

Nilpotency of the b ghost in the non-minimal pure spinor formalism

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ABSTRACT: The b ghost in the non-minimal pure spinor formalism is not a fundamental field. It is based on a complicated chain of operators and proving its nilpotency is nontrivial. Chandia proved this property in arXiv:1008.1778, but with an assumption on the non-minimal variables that is not valid in general. In this work, the b ghost is demonstrated to be nilpotent without this assumption.

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1 Introduction

The super Poincaré covariant quantization of the superstring was achieved in the year 2000, with the development of the pure spinor formalism [1]. One of its oddest features is the absence of a natural prescription for string loop amplitudes, that is manifest in the other formalisms of bosonic and supersymmetric strings, due to the existence of the world-sheet reparametrization invariance.

It is a well known fact that in gauge fixing the reparametrization symmetry, a (b, c) system rises as the ghost-antighost pair. The c ghost is a conformal weight -1 field, as it comes from the general coordinate transformation parameter, and the b ghost, the conjugate of c , is a conformal weight $+2$ field.

Concerning amplitudes, the fundamental objects of study in quantum strings, the c ghost appears at tree and 1-loop level. In these world-sheet topologies (respectively, the sphere and the torus), the conformal Killing symmetries can be removed by fixing some vertex positions. For the pure spinor formalism, Berkovits developed a prescription [1–3] that successfully described superstring amplitudes, where a possible c ghost played no role at all.

For the b ghost, however, the story is different. In a BRST-like description, b ghost insertions lie in the heart of the BRST invariance of string loop amplitudes. The fundamental property is $\{Q, b\} = T$, where T is the energy-momentum tensor (since the BRST charge has ghost number $+1$, the b ghost must have ghost number -1). Combined with the Beltrami differentials, this property induces only a surface contribution in the *moduli* space integration.

In the minimal pure spinor formalism, where the available ghost variables are the pure spinor λ^α and its conjugate ω_α , the b ghost is based upon a complicated chain of operators

and can be implemented only in a picture raised manner [2], as there are no suitable ghost number -1 fields.

With the addition of the ghost fields $(\bar{\lambda}_\alpha, r_\alpha)$ and their conjugates $(\bar{\omega}^\alpha, s^\alpha)$, the non-minimal pure spinor formalism enables a much simpler construction of the b ghost [3]. More than that, the theory can be interpreted as a twisted $\mathcal{N} = 2$ $\hat{c} = 3$ topological string, where the BRST charge and the b ghost are the fermionic generators, while the ghost number current and the energy-momentum tensor are the bosonic ones. This fact allowed the covariant computation of multiloop superstring amplitudes without picture changing operators, making the super Poincaré symmetry explicit in all the steps.

Since the b ghost is a composite field, its nilpotency, a crucial property in the topological string interpretation, is not evident. In [4], the regularity of the b ghost OPE with itself was derived, but in an incomplete manner,¹ as will be explained here. Therefore, a rigorous proof of such a fundamental property was still lacking.

This paper is organized as follows. Section 2 contains a review of the pure spinor formalisms and section 3 presents the construction of the b ghost, its basic properties and a rigorous derivation of the b ghost OPE with itself, proving its regularity. Appendix A contains the conventions that are being used in this work and some ordering considerations.

2 Review of the pure spinor formalism

The pure spinor formalism will be reviewed here, establishing the fundamental fields that will be used in the remaining sections.

2.1 Matter fields

The matter part of the action is

$$S_m = \frac{1}{2\pi} \int d^2z \left(\frac{1}{\alpha'} \partial X^m \bar{\partial} X_m + p_\beta \bar{\partial} \theta^\beta \right), \quad (2.1)$$

and the free field propagators are just

$$X^m(z, \bar{z}) X^n(y, \bar{y}) \sim -\frac{\alpha'}{2} \eta^{mn} \ln |z - y|^2 \quad (2.2a)$$

$$p_\alpha(z) \theta^\beta(y) \sim \frac{\delta_\alpha^\beta}{z - y}. \quad (2.2b)$$

The action S_m is invariant under the supersymmetric charge, defined as

$$q_\alpha = \oint \left[p_\alpha + \frac{1}{\alpha'} \partial X^m (\theta \gamma_m)_\alpha + \frac{1}{12\alpha'} (\theta \gamma_m \partial \theta) (\theta \gamma_m)_\alpha \right]. \quad (2.3)$$

Note that

$$\alpha' \{q_\alpha, q_\beta\} = 2\gamma_{\alpha\beta}^m \oint \partial X_m. \quad (2.4)$$

¹The flaw in the proof of [4] was pointed out by N. Berkovits, in a private communication.

The construction of the supersymmetric invariants follows:

$$\Pi^m = \partial X^m + \frac{1}{2} (\theta \gamma^m \partial \theta), \quad (2.5a)$$

$$d_\alpha = p_\alpha - \frac{1}{\alpha'} \partial X^m (\theta \gamma_m)_\alpha - \frac{1}{4\alpha'} (\theta \gamma^m \partial \theta) (\theta \gamma_m)_\alpha. \quad (2.5b)$$

So far, this is nothing but the left-moving sector of the Green-Schwarz-Siegel action in the conformal gauge. The Virasoro constraint is $\Pi^m \Pi_m + \alpha' d_\alpha \partial \theta^\alpha = 0$ and the fermionic constraints (related to *kappa* symmetry) are $d_\alpha = 0$.

The related OPE's are given by:

$$\Pi^m(z) \Pi^n(y) \sim -\frac{\alpha'}{2} \frac{\eta^{mn}}{(z-y)^2}, \quad (2.6a)$$

$$d_\alpha(z) \Pi^m(y) \sim \frac{\gamma_{\alpha\beta}^m \partial \theta^\beta}{(z-y)}, \quad (2.6b)$$

$$d_\alpha(z) d_\beta(y) \sim -\frac{2}{\alpha'} \frac{\gamma_{\alpha\beta}^m \Pi_m}{(z-y)}. \quad (2.6c)$$

The matter energy-momentum tensor (Virasoro constraint) is

$$T_{\text{matter}} = -\frac{1}{\alpha'} \partial X^m \partial X_m - p_\alpha \partial \theta^\alpha, \quad (2.7)$$

from which it follows that

$$T_{\text{matter}}(z) T_{\text{matter}}(y) \sim -\frac{11}{(z-y)^4} + 2 \frac{T_{\text{matter}}}{(z-y)^2} + \frac{\partial T_{\text{matter}}}{(z-y)}. \quad (2.8)$$

Therefore, the free matter action yields a negative central charge, that will be cancelled with the contribution coming from the ghost sector.

2.2 Ghost fields

Introducing a pure spinor λ^α variable and its conjugate, ω_α , one is able to define

$$J_{\text{BRST}} \equiv \lambda^\alpha d_\alpha, \quad (2.9)$$

and construct a BRST like charge,

$$Q = \oint J_{\text{BRST}}, \quad (2.10)$$

where

$$\{Q, Q\} = -\frac{2}{\alpha'} \oint (\lambda \gamma^m \lambda) \Pi_m = 0, \quad (2.11)$$

and

$$\lambda \gamma^m \lambda = 0 \quad (2.12)$$

is the $D = 10$ pure spinor constraint, which implies that only 11 components of λ^α are independent. Observe that an explicitly Lorentz invariant action for the ghost sector,

$$S_\lambda = \frac{1}{2\pi} \int d^2 z (\omega_\alpha \bar{\partial} \lambda^\alpha), \quad (2.13)$$

must be gauge invariant under $\delta_\epsilon \omega_\alpha = \epsilon_m (\gamma^m \lambda)_\alpha$, due to (2.12).

The simplest gauge invariant objects are

$$T_\lambda = -\omega \partial \lambda, \quad N^{mn} = -\frac{1}{2} \omega \gamma^{mn} \lambda, \quad J_\lambda = -\omega \lambda,$$

respectively, the energy-momentum tensor, the Lorentz current and the ghost number current.

The full set of OPE's of the ghost sector is:

$$\begin{aligned} T_\lambda(z) T_\lambda(y) &\sim \frac{11}{(z-y)^4} + 2 \frac{T_\lambda}{(z-y)^2} + \frac{\partial T_\lambda}{(z-y)}, \\ J_\lambda(z) T_\lambda(y) &\sim -\frac{8}{(z-y)^3} + \frac{J_\lambda}{(z-y)^2}, \\ N^{mn}(z) T_\lambda(y) &\sim \frac{N^{mn}}{(z-y)^2}, \quad T_\lambda(z) \lambda^\alpha(y) \sim \frac{\partial \lambda^\alpha}{(z-y)}, \quad N^{mn}(z) \lambda^\alpha(y) \sim \frac{1}{2} \frac{(\gamma^{mn} \lambda)^\alpha}{(z-y)}, \\ N^{mn}(z) J_\lambda(y) &\sim \text{regular}, \quad J_\lambda(z) \lambda^\alpha(y) \sim \frac{\lambda^\alpha}{(z-y)}, \quad J_\lambda(z) J_\lambda(y) \sim -\frac{4}{(z-y)^2}, \\ N^{mn}(z) N^{pq}(y) &\sim 6 \frac{\eta^{m[p} \eta^{q]n}}{(z-y)^2} + 2 \frac{\eta^{m[q} N^{p]n} + \eta^{n[p} N^{q]m}}{(z-y)}. \end{aligned}$$

The non-minimal version of the pure spinor formalism includes a new set of fields, $(\bar{\lambda}_\alpha, r_\alpha)$. The former is also a pure spinor, that is

$$\bar{\lambda} \gamma^m \bar{\lambda} = 0, \tag{2.14}$$

whereas the latter is a fermionic spinor constrained through

$$\bar{\lambda} \gamma^m r = 0. \tag{2.15}$$

Both constraints imply that there are only 11 independent components in each spinor. Denoting their conjugates as $(\bar{\omega}^\alpha, s^\alpha)$, the action for the non-minimal sector is

$$S_{\bar{\lambda}} = \frac{1}{2\pi} \int d^2 z \left(\bar{\omega}^\alpha \bar{\partial} \bar{\lambda}_\alpha + s^\alpha \bar{\partial} r_\alpha \right), \tag{2.16}$$

which is gauge invariant by the following transformations

$$\begin{aligned} \delta_{\epsilon, \phi} \bar{\omega}^\alpha &= \epsilon^m (\gamma_m \bar{\lambda})^\alpha + \phi^m (\gamma_m r)^\alpha, \\ \delta_\phi s^\alpha &= \phi^m (\gamma_m \bar{\lambda})^\alpha. \end{aligned} \tag{2.17}$$

There are several gauge invariant quantities that can be built out of $\bar{\omega}^\alpha$ and s^α .

$$\begin{aligned} \bar{N}^{mn} &= \frac{1}{2} (\bar{\lambda} \gamma^{mn} \bar{\omega} - r \gamma^{mn} s), \\ J_{\bar{\lambda}} &= -\bar{\lambda} \bar{\omega}, & T_{\bar{\lambda}} &= -\bar{\omega} \partial \bar{\lambda} - s \partial r, & \Phi &= r \bar{\omega}, \\ S &= \bar{\lambda} s, & S^{mn} &= \frac{1}{2} \bar{\lambda} \gamma^{mn} s, & J_r &= r s. \end{aligned} \tag{2.18}$$

Here, $\bar{N}^{mn}$ is the Lorentz generator, $T_{\bar{\lambda}}$ is the energy-momentum tensor, and $J_{\bar{\lambda}}$ and J_r are the ghost number currents. Note that they are not all independent,² since

$$S^{mn} \left(\frac{r\gamma_{mn}\lambda}{\bar{\lambda}\lambda} \right) + S \left(\frac{r\lambda}{\bar{\lambda}\lambda} \right) - 4J_r = 0, \quad (2.19)$$

$$\bar{N}^{mn} \left(\frac{r\gamma_{mn}\lambda}{\bar{\lambda}\lambda} \right) - J_{\bar{\lambda}} \left(\frac{r\lambda}{\bar{\lambda}\lambda} \right) + 3J_r \left(\frac{r\lambda}{\bar{\lambda}\lambda} \right) + 4\Phi = 0. \quad (2.20)$$

The OPE's between them can be summarized as follows:

$$\begin{aligned} T_{\bar{\lambda}}(z) T_{\bar{\lambda}}(y) &\sim 2 \frac{T_{\bar{\lambda}}}{(z-y)^2} + \frac{\partial T_{\bar{\lambda}}}{(z-y)}, & \bar{N}^{mn}(z) T_{\bar{\lambda}}(y) &\sim \frac{\bar{N}^{mn}}{(z-y)^2}, \\ S^{mn}(z) T_{\bar{\lambda}}(y) &\sim \frac{S^{mn}}{(z-y)^2}, \\ J_{\bar{\lambda}}(z) T_{\bar{\lambda}}(y) &\sim -\frac{11}{(z-y)^3} + \frac{\bar{J}_{\bar{\lambda}}}{(z-y)^2}, & \Phi(z) T_{\bar{\lambda}}(y) &\sim \frac{\Phi}{(z-y)^2}, \\ S(z) T_{\bar{\lambda}}(y) &\sim \frac{S}{(z-y)^2}, \\ J_r(z) T_{\bar{\lambda}}(y) &\sim \frac{11}{(z-y)^3} + \frac{J_r}{(z-y)^2}, & T_{\bar{\lambda}}(z) \bar{\lambda}_{\alpha}(y) &\sim \frac{\partial \bar{\lambda}_{\alpha}}{(z-y)}, \\ T_{\bar{\lambda}}(z) r_{\alpha}(y) &\sim \frac{\partial r_{\alpha}}{(z-y)}, \\ \Phi(z) S(y) &\sim -\frac{8}{(z-y)^2} - \frac{J_{\bar{\lambda}} + J_r}{(z-y)}, & \Phi(z) S^{mn}(y) &\sim \frac{\bar{N}^{mn}}{(z-y)}, \\ \Phi(z) \bar{\lambda}_{\alpha}(y) &\sim -\frac{r_{\alpha}}{(z-y)}, \\ \Phi(z) \Phi(y) &\sim \text{regular}, & \bar{N}^{mn}(z) J_{\bar{\lambda}}(y) &\sim \text{regular}, \\ \bar{N}^{mn}(z) \Phi(y) &\sim \text{regular}, \\ \bar{N}^{mn}(z) \bar{N}^{pq}(y) &\sim 2 \frac{\eta^{m[q} \bar{N}^{p]n} + \eta^{n[p} \bar{N}^{q]m}}{(z-y)}, & J_{\bar{\lambda}}(z) J_r(y) &\sim -\frac{3}{(z-y)^2}, \\ J_{\bar{\lambda}}(z) J_{\bar{\lambda}}(y) &\sim -\frac{5}{(z-y)^2}, \\ \bar{N}^{mn}(z) J_r(y) &\sim \text{regular}, & \bar{N}^{mn}(z) S(y) &\sim \text{regular}, \\ J_r(z) J_r(y) &\sim \frac{11}{(z-y)^2}, \\ \bar{N}^{mn}(z) \bar{\lambda}_{\alpha}(y) &\sim -\frac{1}{2} \frac{(\bar{\lambda}\gamma^{mn})_{\alpha}}{(z-y)}, & \bar{N}^{mn}(z) r_{\alpha}(y) &\sim -\frac{1}{2} \frac{(r\gamma^{mn})_{\alpha}}{(z-y)}, \\ J_{\bar{\lambda}}(z) \bar{\lambda}_{\alpha}(y) &\sim \frac{\bar{\lambda}_{\alpha}}{(z-y)}, \end{aligned}$$

²Some numerical coefficients of the corresponding relations in [3] are incorrect, as can be promptly verified by the definitions in (2.18) and the gamma matrices identity (A.8).

$$J_r(z) r_\alpha(y) \sim \frac{r_\alpha(y)}{(z-y)}, \quad J_{\bar{\lambda}}(z) r_\alpha(y) \sim \text{regular},$$

$$J_r(z) \bar{\lambda}_\alpha(y) \sim \text{regular}.$$

Note that there are no contributions to the central charge or to the level of the Lorentz algebra.³

The non-minimal variables enter the formalism in a very simple way, as the BRST charge is defined to be

$$Q \equiv \oint \underbrace{(\lambda^\alpha d_\alpha + \Phi)}_{J_{BRST}(z)}. \quad (2.21)$$

The same notation was used for the BRST charge in the minimal formalism, but from now on, only (2.21) will be referred to as Q . The cohomology of (2.21) is independent of $(\bar{\lambda}, \bar{\omega}, r, s)$, as can be seen from the quartet argument, and there is a state ξ that trivializes it:

$$\xi = \frac{\bar{\lambda} \cdot \theta}{\bar{\lambda} \cdot \lambda - r \cdot \theta}, \quad \{Q, \xi\} = 1. \quad (2.22)$$

Since r_α and θ^α are grassmannian variables, ξ can be expanded as a finite power series in terms of $r \cdot \theta$. Besides, r_α has only 11 independent components, in such a way that

$$\xi = \frac{\bar{\lambda} \cdot \theta}{\bar{\lambda} \cdot \lambda} \sum_{n=0}^{11} \left(\frac{r \cdot \theta}{\bar{\lambda} \cdot \lambda} \right)^n. \quad (2.23)$$

Therefore, one way of avoiding the appearance of ξ is limiting the amount of inverse powers of $\bar{\lambda}\lambda$. However, this is a fundamental ingredient in the construction of the b ghost, constituting the main obstruction for loop amplitude calculations in the pure spinor formalism [3, 5].

3 The b ghost

The b ghost is a central field in string perturbation theory, being related to the BRST invariance of string loop amplitudes. Its basic property is

$$\{Q, b\} = T. \quad (3.1)$$

As there is not such a fundamental object in the pure spinor formalism, it must be build out of the available fields of the theory.

As introduced in [3], the construction of the non-minimal b ghost is based on a chain of operators satisfying some special relations, that will be reviewed below.

³The quadratic pole in $\Phi(z) S(y)$ is also absent in [3].

3.1 Definition and properties

The full quantum version of the b ghost can be cast as

$$b = b_{-1} + b_0 + b_1 + b_2 + b_3, \quad (3.2)$$

where

$$b_{-1} \equiv -s^\alpha \partial \bar{\lambda}_\alpha, \quad (3.3a)$$

$$b_0 \equiv \left(\frac{\bar{\lambda}_\alpha}{(\bar{\lambda}\lambda)}, G^\alpha \right) + O, \quad (3.3b)$$

$$b_1 \equiv -2! \left(\frac{\bar{\lambda}_\alpha r_\beta}{(\bar{\lambda}\lambda)^2}, H^{\alpha\beta} \right), \quad (3.3c)$$

$$b_2 \equiv -3! \left(\frac{\bar{\lambda}_\alpha r_\beta r_\gamma}{(\bar{\lambda}\lambda)^3}, K^{\alpha\beta\gamma} \right), \quad (3.3d)$$

$$b_3 \equiv 4! \left(\frac{\bar{\lambda}_\alpha r_\beta r_\gamma r_\lambda}{(\bar{\lambda}\lambda)^4}, L^{\alpha\beta\gamma\lambda} \right), \quad (3.3e)$$

and

$$O \equiv -\partial \left(\frac{\bar{\lambda}_\alpha \bar{\lambda}_\beta}{(\bar{\lambda}\lambda)^2} \right) \lambda^\alpha \partial \theta^\beta, \quad (3.4a)$$

$$G^\alpha = \frac{1}{2} \gamma_m^{\alpha\beta} (\Pi^m, d_\beta) - \frac{1}{4} N_{mn} (\gamma^{mn} \partial \theta)^\alpha - \frac{1}{4} J_\lambda \partial \theta^\alpha + 4 \partial^2 \theta^\alpha, \quad (3.4b)$$

$$H^{\alpha\beta} = \frac{1}{4 \cdot 96} \gamma_{mnp}^{\alpha\beta} \left(\frac{\alpha'}{2} d \gamma^{mnp} d + 24 N^{mn} \Pi^p \right), \quad (3.4c)$$

$$K^{\alpha\beta\gamma} = -\frac{1}{96} \left(\frac{\alpha'}{2} \right) N_{mn} \gamma_{mnp}^{[\alpha\beta} (\gamma^p d)^\gamma], \quad (3.4d)$$

$$L^{\alpha\beta\gamma\lambda} = -\frac{3}{(96)^2} \left(\frac{\alpha'}{2} \right) (N^{mn}, N^{rs}) \eta^{pq} \gamma_{mnp}^{[\alpha\beta} \gamma_{rs}^{\gamma]\lambda}. \quad (3.4e)$$

Note that the subscript n in b_n is the r charge q_r of the operators, defined as

$$\int dz \{ J_r(z) \mathcal{O}(y) \} = q_r(\mathcal{O}) \mathcal{O}(y). \quad (3.5)$$

The building blocks of b_n satisfy:

$$\begin{aligned} \{Q, -s^\alpha \partial \bar{\lambda}_\alpha\} &= T_{\bar{\lambda}}, & \{Q, G^\alpha\} &= (\lambda^\alpha, T_\lambda + T_{\text{matter}}), \\ [Q, H^{\alpha\beta}] &= (\lambda^{[\alpha}, G^{\beta]}), & \{Q, K^{\alpha\beta\gamma}\} &= (\lambda^{[\alpha}, H^{\beta\gamma]}), \\ [Q, L^{\alpha\beta\gamma\lambda}] &= (\lambda^{[\alpha}, K^{\beta\gamma\lambda]}), & (\lambda^{[\alpha}, L^{\beta\gamma\lambda\sigma]} &= 0. \end{aligned} \quad (3.6)$$

Some observations should be made concerning the above operators:

- the ordering here, implemented through

$$(A, B)(y) \equiv \frac{1}{2\pi i} \oint \frac{dz}{z-y} A(z) B(y), \quad (3.7)$$

plays a major role, allowing a correct manipulation of the quantum corrections to the b ghost. Obviously, a different ordering prescription must not conflict with $\{Q, b\} = T$.

- the operator O defined above is required because

$$\left(\frac{(\bar{\lambda}_\alpha r_\beta - \bar{\lambda}_\beta r_\alpha) \lambda^\alpha}{(\bar{\lambda} \lambda)^2}, G^\beta \right) - \left(\frac{\bar{\lambda}_\alpha r_\beta}{(\bar{\lambda} \lambda)^2}, (\lambda^\alpha, G^\beta) - (\lambda^\beta, G^\alpha) \right) \neq 0, \quad (3.8)$$

and

$$\left(\frac{\bar{\lambda}_\alpha}{(\bar{\lambda} \lambda)}, (\lambda^\alpha, T_\lambda) \right) - T_\lambda \neq 0. \quad (3.9)$$

One can see that $\{Q, O\}$ precisely matches the above inequalities. In [6], besides (3.7), an alternative prescription was used, that conveniently absorbs the operator O .

- the quantum contribution to G^α is proportional to $\partial^2 \theta^\alpha$. The coefficient can be fixed by comparing the cubic pole in the OPE of the energy-momentum tensor with both sides of the equation $\{Q, G^\alpha\} = (\lambda^\alpha, T)$, or directly through the usual $U(5)$ decomposition.⁴

The last observation is directly related to the fact that the b ghost is a conformal weight 2 primary field [6],

$$T(z) b(y) \sim 2 \frac{b}{(z-y)^2} + \frac{\partial b}{(z-y)}. \quad (3.10)$$

This result is reproduced in appendix A. Note also that b is a Lorentz scalar and manifestly supersymmetric.

Another interesting property of the b ghost is the pole structure of its OPE with the BRST current:

$$J_{BRST}(z) b(y) \sim \frac{3}{(z-y)^3} + \frac{J}{(z-y)^2} + \frac{T}{(z-y)}, \quad (3.11)$$

where

$$\begin{aligned} J &= J_\lambda + J_r - 2 \frac{\bar{\lambda} \partial \lambda}{\bar{\lambda} \lambda} + 2 \frac{r \partial \theta}{\bar{\lambda} \lambda} - 2 \frac{(r \lambda) (\bar{\lambda} \partial \theta)}{(\bar{\lambda} \lambda)^2} \\ &= J_\lambda - J_{\bar{\lambda}} - \left\{ Q, \left(S + 2 \frac{\bar{\lambda} \partial \theta}{\bar{\lambda} \lambda} \right) \right\}. \end{aligned} \quad (3.12)$$

⁴That explains why the coefficient used here differs from the one used in [2, 3], where G^α was required to be primary. Equation (A.24) shows that this is not the case, since (λ^α, T) is not a primary field.

[6] contains a detailed discussion on this subject. There, G^α and $\hat{G}^\alpha$ denote the primary and the non-primary constructions, respectively.

With a BRST transformation, the $U(1)$ current can be brought into a more natural form, without changing the ghost numbers of the BRST charge and the b ghost.

To verify the interpretation of J as the ghost number current, it is worth noting that,

$$T(z)J(y) \sim -\frac{3}{(z-y)^3} + \frac{J}{(z-y)^2} + \frac{\partial J}{(z-y)}, \quad (3.13)$$

$$J(z)J_{BRST}(y) \sim \frac{J_{BRST}}{(z-y)}, \quad (3.14)$$

$$J(z)b(y) \sim -\frac{b}{(z-y)}. \quad (3.15)$$

Together, b , T , J_{BRST} and J may describe a twisted $\mathcal{N} = 2$ $\hat{c} = 3$ critical topological string [3]. The untwisted version would satisfy

$$\begin{aligned} T'(z)T'(y) &\sim \frac{(9/2)}{(z-y)^4} + 2\frac{T'}{(z-y)^2} + \frac{\partial T'}{(z-y)}, & T'(z)J(y) &\sim \frac{J}{(z-y)^2} + \frac{\partial J}{(z-y)}, \\ T'(z)G^+(y) &\sim \frac{3}{2}\frac{G^+}{(z-y)^2} + \frac{\partial G^+}{(z-y)}, & T'(z)G^-(y) &\sim \frac{3}{2}\frac{G^-}{(z-y)^2} + \frac{\partial G^-}{(z-y)}, \\ J(z)G^+(y) &\sim \frac{G^+(y)}{(z-y)}, & J(z)G^-(y) &\sim -\frac{G^-(y)}{(z-y)}, \\ J(z)J(y) &\sim \frac{3}{(z-y)^2}, \\ G^+(z)G^-(y) &\sim \frac{3}{(z-y)^3} + \frac{J}{(z-y)^2} + \frac{T' + \frac{1}{2}\partial J}{(z-y)}, \\ G^+(z)G^+(y) &\sim \text{regular}, & G^-(z)G^-(y) &\sim \text{regular}, \end{aligned}$$

where $G^+ = J_{BRST}$, $G^- = b$ and $T' = T - \frac{1}{2}\partial J$. The twist here means $T' \rightarrow T' - \frac{1}{2}\partial J$, which modifies the conformal weights of the ghosts λ and r from $\frac{1}{2}$ to 0 and turns the central charge off.

By examining this set of OPE's, one notes that b must be nilpotent in order for the non-minimal pure spinor formalism to be viewed as a topological string. This property will now be rigorously demonstrated.

3.2 Nilpotency

The OPE of the b ghost with itself can be cast as

$$b(z)b(y) \sim \frac{O_0}{(z-y)^4} + \frac{O_1}{(z-y)^3} + \frac{O_2}{(z-y)^2} + \frac{O_3}{(z-y)}, \quad (3.16)$$

for there are no (covariant, supersymmetric) negative conformal weight fields in the theory. Due to its anticommuting character, $b(z)b(y) = -b(y)b(z)$, implying that

$$b(z)b(y) \sim \frac{O_1}{(z-y)^3} + \frac{1}{2}\frac{\partial O_1}{(z-y)^2} + \frac{O_3}{(z-y)}. \quad (3.17)$$

Furthermore, since $\{Q, b\} = T$ and b is a primary field of conformal weight 2,

$$\begin{aligned} \{Q, b(z)\} b(y) - b(z) \{Q, b(y)\} &= T(z) b(y) - b(z) T(y) \\ &\sim \text{regular}, \end{aligned} \quad (3.18)$$

or, equivalently,

$$\{Q, b(z) b(y)\} \sim \frac{\{Q, O_1\}}{(z-y)^3} + \frac{1}{2} \frac{\partial \{Q, O_1\}}{(z-y)^2} + \frac{\{Q, O_3\}}{(z-y)}. \quad (3.19)$$

Comparing both expressions, one concludes that O_1 and O_3 are BRST closed.

Taking now into account the specific form of the b ghost for the non-minimal pure spinor formalism, given in (3.2), it is a simple task to verify that the cubic poles are all proportional to the constraints (2.14) and (2.15). The possible terms will be listed below:

- b_{-1} may give rise to cubic poles only in the OPE with b_3 , due to ordering effects. The different terms are proportional to

$$\begin{aligned} (\bar{\lambda} \gamma_{mnp} r) (\partial \bar{\lambda} \gamma^{pqr} r) (\bar{\lambda} \gamma^{mn} \lambda) (\bar{\lambda} \gamma_{qr} \lambda), \quad & (\bar{\lambda} \gamma_{mnp} r) (\partial \bar{\lambda} \gamma^{pqr} r) (\bar{\lambda} \gamma^{mn} \gamma_{qr} \lambda), \\ (\bar{\lambda} \gamma_{mnp} \partial \bar{\lambda}) (r \gamma^{pqr} r) (\bar{\lambda} \gamma^{mn} \lambda) (\bar{\lambda} \gamma_{qr} \lambda), \quad & (\bar{\lambda} \gamma_{mnp} \partial \bar{\lambda}) (r \gamma^{pqr} r) (\bar{\lambda} \gamma^{mn} \gamma_{qr} \lambda). \end{aligned} \quad (3.20)$$

- b_0 has cubic poles with itself, b_1 , b_2 and b_3 :
 - in $b_0(z) b_0(y)$, it comes from the multiple contractions of $\Pi^m (\gamma_m d)^\alpha$ with itself and from its single contraction with $\partial^2 \theta^\beta$, both proportional to $\Pi^m (\bar{\lambda} \gamma_m \bar{\lambda})$.
 - for $b_0(z) b_1(y)$, it will arise in the contractions of $(d \gamma^{mnp} d) (\bar{\lambda} \lambda)^{-2}$ with all the terms in b_0 , being proportional to $(\bar{\lambda} \gamma_{mnp} r) (\bar{\lambda} \gamma^{mnp} d)$.
 - in the OPE $b_0(z) b_2(y)$, the multiple contractions of $N^{mn} (\gamma^p d)^\alpha$ will give cubic poles like:

$$\begin{aligned} &(\bar{\lambda} \gamma_{mnp} r) N^{mn} (\bar{\lambda} \gamma^p r), \\ &(\bar{\lambda} \gamma_{mnp} r) J (\bar{\lambda} \gamma^{mn} \lambda) (\bar{\lambda} \gamma^p r), \\ &\partial [(\bar{\lambda} \gamma^{mn} \lambda) \bar{\lambda}_\alpha] (\bar{\lambda} \gamma_{mnp} r) (\gamma^p r)^\alpha. \end{aligned} \quad (3.21)$$

- finally, in $b_0(z) b_3(y)$, the cubic poles are of the form:

$$\begin{aligned} &(\bar{\lambda} \partial \theta) (\bar{\lambda} \gamma^{mn} \lambda) (\bar{\lambda} \gamma_{qr} \lambda) (\bar{\lambda} \gamma_{mnp} r) (r \gamma^{pqr} r), \\ &(\bar{\lambda} \gamma^{mn} \partial \theta) (\bar{\lambda} \gamma_{qr} \lambda) (\bar{\lambda} \gamma_{mnp} r) (r \gamma^{pqr} r). \end{aligned} \quad (3.22)$$

- b_1 has cubic poles with itself, b_2 and b_3 :

- in $b_1(z) b_1(y)$, they are of the form

$$\begin{aligned} &\partial (\bar{\lambda} \gamma_{mnp} r) (\bar{\lambda} \gamma^{mnp} r), \\ &(\bar{\lambda} \gamma_{mnp} r) (\bar{\lambda} \gamma^{mnq} r) N^p_q, \\ &\partial (\bar{\lambda} \gamma_{mnp} r) (\bar{\lambda} \gamma^{pqr} r) (\bar{\lambda} \gamma^{mn} \lambda) (\bar{\lambda} \gamma_{qr} \lambda). \end{aligned} \quad (3.23)$$

- for $b_1(z)b_2(y)$, the only possible cubic poles are proportional to

$$\begin{aligned} & (\bar{\lambda}\gamma_{mnp}r) (\bar{\lambda}\gamma_{qrs}r) (\bar{\lambda}\gamma^{mn}\lambda) (r\gamma^{qrs}\gamma^p\partial\theta), \\ & (\bar{\lambda}\gamma_{mnp}r) (\bar{\lambda}\gamma^{mnq}r) (r\gamma_q\gamma^p\partial\theta). \end{aligned} \quad (3.24)$$

- the cubic poles arising in $b_1(z)b_3(y)$ come from the multiple contractions of $N^{mn}\Pi^p(\bar{\lambda}\lambda)^{-2}$ with b_3 , and are given by

$$\begin{aligned} & (\bar{\lambda}\gamma_{mnp}r) (\bar{\lambda}\gamma^{mnq}r) (r\gamma_{qrs}r) (\bar{\lambda}\gamma^{rs}\lambda) \Pi^p, \\ & (\bar{\lambda}\gamma_{mnp}r) (\bar{\lambda}\gamma^{mn}\lambda) \Pi^p (\bar{\lambda}\gamma_{qrs}r) (\bar{\lambda}\gamma^{qr}\lambda) (r\gamma^{stu}r) (\bar{\lambda}\gamma_{tu}\lambda), \\ & (\bar{\lambda}\gamma_{mnp}r) (\bar{\lambda}\gamma^{mn}\lambda) \Pi^p (\bar{\lambda}\gamma_{qrs}r) (\bar{\lambda}\gamma^{qr}\gamma_{tu}\lambda) (r\gamma^{stu}r). \end{aligned} \quad (3.25)$$

- b_2 has cubic poles with itself and with b_3 :

- in $b_2(z)b_2(y)$, they are of the form

$$\begin{aligned} & (\bar{\lambda}\gamma_{mnp}r) (\bar{\lambda}\gamma_{qrs}r) (r\gamma^{pst}r) \Pi_t \eta^{mq} \eta^{nr}, \\ & (\bar{\lambda}\gamma_{mnp}r) (\bar{\lambda}\gamma^{mn}\lambda) (\bar{\lambda}\gamma_{qrs}r) (\bar{\lambda}\gamma^{qr}\lambda) (r\gamma^{pst}r) \Pi_t. \end{aligned} \quad (3.26)$$

- for $b_2(z)b_3(y)$, d_α appearing in b_2 is inert and there are only contractions involving the ghost Lorentz currents:

$$\begin{aligned} & (\bar{\lambda}\gamma_{mnp}r) (r\gamma^p d) (\bar{\lambda}\gamma^{mnq}r) (r\gamma_{qrs}r) (\bar{\lambda}\gamma^{rs}\lambda), \\ & (\bar{\lambda}\gamma_{mnp}r) (\bar{\lambda}\gamma^{mn}\lambda) (r\gamma^p d) (\bar{\lambda}\gamma_{qrs}r) (\bar{\lambda}\gamma^{qr}\lambda) (r\gamma^{stu}r) (\bar{\lambda}\gamma_{tu}\lambda). \end{aligned} \quad (3.27)$$

- the cubic poles of $b_3(z)b_3(y)$ involve all possible contractions of the the Lorentz generators and will give similar results to the ones above, only with more r 's.

Due to the pure spinor constraints,

$$(\bar{\lambda}\gamma^{mn})_\alpha (\bar{\lambda}\gamma_{mnp}r) = (\bar{\lambda}\gamma^{mn})_\alpha (\bar{\lambda}\gamma_{mnp}\partial\bar{\lambda}) = (\bar{\lambda}\gamma_{mnp}r) (\bar{\lambda}\gamma^{mnp})^\alpha = 0, \quad (3.28)$$

and every expression listed contains at least one of these types of contractions. Consequently, $O_1 = 0$ and

$$b(z)b(y) \sim \frac{O_3}{(z-y)}. \quad (3.29)$$

It is clear from (3.2), that O_3 can only be composed with supersymmetric invariants: matter fields ($\Pi^m, d_\alpha, \partial\theta^\alpha$); ghost currents from the minimal sector (N^{mn}, J); ghost fields ($\lambda^\alpha, \bar{\lambda}_\alpha, r_\alpha$); and, in principle, their partial derivatives.

In [4], the vanishing of O_3 has been argued as follows. The author assumed that all partial derivatives of r_α that may appear in the OPE (3.29) can be removed due to the pure spinor constraint, since

$$\bar{\lambda}\gamma^m\partial r = -\partial\bar{\lambda}\gamma^m r. \quad (3.30)$$

Based on that assumption, all the r_α dependence of O_3 could be made explicitly through

$$O_3 = \Omega + r_\alpha\Omega^\alpha + r_\alpha r_\beta\Omega^{\alpha\beta} + \dots \quad (3.31)$$

where the Ω 's are supersymmetric, ghost number -2 , conformal weight 3 , BRST closed operators. Since the BRST charge can be split into two pieces according to the r -charge

$$Q = Q_0 + Q_1, \quad (3.32a)$$

$$Q_0 = \oint (\lambda^\alpha d_\alpha), \quad (3.32b)$$

$$Q_1 = \oint (\bar{\omega}^\alpha r_\alpha), \quad (3.32c)$$

requiring $[Q, O_3] = 0$, implies $[Q_0, \Omega] = 0$. Then, it has been shown that there are no Ω with the above requisites satisfying $[Q_0, \Omega] = 0$, so it vanishes identically. Then, $\Omega = 0$ implies $[Q_0, \Omega^\alpha] = 0$. Again, this can be demonstrated to vanish. Pursuing this argument, the nilpotency of the b ghost was obtained in [4].

However, the absence of $\partial^n r_\alpha$ in O_3 is incorrect, as will be illustrated soon, which means that the cohomology argument of [4], summarized above, must be extended, as will now be done.

The computation of (3.29) is organized according to the r -charge of the operators, that is

$$O_3 = (bb)_0 + (bb)_1 + (bb)_2 + (bb)_3 + (bb)_4 + (bb)_5 + (bb)_6. \quad (3.33)$$

To make the expressions more clear, the ordering notation will be dropped and α' will be set to 2.

The first term, $(bb)_0$, is given by

$$(bb)_0 \equiv \int dz \{b_0(z) b_0(y) + b_{-1}(z) b_1(y) + b_1(z) b_{-1}(y)\} \quad (3.34)$$

$$\begin{aligned} &= \alpha_{01} \frac{N^{mn} (\bar{\lambda} \gamma_{mn} \partial \theta) (\bar{\lambda} \partial \theta)}{(\bar{\lambda} \lambda)^2} + \alpha_{02} \frac{(\bar{\lambda} \gamma_{mnp} \partial \bar{\lambda}) N^{mn} \Pi^p}{(\bar{\lambda} \lambda)^2} \\ &+ \alpha_{03} \frac{\Pi^m (\bar{\lambda} \partial \theta) (\bar{\lambda} \gamma_m d)}{(\bar{\lambda} \lambda)^2} + \alpha_{04} \frac{(\bar{\lambda} \gamma_{mnp} \partial \bar{\lambda}) (d \gamma^{mnp} d)}{(\bar{\lambda} \lambda)^2} \\ &+ \alpha_{05} \frac{\Pi^m (\bar{\lambda} \gamma_m \partial^2 \bar{\lambda})}{(\bar{\lambda} \lambda)^2} + \alpha_{06} \frac{(\bar{\lambda} \partial \theta) (\bar{\lambda} \partial^2 \theta)}{(\bar{\lambda} \lambda)^2} + \alpha_{07} \frac{(\bar{\lambda} \partial \theta) (\partial \bar{\lambda} \partial \theta)}{(\bar{\lambda} \lambda)^2}, \end{aligned} \quad (3.35)$$

where α_{0n} are just numerical coefficients. By a direct computation, it is relatively simple to show the vanishing of $(bb)_0$. It is enough to compute $[Q, (bb)_0]$ and use the BRST argument mentioned above. Note that $[Q, O_3] = 0$ implies the vanishing of

$$\begin{aligned} [Q_0, (bb)_0] &= \alpha_{01} \frac{N^{mn} (\bar{\lambda} \gamma_{mn} \partial \lambda) (\bar{\lambda} \partial \theta)}{(\bar{\lambda} \lambda)^2} - \alpha_{01} \frac{\frac{1}{2} (d \gamma^{mn} \lambda) (\bar{\lambda} \gamma_{mn} \partial \theta) (\bar{\lambda} \partial \theta)}{(\bar{\lambda} \lambda)^2} \\ &- \alpha_{01} \frac{N^{mn} (\bar{\lambda} \gamma_{mn} \partial \theta) (\bar{\lambda} \partial \lambda)}{(\bar{\lambda} \lambda)^2} - \alpha_{02} \frac{(\bar{\lambda} \gamma_{mnp} \partial \bar{\lambda}) \frac{1}{2} (d \gamma^{mn} \lambda) \Pi^p}{(\bar{\lambda} \lambda)^2} \\ &+ \alpha_{02} \frac{(\bar{\lambda} \gamma_{mnp} \partial \bar{\lambda}) N^{mn} (\lambda \gamma^p \partial \theta)}{(\bar{\lambda} \lambda)^2} - \alpha_{03} \frac{\Pi^m (\bar{\lambda} \partial \theta) (\bar{\lambda} \gamma_m \gamma_n \partial \theta) \Pi^n}{(\bar{\lambda} \lambda)^2} \end{aligned}$$

$$\begin{aligned}
 & +\alpha_{03} \frac{\Pi^m (\bar{\lambda} \partial \lambda) (\bar{\lambda} \gamma_m d)}{(\bar{\lambda} \lambda)^2} + \alpha_{03} \frac{(\lambda \gamma^m \partial \theta) (\bar{\lambda} \partial \theta) (\bar{\lambda} \gamma_m d)}{(\bar{\lambda} \lambda)^2} \\
 & +\alpha_{04} \frac{2 (\bar{\lambda} \gamma_{mnp} \partial \bar{\lambda}) (d \gamma^{mnp} \gamma^q \lambda) \Pi_q}{(\bar{\lambda} \lambda)^2} + \alpha_{06} \frac{(\bar{\lambda} \partial \lambda) (\bar{\lambda} \partial^2 \theta)}{(\bar{\lambda} \lambda)^2} - \alpha_{06} \frac{(\bar{\lambda} \partial \theta) (\bar{\lambda} \partial^2 \lambda)}{(\bar{\lambda} \lambda)^2} \\
 & +\alpha_{05} \frac{(\lambda \gamma^m \partial \theta) (\bar{\lambda} \gamma_m \partial^2 \bar{\lambda})}{(\bar{\lambda} \lambda)^2} + \alpha_{07} \frac{(\bar{\lambda} \partial \lambda) (\partial \bar{\lambda} \partial \theta)}{(\bar{\lambda} \lambda)^2} - \alpha_{07} \frac{(\bar{\lambda} \partial \theta) (\partial \bar{\lambda} \partial \lambda)}{(\bar{\lambda} \lambda)^2}.
 \end{aligned}$$

The Lorentz generators N^{mn} appear in three terms. It is straightforward to check that they are not related by a Fierz decomposition of the spinors, implying that $\alpha_{01} = \alpha_{02} = 0$. Now, there is only one term that contributes with one d_α and two $\partial \theta^\alpha$, so $\alpha_{03} = 0$, which, on the other hand, imply that $\alpha_{04} = 0$, since the term with one d_α and one Π^m cannot be cancelled anymore. The vanishing of α_{05} , α_{06} and α_{07} is evident, since they do not possibly cancel each other. *There is no linear combination of the above operators that can be annihilated by Q_0 , therefore $(bb)_0 = 0$.*

The second term, $(bb)_1$, is

$$(bb)_1 \equiv \int dz \{b_0(z) b_1(y) + b_1(z) b_0(y) + b_{-1}(z) b_2(y) + b_2(z) b_{-1}(y)\} \quad (3.36)$$

$$\begin{aligned}
 & = \alpha_{11} \frac{(\bar{\lambda} \gamma_{mnp} r) N^{mn} \Pi^p (\bar{\lambda} \partial \theta)}{(\bar{\lambda} \lambda)^3} + \alpha_{12} \frac{(\bar{\lambda} \gamma_{mnp} r) N^{mn} (\partial \bar{\lambda} \gamma^p d)}{(\bar{\lambda} \lambda)^3} \\
 & + \alpha_{13} \frac{(\bar{\lambda} \gamma_{mnp} \partial \bar{\lambda}) N^{mn} (r \gamma^p d)}{(\bar{\lambda} \lambda)^3} + \alpha_{14} \frac{(\bar{\lambda} \gamma_{mnp} r) (d \gamma^{mnp} d) (\bar{\lambda} \partial \theta)}{(\bar{\lambda} \lambda)^3} \\
 & + \alpha_{15} \frac{(\bar{\lambda} \gamma_{mnp} r) \Pi^m (\partial \bar{\lambda} \gamma^{np} \partial \theta)}{(\bar{\lambda} \lambda)^3} + \alpha_{16} \frac{(\bar{\lambda} \gamma_m \partial^2 \bar{\lambda}) (r \gamma^m d)}{(\bar{\lambda} \lambda)^3}. \quad (3.37)
 \end{aligned}$$

Since $[Q_1, (bb)_0] = 0$, $[Q_0, (bb)_1]$ must also vanish:

$$\begin{aligned}
 [Q_0, (bb)_1] & = \alpha_{11} \frac{\frac{1}{2} (\bar{\lambda} \gamma_{mnp} r) (d \gamma^{mn} \lambda) \Pi^p (\bar{\lambda} \partial \theta)}{(\bar{\lambda} \lambda)^3} - \alpha_{11} \frac{(\bar{\lambda} \gamma_{mnp} r) N^{mn} (\lambda \gamma^p \partial \theta) (\bar{\lambda} \partial \theta)}{(\bar{\lambda} \lambda)^3} \\
 & - \alpha_{11} \frac{(\bar{\lambda} \gamma_{mnp} r) N^{mn} \Pi^p (\bar{\lambda} \partial \lambda)}{(\bar{\lambda} \lambda)^3} + \alpha_{12} \frac{\frac{1}{2} (\bar{\lambda} \gamma_{mnp} r) (d \gamma^{mn} \lambda) (\partial \bar{\lambda} \gamma^p d)}{(\bar{\lambda} \lambda)^3} \\
 & + \alpha_{12} \frac{(\bar{\lambda} \gamma_{mnp} r) N^{mn} (\partial \bar{\lambda} \gamma^p \gamma^q \lambda) \Pi_q}{(\bar{\lambda} \lambda)^3} - \alpha_{13} \frac{\frac{1}{2} (\bar{\lambda} \gamma_{mnp} \partial \bar{\lambda}) (d \gamma^{mn} \lambda) (r \gamma^p d)}{(\bar{\lambda} \lambda)^3} \\
 & + \alpha_{13} \frac{(\bar{\lambda} \gamma_{mnp} \partial \bar{\lambda}) N^{mn} (r \gamma^p \gamma^q \lambda) \Pi_q}{(\bar{\lambda} \lambda)^3} - \alpha_{14} \frac{2 (\bar{\lambda} \gamma_{mnp} r) (d \gamma^{mnp} \gamma^q \lambda) \Pi_q (\bar{\lambda} \partial \theta)}{(\bar{\lambda} \lambda)^3} \\
 & - \alpha_{14} \frac{(\bar{\lambda} \gamma_{mnp} r) (d \gamma^{mnp} d) (\bar{\lambda} \partial \lambda)}{(\bar{\lambda} \lambda)^3} - \alpha_{15} \frac{(\bar{\lambda} \gamma_{mnp} r) (\lambda \gamma^m \partial \theta) (\partial \bar{\lambda} \gamma^{np} \partial \theta)}{(\bar{\lambda} \lambda)^3} \\
 & - \alpha_{15} \frac{(\bar{\lambda} \gamma_{mnp} r) \Pi^m (\partial \bar{\lambda} \gamma^{np} \partial \lambda)}{(\bar{\lambda} \lambda)^3} + \alpha_{16} \frac{(\bar{\lambda} \gamma_m \partial^2 \bar{\lambda}) (r \gamma^m \gamma^n \lambda) \Pi_n}{(\bar{\lambda} \lambda)^3}.
 \end{aligned}$$

There is only one term that contains one Lorentz generator N^{mn} and two $\partial \theta^\alpha$, so $\alpha_{11} = 0$. Now, there are two other terms that contain N^{mn} , but they are unrelated to any

Fierz decomposition, implying that $\alpha_{12} = \alpha_{13} = 0$. The remaining terms are obviously independent: $\alpha_{14} = 0$, since it is the only one with $(d\gamma^{mnp}d)$; $\alpha_{15} = 0$, as no other term contains two $\partial\theta^\alpha$; and $\alpha_{16} = 0$, for there is nothing else to cancel it. As $(bb)_0$, $(bb)_1$ is not BRST closed for any set of coefficients α_{1n} and $(bb)_1 = 0$ is the single possibility left.

Going on,

$$(bb)_2 \equiv \int dz \{b_0(z) b_2(y) + b_2(z) b_0(y) + b_1(z) b_1(y) + b_{-1}(z) b_3(y) + b_3(z) b_{-1}(y)\} \quad (3.38)$$

can be written as

$$\begin{aligned} (bb)_2 = & \alpha_{21} \frac{(\bar{\lambda}\gamma_{mnp}r)(r\gamma^pd)N^{mn}(\bar{\lambda}\partial\theta)}{(\bar{\lambda}\lambda)^4} + \alpha_{22} \frac{(\bar{\lambda}\gamma_m\partial r)(r\gamma^md)(\bar{\lambda}\partial\theta)}{(\bar{\lambda}\lambda)^4} \\ & + \alpha_{23} \frac{(\bar{\lambda}\gamma_{mnp}r)(\partial\bar{\lambda}\gamma^{pqr}r)N^{mn}N_{qr}}{(\bar{\lambda}\lambda)^4} + \alpha_{24} \frac{(\bar{\lambda}\gamma_m\partial r)(\bar{\lambda}\gamma_nd)(r\gamma^{mn}\partial\theta)}{(\bar{\lambda}\lambda)^4} \\ & + \alpha_{25} \frac{(\bar{\lambda}\gamma_{mnp}r)(\partial\bar{\lambda}\gamma_qr)N^{mn}N^{pq}}{(\bar{\lambda}\lambda)^4} + \alpha_{26} \frac{(\bar{\lambda}\gamma_{mnp}r)(r\gamma^p\partial^2\bar{\lambda})N^{mn}}{(\bar{\lambda}\lambda)^4} \\ & + \alpha_{27} \frac{(\bar{\lambda}\gamma_m\partial r)(r\gamma^m\partial^2\bar{\lambda})}{(\bar{\lambda}\lambda)^4} + \alpha_{28} \frac{(r\gamma_m\partial^2\bar{\lambda})(\bar{\lambda}\gamma^m\partial r)}{(\bar{\lambda}\lambda)^4}. \end{aligned} \quad (3.39)$$

The last line of the expression is Q_0 -closed. In computing $[Q_0, (bb)_2]$,

$$\begin{aligned} [Q_0, (bb)_2] = & \alpha_{21} \frac{\frac{1}{2}(\bar{\lambda}\gamma_{mnp}r)(r\gamma^pd)(d\gamma^{mn}\lambda)(\bar{\lambda}\partial\theta)}{(\bar{\lambda}\lambda)^4} - \alpha_{21} \frac{(\bar{\lambda}\gamma_{mnp}r)(r\gamma^pd)N^{mn}(\bar{\lambda}\partial\lambda)}{(\bar{\lambda}\lambda)^4} \\ & - \alpha_{21} \frac{(\bar{\lambda}\gamma_{mnp}r)(r\gamma^p\gamma^q\lambda)\Pi_q N^{mn}(\bar{\lambda}\partial\theta)}{(\bar{\lambda}\lambda)^4} - \alpha_{22} \frac{(\bar{\lambda}\gamma_m\partial r)(r\gamma^m\gamma^n\lambda)\Pi_n(\bar{\lambda}\partial\theta)}{(\bar{\lambda}\lambda)^4} \\ & - \alpha_{22} \frac{(\bar{\lambda}\gamma_m\partial r)(r\gamma^md)(\bar{\lambda}\partial\lambda)}{(\bar{\lambda}\lambda)^4} - \alpha_{23} \frac{\frac{1}{2}(\bar{\lambda}\gamma_{mnp}r)(\partial\bar{\lambda}\gamma^{pqr}r)(d\gamma^{mn}\lambda)N_{qr}}{(\bar{\lambda}\lambda)^4} \\ & - \alpha_{23} \frac{\frac{1}{2}(\bar{\lambda}\gamma_{mnp}r)(\partial\bar{\lambda}\gamma^{pqr}r)N^{mn}(d\gamma_{qr}\lambda)}{(\bar{\lambda}\lambda)^4} - \alpha_{24} \frac{(\bar{\lambda}\gamma_m\partial r)(\bar{\lambda}\gamma_nd)(r\gamma^{mn}\partial\lambda)}{(\bar{\lambda}\lambda)^4} \\ & + \alpha_{24} \frac{(\bar{\lambda}\gamma_m\partial r)(\bar{\lambda}\gamma_n\gamma_p\lambda)\Pi^p(r\gamma^{mn}\partial\theta)}{(\bar{\lambda}\lambda)^4} - \alpha_{25} \frac{\frac{1}{2}(\bar{\lambda}\gamma_{mnp}r)(\partial\bar{\lambda}\gamma_qr)N^{mn}(d\gamma^{pq}\lambda)}{(\bar{\lambda}\lambda)^4} \\ & - \alpha_{25} \frac{\frac{1}{2}(\bar{\lambda}\gamma_{mnp}r)(\partial\bar{\lambda}\gamma_qr)(d\gamma^{mn}\lambda)N^{pq}}{(\bar{\lambda}\lambda)^4} - \alpha_{26} \frac{\frac{1}{2}(\bar{\lambda}\gamma_{mnp}r)(r\gamma^p\partial^2\bar{\lambda})(d\gamma^{mn}\lambda)}{(\bar{\lambda}\lambda)^4}, \end{aligned}$$

the terms that contain matter fields or the Lorentz current do not vanish for any set α_{2n} of coefficients: $\alpha_{21} = 0$, for it is the single term that contains N^{mn} and Π^m ; $\alpha_{22} = \alpha_{24} = 0$, since they are the only ones that contribute with one Π^m and one $\partial\theta^\alpha$, but independently; $\alpha_{23} = \alpha_{25} = 0$, because they are the remaining (and also independent) terms containing the Lorentz generator; and $\alpha_{26} = 0$, for it is not BRST closed.

$(bb)_3$ can be cast as:

$$\begin{aligned}
 (bb)_3 &\equiv \int dz \{b_0(z) b_3(y) + b_3(z) b_0(y) + b_1(z) b_2(y) + b_2(z) b_1(y)\} \\
 &= \alpha_{31} \frac{(\bar{\lambda} \gamma_{mnp} r) (r \gamma^{pq} r) N^{mn} N_{qr} (\bar{\lambda} \partial \theta)}{(\bar{\lambda} \lambda)^5} + \alpha_{32} \frac{(r \gamma_{mnp} r) (\bar{\lambda} \gamma^p \partial r) N^{mn} (\bar{\lambda} \partial \theta)}{(\bar{\lambda} \lambda)^5} \\
 &\quad + \alpha_{33} \frac{(\bar{\lambda} \gamma_m \partial r) (r \gamma^m \partial r) (\bar{\lambda} \partial \theta)}{(\bar{\lambda} \lambda)^5} + \alpha_{34} \frac{(\bar{\lambda} \gamma_m \partial r) (\bar{\lambda} \gamma_n \partial r) (r \gamma^{mn} \partial \theta)}{(\bar{\lambda} \lambda)^5} \\
 &\quad + \alpha_{35} \frac{(\bar{\lambda} \gamma_m \partial r) (\bar{\lambda} \gamma_n \partial r) (r \gamma^{mn} \lambda) (\bar{\lambda} \partial \theta)}{(\bar{\lambda} \lambda)^6}.
 \end{aligned} \tag{3.41}$$

It is straightforward to see that the first two terms are not BRST closed. One of the contributions of the first one contains two Lorentz generators, that cannot be cancelled, so $\alpha_{31} = 0$. The same happens for the second one, which has a contribution in $[Q_0, (bb)_3]$ with one Lorentz generator, not balanced by any other, thus $\alpha_{32} = 0$. The result of the computation of $[Q_1, (bb)_2] + [Q_0, (bb)_3]$ with the remaining terms is

$$\begin{aligned}
 [Q_1, (bb)_2] + [Q_0, (bb)_3] &= \alpha_{27} \frac{4 (\bar{\lambda} \gamma_m \partial r) (r \gamma^m \partial^2 \bar{\lambda}) (r \lambda)}{(\bar{\lambda} \lambda)^5} - \alpha_{27} \frac{(r \gamma_m \partial r) (r \gamma^m \partial^2 \bar{\lambda})}{(\bar{\lambda} \lambda)^4} \\
 &\quad - \alpha_{27} \frac{(\bar{\lambda} \gamma_m \partial r) (r \gamma^m \partial^2 r)}{(\bar{\lambda} \lambda)^4} + \alpha_{28} \frac{4 (\bar{\lambda} \gamma_m \partial^2 \bar{\lambda}) (r \gamma^m \partial r) (r \lambda)}{(\bar{\lambda} \lambda)^5} \\
 &\quad - \alpha_{28} \frac{(r \gamma_m \partial^2 \bar{\lambda}) (r \gamma^m \partial r)}{(\bar{\lambda} \lambda)^4} - \alpha_{34} \frac{(\bar{\lambda} \gamma_m \partial r) (\bar{\lambda} \gamma_n \partial r) (r \gamma^{mn} \partial \lambda)}{(\bar{\lambda} \lambda)^5} \\
 &\quad - \alpha_{28} \frac{(\bar{\lambda} \gamma_m \partial^2 r) (r \gamma^m \partial r)}{(\bar{\lambda} \lambda)^4} - \alpha_{33} \frac{(\bar{\lambda} \gamma_m \partial r) (r \gamma^m \partial r) (\bar{\lambda} \partial \lambda)}{(\bar{\lambda} \lambda)^5} \\
 &\quad - \alpha_{35} \frac{(\bar{\lambda} \gamma_m \partial r) (\bar{\lambda} \gamma_n \partial r) (r \gamma^{mn} \lambda) (\bar{\lambda} \partial \lambda)}{(\bar{\lambda} \lambda)^6}.
 \end{aligned}$$

Obviously, there is no nontrivial solution to $\{\alpha_{27}, \alpha_{28}, \alpha_{33}, \alpha_{34}, \alpha_{35}\}$ that may lead to the vanishing of this equation, thus $(bb)_2 = (bb)_3 = 0$. Note that

$$\frac{(\bar{\lambda} \gamma_m \partial r) (r \gamma^m \partial r) (\bar{\lambda} \partial \theta)}{(\bar{\lambda} \lambda)^5} \tag{3.42}$$

does not allow the removal of partial derivatives acting on r , which contradicts the assumption of [4].

So far, the pure spinor constraints only have been used to reduce the number of independent terms in the OPE computation. It turns out that for $(bb)_4$, $(bb)_5$ and $(bb)_6$, all possible terms being generated vanish due to the constraints.

For

$$(bb)_4 \equiv \int dz \{b_1(z) b_3(y) + b_3(z) b_1(y) + b_2(z) b_2(y)\}, \tag{3.43}$$

the simple poles are given by:

- terms with two N 's and one Π , like

$$\frac{(\bar{\lambda}\gamma_{mnp}r)(\bar{\lambda}\gamma_{qrs}r)(r\gamma^{pqt}r)N^{mn}N_t^r\Pi^s}{(\bar{\lambda}\lambda)^6}. \quad (3.44)$$

Since $(r\gamma^{mnp}r) = (r\gamma^m\gamma^n\gamma^p r)$ and $(\bar{\lambda}\gamma^{mnp}r)(r\gamma_p)^\alpha = (r\gamma^{mnp}r)(\bar{\lambda}\gamma_p)^\alpha$,

$$(\bar{\lambda}\gamma_{mnp}r)(\bar{\lambda}\gamma_{qrs}r)(r\gamma^{pqt}r) = (r\gamma_{mnp}r)(r\gamma_{qrs}r)(\bar{\lambda}\gamma^p\gamma^t\gamma^q\bar{\lambda}), \quad (3.45)$$

which vanishes because $(\bar{\lambda}\gamma^{mnp}\bar{\lambda}) = 0$.

- terms with one N , one Π and one partial derivative (Taylor expansion of a quadratic pole), as

$$\frac{(\partial\bar{\lambda}\gamma_{mnp}r)(\bar{\lambda}\gamma_{qrs}r)(r\gamma^{pqr}r)N^{mn}\Pi^s}{(\bar{\lambda}\lambda)^6}, \quad (3.46)$$

which vanishes, since

$$\begin{aligned} (\bar{\lambda}\gamma_{qrs}r)(r\gamma^{pqr}r) &= (\bar{\lambda}\gamma_{qr}\gamma_s r)(r\gamma^{qr}\gamma^p r) \\ &= 4(\bar{\lambda}\gamma^m r)(r\gamma_s\gamma_m\gamma^p r) \\ &\quad - 2(\bar{\lambda}\gamma_s r)(r\gamma^p r) - 8(r\gamma_s r)(\bar{\lambda}\gamma^p r) \\ &= 0. \end{aligned} \quad (3.47)$$

- terms with one N and two d 's, like

$$\frac{(\bar{\lambda}\gamma_{mnp}r)(\bar{\lambda}\gamma^{mqr}r)N_r^n(r\gamma^p d)(r\gamma_q d)}{(\bar{\lambda}\lambda)^6}. \quad (3.48)$$

Since $\bar{\lambda}\gamma^{mnp}r$ is equal to $\bar{\lambda}\gamma^m\gamma^n\gamma^p r$, this term is proportional to $(\bar{\lambda}\gamma^m)^\alpha(\bar{\lambda}\gamma_m)^\beta$, and, according to equation (A.15), it vanishes.

- terms with two d 's and one partial derivative, such as

$$\frac{(\bar{\lambda}\gamma_{mnp}r)(\partial\bar{\lambda}\gamma^{mnq}r)(r\gamma^p d)(r\gamma_q d)}{(\bar{\lambda}\lambda)^6}. \quad (3.49)$$

Decomposing $(\partial\bar{\lambda}\gamma^{mnp}r)$ as $(\partial\bar{\lambda}\gamma^{mn}\gamma^p r) + \eta^{np}(\bar{\lambda}\gamma^m\partial r) - \eta^{mp}(\bar{\lambda}\gamma^n\partial r)$, it is possible to rewrite the expression as follows,

$$\begin{aligned} (\bar{\lambda}\gamma_{mnp}r)(\partial\bar{\lambda}\gamma^{mnq}r) &= (\bar{\lambda}\gamma_{mn}\gamma_p r)(\partial\bar{\lambda}\gamma^{mn}\gamma^q r) + 2\eta^{nq}(\bar{\lambda}\gamma_{mnp}r)(\bar{\lambda}\gamma^m\partial r) \\ &= (\bar{\lambda}\gamma_m\partial\bar{\lambda})(r\gamma_p\gamma_m\gamma^q r) - 2(\bar{\lambda}\gamma_p r)(\partial\bar{\lambda}\gamma^q r) \\ &\quad - 8(\bar{\lambda}\gamma^q r)(\partial\bar{\lambda}\gamma_p r) + 2\eta^{nq}(\bar{\lambda}\gamma_m\gamma_{np}r)(\bar{\lambda}\gamma^m\partial r) \\ &= 0, \end{aligned} \quad (3.50)$$

showing that this term also vanishes.

- and terms with one Π and two partial derivatives (Taylor expansion of a cubic pole), like

$$\frac{(\partial \bar{\lambda} \gamma_{mnp} \partial r) (\bar{\lambda} \gamma^{mnq} r) (r \gamma^p_{qr} r) \Pi^r}{(\bar{\lambda} \lambda)^6}. \quad (3.51)$$

Decomposing $(\partial \bar{\lambda} \gamma^{mnp} \partial r)$ as $(\partial \bar{\lambda} \gamma^{mn} \gamma^p \partial r) - \eta^{np} (\partial \bar{\lambda} \gamma^m \partial r) + \eta^{mp} (\partial \bar{\lambda} \gamma^n \partial r)$, the expression

$$(\partial \bar{\lambda} \gamma_{mnp} \partial r) (\bar{\lambda} \gamma^{mnq} r) (r \gamma^p_{qr} r) \quad (3.52)$$

can be split into two pieces. One of them is similar to the ones presented before and also vanishes. The other one is proportional to

$$\begin{aligned} (\bar{\lambda} \gamma_m \partial r) (\bar{\lambda} \gamma_n \partial r) (r \gamma^{mnp} r) &= (r \gamma_m \partial r) (\bar{\lambda} \gamma_n \partial r) (\bar{\lambda} \gamma^{mnp} r) \\ &= - (r \gamma_m \partial r) (\bar{\lambda} \gamma_n \partial r) (\bar{\lambda} \gamma^n \gamma^{mp} r), \end{aligned} \quad (3.53)$$

and vanishes, since $(\bar{\lambda} \gamma^m)^\alpha (\bar{\lambda} \gamma_m)^\beta = 0$.

For

$$(bb)_5 \equiv \int dz \{b_3(z) b_2(y) + b_2(z) b_3(y)\}, \quad (3.54)$$

all contributions to the simple pole will have d_α :

- there are terms with two N 's, as

$$\frac{(\bar{\lambda} \gamma_{mnp} r) (r \gamma^p d) (\bar{\lambda} \gamma_{qrs} r) (r \gamma^{stu} r) N^{mq} \eta^{nr} N_{tu}}{(\bar{\lambda} \lambda)^7}. \quad (3.55)$$

Note that

$$\begin{aligned} (\bar{\lambda} \gamma_{mnp} r) (\bar{\lambda} \gamma_{qrs} r) \eta^{nr} &= (\bar{\lambda} \gamma_n \gamma_{mp} r) (\bar{\lambda} \gamma_r \gamma_{qs} r) \eta^{nr} \\ &= (\bar{\lambda} \gamma_m)^\alpha (\bar{\lambda} \gamma^m)^\beta (\dots)_{\alpha\beta} \\ &= 0, \end{aligned} \quad (3.56)$$

gives a vanishing contribution.

- terms with one N and one partial derivative, as

$$\frac{(\bar{\lambda} \gamma_{mnp} \partial r) (r \gamma^p d) (\bar{\lambda} \gamma^{mnq} r) (r \gamma_{qrs} r) N^{rs}}{(\bar{\lambda} \lambda)^7}. \quad (3.57)$$

It is easy to extract the pure spinor constraint out of this expression:

$$\begin{aligned} (\bar{\lambda} \gamma_{mnp} \partial r) (\bar{\lambda} \gamma^{mnq} r) &= (\bar{\lambda} \gamma_{mn} \gamma_p \partial r) (\bar{\lambda} \gamma^{mn} \gamma^q r) \\ &\quad - 2 (\bar{\lambda} \gamma_m \partial r) (\bar{\lambda} \gamma^m \gamma^{nq} r) \eta_{np} \\ &= 4 (\bar{\lambda} \gamma^m \bar{\lambda}) (\partial r \gamma_p \gamma_m \gamma^q r) - 10 (\bar{\lambda} \gamma_p \partial r) (\bar{\lambda} \gamma^q r) \\ &\quad - 2 (\bar{\lambda} \gamma_m \partial r) (\bar{\lambda} \gamma^{m0} \gamma^{nq} r) \eta_{mp}. \\ &= 0. \end{aligned} \quad (3.58)$$

- and terms with two partial derivatives, coming from the cubic poles, like

$$\frac{(\partial \bar{\lambda} \gamma_{mnp} \partial r) (r \gamma^p d) (\bar{\lambda} \gamma^{mnq} r) (r \gamma_{qrs} r) (\bar{\lambda} \gamma^{rs} \lambda)}{(\bar{\lambda} \lambda)^8}. \quad (3.59)$$

Note that $(r \gamma_{qrs} r) (\bar{\lambda} \gamma^{rs} \lambda)$ has the same structure of (3.28) and also vanishes.

Finally, for the last term in the $b(z) b(y)$ OPE, where only the ghost fields appear,

$$(bb)_6 \equiv \int dz \{b_3(z) b_3(y)\}, \quad (3.60)$$

- there are terms with three N 's, like

$$\frac{(\bar{\lambda} \gamma_{mnp} r) (r \gamma^{pqr} r) (\bar{\lambda} \gamma^{mst} r) (r \gamma_{tuv} r) N_{qr} N^s N^{uv}}{(\bar{\lambda} \lambda)^8}. \quad (3.61)$$

Since $\bar{\lambda} \gamma^{mnp} r = \bar{\lambda} \gamma^m \gamma^n \gamma^p r$, $(\bar{\lambda} \gamma_{mnp} r) (\bar{\lambda} \gamma^{mqr} r)$ vanishes, as shown above.

- terms with two N 's and one partial derivative, like

$$\frac{\partial (\bar{\lambda} \gamma_{mnp} r) (r \gamma^{pqr} r) (\bar{\lambda} \gamma^{mns} r) (r \gamma_{stu} r) N_{qr} N^{tu}}{(\bar{\lambda} \lambda)^8}, \quad (3.62)$$

which has the same structure presented before, being proportional to the pure spinor constraints.

- terms with one N and two partial derivatives, coming from triple poles, such as

$$\frac{\partial^2 (\bar{\lambda} \gamma_{mnp} r) (r \gamma^{pqr} r) (\bar{\lambda} \gamma^{mns} r) (r \gamma_{qst} r) N_r^t}{(\bar{\lambda} \lambda)^8}, \quad (3.63)$$

which are similar to the above ones and vanish.

- and terms with three partial derivatives, like

$$\frac{(\partial \bar{\lambda} \gamma_{mnp} \partial r) (r \gamma^{pqr} \partial r) (\bar{\lambda} \gamma^{mns} r) (r \gamma_{qrs} r)}{(\bar{\lambda} \lambda)^8}, \quad (3.64)$$

that can be rewritten as

$$\frac{(\partial \bar{\lambda} \gamma_m \partial r) (r \gamma^q \partial r) (\bar{\lambda} \gamma^{mnp} r) (r \gamma_{npq} r)}{(\bar{\lambda} \lambda)^8} \quad (3.65)$$

and vanish, since

$$\begin{aligned} (\bar{\lambda} \gamma^{mnp} r) (r \gamma_{npq} r) &= (\bar{\lambda} \gamma^{np} \gamma^m r) (r \gamma_{np} \gamma_q r) \\ &= (\bar{\lambda} \gamma^n r) (r \gamma^m \gamma_n \gamma_q r) \\ &\quad - 2 (\bar{\lambda} \gamma^m r) (r \gamma_q r) - 8 (r \gamma^m r) (\bar{\lambda} \gamma_q r) \\ &= 0. \end{aligned} \quad (3.66)$$

Summarizing, in the OPE computation several terms vanish identically due to the pure spinor constraints (in particular, $(bb)_4$, $(bb)_5$ and $(bb)_6$ do not present nontrivial contributions). The remaining terms are excluded through the BRST argument, since they were shown to be not BRST closed. Therefore,

$$(bb)_1 = (bb)_2 = (bb)_3 = (bb)_4 = (bb)_5 = (bb)_6 = 0, \quad (3.67)$$

and the pure spinor b ghost is, indeed, nilpotent:

$$b(z)b(y) \sim \text{regular}. \quad (3.68)$$

4 Conclusion

In this work, some properties of the b ghost in the non-minimal pure spinor formalism were reviewed and confirmed. The main object of study was the nilpotency of the non-minimal b ghost.

From general arguments, the $b(z)b(y)$ OPE is reduced to

$$b(z)b(y) \sim \frac{O_1}{(z-y)^3} + \frac{1}{2} \frac{\partial O_1}{(z-y)^2} + \frac{O_3}{(z-y)},$$

where O_1 and O_3 are BRST closed.

As was already known from [4], the different terms in the cubic pole, O_1 , are all proportional to the pure spinor constraints

$$\bar{\lambda}\gamma^m r = \bar{\lambda}\gamma^m \bar{\lambda} = 0.$$

However, the demonstration that the simple pole (O_3) vanishes, was incomplete, due to a wrong assumption on the absence of r_α derivatives.

A counter-example to that assumption is very simple,

$$\frac{(\bar{\lambda}\gamma_m \partial r)(r\gamma^m \partial r)(\bar{\lambda}\partial\theta)}{(\bar{\lambda}\lambda)^5}.$$

Note that the fundamental ingredient here is $(r\gamma^m \partial r)$, an object that does not allow, in general, the removal of the partial derivatives acting on r_α .

Knowing this flaw, the proof that $O_3 = 0$ was carried out in a straightforward manner. First, a careful analysis was made, obtaining all terms that could be generated in the OPE computation. For some of them, the cancellation is very simple to obtain and no BRST argument is needed. However, for most of the terms (those that appear in ordering rearrangements, for example), a direct check is very hard to perform. However, they were shown to be not BRST closed. Since the possible poles appearing in $b(z)b(y)$ must commute with the BRST charge, O_3 vanishes, and this constitutes a rigorous proof of the b ghost nilpotency in the non-minimal pure spinor formalism, confirming the interpretation of the theory as a $\hat{c} = 3$ $\mathcal{N} = 2$ topological string.

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A Conventions and useful formulas

Conventions. Indices:

$$\begin{cases} m, n, \dots = 0, \dots, 9 & \text{space-time vector indices,} \\ \alpha, \beta, \dots = 1, \dots, 16 & \text{space-time spinor indices,} \end{cases}$$

The indices antisymmetrization is represented by the square brackets, meaning

$$[I_1 \dots I_n] = \frac{1}{n!} (I_1 \dots I_n + \text{all antisymmetric permutations}). \quad (\text{A.1})$$

For example,

$$\gamma^{[m} \gamma^{n]} = \frac{1}{2} (\gamma^m \gamma^n - \gamma^n \gamma^m) = \gamma^{mn}, \quad (\text{A.2})$$

or,

$$\lambda^{[\alpha} H^{\beta\gamma]} = \frac{1}{3!} \left(\lambda^\alpha H^{\beta\gamma} - \lambda^\alpha H^{\gamma\beta} + \lambda^\beta H^{\gamma\alpha} - \lambda^\beta H^{\alpha\gamma} + \lambda^\gamma H^{\alpha\beta} - \lambda^\gamma H^{\beta\alpha} \right). \quad (\text{A.3})$$

Concerning OPE's, the right-hand sides of the equations are always evaluated at the coordinate of the second entry, that is,

$$A(z) B(y) \sim \frac{C}{(z-y)^2} + \frac{D}{(z-y)} \quad (\text{A.4})$$

means $C = C(y)$ and $D = D(y)$.

Gamma matrices. The gamma matrices $\gamma_{\alpha\beta}^m$ and $\gamma_m^{\alpha\beta}$ satisfy

$$\{\gamma^m, \gamma^n\}^\alpha_\beta = (\gamma^m)^{\alpha\sigma} \gamma_{\sigma\beta}^n + (\gamma^n)^{\alpha\sigma} \gamma_{\sigma\beta}^m = 2\eta^{mn} \delta^\alpha_\beta. \quad (\text{A.5})$$

The Fierz decompositions of bispinors are given by

$$\chi^\alpha \psi^\beta = \frac{1}{16} \gamma_m^{\alpha\beta} (\chi \gamma^m \psi) + \frac{1}{3!16} \gamma_{mnp}^{\alpha\beta} (\chi \gamma^{mnp} \psi) + \frac{1}{5!16} \left(\frac{1}{2} \right) \gamma_{mnpqr}^{\alpha\beta} (\chi \gamma^{mnpqr} \psi), \quad (\text{A.6a})$$

$$\chi_\alpha \psi^\beta = \frac{1}{16} \delta_\alpha^\beta (\chi \psi) - \frac{1}{2!16} (\gamma_{mn})^\beta_\alpha (\chi \gamma^{mn} \psi) + \frac{1}{4!16} (\gamma_{mnpq})^\beta_\alpha (\chi \gamma^{mnpq} \psi), \quad (\text{A.6b})$$

where

$$\gamma_m^{\alpha\beta} = \gamma_m^{\beta\alpha}, \gamma_{mnp}^{\alpha\beta} = -\gamma_{mnp}^{\beta\alpha}, \gamma_{mnpqr}^{\alpha\beta} = \gamma_{mnpqr}^{\beta\alpha}. \quad (\text{A.7})$$

The main gamma matrix identity that is being used in this work is

$$(\gamma^{mn})^\alpha_\beta (\gamma_{mn})^\gamma_\lambda = 4\gamma_{\beta\lambda}^m \gamma_m^{\alpha\gamma} - 2\delta_\beta^\alpha \delta_\lambda^\gamma - 8\delta_\lambda^\alpha \delta_\beta^\gamma, \quad (\text{A.8})$$

which can be deduced from (A.6). The other relevant one is given by

$$\eta_{mn} (\gamma_{\alpha\beta}^m \gamma_{\gamma\lambda}^n + \gamma_{\alpha\gamma}^m \gamma_{\beta\lambda}^n + \gamma_{\alpha\lambda}^m \gamma_{\gamma\beta}^n) = 0. \quad (\text{A.9})$$

There are several other identities that can be derived from (A.8):

$$(\gamma^{mn})^\alpha{}_\beta \gamma_{mnp}^\lambda = 2 (\gamma^m)^{\alpha\gamma} (\gamma_{pm})^\lambda{}_\beta + 6 \gamma_p^{\alpha\gamma} \delta_\beta^\lambda - (\gamma \leftrightarrow \lambda), \quad (\text{A.10})$$

$$(\gamma_{mn})^\alpha{}_\beta \gamma_{\gamma\lambda}^{mnp} = -2 (\gamma_m)_{\beta\lambda} (\gamma^{pm})^\alpha{}_\gamma + 6 \gamma_{\beta\lambda}^p \delta_\gamma^\alpha - (\gamma \leftrightarrow \lambda), \quad (\text{A.11})$$

$$\gamma_{mnp}^{\alpha\beta} (\gamma^{mnp})^{\gamma\lambda} = 12 \left[\gamma_m^{\alpha\lambda} (\gamma^m)^{\beta\gamma} - \gamma_m^{\alpha\gamma} (\gamma^m)^{\beta\lambda} \right], \quad (\text{A.12})$$

$$\gamma_{mnp}^{\alpha\beta} \gamma_{\gamma\lambda}^{mnp} = 48 \left(\delta_\gamma^\alpha \delta_\lambda^\beta - \delta_\lambda^\alpha \delta_\gamma^\beta \right). \quad (\text{A.13})$$

All of them are very helpful in extracting the pure spinor constraints out of product of bispinors containing space-time vector indices contracted. For example:

$$\begin{aligned} (\bar{\lambda} \gamma_{mnp} r) (\bar{\lambda} \gamma^{mn} \lambda) &= 2 (\bar{\lambda} \gamma^m \bar{\lambda}) (r \gamma_{pm} \lambda) + 6 (r \lambda) (\bar{\lambda} \gamma_p \bar{\lambda}) \\ &\quad - 2 (\bar{\lambda} \gamma^m r) (\bar{\lambda} \gamma_{pm} \lambda) - 6 (\bar{\lambda} \lambda) (\bar{\lambda} \gamma_p r) \\ &= 0. \end{aligned} \quad (\text{A.14})$$

The last identity that is often used in the calculations is

$$\gamma^m \gamma^{n_1 \dots n_k} \gamma_m = (-1)^k (10 - 2k) \gamma^{n_1 \dots n_k}, \quad (\text{A.15})$$

which is particularly useful since it implies that $(\gamma^m \lambda)_\alpha (\gamma_m \lambda)_\beta = 0$ for λ being a pure spinor.

Ordering considerations. This part intended to present some aspects of the ordering prescription that is being used in this work.

Classical relations between currents are now corrected with ordering contributions. For example,

$$N_{\text{cl}}^{mn} (\gamma_n \lambda)_\alpha = \frac{1}{2} J_{\text{cl}} (\gamma^m \lambda)_\alpha \quad (\text{A.16})$$

is valid for any pure spinor λ . Its quantum version is given by

$$\left(N^{mn}, \lambda^\beta \right) \gamma_{\alpha\beta}^p \eta_{np} - \frac{1}{2} \left(J_\lambda, \lambda^\beta \right) \gamma_{\alpha\beta}^m = 2 (\gamma^m \partial \lambda)_\alpha, \quad (\text{A.17})$$

showing that some of the 45 Lorentz generators can be written in terms of the others (in fact, only 10 are independent components).

Another important example is the equation

$$4 \lambda^\alpha T_{\text{cl}} + J_{\text{cl}} \partial \lambda^\alpha + N_{\text{cl}}^{mn} (\gamma_{mn} \partial \lambda)^\alpha = 0, \quad (\text{A.18})$$

which establishes a connection between the energy-momentum tensor and the other currents. Implementing the ordering leads to

$$(\lambda^\alpha, T) + 4 \partial^2 \lambda^\alpha = -\frac{1}{4} (J_\lambda, \partial \lambda^\alpha) - \frac{1}{4} (N_{mn}, (\gamma^{mn} \partial \lambda)^\alpha). \quad (\text{A.19})$$

This relation appears in the construction of the quantum b ghost, as well as

$$\left(\frac{1}{4}\right) \gamma_{mnp}^{\beta\alpha} (N^{mn}, \lambda \gamma^p \partial \theta) = 8 \partial \lambda^{[\alpha} \partial \theta^{\beta]} + \left(\lambda^{[\alpha}, N_{mn} (\gamma^{mn} \partial \theta)^{\beta]}\right) + \left(\lambda^{[\alpha}, J_\lambda \partial \theta^{\beta]}\right), \quad (\text{A.20})$$

which is the ordered version of

$$\gamma_{mnp}^{\alpha\beta} N_{\text{cl}}^{mn} (\lambda \gamma^p \partial \theta) + 4 \lambda^{[\alpha} N_{\text{cl}}^{mn} (\gamma_{mn} \partial \theta)^{\beta]} + 4 \lambda^{[\alpha} J_{\text{cl}} \partial \theta^{\beta]} = 0. \quad (\text{A.21})$$

A further application is the Sugawara construction of the energy-momentum tensor for the minimal ghost sector,

$$T_\lambda = -\frac{1}{20} (N^{mn}, N_{mn}) - \frac{1}{8} (J_\lambda, J_\lambda) + \partial J_\lambda, \quad (\text{A.22})$$

which correctly reproduces the related OPE's.

OPE computations are more systematic⁵ within the prescription (3.7). As an example, it will be shown here that the b ghost for the non-minimal formalism is a primary field.

Concerning b_{-1} , the ordering does not matter and it is straightforward to see that

$$T(z) b_{-1}(y) \sim 2 \frac{b_{-1}}{(z-y)^2} + \frac{\partial b_{-1}}{(z-y)}. \quad (\text{A.23})$$

For b_0 , however, there are some subtleties. Analyzing G^α first,

$$T(z) G^\alpha(y) \sim 2 \frac{G^\alpha}{(z-y)^2} + \frac{\partial G^\alpha}{(z-y)} + \frac{\partial \theta^\alpha}{(z-y)^3}. \quad (\text{A.24})$$

Note that the cubic pole receives contributions from J_λ (the ghost current anomaly), $\partial^2 \theta^\alpha$ and $(\Pi^m, \gamma_m^{\alpha\beta} d_\beta)$:

$$T(z) J_\lambda(y) \sim \frac{8}{(z-y)^3} + \frac{J_\lambda}{(z-y)^2} + \frac{\partial J_\lambda}{(z-y)}, \quad (\text{A.25})$$

$$T(z) \partial^2 \theta^\alpha(y) \sim 2 \frac{\partial \theta^\alpha}{(z-y)^3} + 2 \frac{\partial^2 \theta^\alpha}{(z-y)^2} + \frac{\partial^3 \theta^\alpha}{(z-y)}, \quad (\text{A.26})$$

$$\begin{aligned} T(z) (\Pi^m, \gamma_m^{\alpha\beta} d_\beta)(y) &\sim \left(\frac{\Pi^m}{(z-y)^2} + \frac{\partial \Pi^m}{(z-y)}, \gamma_m^{\alpha\beta} d_\beta \right) \\ &\quad + \left(\Pi^m, \frac{\gamma_m^{\alpha\beta} d_\beta}{(z-y)^2} + \frac{\gamma_m^{\alpha\beta} \partial d_\beta}{(z-y)} \right). \end{aligned} \quad (\text{A.27})$$

According to the ordering prescription, the first term in the last OPE can be rewritten as

$$\begin{aligned} \frac{1}{2\pi i} \oint \frac{dw}{(w-y)} \left\{ \frac{1}{(z-w)^2} \Pi^m(w) + \frac{1}{(z-w)} \partial \Pi^m(w) \right\} \gamma_m^{\alpha\beta} d_\beta(y) = \\ -10 \frac{\partial \theta^\alpha}{(z-y)^3} + \frac{(\Pi^m, \gamma_m^{\alpha\beta} d_\beta)}{(z-y)^2} + \frac{(\partial \Pi^m, \gamma_m^{\alpha\beta} d_\beta)}{(z-y)}. \end{aligned} \quad (\text{A.28})$$

⁵See chapter 6 of [7], where the normal ordering is presented in details.

Therefore,

$$T(z) \left(\Pi^m, \gamma_m^{\alpha\beta} d_\beta \right) (y) \sim -10 \frac{\partial \theta^\alpha}{(z-y)^3} + 2 \frac{\left(\Pi^m, \gamma_m^{\alpha\beta} d_\beta \right)}{(z-y)^2} + \frac{\partial \left(\Pi^m, \gamma_m^{\alpha\beta} d_\beta \right)}{(z-y)}. \quad (\text{A.29})$$

Adding up all the contributions, equation (A.24) is reproduced. For the whole b_0 ,

$$T(z) b_0(y) \sim \left(\frac{1}{(z-y)} \partial \left(\frac{\bar{\lambda}_\alpha}{\bar{\lambda}\lambda} \right), G^\alpha \right) + \left(\frac{\bar{\lambda}_\alpha}{\bar{\lambda}\lambda}, 2 \frac{G^\alpha}{(z-y)^2} + \frac{\partial G^\alpha}{(z-y)} + \frac{\partial \theta^\alpha}{(z-y)^3} \right) + 2 \frac{O}{(z-y)^2} + \frac{\partial O}{(z-y)}. \quad (\text{A.30})$$

Again, the first term on the right-hand side can be rewritten as

$$\begin{aligned} \frac{1}{2\pi i} \oint \frac{dw}{(w-y)} \frac{1}{(z-w)} \partial_w \left(\frac{\bar{\lambda}_\alpha}{\bar{\lambda}\lambda} \right) (w) G^\alpha(y) \\ = \frac{1}{(z-y)} \left(\partial \left(\frac{\bar{\lambda}_\alpha}{\bar{\lambda}\lambda} \right), G^\alpha \right) - \frac{1}{(z-y)^3} \left(\frac{\bar{\lambda}_\alpha \partial \theta^\alpha}{\bar{\lambda}\lambda} \right). \end{aligned} \quad (\text{A.31})$$

Replacing this equation in (A.30), the cubic pole disappears, yielding a primary field.

For b_1 , b_2 and b_3 , there are no contributions like the one in b_0 (they are all proportional to the pure spinor constraints), therefore the b ghost, given by equation (3.2), is a primary field:

$$T(z) b(y) \sim 2 \frac{b}{(z-y)^2} + \frac{\partial b}{(z-y)}. \quad (\text{A.32})$$

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