)^3 \rangle - \langle (\Delta N_{\bar{p}})^3 \rangle - 3\langle (\Delta N_p)^2 \Delta N_{\bar{p}} \rangle + 3\langle \Delta N_p (\Delta N_{\bar{p}})^2 \rangle, \quad (5.67)$$

$$\begin{aligned} \langle (\Delta N_{p-\bar{p}})^4 \rangle - 3\langle (\Delta N_{p-\bar{p}})^2 \rangle^2 &= \langle (\Delta N_p)^4 \rangle - 3\langle (\Delta N_p)^2 \rangle^2 + \langle (\Delta N_{\bar{p}})^4 \rangle - 3\langle (\Delta N_{\bar{p}})^2 \rangle^2 \\ &- 4(\langle (\Delta N_p)^3 \Delta N_{\bar{p}} \rangle - 3\langle (\Delta N_p)^2 \rangle \langle \Delta N_p \Delta N_{\bar{p}} \rangle) \\ &+ 6(\langle (\Delta N_p)^2 (\Delta N_{\bar{p}})^2 \rangle - 2\langle \Delta N_p \Delta N_{\bar{p}} \rangle^2 - \langle (\Delta N_p)^2 \rangle \langle (\Delta N_{\bar{p}})^2 \rangle) \\ &- 4(\langle (\Delta N_{\bar{p}})^3 \Delta N_p \rangle - 3\langle (\Delta N_{\bar{p}})^2 \rangle \langle \Delta N_p \Delta N_{\bar{p}} \rangle). \end{aligned} \quad (5.68)$$

In the case of ideal Poissonian distribution, it is straightforward to show that

$$\begin{aligned} \frac{\sigma_{p-\bar{p}}}{M_{p-\bar{p}}} &\rightarrow \frac{\langle N_p \rangle + \langle N_{\bar{p}} \rangle}{\langle N_p \rangle - \langle N_{\bar{p}} \rangle}, \\ S_{p-\bar{p}} \sigma_{p-\bar{p}} &\rightarrow \frac{\langle N_p \rangle - \langle N_{\bar{p}} \rangle}{\langle N_p \rangle + \langle N_{\bar{p}} \rangle}, \\ \kappa_{p-\bar{p}} \sigma_{p-\bar{p}}^2 &\rightarrow 1. \end{aligned} \quad (5.69)$$

while the emissions of protons and anti-protons are treated as independent.

In Figure 26, we present various cumulant ratios at different energies. Again, the results obtained by SPheRIO are presented together with those by UrQMD and HRG models. These model calculations are then compared against the experimental data. The calculations have been carried out for averaged ICs. The STAR measurements (THäDER, 2016) are for 0 - 5% Au+Au collisions at various energies from $\sqrt{s_{NN}} = 7.7$ to 200 GeV. As discussed above, the products $\kappa\sigma^2$ and $S\sigma$ are related to the ratios of particle number cumulants, which are the same as ratios of susceptibilities for a homogeneous system. In particular, for an ideal Poissonian distribution, $S\sigma/\text{Skellam}$ and $\kappa\sigma^2$ are both expected to be identical to the unit. For a hydrodynamic approach, the system is assumed to be in local equilibrium but not homogeneous. In general, the numerical results from SPheRIO demonstrate a consistent trend with respect to those obtained by employing the HRG model. While the effect of hadronic decay is also observed to possess a similar impact to the cumulant ratios, in comparison to that in the HRG approach, the former is found to be more significant numerically. For net-proton and net-kaon fluctuations, the calculated $\kappa\sigma^2$ almost keeps at the value 1 which is in agreement with the UrQMD ones. For the case of net-charge fluctuations, the calculated $\kappa\sigma^2$ also keeps at one while the UrQMD model predicts an opposite trend. Both of them fail to describe the nonmonotonous behavior shown in the data of net-proton, net-kaon and net-charge.

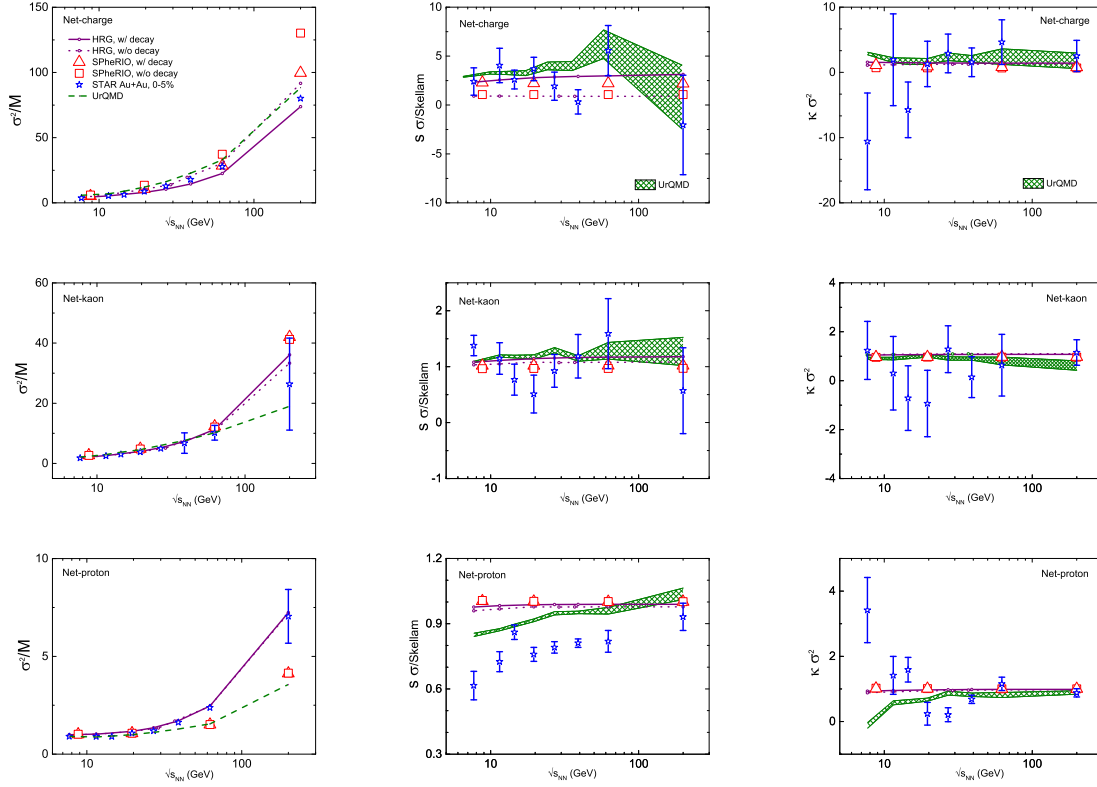


Figure 26 – The multiplicity of net-charged particles, net-proton and net-kaon at the different energies. The corresponding results are given from the HRG model (purple solid and dashed lines), the SPheRIO with an average IC (red open triangle and square), the STAR collaboration (blue open star with the error bar) (THäder, 2016) and the UrQMD model calculations (olive dashed line or area) (BASS et al., 1998; BLEICHER et al., 1999). “w/” and “w/o” represent for “with” and “without” respectively.

6 CONCLUSION

In this thesis, we investigated a few aspects regarding the thermodynamical properties of the matter created in relativistic heavy-ion collisions. To this end, we developed the quasi-particle model inspired by Gorenstein and Yang for describing the thermodynamic properties of QGP and proposed a new possible solution where the quasi-particle mass is a function of T , μ and k . Lattice QCD and the other theoretical studies stressed that there exists CEP at the QCD phase diagram, although its existence has not been experimentally proved. To show this property, an EoS with a phenomenological CEP is implemented. The EoS is essential to hydrodynamic simulations, in other words the effects of EoS on hydrodynamic analysis are significant. At present, we focus on the noncritical aspects of the multiplicity fluctuations in heavy-ion collisions by employing a hydrodynamic simulation code SPheRIO and then compare the results with the one from STAR, NA48 Collaboration, HRG model and UrQMD model. We find that the present approach inherits the main characteristics of the HRG model, which take into consideration thermal fluctuations, finite volume correction, and resonance decay. However, if we adopt an EoS with the QCD CEP for SPheRIO, the relevant results may show some evidences for the existence of the QCD CEP and this will be done in the near future. The purpose of those works is to provide the theoretical support and predictions for searching for the QCD CEP in heavy ion colliders.

In chapter 4, we introduced some models or theoretical tools to characterize the thermodynamic properties of QCD matter. It is understood that the hadronic matter will be switched to an excited state, called QGP with the increase of temperature and/or density. Due to the asymptotic freedom, pQCD calculations can be used to characterize the QGP phase at extremely high temperature. However this perturbative method can not provide a reliable results when $T \sim T_c$, seen Figure 9 and Figure 10. As a phenomenological model, quasi-particle model is then adopted because it can indicate the thermodynamic properties of QGP over a wide range of temperature and baryon chemical potential. The free parameters in this phenomenological model could be determined by Lattice data. For completeness, we discuss the thermodynamic self-consistency of HRG model with excluded volumes and present thermodynamic inconsistency problem for the primordial quasi-particle model. In order to evade this inconsistency problem, a lot of theoretical researchs had been done. In this paper, we proposed the general solutions following Gorenstein and Yang's idea and we also suggested that the solution given by Peshier *et al* is just a special one. In order to echo this statement, another special solution with a T -, μ - and k -dependent mass was given. As shown in Figure 16 and Figure 17, our special solution can reproduce Lattice data at both zero and finite chemical potential over the range of temperature, $[0.9T_c, 2.4T_c]$. From Figure 18, the quasi-particle masses we obtained are consistent with the ones from the perturbative calculations. Besides, we show that the discrepancy between Peshier's and our solution is not so great which can be seen in Figure 18.

In chapter 5, we turned towards another two goals: understanding the QCD phase diagram and searching for the QCD CEP. For the first goal, in order to implement a QCD CEP, a hybrid parameterization of a quasi-particle EoS is adopted here. Such a parameterization is based on the

interpolation scheme proposed by Hama *et al* with a modified Gibbs condition. To be more specific, the interpolation scheme is used to combine the quasi-particle model fitted with Lattice data at high temperature and HRG model at low temperature. As shown in Figure 22, in this approach, the QCD phase transition is a “cross-over” at vanishing chemical potential and a first-order when $\mu \gtrsim 0.5\text{GeV}$. For the second one, we adopt an EoS without CEP and an averaged IC for the hydrodynamic code SPheRIO. Then the calculated net-proton, net-kaon and net charged multiplicity fluctuations are compared with the ones from STAR, NA48, the HRG model with exclude volume corrections and the transport model, UrQMD. The comparisons are shown in Figure 26. We found that the hydrodynamic model we used is consistent with the one from the HRG model. However, to avoid the effects of volume fluctuations, the K/π , p/π and K/p particle ratio fluctuations are also calculated by the SPheRIO with or without fluctuations of ICs. Those results are also compared with the ones from STAR, NA48, the HRG model with exclude volume corrections and the UrQMD model, shown in Figure 25. We found that the EbE fluctuating of ICs may have great influences on the final results.

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APPENDIX A – PARTICLE LIST FOR HRG MODEL

Table 5 – Particle list where the anti-particles are not included

hadrons	mass(GeV)	g	B	Q	S	hadrons	mass	g	B	Q	S
π^+	0.14	1	0	1	0	$N^0(2220)$	2.22	10	1	0	0
π^0	0.14	1	0	0	0	$N^+(2250)$	2.25	10	1	1	0
η	0.55	1	0	0	0	$N^0(2250)$	2.25	10	1	0	0
ρ^+	0.78	3	0	1	0	$N^+(2600)$	2.6	12	1	1	0
ρ^0	0.77	3	0	0	0	$N^0(2600)$	2.6	12	1	0	0
ω	0.78	3	0	0	0	$\Delta^{++}(1232)$	1.23	4	1	2	0
η'	0.96	1	0	0	0	$\Delta^+(1232)$	1.23	4	1	1	0
$f_0(980)$	0.99	1	0	0	0	$\Delta^0(1232)$	1.23	4	1	0	0
$a_0^+(980)$	0.99	1	0	1	0	$\Delta^-(1232)$	1.23	4	1	-1	0
$a_0^0(980)$	0.99	1	0	0	0	$\Delta^{++}(1600)$	1.6	4	1	2	0
$\phi(1020)$	1.02	3	0	0	0	$\Delta^+(1600)$	1.6	4	1	1	0
h_1	1.17	3	0	0	0	$\Delta^0(1600)$	1.6	4	1	0	0
b_1^+	1.23	3	0	1	0	$\Delta^-(1600)$	1.6	4	1	-1	0
b_1^0	1.23	3	0	0	0	$\Delta^{++}(1620)$	1.62	2	1	2	0
$a_1^+(1260)$	1.23	3	0	1	0	$\Delta^+(1620)$	1.62	2	1	1	0
$a_1^0(1260)$	1.23	3	0	0	0	$\Delta^0(1620)$	1.62	2	1	0	0
$f_2(1270)$	1.27	5	0	0	0	$\Delta^-(1620)$	1.62	2	1	-1	0
$f_1(1285)$	1.28	3	0	0	0	$\Delta^{++}(1700)$	1.7	4	1	2	0
$\eta(1295)$	1.3	1	0	0	0	$\Delta^+(1700)$	1.7	4	1	1	0
$\pi^+(1300)$	1.3	1	0	1	0	$\Delta^0(1700)$	1.7	4	1	0	0
$\pi^0(1300)$	1.3	1	0	0	0	$\Delta^-(1700)$	1.7	4	1	-1	0
$a_2^+(1320)$	1.32	5	0	1	0	$\Delta^{++}(1905)$	1.9	6	1	2	0
$a_2^0(1320)$	1.32	5	0	0	0	$\Delta^+(1905)$	1.9	6	1	1	0
$f_1(1420)$	1.43	3	0	0	0	$\Delta^0(1905)$	1.9	6	1	0	0
$\omega(1420)$	1.42	3	0	0	0	$\Delta^-(1905)$	1.9	6	1	-1	0
$a_0^+(1450)$	1.47	1	0	1	0	$\Delta^{++}(1910)$	1.91	2	1	2	0
$a_0^0(1450)$	1.47	1	0	0	0	$\Delta^+(1910)$	1.91	2	1	1	0
$\rho^+(1450)$	1.47	3	0	1	0	$\Delta^0(1910)$	1.91	2	1	0	0
$\rho^0(1450)$	1.47	3	0	0	0	$\Delta^-(1910)$	1.91	2	1	-1	0
$f_0(1500)$	1.5	1	0	0	0	$\Delta^{++}(1920)$	1.92	4	1	2	0
f_2'	1.52	5	0	0	0	$\Delta^+(1920)$	1.92	4	1	1	0
$\omega(1650)$	1.65	3	0	0	0	$\Delta^0(1920)$	1.92	4	1	0	0
$\omega 3(1670)$	1.67	7	0	0	0	$\Delta^-(1920)$	1.92	4	1	-1	0
$\pi_2^+(1670)$	1.67	5	0	1	0	$\Delta^{++}(1930)$	1.93	6	1	2	0
$\pi_2^0(1670)$	1.67	5	0	0	0	$\Delta^+(1930)$	1.93	6	1	1	0
$\phi(1680)$	1.68	3	0	0	0	$\Delta^0(1930)$	1.93	6	1	0	0
$\rho_3^+(1690)$	1.69	7	0	1	0	$\Delta^-(1930)$	1.93	6	1	-1	0
$\rho_3^0(1690)$	1.69	7	0	0	0	$\Delta^{++}(1950)$	1.95	8	1	2	0

Table 6 – Particle list where the anti-particles are not included

hadrons	mass(GeV)	g	B	Q	S	hadrons	mass	g	B	Q	S
$\rho^+(1700)$	1.7	3	0	1	0	$\Delta^+(1950)$	1.95	8	1	1	0
$\rho^0(1700)$	1.7	3	0	0	0	$\Delta^0(1950)$	1.95	8	1	0	0
$f_0(1710)$	1.72	1	0	0	0	$\Delta^-(1950)$	1.95	8	1	-1	0
$\pi^+(1800)$	1.8	1	0	1	0	$\Delta^{++}(2420)$	2.42	12	1	2	0
$\pi^0(1800)$	1.8	1	0	0	0	$\Delta^+(2420)$	2.42	12	1	1	0
$\phi_3(1850)$	1.85	7	0	0	0	$\Delta^0(2420)$	2.42	12	1	0	0
$f_2(2010)$	2.01	5	0	0	0	$\Delta^-(2420)$	2.42	12	1	-1	0
$a_4^+(2040)$	2.01	9	0	1	0	Λ	1.12	2	1	0	-1
$a_4^0(2040)$	2.01	9	0	0	0	$\Lambda(1405)$	1.41	2	1	0	-1
$f_4(2050)$	2.03	9	0	0	0	$\Lambda(1520)$	1.52	4	1	0	-1
$f_2(2300)$	2.3	5	0	0	0	$\Lambda(1600)$	1.6	2	1	0	-1
$f_2(2340)$	2.34	5	0	0	0	$\Lambda(1670)$	1.67	2	1	0	-1
$f_0(1370)$	1.35	1	0	0	0	$\Lambda(1690)$	1.69	4	1	0	-1
$\eta(1440)$	1.43	1	0	0	0	$\Lambda(1800)$	1.8	2	1	0	-1
K^+	0.49	1	0	1	1	$\Lambda(1810)$	1.81	2	1	0	-1
K^0	0.5	1	0	0	1	$\Lambda(1820)$	1.82	6	1	0	-1
$K^{*+}(892)$	0.89	3	0	1	1	$\Lambda(1830)$	1.83	6	1	0	-1
$K^{*0}(892)$	0.9	3	0	0	1	$\Lambda(1890)$	1.89	4	1	0	-1
$K_1^+(1270)$	1.27	3	0	1	1	$\Lambda(2100)$	2.1	8	1	0	-1
$K_1^0(1270)$	1.27	3	0	0	1	$\Lambda(2110)$	2.11	6	1	0	-1
$K_1^+(1400)$	1.4	3	0	1	1	$\Lambda(2350)$	2.35	10	1	0	-1
$K_1^0(1400)$	1.4	3	0	0	1	Σ^+	1.19	2	1	1	-1
$K^{*+}(1410)$	1.41	3	0	1	1	Σ^0	1.19	2	1	0	-1
$K^{*0}(1410)$	1.41	3	0	0	1	Σ^-	1.2	2	1	-1	-1
$K_0^{*+}(1430)$	1.41	1	0	1	1	$\Sigma^+(1385)$	1.38	4	1	1	-1
$K_0^{*0}(1430)$	1.41	1	0	0	1	$\Sigma^0(1385)$	1.38	4	1	0	-1
$K_2^{*+}(1430)$	1.42	5	0	1	1	$\Sigma^-(1385)$	1.39	4	1	-1	-1
$K_2^{*0}(1430)$	1.43	5	0	0	1	$\Sigma^+(1660)$	1.66	2	1	1	-1
$K^{*+}(1680)$	1.72	3	0	1	1	$\Sigma^0(1660)$	1.66	2	1	0	-1
$K^{*0}(1680)$	1.72	3	0	0	1	$\Sigma^-(1660)$	1.66	2	1	-1	-1
$K_2^+(1770)$	1.77	5	0	1	1	$\Sigma^+(1670)$	1.67	2	1	1	-1
$K_2^0(1770)$	1.77	5	0	0	1	$\Sigma^0(1670)$	1.67	2	1	0	-1
$K_3^{*+}(1780)$	1.78	7	0	1	1	$\Sigma^-(1670)$	1.67	2	1	-1	-1
$K_3^{*0}(1780)$	1.78	7	0	0	1	$\Sigma^+(1750)$	1.75	2	1	1	-1
$K_2^+(1820)$	1.82	5	0	1	1	$\Sigma^0(1750)$	1.75	2	1	0	-1
$K_2^0(1820)$	1.82	5	0	0	1	$\Sigma^-(1750)$	1.75	2	1	-1	-1
$K_4^{*+}(2045)$	2.05	9	0	1	1	$\Sigma^+(1775)$	1.77	6	1	1	-1
$K_4^{*0}(2045)$	2.05	9	0	0	1	$\Sigma^0(1775)$	1.77	6	1	0	-1
p	0.94	2	1	1	0	$\Sigma^-(1775)$	1.77	6	1	-1	-1
n	0.94	2	1	0	0	$\Sigma^+(1915)$	1.91	6	1	1	-1
$N^+(1440)$	1.44	2	1	1	0	$\Sigma^0(1915)$	1.91	6	1	0	-1
$N^0(1440)$	1.44	2	1	0	0	$\Sigma^-(1915)$	1.91	6	1	-1	-1
$N^+(1520)$	1.52	4	1	1	0	$\Sigma^+(1940)$	1.94	4	1	1	-1
$N^0(1520)$	1.52	4	1	0	0	$\Sigma^0(1940)$	1.94	4	1	0	-1
$N^+(1535)$	1.53	2	1	1	0	$\Sigma^-(1940)$	1.94	4	1	-1	-1
$N^0(1535)$	1.53	2	1	0	0	$\Sigma^+(2030)$	2.03	8	1	1	-1
$N^+(1650)$	1.65	2	1	1	0	$\Sigma^0(2030)$	2.03	8	1	0	-1
$N^0(1650)$	1.65	2	1	0	0	$\Sigma^-(2030)$	2.03	8	1	-1	-1
$N^+(1675)$	1.67	6	1	1	0	Ξ^0	1.32	2	1	0	-2

Table 7 – Particle list where the anti-particles are not included

hadrons	mass(GeV)	g	B	Q	S	hadrons	mass	g	B	Q	S
$N^0(1675)$	1.67	6	1	0	0	Ξ^-	1.32	2	1	-1	-2
$N^+(1680)$	1.68	6	1	1	0	$\Xi^0(1530)$	1.53	4	1	0	-2
$N^0(1680)$	1.68	6	1	0	0	$\Xi^-(1530)$	1.53	4	1	-1	-2
$N^+(1700)$	1.7	4	1	1	0	$\Xi^0(1820)$	1.82	4	1	0	-2
$N^0(1700)$	1.7	4	1	0	0	$\Xi^-(1820)$	1.82	4	1	-1	-2
$N^+(1710)$	1.71	2	1	1	0	$\Xi^0(1690)$	1.69	4	1	0	-2
$N^0(1710)$	1.71	2	1	0	0	$\Xi^-(1690)$	1.69	4	1	-1	-2
$N^+(1720)$	1.72	4	1	1	0	Ω	1.67	4	1	-1	-3
$N^0(1720)$	1.72	4	1	0	0	$\Xi^0(1950)$	1.95	4	1	0	-2
$N^+(2190)$	2.19	8	1	1	0	$\Xi^-(1950)$	1.95	4	1	-1	-2
$N^0(2190)$	2.19	8	1	0	0	$\Xi^0(2030)$	2.03	6	1	0	-2
$N^+(2220)$	2.22	10	1	1	0	$\Xi^-(2030)$	2.03	6	1	-1	-2

APPENDIX B – THERMODYNAMIC CONSISTENCY WITH EXCLUDED VOLUMES

For the simplicity, we rewrite the Eq.(4.26) as

$$\begin{aligned} p(T, \mu) &= p^{id}(T, \tilde{\mu}) \\ \tilde{\mu} &\equiv \mu - vp. \end{aligned} \quad (\text{B.1})$$

The total number density can be derived from the following thermodynamica relation,

$$n = \left. \frac{\partial p}{\partial \mu} \right|_T. \quad (\text{B.2})$$

Accordingly

$$n = \left. \frac{\partial p^{id}}{(\partial \mu - pv)} \right|_T \left. \frac{\partial (\mu - pv)}{\partial \mu} \right|_T, \quad (\text{B.3})$$

or

$$n = \left. \frac{\partial p^{id}}{\partial (\mu - pv)} \right|_T \left[1 - v \left. \frac{\partial p}{\partial \mu_i} \right|_T \right]. \quad (\text{B.4})$$

For a ideal gas of point-like hadrons with an intrinsic volume v , its number density can be expressed as n^{id} , where

$$n^{id} = \frac{\partial p^{id}(T, \tilde{\mu})}{\partial \tilde{\mu}} = \frac{\partial p^{id}}{\partial (\mu - pv)}. \quad (\text{B.5})$$

Then we have

$$n = n^{id}(T, \mu - pv) [1 - vn], \quad (\text{B.6})$$

or

$$n = \frac{n^{id}}{1 + vn^{id}}. \quad (\text{B.7})$$

Thus, the Eq.(4.27) is proved.

APPENDIX C – THE METHOD OF CHARACTERISTICS

In this Appendix, we show how to obtain the solutions of Eqs.(4.122) and (4.129). As a matter of fact, the procedure to solve the above equations is very similar, while the latter is slightly more complicated. Therefore, in what follows, we explicitly derive the solution of Eq.(4.129) and briefly discuss how that of Eq.(4.122) is obtained. One first rewrites Eq.(4.129) by defining

$$w = \omega^* - \mu. \quad (\text{C.1})$$

Since $m = \sqrt{(w + \mu)^2 - k^2}$, considering k merely as a parameter in $m = m(k, T, \mu)$, and $(w + \mu)$ as an intermediate variable, one has

$$\begin{aligned} \frac{\partial m}{\partial \mu} &= \frac{\partial m}{\partial(w + \mu)} \frac{\partial(w + \mu)}{\partial \mu}, \\ \frac{\partial m}{\partial T} &= \frac{\partial m}{\partial(w + \mu)} \frac{\partial(w + \mu)}{\partial T} = \frac{\partial m}{\partial(w + \mu)} \frac{\partial w}{\partial T}. \end{aligned}$$

Thus Eq.(4.129) implies

$$\frac{\partial(w + \mu)}{\partial \mu} = \frac{T}{w} \frac{\partial w}{\partial T}, \quad (\text{C.2})$$

or equivalently,

$$w \frac{\partial w}{\partial \mu} - T \frac{\partial w}{\partial T} + w = 0, \quad (\text{C.3})$$

whose solution can be obtained by using the method of characteristics (COURANT; HILBERT, 1989). To be specific, the above partial different equation can be fit into the formal form

$$a(\mu, T, w) \frac{\partial w}{\partial \mu} + b(\mu, T, w) \frac{\partial w}{\partial T} = c(\mu, T, w), \quad (\text{C.4})$$

with

$$\begin{aligned} a(\mu, T, w) &= w, \\ b(\mu, T, w) &= -T, \\ c(\mu, T, w) &= -w. \end{aligned} \quad (\text{C.5})$$

whose formal solution is the surface, defined by $f(\mu, T, w) = w - w(\mu, T) = 0$, tangent to the vector field $(a(\mu, T, w), b(\mu, T, w), c(\mu, T, w))$, namely,

$$\frac{d\mu}{w} = \frac{dT}{-T} = \frac{dw}{-w}. \quad (\text{C.6})$$

As it contains two independent equations, one may conveniently select

$$d(\mu + w) = 0,$$

and

$$d \left[\ln \left(\frac{w}{T} \right) \right] = d \left(\frac{w}{T} \right) = 0.$$

This indicates that, for any function $F(u, v)$, the desired solution w satisfies

$$F \left(\frac{w}{T}, (w + \mu) \right) = 0. \quad (\text{C.7})$$

Now, as discussed in the above text, the solution of the equation is determined by the boundary condition at $\mu = 0$, where $m(k, T, \mu = 0) \equiv f(T)$. In other words, the form of F shall be determined by the boundary condition. If one defines $F_0(u, v) \equiv F(\mu = 0)$, it is readily to verify that¹

$$F(u, v) = \sqrt{f \left(\frac{v}{u} \right)^2 + k^2} - v \quad (\text{C.8})$$

indeed satisfies Eq.(4.129). Subsequently, the general solution of $\omega^*(k, T, \mu)$ for finite chemical potential is given by

$$\sqrt{f \left(\frac{T\omega^*}{\omega^* - \mu} \right)^2 + k^2} - \omega^* = 0, \quad (\text{C.9})$$

or

$$m = f \left(\frac{T\omega^*}{\omega^* - \mu} \right), \quad (\text{C.10})$$

which is Eq.(4.130).

As for Eq.(4.122), one may immediately recognize that the equation possesses the same form of Eq.(C.4) with

$$\begin{aligned} a(\mu, T, m) &= \langle \langle 1 \rangle \rangle_+, \\ b(\mu, T, m) &= -\langle \langle 1 \rangle \rangle_-, \\ c(\mu, T, m) &= 0. \end{aligned} \quad (\text{C.11})$$

Therefore, the formal solution reads

$$\frac{d\mu}{\langle \langle 1 \rangle \rangle_+} = \frac{dT}{\langle \langle 1 \rangle \rangle_-}, \quad (\text{C.12})$$

which is the solution presented in the main text.

¹ It is in fact one of many equivalent choices, *e.g.*, another possibility is $F(u, v) = uf^{-1}(\sqrt{v^2 - k^2}) - v$.

APPENDIX D – FLUCTUATIONS OF ICS

Considering the effect of EbE fluctuating ICs, we suppose that there are two events A and B . For the particle specie i , the event A who has m elements can be denoted as,

$$A = \{\mathcal{A}_{1,i}, \mathcal{A}_{2,i}, \dots, \mathcal{A}_{m,i}\}. \quad (\text{D.1})$$

As shown in the Eqs.(5.55) and (5.57), the average and variance for event A should be,

$$\bar{A}_i = \langle N_i \rangle_A = \frac{\mathcal{A}_{1,i} + \mathcal{A}_{2,i} + \dots + \mathcal{A}_{m,i}}{m}, \quad (\text{D.2})$$

$$\langle (\Delta A_i)^2 \rangle = \langle (\Delta N)^2 \rangle_A = \frac{\langle (\mathcal{A}_1 - \bar{A}_i)^2 \rangle + \dots + \langle (\mathcal{A}_m - \bar{A}_i)^2 \rangle}{m}. \quad (\text{D.3})$$

For another event B who includes n elements,

$$B = \{\mathcal{B}_{1,i}, \mathcal{B}_{2,i}, \dots, \mathcal{B}_{n,i}\} \quad (\text{D.4})$$

The average and variance for event B could be,

$$\bar{B}_i = \langle N_i \rangle_B = \frac{\mathcal{B}_{1,i} + \mathcal{B}_{2,i} + \dots + \mathcal{B}_{n,i}}{n} \quad (\text{D.5})$$

$$\langle (\Delta B_i)^2 \rangle = \langle (\Delta N_i)^2 \rangle_B = \frac{\langle (\mathcal{B}_{1,i} - \bar{B}_i)^2 \rangle + \dots + \langle (\mathcal{B}_{n,i} - \bar{B}_i)^2 \rangle}{n} \quad (\text{D.6})$$

After considering the effect of EbE, The total elements in those two events are $m + n$. And the final ariance can be expressed as for the particle species i and j ,

$$\begin{aligned} \langle (\Delta N_i)(\Delta N_j) \rangle_{A+B} &= \frac{1}{m+n} \left(\langle (\mathcal{A}_{1,i} - \bar{N}_i)(\mathcal{A}_{1,j} - \bar{N}_j) \rangle + \dots + \langle (\mathcal{A}_{m,i} - \bar{N}_i)(\mathcal{A}_{m,j} - \bar{N}_j) \rangle \right) \\ &+ \frac{1}{m+n} \left(\langle (\mathcal{B}_{1,i} - \bar{N}_i)(\mathcal{B}_{1,j} - \bar{N}_j) \rangle + \dots + \langle (\mathcal{B}_{n,i} - \bar{N}_i)(\mathcal{B}_{n,j} - \bar{N}_j) \rangle \right) \end{aligned} \quad (\text{D.7})$$

The first term in above equation (D.7) can be decomposed into,

$$\begin{aligned} \langle (\mathcal{A}_{1,i} - \bar{N}_i)(\mathcal{A}_{1,j} - \bar{N}_j) \rangle &= \langle (\mathcal{A}_{1,i} - \bar{A}_i + \bar{A}_i - \bar{N}_i)(\mathcal{A}_{1,j} - \bar{A}_j + \bar{A}_j - \bar{N}_j) \rangle \\ &= \langle (\mathcal{A}_{1,i} - \bar{A}_i)(\mathcal{A}_{1,j} - \bar{A}_j) \rangle + \langle (\bar{A}_i - \bar{N}_i)(\bar{A}_j - \bar{N}_j) \rangle \\ &+ (\bar{A}_j - \bar{N}_j) \langle (\mathcal{A}_{1,i} - \bar{A}_i) \rangle + (\bar{A}_i - \bar{N}_i) \langle (\mathcal{A}_{1,j} - \bar{A}_j) \rangle \end{aligned} \quad (\text{D.8})$$

Since $\sum_{l=1}^m \langle (\mathcal{A}_{l,i(j)} - \bar{A}_{i(j)}) \rangle \equiv 0$ and $\langle \Delta A_i \Delta A_j \rangle = \frac{1}{m} \sum_{l=1}^m \langle (\mathcal{A}_{l,i} - \bar{A}_i)(\mathcal{A}_{l,j} - \bar{A}_j) \rangle$, it is easy to obtain,

$$\begin{aligned}
\langle(\Delta N_i)(\Delta N_j)\rangle_{A+B} = & \frac{1}{m+n} (m\langle\Delta A_i\Delta A_j\rangle + n\langle\Delta B_i\Delta B_j\rangle) \\
& + \frac{1}{m+n} (m(\bar{A}_i - \bar{N}_i)(\bar{A}_j - \bar{N}_j) + n(\bar{B}_i - \bar{N}_i)(\bar{B}_j - \bar{N}_j)) \quad (\text{D.9})
\end{aligned}$$

If we set $m = n$,

$$\begin{aligned}
\langle(\Delta N_i)(\Delta N_j)\rangle_{A+B} = & \frac{1}{2} (\langle\Delta A_i\Delta A_j\rangle + \langle\Delta B_i\Delta B_j\rangle) \\
& + \frac{1}{2} ((\bar{A}_i - \bar{N}_i)(\bar{A}_j - \bar{N}_j) + (\bar{B}_i - \bar{N}_i)(\bar{B}_j - \bar{N}_j)) \quad (\text{D.10})
\end{aligned}$$

Considering that there are N events with the same numbers of elements, Eq.(5.61) is revisited.

APPENDIX E – THE HIGHER MOMENTS OF NET-CHARGE, NET-PROTON AND NET-KAON MULTIPLICITY

The net-charge, net-proton and net-kaon are expressed as, $N_{net-charge} = N_+ - N_-$, $N_{p-\bar{p}} = N_p - N_{\bar{p}}$ and $N_{K^+-K^-} = N_{K^+} - N_{K^-}$ respectively. The cumulants of net-particle multiplicity can be derived from the cumulants of particle(h) and anti-particle($\bar{h}$) multiplicity and their correlations,

$$M_{h-\bar{h}} = \langle N_{h-\bar{h}} \rangle = \langle N_h \rangle - \langle N_{\bar{h}} \rangle, \quad (\text{E.1})$$

$$\begin{aligned} \sigma_{h-\bar{h}}^2 &= \langle (\Delta N_{h-\bar{h}})^2 \rangle = \langle ((N_h - N_{\bar{h}}) - (\langle N_h \rangle - \langle N_{\bar{h}} \rangle))^2 \rangle \\ &= \langle (N_h - N_{\bar{h}})^2 \rangle - (\langle N_h \rangle - \langle N_{\bar{h}} \rangle)^2 \\ &= \langle (\Delta N_h)^2 \rangle + \langle (\Delta N_{\bar{h}})^2 \rangle - 2\langle \Delta N_h \Delta N_{\bar{h}} \rangle, \end{aligned} \quad (\text{E.2})$$

$$\begin{aligned} \langle (\Delta N_{h-\bar{h}})^3 \rangle &= \langle ((N_h - N_{\bar{h}}) - (\langle N_h \rangle - \langle N_{\bar{h}} \rangle))^3 \rangle \\ &= \langle (N_h - N_{\bar{h}})^3 \rangle - 3\langle (N_h - N_{\bar{h}})^2 (\langle N_h \rangle - \langle N_{\bar{h}} \rangle) \rangle + 2(\langle N_h \rangle - \langle N_{\bar{h}} \rangle)^3 \\ &= \langle N_h^3 \rangle - 3\langle N_h^2 N_{\bar{h}} \rangle + 3\langle N_{\bar{h}}^2 N_h \rangle - \langle N_{\bar{h}}^3 \rangle \\ &\quad - 3\langle N_h \rangle \langle N_h^2 - 2N_h N_{\bar{h}} + N_{\bar{h}}^2 \rangle + 3\langle N_{\bar{h}} \rangle \langle N_h^2 - 2N_h N_{\bar{h}} + N_{\bar{h}}^2 \rangle \\ &\quad + 2\langle N_h \rangle^3 - 6\langle N_h \rangle^2 \langle N_{\bar{h}} \rangle + 6\langle N_{\bar{h}} \rangle^2 \langle N_h \rangle - 2\langle N_{\bar{h}} \rangle^3 \\ &= \langle (\Delta N_h)^3 \rangle - \langle (\Delta N_{\bar{h}})^3 \rangle - 3\langle N_h^2 N_{\bar{h}} \rangle + 3\langle N_{\bar{h}}^2 N_h \rangle + 6\langle N_h \rangle \langle N_h N_{\bar{h}} \rangle \\ &\quad - 6\langle N_{\bar{h}} \rangle \langle N_h N_{\bar{h}} \rangle - 3\langle N_h \rangle \langle N_{\bar{h}}^2 \rangle + 3\langle N_{\bar{h}} \rangle \langle N_h^2 \rangle \\ &\quad - 6\langle N_h \rangle^2 \langle N_{\bar{h}} \rangle + 6\langle N_{\bar{h}} \rangle^2 \langle N_h \rangle \\ &= \langle (\Delta N_h)^3 \rangle - \langle (\Delta N_{\bar{h}})^3 \rangle - 3\langle (\Delta N_h)^2 \Delta N_{\bar{h}} \rangle + 3\langle \Delta N_h (\Delta N_{\bar{h}})^2 \rangle, \end{aligned} \quad (\text{E.3})$$

$$\begin{aligned} \langle (\Delta N_{h-\bar{h}})^4 \rangle - 3\langle (\Delta N_{h-\bar{h}})^2 \rangle^2 &= \langle ((N_h - N_{\bar{h}}) - (\langle N_h \rangle - \langle N_{\bar{h}} \rangle))^4 \rangle - 3\langle ((N_h - N_{\bar{h}}) - (\langle N_h \rangle - \langle N_{\bar{h}} \rangle))^2 \rangle^2 \\ &= \langle (N_h - N_{\bar{h}})^4 \rangle - 4\langle (N_h - N_{\bar{h}})^3 (\langle N_h \rangle - \langle N_{\bar{h}} \rangle) \rangle \\ &\quad + 6\langle (N_h - N_{\bar{h}})^2 (\langle N_h \rangle - \langle N_{\bar{h}} \rangle)^2 \rangle - 3(\langle N_h \rangle - \langle N_{\bar{h}} \rangle)^4 \\ &\quad - 3\langle (\Delta N_h)^2 \rangle + \langle (\Delta N_{\bar{h}})^2 \rangle - \langle \Delta N_h \Delta N_{\bar{h}} \rangle^2 \\ &= \langle N_h^4 \rangle - 4\langle N_h^3 N_{\bar{h}} \rangle + 6\langle N_{\bar{h}}^2 N_h^2 \rangle - 4\langle N_{\bar{h}}^3 N_h \rangle + \langle N_{\bar{h}}^4 \rangle \\ &\quad - 4\langle N_h^3 \rangle + 3\langle N_h^2 N_{\bar{h}} \rangle + 3\langle N_{\bar{h}}^2 N_h \rangle - \langle N_{\bar{h}}^3 \rangle (\langle N_h \rangle - \langle N_{\bar{h}} \rangle) \\ &\quad + 6\langle N_h^2 \rangle - 2\langle N_h N_{\bar{h}} \rangle + \langle N_{\bar{h}}^2 \rangle (\langle N_h \rangle^2 - 2\langle N_h \rangle \langle N_{\bar{h}} \rangle + \langle N_{\bar{h}} \rangle^2) \\ &\quad - 3(\langle N_h \rangle^4 - 4\langle N_h \rangle^3 \langle N_{\bar{h}} \rangle + 6\langle N_{\bar{h}} \rangle^2 \langle N_h \rangle^2 - 4\langle N_{\bar{h}} \rangle^3 \langle N_h \rangle + \langle N_{\bar{h}} \rangle^4) \\ &\quad - 3\langle (\Delta N_h)^2 \rangle + \langle (\Delta N_{\bar{h}})^2 \rangle - \langle \Delta N_h \Delta N_{\bar{h}} \rangle^2 \\ &= \langle (\Delta N_h)^4 \rangle - 3\langle (\Delta N_h)^2 \rangle^2 \\ &\quad + \langle (\Delta N_{\bar{h}})^4 \rangle - 3\langle (\Delta N_{\bar{h}})^2 \rangle^2 \\ &\quad - 4\langle (\Delta N_h)^3 \Delta N_{\bar{h}} \rangle - 3\langle (\Delta N_h)^2 \rangle \langle (\Delta N_h) \Delta N_{\bar{h}} \rangle \\ &\quad + 6\langle (\Delta N_h)^2 \Delta N_{\bar{h}}^2 \rangle - 2\langle (\Delta N_h) \Delta N_{\bar{h}} \rangle^2 - \langle (\Delta N_h)^2 \rangle \langle (\Delta N_{\bar{h}})^2 \rangle \\ &\quad - 4\langle (\Delta N_{\bar{h}})^3 \Delta N_h \rangle - 3\langle (\Delta N_{\bar{h}})^2 \rangle \langle (\Delta N_h) \Delta N_{\bar{h}} \rangle. \end{aligned} \quad (\text{E.4})$$

For simplify, the charge "+" particles can be expressed as $\sum_i h_+^i$ and $\sum_i h_-^i$ denotes the the charge "-" particles. The formulas (E.1)-(E.4) can be rearranged as,

$$M_{h-\bar{h}} = \left\langle \sum_i N_{h_+^i} \right\rangle - \left\langle \sum_j N_{h_-^j} \right\rangle = \sum_i \langle N_{h_+^i} \rangle - \sum_j \langle N_{h_-^j} \rangle, \quad (\text{E.5})$$

$$\begin{aligned} \sigma_{h-\bar{h}}^2 &= \left\langle \left(\sum_i \Delta N_{h_+^i} \right)^2 \right\rangle + \left\langle \left(\sum_j \Delta N_{h_-^j} \right)^2 \right\rangle - 2 \left\langle \sum_i \Delta N_{h_+^i} \sum_j \Delta N_{h_-^j} \right\rangle \\ &= \sum_i \left\langle \left(\Delta N_{h_+^i} \right)^2 \right\rangle + \sum_{i \neq i'} \left\langle \Delta N_{h_+^i} \Delta N_{h_+^{i'}} \right\rangle + \sum_j \left\langle \left(\Delta N_{h_-^j} \right)^2 \right\rangle + \sum_{j \neq j'} \left\langle \Delta N_{h_-^j} \Delta N_{h_-^{j'}} \right\rangle \\ &\quad - 2 \sum_{i,j} \left\langle \Delta N_{h_+^i} \Delta N_{h_-^j} \right\rangle, \end{aligned} \quad (\text{E.6})$$

$$\begin{aligned} \langle (\Delta N_{h-\bar{h}})^3 \rangle &= \left\langle \left(\sum_i \Delta N_{h_+^i} \right)^3 \right\rangle - \left\langle \left(\sum_j \Delta N_{h_-^j} \right)^3 \right\rangle - 3 \left\langle \left(\sum_i \Delta N_{h_+^i} \right)^2 \left(\sum_j \Delta N_{h_-^j} \right) \right\rangle \\ &\quad + 3 \left\langle \left(\sum_i \Delta N_{h_+^i} \right) \left(\sum_j \Delta N_{h_-^j} \right)^2 \right\rangle \\ &= \sum_i \left\langle \left(\Delta N_{h_+^i} \right)^3 \right\rangle + 3 \sum_{i \neq i'} \left\langle \left(\Delta N_{h_+^i} \right)^2 \Delta N_{h_+^{i'}} \right\rangle + \sum_{i \neq i' \neq i''} \left\langle \Delta N_{h_+^i} \Delta N_{h_+^{i'}} \Delta N_{h_+^{i''}} \right\rangle \\ &\quad - \sum_j \left\langle \left(\Delta N_{h_-^j} \right)^3 \right\rangle - 3 \sum_{j \neq j'} \left\langle \left(\Delta N_{h_-^j} \right)^2 \Delta N_{h_-^{j'}} \right\rangle - \sum_{j \neq j' \neq j''} \left\langle \Delta N_{h_-^j} \Delta N_{h_-^{j'}} \Delta N_{h_-^{j''}} \right\rangle \\ &\quad - 3 \sum_{i,j} \left\langle \left(\Delta N_{h_+^i} \right)^2 \left(\Delta N_{h_-^j} \right) \right\rangle - 3 \sum_{i \neq i', j} \left\langle \left(\Delta N_{h_+^i} \right) \left(\Delta N_{h_+^{i'}} \right) \left(\Delta N_{h_-^j} \right) \right\rangle \\ &\quad + 3 \sum_{i,j} \left\langle \left(\Delta N_{h_-^j} \right)^2 \left(\Delta N_{h_+^i} \right) \right\rangle + 3 \sum_{j \neq j', i} \left\langle \left(\Delta N_{h_-^j} \right) \left(\Delta N_{h_-^{j'}} \right) \left(\Delta N_{h_+^i} \right) \right\rangle, \end{aligned} \quad (\text{E.7})$$

$$\begin{aligned}
& \langle (\Delta N_{h-\bar{h}})^4 \rangle - 3 \langle (\Delta N_{h-\bar{h}})^2 \rangle = \left\langle \left(\sum_i \Delta N_{h_+^i} \right)^4 \right\rangle - 3 \left\langle \left(\sum_i \Delta N_{h_+^i} \right)^2 \right\rangle^2 \\
& + \left\langle \left(\sum_j \Delta N_{h_-^j} \right)^4 \right\rangle - 3 \left\langle \left(\sum_j \Delta N_{h_-^j} \right)^2 \right\rangle^2 \\
& - 4 \left(\left\langle \left(\sum_i \Delta N_{h_+^i} \right)^3 \left(\sum_j \Delta N_{h_-^j} \right) \right\rangle - 3 \left\langle \left(\sum_i \Delta N_{h_+^i} \right)^2 \right\rangle \left\langle \left(\sum_j \Delta N_{h_-^j} \right) \right\rangle \right) \\
& + 6 \left(\left\langle \left(\sum_i \Delta N_{h_+^i} \right)^2 \left(\sum_j \Delta N_{h_-^j} \right)^2 \right\rangle - 2 \left\langle \left(\sum_i \Delta N_{h_+^i} \right) \left(\sum_j \Delta N_{h_-^j} \right) \right\rangle^2 \right. \\
& \quad \left. - \left\langle \left(\sum_i \Delta N_{h_+^i} \right)^2 \right\rangle \left\langle \left(\sum_j \Delta N_{h_-^j} \right)^2 \right\rangle \right) \\
& - 4 \left(\left\langle \left(\sum_j \Delta N_{h_-^j} \right)^3 \sum_i \Delta N_{h_+^i} \right\rangle - 3 \left\langle \left(\sum_j \Delta N_{h_-^j} \right)^2 \right\rangle \left\langle \left(\sum_i \Delta N_{h_+^i} \right) \right\rangle \right) \\
= & \sum_i \left\langle (\Delta N_{h_+^i})^4 \right\rangle + 4 \sum_{i \neq i'} \left\langle (\Delta N_{h_+^i})^3 \Delta N_{h_+^{i'}} \right\rangle + 3 \sum_{i \neq i'} \left\langle (\Delta N_{h_+^i})^2 (\Delta N_{h_+^{i'}})^2 \right\rangle \\
& + 6 \sum_{i \neq i' \neq i''} \left\langle (\Delta N_{h_+^i})^2 \Delta N_{h_+^{i'}} \Delta N_{h_+^{i''}} \right\rangle \\
& + \sum_{i \neq i' \neq i'' \neq i'''} \left\langle \Delta N_{h_+^i} \Delta N_{h_+^{i'}} \Delta N_{h_+^{i''}} \Delta N_{h_+^{i'''}} \right\rangle - 3 \left\langle \sum_i (\Delta N_{h_+^i})^2 + \sum_{i \neq i'} \Delta N_{h_+^i} \Delta N_{h_+^{i'}} \right\rangle^2 \\
& + \sum_j \left\langle (\Delta N_{h_-^j})^4 \right\rangle + 4 \sum_{j \neq j'} \left\langle (\Delta N_{h_-^j})^3 \Delta N_{h_-^{j'}} \right\rangle + 3 \sum_{j \neq j'} \left\langle (\Delta N_{h_-^j})^2 (\Delta N_{h_-^{j'}})^2 \right\rangle \\
& + 6 \sum_{j \neq j' \neq j''} \left\langle (\Delta N_{h_-^j})^2 \Delta N_{h_-^{j'}} \Delta N_{h_-^{j''}} \right\rangle \\
& + \sum_{j \neq j' \neq j'' \neq j'''} \left\langle \Delta N_{h_-^j} \Delta N_{h_-^{j'}} \Delta N_{h_-^{j''}} \Delta N_{h_-^{j'''}} \right\rangle - 3 \left\langle \sum_j (\Delta N_{h_-^j})^2 + \sum_{j \neq j'} \Delta N_{h_-^j} \Delta N_{h_-^{j'}} \right\rangle^2 \\
& + \text{etc} .
\end{aligned} \tag{E.8}$$

Here we take the positive charged particles as an example,

$$\begin{aligned}
& \left\langle \left(\sum_i \Delta N_{h_+^i} \right)^4 \right\rangle - 3 \left\langle \left(\sum_i \Delta N_{h_+^i} \right)^2 \right\rangle^2 \\
&= \sum_i \left\langle \left(\Delta N_{h_+^i} \right)^4 \right\rangle + 4 \sum_{i \neq i'} \left\langle \left(\Delta N_{h_+^i} \right)^3 \Delta N_{h_+^{i'}} \right\rangle + 3 \sum_{i \neq i'} \left\langle \left(\Delta N_{h_+^i} \right)^2 \left(\Delta N_{h_+^{i'}} \right)^2 \right\rangle \\
&+ 6 \sum_{i \neq i' \neq i''} \left\langle \left(\Delta N_{h_+^i} \right)^2 \Delta N_{h_+^{i'}} \Delta N_{h_+^{i''}} \right\rangle + \sum_{i \neq i' \neq i'' \neq i'''} \left\langle \Delta N_{h_+^i} \Delta N_{h_+^{i'}} \Delta N_{h_+^{i''}} \Delta N_{h_+^{i'''}} \right\rangle \\
&- 3 \left\langle \sum_i \left(\Delta N_{h_+^i} \right)^2 + \sum_{i \neq i'} \Delta N_{h_+^i} \Delta N_{h_+^{i'}} \right\rangle \left\langle \sum_{i^*} \left(\Delta N_{h_+^{i^*}} \right)^2 + \sum_{i^* \neq i'^*} \Delta N_{h_+^{i^*}} \Delta N_{h_+^{i'^*}} \right\rangle \\
&= \sum_i \left\langle \left(\Delta N_{h_+^i} \right)^4 \right\rangle + 4 \sum_{i \neq i'} \left\langle \left(\Delta N_{h_+^i} \right)^3 \Delta N_{h_+^{i'}} \right\rangle + 3 \sum_{i \neq i'} \left\langle \left(\Delta N_{h_+^i} \right)^2 \left(\Delta N_{h_+^{i'}} \right)^2 \right\rangle \\
&+ 6 \sum_{i \neq i' \neq i''} \left\langle \left(\Delta N_{h_+^i} \right)^2 \Delta N_{h_+^{i'}} \Delta N_{h_+^{i''}} \right\rangle + \sum_{i \neq i' \neq i'' \neq i'''} \left\langle \Delta N_{h_+^i} \Delta N_{h_+^{i'}} \Delta N_{h_+^{i''}} \Delta N_{h_+^{i'''}} \right\rangle \\
&- 3 \sum_i \left\langle \left(\Delta N_{h_+^i} \right)^2 \right\rangle^2 - 3 \sum_{i \neq i'} \left\langle \left(\Delta N_{h_+^i} \right)^2 \right\rangle \left\langle \left(\Delta N_{h_+^{i'}} \right)^2 \right\rangle \\
&- 6 \sum_{i \neq i'} \left\langle \Delta N_{h_+^i} \Delta N_{h_+^{i'}} \right\rangle \left\langle \left(\Delta N_{h_+^i} \right)^2 \right\rangle - 6 \sum_{i \neq i' \neq i''} \left\langle \Delta N_{h_+^i} \Delta N_{h_+^{i''}} \right\rangle \left\langle \left(\Delta N_{h_+^i} \right)^2 \right\rangle \\
&- 6 \sum_{i \neq i'} \left\langle \Delta N_{h_+^i} \Delta N_{h_+^{i'}} \right\rangle^2 - 6 \sum_{i \neq i' \neq i''} \left\langle \Delta N_{h_+^i} \Delta N_{h_+^{i'}} \right\rangle \left\langle \Delta N_{h_+^i} \Delta N_{h_+^{i''}} \right\rangle \\
&- 3 \sum_{i \neq i' \neq i'' \neq i'''} \left\langle \Delta N_{h_+^i} \Delta N_{h_+^{i'}} \right\rangle \left\langle \Delta N_{h_+^{i''}} \Delta N_{h_+^{i'''}} \right\rangle. \tag{E.9}
\end{aligned}$$

To be specific, we choose

$$C_2(N_1, N_2) = \langle \Delta N_1 \Delta N_2 \rangle, \tag{E.10}$$

$$C_3(N_1, N_2, N_3) = \langle \Delta N_1 \Delta N_2 \Delta N_3 \rangle, \tag{E.11}$$

$$\begin{aligned}
C_4(N_1, N_2, N_3, N_4) &= \langle \Delta N_1 \Delta N_2 \Delta N_3 \Delta N_4 \rangle - \langle \Delta N_1 \Delta N_2 \rangle \langle \Delta N_3 \Delta N_4 \rangle \\
&- \langle \Delta N_1 \Delta N_3 \rangle \langle \Delta N_2 \Delta N_4 \rangle - \langle \Delta N_1 \Delta N_4 \rangle \langle \Delta N_2 \Delta N_3 \rangle. \tag{E.12}
\end{aligned}$$

Then, we have

$$\left\langle \left(\sum_i \Delta N_{h_+^i} \right)^2 \right\rangle = \sum_i C_2(N_i, N_i) + \sum_{i \neq i'} C_2(N_i, N_{i'}), \tag{E.13}$$

$$\begin{aligned}
\left\langle \left(\sum_i \Delta N_{h_+^i} \right)^3 \right\rangle &= \sum_i C_3(N_i, N_i, N_i) + 3 \sum_{i \neq i'} C_3(N_i, N_i, N_{i'}) \\
&+ \sum_{i \neq i' \neq i''} C_3(N_i, N_{i'}, N_{i''}), \tag{E.14}
\end{aligned}$$

$$\begin{aligned}
\left\langle \left(\sum_i \Delta N_{h_+^i} \right)^4 \right\rangle - 3 \left\langle \left(\sum_i \Delta N_{h_+^i} \right)^2 \right\rangle^2 &= \sum_i C_4(N_i, N_i, N_i, N_i) + 4 \sum_{i \neq i'} C_4(N_i, N_i, N_i, N_{i'}) \\
&+ 3 \sum_{i \neq i'} C_4(N_i, N_i, N_{i'}, N_{i'}) + 6 \sum_{i \neq i' \neq i''} C_4(N_i, N_i, N_{i''}, N_{i'}) \\
&+ \sum_{i \neq i' \neq i'' \neq i'''} C_4(N_i, N_{i'}, N_{i''}, N_{i'''}). \tag{E.15}
\end{aligned}$$

After considering the symmetry, Eqs. (E.13)-(E.15) can be rewritten as

$$\left\langle \left(\sum_i \Delta N_{h_+^i} \right)^2 \right\rangle = \sum_i C_2(N_i, N_i) + 2 \sum_{i > i'} C_2(N_i, N_{i'}) \tag{E.16}$$

$$\begin{aligned}
\left\langle \left(\sum_i \Delta N_{h_+^i} \right)^3 \right\rangle &= \sum_i C_3(N_i, N_i, N_i) + 3 \sum_{i \neq i'} C_3(N_i, N_i, N_{i'}) \\
&+ 6 \sum_{i > i' > i''} C_3(N_i, N_{i'}, N_{i''}) \tag{E.17}
\end{aligned}$$

$$\begin{aligned}
\left\langle \left(\sum_i \Delta N_{h_+^i} \right)^4 \right\rangle - 3 \left\langle \left(\sum_i \Delta N_{h_+^i} \right)^2 \right\rangle^2 &= \sum_i C_4(N_i, N_i, N_i, N_i) + 4 \sum_{i \neq i'} C_4(N_i, N_i, N_i, N_{i'}) \\
&+ 6 \sum_{i > i'} C_4(N_i, N_i, N_{i'}, N_{i'}) + 12 \sum_{i \neq i' \neq i'', i' > i''} C_4(N_i, N_i, N_{i'}, N_{i''}) \\
&+ 24 \sum_{i > i' > i'' > i'''} C_4(N_i, N_{i'}, N_{i''}, N_{i'''}) \tag{E.18}
\end{aligned}$$