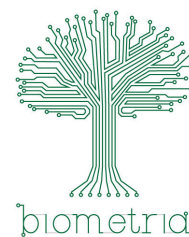




Universidade Estadual Paulista “Júlio de Mesquita Filho”
Instituto de Biociências – Câmpus de Botucatu
Programa de Pós-graduação em Biometria



Mathematical modeling of dengue transmission dynamics in *Wolbachia*-infected mosquitoes

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Botucatu
2026

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Resumo

A dengue, doença causada pelo vírus da dengue (DENV), é endêmica nas Américas e constitui a arbovirose de maior prevalência mundial, apresentando índices crescentes de morbidade e mortalidade. Trata-se de uma doença sistêmica, sazonal, infecciosa, de caráter agudo e febril, transmitida à população humana pela picada de fêmeas do mosquito *Aedes aegypti* ou *Aedes albopictus* infectadas pelo DENV. A ineficiência dos métodos usuais de controle, químicos e mecânicos, a expansão do mosquito para novas áreas e o impacto dessa arbovirose sobre o sistema de saúde tornam urgente a busca por novas estratégias de controle. Nesse contexto, a liberação de mosquitos infectados com *Wolbachia*, uma bactéria intracelular capaz de inibir a replicação do vírus no organismo do vetor, tem sido utilizada em larga escala. Adicionalmente, a vacinação da população — atualmente em curso — pode atuar como uma peça-chave na redução e eventual supressão dos casos de infecção por dengue. Neste trabalho, propõe-se um modelo de equações diferenciais ordinárias para descrever a interação entre as populações de vetores infectados e não infectados por *Wolbachia*, a população humana e o vírus da dengue. Considera-se ainda a vacinação da população humana. Os resultados mostram como o efeito combinado da vacinação e da redução da capacidade vetorial altera a prevalência da doença, destacando regiões no espaço de parâmetros nas quais uma ou ambas as medidas de controle podem alcançar reduções substanciais ou até a eliminação completa da transmissão da dengue.

Palavras-chave: controle biológico; vacinação; equações diferenciais ordinárias; limiares.

Abstract

Dengue, a disease caused by the dengue virus (DENV), is endemic in the Americas and constitutes the most prevalent arboviral disease worldwide, with increasing rates of morbidity and mortality. It is a systemic, seasonal, acute febrile infectious disease transmitted to the human population through the bite of female *Aedes aegypti* or *Aedes albopictus* mosquitoes infected with DENV. The inefficiency of conventional chemical and mechanical control methods, the expansion of the mosquito into new areas, and the impact of this arboviral disease on public health systems make the search for new control strategies urgent. In this context, the release of *Wolbachia*-infected mosquitoes—harboring an intracellular bacterium capable of inhibiting viral replication within the vector—has been implemented on a large scale. Additionally, vaccination of the population, which is currently underway, may play a key role in reducing and potentially suppressing dengue cases. In this work, we propose a system of ordinary differential equations to describe the interaction between *Wolbachia*-infected and uninfected vector populations, the human population, and the dengue virus. Vaccination of the human population is also incorporated into the model. The results illustrate how the combined effects of vaccination and reduced vectorial capacity modify disease prevalence, highlighting regions of parameter space in which one or both control measures can achieve substantial reductions or even complete elimination of dengue transmission.

Key-words: biological control; vaccination; ordinary differential equations; thresholds.

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1 Introduction

Dengue is currently the most prevalent arboviral disease worldwide, being reported in more than 100 tropical and subtropical countries. The dengue virus (DENV) is an arbovirus belonging to the *Flaviviridae* family and the *Flavivirus* genus, and it comprises four distinct serotypes (DENV-1, DENV-2, DENV-3, and DENV-4), with DENV-2 being the most prevalent serotype in Brazil (MENEZES et al., 2021). Transmission occurs through the bite of the mosquito *Aedes aegypti* Linnaeus, 1762 (Diptera: Culicidae), which is considered the primary vector of the disease in Brazil. It is known that *A. aegypti* prefers tropical, warm, and humid environments, which also characterizes dengue as a seasonal disease. These mosquitoes typically breed in stagnant water habitats (BIASSOTI; ORTIZ, 2017).

In 2023, a total of 1.408.684 dengue cases were confirmed in Brazil, of which 1.179 were fatal. In 2024, dengue virus infection in Brazil reached a historical record, with 5.973.343 reported cases nationwide - an increase of 324.04% compared to 2023 - and 6.321 deaths attributed to the disease. In 2025, 1.469.018 dengue cases had been confirmed in Brazil, representing a 75.41% decrease relative to the previous year, with 1.776 deaths caused by the virus. In the same year, 877.795 dengue cases were confirmed in the State of São Paulo, including 1.480 cases in the city of Botucatu (Governo Federal, 2025).

Currently, the main methods for controlling the disease include public education and prophylaxis (reducing sites with standing water, where the vector lays its eggs), mosquito population control (with insecticides and larvicides), and medical intervention after infection (mainly hydration and symptomatic relief). During outbreaks, the most direct and effective approach to ensure vector population control is the use of insecticides; however, this is not a sustainable method, since *A. aegypti* has developed resistance to insecticides, and their continuous use can cause severe environmental damage (BALDACCHINO et al., 2015). Thus, the search for alternative control strategies, such as biological control, has become increasingly urgent.

Wolbachia is an intracellular symbiotic bacterium found in more than half of all insect species worldwide, and its application in biological control has attracted considerable attention in recent years. It has been demonstrated that the use of *Wolbachia* can inhibit dengue virus replication in *A. aegypti* (HOFFMANN et al., 2011; WALKER et al., 2011). Different *Wolbachia* strains, such as wMel, wAlbA, wAlbB, wAu, and wMelPop, produce distinct effects on the mosquito, including a reduction in the vector's lifespan, preventing the dengue virus from becoming

infectious within its lifetime (WALKER et al., 2011), or the deformation of the proboscis (mouth-part structure), rendering the vector ineffective (ITURBE-ORMAETXE; WALKER; O'NEILL, 2011). Another well-documented effect is the reduction of dengue virus particles present in the mosquito's saliva (TURLEY et al., 2009). One of the most significant effects induced by *Wolbachia* is the phenomenon known as Cytoplasmic Incompatibility (CI), which occurs between the gametes of infected males and the eggs of uninfected females, leading to embryo death. In contrast, females infected with *Wolbachia* produce viable infected offspring when mating with either infected or uninfected males through maternal transmission, resulting in a reproductive advantage over uninfected females (WALKER et al., 2011). These factors demonstrate that the use of this bacterium is indeed effective as a biological control strategy.

In Brazil, the *Wolbachia* Method, patented by the World Mosquito Program (WMP), is implemented by the Oswaldo Cruz Foundation (Fiocruz) with funding from the Brazilian Ministry of Health. The project aims to promote the release of *Aedes aegypti* mosquitoes infected with *Wolbachia* as a strategy for vector control and for the suppression or interruption of the transmission of arboviruses such as Dengue, Zika, and Chikungunya. The first field trials began in 2014 in the State of Rio de Janeiro, in Tubiacanga, a neighborhood located on Ilha do Governador Island. In Niterói, Rio de Janeiro, the release of *Wolbachia*-infected mosquitoes started as a pilot project in the Jurujuba neighborhood in 2015. In 2017, the program was expanded for the first time within the municipality, and by 2021 the method already covered 75% of the city. Currently, Niterói is the first city in Brazil to be protected by the method; the 158 confirmed dengue cases reported in the year of project implementation decreased to 67 in 2023, representing a reduction of approximately 64%. Data demonstrating the effectiveness of the protection provided by the *Wolbachia* Method were released by WMP Brazil. In addition to the decrease in dengue cases, these data also indicate a 60% reduction in Chikungunya and a 40% reduction in Zika in the areas where the entomological intervention was carried out. The *Wolbachia* Method is also being applied in Belo Horizonte, state of Minas Gerais. Its implementation in the state capital took place in stages. The first stage, in the Venda Nova region, was completed in January 2021. In 2022, a new expansion phase began, and the releases were concluded in 2023. Currently, a randomized controlled clinical trial, of great epidemiological importance, is underway to assess the method's efficacy and its impact in terms of population protection. Belo Horizonte is also home to the *Wolbachia* Biofactory, responsible for maintaining the complete life cycle of *A. aegypti* mosquitoes infected with *Wolbachia* (Wolbitos), from egg production to adult mosquitoes, as well as the distribution of these biological materials and administrative coordination. The facility is expected to reach a weekly production capacity of approximately two million mosquitoes (Fundação Oswaldo Cruz, 2024).

In the municipality of Joinville, in the State of Santa Catarina, the program began in December 2023. In 2024, the community engagement phase was initiated, consisting of outreach activities to promote the *Wolbachia* Method. These actions involved public and state schools, including teacher training, as well as the participation of teams engaged in the process. This

was the final phase before the release of mosquitoes, which started in August 2024. For the first release phase, 20 weeks of mosquito releases and 70 routes were planned, meaning that the releases continued until January 2025 (DIVESC, 2024). For the second release phase, technical teams established 3.844 release points distributed across 84 routes. The strategy provides for 26 weeks of field releases through early 2026. The selection of neighborhoods considered criteria such as the number of breeding sites and confirmed cases. The selected neighborhoods comprise approximately 75% of Joinville's population (Prefeitura de Joinville, 2024). In Joinville, at the beginning of 2025, a 98% reduction in dengue infection cases was observed compared to the previous year. According to the Joinville Department of Health, 58.322 cases and 66 deaths from the disease were recorded in the first four months of 2024. During the same period in 2025, 622 cases and no deaths were confirmed (G1 Santa Catarina, 2025).

Other Brazilian cities where the technology is being applied include Campo Grande (MS) and Petrolina (PE). Community engagement activities are also planned to be carried out in Uberlândia (MG), Presidente Prudente (SP), and Natal (RN) (FIOTEC, 2024).

Developed by the Butantan Institute in Brazil, the tetravalent dengue vaccine Butantan-DV has been formally approved by the National Health Surveillance Agency (Anvisa), on 26 November 2025, for use in the Brazilian population aged 12 to 59 years and represents the first single-dose dengue vaccine in the world protecting against all four dengue virus serotypes. Clinical trials conducted with over 16 000 volunteers demonstrated an overall efficacy of approximately 74% against symptomatic dengue, more than 91% effectiveness against severe disease, and 100% protection against hospitalization due to dengue, with robust safety profiles in both previously exposed and previously unexposed individuals (Instituto Butantan, 2026). Following the regulatory approval, a first batch of doses has already been manufactured and delivered for early deployment through the Brazilian Unified Health System (SUS). The national vaccination strategy launched in January 2026 includes a pilot mass vaccination campaign in Botucatu (São Paulo), where the immunization program targets eligible residents aged 15 to 59 years in order to evaluate the real-world impact of vaccination on dengue transmission and disease burden (Ministério da Saúde, 2026). The federal Ministry of Health and Instituto Butantan plan to scale production and distribution throughout 2026 and beyond, in coordination with the National Immunization Program (PNI), with the aim of progressively expanding coverage and integrating Butantan-DV into routine public health vaccination schedules (Presidência da República – Casa Civil, 2025; Agência Brasil, 2025).

Another immunizer, produced by the Japanese pharmaceutical company Takeda, the Qdenga vaccine, made available through the Brazilian Unified Health System (SUS), is a tetravalent immunizer developed from a live attenuated virus, that is, a weakened form of the infectious microorganism. This condition enhances the immune system's response, acting similarly to the body's natural defense during dengue infection (CAMBAUVA, 2024). By the end of 2025, Brazil is expected to receive 5.2 million doses, in addition to a donation of 1.3 million doses, allowing for

the vaccination of 3.2 million people with the two doses required to complete the immunization schedule (SOCIAL, 2024).

Mathematical modeling in the context of biological control applied to population dynamics brings us back to discrete-time models, such as the work of Caspari and Watson (1959), which proposed one of the first mathematical models to study the effects of cytoplasmic incompatibility caused by infections on the population dynamics of mosquitoes. It was established that if the infection does not generate fitness costs, its frequency should increase; however, in a scenario where the infection reduces insect fecundity, for the infection to continue spreading, its frequency in the population must exceed a critical threshold, corresponding to an unstable equilibrium point of the system. This study concluded that cytoplasmic incompatibility has a strong influence on mosquito population dynamics, as it promotes a reproductive advantage for the infected population.

From the perspective of discrete models, several authors have modeled the effects of *Wolbachia* on vector population dynamics and its impact on dengue transmission. Yu and Zheng (2019) proposed a discrete model to study the persistence of *Wolbachia* infection that exhibits various dynamic properties, including bistable, semi-stable, and stable equilibrium points. Their model describes the infection frequency of *Wolbachia* in mosquitoes in terms of cytoplasmic incompatibility induced by the bacterium, fecundity cost, and maternal transmission failure of *Wolbachia* from mothers to offspring. The model also considers competition between *Wolbachia*-infected and uninfected mosquito populations. It was observed that with a high cytoplasmic incompatibility rate and a low maternal transmission failure rate, the population size of *Wolbachia*-infected mosquitoes rapidly becomes greater than or equal to that of wild mosquitoes, provided that the impact of infection on mosquito fitness is not substantial. Numerical simulations showed that there exists a lower threshold for the initial number of released *Wolbachia*-infected mosquitoes required for the infection to establish in the population, and that successive releases of infected mosquitoes accelerate its persistence. Another model investigating the introduction of infection into wild mosquito populations was proposed by Li, Guo and Xing (2020). This model identifies four equilibrium points — the absence and prevalence of infected mosquitoes, population coexistence, and the existence of a small number of infected mosquitoes — whose stability depends on the birth and death rates of uninfected and infected vectors, the transmission rate of *Wolbachia*, and the fecundity cost associated with inter-population competition.

Other authors have used systems of ordinary differential equations to study the use of *Wolbachia* as a means of suppressing dengue virus transmission. Li and Liu (2021) proposed a compartmental SEI model for *Wolbachia*-free and *Wolbachia*-infected mosquitoes, coupled with a compartmental SIS model for the human population. The dengue-free equilibrium points depend on the parameter that measures the fitness cost of infection in mosquitoes: if there is no cost, three equilibrium points exist; if there is a cost, four equilibrium points are found. Two additional endemic equilibria (with dengue circulation), corresponding to the extinction and persistence of

the *Wolbachia*-infected mosquito population, are also observed. The stability of these equilibrium points depends on the fitness-related parameter and on the basic reproduction number, calculated using the next-generation matrix method. A sensitivity analysis was performed to discuss how the parameters affect the basic reproduction number and the endemic equilibria. This analysis highlighted the importance of the mosquito biting rate, revealing that the endemic equilibrium is most sensitive to this parameter and positively correlated with it. Numerical analyses demonstrated that the introduction of *Wolbachia* drastically reduces dengue virus transmission and that the initial proportion of infected mosquitoes determines whether the infection will successfully establish and spread within the mosquito population. It was concluded that during outbreaks, the release of *Wolbachia*-infected mosquitoes must occur in a sufficiently large fraction to suppress dengue transmission. The study also emphasized the importance of reducing the mosquito biting rate, achievable through the use of protective nets and repellents.

Ogunlade et al. (2023) investigated the effect of releasing *A. aegypti* mosquitoes infected with *Wolbachia* on reducing dengue virus transmission in Townsville, Australia. The model distinguishes between locally acquired and imported human dengue cases. To quantify the impact of releasing infected mosquitoes, the author examined two scenarios: pre-*Wolbachia* and post-*Wolbachia*. The human population is divided into SEIR compartments, while the mosquito population, composed of *Wolbachia*-free and *Wolbachia*-infected mosquitoes, is structured into SEI compartments, in addition to a compartment representing the aquatic stages of the vector. The model also distinguishes between male and female mosquitoes, such that mating between uninfected females and infected males produces no viable offspring due to cytoplasmic incompatibility. The influence of climatic factors on the mosquito life cycle is also incorporated; the mosquito's maximum environmental carrying capacity, related to the total human population, is modeled as a periodic function. The effectiveness of the *Wolbachia* intervention is evaluated by comparing the cumulative dengue cases in the presence and absence of *Wolbachia*-infected populations. With respect to imported dengue cases, it was observed that they were significantly lower during the pre-*Wolbachia* period, reflecting an increase in dengue cases in nearby regions. When considering only locally transmitted dengue infections, this number decreased drastically after the introduction of infected mosquitoes—highlighting the importance of distinguishing between local and imported cases. Finally, parameter estimation revealed that the probability of dengue transmission from a *Wolbachia*-infected mosquito to a susceptible human is approximately twenty times lower than that of an uninfected mosquito. Based on the estimated transmission rates, *Wolbachia* proved highly effective in substantially reducing dengue incidence in Townsville.

Within the scope of mathematical models addressing *Wolbachia* based dengue control, Escobar-Lasso and Vasilieva (2020) formulated a two-dimensional monotone model of ordinary differential equations describing the competition between *Wolbachia*-infected and wild mosquito populations. The model retains the essential biological mechanisms, maternal transmission and cytoplasmic incompatibility, while remaining analytically tractable through the theory of monotone dynamical systems. Four equilibria were identified: the trivial equilibrium corresponding to

total extinction of mosquitoes, two boundary equilibria representing the exclusive persistence of each mosquito type, and one coexistence equilibrium. Under determined parameter conditions, the boundary equilibria are asymptotically stable, the coexistence equilibrium is unstable, and the extinction state is a nodal repeller. This configuration defines a bistable system where establishment of *Wolbachia*-infected mosquitoes requires initial values above a threshold manifold separating the attraction basins of the two boundary equilibria.

Building upon the previous monotone formulation, [Orozco-Gonzales et al. \(2024\)](#) developed a bidimensional dynamical system to describe the long-term persistence of different *Wolbachia* strains in competing mosquito populations. The proposed model incorporates imperfect maternal transmission, incomplete cytoplasmic incompatibility, and direct loss of *Wolbachia* due to thermal stress. The system can exhibit up to four equilibrium points: a trivial equilibrium representing total extinction of mosquitoes, a *Wolbachia*-free equilibrium corresponding to infection loss, and two coexistence equilibria where infected and uninfected populations persist simultaneously. Under determined conditions, one coexistence state is unstable while the other is locally asymptotically stable, shaping a bistable regime in which the long-term outcome depends on the initial densities of each mosquito population. This qualitative behavior explains why some strains, despite being initially introduced in high numbers, may fail to persist in the environment. Numerical analyses confirmed these results and compared the performance of four *Wolbachia* strains (*wMel*, *wMelPop*, *wAlbB*, and *wAu*), revealing distinct persistence capacities and temperature-dependent stability patterns among them.

In this context, the present study proposes an extended compartmental framework that integrates both human and mosquito dynamics, in order to explore the combined effects of human population vaccination and *Wolbachia* based control on dengue transmission. By considering the human vaccination process within the interaction dynamics of mosquito populations, the model enables a direct comparison between two major dengue suppression strategies: biological control through *Wolbachia* dissemination and immunization of the human population. The analysis focuses on how the interplay between these interventions can modify disease prevalence and the persistence of infection, highlighting parameter regimes under which either or both control methods can lead to a significant reduction or elimination of dengue cases.

Objective

Mathematical modeling of the population dynamics of *Wolbachia*-infected and wild (non-*Wolbachia*-infected) *Aedes aegypti* mosquitoes, as well as the human population, under conditions of dengue virus circulation, considering a scenario in which vaccination of the human population is implemented.

Specific Objectives:

-
- i. To construct a compartmental mathematical model describing the population dynamics of wild and *Wolbachia*-infected *Aedes aegypti* mosquitoes, along with the human population, under dengue virus transmission conditions.
 - ii. To incorporate the effects of human vaccination into the model and evaluate its impact on the transmission dynamics of dengue.
 - iii. To perform a theoretical analysis of the model, including identification of equilibrium points and examination of their stability.
 - iv. To simulate various scenarios using numerical methods to assess the effectiveness of *Wolbachia* deployment and human vaccination, both individually and in combination, in reducing dengue transmission.

2 Methods

The present work is developed in the format of mathematical propositions and proofs. For clarity and readability, the main text presents only the results and their interpretations in continuous form, while the complete statements of all propositions and their corresponding proofs are presented in **Appendix A**, where **Appendix A.1** addresses the mosquito subsystem and **Appendix A.2** the human subsystem.

2.1 Proposed Model

We model the dynamics of *Aedes aegypti* using a compartmental SI (Susceptible–Infected) framework, structured to distinguish two vector subpopulations: mosquitoes free of *Wolbachia* and mosquitoes infected with *Wolbachia*. For the human population, assumed to remain constant, we adopt a compartmental SIR (Susceptible–Infected–Recovered) model to describe the host epidemiological dynamics. Furthermore, a vaccination scenario for the human population is explicitly considered, and immunity loss is not incorporated. Transmission occurs between susceptible vectors and infected humans through mosquito bites, and likewise between infected vectors and susceptible humans, also through the mosquito bite. We assume that the mosquito-human transmission rate is the same for both mosquito populations, whereas the human-mosquito transmission rate is lower for *Wolbachia*-infected mosquitoes. The model assumes a 1:1 mosquito sex ratio, and all vector variables represent female mosquitoes. Lastly, the aquatic stages of the vector are not considered.

Table 1 summarizes the state variables and their definitions, whereas Table 2 presents the parameters used in the formulation of the model and their respective definitions. Figure 1 provides a schematic representation of the model structure corresponding to the system of ordinary differential equations defined by (1)–(7).

Table 1 – State variables and definitions.

Variable	Definition
$S_h(t)$	Susceptible human population
$I_h(t)$	Infected human population
$R_h(t)$	Recovered human population
$S_f(t)$	Susceptible wild mosquito population
$I_f(t)$	Infected wild mosquito population
$S_w(t)$	Susceptible <i>Wolbachia</i> mosquito population
$I_w(t)$	Infected <i>Wolbachia</i> mosquito population

Table 2 – Parameters and definitions.

Parameter	Definition	Unit
μ_h	Human mortality rate	day ⁻¹
γ_h	Human recovery rate	day ⁻¹
ν_h	Vaccination rate of the human population	day ⁻¹
σ	Factor related to competition between mosquitoes	individual ⁻¹
$\rho_i, i \in \{f, w\}$	Mosquito fertility rates ($\rho_f > \rho_w$)	day ⁻¹
$\mu_i, i \in \{f, w\}$	Mosquito mortality rates ($\mu_f < \mu_w$)	day ⁻¹
$\bar{\beta}_i, i \in \{f, w\}$	Human-mosquito transmission rates ($\bar{\beta}_f > \bar{\beta}_w$)	day ⁻¹
$\beta_i, i \in \{f, w\}$	Mosquito-human transmission rates ($\beta_f = \beta_w$)	day ⁻¹
$\Lambda_i, i \in \{f, w\}$	Recruitment rates	individual · day ⁻¹
$\delta \in (0, 1)$	Probability of maternal transmission of <i>Wolbachia</i>	-
$\eta \in (0, 1)$	Strength of cytoplasmic incompatibility	-

Human Population

$$\frac{dS_h}{dt} = \mu_h N_h - (\beta_f I_f + \beta_w I_w) \left(\frac{S_h}{N_h} \right) - (\mu_h + \nu_h) S_h \quad (1)$$

$$\frac{dI_h}{dt} = (\beta_f I_f + \beta_w I_w) \left(\frac{S_h}{N_h} \right) - (\gamma_h + \mu_h) I_h \quad (2)$$

$$\frac{dR_h}{dt} = \nu_h S_h + \gamma_h I_h - \mu_h R_h \quad (3)$$

where $N_h(t) = S_h(t) + I_h(t) + R_h(t)$; with $N_h(t) \equiv N_h$. Because the human population is constant, we can consider only two equations (e.g., (1) and (2)), since we can obtain R_h as $R_h = N_h - (S_h + I_h)$.

Wild Mosquitoes

$$\frac{dS_f}{dt} = \Lambda_f e^{-\sigma M(t)} - \bar{\beta}_f S_f \left(\frac{I_h}{N_h} \right) - \mu_f S_f \quad (4)$$

$$\frac{dI_f}{dt} = \bar{\beta}_f S_f \left(\frac{I_h}{N_h} \right) - \mu_f I_f \quad (5)$$

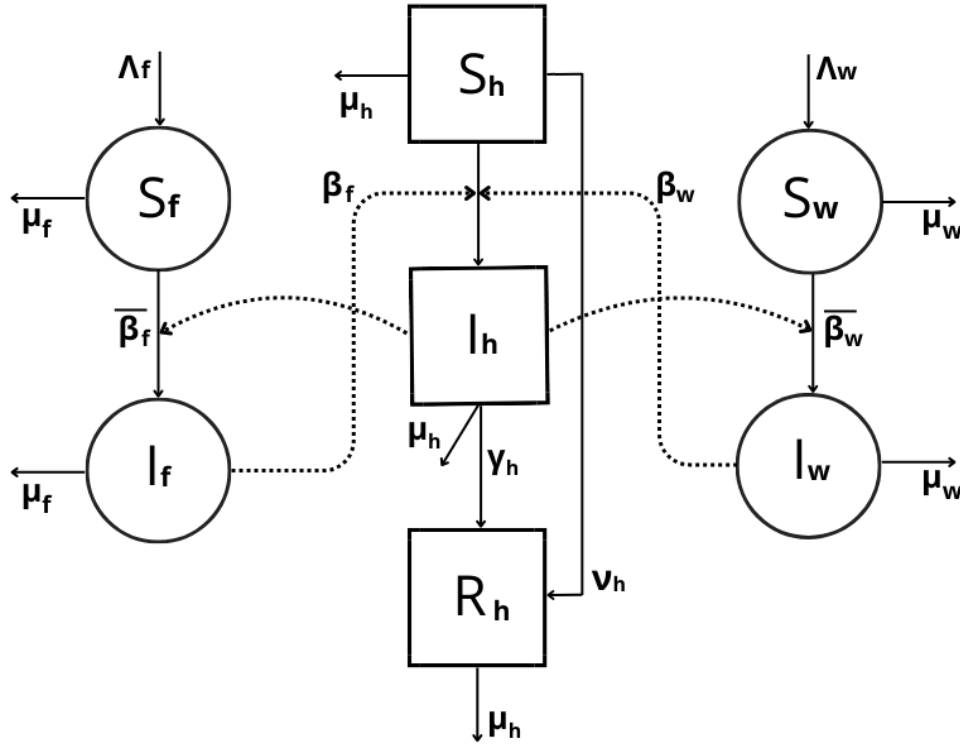


Figure 1 – Schematic representation of the mathematical model. Source: author's own elaboration.

***Wolbachia*-Infected Mosquitoes**

$$\frac{dS_w}{dt} = \Lambda_w e^{-\sigma M(t)} - \bar{\beta}_w S_w \left(\frac{I_h}{N_h} \right) - \mu_w S_w \quad (6)$$

$$\frac{dI_w}{dt} = \bar{\beta}_w S_w \left(\frac{I_h}{N_h} \right) - \mu_w I_w \quad (7)$$

where

$$\Lambda_f = \rho_f (S_f + I_f) \frac{(S_f + I_f) + (1-\eta)(S_w + I_w)}{M(t)} + \rho_w (1 - \delta) (S_w + I_w),$$

$$\Lambda_w = \rho_w \delta (S_w + I_w),$$

$$M(t) = S_f(t) + I_f(t) + S_w(t) + I_w(t).$$

Representing the biological features of *Wolbachia*-infected mosquitoes, we consider $\rho_w < \rho_f$, for mosquito fertility rates, and $\mu_f < \mu_w$, for mosquito mortality rates. One can observe that both recruitment rates, Λ_f , Λ_w , are multiplied by $e^{-\sigma M(t)}$, where σ is the density-dependent competition parameter. Furthermore, the mosquito and human dynamics are coupled through the transmission parameters, which is reduced for *Wolbachia*-infected mosquitoes, that is, $\bar{\beta}_w < \bar{\beta}_f$.

Observing the recruitment rate Λ_f , the first term, $\rho_f (S_f + I_f)$, represents the reproductive contribution of wild female mosquitoes to the growth of the wild mosquito population. These females may mate either with wild male mosquitoes or with *Wolbachia*-infected male mosquitoes, accounting for incomplete cytoplasmic incompatibility $(1 - \eta)$. Such interactions are governed by the encounter rates $(S_f + I_f)M(t)^{-1}$ and $(S_w + I_w)M(t)^{-1}$, respectively. The second term,

$\rho_w(S_w + I_w)$, represents the reproductive contribution of *Wolbachia*-infected female mosquitoes to the growth of the wild mosquito population, where $(1 - \delta)$ accounts for imperfect maternal transmission.

Looking at the recruitment rate Λ_w , the quantity $\rho_w(S_w + I_w)$ represents the reproductive contribution of female *Wolbachia*-infected mosquitoes to the growth of the *Wolbachia*-infected mosquito population, where $\delta < 1$ denotes the maternal transmission parameter, which governs the introduction of *Wolbachia*-infected offspring, since infected females transmit the bacterium to their progeny regardless of the male mosquito's infection condition.

Lastly, we present Table 3, which summarizes the biologically relevant parameter sets considered throughout this work, representing the life-history traits of wild mosquitoes and the *wMel*, *wMelPop*, *wAlbB*, and *wAu* *Wolbachia* strains (OROZCO-GONZALES et al., 2024).

Table 3 – Numerical values of the mosquito life traits under favorable ambient temperatures.

Wolbachia strain	None	wMel	wMelPop	wAlbB	wAu
Female fecundity	$\rho_f = 4.55$	$\rho_w = 4.55 \times 0.9$	$\rho_w = 4.55 \times 0.5$	$\rho_w = 4.55 \times 0.8$	$\rho_w = 4.55 \times 0.98$
Mortality	$\mu_f = 0.03$	$\mu_w = 0.03/0.9$	$\mu_w = 0.03/0.5$	$\mu_w = 0.03/0.8$	$\mu_w = 0.03/0.95$
CI degree	N/A	$\eta \in [0.9, 1]$	$\eta \in [0.9, 1]$	$\eta \in [0.98, 1]$	$\eta = 0$
Maternal transmission	N/A	$\delta \in [0.9, 1]$	$\delta \in [0.9, 1]$	$\delta \in [0.98, 1]$	$\delta \in [0.98, 1]$
Virus blockage	None	Moderate	Very high	High	High

2.2 Fast Vector Dynamics

Mosquitoes have a reasonably shorter lifespan than humans, so we can consider the human population to be constant while observing changes in mosquitoes populations. At first, we solve the system of mosquito differential equations while treating the human state variables as constant values. Consequently, mosquito populations are modeled independently, with human variables entering the equations only as fixed values rather than dynamic components. Once the mosquito subsystem has been analyzed and its equilibrium points determined, these solutions can be substituted into the human equations. The resulting reduced system describes the long-term behavior of the human population.

3 General Mosquito Subsystem

The mosquitoes subsystem of ordinary differential equations is written as

$$\begin{aligned}
 \frac{dS_f}{dt} &= \Lambda_f e^{-\sigma M(t)} - \bar{\bar{\beta}}_f S_f I_h - \mu_f S_f \\
 \frac{dI_f}{dt} &= \bar{\bar{\beta}}_f S_f I_h - \mu_f I_f \\
 \frac{dS_w}{dt} &= \Lambda_w e^{-\sigma M(t)} - \bar{\bar{\beta}}_w S_w I_h - \mu_w S_w \\
 \frac{dI_w}{dt} &= \bar{\bar{\beta}}_w S_w I_h - \mu_w I_w
 \end{aligned} \tag{A}$$

with recruitment rates

$$\Lambda_f = \rho_f (S_f + I_f) \frac{(S_f + I_f) + (1-\eta)(S_w + I_w)}{M(t)} + \rho_w (1 - \delta) (S_w + I_w),$$

$\Lambda_w = \rho_w \delta (S_w + I_w)$, alongside with

$$\bar{\bar{\beta}}_f = \frac{\bar{\beta}_f}{N_h}, \quad \bar{\bar{\beta}}_w = \frac{\bar{\beta}_w}{N_h} \quad \text{and} \quad M(t) = S_f(t) + I_f(t) + S_w(t) + I_w(t).$$

3.1 Positiveness

Define the nonnegative set

$$\Omega_m = \{(S_f, I_f, S_w, I_w) \in \mathbb{R}^4 : S_f \geq 0, I_f \geq 0, S_w \geq 0, I_w \geq 0\}.$$

Assume that the initial condition satisfies $(S_f(0), I_f(0), S_w(0), I_w(0)) \in \Omega_m$. We prove that Ω_m is positively invariant for the mosquito subsystem by verifying that, on each boundary of Ω_m , the corresponding component of the vector field is nonnegative. This guarantees that the vector field is inward-pointing or tangent on the boundary and prevents trajectories from crossing Ω_m .

Boundary $S_f = 0$. If $S_f = 0$, then

$$\left. \frac{dS_f}{dt} \right|_{S_f=0} = \Lambda_f e^{-\sigma M(t)} \geq 0,$$

since $e^{-\sigma M(t)} > 0$ and $\Lambda_f \geq 0$ for nonnegative mosquito variables and parameters.

Boundary $I_f = 0$. If $I_f = 0$, then

$$\left. \frac{dI_f}{dt} \right|_{I_f=0} = \bar{\beta}_f S_f I_h \geq 0,$$

because $S_f \geq 0$ and $I_h \geq 0$.

Boundary $S_w = 0$. If $S_w = 0$, then

$$\left. \frac{dS_w}{dt} \right|_{S_w=0} = \Lambda_w e^{-\sigma M(t)} \geq 0,$$

since $\Lambda_w = \rho_w \delta (S_w + I_w) \geq 0$.

Boundary $I_w = 0$. If $I_w = 0$, then

$$\left. \frac{dI_w}{dt} \right|_{I_w=0} = \bar{\beta}_w S_w I_h \geq 0,$$

because $S_w \geq 0$ and $I_h \geq 0$.

Thus, on each boundary of Ω_m , the corresponding component of the vector field is nonnegative, and the trajectory cannot cross Ω_m through the boundary. Therefore, Ω_m is positively invariant for the mosquito subsystem. Consequently, for any initial condition $(S_f(0), I_f(0), S_w(0), I_w(0)) \in \Omega_m$, the solution satisfies

$$S_f(t) \geq 0, \quad I_f(t) \geq 0, \quad S_w(t) \geq 0, \quad I_w(t) \geq 0, \quad \forall t \geq 0.$$

3.2 Jacobian Matrix of Mosquito Subsystem

To analyze the local stability of the mosquito equilibrium points, we calculated the Jacobian matrix (J) of the mosquito subsystem defined in (A). Furthermore, we shall evaluate the matrix J at each corresponding equilibrium.

Let $J_f^\dagger := \sigma \Lambda_f e^{-\sigma M}$, $J_w^\dagger := \sigma \Lambda_w e^{-\sigma M}$, $\frac{\partial \Lambda_f}{\partial S_f} := \dot{\Lambda}_{Sf}$, $\frac{\partial \Lambda_f}{\partial I_f} := \dot{\Lambda}_{If}$, $\frac{\partial \Lambda_f}{\partial S_w} := \dot{\Lambda}_{Sw}$, $\frac{\partial \Lambda_f}{\partial I_w} := \dot{\Lambda}_{Iw}$ and $z := e^{-\sigma M}$, for brevity.

The Jacobian matrix of the mosquito subsystem is written as

$$J = \begin{bmatrix} \dot{\Lambda}_{Sf} z - J_f^\dagger - \bar{\beta}_f I_h - \mu_f & \dot{\Lambda}_{If} z - J_f^\dagger & \dot{\Lambda}_{Sw} z - J_f^\dagger & \dot{\Lambda}_{Iw} z - J_f^\dagger \\ \bar{\beta}_f I_h & -\mu_f & 0 & 0 \\ -J_w^\dagger & -J_w^\dagger & \rho_w \delta z - J_w^\dagger - \bar{\beta}_w I_h - \mu_w & \rho_w \delta z - J_w^\dagger \\ 0 & 0 & \bar{\beta}_w I_h & -\mu_w \end{bmatrix}.$$

The complete derivation of the Jacobian matrix for the mosquito subsystem is available in **Appendix A**.

3.3 Mosquito Subsystem Equilibrium Points

Throughout the algebraic analysis, the following quantities are introduced:

1. $Q_f := \frac{\rho_f}{\mu_f}$ denotes the basic offspring number of wild mosquitoes, defined as the mean number of descendants produced by a single wild mosquito over its lifetime.
2. $Q_w := \frac{\rho_w \delta}{\mu_w}$ denotes the basic offspring number of *Wolbachia*-infected mosquitoes, representing the mean number of infected descendants produced by one infected mosquito during its lifespan, accounting for maternal transmission.
3. $Q_{w,f} := \frac{\rho_w(1-\delta)}{\mu_f}$ represents the expected number of wild mosquitoes produced by a *Wolbachia*-infected mosquito. This quantity captures the contribution of imperfect maternal transmission, whereby a fraction of the offspring emerges uninfected.
4. $Q_c := \frac{Q_{w,f} + Q_w + \eta Q_f}{Q_f}$ defines the coexistence threshold for the mosquito populations. In particular, the condition $Q_c > 1$ characterizes the regime under which wild and *Wolbachia*-infected mosquitoes can coexist.
5. $M = S_f + I_f + S_w + I_w = \frac{1}{\sigma} \ln(Q_w)$ denotes the total mosquito population size at the coexistence equilibrium.

Within our analytical work, we have determined four equilibrium points for the mosquito subsystem, three of which are non-trivial. Complete derivations are provided in **Appendix A** (**Proposition A.1.1**).

3.3.1 Trivial Equilibrium

The trivial equilibrium

$$E_0 = (0, 0, 0, 0)$$

corresponds to the extinction of both wild and *Wolbachia*-infected mosquito populations.

Setting

$$\frac{dS_f}{dt} = \frac{dI_f}{dt} = \frac{dS_w}{dt} = \frac{dI_w}{dt} = 0$$

and solving the resulting algebraic system yields $S_f = I_f = S_w = I_w = 0$, which defines the equilibrium point E_0 .

3.3.1.1 Local Stability of Equilibrium Point E_0

At the trivial equilibrium $E_0 = (0, 0, 0, 0)$, the Jacobian matrix reduces to

$$J(E_0) = \begin{bmatrix} \rho_f - \bar{\beta}_f I_h - \mu_f & \rho_f & \rho_w(1 - \delta) & \rho_w(1 - \delta) \\ \bar{\beta}_f I_h & -\mu_f & 0 & 0 \\ 0 & 0 & \rho_w \delta - \bar{\beta}_w I_h - \mu_w & \rho_w \delta \\ 0 & 0 & \bar{\beta}_w I_h & -\mu_w \end{bmatrix}.$$

This matrix is block diagonal, with wild-type and *Wolbachia* blocks

$$J_f = \begin{bmatrix} \rho_f - \bar{\beta}_f I_h - \mu_f & \rho_f \\ \bar{\beta}_f I_h & -\mu_f \end{bmatrix}, \quad J_w = \begin{bmatrix} \rho_w \delta - \bar{\beta}_w I_h - \mu_w & \rho_w \delta \\ \bar{\beta}_w I_h & -\mu_w \end{bmatrix}.$$

At mosquito extinction, $I_h = 0$, yielding

$$J_f = \begin{bmatrix} \rho_f - \mu_f & \rho_f \\ 0 & -\mu_f \end{bmatrix}, \quad J_w = \begin{bmatrix} \rho_w \delta - \mu_w & \rho_w \delta \\ 0 & -\mu_w \end{bmatrix}.$$

The eigenvalues of the wild-type block are given by

$$\lambda_f^- = -\mu_f, \quad \lambda_f^+ = \rho_f - \mu_f = \mu_f(Q_f - 1).$$

On the other side, the eigenvalues of the *Wolbachia* block are written as

$$\lambda_w^- = -\mu_w, \quad \lambda_w^+ = \rho_w \delta - \mu_w = \mu_w(Q_w - 1).$$

Hence, all eigenvalues have negative real parts if and only if

$$Q_f < 1 \quad \text{and} \quad Q_w < 1.$$

Therefore, the trivial equilibrium $E_0 = (0, 0, 0, 0)$, corresponding to complete mosquito extinction, is locally asymptotically stable if, and only if, both basic offspring numbers satisfy $Q_f < 1$ and $Q_w < 1$. The biological interpretation is that if neither mosquito population can reproduce at an average rate of at least one mosquito per generation, both populations will go extinct (**Proposition A.1.2**).

3.3.2 *Wolbachia* Extinction Equilibrium Point

This equilibrium represents a scenario in which the *Wolbachia*-infected mosquito population becomes extinct, while the wild mosquito population persists,

$$E_f = \left(\frac{\mu_f}{\sigma} \frac{\ln(Q_f)}{\mu_f + \bar{\beta}_f I_h}, \frac{\bar{\beta}_f I_h}{\sigma} \frac{\ln(Q_f)}{\mu_f + \bar{\beta}_f I_h}, 0, 0 \right).$$

From a biological perspective, this configuration reflects the possibility that *Wolbachia*-infected mosquitoes fail to establish within wild mosquito population. Moreover, from a mathematical standpoint, it arises because the state

$$(S_f, I_f, S_w, I_w) = (\overline{S}_f, \overline{I}_f, 0, 0)$$

constitutes a particular solution of the algebraic system obtained by setting all right-hand sides of the mosquito subsystem equal to zero.

3.3.2.1 Local Stability of Equilibrium Point E_f

We analyze the local stability of the *Wolbachia* extinction equilibrium point

$$E_f = \left(\frac{\mu_f}{\sigma} \frac{\ln(Q_f)}{\mu_f + \overline{\beta}_f I_h}, \frac{\overline{\beta}_f I_h}{\sigma} \frac{\ln(Q_f)}{\mu_f + \overline{\beta}_f I_h}, 0, 0 \right) : \exists \text{ if } Q_f > 1.$$

Substituting E_f equilibrium values into the Jacobian Matrix leads to

$$J(E_f) = \begin{bmatrix} -\overline{\beta}_f I_h - \mu_f \ln(Q_f) & \mu_f (1 - \ln(Q_f)) & \frac{\rho_w(1 - \delta) - \rho_f(\eta + \ln(Q_f))}{Q_f} & \frac{\rho_w(1 - \delta) - \rho_f(\eta + \ln(Q_f))}{Q_f} \\ \overline{\beta}_f I_h & -\mu_f & 0 & 0 \\ 0 & 0 & \frac{\rho_w \delta}{Q_f} - (\overline{\beta}_w I_h + \mu_w) & \frac{\rho_w \delta}{Q_f} \\ 0 & 0 & \overline{\beta}_w I_h & -\mu_w \end{bmatrix}.$$

The matrix $J(E_f)$ is upper triangular, in block structure, and can be written as

$$J(E_f) = \begin{bmatrix} J_f & B \\ 0 & J_w \end{bmatrix},$$

where J_f and J_w denote the wild-type and *Wolbachia* mosquito blocks, respectively. The eigenvalues of $J(E_f)$ are therefore given by the eigenvalues of J_f and J_w .

For the wild mosquito block J_f , the trace and determinant satisfy

$$\text{tr}(J_f) = -\mu_f [\ln(Q_f) + 1] < 0, \quad \det(J_f) = \mu_f \ln(Q_f) (\mu_f + \overline{\beta}_f I_h) > 0,$$

for all $Q_f > 1$. Hence, both eigenvalues of J_f are real and strictly negative.

Since steady-state E_f corresponds to *Wolbachia* extinction, where the state variables $(S_w, I_w) = (0, 0)$, we assume $\overline{\beta}_w = 0$. Under this assumption, the *Wolbachia* block reduces to

$$J_w = \begin{bmatrix} \frac{\rho_w \delta}{Q_f} - \mu_w & \frac{\rho_w \delta}{Q_f} \\ 0 & -\mu_w \end{bmatrix},$$

which is a triangular matrix, with eigenvalues located on the principal diagonal,

$$\lambda_w^- = -\mu_w < 0, \text{ for all parameter values, and}$$

$$\lambda_w^+ = \mu_w \left(\frac{Q_w}{Q_f} - 1 \right) < 0 \iff \frac{Q_w}{Q_f} - 1 < 0 \iff Q_w < Q_f.$$

One can observe that $J(E_f)$ is an upper-triangular matrix, meaning its eigenvalues are the calculated eigenvalues of J_f and J_w blocks. Under the existence condition of the equilibrium point, $Q_f > 1$, the wild block eigenvalues presents negative real parts, and the stability of E_f is determined by the sign of λ_w^+ :

$$\lambda_w^+ < 0 \iff Q_w < Q_f.$$

Therefore, the *Wolbachia* extinction equilibrium E_f is locally asymptotically stable if, and only if, $Q_f > \max\{1, Q_w\}$. As Q_f and Q_w can be interpreted as mosquito fitness, the biological meaning is that wild mosquito persistence and *Wolbachia* extinction are possible in a scenario where the wild mosquito can produce at least one offspring per generation and its fitness is greater than that of the *Wolbachia*-infected mosquito (**Proposition A.1.3**).

3.3.3 Mosquito Coexistence Equilibria

These equilibria, denoted by $E_\pm = (S_f^\pm, I_f^\pm, S_w^\pm, I_w^\pm)$, represent the persistence of *Wolbachia*-infected mosquitoes within wild mosquito population. Biologically, it corresponds to a scenario in which the population infected with the bacterium becomes established and actively participates in the reproductive dynamics of susceptible wild mosquito populations, producing new *Wolbachia*-infected offspring through maternal transmission while simultaneously inducing cytoplasmic incompatibility.

Let us define the constant quantity $\beta^f := \frac{\bar{\beta}_f I_h}{\mu_f}$. The equilibrium components are given explicitly by

$$S_f^\pm = \frac{M(\beta^f + 1)(Q_c - 1) \pm \sqrt{[M(\beta^f + 1)]^2 - 4\eta[1 + \beta^f(2 + \beta^f)] \frac{Q_{w,f}}{Q_f} M^2}}{2\eta[1 + \beta^f(2 + \beta^f)]},$$

$$I_f^\pm = \beta^f S_f^\pm, \quad S_w^\pm = \frac{\mu_w}{\mu_w + \bar{\beta}_w I_h} \left[\frac{1}{\sigma} \ln(Q_w) - (S_f^\pm + I_f^\pm) \right],$$

$$I_w^\pm = \frac{1}{\sigma} \ln(Q_w) - (S_f^\pm + I_f^\pm + S_w^\pm).$$

We recall that

$$M = S_f + I_f + S_w + I_w = \frac{1}{\sigma} \ln(Q_w)$$

denotes the total mosquito population size at the coexistence equilibria. Since all the state variables are nonnegative, admissibility requires

$$I_w^\pm > 0 \iff \frac{1}{\sigma} \ln(Q_w) > S_f^\pm + I_f^\pm + S_w^\pm \iff Q_w > 1.$$

Hence, we impose the condition $Q_w > 1$ as a necessary requirement for the coexistence equilibria to be positive and admissible, and therefore biologically meaningful.

3.3.3.1 Existence of Mosquito Coexistence Equilibria $E_{\pm}$

The coexistence equilibria arise from the steady-state condition of the susceptible wild mosquito state variable, which reduces to a quadratic equation in S_f of the form

$$AS_f^2 + BS_f + C = 0,$$

with coefficients given explicitly by

$$A = \eta(\beta^f + 1)^2, \quad B = -M(\beta^f + 1)(Q_c - 1), \quad C = \frac{Q_{w,f}}{Q_f} M^2.$$

It is easy to see that $A > 0$, and we recall the previously defined quantities,

$$Q_{w,f} := \frac{\rho_w(1 - \delta)}{\mu_f} \quad \text{e} \quad Q_c := \frac{Q_{w,f} + Q_w + \eta Q_f}{Q_f}.$$

We observe that

$$0 < \delta < 1 \implies C > 0, \quad B < 0 \iff Q_c > 1.$$

Accordingly with Descartes' rule of signs, and considering biologically admissible parameter values, the coefficients satisfy $A > 0$ and $C > 0$. Moreover, when condition $Q_c > 1$ is satisfied, the coefficient B is negative. Consequently, the quadratic equation admits two positive real solutions for S_f , which result in the two coexistence branches denoted by $S_f^{\pm}$ and, in turn, the coexistence equilibria $E_{\pm}$ (**Proposition A.1.4**).

Furthermore, coexistence equilibria of the mosquito subsystem are characterized by the existence of real solutions of the presented quadratic equation in the wild mosquito population S_f . The corresponding discriminant can be written as a function of the maternal transmission probability of *Wolbachia* infection (δ) (**Proposition A.1.5**) and the strength of cytoplasmic incompatibility due to *Wolbachia* infection (η) (**Proposition A.1.6**).

Maternal transmission probability (δ).

The discriminant of the coexistence quadratic expression can be rewritten as a function of the maternal transmission parameter parameter δ .

$$\Delta(\delta) = M^2(\beta^f + 1)^2 \Phi(\delta), \quad \Phi(\delta) = (Q_c - 1)^2 - 4\eta \frac{Q_{w,f}}{Q_f}.$$

where Q_c and $Q_{w,f}$ depend on the δ parameter. Since $M^2(\beta^f + 1)^2 > 0$, real and distinct coexistence equilibria exist if, and only if, $\Phi(\delta) > 0$.

Introducing the constants

$$a_1 = \frac{\rho_w}{\mu_f}, \quad a_2 = \frac{\rho_w}{\mu_w}, \quad Q_f = \frac{\rho_f}{\mu_f},$$

and defining

$$s_0 = a_1 + (\eta - 1)Q_f, \quad s_1 = a_2 - a_1,$$

the condition $\Phi(\delta) = 0$ can be written as a quadratic equation in δ ,

$$s_1^2 \delta^2 + (2s_0 s_1 + 4\eta a_1 Q_f) \delta + (s_0^2 - 4\eta a_1 Q_f) = 0.$$

Solving this equation yields two critical values,

$$\delta_{\pm} = \frac{-(2s_0 s_1 + 4\eta a_1 Q_f) \pm \sqrt{(2s_0 s_1 + 4\eta a_1 Q_f)^2 - 4s_1^2(s_0^2 - 4\eta a_1 Q_f)}}{2s_1^2}.$$

In regard of biological viability, we impose the condition that assures the discriminant $\Delta(\delta)$ is strictly positive, which is satisfied when

$$\Phi(\delta) > 0 \iff \delta \in (-\infty, \delta_-) \cup (\delta_+, \infty).$$

Furthermore, considering the biological restriction $0 < \delta < 1$, the admissible region reduces to

$$\delta \in (0, 1) \cap \left((-\infty, \delta_-) \cup (\delta_+, \infty) \right).$$

Therefore, the maternal transmission probability determines the existence of the coexistence equilibria, meaning that only admissible values of δ yield real solutions.

Figure 2 shows the equilibrium values of the wild susceptible mosquito S_f as a function of the maternal transmission (δ). The green horizontal line represents the *Wolbachia* extinction equilibrium E_f . The purple solid curve corresponds to the stable coexistence equilibrium E_- , and the purple dashed curve to the unstable coexistence equilibrium E_+ . The red vertical dashed line indicates the coexistence threshold δ_+ , above which the coexistence equilibria exist.

Figure 3 shows the parabola-shaped plot of the discriminant factor $\Phi(\delta)$, over $\delta \in [0, 1]$. The curve shows $\Phi(\delta)$, the horizontal dashed line marks $\Phi(\delta) = 0$, green vertical dashed lines indicate δ_{param} , calculated considering the *wMel* parameter set, and the positive discriminant root δ_+ , in red dashed lines. The negative discriminant root, calculated as $\delta_- = -404.1573$, is not shown on the Figure.

Strength of cytoplasmic incompatibility (η).

The discriminant of the coexistence quadratic expression can also be written as a function of the cytoplasmic incompatibility parameter η .

$$\Delta(\eta) = M^2(\beta^f + 1)^2 \Phi(\eta), \quad \Phi(\eta) = (Q_c - 1)^2 - 4\eta \frac{Q_{w,f}}{Q_f}.$$

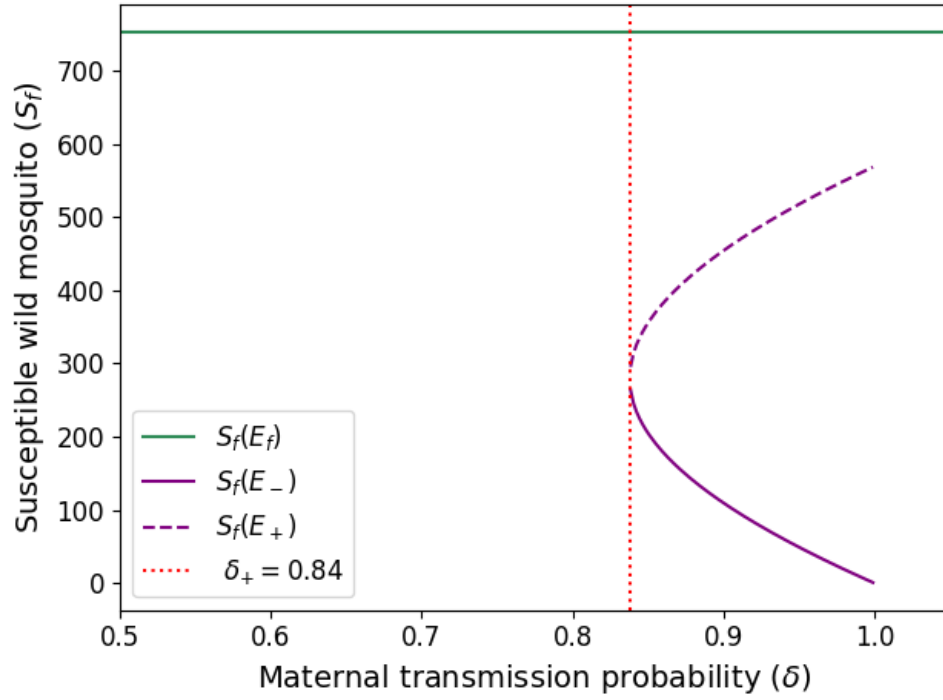


Figure 2 – Bifurcation diagram of the wild susceptible mosquito S_f versus the maternal transmission probability δ . The *Wolbachia*-extinction equilibrium E_f is shown in green, the stable and unstable coexistence equilibria E_- and E_+ are shown in solid and dashed purple, and the coexistence threshold δ_+ is marked by a red vertical dashed line.

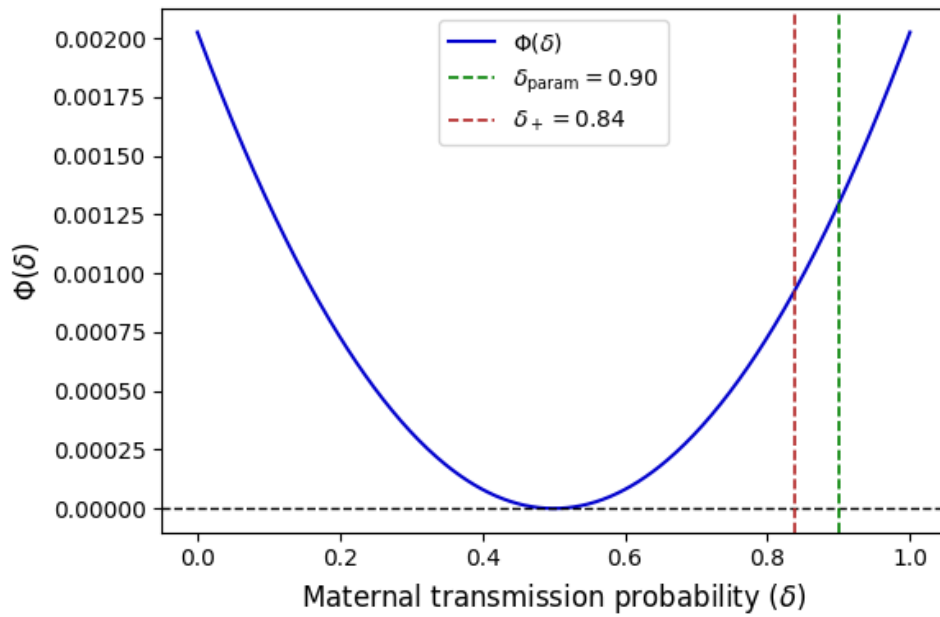


Figure 3 – Plot of the discriminant factor $\Phi(\delta)$, over $\delta \in [0, 1]$. The curve shows $\Phi(\delta)$, the horizontal dashed line marks $\Phi(\delta) = 0$, green vertical dashed lines indicate δ_{param} , and red dashed lines show δ_+ coexistence threshold.

Since $M^2(\beta^f + 1)^2 > 0$, real coexistence equilibria exist whenever $\Phi(\eta) > 0$.

Substituting the expression for Q_c yields

$$Q_c - 1 = \eta + \frac{Q_{w,f} + Q_w - Q_f}{Q_f},$$

so that $\Phi(\eta)$ is a quadratic polynomial in η . Solving $\Phi(\eta) = 0$ gives two critical values,

$$\eta_{\pm} = 1 + \frac{Q_{w,f}}{Q_f} - \frac{Q_w}{Q_f} \pm 2\sqrt{\frac{Q_{w,f}}{Q_f} \left(1 - \frac{Q_w}{Q_f}\right)}.$$

Hence, for the discriminant $\Delta(\eta)$ to be strictly positive, it suffices that

$$\Phi(\eta) > 0 \iff \eta \in (-\infty, \eta_-) \cup (\eta_+, \infty).$$

Imposing the biological restriction $0 < \eta < 1$, the admissible values of η reduce to

$$\eta \in (0, 1) \cap \left((-\infty, \eta_-) \cup (\eta_+, \infty) \right).$$

Similarly as before, we observe that the strength of cytoplasmic incompatibility also controls the existence of the coexistence equilibria, concluding that only admissible values of η yield real solutions.

Figure 4 shows equilibrium values of the wild-type susceptible mosquito population S_f as a function of the cytoplasmic incompatibility (η). The horizontal green line corresponds to the *Wolbachia*-extinction equilibrium E_f . The solid purple curve denotes the stable coexistence equilibrium E_- , while the dashed purple curve denotes the unstable coexistence equilibrium E_+ . The red vertical dashed line marks the threshold η_+ , above which the coexistence equilibria exist.

3.3.3.2 Local Stability of Mosquito Coexistence Equilibria $E_{\pm}$

To analyze the local stability of the coexistence equilibria $E_{\pm}$, we examine the Jacobian matrix of the mosquito subsystem. Evaluated at $E_{\pm}$, the Jacobian is

$$J(E_{\pm}) = \begin{bmatrix} J_{11}^{(\pm)} & J_{12}^{(\pm)} & J_{13}^{(\pm)} & J_{14}^{(\pm)} \\ \bar{\beta}_f I_h & -\mu_f & 0 & 0 \\ -\sigma(\bar{\beta}_w I_h + \mu_w) S_w^{\pm} & -\sigma(\bar{\beta}_w I_h + \mu_w) S_w^{\pm} & -\sigma(\bar{\beta}_w I_h + \mu_w) S_w^{\pm} - \bar{\beta}_w I_h & \mu_w - \sigma(\bar{\beta}_w I_h + \mu_w) S_w^{\pm} \\ 0 & 0 & \bar{\beta}_w I_h & -\mu_w \end{bmatrix}.$$

with

$$J_{11}^{(\pm)} = \frac{1}{Q_w} \frac{\partial \Lambda_f}{\partial S_f} \Big|_{E_{\pm}} - \sigma(\bar{\beta}_f I_h + \mu_f) S_f^{\pm} - \bar{\beta}_f I_h - \mu_f,$$

$$J_{12}^{(\pm)} = \frac{1}{Q_w} \frac{\partial \Lambda_f}{\partial I_f} \Big|_{E_{\pm}} - \sigma(\bar{\beta}_f I_h + \mu_f) S_f^{\pm},$$

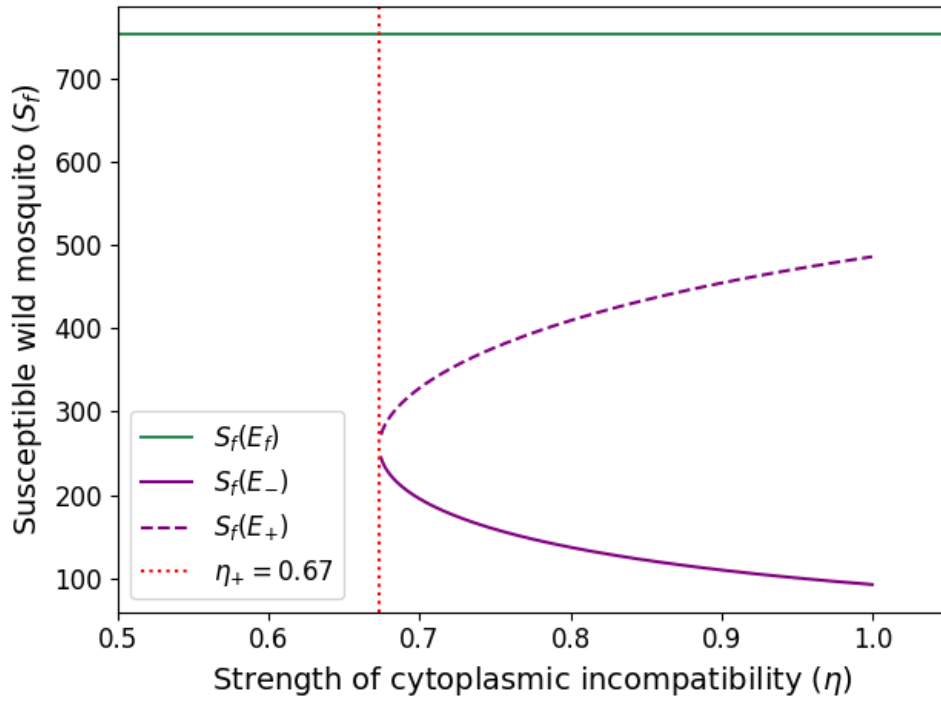


Figure 4 – Bifurcation diagram of susceptible wild mosquito S_f versus the strength of cytoplasmic incompatibility η . The *Wolbachia*-extinction equilibrium E_f is shown in green, the stable and unstable coexistence equilibria E_- and E_+ are shown in solid and dashed purple, and the coexistence threshold η_+ is marked by a red vertical dashed line.

$$J_{13}^{(\pm)} = J_{14}^{(\pm)} = \frac{1}{Q_w} \left[\rho_w(1 - \delta) - \rho_f \eta \frac{(S_f^\pm + I_f^\pm)^2}{M^2} \right] - \sigma(\bar{\beta}_f I_h + \mu_f) S_f^\pm.$$

To determine whether $E_\pm$ are locally asymptotically stable, we apply the Routh–Hurwitz criterion to the characteristic polynomial, instead of computing eigenvalues directly (GANTMA-CHER, 1959).

$$p(\lambda) = \det(\lambda I - J) = \lambda^4 + a_1 \lambda^3 + a_2 \lambda^2 + a_3 \lambda + a_4.$$

For a quartic polynomial of this form, the Routh–Hurwitz conditions are

$$a_1 > 0, \quad a_2 > 0, \quad a_3 > 0, \quad a_4 > 0, \quad a_1 a_2 a_3 > a_3^2 + a_1^2 a_4.$$

If these inequalities are satisfied, then all eigenvalues of J have strictly negative real parts, and the corresponding equilibrium is locally asymptotically stable.

Coefficient $a_1^{(\pm)}$ is the negative of the trace of $J(E^\pm)$, $a_1^{(\pm)} = -\text{tr } J(E_\pm)$,

Coefficient $a_2^{(\pm)}$ is the sum of the principal 2×2 minors of $J(E_\pm)$,

Coefficient $a_3^{(\pm)}$ is minus the sum of the principal 3×3 minors of $J(E_\pm)$,

Coefficient $a_4^{(\pm)}$ is the determinant of the Jacobian, $a_4^{(\pm)} = \det J(E_\pm)$.

Computationally, we substituted the explicit symbolic expressions of $J_{1j}^{(\pm)}$, $S_w^\pm$, $I_f^\pm$ and $I_w^\pm$, corresponding to the negative coexistence equilibrium (E_-), and the positive coexistence

equilibrium (E_+). Then, we calculated a_1, a_2, a_3, a_4 coefficients of the characteristic polynomial, considering parameter sets referred to the biological features of wild mosquitoes, $wMel$, $wMelPop$, $wAlbB$ and wAu *Wolbachia* strains, presented on Table 3.

Substituting the negative coexistence equilibrium values (E_-), all the coefficients conditions are satisfied, for the biologically relevant parameter sets. On the other side, when substituting the positive coexistence equilibrium values (E_+), condition $a_4 > 0$ is not satisfied. By the definition of the coefficient expression, $a_4^{(\pm)} = \det J(E_{\pm})$, we see that the determinant of the Jacobian matrix is negative when considering the E_+ equilibrium point.

Therefore, the equilibrium point E_- is locally asymptotically stable, and the equilibrium point E_+ is unstable, whenever conditions $Q_w > 1$ and $Q_c > 1$ holds, alongside the conditions $\delta \in (0, 1) \cap ((-\infty, \delta_-) \cup (\delta_+, \infty))$ and $\eta \in (0, 1) \cap ((-\infty, \eta_-) \cup (\eta_+, \infty))$ are satisfied (**Proposition A.1.7**).

3.4 Chapter Conclusion

So far, we have shown that the mosquito subsystem admits up to four equilibrium points. Following the fast vector dynamics approach ([KEELING; ROHANI, 2008](#)), we now substitute the mosquito equilibrium values into the human equations, formulating reduced human subsystems associated with each mosquito equilibrium.

Our analysis shows that the coexistence equilibrium branches arise as $E_{\pm}$, of which the negative branch E_- is locally asymptotically stable, while the positive branch E_+ is unstable. Accordingly, we define

$$E_c := E_-,$$

and henceforth refer to the coexistence equilibrium exclusively as E_c .

We further established that the *Wolbachia* extinction equilibrium E_f is locally asymptotically stable whenever

$$Q_f > \max\{1, Q_w\},$$

and that the coexistence equilibrium E_c is locally asymptotically stable for parameter values satisfying

$$Q_w > 1, \quad Q_c > 1, \\ \delta \in (0, 1) \cap ((-\infty, \delta_-) \cup (\delta_+, \infty)), \quad \eta \in (0, 1) \cap ((-\infty, \eta_-) \cup (\eta_+, \infty)).$$

These results imply that the mosquito subsystem exhibits bistability, in which both equilibria E_f and E_c are mathematically admissible. In this regime, the long-term mosquito dynamics are determined by the initial condition

$$(S_f(0), I_f(0), S_w(0), I_w(0)) \in \Omega_m.$$

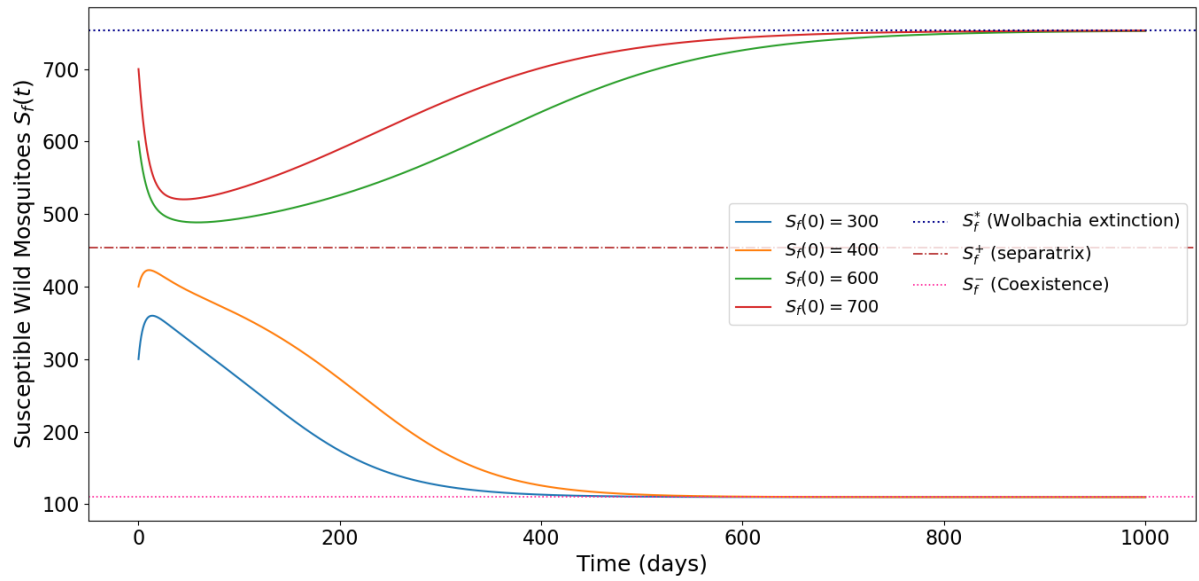


Figure 5 – Susceptible wild mosquito (S_f) dynamics, under varying initial conditions of $S_f(0)$, while keeping $(I_f(0), S_w(0), I_w(0))$ fixed.

Finally, the trivial equilibrium E_0 , corresponding to the extinction of both mosquito populations, is not biologically relevant, and the coexistence equilibrium E_+ is unstable. Consequently, only the equilibria E_f and E_c are retained for further analysis.

Therefore, in the following chapter, which focuses on analyzing the general human subsystem, we restrict attention to the reduced systems associated with the *Wolbachia* extinction equilibrium E_f and the stable mosquito coexistence equilibrium E_c .

At last, we highlight the critical role played by the initial conditions of the mosquito subsystem, as illustrated in Figure 5, where horizontal dashed lines show the equilibrium value for *Wolbachia* extinction steady-state E_f , whereas pink dashed line is the locally asymptotically stable equilibrium E_- and red dashed lines represent the separatrix, or the unstable equilibrium E_+ . This demonstrates that the mosquito subsystem exhibits bistability, in the sense that long-term dynamics depends on the initial condition $(S_f(0), I_f(0), S_w(0), I_w(0)) \in \Omega_m$, leading to convergence toward distinct equilibrium regimes.

4 General Human Subsystem

In this setting, mosquito populations are assumed to rapidly converge to an equilibrium (KEELING; ROHANI, 2008) and the human dynamics are governed by a reduced two dimensional system whose form depends on the equilibrium regime established in the mosquito subsystem.

Considering the existence of two distinct possible equilibrium points, namely *Wolbachia* extinction and mosquito coexistence, let us now define the two vector fields

$$H_f(S_h^f, I_h^f) = \begin{pmatrix} \frac{dS_h^f}{dt} \\ \frac{dI_h^f}{dt} \end{pmatrix}, \quad H_c(S_h^c, I_h^c) = \begin{pmatrix} \frac{dS_h^c}{dt} \\ \frac{dI_h^c}{dt} \end{pmatrix} \quad (\text{B})$$

where, corresponding to *Wolbachia* extinction equilibrium,

$$\begin{aligned} \frac{dS_h^f}{dt} &= \mu_h N_h - G_f(I_h^f) S_h^f - (\mu_h + \nu_h) S_h^f, \\ \frac{dI_h^f}{dt} &= G_f(I_h^f) S_h^f - (\gamma_h + \mu_h) I_h^f, \end{aligned}$$

$$\text{with } G_f(I_h^f) = \frac{\hat{\beta} \bar{\beta}_f I_h^f}{\sigma} \frac{\ln(Q_f)}{\mu_f + \bar{\beta}_f I_h^f}.$$

Whereas, associated with the mosquito coexistence equilibrium,

$$\begin{aligned} \frac{dS_h^c}{dt} &= \mu_h N_h - G_c(I_h^c) S_h^c - (\mu_h + \nu_h) S_h^c, \\ \frac{dI_h^c}{dt} &= G_c(I_h^c) S_h^c - (\gamma_h + \mu_h) I_h^c, \end{aligned}$$

$$\text{with } G_c(I_h^c) = I_h^c \hat{\beta} \left\{ \frac{\bar{\beta}_w \ln(Q_w)}{\sigma(\bar{\beta}_w I_h^c + \mu_w)} + \frac{M^-(\mu_w \bar{\beta}_f - \mu_f \bar{\beta}_w)}{2\eta(\bar{\beta}_w I_h^c + \mu_w)(\bar{\beta}_f I_h^c + \mu_f)} \right\}.$$

Since we consider the stable coexistence equilibrium $E_c = E_-$, the expression M^- on equations is given by

$$M^- = M \left[(Q_c - 1) - \sqrt{(Q_c - 1)^2 - 4\eta \frac{Q_{w,f}}{Q_f}} \right].$$

The defined vector fields represent the distinct dynamical regimes induced by the mosquito population, as they generate the corresponding human subsystem of ordinary differential equations (GUIDORIZZI, 2001; TESCHL, 2000). Let H denote the set of admissible human vector fields generated by the established regime of the mosquitoes subsystem. In particular, $H := \{ H_f, H_c \}$, where H_f and H_c are the vector fields corresponding with *Wolbachia* extinction and stable mosquito coexistence, respectively.

Our analysis allows us to state that the general human subsystem is governed by one of the vector fields $H_f, H_c \in H$, and therefore described by a reduced two dimensional system, whose form depends on the equilibrium regime established in the mosquito subsystem. Accordingly, we define the following two conditions:

- (i) $H_f \in H \iff Q_f > \max\{1, Q_w\}$, for the wild mosquito basic offspring number.
- (ii) $H_c \in H \iff Q_w > 1, Q_c > 1$ alongside the conditions $\delta \in (0, 1) \cap ((-\infty, \delta_-) \cup (\delta_+, \infty))$, $\eta \in (0, 1) \cap ((-\infty, \eta_-) \cup (\eta_+, \infty))$ are satisfied.

We conducted a stability analysis and showed that the mosquito subsystem exhibits bistability, with both equilibrium points E_f and E_c being locally asymptotically stable. Consequently, the long-term dynamics of the system depend on the initial conditions of the mosquito population. Similarly as before, in order to describe mosquitoes dynamics, we have defined the nonnegative set

$$\Omega_m = \{(S_f, I_f, S_w, I_w) \in \mathbb{R}^4 : S_f \geq 0, I_f \geq 0, S_w \geq 0, I_w \geq 0\}.$$

In particular, we have proved that trajectories starting in Ω_m remain in Ω_m , for all $t \geq 0$.

Thus, the initial conditions of the mosquito subsystem, i.e.: $(S_f(0), I_f(0), S_w(0), I_w(0)) \in \Omega_m$, are the key factor in determining which locally asymptotically stable equilibrium point, E_f or E_c is selected, and therefore which vector field, H_f or H_c , generates the corresponding subsystem of ordinary differential equations.

4.1 Positiveness

Define the nonnegative set

$$\Omega_h = \{(S_h, I_h) \in \mathbb{R}_+^2 : S_h \geq 0, I_h \geq 0\}.$$

Assume that the initial condition satisfies $(S_h(0), I_h(0)) \in \Omega_h$. We show that Ω_h is positively invariant for each admissible human subsystem defined in (B).

Case 1: *Wolbachia* extinction regime.

When condition (i) is satisfied, the human dynamics are governed by

$$\frac{dS_h^f}{dt} = \mu_h N_h - G_f(I_h^f) S_h^f - (\mu_h + \nu_h) S_h^f, \quad \frac{dI_h^f}{dt} = G_f(I_h^f) S_h^f - (\gamma_h + \mu_h) I_h^f,$$

where

$$G_f(I_h^f) = \frac{\hat{\beta} \bar{\bar{\beta}}_f I_h^f}{\sigma} \frac{\ln(Q_f)}{\mu_f + \bar{\bar{\beta}}_f I_h^f}.$$

Note that $G_f(I_h^f) \geq 0$ for all $I_h^f \geq 0$.

Boundary $S_h^f = 0$. If $S_h^f = 0$, then

$$\left. \frac{dS_h^f}{dt} \right|_{S_h^f=0} = \mu_h N_h \geq 0.$$

Boundary $I_h^f = 0$. If $I_h^f = 0$, then $G_f(I_h^f) = 0$ and

$$\left. \frac{dI_h^f}{dt} \right|_{I_h^f=0} = 0.$$

Hence, the vector field is inward pointing or tangent on each boundary of Ω_h , and Ω_h is positively invariant under the dynamics of the *Wolbachia* extinction regime.

Case 2: mosquito coexistence regime.

When condition (ii) holds, the mosquito subsystem converges to the stable coexistence equilibrium point E_c , and the human dynamics reduce to

$$\begin{aligned} \frac{dS_h^c}{dt} &= \mu_h N_h - G_c(I_h^c) S_h^c - (\mu_h + \nu_h) S_h^c, \\ \frac{dI_h^c}{dt} &= G_c(I_h^c) S_h^c - (\gamma_h + \mu_h) I_h^c, \end{aligned}$$

where

$$G_c(I_h^c) = I_h^c \hat{\beta} \left\{ \frac{\bar{\bar{\beta}}_w \ln(Q_w)}{\sigma(\bar{\bar{\beta}}_w I_h^c + \mu_w)} + \frac{M^-(\mu_w \bar{\bar{\beta}}_f - \mu_f \bar{\bar{\beta}}_w)}{2\eta(\bar{\bar{\beta}}_w I_h^c + \mu_w)(\bar{\bar{\beta}}_f I_h^c + \mu_f)} \right\}.$$

All parameters and equilibrium quantities involved in G_c are nonnegative and moreover, $\bar{\bar{\beta}}_w < \bar{\bar{\beta}}_f$ and $\mu_f < \mu_w \implies (\mu_w \bar{\bar{\beta}}_f - \mu_f \bar{\bar{\beta}}_w) > 0$. Hence, $G_c(I_h^c) \geq 0$.

Boundary $S_h^c = 0$. If $S_h^c = 0$, then

$$\left. \frac{dS_h^c}{dt} \right|_{S_h^c=0} = \mu_h N_h \geq 0.$$

Boundary $I_h^c = 0$. If $I_h^c = 0$, then $G_c(0) = 0$ and

$$\left. \frac{dI_h^c}{dt} \right|_{I_h^c=0} = 0.$$

Thus, the vector field is inward-pointing or tangent on the boundary of Ω_h , implying that Ω_h is positively invariant under the coexistence regime.

Therefore, in both admissible regimes, the human subsystem preserves nonnegativity. Hence, for any initial condition $(S_h(0), I_h(0)) \in \Omega_h$, the corresponding solution satisfies

$$S_h(t) \geq 0, \quad I_h(t) \geq 0, \quad \forall t \geq 0.$$

In the following sections, we analyze each admissible human vector field belonging to $H = \{H_f, H_c\}$, separately. For each regime, we explicitly state the corresponding reduced two-dimensional human system, determine its equilibrium points, and establish the conditions for existence and admissibility. Furthermore, we characterize the local stability of these equilibria, deriving parameter-based thresholds that specify when the equilibrium is locally asymptotically stable.

4.2 *Wolbachia* Extinction Regime

4.2.1 *Wolbachia* Extinction Equilibrium

In this setting, we analyze the human subsystem of ordinary differential equations induced by the mosquito population regime corresponding to the extinction of *Wolbachia*-infected mosquitoes. This set of equations is obtained after substituting the equilibrium values associated with the mosquito subsystem under the *Wolbachia* extinction scenario. Whenever condition (i) is satisfied, the resulting human dynamics are generated by the vector field $H_f \in H$, given by

$$H_f(S_h^f, I_h^f) = \begin{pmatrix} \frac{dS_h^f}{dt} \\ \frac{dI_h^f}{dt} \end{pmatrix},$$

which defines the corresponding subsystem of equations, written as

$$\begin{aligned} \frac{dS_h^f}{dt} &= \mu_h N_h - G_f(I_h^f) S_h^f - (\mu_h + \nu_h) S_h^f, \\ \frac{dI_h^f}{dt} &= G_f(I_h^f) S_h^f - (\gamma_h + \mu_h) I_h^f, \\ \text{with } G_f(I_h^f) &= \frac{\hat{\beta} \bar{\beta}_f I_h^f}{\sigma} \frac{\ln(Q_f)}{\mu_f + \bar{\beta}_f I_h^f}. \end{aligned}$$

We observed that the human subsystem reaches an equilibrium that depends entirely on the parameters related to the population of wild mosquitoes. The resulting equilibrium point for the human population, denoted by $E_f^h = (S_h^f, I_h^f)$, is expressed as

$$E_f^h = \left(\frac{\sigma[\mu_f(\gamma_h + \mu_h) + \bar{\beta}_f \mu_h N_h]}{\bar{\beta}_f [\hat{\beta} \ln(Q_f) + \sigma(\mu_h + \nu_h)]}, S_h^f \left[\frac{\hat{\beta} \ln(Q_f)}{\sigma(\gamma_h + \mu_h)} \right] - \frac{\mu_f}{\bar{\beta}_f} \right),$$

which corresponds to the steady-state solution of the human subsystem under the *Wolbachia* extinction regime (**Proposition A.2.3**).

Since the second component represents the equilibrium number of infected humans, biological feasibility requires this quantity to be positive. In this sense, setting $I_h^f > 0$ ensures that negative population values are not admissible. Imposing this condition yields the following threshold for the existence of this equilibrium:

$$Q_f > \psi_f := \exp \left\{ \frac{\mu_f \sigma(\mu_h + \nu_h)(\gamma_h + \mu_h)}{\hat{\beta} \bar{\beta}_f \mu_h N_h} \right\}.$$

Therefore, the equilibrium E_f^h exists only when the defined quantity Q_f exceeds the critical value ψ_f , due to biological viability (**Proposition A.2.4**).

4.2.2 Local Stability of Equilibrium Point E_f^h

The local stability of the equilibrium point E_f^h is established through the analysis of the corresponding Jacobian matrix. Two classical conditions are verified: the positivity of the determinant, which provides a necessary condition for local stability, and the negativity of the trace, which ensures sufficiency.

The determinant of the Jacobian is strictly positive whenever

$$\det(J) > 0 \iff Q_f > \psi_f \iff R^f > 1,$$

where

$$\psi_f = \exp \left\{ \frac{\mu_f \sigma(\mu_h + \nu_h)(\gamma_h + \mu_h)}{\hat{\beta} \bar{\beta}_f \mu_h N_h} \right\}, \quad R^f = \frac{\hat{\beta} \bar{\beta}_f \ln(Q_f) \mu_h N_h}{\mu_f \sigma(\gamma_h + \mu_h)(\mu_h + \nu_h)}.$$

This quantity R^f works as a threshold parameter for the *Wolbachia* extinction subsystem, determining the parameter region in which E_f^h is locally asymptotically stable.

Furthermore, the analysis of the trace of the Jacobian shows that

$$\text{tr}(J) < 0 \iff Q_f > \psi_f \iff R^f > 1.$$

Therefore, the sufficient condition for local asymptotic stability coincides with the necessary condition, indicating that whenever $\det(J) > 0$, the trace condition $\text{tr}(J) < 0$ is also satisfied. Consequently, the equilibrium E_f^h is locally asymptotically stable whenever condition $Q_f > \psi_f$ is satisfied, or equivalently, $R^f > 1$ holds. On the other side, $Q_f < \psi_f \iff R^f < 1$ implies that this equilibrium point is unstable (**Proposition A.2.5**).

4.2.2.1 R^f Under *Wolbachia* Extinction

Under *Wolbachia* extinction conditions, we computed the threshold R^f for the human subsystem using the next-generation matrix method (**DRIESSCHE; WATMOUGH, 2002**). Line-

arization of the infection dynamics at the disease-free equilibrium

$$S_h^{DFE} = \frac{\mu_h N_h}{\mu_h + \nu_h}, \quad I_h^{DFE} = 0$$

yields the next-generation operator associated with new infections and transitions. The resulting threshold quantity is

$$R^f = \frac{\hat{\beta} \mu_h N_h}{(\mu_h + \nu_h)(\gamma_h + \mu_h)} \frac{\bar{\beta}_f \ln(Q_f)}{\sigma \mu_f}.$$

It is evident that this expression matches identically with the previously calculated result.

Therefore, the *Wolbachia* extinction equilibrium point is locally asymptotically stable if $R^f > 1$, and unstable if $R^f < 1$ (**Proposition A.2.6**).

One can rewrite R^f expression in order to identify mosquitoes and human population components. Since $\bar{\beta}_f = \frac{\bar{\beta}_f}{N_h}$, $\bar{\beta}_w = \frac{\bar{\beta}_w}{N_h}$ and $\beta = \frac{\hat{\beta}}{N_h}$,

$$R^f = \left[\frac{\beta}{(\mu_h + \nu_h)(\gamma_h + \mu_h)} \frac{\mu_h}{N_h} \right] \left[\frac{\bar{\beta}_f \ln(Q_f)}{\mu_f \sigma} \right].$$

Moreover, we consider $v_h = 0$ to clarify the relation between humans and mosquitoes.

$$(R^f)_{v_h=0} = \left[\frac{\beta}{N_h(\gamma_h + \mu_h)} \right] \left[\frac{\bar{\beta}_f \ln(Q_f)}{\mu_f \sigma} \right]$$

At last, one can see that $(R^f)_{v_h=0}$ expression presents biological interpretation. The first term represents human population, where $(\gamma_h + \mu_h)^{-1}$ represents the mean infectious period, according to β transmission parameter. The second term presents mosquitoes components, with $\frac{\bar{\beta}_f}{\mu_f}$ representing the contribution of mosquitoes to dengue transmission dynamics through parameter $\bar{\beta}_f$, limited by mosquito mortality μ_f , whereas $\frac{\ln(Q_f)}{\sigma}$ represents the total mosquito population (i.e.: $S_f + I_f$) at the *Wolbachia* extinction equilibrium.

4.2.3 Disease-Free Equilibrium as a Particular Solution of H_f

We next show that the disease-free equilibrium (DFE) is a particular solution of the human subsystem of ordinary differential equations generated by the vector field $H_f \in H$, associated with the *Wolbachia* extinction scenario.

Given that condition (i) is satisfied, thus *Wolbachia* extinction equilibrium point is locally asymptotically stable, we recall that is defined

$$G_f(I_h^f) := \frac{\hat{\beta} \bar{\beta}_f I_h^f}{\sigma} \frac{\ln(Q_f)}{\mu_f + \bar{\beta}_f I_h^f},$$

so that the human subsystem under *Wolbachia* extinction is

$$\begin{aligned} \frac{dS_h}{dt} &= \mu_h N_h - G_f(I_h^f) S_h^f - (\mu_h + \nu_h) S_h^f, \\ \frac{dI_h}{dt} &= G_f(I_h^f) S_h^f - (\gamma_h + \mu_h) I_h^f. \end{aligned}$$

At a disease-free equilibrium we impose $I_h^f = 0$. Since $G_f(0) = 0$, the second equation becomes

$$0 = \frac{dI_h^f}{dt} = G_f(0) S_h^f$$

so it is identically satisfied for any $S_h^f \geq 0$. The first equation reduces to

$$0 = \frac{dS_h^f}{dt} = \mu_h N_h - (\mu_h + \nu_h) S_h^f,$$

hence

$$S_h^f = \frac{\mu_h N_h}{\mu_h + \nu_h}.$$

Therefore, the disease-free equilibrium,

$$(S_h^{DFE}, I_h^{DFE}) = \left(\frac{\mu_h N_h}{\mu_h + \nu_h}, 0 \right),$$

is a steady-state solution of the dynamical system generated by the vector field H_f , representing the *Wolbachia* extinction regime.

4.3 Mosquito Coexistence Regime

We now analyze the human subsystem of ordinary differential equations resulted from the mosquito population regime corresponding to coexistence between *Wolbachia*-infected and wild mosquitoes. As shown previously, the coexistence equilibrium admits two branches, E_+ and E_- ; however, assuring that conditions (ii) holds, only the negative branch E_- is locally asymptotically stable, whereas E_+ is unstable. Consequently, we restrict our analysis to the stable coexistence equilibrium, $E_c = E_-$.

The corresponding human dynamics are generated by the previously defined vector field $H_c \in H$,

$$H_c(S_h^c, I_h^c) = \begin{pmatrix} \frac{dS_h^c}{dt} \\ \frac{dI_h^c}{dt} \end{pmatrix},$$

with

$$\begin{aligned} \frac{dS_h^c}{dt} &= \mu_h N_h - G_c(I_h^c) S_h^c - (\mu_h + \nu_h) S_h^c, \\ \frac{dI_h^c}{dt} &= G_c(I_h^c) S_h^c - (\gamma_h + \mu_h) I_h^c, \end{aligned}$$

where

$$G_c(I_h^c) = I_h^c \hat{\beta} \left\{ \frac{\bar{\beta}_w \ln(Q_w)}{\sigma(\bar{\beta}_w I_h^c + \mu_w)} + \frac{M^-(\mu_w \bar{\beta}_f - \mu_f \bar{\beta}_w)}{2\eta(\bar{\beta}_w I_h^c + \mu_w)(\bar{\beta}_f I_h^c + \mu_f)} \right\}.$$

One requires the force of infection $G_c(I_h^c)$ to be nonnegative. In this sense, assuming $\bar{\beta}_w < \bar{\beta}_f$ and $\mu_f < \mu_w$, we have $(\mu_w \bar{\beta}_f - \mu_f \bar{\beta}_w) > 0$, and since the remaining factors in $G_c(I_h^c)$ are nonnegative, it follows that $G_c(I_h^c) \geq 0$, whenever $I_h^c \geq 0$.

Since we consider the stable coexistence equilibrium $E_c = E_-$, the term M^- shown is

$$M^- = M \left[(Q_c - 1) - \sqrt{(Q_c - 1)^2 - 4\eta \frac{Q_{w,f}}{Q_f}} \right].$$

Under conditions in which both mosquito populations coexist, the human subsystem reaches a nontrivial equilibrium characterized by positive values of both S_h^c and I_h^c . At this equilibrium, the number of infected individuals in the human population is obtained as the solution of a quadratic equation in I_h^c , expressed as

$$I_h^c = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

with coefficients a , b , and c given by

$$\begin{aligned} a &= \frac{\bar{\beta}_w \bar{\beta}_f \ln(Q_w)}{\sigma} + \frac{\bar{\beta}_w \bar{\beta}_f (\mu_h + \nu_h)}{\hat{\beta}}, \\ b &= \frac{(\mu_h + \nu_h) (\bar{\beta}_w \mu_f + \mu_w \bar{\beta}_f)}{\hat{\beta}} + \frac{\bar{\beta}_w \mu_f \ln(Q_w)}{\sigma} + \frac{M^- (\mu_w \bar{\beta}_f - \mu_f \bar{\beta}_w)}{2\eta} - \frac{\bar{\beta}_w \bar{\beta}_f \ln(Q_w)}{\sigma} \frac{\mu_h N_h}{\gamma_h + \mu_h}, \\ c &= \frac{(\mu_h + \nu_h) \mu_w \mu_f}{\hat{\beta}} - \frac{\mu_h N_h}{\gamma_h + \mu_h} \left[\frac{\bar{\beta}_w \mu_f \ln(Q_w)}{\sigma} + \frac{M^- (\mu_w \bar{\beta}_f - \mu_f \bar{\beta}_w)}{2\eta} \right]. \end{aligned}$$

The corresponding number of susceptible individuals is expressed as

$$S_h^c = \frac{\mu_h N_h}{\mu_h + \nu_h} - \frac{\gamma_h + \mu_h}{\mu_h + \nu_h} I_h^c.$$

Hence, we define $E_c^h = (S_h^c, I_h^c)$ as the equilibrium point corresponding to the human dynamics in a scenario of wild and *Wolbachia*-infected mosquito coexistence (**Proposition A.2.7**).

Positivity of the Solutions

Under mosquito coexistence, the endemic human equilibrium I_h^c is determined by the quadratic equation

$$a (I_h^c)^2 + b I_h^c + c = 0, \quad a > 0.$$

In order to obtain explicit admissibility conditions in terms of the human-*Wolbachia* mosquito transmission rate, $\bar{\beta}_w$, we solve the equations $c = 0$ and $b = 0$ with respect to $\bar{\beta}_w$ and then determine the corresponding sign regions for the coefficients.

Imposing the condition $c = 0$ and solving for the transmission parameter $\bar{\beta}_w$, we obtain

$$c = 0 \iff$$

$$\bar{\bar{\beta}}_w = \psi_c := \frac{\sigma \mu_w [2\eta \mu_f (\gamma_h + \mu_h)(\mu_h + \nu_h) - \hat{\beta} \mu_h N_h M^- \bar{\bar{\beta}}_f]}{\hat{\beta} \mu_h N_h \mu_f (2\eta \ln(Q_w) - \sigma M^-)}.$$

Since c depends linearly on the transmission parameter $\bar{\bar{\beta}}_w$, with slope proportional to $\left(\frac{M^-}{2\eta} - \frac{\ln(Q_w)}{\sigma}\right)$, the coefficient sign regions are:

$$\text{If } \frac{\ln(Q_w)}{\sigma} > \frac{M^-}{2\eta}, \quad \bar{\bar{\beta}}_w > \psi_c \Rightarrow c < 0, \quad \bar{\bar{\beta}}_w = \psi_c \Rightarrow c = 0, \quad \bar{\bar{\beta}}_w < \psi_c \Rightarrow c > 0,$$

$$\text{If } \frac{\ln(Q_w)}{\sigma} < \frac{M^-}{2\eta}, \quad \bar{\bar{\beta}}_w > \psi_c \Rightarrow c > 0, \quad \bar{\bar{\beta}}_w = \psi_c \Rightarrow c = 0, \quad \bar{\bar{\beta}}_w < \psi_c \Rightarrow c < 0.$$

Let us define the quantity

$$\Lambda_b := \mu_f \left(\frac{\mu_h + \nu_h}{\hat{\beta}} + \frac{\ln(Q_w)}{\sigma} - \frac{M^-}{2\eta} \right) - \frac{\mu_h N_h}{\gamma_h + \mu_h} \bar{\bar{\beta}}_f \frac{\ln(Q_w)}{\sigma}.$$

Similarly as before, setting the coefficient $b = 0$ and solving for $\bar{\bar{\beta}}_w$ yields

$$b = 0 \iff$$

$$\bar{\bar{\beta}}_w = \psi_b := \frac{\mu_w \bar{\bar{\beta}}_f \sigma (\gamma_h + \mu_h) [2\eta (\mu_h + \nu_h) + \hat{\beta} M^-]}{2\eta \hat{\beta} \bar{\bar{\beta}}_f \mu_h N_h \ln(Q_w) + \mu_f (\gamma_h + \mu_h) [\hat{\beta} \sigma M^- - 2\eta \sigma (\mu_h + \nu_h) - 2\eta \hat{\beta} \ln(Q_w)]}.$$

Since b depends linearly on $\bar{\bar{\beta}}_w$, the sign regions are determined by the sign of Λ_b :

$$\text{If } \Lambda_b > 0, \quad \bar{\bar{\beta}}_w > \psi_b \Rightarrow b > 0, \quad \bar{\bar{\beta}}_w = \psi_b \Rightarrow b = 0, \quad \bar{\bar{\beta}}_w < \psi_b \Rightarrow b < 0,$$

$$\text{If } \Lambda_b < 0, \quad \bar{\bar{\beta}}_w > \psi_b \Rightarrow b < 0, \quad \bar{\bar{\beta}}_w = \psi_b \Rightarrow b = 0, \quad \bar{\bar{\beta}}_w < \psi_b \Rightarrow b > 0.$$

At last, we determine the positivity regimes for the solutions of the quadratic in I_h^c . Under Descartes' rule of signs, the sign of coefficient c determines the number of positive and distinct roots, with respect to three defined regions (**Proposition A.2.8**). Accordingly,

(I) $c < 0 \implies$ the quadratic has exactly one positive root, for any sign of b .

(II) $c = 0 \implies$ the roots are 0 and $-b/a$, hence one positive root $\Leftrightarrow b < 0$.

(III) $c > 0 \implies$ two distinct and positive roots $\Leftrightarrow b < 0$ and $\Delta := b^2 - 4ac \geq 0$.

Discriminant as a Function of $\bar{\bar{\beta}}_w$

In order to characterize the existence of real solutions for the quadratic expression I_h^c , we study the discriminant as a function of the human-*Wolbachia* mosquito transmission rate, $\bar{\bar{\beta}}_w$. One can express $\Delta(\bar{\bar{\beta}}_w)$ as a quadratic polynomial, written as

$$\Delta(\bar{\beta}_w) = A_2 \bar{\beta}_w^2 + A_1 \bar{\beta}_w + A_0, \quad A_0 = b_0^2 = \left[\mu_w \bar{\beta}_f \left(\frac{\mu_h + \nu_h}{\hat{\beta}} + \frac{M^-}{2\eta} \right) \right]^2 > 0.$$

Assume that $A_2 > 0$ and $A_1 \geq 0$. Then, for all $\bar{\beta}_w \geq 0$,

$$\frac{d}{d\bar{\beta}_w} [\Delta(\bar{\beta}_w)] = 2A_2 \bar{\beta}_w + A_1 \geq A_1 \geq 0,$$

so Δ is nondecreasing on $[0, \infty)$. Consequently,

$$\Delta(\bar{\beta}_w) \geq \Delta(0) = A_0 = b_0^2 > 0, \quad \forall \bar{\beta}_w \geq 0.$$

Thus,

$$\Delta(\bar{\beta}_w) > 0 \quad \text{for every admissible } \bar{\beta}_w \geq 0,$$

Therefore, the quadratic equilibrium equation in I_h^c admits two distinct and real roots, for all admissible values of the human-*Wolbachia* mosquito transmission rate $\bar{\beta}_w$.

Within **Proposition A.2.9**, we show that, for the parameter set representing the biological features of the *wMel Wolbachia* strain, the discriminant is strictly positive, that is, $\Delta(\bar{\beta}_w) > 0$. Moreover, we verify that the system lies in region (I), so that the quadratic equation for I_h^c admits exactly one positive root.

Figure 6 illustrates the discriminant $\Delta(\bar{\beta}_w)$ plotted against the transmission rate $\bar{\beta}_w$, where purple dashed lines represent the threshold ψ_c and green dashed lines show $\bar{\beta}_w$, calculated for the current parameter set.

4.3.1 Local Stability of Equilibrium Point E_c^h

4.3.1.1 R^c Under Mosquito Coexistence

Linearizing the infection dynamics at the disease-free equilibrium

$$S_h^{DFE} = \frac{\mu_h N_h}{\mu_h + \nu_h}, \quad I_h^{DFE} = 0,$$

yields the next-generation operator associated with new infections and transitions.

Evaluating $G_c(I_h^c)$ at the disease-free equilibrium gives

$$G_c(0) = \frac{\hat{\beta} \bar{\beta}_w \ln(Q_w)}{\sigma \mu_w} + \frac{\hat{\beta} M^- (\mu_w \bar{\beta}_f - \mu_f \bar{\beta}_w)}{2\eta \mu_w \mu_f}.$$

We are choosing the stable equilibrium point E_- , so the minus branch, M^- , is given by

$$M^- = M \left[(Q_c - 1) - \sqrt{(Q_c - 1)^2 - 4\eta \frac{Q_{w,f}}{Q_f}} \right].$$

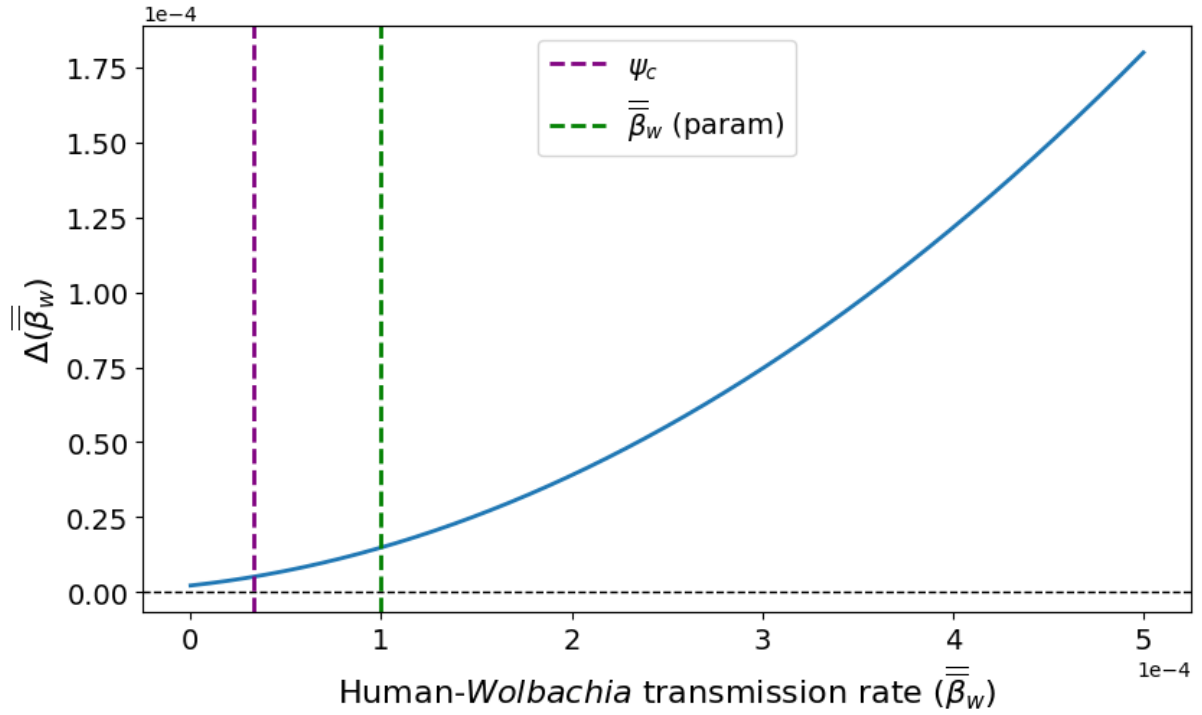


Figure 6 – Discriminant $\Delta(\bar{\beta}_w)$ versus $\bar{\beta}_w$ for the baseline wMel parameter set. The purple dashed line indicates the threshold $\psi_c = 3.3041 \times 10^{-05}$ and the green dashed line marks the baseline value $\bar{\beta}_w = 10^{-4}$. Since $\bar{\beta}_w > \psi_c$, it follows that $c < 0$, placing the system in region (I) and implying that the quadratic equation for I_h^c has exactly one positive root.

The resulting threshold quantity is

$$R^c = \frac{\hat{\beta} \mu_h N_h}{(\mu_h + \nu_h)(\gamma_h + \mu_h)} \left[\frac{\bar{\beta}_w \ln(Q_w)}{\sigma \mu_w} + \frac{M^- (\mu_w \bar{\beta}_f - \mu_f \bar{\beta}_w)}{2\eta \mu_w \mu_f} \right].$$

Therefore, the human population equilibrium under mosquito coexistence conditions is locally asymptotically stable if $R^c > 1$, and unstable if $R^c < 1$. Moreover, we can conclude that E_c^h equilibrium point is LAS whenever $R^c > \max \{1, R^f\}$ (**Proposition A.2.10**).

Thus, we rewrite R^c expression in order to separate mosquitoes and human population components. Since $\bar{\beta}_f = \frac{\bar{\beta}_f}{N_h}$, $\bar{\beta}_w = \frac{\bar{\beta}_w}{N_h}$ and $\beta = \frac{\hat{\beta}}{N_h}$,

$$R^c = \frac{\beta}{(\mu_h + \nu_h)(\gamma_h + \mu_h)} \frac{\mu_h}{N_h} \left[\frac{\bar{\beta}_w \ln(Q_w)}{\sigma \mu_w} + \frac{M^- (\mu_w \bar{\beta}_f - \mu_f \bar{\beta}_w)}{2\eta \mu_w \mu_f} \right].$$

At last, we consider $v_h = 0$, and establish mosquitoes and humans contributions, through its components on $(R^c)_{v_h=0}$,

$$(R^c)_{v_h=0} = \frac{\beta}{N_h(\gamma_h + \mu_h)} \left[\frac{\bar{\beta}_w \ln(Q_w)}{\sigma \mu_w} + \frac{M^- (\mu_w \bar{\beta}_f - \mu_f \bar{\beta}_w)}{2\eta \mu_w \mu_f} \right].$$

The expression has biological meaning. The first term represents human population, with the same mean infectious period and transmission interpretation as $(R^f)_{v_h=0}$. The second term

presents mosquitoes components, where the first term inside square brackets, given by $\frac{\bar{\beta}_w}{\mu_w}$ represents the contribution of *Wolbachia*-infected mosquitoes to dengue transmission dynamics within parameter $\bar{\beta}_w$, limited by mosquito mortality μ_w , whereas $\frac{\ln(Q_w)}{\sigma}$ represents the total mosquito population (i.e.: $S_f + I_f + S_w + I_w$) at equilibrium. Furthermore, the second term represents the wild mosquitoes contribution. Those are reasonable representations of each mosquito population, as one can see when looking at Λ_f and Λ_w recruitment rates:

$$\Lambda_f = \rho_f (S_f + I_f) \frac{(S_f + I_f) + (1 - \eta)(S_w + I_w)}{M(t)} + \rho_w(1 - \delta) (S_w + I_w),$$

$$\Lambda_w = \rho_w \delta (S_w + I_w).$$

The recruitment of *Wolbachia*-infected mosquitoes, Λ_w , is determined by *Wolbachia*-infected individuals that do not lose the infection. In contrast, the recruitment rate of wild mosquitoes, Λ_f , also includes a contribution from *Wolbachia*-infected mosquitoes due to the loss of infection. Consequently, we expect R^c to contain an individual component associated with *Wolbachia*-infected mosquitoes and a composite term associated with wild mosquitoes.

Additionally, we expect to glance at R -quantities expressed as the product between human and mosquitoes components, as much as it is for other vector-bone diseases (KEELING; ROHANI, 2008).

4.3.2 Disease-Free Equilibrium as a Particular Solution of H_c

Lastly, we demonstrate that the disease-free equilibrium is also a particular solution of the human subsystem of ordinary differential equations generated by the vector field $H_c \in H$, corresponding to the mosquito coexistence regime.

Assuming that condition (ii) is satisfied, hence mosquitoes coexistence is locally asymptotically stable, we have defined

$$G_c(I_h^c) := I_h^c \hat{\beta} \left\{ \frac{\bar{\beta}_w \ln(Q_w)}{\sigma(\bar{\beta}_w I_h^c + \mu_w)} + \frac{M^-(\mu_w \bar{\beta}_f - \mu_f \bar{\beta}_w)}{2\eta(\bar{\beta}_w I_h^c + \mu_w)(\bar{\beta}_f I_h^c + \mu_f)} \right\}.$$

Then the human subsystem associated with mosquito coexistence can be written as

$$\frac{dS_h^c}{dt} = \mu_h N_h - G_c(I_h^c) S_h^c - (\mu_h + \nu_h) S_h^c,$$

$$\frac{dI_h^c}{dt} = G_c(I_h^c) S_h^c - (\gamma_h + \mu_h) I_h^c.$$

At a disease-free equilibrium we impose $I_h^c = 0$. Since $G_c(0) = 0$, the second equation becomes

$$0 = \frac{dI_h^c}{dt} = G_c(0) S_h^c,$$

so it is identically satisfied for any $S_h^c \geq 0$, and the first equation reduces to

$$0 = \frac{dS_h^c}{dt} = \mu_h N_h - (\mu_h + \nu_h) S_h^c,$$

hence

$$S_h^c = \frac{\mu_h N_h}{\mu_h + \nu_h}.$$

Thus, the disease-free equilibrium point for the human subsystem under mosquito coexistence is

$$(S_h^{DFE}, I_h^{DFE}) = \left(\frac{\mu_h N_h}{\mu_h + \nu_h}, 0 \right).$$

Therefore, the disease-free equilibrium is also a steady-state solution of the dynamical system generated by the vector field H_c , corresponding to mosquito coexistence.

4.4 Simulations

For mosquitoes, we consider the parameter set representing biological features of *wMel* *Wolbachia* strain, presented on Table 3. Additionally, the transmission rates are taken as $\bar{\beta}_f = 0.3$, $\bar{\beta}_w = 0.1$, and $\beta = 0.2$. To represent the life-traits of the human population, we use $\mu_h = 4.0 \times 10^{-5}$, $\gamma_h = 0.2$, with initial conditions $(S_h, I_h, R_h) = (950, 25, 25)$, which implies $N_h = 1000$.

Using this numerical setup, we sweep the maternal transmission probability, $\delta \in [0.8, 1.0]$ (Figure 8), and the strength of cytoplasmic incompatibility, $\eta \in [0.6, 1.0]$ (Figure 9), within 50 points, and register the peak of infected humans I_h , representing human dynamics under *Wolbachia* extinction conditions, E_f^h , with blue circles, whereas using orange squares to represent human population under mosquitoes coexistence, E_c^h (see Figure 7).

In view of a scenario where vaccine is absent, on Figures 8 and 9, we use $\nu_h = 0 \implies$

$$\begin{aligned} (R^f)_{\nu_h=0} \big|_{\text{param}} &= \left[\frac{(0.2)}{(1000)((0.2) + (4.0e - 05))} \right] \left[\frac{(0.3) (5.021)}{(0.03) (0.005)} \right] \\ &= 10.0413 > 1, \end{aligned}$$

meaning that we are simulating over a scenario where the human equilibrium point corresponding to the *Wolbachia* extinction regime, E_f^h , is locally asymptotically stable. Similarly, when calculating $(R^c)_{\nu_h=0}$,

$$\begin{aligned} \frac{(0.2)}{(1000)((0.2) + (4.0e - 05))} &= 0.0009 \quad \text{and} \quad M^- = 8.4281 \implies \\ (R^c)_{\nu_h=0} \big|_{\text{param}} (0.0009) &\left[\frac{\left(\frac{0.1}{(1000)} \right) (4.70)}{(0.005) \left(\frac{0.03}{0.9} \right)} + \frac{M^- \left[\left(\frac{0.03}{0.9} \right) \left(\frac{0.3}{(1000)} \right) - (0.03) \left(\frac{0.1}{(1000)} \right) \right]}{2(0.9) \left(\frac{0.03}{0.9} \right) (0.03)} \right] \\ &= 2.8555 > 1, \end{aligned}$$

showing that the equilibrium point representing coexistence within *Wolbachia*-infected and wild mosquitoes populations, E_c^h , is also locally asymptotically stable.

This brief numerical analysis shows that, when considering biological relevant parameter sets, describing biological features of wild mosquitoes, $w\text{Mel}$, $w\text{MelPop}$, $w\text{AlbB}$, $w\text{Au}$ and *Wolbachia* strains, presented on Table 3, we are simulating scenarios where both equilibrium points for the human population dynamics, representing *Wolbachia* extinction, E_f^h , and coexistence between wild and *Wolbachia*-infected mosquitoes, E_c^h , are locally asymptotically stable.

On Figure 8, we perform a one-parameter sweep of the maternal transmission probability $\delta \in [0.8, 1.0)$, with the strength of cytoplasmic incompatibility fixed, and record the peak number of infected humans I_h^{peak} . The vertical red dashed line marks the coexistence threshold δ_+ : for $\delta > \delta_+$, coexistence becomes admissible and $I_h^{\text{peak}}(E_c^h)$ decreases monotonically as δ increases. For the baseline value used in the simulations ($\delta = 0.90$, $\eta = 0.90$), we obtain $I_h^{\text{peak}}(E_c^h) = 221.338$, while for $\delta = 0.99$ and $\eta = 0.90$, we obtain $I_h^{\text{peak}}(E_c^h) = 167.886$. This yields an absolute reduction of 53.452 cases, i.e., a percentage reduction of 24.15%, highlighting that higher maternal transmission mitigates the outbreak peak under the coexistence regime.

On Figure 9, we vary the strength of cytoplasmic incompatibility $\eta \in [0.6, 1.0)$ while keeping the maternal transmission probability fixed, and compute the peak number of infected humans I_h^{peak} . The vertical red dashed line indicates the coexistence threshold η_+ : for $\eta > \eta_+$, mosquito coexistence is admissible and $I_h^{\text{peak}}(E_c^h)$ decreases monotonically as η increases. For the baseline values adopted in the simulations ($\delta = 0.90$, $\eta = 0.90$), we obtain $I_h^{\text{peak}}(E_c^h) = 221.338$, whereas for $\delta = 0.90$ and $\eta = 0.99$, we obtain $I_h^{\text{peak}}(E_c^h) = 188.780$. This corresponds to an absolute reduction of 32.558 cases, i.e., a percentage reduction of 14.71%, showing that stronger cytoplasmic incompatibility reduces the outbreak peak under the coexistence regime.

On Figure 10, we report the vaccination–threshold analysis obtained from the two admissible reduced human subsystems, each one induced by a distinct equilibrium regime in the mosquito dynamics. In particular, we compute two thresholds: R^f , associated with the human dynamics when the mosquito subsystem converges to the *Wolbachia* extinction equilibrium, and R^c , associated with the human dynamics when wild and *Wolbachia*-infected mosquitoes coexist.

These thresholds characterize local stability of the corresponding human equilibria: $R_f < 1$ implies that the equilibrium E_f^h is unstable, whereas $R^c < 1$ implies that the equilibrium E_c^h is unstable. Consequently, the critical vaccination rates are defined by the crossings $R^f(\nu_h) = 1$ and $R_c(\nu_h) = 1$. For the parameter set considered, with $N_h = 1000$, we obtain

$$\nu_h^{(f)} \approx 3.62 \times 10^{-4} \quad \text{and} \quad \nu_h^{(c)} \approx 1.17 \times 10^{-4},$$

where $\nu_h^{(f)}$ is the vaccination rate required to bring R^f down to one, and $\nu_h^{(c)}$ is the corresponding threshold for R^c . The comparison $\nu_h^{(c)} < \nu_h^{(f)}$ indicates that, within the coexistence regime, a smaller vaccination effort is sufficient to reach the stability threshold. In practical terms, this means that the presence of *Wolbachia*-infected mosquitoes reduces the vaccination intensity needed to control transmission, when compared to the regime in which *Wolbachia* fails to persist.

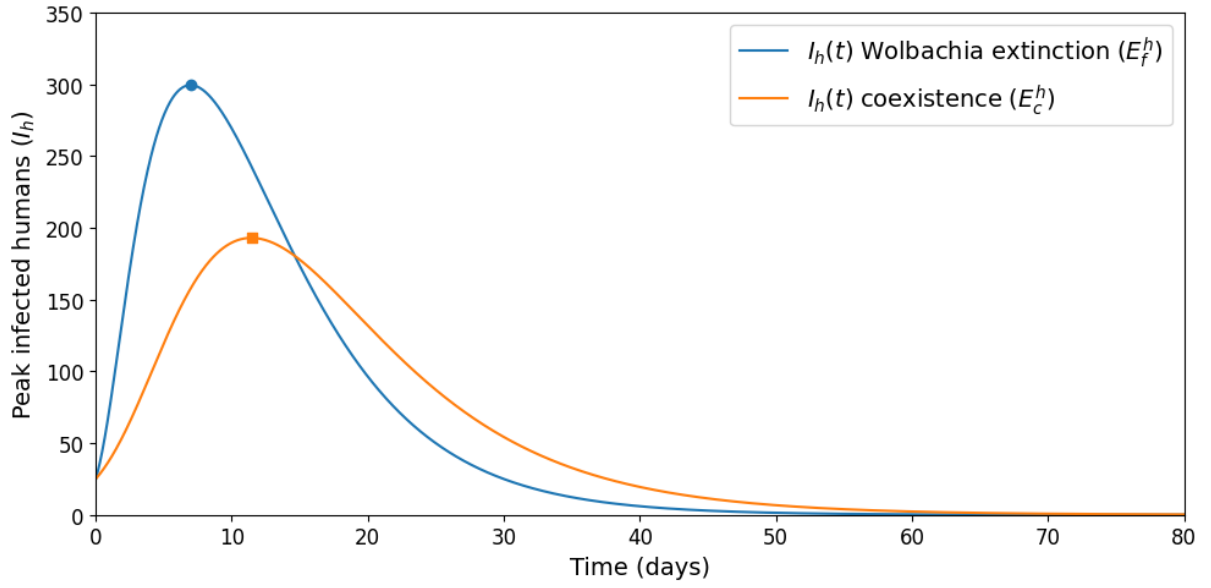


Figure 7 – Registered peak infected humans I_h , represented on blue circles for the human dynamics corresponding to *Wolbachia* extinction regime, E_f^h , and orange squares for mosquito coexistence, E_c^h .

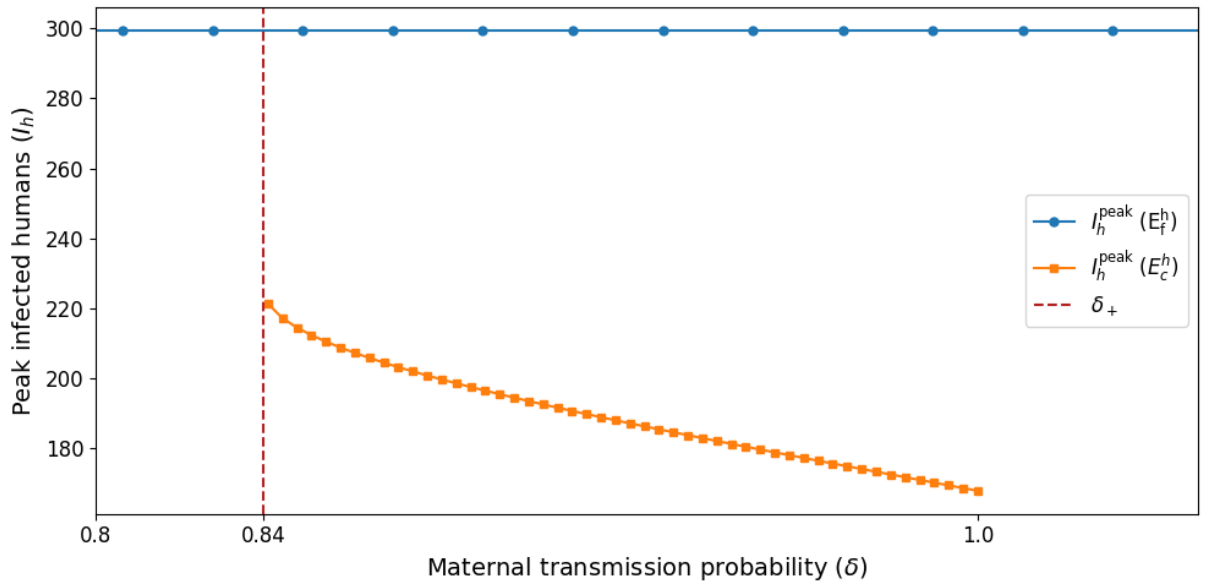


Figure 8 – Peak human infection I_h as a function of the maternal transmission probability δ , with the cytoplasmic incompatibility parameter fixed. Red dashed line shows δ_+ threshold for coexistence.

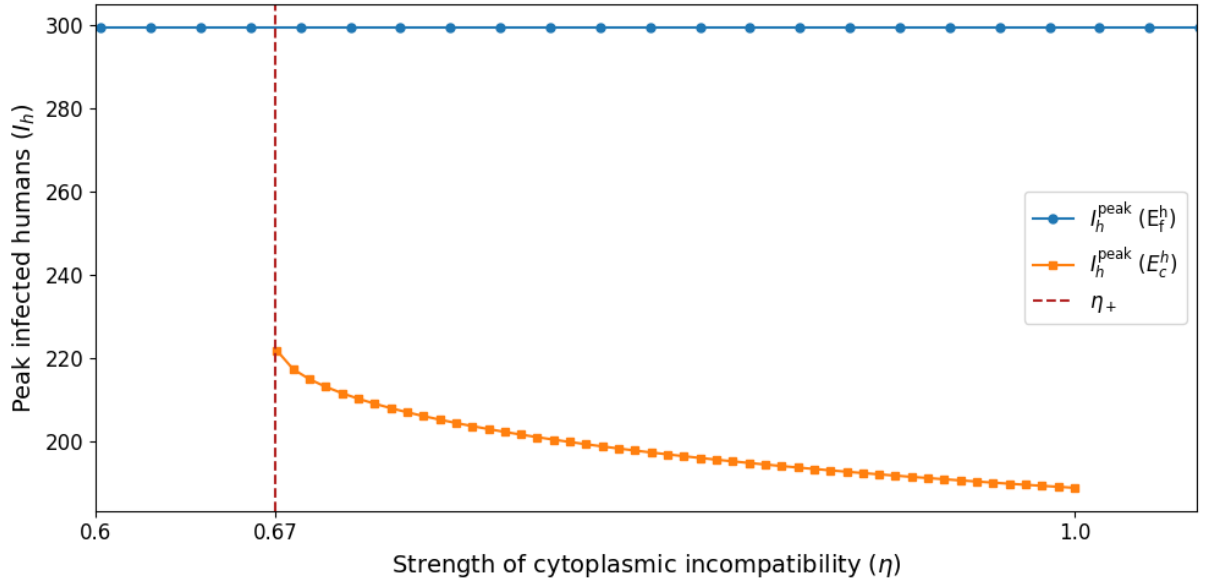


Figure 9 – Peak human infection I_h as a function of the strength of cytoplasmic incompatibility η , with the maternal transmission parameter fixed. Red dashed line shows η_+ threshold for coexistence.

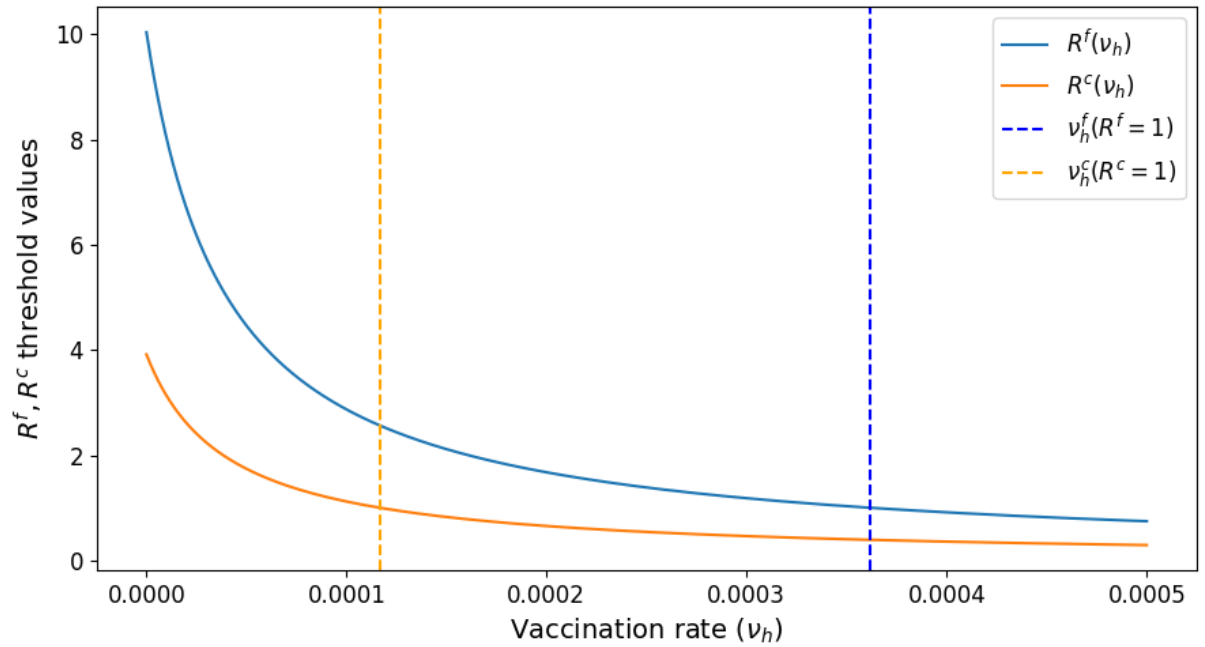


Figure 10 – Vaccination thresholds for the reduced human subsystems under distinct mosquito regimes. The curves show the threshold $R^f(\nu_h)$, corresponding to human dynamics under *Wolbachia* extinction and $R^c(\nu_h)$, representing the human dynamics under mosquito coexistence, with critical rates $\nu_h^{(f)} \approx 3.62 \times 10^{-4}$ and $\nu_h^{(c)} \approx 1.17 \times 10^{-4}$, defined by $R^f = 1$ and $R^c = 1$, respectively.

5 Discussion

Under the fast vector dynamics assumption, mosquito populations are assumed to reach equilibrium reasonably faster than the human population (KEELING; ROHANI, 2008). This separation of timescales allows treating human variables as constants when analyzing the mosquito subsystem. By solving this reduced subsystem, four equilibrium points were obtained, corresponding to distinct scenarios for mosquitoes populations: the trivial equilibrium, the *Wolbachia*-mosquitoes extinction equilibrium, and two mosquitoes coexistence branches.

Our analysis reveals the simultaneous existence of two locally asymptotically stable equilibria, the *Wolbachia* extinction equilibrium and the stable coexistence equilibrium, which characterizes a bistable dynamical regime. In particular, we have shown that *Wolbachia* extinction regime is admissible only when the basic offspring number of wild mosquitoes exceeds both unity and the basic offspring number of *Wolbachia*-infected mosquitoes, that is, when $Q_f > \max\{1, Q_w\}$, whereas the coexistence regime is possible when the maternal transmission and the cytoplasmic incompatibility parameters exceed the established critical thresholds, illustrating their relevance within *Wolbachia*-infected mosquitoes release strategies.

In this setting, the steady-state of the mosquito population is not uniquely determined by parameter values alone, but also depends critically on the initial conditions of wild and *Wolbachia*-infected mosquitoes. Consequently, the success or failure of *Wolbachia* establishment is inherently sensitive to initial condition, with invasion outcomes determined by the basin of attraction in which the system is initialized.

The mosquito subsystem constitutes the central component of the coupled host–vector model, as it determines the the long-term dynamics of vector populations that drive dengue virus transmission. By explicitly incorporating density-dependent recruitment rates, maternal transmission probability of *Wolbachia* infection and the strength of cytoplasmic incompatibility due to *Wolbachia* infection, we were able to identify and describe the biologically and mathematically viable configurations of the vector populations, namely the *Wolbachia* extinction scenario and the stable coexistence between wild and *Wolbachia*-infected mosquitoes.

Following the fast vector dynamics assumption, substituting the equilibrium values of the admissible mosquito steady states into the human equations yields a simplified description of the human population dynamics. This methodology allowed us to determine two distinct human

vector fields, each generating a corresponding system of ordinary differential equations. The resulting human subsystems are therefore regime-dependent, with their structure determined by whether the mosquito population converges to the *Wolbachia* extinction equilibrium or to the stable coexistence equilibrium.

In the *Wolbachia* extinction regime, dengue transmission is mediated exclusively by wild mosquitoes, and the corresponding human subsystem is characterized by an epidemiological threshold denoted by R^f . In contrast, under stable mosquito coexistence, both wild and *Wolbachia*-infected mosquitoes contribute to the force of infection, leading to a modified transmission structure and an associated threshold R^c . In each regime, the disease-free equilibrium of the human subsystem is locally asymptotically stable whenever the corresponding threshold satisfies $R^f < 1$ or $R^c < 1$, respectively.

Focusing on the impact of *Wolbachia* infection traits on the peak of dengue infection, increasing either maternal transmission probability or strength of cytoplasmic incompatibility reduces the infection peaks, with maternal transmission having the most significant influence (Figures 8 and 9). This behavior is consistent with the underlying mechanisms, since maternal transmission directly sustains and increases the *Wolbachia*-infected mosquito population, whereas cytoplasmic incompatibility induces a reproductive advantage for mosquitoes carrying the bacterium. Furthermore, under a scenario of coexistence between wild and *Wolbachia*-infected mosquitoes (Figures 7–10), dengue infection can be eliminated with a lower vaccination rate compared to the scenario in which only wild mosquitoes persist. Moreover, when dengue is established, the prevalence of infection is lower in the mosquito coexistence scenario.

In synthesis, the present analysis focused on modelling the dynamics of dengue transmission within human population and mosquitoes, and our results shows that both scenarios, representing *Wolbachia*-infected mosquitoes extinction, and its coexistence within wild mosquito population are

(i) Biologically viable, since alongside nonnegative initial conditions, the vector solution remains nonnegative, for all time $t \geq 0$, i.e.:

$$X = X(t), \quad X \in \mathbb{R}^7, \quad X(0) \geq 0 \implies$$

$$X : \{S_f(t), I_f(t), S_w(t), I_w(t), S_h(t), I_h(t), R_h(t)\} \geq 0, \quad \forall t \geq 0.$$

(ii) Mathematically admissible, seeing that both equilibrium points E_f and E_c are locally asymptotically stable, when condition $Q_f > \max\{1, Q_w\}$ is satisfied, for the *Wolbachia* extinction equilibrium, and for the coexistence equilibrium, when conditions $Q_w > 1$ and $Q_c > 1$ holds, alongside the conditions $\delta \in (0, 1) \cap ((-\infty, \delta_-) \cup (\delta_+, \infty))$ and $\eta \in (0, 1) \cap ((-\infty, \eta_-) \cup (\eta_+, \infty))$ are satisfied.

Within time, the mosquito population dynamics converge to one of the admissible equili-

brium regimes, depending on the initial condition of the mosquito subsystem,

$$(S_f(0), I_f(0), S_w(0), I_w(0)) \in \Omega_m.$$

This observation allows a straightforward interpretation: the success or failure of *Wolbachia*-infected mosquito releases is intrinsically dependent on the chosen release strategy. In particular, the timing, frequency, and magnitude of releases - such as weekly or monthly release schemes - determine the effective initial conditions of the mosquito population and, consequently, the long-term dynamics of dengue virus transmission between human and mosquito populations.

6 Overview of Master's Research and Activities

Additional training includes participation in the Training Program in Quantitative Ecology, funded by the Serrapilheira Institute. The program aims to prepare undergraduate and graduate students for interdisciplinary research in ecology, emphasizing quantitative, mathematical, computational, and statistical approaches, and integrating theoretical coursework with field practice across Brazilian biomes. It is structured as an intensive summer course, taught entirely in English, followed by a field module in Atlantic Forest and Amazon reserves, in which participants developed research projects grounded in quantitative analysis and ecological modeling.

Teaching experience includes participation in the Programa de Aperfeiçoamento e Apoio à Docência no Ensino Superior (PAADES), an institutional program that regulates teaching internships at UNESP and aims to train students for academic instruction. Served as a teaching assistant in Mathematics for the Biological Sciences Licenciature course, delivering theoretical lectures, preparing exercise sets, and conducting computer lab sessions involving Python-based algorithms to explore topics related to the discipline. Also served as a teaching assistant in Calculus II for the Medical Physics course, designing and teaching theoretical classes and organizing weekly supplementary sessions focused on problem-solving and discussion of core theoretical concepts.

Extension activities comprise continuous involvement in the Atena Pre-University Program, as Physics Professor and as General Student Coordinator. Additional outreach included the Winter School – SignalSpark: Exploration in Arduino, Data, and Programming, designing interactive sessions on programming and prototyping for elementary students.

Participated in and served on the organizing committees of several academic events, including the XVI Workshop de Biometria and I Jornada à Pós-Graduação. Presented research through both oral and poster sessions at other events, such as the XLI and XLIV Congresso Nacional de Matemática Aplicada e Computacional, XI Congresso de Biociências, XIV Um Dia de Biometria and XVII Encontro Acadêmico de Modelagem Computacional.

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Appendices

A Formal Mathematical Statements – Complete Statements of Results and Proofs

A.1 Mosquito General Subsystem

Mosquitoes have a much shorter lifespan than humans, so we can reasonably consider the human population to be constant while observing changes in mosquitoes populations. Alongside the following algebraic work, we set $\bar{\beta}_f = \frac{\beta_f}{N_h}$, $\bar{\beta}_w = \frac{\beta_w}{N_h}$ and $\hat{\beta} = \frac{\beta}{N_h}$, with $\beta_f = \beta_w = \beta$.

Let us now define the Jacobian matrix of the system. We shall begin by calculating the recruitment rates derivatives.

$$\frac{\partial \Lambda_f}{\partial S_f} = \frac{\partial \Lambda_f}{\partial I_f} = \rho_f \left[\frac{(S_f + I_f) + (1 - \eta)(S_w + I_w)}{M} + \eta \frac{(S_f + I_f)(S_w + I_w)}{M^2} \right]$$

$$\frac{\partial \Lambda_f}{\partial S_w} = \frac{\partial \Lambda_f}{\partial I_w} = \rho_w(1 - \delta) - \rho_f \eta \frac{(S_f + I_f)^2}{M^2}, \quad \frac{\partial \Lambda_w}{\partial S_f} = \frac{\partial \Lambda_w}{\partial I_f} = 0, \quad \frac{\partial \Lambda_w}{\partial S_w} = \frac{\partial \Lambda_w}{\partial I_w} = \rho_w \delta$$

$$\frac{\partial}{\partial x} e^{-\sigma M(x)} = e^{-\sigma M(x)} \frac{\partial}{\partial x} (-\sigma M(x)) = -\sigma e^{-\sigma M}, \quad \text{for } x \in (S_f, I_f, S_w, I_w)$$

Now we define

$$\frac{dS_f}{dt} = \Lambda_f e^{-\sigma M(t)} - \bar{\beta}_f S_f \left(\frac{I_h}{N_h} \right) - \mu_f S_f := F_1(S_f, I_f, S_w, I_w)$$

$$\frac{dI_f}{dt} = \bar{\beta}_f S_f \left(\frac{I_h}{N_h} \right) - \mu_f I_f := F_2(S_f, I_f, S_w, I_w)$$

$$\frac{dS_w}{dt} = \Lambda_w e^{-\sigma M(t)} - \bar{\beta}_w S_w \left(\frac{I_h}{N_h} \right) - \mu_w S_w := F_3(S_f, I_f, S_w, I_w)$$

$$\frac{dI_w}{dt} = \bar{\beta}_w S_w \left(\frac{I_h}{N_h} \right) - \mu_w I_w := F_4(S_f, I_f, S_w, I_w)$$

Such that the J_{ij} entries of the Jacobian Matrix are given by:

$$\begin{aligned}
J_{11} &= \frac{\partial F_1}{\partial S_f} = \left[\frac{\partial \Lambda_f}{\partial S_f} e^{-\sigma M} + \Lambda_f \frac{\partial}{\partial S_f} e^{-\sigma M} \right] - \frac{\partial}{\partial S_f} (\bar{\beta}_f S_f I_h - \mu_f S_f) \\
&= \left[e^{-\sigma M} \frac{\partial \Lambda_f}{\partial S_f} - \sigma e^{-\sigma M} \Lambda_f \right] - \bar{\beta}_f I_h - \mu_f = e^{-\sigma M} \left[\frac{\partial \Lambda_f}{\partial S_f} - \sigma \Lambda_f \right] - \bar{\beta}_f I_h - \mu_f \\
&= e^{-\sigma M} \left\{ \rho_f \left[\frac{(S_f + I_f) + (1 - \eta)(S_w + I_w)}{M} + \eta \frac{(S_f + I_f)(S_w + I_w)}{M^2} \right] - \sigma \Lambda_f \right\} - \bar{\beta}_f I_h - \mu_f
\end{aligned}$$

$$\begin{aligned}
J_{12} &= \frac{\partial F_1}{\partial I_f} = e^{-\sigma M} \left[\frac{\partial \Lambda_f}{\partial I_f} - \sigma \Lambda_f \right] - \frac{\partial}{\partial I_f} (\bar{\beta}_f S_f I_h - \mu_f S_f) \\
&= e^{-\sigma M} \left\{ \rho_f \left[\frac{(S_f + I_f) + (1 - \eta)(S_w + I_w)}{M} + \eta \frac{(S_f + I_f)(S_w + I_w)}{M^2} \right] - \sigma \Lambda_f \right\}
\end{aligned}$$

$$\begin{aligned}
J_{13} &= \frac{\partial F_1}{\partial S_w} = e^{-\sigma M} \left[\frac{\partial \Lambda_f}{\partial S_w} - \sigma \Lambda_f \right] - \frac{\partial}{\partial S_w} (\bar{\beta}_f S_f I_h - \mu_f S_f) \\
&= e^{-\sigma M} \left[\rho_w (1 - \delta) - \rho_f \eta \frac{(S_f + I_f)^2}{M^2} - \sigma \Lambda_f \right]
\end{aligned}$$

$$\begin{aligned}
J_{14} &= \frac{\partial F_1}{\partial I_w} = e^{-\sigma M} \left[\frac{\partial \Lambda_f}{\partial I_w} - \sigma \Lambda_f \right] - \frac{\partial}{\partial I_w} (\bar{\beta}_f S_f I_h - \mu_f S_f) \\
&= e^{-\sigma M} \left[\rho_w (1 - \delta) - \rho_f \eta \frac{(S_f + I_f)^2}{M^2} - \sigma \Lambda_f \right]
\end{aligned}$$

$$J_{21} = \frac{\partial F_2}{\partial S_f} = \bar{\beta}_f I_h - \mu_f \quad , \quad J_{22} = \frac{\partial F_2}{\partial I_f} = -\mu_f \quad , \quad J_{23} = \frac{\partial F_2}{\partial S_w} = 0 \quad , \quad J_{24} = \frac{\partial F_2}{\partial I_w} = 0$$

$$J_{31} = \frac{\partial F_3}{\partial S_f} = \left[\frac{\partial \Lambda_w}{\partial S_f} e^{-\sigma M} + \Lambda_w \frac{\partial}{\partial S_f} e^{-\sigma M} \right] - \frac{\partial}{\partial S_f} (\bar{\beta}_w S_w I_h - \mu_w S_w) = -\Lambda_w \sigma e^{-\sigma M}$$

$$J_{32} = \frac{\partial F_3}{\partial I_f} = \left[\frac{\partial \Lambda_w}{\partial I_f} e^{-\sigma M} + \Lambda_w \frac{\partial}{\partial I_f} e^{-\sigma M} \right] - \frac{\partial}{\partial I_f} (\bar{\beta}_w S_w I_h - \mu_w S_w) = -\Lambda_w \sigma e^{-\sigma M}$$

$$J_{33} = \frac{\partial F_3}{\partial S_w} = e^{-\sigma M} \left[\frac{\partial \Lambda_w}{\partial S_w} - \sigma \Lambda_w \right] - \frac{\partial}{\partial S_w} (\bar{\beta}_w S_w I_h - \mu_w S_w) = e^{-\sigma M} (\rho_w \delta - \sigma \Lambda_w) - \bar{\beta}_w I_h - \mu_w$$

$$J_{34} = \frac{\partial F_3}{\partial I_w} = e^{-\sigma M} \left[\frac{\partial \Lambda_w}{\partial I_w} - \sigma \Lambda_w \right] - \frac{\partial}{\partial I_w} (\bar{\beta}_w S_w I_h - \mu_w S_w) = e^{-\sigma M} (\rho_w \delta - \sigma \Lambda_w)$$

$$J_{41} = \frac{\partial F_4}{\partial S_f} = 0 \quad , \quad J_{42} = \frac{\partial F_4}{\partial I_f} = 0 \quad , \quad J_{43} = \frac{\partial F_4}{\partial S_w} = \bar{\beta}_w I_h \quad , \quad J_{44} = \frac{\partial F_4}{\partial I_w} = -\mu_w$$

Finally, we build the Jacobian Matrix of the general mosquito subsystem as:

$$J(S_f, I_f, S_w, I_w) = \begin{bmatrix} J_{11} & J_{12} & J_{13} & J_{14} \\ J_{21} & J_{22} & J_{23} & J_{24} \\ J_{31} & J_{32} & J_{33} & J_{34} \\ J_{41} & J_{42} & J_{43} & J_{44} \end{bmatrix}.$$

□

Proposition A.1.1. The general mosquito subsystem presents up to four equilibrium points, given by $(S_f^*, I_f^*, S_w^*, I_w^*)$. The equilibrium points are E_0 : trivial equilibrium, E_f : *Wolbachia* extinction equilibrium and $E_{\pm}$: mosquitoes coexistence equilibria.

Proof A.1.1. Accordingly to the fast vector dynamics, we consider the human state variables, (S_h, I_h, R_h) , as constants.

From Equation (5):

$$0 = \bar{\beta}_f S_f I_h - \mu_f I_f \implies \bar{I}_f = \frac{\bar{\beta}_f S_f I_h}{\mu_f} \quad (\text{i})$$

From Equation (7):

$$0 = \bar{\beta}_w S_w I_h - \mu_w I_w \implies \bar{I}_w = \frac{\bar{\beta}_w S_w I_h}{\mu_w} \quad (\text{ii})$$

(ii) into Equation (6):

$$\begin{aligned} 0 &= \rho_w \delta \left(S_w + \frac{\bar{\beta}_w S_w I_h}{\mu_w} \right) e^{-\sigma M(t)} - \bar{\beta}_w S_w I_h - \mu_w S_w \\ 0 &= S_w \left[\rho_w \delta \left(1 + \frac{\bar{\beta}_w I_h}{\mu_w} \right) e^{-\sigma M(t)} - \bar{\beta}_w I_h - \mu_w \right] \\ \implies &\begin{cases} \bar{S}_w = 0 & (\text{iii}) \\ \rho_w \delta \left(1 + \frac{\bar{\beta}_w I_h}{\mu_w} \right) e^{-\sigma M(t)} - \bar{\beta}_w I_h - \mu_w & (\text{iv}) \end{cases} \end{aligned}$$

(iii) $\bar{S}_w = 0$ into Equation (6):

$$0 = \rho_w \delta (S_w + I_w) e^{-\sigma M} - \bar{\beta}_w S_w I_h - \mu_w S_w \implies \bar{I}_w = 0$$

(iii) $\bar{S}_w = 0$ into Equation (7):

$$0 = \bar{\beta}_w S_w I_h - \mu_w I_w \implies \bar{I}_w = 0$$

Therefore, $\overline{S_w} = 0$ leads to $\overline{I_w} = 0$: *Wolbachia* extinction.

(iii) $\overline{S_w} = 0 \implies \overline{I_w} = 0$ into **Equation (4)**:

$$\Lambda_f = \rho_f(S_f + I_f) \implies 0 = \rho_f(S_f + I_f)e^{-\sigma(S_f + I_f)} - \overline{\beta}_f S_f I_h - \mu_f S_f$$

Substituting (i) $I_f = \overline{I_f} = \frac{\overline{\beta}_f S_f I_h}{\mu_f}$:

$$0 = \rho_f \left(S_f + \frac{\overline{\beta}_f S_f I_h}{\mu_f} \right) e^{-\sigma(S_f + \overline{I_f})} - \overline{\beta}_f S_f I_h - \mu_f S_f$$

$$0 = S_f \left[\rho_f \left(1 + \frac{\overline{\beta}_f I_h}{\mu_f} \right) e^{-\sigma(S_f + \overline{I_f})} - \overline{\beta}_f I_h - \mu_f \right]$$

$$\implies \begin{cases} \overline{S_f} = 0 & \text{(v)} \\ \rho_f \left(1 + \frac{\overline{\beta}_f I_h}{\mu_f} \right) e^{-\sigma(S_f + \overline{I_f})} - \overline{\beta}_f I_h - \mu_f & \text{(vi)} \end{cases}$$

$$\text{(vi): } \rho_f \left(1 + \frac{\overline{\beta}_f I_h}{\mu_f} \right) e^{-\sigma(S_f + \overline{I_f})} = \rho_f \left(\frac{\mu_f + \overline{\beta}_f I_h}{\mu_f} \right) e^{-\sigma(S_f + \overline{I_f})} = \overline{\beta}_f I_h + \mu_f$$

$$\iff e^{-\sigma(S_f + \overline{I_f})} = \frac{\mu_f(\overline{\beta}_f I_h + \mu_f)}{\rho_f(\mu_f + \overline{\beta}_f I_h)} = \frac{\mu_f}{\rho_f} \iff S_f + \overline{I_f} = \frac{1}{\sigma} \ln \left(\frac{\rho_f}{\mu_f} \right)$$

$$Q_f := \frac{\rho_f}{\mu_f} \implies S_f + \overline{I_f} = \frac{1}{\sigma} \ln(Q_f)$$

$$\iff S_f + \frac{\overline{\beta}_f S_f I_h}{\mu_f} = S_f \left(1 + \frac{\overline{\beta}_f I_h}{\mu_f} \right) = S_f \left(\frac{\mu_f + \overline{\beta}_f I_h}{\mu_f} \right) = \frac{1}{\sigma} \ln(Q_f)$$

$$\therefore \overline{S_f} = \frac{\mu_f}{\sigma} \frac{\ln(Q_f)}{\mu_f + \overline{\beta}_f I_h}, \text{ where } I_h(t) := I_h \in \mathbb{R} \text{ is a constant and } Q_f > 1.$$

(v) $\overline{S_f} = 0$ into **Equation (4)**:

$$0 = \overline{\beta}_f \overline{S_f} I_h - \mu_f \overline{I_f} \implies \overline{I_f} = 0$$

$\therefore \overline{S_w} = 0 \implies \overline{I_w} = 0$ and $\overline{S_f} = 0 \implies \overline{I_f} = 0$: Trivial equilibria.

$E_0 = (0, 0, 0, 0)$: always exists.

(vi) $\overline{S}_f = \frac{\mu_f}{\sigma} \frac{\ln(Q_f)}{\mu_f + \overline{\beta}_f I_h}$ **into Equation (4):**

$$0 = \overline{\beta}_f \left(\frac{\mu_f}{\sigma} \frac{\ln(Q_f)}{\mu_f + \overline{\beta}_f I_h} \right) I_h - \mu_f \overline{I}_f \implies \mu_f \overline{I}_f = \frac{\overline{\beta}_f I_h \mu_f}{\sigma} \frac{\ln(Q_f)}{\mu_f + \overline{\beta}_f I_h}$$

$$\iff \overline{I}_f = \frac{\overline{\beta}_f I_h}{\sigma} \frac{\ln(Q_f)}{\mu_f + \overline{\beta}_f I_h}, \text{ where } I_h(t) := I_h \in \mathbb{R} \text{ is a constant and } Q_f > 1.$$

$\therefore \overline{S}_w = 0 \implies \overline{I}_w = 0$ and $\overline{S}_f = \frac{\mu_f}{\sigma} \frac{\ln(Q_f)}{\mu_f + \overline{\beta}_f I_h} \implies \overline{I}_f = \frac{\overline{\beta}_f I_h}{\sigma} \frac{\ln(Q_f)}{\mu_f + \overline{\beta}_f I_h}$: *Wolbachia* extinction.

$$E_f = \left(\frac{\mu_f}{\sigma} \frac{\ln(Q_f)}{\mu_f + \overline{\beta}_f I_h}, \frac{\overline{\beta}_f I_h}{\sigma} \frac{\ln(Q_f)}{\mu_f + \overline{\beta}_f I_h}, 0, 0 \right) : \exists \text{ if } Q_f > 1.$$

At last, expression (iv) will lead us to coexistence.

$$\text{(iv): } \rho_w \delta \left(1 + \frac{\overline{\beta}_w I_h}{\mu_w} \right) e^{-\sigma M(t)} = \rho_w \delta \left(\frac{\mu_w + \overline{\beta}_w I_h}{\mu_w} \right) e^{-\sigma M(t)} = \overline{\beta}_w I_h + \mu_w$$

$$\iff e^{-\sigma M(t)} = \frac{\mu_w (\overline{\beta}_w I_h + \mu_w)}{\rho_w \delta (\mu_w + \overline{\beta}_w I_h)} = \frac{\mu_w}{\rho_w \delta}$$

$$\iff -\sigma M(t) = \ln \left(\frac{\mu_w}{\rho_w \delta} \right) \iff M(t) = \frac{1}{\sigma} \ln \left(\frac{\rho_w \delta}{\mu_w} \right), \text{ with } Q_w := \frac{\rho_w \delta}{\mu_w}$$

$$\therefore M = S_f + I_f + S_w + I_w = \frac{1}{\sigma} \ln(Q_w), \quad Q_w > 1.$$

$$\iff S_w + I_w = M - (S_f + I_f)$$

and also $e^{-\sigma M(t)} = \frac{1}{Q_w}$, $Q_w > 1$.

(iv) **into Equation (3):**

Substituting (i) $I_f = \overline{I}_f = \frac{\overline{\beta}_f I_h}{\mu_f} S_f = C_2 S_f$, where $C_2 = \frac{\overline{\beta}_f I_h}{\mu_f}$,

also using $\overline{\delta} = (1 - \delta)$ and $\overline{\eta} = (1 - \eta)$.

$$\Lambda_f = \rho_f (S_f + I_f) \frac{\eta(S_f + I_f) + (1 - \eta)M}{M} + \rho_w \overline{\delta} [M - (S_f + I_f)]$$

$$= \frac{\rho_f \eta}{M} (S_f + I_f)^2 + \rho_f (1 - \eta) (S_f + I_f) + \rho_w \overline{\delta} (S_f + I_f)$$

$$\begin{aligned}
&= \frac{\rho_f \eta}{M} (S_f + I_f)^2 + (S_f + I_f) [\rho_f (1 - \eta) - \rho_w \bar{\delta}] + \rho_w \bar{\delta} M \\
&= \frac{\rho_f \eta}{M} (S_f + C_2 S_f)^2 + (S_f + C_2 S_f) (\rho_f \bar{\eta} - \rho_w \bar{\delta}) + \rho_w \bar{\delta} M \\
&= \frac{\rho_f \eta}{M} (S_f + C_2 S_f)^2 + (S_f + C_2 S_f) (\rho_f \bar{\eta} - \rho_w \bar{\delta}) + \rho_w \bar{\delta} M \\
&= \frac{\rho_f \eta}{M} (S_f^2 + 2S_f^2 C_2 + C_2^2) + S_f [(1 + C_2)(\rho_f \bar{\eta} - \rho_w \bar{\delta})] + \rho_w \bar{\delta} M \\
&= \rho_f \eta S_f^2 (1 + 2C_2 + C_2^2) + M S_f [(1 + C_2)(\rho_f \bar{\eta} - \rho_w \bar{\delta})] + M^2 \rho_w \bar{\delta} \\
\implies \frac{dS_f}{dt} &= \left\{ \rho_f \eta S_f^2 (1 + 2C_2 + C_2^2) + M S_f [(1 + C_2)(\rho_f \bar{\eta} - \rho_w \bar{\delta})] + M^2 \rho_w \bar{\delta} \right\} \frac{1}{Q_w} \\
&\quad - M (\bar{\beta}_f I_h S_f + \mu_f S_f) = 0
\end{aligned}$$

We can also rewrite the expression $M(\bar{\beta}_f I_h S_f + \mu_f S_f) = 0$ in terms of the constant C_2 :

$$\begin{aligned}
C_2 &= \frac{\bar{\beta}_f I_h}{\mu_f} \iff \bar{\beta}_f I_h = \mu_f C_2 \iff M(\bar{\beta}_f I_h S_f + \mu_f S_f) = M(\mu_f C_2 S_f + \mu_f S_f) \\
&= M S_f \mu_f (C_2 + 1)
\end{aligned}$$

$$\implies \frac{dS_f}{dt} = \left\{ \rho_f \eta S_f^2 (1 + 2C_2 + C_2^2) + M S_f [(1 + C_2)(\rho_f \bar{\eta} - \rho_w \bar{\delta})] + M^2 \rho_w \bar{\delta} \right\} \frac{1}{Q_w} - M S_f \mu_f (C_2 + 1) = 0$$

We shall define new constants $K_1 = (C_2 + 1)$ and $K_2 = (1 + 2C_2 + C_2^2)$

$$\begin{aligned}
0 &= \rho_f \eta K_2 S_f^2 + S_f [M K_1 (\rho_f \bar{\eta} - \rho_w \bar{\delta}) - M K_1 \mu_f Q_w] + M^2 \rho_w \bar{\delta} \\
&= \rho_f \eta K_2 S_f^2 + S_f [M K_1 (\rho_f \bar{\eta} - \rho_w \bar{\delta} - \mu_f Q_w)] + M^2 \rho_w \bar{\delta} \\
&= \frac{\rho_f}{\mu_f} \eta K_2 S_f^2 + S_f \left[M K_1 \left(\frac{\rho_f}{\mu_f} \bar{\eta} - \frac{\rho_w \bar{\delta}}{\mu_f} - \frac{\mu_f}{\mu_f} Q_w \right) \right] + \frac{\rho_w \bar{\delta}}{\mu_f} M^2
\end{aligned}$$

Now we define the quantity $Q_{w,f} := \frac{\rho_w \bar{\delta}}{\mu_f} = \frac{\rho_w (1 - \delta)}{\mu_f}$ and also use $Q_f = \frac{\rho_f}{\mu_f}$

$$= Q_f \eta K_2 S_f^2 + S_f [M K_1 (Q_f \bar{\eta} - Q_{w,f} - Q_w)] + Q_{w,f} M^2$$

$$\begin{aligned}
&= \eta K_2 S_f^2 + S_f \left[MK_1 \left(\bar{\eta} - \frac{Q_{w,f}}{Q_f} - \frac{Q_w}{Q_f} \right) \right] + \frac{Q_{w,f}}{Q_f} M^2 \\
&= \eta K_2 S_f^2 - MK_1 S_f \left[\frac{Q_{w,f}}{Q_f} + \frac{Q_w}{Q_f} - (1 - \eta) \right] + \frac{Q_{w,f}}{Q_f} M^2
\end{aligned}$$

We can rewrite the expression $\frac{Q_{w,f}}{Q_f} + \frac{Q_w}{Q_f} - (1 - \eta) = \frac{Q_{w,f} + Q_w + \eta Q_f - Q_f}{Q_f} = \frac{Q_{w,f} + Q_w + \eta Q_f}{Q_f} - \frac{Q_f}{Q_f}$ and define the quantity $Q_c := \frac{Q_{w,f} + Q_w + \eta Q_f}{Q_f} \implies \frac{Q_{w,f} + Q_w + \eta Q_f}{Q_f} - \frac{Q_f}{Q_f} = Q_c - 1$

$$\implies \frac{dS_f}{dt} = \eta K_2 S_f^2 - MK_1 S_f (Q_c - 1) + \frac{Q_{w,f}}{Q_f} M^2 = 0 \quad : \text{quadratic in } S_f$$

$$\eta K_2 S_f^2 - MK_1 S_f (Q_c - 1) + \frac{Q_{w,f}}{Q_f} M^2 = 0 \Leftrightarrow A S_f^2 + B S_f + C = 0$$

where $A = \eta K_2$, $B = -MK_1(Q_c - 1)$ and $C = \frac{Q_{w,f}}{Q_f} M^2$

At last, roots are given by:

$$\begin{aligned}
S_f^\pm &= \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \implies \\
S_f^\pm &= \frac{M \left(\frac{\bar{\beta}_f I_h}{\mu_f} + 1 \right) (Q_c - 1) \pm \sqrt{\left[M \left(\frac{\bar{\beta}_f I_h}{\mu_f} + 1 \right) \right]^2 - 4\eta \left[1 + \frac{\bar{\beta}_f I_h}{\mu_f} \left(2 + \frac{\bar{\beta}_f I_h}{\mu_f} \right) \right] \frac{Q_{w,f}}{Q_f} M^2}}{2\eta \left[1 + \frac{\bar{\beta}_f I_h}{\mu_f} \left(2 + \frac{\bar{\beta}_f I_h}{\mu_f} \right) \right]},
\end{aligned}$$

$$I_f^\pm = \frac{\bar{\beta}_f I_h}{\mu_f} S_f^\pm.$$

$$M = S_f + I_f + S_w + I_w = \frac{1}{\sigma} \ln(Q_w) \iff S_w + I_w = \frac{1}{\sigma} \ln(Q_w) - (S_f + I_f)$$

$$(ii) \bar{I}_w = \frac{\bar{\beta}_w I_h}{\mu_w} S_w = C_1 S_w, \quad \text{where } C_1 = \frac{\bar{\beta}_w I_h}{\mu_w}$$

$$\implies S_w + \frac{\bar{\beta}_w I_h}{\mu_w} S_w = S_w \left(1 + \frac{\bar{\beta}_w I_h}{\mu_w} \right) = \frac{1}{\sigma} \ln(Q_w) - (S_f + I_f)$$

$$S_w^\pm = \frac{\mu_w}{\mu_w + \bar{\beta}_w I_h} \left[\frac{1}{\sigma} \ln(Q_w) - (S_f^\pm + I_f^\pm) \right]$$

And finally,

$$I_w^\pm = \frac{1}{\sigma} \ln(Q_w) - (S_f^\pm + I_f^\pm + S_w^\pm).$$

Therefore, we calculated four steady-states representing the mosquitoes populations:

$$E_0 = (0, 0, 0, 0) : \text{always exists,}$$

$$E_f = \left(\frac{\mu_f}{\sigma} \frac{\ln(Q_f)}{\mu_f + \bar{\beta}_f I_h}, \frac{\bar{\beta}_f I_h}{\sigma} \frac{\ln(Q_f)}{\mu_f + \bar{\beta}_f I_h}, 0, 0 \right) : \exists \iff Q_f > 1,$$

$$E_- = (S_f^-, I_f^-, S_w^-, I_w^-) : \exists \iff Q_w > 1,$$

$$E_+ = (S_f^+, I_f^+, S_w^+, I_w^+) : \exists \iff Q_w > 1.$$

□

Proposition A.1.2. The trivial steady state $E_0 = (0, 0, 0, 0)$ is locally asymptotically stable (LAS) if, and only if the conditions $Q_f < 1$ and $Q_w < 1$ are satisfied.

Proof A.1.2. Substituing the trivial equilibrium values $E_0 = (0, 0, 0, 0)$ into the Jacobian Matrix leads to

$$J(E_0) = \begin{bmatrix} \rho_f - \bar{\beta}_f I_h - \mu_f & \rho_f & \rho_w(1 - \delta) & \rho_w(1 - \delta) \\ \bar{\beta}_f I_h & -\mu_f & 0 & 0 \\ 0 & 0 & \rho_w \delta - \bar{\beta}_w I_h - \mu_w & \rho_w \delta \\ 0 & 0 & \bar{\beta}_w I_h & -\mu_w \end{bmatrix}.$$

Which can be written as two block matrices:

$$J_f = \begin{bmatrix} \rho_f - \bar{\beta}_f I_h - \mu_f & \rho_f \\ \bar{\beta}_f I_h & -\mu_f \end{bmatrix} \quad \text{and} \quad J_w = \begin{bmatrix} \rho_w \delta - \bar{\beta}_w I_h - \mu_w & \rho_w \delta \\ \bar{\beta}_w I_h & -\mu_w \end{bmatrix}$$

But this is trivial equilibrium - mosquitoes extinction, so it is reasonable to consider that $(S_f, I_f, S_w, I_w) = (0, 0, 0, 0) \implies I_h = 0$. With this, matrices J_f and J_w are rewritten as

$$J_f = \begin{bmatrix} \rho_f - \mu_f & \rho_f \\ 0 & -\mu_f \end{bmatrix} \quad \text{and} \quad J_w = \begin{bmatrix} \rho_w \delta - \mu_w & \rho_w \delta \\ 0 & -\mu_w \end{bmatrix}$$

Starting of by calculating the eigenvalues of the Wild Mosquitoes Block

$$\text{tr}(J_f) = \rho_f - 2\mu_f, \quad \det(J_f) = -\mu_f(\rho_f - \mu_f) \implies$$

$$\implies \Delta_f = [\text{tr}(J_f)]^2 - 4\det(J_f) = (\rho_f - 2\mu_f)^2 + 4\mu_f(\rho_f - \mu_f) = \rho_f^2 - 4\rho_f\mu_f + 4\mu_f^2 + 4\rho_f\mu_f - 4\mu_f^2$$

$$\Delta_f = \rho_f^2 \implies \Delta_f^{1/2} = \rho_f$$

$$\lambda_f^\pm = \frac{\text{tr}(J_f) \pm \sqrt{\Delta_f}}{2} = \frac{(\rho_f - 2\mu_f) \pm \rho_f}{2} = \begin{cases} \lambda_f^- = -\mu_f \\ \lambda_f^+ = \rho_f - \mu_f = \mu_f(Q_f - 1) \end{cases}$$

On the other side, the eigenvalues of the *Wolbachia* Mosquitoes Block

$$\begin{aligned} \text{tr}(J_w) &= \rho_w \delta - 2\mu_w, \quad \det(J_w) = -\mu_w(\rho_w \delta - \mu_w) \implies \\ \implies \Delta_f &= [\text{tr}(J_w)]^2 - 4\det(J_w) = (\rho_w \delta - 2\mu_w)^2 + 4\mu_w(\rho_w \delta - \mu_w) \\ &= (\rho_w \delta)^2 - 4\mu_w \rho_w \delta + 4\mu_w^2 + 4\mu_w \rho_w \delta - 4\mu_w^2 \\ \Delta_f &= (\rho_w \delta)^2 \implies \Delta_f^{1/2} = \rho_w \delta \end{aligned}$$

$$\lambda_w^\pm = \frac{\text{tr}(J_w) \pm \sqrt{\Delta_w}}{2} = \frac{\rho_w \delta - 2\mu_w \pm \rho_w \delta}{2} = \begin{cases} \lambda_w^- = -\mu_w \\ \lambda_w^+ = \rho_w \delta - \mu_w = \mu_w(Q_w - 1) \end{cases}$$

Therefore, the Jacobian Matrix has four eigenvalues:

$$\lambda_f^- = -\mu_f < 0, \quad \lambda_w^- = -\mu_w < 0$$

$$\lambda_f^+ = \mu_f(Q_f - 1) < 0 \iff Q_f < 1, \quad \lambda_w^+ = \mu_w(Q_w - 1) < 0 \iff Q_w < 1.$$

At last, when both offspring number Q_f, Q_w are negative, λ_f^+ and λ_f^- are also negative, meaning that the steady state E_0 is LAS if, and only if $Q_f < 1, Q_w < 1$.

□

Proposition A.1.3. The *Wolbachia* extinction equilibrium point, E_f , is locally asymptotically stable if, and only if, the condition $Q_f > \max\{1, Q_w\}$ is satisfied.

Proof A.1.3. Substituting the *Wolbachia* extinction equilibrium values,

$$E_f = (\overline{S}_f, \overline{I}_f, 0, 0), \quad \overline{S}_f = \frac{\mu_f}{\sigma} \frac{\ln(Q_f)}{\mu_f + \overline{\beta}_f I_h}, \quad \overline{I}_f = \frac{\overline{\beta}_f I_h}{\sigma} \frac{\ln(Q_f)}{\mu_f + \overline{\beta}_f I_h},$$

into the Jacobian Matrix corresponding to the mosquito subsystem leads to

$$J(E_f) = \begin{bmatrix} -\overline{\beta}_f I_h - \mu_f \ln(Q_f) & \mu_f(1 - \ln(Q_f)) & \frac{\rho_w(1 - \delta) - \rho_f(\eta + \ln(Q_f))}{Q_f} & \frac{\rho_w(1 - \delta) - \rho_f(\eta + \ln(Q_f))}{Q_f} \\ \overline{\beta}_f I_h & -\mu_f & 0 & 0 \\ 0 & 0 & \frac{\rho_w \delta}{Q_f} - (\overline{\beta}_w I_h + \mu_w) & \frac{\rho_w \delta}{Q_f} \\ 0 & 0 & \overline{\beta}_w I_h & -\mu_w \end{bmatrix}.$$

This matrix can be written in block form as

$$J(E_f) = \begin{bmatrix} J_f & B \\ 0 & J_w \end{bmatrix},$$

where

$$J_f = \begin{bmatrix} -\bar{\beta}_f I_h - \mu_f \ln(Q_f) & \mu_f (1 - \ln(Q_f)) \\ \bar{\beta}_f I_h & -\mu_f \end{bmatrix}, \quad J_w = \begin{bmatrix} \frac{\rho_w \delta}{Q_f} - (\bar{\beta}_w I_h + \mu_w) & \frac{\rho_w \delta}{Q_f} \\ \bar{\beta}_w I_h & -\mu_w \end{bmatrix},$$

and

$$B = \begin{bmatrix} \frac{\rho_w(1 - \delta) - \rho_f(\eta + \ln(Q_f))}{Q_f} & \frac{\rho_w(1 - \delta) - \rho_f(\eta + \ln(Q_f))}{Q_f} \\ 0 & 0 \end{bmatrix}.$$

We shall start with the eigenvalues of the wild mosquitoes block J_f .

$$\text{tr}(J_f) = -\mu_f \ln(Q_f) - \mu_f = -\mu_f [\ln(Q_f) + 1] < 0.$$

$$\det(J_f) = [-\bar{\beta}_f I_h - \mu_f \ln(Q_f)(-\mu_f)] - [\mu_f(1 - \ln(Q_f))(\bar{\beta}_f I_h)] = \mu_f \ln(Q_f)(\mu_f + \bar{\beta}_f I_h) > 0$$

Hence,

$$Q_f > 1 \implies \text{tr}(J_f) < 0 \text{ and } \det(J_f) > 0.$$

which means both eigenvalues $\lambda_f^\pm$ are real and strictly negative. Therefore, the wild mosquitoes block J_f is locally asymptotically stable whenever E_f exists ($Q_f > 1$).

On the other side, we shall calculate the eigenvalues of the *Wolbachia* block J_w .

Since E_f represents *Wolbachia* extinction, it is reasonable to assume the approximation

$$(S_w, I_w) = 0 \implies \bar{\beta}_w = 0.$$

Under which, the *Wolbachia* Mosquitoes Block J_w simplifies to

$$J_w = \begin{bmatrix} \frac{\rho_w \delta}{Q_f} - \mu_w & \frac{\rho_w \delta}{Q_f} \\ 0 & -\mu_w \end{bmatrix}.$$

$$\text{tr}(J_w) = \frac{\rho_w \delta}{Q_f} - 2\mu_w, \quad \det(J_w) = -\mu_w \left(\frac{\rho_w \delta}{Q_f} - \mu_w \right) \implies$$

$$\Delta_w = [\text{tr}(J_w)]^2 - 4C_w = \left(\frac{\rho_w \delta}{Q_f} - 2\mu_w \right)^2 + 4\mu_w \left(\frac{\rho_w \delta}{Q_f} - \mu_w \right)$$

$$\Delta_w = \left(\frac{\rho_w \delta}{Q_f} \right)^2 \implies \Delta_w^{1/2} = \frac{\rho_w \delta}{Q_f}$$

and the eigenvalues of J_w are

$$\lambda_w^\pm = \frac{\text{tr}(J_w) \pm \sqrt{\Delta_w}}{2} = \frac{\frac{\rho_w \delta}{Q_f} - 2\mu_w \pm \frac{\rho_w \delta}{Q_f}}{2} = \begin{cases} \lambda_w^- = -\mu_w \\ \lambda_w^+ = \frac{\rho_w \delta}{Q_f} - \mu_w = \mu_w \left(\frac{Q_w}{Q_f} - 1 \right) \end{cases}$$

Hence,

$$\lambda_w^- = -\mu_w < 0, \text{ for all parameter values, and}$$

$$\lambda_w^+ < 0 \iff \frac{Q_w}{Q_f} - 1 < 0 \iff Q_w < Q_f,$$

Since $J(E_f)$ is a upper-triangular matrix, its eigenvalues are the calculated eigenvalues of J_f and J_w blocks ($\lambda_f^\pm$ and $\lambda_w^\pm$). Under the existence condition $Q_f > 1$, the wild block J_f is always stable (both $\lambda_f^\pm < 0$), and the stability of E_f is determined by the sign of λ_w^+ :

$$\lambda_w^+ < 0 \iff Q_w < Q_f.$$

Therefore, E_f is LAS if, and only if $Q_f > \max\{1, Q_w\}$.

□

Proposition A.1.4. The equilibrium points corresponding to wild and *Wolbachia*-infected mosquitos equilibrium points exists if, and only if, the condition $Q_c > 1$ is satisfied.

Proof A.1.4. We recall that $Q_c := \frac{Q_{w,f} + Q_w + \eta Q_f}{Q_f}$ is the coexistence threshold, and define the constant quantity $\beta^f = \frac{\bar{\beta}_f I_h}{\mu_f}$.

When solving for the steady-state solutions, we defined the quadratic $A S_f^2 + B S_f + C = 0$, where $A = \eta (\beta^f + 1)^2$, $B = -M(\beta^f + 1)(Q_c - 1)$ and $C = \frac{Q_{w,f}}{Q_f} M^2$.

According to **Descartes' Rule of Signs**, the number of positive roots of the polynomial is equal to the number of variations in the sign of its coefficients, so let us check the expressions signs.

We observe that

$$\beta^f = \frac{\bar{\beta}_f I_h}{\mu_f} > 0 \implies (\beta^f + 1) > 0,$$

$$\text{and } (\beta^f + 1)^2 = 1 + 2\beta^f + (\beta^f)^2 = 1 + \beta^f(2 + \beta^f) > 0.$$

In such way that

$$A = \eta(\beta^f + 1)^2 = \eta[1 + \beta^f(2 + \beta^f)] > 0,$$

$$B = -M(\beta^f + 1)(Q_c - 1) = -\frac{1}{\sigma} \ln(Q_w) (\beta^f + 1) (Q_c - 1) \implies \begin{cases} B < 0, & \text{if } Q_c > 1 \\ B > 0, & \text{if } Q_c < 1 \end{cases},$$

$$C = \frac{Q_{w,f}}{Q_f} M^2 = \frac{\rho_w(1-\delta)}{\mu_w} \left(\frac{1}{\sigma} \ln(Q_w) \right)^2 > 0, \text{ if } \delta < 1$$

$$0 < \delta < 1 \implies C > 0.$$

Notably, when condition $Q_c > 1$ is satisfied, the quadratic equation admits two positive roots, $S_f^\pm$, assuring the existence of two coexistence equilibria, $E_\pm$. Therefore, $Q_c > 1$ is a necessary and sufficient condition for the existence of the coexistence equilibrium points.

□

Proposition A.1.5. There exist critical thresholds, imposing conditions on maternal transmission parameter δ . When δ satisfy the threshold conditions, the coexistence equilibria are admissible.

Proof A.1.5. The discriminant of the coexistence quadratic expression can be rewritten as a function of the maternal transmission parameter δ .

$$\Delta(\delta) = M^2(\beta^f + 1)^2 \Phi(\delta), \quad \Phi(\delta) = (Q_c - 1)^2 - 4\eta \frac{Q_{w,f}}{Q_f}.$$

where Q_c and $Q_{w,f}$ depend on the δ parameter. Since $M^2(\beta^f + 1)^2 > 0$, real and distinct coexistence equilibria exist if, and only if, $\Phi(\delta) > 0$.

Introduce the constants

$$a_1 := \frac{\rho_w}{\mu_f}, \quad a_2 := \frac{\rho_w}{\mu_w}, \quad Q_f = \frac{\rho_f}{\mu_f},$$

so that

$$Q_{w,f} = a_1(1 - \delta), \quad Q_w = a_2\delta.$$

Hence,

$$Q_c - 1 = \frac{a_1(1 - \delta) + a_2\delta + (\eta - 1)Q_f}{Q_f}.$$

Define

$$s_0 := a_1 + (\eta - 1)Q_f, \quad s_1 := a_2 - a_1.$$

Then

$$Q_c - 1 = \frac{s_0 + s_1\delta}{Q_f}, \quad \frac{Q_{w,f}}{Q_f} = \frac{a_1}{Q_f}(1 - \delta).$$

Substituting these into $\Phi(\delta)$ yields

$$\left(\frac{s_0 + s_1\delta}{Q_f} \right)^2 = 4\eta \frac{a_1}{Q_f}(1 - \delta),$$

which, after multiplying by Q_f^2 , becomes

$$(s_0 + s_1\delta)^2 = 4\eta a_1 Q_f(1 - \delta).$$

Expanding terms gives

$$s_1^2\delta^2 + (2s_0s_1 + 4\eta a_1 Q_f)\delta + (s_0^2 - 4\eta a_1 Q_f) = 0.$$

Thus, the discriminant expression $\Phi(\delta) = 0$ is the quadratic

$$A_\delta \delta^2 + B_\delta \delta + C_\delta = 0,$$

where

$$A_\delta = s_1^2, \quad B_\delta = 2s_0s_1 + 4\eta a_1 Q_f, \quad C_\delta = s_0^2 - 4\eta a_1 Q_f.$$

The values of δ for which the coexistence equilibria is possible (i.e. $\Delta = 0$) are given explicitly by the roots of:

$$\delta_\pm = \frac{-B_\delta \pm \sqrt{B_\delta^2 - 4A_\delta C_\delta}}{2A_\delta}.$$

$$\delta_\pm = \frac{-(2s_0s_1 + 4\eta a_1 Q_f) \pm \sqrt{(2s_0s_1 + 4\eta a_1 Q_f)^2 - 4s_1^2(s_0^2 - 4\eta a_1 Q_f)}}{2s_1^2}.$$

Imposing the condition that assures the discriminant $\Delta(\delta)$ is strictly positive, we obtain

$$\Phi(\delta) > 0 \iff \delta \in (-\infty, \delta_-) \cup (\delta_+, \infty).$$

Furthermore, considering the biological restriction $0 < \delta < 1$, the admissible region reduces to

$$\delta \in (0, 1) \cap \left((-\infty, \delta_-) \cup (\delta_+, \infty) \right).$$

At last, we observe that the coexistence equilibria exist (i.e., are real and admissible) whenever the maternal transmission parameter δ lies in the admissible region.

□

Proposition A.1.6. There exist critical thresholds, imposing conditions on cytoplasmic incompatibility parameter η . When η satisfy the threshold conditions, the coexistence equilibria are admissible.

Proof A.1.6. The discriminant of the coexistence quadratic expression can be written as a function of the cytoplasmic incompatibility parameter parameter η .

$$\Delta(\eta) = M^2(\beta^f + 1)^2 \Phi(\eta), \quad \Phi(\eta) = (Q_c - 1)^2 - 4\eta \frac{Q_{w,f}}{Q_f}.$$

Since $M^2(\beta^f + 1)^2 > 0$, real coexistence equilibria exist whenever $\Phi(\eta) > 0$.

Using

$$Q_c = \frac{Q_{w,f} + Q_w + \eta Q_f}{Q_f}, \quad Q_c - 1 = \eta + \frac{Q_{w,f} + Q_w - Q_f}{Q_f},$$

define

$$q_0 := \frac{Q_{w,f}}{Q_f}, \quad q_1 := \frac{Q_w}{Q_f}, \quad q := q_0 + q_1 - 1.$$

Then

$$\Phi(\eta) = (\eta + q)^2 - 4\eta q_0 = \eta^2 + (2q - 4q_0)\eta + q^2.$$

The discriminant of this quadratic polynomial in η is

$$\Delta_\eta = (2q - 4q_0)^2 - 4q^2 = 16q_0(1 - q_1).$$

Provided $\Delta_\eta \geq 0$ (i.e. $q_1 \leq 1 \Leftrightarrow Q_w < Q_f$), the roots that satisfies $\Delta(\eta) = 0$ are given by

$$\eta_{\pm} = \frac{-(2q - 4q_0) \pm \sqrt{\Delta_\eta}}{2} = 1 + q_0 - q_1 \pm 2\sqrt{q_0(1 - q_1)}.$$

Equivalently, written purely in terms of the previously defined quantities, $(Q_f, Q_w, Q_{w,f})$,

$$\eta_{\pm} = 1 + \frac{Q_{w,f}}{Q_f} - \frac{Q_w}{Q_f} \pm 2\sqrt{\frac{Q_{w,f}}{Q_f} \left(1 - \frac{Q_w}{Q_f}\right)}.$$

Our analysis show that parameter values satisfying $\Phi(\eta) > 0$ determine the existence of real coexistence equilibria $E_{\pm}$, subject to the biological constraints $0 < \eta < 1$. Therefore, for the discriminant $\Delta(\eta)$ to be strictly positive, it suffices that

$$\eta \in (0, 1) \cap \left((-\infty, \eta_-) \cup (\eta_+, \infty) \right).$$

Similarly as before, coexistence equilibria exist whenever the cytoplasmic incompatibility parameter η belongs to the admissible region.

□

Proposition A.1.7. The negative branch of coexistence equilibrium, E_- , is locally asymptotically stable, whereas the positive branch, E_+ is unstable.

Proof A.1.7. To analyse the local stability of the coexistence equilibria $E^{\pm}$, we examine the Jacobian matrix of the mosquito subsystem. Collecting the expressions derived previously, the Jacobian evaluated at $E^{\pm}$ is

$$J(E^{\pm}) = \begin{bmatrix} J_{11}^{(\pm)} & J_{12}^{(\pm)} & J_{13}^{(\pm)} & J_{14}^{(\pm)} \\ \bar{\beta}_f I_h & -\mu_f & 0 & 0 \\ -\sigma(\bar{\beta}_w I_h + \mu_w) S_w^{\pm} & -\sigma(\bar{\beta}_w I_h + \mu_w) S_w^{\pm} & -\sigma(\bar{\beta}_w I_h + \mu_w) S_w^{\pm} - \bar{\beta}_w I_h & \mu_w - \sigma(\bar{\beta}_w I_h + \mu_w) S_w^{\pm} \\ 0 & 0 & \bar{\beta}_w I_h & -\mu_w \end{bmatrix}.$$

with

$$J_{11}^{(\pm)} = \frac{1}{Q_w} \frac{\partial \Lambda_f}{\partial S_f} \Big|_{E^{\pm}} - \sigma(\bar{\beta}_f I_h + \mu_f) S_f^{\pm} - \bar{\beta}_f I_h - \mu_f,$$

$$J_{12}^{(\pm)} = \frac{1}{Q_w} \frac{\partial \Lambda_f}{\partial I_f} \Big|_{E^\pm} - \sigma(\bar{\beta}_f I_h + \mu_f) S_f^\pm,$$

$$J_{13}^{(\pm)} = J_{14}^{(\pm)} = \frac{1}{Q_w} \left[\rho_w(1 - \delta) - \rho_f \eta \frac{(S_f^\pm + I_f^\pm)^2}{M^2} \right] - \sigma(\bar{\beta}_f I_h + \mu_f) S_f^\pm,$$

and

$$M = \frac{1}{\sigma} \ln(Q_w), \quad I_f^\pm = \frac{\bar{\beta}_f I_h}{\mu_f} S_f^\pm, \quad I_w^\pm = \frac{\bar{\beta}_w I_h}{\mu_w} S_w^\pm.$$

To determine whether $E^\pm$ are locally asymptotically stable, we apply the Routh–Hurwitz criterion to the characteristic polynomial, instead of computing eigenvalues directly.

$$p(\lambda) = \det(\lambda I - J) = \lambda^4 + a_1 \lambda^3 + a_2 \lambda^2 + a_3 \lambda + a_4.$$

For a quartic polynomial of this form, the Routh–Hurwitz conditions are

$$a_1 > 0, \quad a_2 > 0, \quad a_3 > 0, \quad a_4 > 0, \quad a_1 a_2 a_3 > a_3^2 + a_1^2 a_4.$$

If these inequalities are satisfied, then all eigenvalues of J have strictly negative real parts, and the corresponding equilibrium is locally asymptotically stable (Gantmacher, 1959).

Coefficient $a_1^{(\pm)}$.

The coefficient $a_1^{(\pm)}$ is the negative of the trace of $J(E_\pm)$.

$$a_1^{(\pm)} = -\operatorname{tr} J(E^\pm).$$

Coefficient $a_2^{(\pm)}$.

The coefficient $a_2^{(\pm)}$ is the sum of the principal 2×2 minors of $J(E^\pm)$. Using the explicit form of $J(E^\pm)$, the six principal minors are:

1. minor (1,2)

$$\det J(E^\pm)_{[1,2]} = \det \begin{pmatrix} J_{11}^{(\pm)} & J_{12}^{(\pm)} \\ \bar{\beta}_f I_h & -\mu_f \end{pmatrix} = -\mu_f J_{11}^{(\pm)} - \bar{\beta}_f I_h J_{12}^{(\pm)}.$$

2. minor(1,3)

$$\det J(E^\pm)_{[1,3]} = \det \begin{pmatrix} J_{11}^{(\pm)} & J_{13}^{(\pm)} \\ -\sigma(\bar{\beta}_w I_h + \mu_w) S_w^\pm & -\sigma(\bar{\beta}_w I_h + \mu_w) S_w^\pm - \bar{\beta}_w I_h \end{pmatrix}$$

$$= -\sigma(\bar{\beta}_w I_h + \mu_w) S_w^\pm J_{11}^{(\pm)} - \bar{\beta}_w I_h J_{13}^{(\pm)} + \sigma(\bar{\beta}_w I_h + \mu_w) S_w^\pm J_{13}^{(\pm)}.$$

3. minor(1,4)

$$\det J(E^\pm)_{[1,4]} = \det \begin{pmatrix} J_{11}^{(\pm)} & J_{14}^{(\pm)} \\ 0 & -\mu_w \end{pmatrix} = -\mu_w J_{11}^{(\pm)}.$$

4. minor(2,3)

$$\begin{aligned}\det J(E^\pm)_{[2,3]} &= \det \begin{pmatrix} -\mu_f & 0 \\ -\sigma(\bar{\beta}_w I_h + \mu_w) S_w^\pm & -\sigma(\bar{\beta}_w I_h + \mu_w) S_w^\pm - \bar{\beta}_w I_h \end{pmatrix} \\ &= \mu_f \left[\sigma(\bar{\beta}_w I_h + \mu_w) S_w^\pm + \bar{\beta}_w I_h \right].\end{aligned}$$

5. minor(2,4)

$$\det J(E^\pm)_{[2,4]} = \det \begin{pmatrix} -\mu_f & 0 \\ 0 & -\mu_w \end{pmatrix} = \mu_f \mu_w.$$

6. minor(3,4)

$$\begin{aligned}\det J(E^\pm)_{[3,4]} &= \det \begin{pmatrix} -\sigma(\bar{\beta}_w I_h + \mu_w) S_w^\pm - \bar{\beta}_w I_h & \mu_w - \sigma(\bar{\beta}_w I_h + \mu_w) S_w^\pm \\ \bar{\beta}_w I_h & -\mu_w \end{pmatrix} \\ &= \sigma(\bar{\beta}_w I_h + \mu_w) S_w^\pm (\mu_w + \bar{\beta}_w I_h).\end{aligned}$$

Thus,

$$a_2^{(\pm)} = \sum_{1 \leq i < j \leq 4} \det J(E^\pm)_{[i,j]},$$

Coefficient $a_3^{(\pm)}$.

The coefficient $a_3^{(\pm)}$ is minus the sum of the principal 3×3 minors of $J(E^\pm)$, i.e.

$$a_3^{(\pm)} = - \sum_{1 \leq i < j < k \leq 4} \det J(E^\pm)_{[i,j,k],[i,j,k]}.$$

There are four such minors, corresponding to the index sets $(1, 2, 3)$, $(1, 2, 4)$, $(1, 3, 4)$, and $(2, 3, 4)$.

1. minor (1,2,3)

$$\begin{aligned}J(E^\pm)_{[1,2,3]} &= \begin{pmatrix} J_{11}^{(\pm)} & J_{12}^{(\pm)} & J_{13}^{(\pm)} \\ \bar{\beta}_f I_h & -\mu_f & 0 \\ -\sigma(\bar{\beta}_w I_h + \mu_w) S_w^\pm & -\sigma(\bar{\beta}_w I_h + \mu_w) S_w^\pm & -\sigma(\bar{\beta}_w I_h + \mu_w) S_w^\pm - \bar{\beta}_w I_h \end{pmatrix} \Rightarrow \\ \det J(E^\pm)_{[1,2,3]} &= \sigma(\bar{\beta}_w I_h + \mu_w) S_w^\pm \left[\mu_f (J_{11}^{(\pm)} - J_{13}^{(\pm)}) + \bar{\beta}_f I_h (J_{12}^{(\pm)} - J_{13}^{(\pm)}) \right] + \bar{\beta}_w I_h \left[\mu_f J_{11}^{(\pm)} + \bar{\beta}_f I_h J_{12}^{(\pm)} \right].\end{aligned}$$

2. minor (1,2,4)

$$\begin{aligned}J(E^\pm)_{[1,2,4]} &= \begin{pmatrix} J_{11}^{(\pm)} & J_{12}^{(\pm)} & J_{14}^{(\pm)} \\ \bar{\beta}_f I_h & -\mu_f & 0 \\ 0 & 0 & -\mu_w \end{pmatrix} \\ \Rightarrow \det J(E^\pm)_{[1,2,4]} &= \mu_w \left(\mu_f J_{11}^{(\pm)} + \bar{\beta}_f I_h J_{12}^{(\pm)} \right).\end{aligned}$$

3. minor (1,3,4)

$$J(E^\pm)_{[1,3,4]} = \begin{pmatrix} J_{11}^{(\pm)} & J_{13}^{(\pm)} & J_{14}^{(\pm)} \\ -\sigma(\bar{\beta}_w I_h + \mu_w) S_w^\pm & -\sigma(\bar{\beta}_w I_h + \mu_w) S_w^\pm - \bar{\beta}_w I_h & \mu_w - \sigma(\bar{\beta}_w I_h + \mu_w) S_w^\pm \\ 0 & \bar{\beta}_w I_h & -\mu_w \end{pmatrix}$$

$$\implies \det J(E^\pm)_{[1,3,4]} = \sigma(\bar{\beta}_w I_h + \mu_w) S_w^\pm \left[\mu_w (J_{11}^{(\pm)} - J_{13}^{(\pm)}) + \bar{\beta}_w I_h (J_{11}^{(\pm)} - J_{14}^{(\pm)}) \right].$$

4. minor (2,3,4)

$$J(E^\pm)_{[2,3,4]} = \begin{pmatrix} -\mu_f & 0 & 0 \\ -\sigma(\bar{\beta}_w I_h + \mu_w) S_w^\pm & -\sigma(\bar{\beta}_w I_h + \mu_w) S_w^\pm - \bar{\beta}_w I_h & \mu_w - \sigma(\bar{\beta}_w I_h + \mu_w) S_w^\pm \\ 0 & \bar{\beta}_w I_h & -\mu_w \end{pmatrix}$$

$$\implies \det J(E^\pm)_{[2,3,4]} = -\mu_f \sigma S_w^\pm (\bar{\beta}_w I_h + \mu_w)^2.$$

The coefficient $a_3^{(\pm)}$ is therefore given by

$$a_3^{(\pm)} = -\left(\det J(E^\pm)_{[1,2,3]} + \det J(E^\pm)_{[1,2,4]} + \det J(E^\pm)_{[1,3,4]} + \det J(E^\pm)_{[2,3,4]} \right),$$

Coefficient $a_4^{(\pm)}$.

This coefficient is the determinant of the Jacobian,

$$a_4^{(\pm)} = \det J(E^\pm).$$

Furthermore, we define $C := a_1 a_2 a_3 - a_3^2 - a_1^2 a_4$ such that $C > 0 \iff a_1 a_2 a_3 > a_3^2 + a_1^2 a_4$, representing the second condition regarding the coefficients.

Because the resulting algebraic expression is too heavy when the explicit formulas for $J_{1j}^{(\pm)}$, $S_w^\pm$, $I_f^\pm$ and $I_w^\pm$ are inserted, all numerical values of the coefficients $a_i^{(\pm)}$ are computed computationally for biologically relevant parameter sets.

In particular, when considering the parameter set representing biological features of wMel *Wolbachia strain*, we calculated

$$a_i^{(\pm)} \mid E^\pm = (a_1, a_2, a_3, a_4, C) \mid E^\pm :$$

$$a_i^{(-)} = (0.2771, 0.0008, 1.046e^{-5}, 4.0638e^{-6}, C^{(-)} = 4.0638e^{-6}),$$

$$a_i^{(+)} = (0.2195, 0.0124, 0.0001, -1.1504e^{-6}, C^{(+)} = 5.1188e^{-7}).$$

Therefore, $a_1 > 0, a_2 > 0, a_3 > 0, a_4 > 0$, evaluated at E_- , alongside with $C > 0$, proving that E_- is locally asymptotically stable. On the other side, when evaluated at E_+ , $a_4 < 0$. From its definition, $a_4^{(+)} = \det J(E_+) < 0$ meaning that E_+ is unstable.

The same analysis was performed considering different *Wolbachia* variants presented on Table 3, namely *wMelPop*, *wAlbB* and *wAu* strains, and the same pattern, with Jacobian presenting negative determinant ($a_4 < 0$) was observed.

□

A.2 Human General Subsystem

Once the mosquito subsystem has been analyzed and its equilibrium points determined, these solutions can be substituted into the human equations. By doing so, the mosquito dynamics enter as fixed contributions, and the resulting reduced system describes the long-term behavior of the human population. This procedure allows us to characterize the equilibria of the complete system by combining the steady states of mosquitoes with the corresponding human dynamics.

The equations are

$$\frac{dS_h}{dt} = \mu_h N_h - \hat{\beta}(I_f + I_w)S_h - (\mu_h + \nu_h)S_h \quad (1)$$

$$\frac{dI_h}{dt} = \hat{\beta}(I_f + I_w)S_h - (\gamma_h + \mu_h)I_h \quad (2)$$

$$\frac{dR_h}{dt} = \nu_h S_h + \gamma_h I_h - \mu_h R_h \quad (3)$$

where $\beta_f = \beta_w = \beta$ and $\hat{\beta} = \frac{\beta}{N_h}$.

The human population is modeled so that it remains constant — birth replaces mortality, i.e.:

$$S_h + I_h + R_h = N_h, \quad N_h \in \mathbb{R} \implies \frac{dS_h}{dt} + \frac{dI_h}{dt} + \frac{dR_h}{dt} = 0.$$

$$\begin{aligned} & \mu_h N_h - \hat{\beta}(I_f + I_w)S_h - (\mu_h + \nu_h)S_h + \hat{\beta}(I_f + I_w)S_h - (\gamma_h + \mu_h)I_h + \nu_h S_h + \gamma_h I_h - \mu_h R_h \\ &= \mu_h N_h - \mu_h S_h - \mu_h I_h - \mu_h R_h = \mu_h(S_h + I_h + R_h) - \mu_h(S_h + I_h + R_h) = 0. \end{aligned}$$

Thus, we can consider only two equations (e.g., (1) and (2)), since we can obtain R_h as $R_h = N_h - (S_h + I_h)$.

Proposition A.2.1. The trivial equilibrium point for the human population is given by

$$E_0^h = \left(\frac{\mu_h N_h}{\mu_h + \nu_h}, 0 \right).$$

Proof A.2.1. $E_0^m = (0, 0, 0, 0)$, for the mosquitoes, such that equations are written as

$$\frac{dS_h}{dt} = \mu_h N_h - (\mu_h + \nu_h) S_h = 0 \quad (1A)$$

$$\implies \bar{S}_h = \frac{\mu_h N_h}{\mu_h + \nu_h}$$

$$\begin{aligned} \frac{dI_h}{dt} &= -(\gamma_h + \mu_h) I_h = 0 \\ \implies \bar{I}_h &= 0 \end{aligned} \tag{2A}$$

Indeed,

$$\begin{aligned} \frac{dI_h}{dt} &= -(\gamma_h + \mu_h) I_h \iff \frac{1}{I_h} dI_h = -(\gamma_h + \mu_h) dt \iff \int \frac{1}{I_h} dI_h = \int -(\gamma_h + \mu_h) dt \\ &\iff \ln|I_h| = -(\gamma_h + \mu_h) t + C, \quad C \in \mathbb{R} \iff I_h = e^{-(\gamma_h + \mu_h)t} e^C \end{aligned}$$

Let $e^C = I_0$, then

$$I_h(t) = I_0 e^{-(\gamma_h + \mu_h)t} : \text{exponential decay}$$

$\therefore \bar{I}_h = 0$ is indeed a fixed point.

Therefore, the trivial equilibrium point for the human population is given by

$$E_0^h = \left(\frac{\mu_h N_h}{\mu_h + \nu_h}, 0 \right).$$

□

Proposition A.2.2. The trivial equilibrium point for the human population E_0^h is locally asymptotically stable, everytime that it exists, which means when condition related to mosquitoes population holds true ($Q_f < 1$).

Proof A.2.2. To analyze its stability, we define the functions:

$$\begin{aligned} F_1(S_h, I_h, R_h) &= \mu_h N_h - (\mu_h + \nu_h) S_h \\ F_2(S_h, I_h, R_h) &= -(\gamma_h + \mu_h) I_h \end{aligned}$$

The Jacobian matrix is given by:

$$J = \begin{bmatrix} \frac{\partial F_1}{\partial S_h} & \frac{\partial F_1}{\partial I_h} \\ \frac{\partial F_2}{\partial S_h} & \frac{\partial F_2}{\partial I_h} \end{bmatrix} = \begin{bmatrix} -(\mu_h + \nu_h) & 0 \\ 0 & -(\gamma_h + \mu_h) \end{bmatrix}$$

Since the matrix is lower triangular, its eigenvalues are on the main diagonal.

$$\lambda_1 = -(\mu_h + \nu_h), \quad \lambda_2 = -(\gamma_h + \mu_h), \quad \lambda_3 = -\mu_h$$

As all eigenvalues are real and negative, we conclude that the equilibrium point E_0^h is locally asymptotically stable.

□

Proposition A.2.3. The equilibrium point for the human population in a scenario representing the extinction of *Wolbachia* infected mosquitoes is given by:

$$E_f^h = \left(\frac{\sigma[\mu_f(\gamma_h + \mu_h) + \bar{\beta}_f \mu_h N_h]}{\bar{\beta}_f [\hat{\beta} \ln(Q_f) + \sigma(\mu_h + \nu_h)]}, \bar{S}_h \left[\frac{\hat{\beta} \ln(Q_f)}{\sigma(\gamma_h + \mu_h)} \right] - \frac{\mu_f}{\bar{\beta}_f} \right).$$

Proof A.2.3. Our analysis with respect to the mosquito subsystem shows that the corresponding equilibrium point, representing the *Wolbachia* extinction regime, is given by

$$E_f = \left(\frac{\mu_f}{\sigma} \frac{\ln(Q_f)}{\mu_f + \bar{\beta}_f I_h}, \frac{\bar{\beta}_f I_h}{\sigma} \frac{\ln(Q_f)}{\mu_f + \bar{\beta}_f I_h}, 0, 0 \right).$$

We substitute the steady-state values on the human population differential equations, obtaining the two-dimensional system:

$$\frac{dS_h}{dt} = \mu_h N_h - \hat{\beta} \left(\frac{\bar{\beta}_f I_h}{\sigma} \frac{\ln(Q_f)}{\mu_f + \bar{\beta}_f I_h} \right) S_h - (\mu_h + \nu_h) S_h \quad (1B)$$

$$\frac{dI_h}{dt} = \hat{\beta} \left(\frac{\bar{\beta}_f I_h}{\sigma} \frac{\ln(Q_f)}{\mu_f + \bar{\beta}_f I_h} \right) S_h - (\gamma_h + \mu_h) I_h \quad (2B)$$

from Equation (2B):

$$\begin{aligned} 0 &= \bar{I}_h \left(\frac{\bar{\beta}_f \hat{\beta}}{\sigma} \frac{\ln(Q_f)}{\mu_f + \bar{\beta}_f \bar{I}_h} \right) \bar{S}_h - (\gamma_h + \mu_h) \bar{I}_h = \bar{I}_h \left[\frac{\bar{\beta}_f \hat{\beta}}{\sigma} \frac{\ln(Q_f)}{\mu_f + \bar{\beta}_f \bar{I}_h} \bar{S}_h - (\gamma_h + \mu_h) \right] \\ &\implies \begin{cases} \bar{I}_h = 0 & \text{(i)} \\ \frac{\bar{\beta}_f \hat{\beta}}{\sigma} \frac{\ln(Q_f)}{\mu_f + \bar{\beta}_f \bar{I}_h} \bar{S}_h - (\gamma_h + \mu_h) = 0 & \text{(ii)} \end{cases} \end{aligned}$$

(i) $\bar{I}_h = 0$ into Equation (1B):

$$0 = \mu_h N_h - (\mu_h + \nu_h) S_h \implies \bar{S}_h = \frac{\mu_h N_h}{\mu_h + \nu_h},$$

which take us back to trivial equilibria $E_0^h = \left(\frac{\mu_h N_h}{\mu_h + \nu_h}, 0 \right)$.

On the other side, manipulating the expression **(ii)**, we have:

$$\frac{\bar{S}_h}{\mu_f + \bar{\beta}_f \bar{I}_h} = \frac{\sigma(\gamma_h + \mu_h)}{\bar{\beta}_f \hat{\beta} \ln(Q_f)} \iff \frac{\mu_f + \bar{\beta}_f \bar{I}_h}{\bar{S}_h} = \frac{\bar{\beta}_f \hat{\beta} \ln(Q_f)}{\sigma(\gamma_h + \mu_h)} \iff \mu_f + \bar{\beta}_f \bar{I}_h = \bar{S}_h \frac{\bar{\beta}_f \hat{\beta} \ln(Q_f)}{\sigma(\gamma_h + \mu_h)} \quad (*)$$

$$\iff \bar{I}_h = \bar{S}_h \frac{\hat{\beta} \ln(Q_f)}{\sigma(\gamma_h + \mu_h)} - \frac{\mu_f}{\bar{\beta}_f} \quad (**)$$

(ii) into Equation (1B):

$$0 = \mu_h N_h - \left[\frac{\bar{\beta}_f \hat{\beta}}{\sigma} \left(\bar{S}_h \frac{\hat{\beta} \ln(Q_f)}{\sigma(\gamma_h + \mu_h)} - \frac{\mu_f}{\bar{\beta}_f} \right) \frac{\ln(Q_f)}{\left(\frac{\bar{S}_h \bar{\beta}_f \hat{\beta} \ln(Q_f)}{\sigma(\gamma_h + \mu_h)} \right)} \right] \bar{S}_h - (\mu_h + \nu_h) \bar{S}_h$$

The expression in square parenthesis can be rewritten as:

$$\begin{aligned} \left[\frac{\bar{\beta}_f \hat{\beta}}{\sigma} \left(\bar{S}_h \frac{\hat{\beta} \ln(Q_f)}{\sigma(\gamma_h + \mu_h)} - \frac{\mu_f}{\bar{\beta}_f} \right) \frac{\ln(Q_f)}{\left(\frac{\bar{S}_h \bar{\beta}_f \hat{\beta} \ln(Q_f)}{\sigma(\gamma_h + \mu_h)} \right)} \right] &= \left[\frac{\bar{\beta}_f \hat{\beta}^2 \ln(Q_f)}{\sigma^2(\gamma_h + \mu_h)} \bar{S}_h - \frac{\mu_f \bar{\beta}_f}{\sigma} \right] \left[\frac{\sigma(\gamma_h + \mu_h)}{\bar{S}_h \bar{\beta}_f \hat{\beta}} \right] \\ &= \frac{\sigma(\gamma_h + \mu_h)}{\bar{S}_h \bar{\beta}_f \hat{\beta}} \frac{\bar{\beta}_f \hat{\beta}^2 \ln(Q_f)}{\sigma^2(\gamma_h + \mu_h)} \bar{S}_h - \frac{\sigma(\gamma_h + \mu_h)}{\bar{S}_h \bar{\beta}_f \hat{\beta}} \frac{\mu_f \bar{\beta}_f}{\sigma} = \frac{\hat{\beta} \ln(Q_f)}{\sigma} - \frac{\mu_f(\gamma_h + \mu_h)}{\bar{S}_h \bar{\beta}_f} \end{aligned}$$

Thus,

$$\begin{aligned} 0 &= \mu_h N_h - \left[\frac{\hat{\beta} \ln(Q_f)}{\sigma} - \frac{\mu_f(\gamma_h + \mu_h)}{\bar{S}_h \bar{\beta}_f} \right] \bar{S}_h - (\mu_h + \nu_h) \bar{S}_h \\ &= \mu_h N_h - \frac{\hat{\beta} \ln(Q_f)}{\sigma} \bar{S}_h + \frac{\mu_f(\gamma_h + \mu_h)}{\bar{\beta}_f} - (\mu_h + \nu_h) \bar{S}_h \\ &= -\bar{S}_h \left[\frac{\hat{\beta} \ln(Q_f)}{\sigma} + (\mu_h + \nu_h) \right] + \frac{\mu_f(\gamma_h + \mu_h)}{\bar{\beta}_f} + \mu_h N_h \\ &\iff \bar{S}_h \left[\frac{\hat{\beta} \ln(Q_f) + \sigma(\mu_h + \nu_h)}{\sigma} \right] = \frac{\mu_f(\gamma_h + \mu_h) + \bar{\beta}_f \mu_h N_h}{\bar{\beta}_f} \\ &\iff \bar{S}_h = \frac{\sigma[\mu_f(\gamma_h + \mu_h) + \bar{\beta}_f \mu_h N_h]}{\bar{\beta}_f [\hat{\beta} \ln(Q_f) + \sigma(\mu_h + \nu_h)]} \end{aligned}$$

Such that

$$\bar{I}_h = \bar{S}_h \left[\frac{\hat{\beta} \ln(Q_f)}{\sigma(\gamma_h + \mu_h)} \right] - \frac{\mu_f}{\bar{\beta}_f} = \left\{ \frac{\sigma[\mu_f(\gamma_h + \mu_h) + \bar{\beta}_f \mu_h N_h]}{\bar{\beta}_f [\hat{\beta} \ln(Q_f) + \sigma(\mu_h + \nu_h)]} \left[\frac{\hat{\beta} \ln(Q_f)}{\sigma(\gamma_h + \mu_h)} \right] \right\} - \frac{\mu_f}{\bar{\beta}_f}$$

And finally,

$$N_h = \bar{S}_h + \bar{I}_h + \bar{R}_h \iff \bar{R}_h = N_h - (\bar{S}_h + \bar{I}_h)$$

$$\iff \bar{R}_h = N_h - \left\{ \frac{\sigma[\mu_f(\gamma_h + \mu_h) + \bar{\beta}_f \mu_h N_h]}{\bar{\beta}_f [\hat{\beta} \ln(Q_f) + \sigma(\mu_h + \nu_h)]} + \left[\frac{\sigma[\mu_f(\gamma_h + \mu_h) + \bar{\beta}_f \mu_h N_h]}{\bar{\beta}_f [\hat{\beta} \ln(Q_f) + \sigma(\mu_h + \nu_h)]} \left[\frac{\hat{\beta} \ln(Q_f)}{\sigma(\gamma_h + \mu_h)} \right] \right] - \frac{\mu_f}{\bar{\beta}_f} \right\}.$$

Therefore, we calculated the equilibrium point for the human population in a scenario representing the extinction of *Wolbachia* infected mosquitoes as:

$$E_f^h = \left(\frac{\sigma[\mu_f(\gamma_h + \mu_h) + \bar{\beta}_f \mu_h N_h]}{\bar{\beta}_f [\hat{\beta} \ln(Q_f) + \sigma(\mu_h + \nu_h)]}, \bar{S}_h \left[\frac{\hat{\beta} \ln(Q_f)}{\sigma(\gamma_h + \mu_h)} \right] - \frac{\mu_f}{\bar{\beta}_f} \right).$$

□

Proposition A.2.4. The human equilibrium point representing the *Wolbachia* extinction regime, E_f^h , exists if, and only if, $Q_f > \psi_f := \exp \left\{ \frac{\mu_f \sigma(\mu_h + \nu_h)(\gamma_h + \mu_h)}{\hat{\beta} \bar{\beta}_f \mu_h N_h} \right\}$, due to biological viability condition.

Proof A.2.4.

$$\begin{aligned} I_h^* > 0 &\implies \\ S_h^* \left[\frac{\hat{\beta} \ln(Q_f)}{\sigma(\gamma_h + \mu_h)} \right] - \frac{\mu_f}{\bar{\beta}_f} &= \left\{ \frac{\sigma[\mu_f(\gamma_h + \mu_h) + \bar{\beta}_f \mu_h N_h]}{\bar{\beta}_f [\hat{\beta} \ln(Q_f) + \sigma(\gamma_h + \mu_h)]} \right\} \left[\frac{\hat{\beta} \ln(Q_f)}{\sigma(\gamma_h + \mu_h)} \right] - \frac{\mu_f}{\bar{\beta}_f} > 0 \\ &\implies \left\{ \frac{\sigma[\mu_f(\gamma_h + \mu_h) + \bar{\beta}_f \mu_h N_h]}{\bar{\beta}_f [\hat{\beta} \ln(Q_f) + \sigma(\gamma_h + \mu_h)]} \right\} \left[\frac{\hat{\beta} \ln(Q_f)}{\sigma(\gamma_h + \mu_h)} \right] > \frac{\mu_f}{\bar{\beta}_f} \end{aligned}$$

$$\hat{\beta} \ln(Q_f) [\mu_f(\gamma_h + \mu_h) + \bar{\beta}_f \mu_h N_h] > \mu_f(\gamma_h + \mu_h) [\hat{\beta} \ln(Q_f) + \sigma(\mu_h + \nu_h)]$$

$$\hat{\beta} \ln(Q_f) \mu_f(\gamma_h + \mu_h) + \hat{\beta} \ln(Q_f) \bar{\beta}_f \mu_h N_h > [\mu_f \hat{\beta} \ln(Q_f) + \mu_f \sigma(\mu_h + \nu_h)] (\gamma_h + \mu_h)$$

$$\hat{\beta} \ln(Q_f) \mu_f(\gamma_h + \mu_h) + \hat{\beta} \ln(Q_f) \bar{\beta}_f \mu_h N_h - \mu_f \hat{\beta} \ln(Q_f) > \mu_f \sigma(\mu_h + \nu_h)(\gamma_h + \mu_h)$$

$$\hat{\beta} \ln(Q_f) [\mu_f(\gamma_h + \mu_h) + \bar{\beta}_f \mu_h N_h - \mu_f(\gamma_h + \mu_h)] > \mu_f \sigma(\mu_h + \nu_h)(\gamma_h + \mu_h)$$

$$\hat{\beta} \ln(Q_f) (\bar{\beta}_f \mu_h N_h) > \mu_f \sigma(\mu_h + \nu_h)(\gamma_h + \mu_h)$$

$$\hat{\beta} \ln(Q_f) (\bar{\beta}_f \mu_h N_h) > \mu_f \sigma(\mu_h + \nu_h)(\gamma_h + \mu_h)$$

$$\hat{\beta} \ln(Q_f) (\bar{\beta}_f \mu_h N_h) > \mu_f \sigma(\mu_h + \nu_h)(\gamma_h + \mu_h)$$

$$\ln(Q_f) > \frac{\mu_f \sigma(\mu_h + \nu_h)(\gamma_h + \mu_h)}{\hat{\beta} \bar{\beta}_f \mu_h N_h}$$

$$Q_f > \exp \left\{ \frac{\mu_f \sigma(\mu_h + \nu_h)(\gamma_h + \mu_h)}{\hat{\beta} \bar{\beta}_f \mu_h N_h} \right\} := \psi_f$$

$Q_f > \psi_f$: biological viability.

□

Proposition A.2.5. The equilibrium point for the human population, considering *Wolbachia*-infected mosquitoes extinction is locally asymptotically stable whenever $Q_f > \psi_f$, or conversely, $R^f > 1$, for the defined threshold quantities, given by

$$\psi_f = \exp \left\{ \frac{\mu_f \sigma(\mu_h + \nu_h)(\gamma_h + \mu_h)}{\hat{\beta} \bar{\beta}_f \mu_h N_h} \right\} \quad R^f = \frac{\hat{\beta} \bar{\beta}_f \ln(Q_f) \mu_h N_h}{\mu_f \sigma(\gamma_h + \mu_h)(\mu_h + \nu_h)}.$$

Proof A.2.5. We calculated the equilibrium point E_f^h as:

$$E_f^h = \left(\frac{\sigma[\mu_f(\gamma_h + \mu_h) + \bar{\beta}_f \mu_h N_h]}{\bar{\beta}_f [\hat{\beta} \ln(Q_f) + \sigma(\mu_h + \nu_h)]}, \bar{S}_h \left[\frac{\hat{\beta} \ln(Q_f)}{\sigma(\gamma_h + \mu_h)} \right] - \frac{\mu_f}{\bar{\beta}_f} \right).$$

We shall define the quantities:

$$g(I_h) = \frac{I_h}{\mu_f + \bar{\beta}_f I_h}, \quad g'(I_h) = \frac{\mu_f}{(\mu_f + \bar{\beta}_f I_h)^2}, \quad \alpha = \frac{\hat{\beta} \bar{\beta}_f \ln(Q_f)}{\sigma}.$$

$$\begin{aligned} \frac{\partial F_1}{\partial S_h} &= -\alpha g(I_h) - (\mu_h + \nu_h), & \frac{\partial F_1}{\partial I_h} &= -\alpha g'(I_h) S_h, \\ \frac{\partial F_2}{\partial S_h} &= \alpha g(I_h), & \frac{\partial F_2}{\partial I_h} &= \alpha g'(I_h) S_h - (\gamma_h + \mu_h). \end{aligned}$$

$$J(S_h, I_h) = \begin{bmatrix} -\alpha g(I_h) - (\mu_h + \nu_h) & -\alpha g'(I_h) S_h \\ \alpha g(I_h) & \alpha g'(I_h) S_h - (\gamma_h + \mu_h) \end{bmatrix}.$$

Using equilibrium conditions, we might rewrite the matrix as:

$$J(S_h^*, I_h^*) = \begin{bmatrix} -\frac{(\gamma_h + \mu_h) I_h^*}{S_h^*} - (\mu_h + \nu_h) & -\frac{\mu_f (\gamma_h + \mu_h)^2}{\alpha S_h^*} \\ \frac{(\gamma_h + \mu_h) I_h^*}{S_h^*} & \frac{\mu_f (\gamma_h + \mu_h)^2}{\alpha S_h^*} - (\gamma_h + \mu_h) \end{bmatrix} \Rightarrow$$

$$tr(J) = \frac{\mu_f (\gamma_h + \mu_h)^2}{\alpha S_h^*} - \frac{\mu_h N_h}{S_h^*} - (\gamma_h + \mu_h),$$

$$det(J) = \frac{\gamma_h + \mu_h}{S_h^*} \left(\mu_h N_h - \frac{(\mu_h + \nu_h) \mu_f (\gamma_h + \mu_h)}{\alpha} \right).$$

Hence, equilibrium point is LAS if $\text{tr}(J) < 0$ and $\det(J) > 0$.

The local stability of the equilibrium point E_f^h will be established through the analysis of its Jacobian matrix. Accordingly, we shall divide **Proposition A.2.5** into two auxiliary propositions: the first one aims to demonstrate that $\det(J) > 0$ (**Proposition A.2.5.1**), and the second to show that $\text{tr}(J) < 0$ (**Proposition A.2.5.2**).

□

Proposition A.2.5.1. The determinant is strictly positive ($\det(J) > 0$) if, and only if, $Q_f > \psi_f$, or conversely, $R^f > 1$ (**Necessary Condition**).

Proof [A.2.5.1](#).

$$\det(J) > 0 \implies$$

$$\frac{\gamma_h + \mu_h}{S_h^*} \left(\mu_h N_h - \frac{(\mu_h + \nu_h) \mu_f (\gamma_h + \mu_h)}{\alpha} \right) > 0$$

$$\iff \mu_h N_h > \frac{(\mu_h + \nu_h) \mu_f (\gamma_h + \mu_h)}{\alpha}$$

$$\alpha > \frac{\mu_f (\gamma_h + \mu_h) (\mu_h + \nu_h)}{\mu_h N_h} := \alpha^*$$

$$\frac{\hat{\beta} \bar{\beta}_f \ln(Q_f)}{\sigma} > \frac{\mu_f (\mu_h + \nu_h) (\gamma_h + \mu_h)}{\mu_h N_h} \quad (R)$$

$$\ln(Q_f) > \frac{\mu_f \sigma (\mu_h + \nu_h) (\gamma_h + \mu_h)}{\hat{\beta} \bar{\beta}_f \mu_h N_h}$$

$$Q_f > \exp \left\{ \frac{\mu_f \sigma (\mu_h + \nu_h) (\gamma_h + \mu_h)}{\hat{\beta} \bar{\beta}_f \mu_h N_h} \right\} = \psi_f.$$

And also, condition (R) holds that

$$\frac{\hat{\beta} \bar{\beta}_f \ln(Q_f) \mu_h N_h}{\mu_f \sigma (\gamma_h + \mu_h) (\mu_h + \nu_h)} > 1$$

from where we can determine

$$R^f := \frac{\hat{\beta} \bar{\beta}_f \ln(Q_f) \mu_h N_h}{\mu_f \sigma (\gamma_h + \mu_h) (\mu_h + \nu_h)}.$$

$$\therefore \det(J) > 0 \iff Q_f > \psi_f \iff R^f > 1.$$

□

Proposition A.2.5.2. Considering a strictly negative trace ($\text{tr}(J) < 0$), the conditions are the same as the conditions that satisfies $\det(J) > 0$, i.e., $Q_f > \psi_w \Leftrightarrow R^f > 1$ (**Sufficient Condition**).

Proof A.2.5.2. From Equation (2B):

$$0 = \hat{\beta} \left(\frac{\bar{\bar{\beta}}_f I_h}{\sigma} \frac{\ln(Q_f)}{\mu_f + \bar{\bar{\beta}}_f I_h} \right) S_h - (\gamma_h + \mu_h) I_h = \alpha g(I_h) S_h - (\gamma_h + \mu_h) I_h$$

$$\alpha g(I_h) S_h = (\gamma_h + \mu_h) I_h \quad (*)$$

$$\alpha \frac{I_h}{\mu_f + \bar{\bar{\beta}}_f I_h} S_h = (\gamma_h + \mu_h) I_h$$

$$\alpha S_h = (\gamma_h + \mu_h) (\mu_f + \bar{\bar{\beta}}_f I_h)$$

$$\mu_f + \bar{\bar{\beta}}_f I_h = \frac{\alpha S_h}{(\gamma_h + \mu_h)}$$

$$I_h = \frac{\alpha S_h}{\bar{\bar{\beta}}_f (\gamma_h + \mu_h)} - \frac{\mu_f}{\bar{\bar{\beta}}_f} \quad (**)$$

Substituting (*) and (**) into (I):

$$0 = \mu_h N_h - \hat{\beta} \left(\frac{\bar{\bar{\beta}}_f I_h}{\sigma} \frac{\ln(Q_f)}{\mu_f + \bar{\bar{\beta}}_f I_h} \right) S_h - (\mu_h + \nu_h) S_h = \mu_h N_h - \alpha g(I_h) S_h - (\mu_h + \nu_h) S_h$$

$$\mu_h N_h = (\gamma_h + \mu_h) I_h + (\mu_h + \nu_h) S_h = (\gamma_h + \mu_h) \left[\frac{\alpha S_h}{\bar{\bar{\beta}}_f (\gamma_h + \mu_h)} - \frac{\mu_f}{\bar{\bar{\beta}}_f} \right] + (\mu_h + \nu_h) S_h$$

$$\mu_h N_h = \frac{\alpha S_h - \mu_f (\gamma_h + \mu_h)}{\bar{\bar{\beta}}_f} + (\mu_h + \nu_h) S_h$$

$$\bar{\bar{\beta}}_f \mu_h N_h = \alpha S_h - \mu_f (\gamma_h + \mu_h) + \bar{\bar{\beta}}_f (\mu_h + \nu_h) S_h$$

$$\alpha S_h + \bar{\bar{\beta}}_f (\mu_h + \nu_h) S_h = \bar{\bar{\beta}}_f \mu_h N_h + \mu_f (\gamma_h + \mu_h)$$

$$S_h \left[\alpha + \bar{\bar{\beta}}_f(\mu_h + \nu_h) \right] = \bar{\bar{\beta}}_f \mu_h N_h + \mu_f(\gamma_h + \mu_h)$$

$$S_h = \frac{\bar{\bar{\beta}}_f \mu_h N_h + \mu_f(\gamma_h + \mu_h)}{\left[\alpha + \bar{\bar{\beta}}_f(\mu_h + \nu_h) \right]} \quad (***)$$

and

$$\alpha S_h = \frac{\alpha \left[\bar{\bar{\beta}}_f \mu_h N_h + \mu_f(\gamma_h + \mu_h) \right]}{\left[\alpha + \bar{\bar{\beta}}_f(\mu_h + \nu_h) \right]} \quad (***).$$

Now, back to the determinant $\det(J)$

$$\det(J) = \frac{\gamma_h + \mu_h}{S_h^*} \left[\mu_h N_h - \frac{(\mu_h + \nu_h) \mu_f(\gamma_h + \mu_h)}{\alpha} \right]$$

$\det(J)$ sign depends on square brackets terms

$$\mu_h N_h < \frac{\mu_f(\gamma_h + \mu_h)(\mu_h + \nu_h)}{\alpha} \iff \alpha < \frac{\mu_f(\gamma_h + \mu_h)(\mu_h + \nu_h)}{\mu_h N_h} \implies \det(J) < 0$$

$$\mu_h N_h = \frac{\mu_f(\gamma_h + \mu_h)(\mu_h + \nu_h)}{\alpha} \iff \alpha^* = \frac{\mu_f(\gamma_h + \mu_h)(\mu_h + \nu_h)}{\mu_h N_h} \implies \det(J) = 0 \quad (*)$$

$$\mu_h N_h > \frac{\mu_f(\gamma_h + \mu_h)(\mu_h + \nu_h)}{\alpha} \iff \alpha > \frac{\mu_f(\gamma_h + \mu_h)(\mu_h + \nu_h)}{\mu_h N_h} \implies \det(J) > 0$$

We might evaluate expressions $(***)$ and $(***)$ under $(*)$ condition.

$$\begin{aligned} (***) \quad S_h &= \frac{\bar{\bar{\beta}}_f \mu_h N_h + \mu_f(\gamma_h + \mu_h)}{\left[\alpha + \bar{\bar{\beta}}_f(\mu_h + \nu_h) \right]} = \frac{\bar{\bar{\beta}}_f \mu_h N_h + \mu_f(\gamma_h + \mu_h)}{\left[\frac{\mu_f(\gamma_h + \mu_h)(\mu_h + \nu_h)}{\mu_h N_h} + \bar{\bar{\beta}}_f(\mu_h + \nu_h) \right]} \\ &= \frac{\mu_h N_h \bar{\bar{\beta}}_f + \mu_f(\gamma_h + \mu_h)}{(\mu_h + \nu_h) \left[\frac{\mu_f(\gamma_h + \mu_h) + \mu_h N_h \bar{\bar{\beta}}_f}{\mu_h N_h} \right]} = \frac{\mu_h N_h \left[\mu_f(\gamma_h + \mu_h) + \mu_h N_h \bar{\bar{\beta}}_f \right]}{(\mu_h + \nu_h) \left[\mu_f(\gamma_h + \mu_h) + \mu_h N_h \bar{\bar{\beta}}_f \right]} \\ &= \frac{\mu_h N_h}{\mu_h + \nu_h} = S_h^* \end{aligned}$$

and

$$(***) \quad \alpha^* S_h^* = \left[\frac{\mu_f(\gamma_h + \mu_h)(\mu_h + \nu_h)}{\mu_h N_h} \right] \left(\frac{\mu_h N_h}{\mu_h + \nu_h} \right) = \mu_f(\gamma_h + \mu_h).$$

Now, back to the trace $tr(J)$, also evaluated under $\star$ condition

$$tr(J) = \frac{\mu_f(\gamma_h + \mu_h)^2}{\alpha S_h^*} - \frac{\mu_h N_h}{S_h^*} - (\gamma_h + \mu_h) \implies$$

$$tr^*(J) = \frac{\mu_f(\gamma_h + \mu_h)^2}{\alpha^* S_h^*} - \frac{\mu_h N_h}{S_h^*} - (\gamma_h + \mu_h) =$$

$$\frac{\mu_f(\gamma_h + \mu_h)^2}{\mu_f(\gamma_h + \mu_h)} - \frac{\mu_h N_h}{\left(\frac{\mu_h N_h}{\mu_h + \nu_h} \right)} - (\gamma_h + \mu_h) = (\gamma_h + \mu_h) - (\mu_h + \nu_h) - (\gamma_h + \mu_h)$$

$$= -(\mu_h + \nu_h) < 0.$$

Remembering that $\star$ condition is

$$\alpha^* = \frac{\mu_f(\gamma_h + \mu_h)(\mu_h + \nu_h)}{\mu_h N_h} \implies \det(J) = 0$$

Hence,

$$(I) \quad \alpha < \alpha^* \implies \det(J) < 0$$

for any $\alpha < \alpha^*$, E_f^h is a saddle point.

$$(II) \quad \alpha = \alpha^* \implies \det(J) = 0 \quad \text{and} \quad tr(J) < 0$$

for $\alpha = \alpha^*$, E_f^h is LAS.

$$(III) \quad \alpha > \alpha^* \implies \det(J) > 0$$

$$S_h = \frac{\bar{\beta}_f \mu_h N_h + \mu_f(\gamma_h + \mu_h)}{[\alpha + \bar{\beta}_f(\mu_h + \nu_h)]} \text{ is decreasing with } \alpha$$

and

$$\alpha S_h = \frac{\alpha \left[\bar{\beta}_f \mu_h N_h + \mu_f (\gamma_h + \mu_h) \right]}{\left[\alpha + \bar{\beta}_f (\mu_h + \nu_h) \right]} \text{ is increasing with } \alpha.$$

Hence,

$$tr(J) = \frac{\mu_f (\gamma_h + \mu_h)^2}{\alpha S_h^*} - \frac{\mu_h N_h}{S_h^*} - (\gamma_h + \mu_h) \text{ is decreasing with } \alpha.$$

One could have noticed that $tr(J)$ evaluated at $\alpha = \alpha^*$, $tr^*(J) = -(\mu_h + \nu_h) < 0$ is negative and $tr(J) < 0$ for any $\alpha > \alpha^*$.

Therefore, for any $\alpha > \alpha^*$, $\det(J) > 0$ and $tr(J) < 0$, showing that the equilibrium point E_f^h is locally asymptotically stable.

At last, we have proved that that same conditions that satisfies $\det(J) > 0$ also satisfies $tr(J) < 0$.

Therefore, E_f^h is locally asymptotically stable if, and only if, $Q_f > \psi_f \iff R^f > 1$,

$$\text{with } \psi_f = \exp \left\{ \frac{\mu_f \sigma (\mu_h + \nu_h) (\gamma_h + \mu_h)}{\hat{\beta} \bar{\beta}_f \mu_h N_h} \right\} \text{ and } R^f = \frac{\hat{\beta} \bar{\beta}_f \ln(Q_f) \mu_h N_h}{\mu_f \sigma (\gamma_h + \mu_h) (\mu_h + \nu_h)}.$$

□

Proposition A.2.6. The human population equilibrium under *Wolbachia* extinction equilibrium point is locally asymptotically stable if $R^f > 1$, and unstable if $R^f < 1$.

Proof A.2.6. The following subsystem represents human population under *Wolbachia* extinction

$$\begin{cases} \frac{dS_h}{dt} = \mu_h N_h - \hat{\beta} \left(\frac{\bar{\beta}_f I_h}{\sigma} \frac{\ln(Q_f)}{\mu_f + \bar{\beta}_f I_h} \right) S_h - (\mu_h + \nu_h) S_h, \\ \frac{dI_h}{dt} = \hat{\beta} \left(\frac{\bar{\beta}_f I_h}{\sigma} \frac{\ln(Q_f)}{\mu_f + \bar{\beta}_f I_h} \right) S_h - (\gamma_h + \mu_h) I_h. \end{cases}$$

One particular solution is the disease-free equilibrium (DFE)

$$S_h^{DFE} = \frac{\mu_h N_h}{\mu_h + \nu_h}, \quad I_h^{DFE} = 0.$$

New infection and transition terms:

$$F(S_h, I_h) = \hat{\beta} S_h I_h \left(\frac{\bar{\beta}_f}{\sigma} \frac{\ln(Q_f)}{\mu_f + \bar{\beta}_f I_h} \right), \quad V(I_h) = (\gamma_h + \mu_h) I_h.$$

Linearization at the DFE:

$$\left. \frac{\partial F}{\partial I_h} \right|_{DFE} = \hat{\beta} S_h^{DFE} \frac{\bar{\beta}_f \ln(Q_f)}{\sigma \mu_f}, \quad \left. \frac{\partial V}{\partial I_h} \right|_{DFE} = \gamma_h + \mu_h.$$

We use the next generation operator:

$$K = F V^{-1}.$$

Yields the epidemiological threshold:

$$\rho(K) = \frac{\hat{\beta} S_h^{DFE} \bar{\beta}_f \ln(Q_f)}{\gamma_h + \mu_h \sigma \mu_f} = \frac{\hat{\beta} \mu_h N_h}{(\mu_h + \nu_h)(\gamma_h + \mu_h)} \frac{\bar{\beta}_f \ln(Q_f)}{\sigma \mu_f} := R^f.$$

$R^f < 1 \implies$ DFE is stable equilibrium.

Therefore, human population equilibrium under *Wolbachia* extinction equilibrium point is locally asymptotically stable if $R^f > 1$, and unstable if $R^f < 1$.

□

Proposition A.2.7. The equilibrium point corresponding to the human dynamics in a scenario of wild and *Wolbachia*-infected mosquito coexistence is given by $E_c^h = (S_h^c, I_h^c)$, with

$$I_h^c = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

$$S_h^c = \frac{\mu_h N_h}{(\mu_h + \nu_h)} - \frac{(\gamma_h + \mu_h)}{(\mu_h + \nu_h)} I_h^c.$$

With the coefficients a, b and c of the quadratic expression in I_h^c given by:

$$a = \frac{\bar{\beta}_w \bar{\beta}_f \ln(Q_w)}{\sigma} + \frac{\bar{\beta}_w \bar{\beta}_f (\mu_h + \nu_h)}{\hat{\beta}},$$

$$b = \frac{(\mu_h + \nu_h)(\bar{\beta}_w \mu_f + \mu_w \bar{\beta}_f)}{\hat{\beta}} + \frac{\bar{\beta}_w \mu_f \ln(Q_w)}{\sigma} + \frac{\bar{M}(\mu_w \bar{\beta}_f - \mu_f \bar{\beta}_w)}{2\eta} - \frac{\bar{\beta}_w \bar{\beta}_f \ln(Q_w)}{\sigma} \frac{\mu_h N_h}{\gamma_h + \mu_h},$$

$$c = \frac{(\mu_h + \nu_h) \mu_w \mu_f}{\hat{\beta}} - \frac{\mu_h N_h}{\gamma_h + \mu_h} \left[\frac{\bar{\beta}_w \mu_f \ln(Q_w)}{\sigma} + \frac{\bar{M}(\mu_w \bar{\beta}_f - \mu_f \bar{\beta}_w)}{2\eta} \right].$$

Proof A.2.7. Equilibrium points for the mosquitoes populations are given by

$$E_+ = (S_f^+, S_w^+, I_f^+, I_w^+) \text{ and } E_- = (S_f^-, S_w^-, I_f^-, I_w^-)$$

such that equations are rewritten as:

$$\frac{dS_h^\pm}{dt} = \mu_h N_h - \hat{\beta}(I_f^\pm + I_w^\pm)S_h - (\mu_h + \nu_h)S_h$$

$$\frac{dI_h^\pm}{dt} = \hat{\beta}(I_f^\pm + I_w^\pm)S_h - (\gamma_h + \mu_h)I_h$$

For now, let us streamline the notation, keeping in mind that these calculations apply to both coexistence equilibria E_+ and E_- .

We might start by manipulating the expression $(I_f + I_w)$. We have defined the constants $C_1 = \frac{\bar{\beta}_w I_h}{\mu_w}$ and $C_2 = \frac{\bar{\beta}_f I_h}{\mu_f}$, and also the relations $I_f = C_2 S_f$ and $S_w = \frac{\ln(Q_w)}{\sigma(C_1+1)} - \frac{S_f + I_f}{(C_1+1)}$. Thus,

$$\begin{aligned}
M = S_f + S_w + I_f + I_w &= \frac{1}{\sigma} \ln(Q_w) \implies I_f + I_w = \frac{1}{\sigma} \ln(Q_w) - (S_f + S_w) \\
\implies I_f + I_w &= \frac{1}{\sigma} \ln(Q_w) - \left[S_f + \left(\frac{\ln(Q_w)}{\sigma(C_1+1)} - \frac{S_f(1+C_2)}{(C_1+1)} \right) \right] \\
\implies \sigma(C_1+1)(I_f + I_w) &= (C_1+1) \ln(Q_w) - \sigma(C_1+1)S_f - \ln(Q_w) + \sigma(C_2+1)S_f \\
= \ln(Q_w)[(C_1+1)-1] + S_f[\sigma(C_2+1) - \sigma(C_1+1)] &= C_1 \ln(Q_w) + \sigma S_f[(C_2+1) - (C_1+1)] \\
\implies \sigma(C_1+1)(I_f + I_w) &= C_1 \ln(Q_w) + \sigma S_f(C_2 - C_1) \\
\implies I_f + I_w &= \frac{C_1 \ln(Q_w) + \sigma(C_2 - C_1)S_f}{\sigma(C_1+1)} = \frac{C_1 \ln(Q_w)}{\sigma(C_1+1)} + \frac{(C_2 - C_1)S_f}{(C_1+1)}
\end{aligned}$$

Substituting the values of the constants C_1 and C_2 :

$$\begin{aligned}
I_f + I_w &= \frac{\bar{\beta}_w \ln(Q_w) I_h}{\mu_w \sigma \left(\frac{\bar{\beta}_w I_h}{\mu_w} + 1 \right)} + \frac{\left[\left(\frac{\bar{\beta}_f}{\mu_f} - \frac{\bar{\beta}_w}{\mu_w} \right) I_h \right] S_f}{\left(\frac{\bar{\beta}_w I_h}{\mu_w} + 1 \right)} = \frac{\bar{\beta}_w \ln(Q_w) I_h}{\sigma(\bar{\beta}_w I_h + \mu_w)} + \frac{\mu_w \left[\left(\frac{\bar{\beta}_f}{\mu_f} - \frac{\bar{\beta}_w}{\mu_w} \right) S_f \right] I_f}{(\bar{\beta}_w I_h + \mu_w)} \\
\implies \sigma(I_f + I_w) &= \frac{\bar{\beta}_w \ln(Q_w) I_h}{(\bar{\beta}_w I_h + \mu_w)} + \frac{\sigma \mu_w \left[\left(\frac{\bar{\beta}_f}{\mu_f} - \frac{\bar{\beta}_w}{\mu_w} \right) S_f \right] I_f}{(\bar{\beta}_w I_h + \mu_w)} \\
\implies I_f + I_w &= I_h \left[\frac{\bar{\beta}_w \ln(Q_w) + \sigma \mu_w S_f \left(\frac{\bar{\beta}_f}{\mu_f} - \frac{\bar{\beta}_w}{\mu_w} \right)}{\sigma(\bar{\beta}_w I_h + \mu_w)} \right] = I_h \left[\frac{\bar{\beta}_w \ln(Q_w) + \sigma S_f \left(\frac{\mu_w \bar{\beta}_f - \mu_f \bar{\beta}_w}{\mu_f} \right)}{\sigma(\bar{\beta}_w I_h + \mu_w)} \right] \\
\implies I_f + I_w &= I_h \left[\frac{\mu_f \bar{\beta}_w \ln(Q_w) + \sigma S_f (\mu_w \bar{\beta}_f - \mu_f \bar{\beta}_w)}{\mu_f \sigma(\bar{\beta}_w I_h + \mu_w)} \right].
\end{aligned}$$

After some manipulations, we can rewrite this expression in the following form:

$$I_f + I_w = I_h \left[\frac{\mu_f \bar{\beta}_w \ln(Q_w) + \sigma S_f (\mu_w \bar{\beta}_f - \mu_f \bar{\beta}_w)}{\mu_f \sigma(\bar{\beta}_w I_h + \mu_w)} \right].$$

But there is a problem, due to the fact that S_f is a function of the infected human population I_h , i.e.: $S_f = g(I_h)$.

$$S_f^\pm = \frac{M \left(\frac{\bar{\beta}_f I_h}{\mu_f} + 1 \right) (Q_c - 1) \pm \sqrt{\left[M \left(\frac{\bar{\beta}_f I_h}{\mu_f} + 1 \right) \right]^2 - 4\eta \left[1 + \frac{\bar{\beta}_f I_h}{\mu_f} \left(2 + \frac{\bar{\beta}_f I_h}{\mu_f} \right) \right] \frac{Q_{w,f}}{Q_f} M^2}}{2\eta \left[1 + \frac{\bar{\beta}_f I_h}{\mu_f} \left(2 + \frac{\bar{\beta}_f I_h}{\mu_f} \right) \right]}$$

So we might rewrite it, in order to remove I_h from the square root. After some manipulations, we can rewrite this expression as:

$$S_f^\pm = \frac{M \left[(Q_c - 1) \pm \sqrt{(Q_c - 1)^2 - 4\eta \frac{Q_{w,f}}{Q_f}} \right]}{2\eta \left(\frac{\bar{\beta}_f I_h}{\mu_f} + 1 \right)}.$$

We might already simplify it by setting $\bar{M} = M \left[(Q_c - 1) \pm \sqrt{(Q_c - 1)^2 - 4\eta \frac{Q_{w,f}}{Q_f}} \right]$.

And now, substituting the new S_f expression into $(I_f + I_w)$:

$$I_f + I_w = I_h \left\{ \frac{\bar{\beta}_w \ln(Q_w)}{\sigma(\bar{\beta}_w I_h + \mu_w)} + \frac{\bar{M}(\mu_w \bar{\beta}_f - \mu_f \bar{\beta}_w)}{2\eta(\bar{\beta}_w I_h + \mu_w)(\bar{\beta}_f I_h + \mu_f)} \right\}.$$

Hence, equations can be written as:

$$\frac{dS_h}{dt} = \mu_h N_h - \hat{\beta} I_h \left\{ \frac{\bar{\beta}_w \ln(Q_w)}{\sigma(\bar{\beta}_w I_h + \mu_w)} + \frac{\bar{M}(\mu_w \bar{\beta}_f - \mu_f \bar{\beta}_w)}{2\eta(\bar{\beta}_w I_h + \mu_w)(\bar{\beta}_f I_h + \mu_f)} \right\} S_h - (\mu_h + \nu_h) S_h$$

$$\frac{dI_h}{dt} = \hat{\beta} I_h \left\{ \frac{\bar{\beta}_w \ln(Q_w)}{\sigma(\bar{\beta}_w I_h + \mu_w)} + \frac{\bar{M}(\mu_w \bar{\beta}_f - \mu_f \bar{\beta}_w)}{2\eta(\bar{\beta}_w I_h + \mu_w)(\bar{\beta}_f I_h + \mu_f)} \right\} S_h - (\gamma_h + \mu_h) I_h$$

Let's define

$$G(I_h) = \frac{\bar{\beta}_w \ln(Q_w)}{\sigma(\bar{\beta}_w I_h + \mu_w)} + \frac{\bar{M}(\mu_w \bar{\beta}_f - \mu_f \bar{\beta}_w)}{2\eta(\bar{\beta}_w I_h + \mu_w)(\bar{\beta}_f I_h + \mu_f)} \quad (i)$$

At last, equations are rewritten as:

$$\frac{dS_h}{dt} = \mu_h N_h - \hat{\beta} I_h G(I_h) S_h - (\mu_h + \nu_h) S_h \quad (1C)$$

$$\frac{dI_h}{dt} = \hat{\beta} I_h G(I_h) S_h - (\gamma_h + \mu_h) I_h \quad (2C)$$

From Equation (2C):

$$0 = \hat{\beta} I_h G(I_h) S_h - (\gamma_h + \mu_h) I_h \implies \begin{cases} \text{(ii)} \quad \overline{I_h} = 0 \\ \text{(iii)} \quad \hat{\beta} G(I_h) \overline{S_h} - (\gamma_h + \mu_h) = 0 \end{cases}$$

$$\hat{\beta} G(I_h) \overline{S_h} = (\gamma_h + \mu_h) \implies \overline{S_h} = \frac{(\gamma_h + \mu_h)}{\hat{\beta} G(I_h)} \quad \text{(iii)}$$

(iii) into Equation (1C):

$$0 = \mu_h N_h - \hat{\beta} I_h G(I_h) \overline{S_h} - (\mu_h + \nu_h) \overline{S_h}$$

$$(\mu_h + \nu_h) \overline{S_h} = \mu_h N_h - \hat{\beta} I_h G(I_h) \left[\frac{(\gamma_h + \mu_h)}{\hat{\beta} G(I_h)} \right]$$

$$(\mu_h + \nu_h) \overline{S_h} = \mu_h N_h - I_h (\gamma_h + \mu_h)$$

$$\overline{S_h} = \frac{\mu_h N_h}{(\mu_h + \nu_h)} - \frac{I_h (\gamma_h + \mu_h)}{(\mu_h + \nu_h)} \quad \text{(iv)}$$

Comparing (iii) and (iv):

$$\frac{(\gamma_h + \mu_h)}{\hat{\beta} G(I_h)} = \frac{\mu_h N_h}{(\mu_h + \nu_h)} - \frac{I_h (\gamma_h + \mu_h)}{(\mu_h + \nu_h)}$$

$$\frac{(\gamma_h + \mu_h)}{G(I_h)} = \hat{\beta} \left[\frac{\mu_h N_h}{(\mu_h + \nu_h)} - \frac{I_h (\gamma_h + \mu_h)}{(\mu_h + \nu_h)} \right]$$

$$G(I_h) = \frac{(\gamma_h + \mu_h)}{\hat{\beta} \left[\frac{\mu_h N_h}{(\mu_h + \nu_h)} - \frac{I_h (\gamma_h + \mu_h)}{(\mu_h + \nu_h)} \right]} = \frac{(\gamma_h + \mu_h)(\mu_h + \nu_h)}{\hat{\beta} [\mu_h N_h - (\gamma_h + \mu_h) I_h]}$$

$$G(I_h) = \frac{(\mu_h + \nu_h)}{\hat{\beta}} \frac{1}{\frac{\mu_h N_h}{\gamma_h + \mu_h} - I_h} \quad \text{(v)}$$

Comparing (i) and (v):

$$\frac{\overline{\beta}_w \ln(Q_w)}{\sigma(\overline{\beta}_w I_h + \mu_w)} + \frac{\overline{M}(\mu_w \overline{\beta}_f - \mu_f \overline{\beta}_w)}{2\eta(\overline{\beta}_w I_h + \mu_w)(\overline{\beta}_f I_h + \mu_f)} = \frac{(\mu_h + \nu_h)}{\hat{\beta}} \frac{1}{\left(\frac{\mu_h N_h}{\gamma_h + \mu_h} \right) - I_h}$$

We shall define

$$k_1 = \frac{\overline{\beta}_w \ln(Q_w)}{\sigma}, \quad k_2 = \frac{\overline{M}(\mu_w \overline{\beta}_f - \mu_f \overline{\beta}_w)}{2\eta}, \quad k_3 = \frac{(\mu_h + \nu_h)}{\hat{\beta}}, \quad \text{and} \quad k_4 = \left(\frac{\mu_h N_h}{\gamma_h + \mu_h} \right).$$

Such that

$$\begin{aligned} \frac{k_1}{(\bar{\beta}_w I_h + \mu_w)} + \frac{k_2}{(\bar{\beta}_w I_h + \mu_w)(\bar{\beta}_f I_h + \mu_f)} &= \frac{k_3}{k_4 - I_h} \\ k_1 + \frac{k_2}{(\bar{\beta}_f I_h + \mu_f)} &= \frac{k_3}{k_4 - I_h} (\bar{\beta}_w I_h + \mu_w) \\ k_1(\bar{\beta}_f I_h + \mu_f) + k_2 &= \frac{k_3}{k_4 - I_h} (\bar{\beta}_w I_h + \mu_w)(\bar{\beta}_f I_h + \mu_f) \\ [k_1(\bar{\beta}_f I_h + \mu_f) + k_2] (k_4 - I_h) &= k_3(\bar{\beta}_w I_h + \mu_w)(\bar{\beta}_f I_h + \mu_f) \\ [k_1 \bar{\beta}_f I_h + k_1 \mu_f + k_2] (k_4 - I_h) &= k_3(\bar{\beta}_w I_h + \mu_w)(\bar{\beta}_f I_h + \mu_f) \end{aligned}$$

Now, let's define $A_0 = k_1 \bar{\beta}_f$ and $B_0 = k_1 \mu_f + k_2$, such that

$$\begin{aligned} [A_0 I_h + B_0] (k_4 - I_h) &= k_3(\bar{\beta}_w I_h + \mu_w)(\bar{\beta}_f I_h + \mu_f) \\ A_0 I_h k_4 + B_0 k_4 - A_0 I_h^2 - B_0 I_h &= k_3 (\bar{\beta}_w \bar{\beta}_f I_h^2 + \bar{\beta}_f I_h \mu_w + \bar{\beta}_w I_h \mu_f + \mu_f \mu_w) \\ A_0 I_h k_4 + B_0 k_4 - A_0 I_h^2 - B_0 I_h &= k_3 [\bar{\beta}_w \bar{\beta}_f I_h^2 + (\bar{\beta}_f \mu_w + \bar{\beta}_w \mu_f) I_h + \mu_f \mu_w] \\ \implies A_0 I_h k_4 + B_0 k_4 - A_0 I_h^2 - B_0 I_h - k_3 [\bar{\beta}_w \bar{\beta}_f I_h^2 + (\bar{\beta}_f \mu_w + \bar{\beta}_w \mu_f) I_h + \mu_f \mu_w] &= 0 \\ (-A_0 - \bar{\beta}_w \bar{\beta}_f k_3) I_h^2 + [A_0 k_4 - B_0 - (\bar{\beta}_f \mu_w + \bar{\beta}_w \mu_f) k_3] I_h + B_0 k_4 - \mu_f \mu_w k_3 & \\ (A_0 + \bar{\beta}_w \bar{\beta}_f k_3) I_h^2 + [B_0 + (\bar{\beta}_f \mu_w + \bar{\beta}_w \mu_f) k_3 - A_0 k_4] I_h + \mu_f \mu_w k_3 - B_0 k_4 & \end{aligned}$$

which is a quadratic expression in I_h , with coefficients:

$$\begin{aligned} a &= A_0 + \bar{\beta}_w \bar{\beta}_f k_3, \\ b &= B_0 + (\bar{\beta}_f \mu_w + \bar{\beta}_w \mu_f) k_3 - A_0 k_4, \\ c &= \mu_f \mu_w k_3 - B_0 k_4. \end{aligned}$$

We might rewrite the coefficients in their original form:

$$\begin{aligned}
a &= \frac{\bar{\beta}_w \bar{\beta}_f \ln(Q_w)}{\sigma} + \frac{\bar{\beta}_w \bar{\beta}_f (\mu_h + \nu_h)}{\hat{\beta}} \\
b &= \frac{(\mu_h + \nu_h)(\bar{\beta}_w \mu_f + \mu_w \bar{\beta}_f)}{\hat{\beta}} + \frac{\bar{\beta}_w \mu_f \ln(Q_w)}{\sigma} + \frac{\bar{M}(\mu_w \bar{\beta}_f - \mu_f \bar{\beta}_w)}{2\eta} - \frac{\bar{\beta}_w \bar{\beta}_f \ln(Q_w)}{\sigma} \left(\frac{\mu_h N_h}{\gamma_h + \mu_h} \right), \\
c &= \frac{(\mu_h + \nu_h) \mu_w \mu_f}{\hat{\beta}} - \frac{\mu_h N_h}{\gamma_h + \mu_h} \left[\frac{\bar{\beta}_w \mu_f \ln(Q_w)}{\sigma} + \frac{\bar{M}(\mu_w \bar{\beta}_f - \mu_f \bar{\beta}_w)}{2\eta} \right].
\end{aligned}$$

Finally, we calculated the equilibrium point (S_h^*, I_h^*) , given by:

$$I_h^* = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \quad S_h^* = \frac{\mu_h N_h}{(\mu_h + \nu_h)} - \frac{(\gamma_h + \mu_h)}{(\mu_h + \nu_h)} I_h^*.$$

Our analysis of the human dynamics under the mosquito coexistence regime is restricted to the locally asymptotically stable steady state $E_c = E_-$. Accordingly, we set $\bar{M} = M^-$ and define the corresponding human equilibrium as $E_c^h = (S_h^c, I_h^c)$, with

$$I_h^c = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \quad S_h^c = \frac{\mu_h N_h}{\mu_h + \nu_h} - \frac{\gamma_h + \mu_h}{\mu_h + \nu_h} I_h^c.$$

This pair represents the steady-state solution of the human subsystem corresponding to the coexistence regime between *Wolbachia*-infected and wild mosquitoes.

□

Proposition A.2.8. The signs of the coefficients b and c determine the number of positive roots of the quadratic equilibrium equation for I_h^c . In particular, the sign changes of b and c occur at the thresholds ψ_b and ψ_c , respectively, obtained by solving $b = 0$ and $c = 0$ for $\bar{\beta}_w$. Moreover, the following positivity regions hold:

- (I) $c < 0 \implies$ the quadratic equation for I_h^c has exactly one positive root, for any b .
- (II) $c = 0 \implies$ the quadratic equation for I_h^c has exactly one positive root if, and only if, $b < 0$.
- (III) $c > 0 \implies$ the quadratic equation for I_h^c has two positive roots if, and only if, $b < 0$ and $\Delta := b^2 - 4ac \geq 0$.

Proof A.2.8. Recall that the quadratic coefficient c can be written as

$$c = \frac{(\mu_h + \nu_h) \mu_w \mu_f}{\hat{\beta}} - \frac{\mu_h N_h}{\gamma_h + \mu_h} \left(\frac{\bar{\beta}_w \mu_f \ln(Q_w)}{\sigma} + \frac{M^-}{2\eta} (\mu_w \bar{\beta}_f - \mu_f \bar{\beta}_w) \right).$$

Expanding the terms in $\bar{\beta}_w$, we obtain

$$c = \frac{(\mu_h + \nu_h) \mu_w \mu_f}{\hat{\beta}} - \frac{\mu_h N_h}{\gamma_h + \mu_h} \left[\frac{M^-}{2\eta} \mu_w \bar{\beta}_f + \mu_f \left(\frac{\ln(Q_w)}{\sigma} - \frac{M^-}{2\eta} \right) \bar{\beta}_w \right].$$

Assuming $\frac{\ln(Q_w)}{\sigma} \neq \frac{M^-}{2\eta}$, solving $c = 0$ for $\bar{\beta}_w$ yields the threshold value ψ_c , given by

$$\psi_c = \frac{\sigma \mu_w [2\eta \mu_f (\gamma_h + \mu_h) (\mu_h + \nu_h) - \hat{\beta} \mu_h N_h M^- \bar{\beta}_f]}{\hat{\beta} \mu_h N_h \mu_f [2\eta \ln(Q_w) - \sigma M^-]}.$$

Since c depends linearly on $\bar{\beta}_w$ with slope proportional to $\left(\frac{M^-}{2\eta} - \frac{\ln(Q_w)}{\sigma} \right)$, the sign regions are determined as follows:

$$\text{If } \frac{\ln(Q_w)}{\sigma} > \frac{M^-}{2\eta}, \quad \bar{\beta}_w > \psi_c \Rightarrow c < 0, \quad \bar{\beta}_w = \psi_c \Rightarrow c = 0, \quad \bar{\beta}_w < \psi_c \Rightarrow c > 0,$$

$$\text{If } \frac{\ln(Q_w)}{\sigma} < \frac{M^-}{2\eta}, \quad \bar{\beta}_w > \psi_c \Rightarrow c > 0, \quad \bar{\beta}_w = \psi_c \Rightarrow c = 0, \quad \bar{\beta}_w < \psi_c \Rightarrow c < 0.$$

Similarly, the coefficient b is given by

$$b = \frac{(\mu_h + \nu_h) (\bar{\beta}_w \mu_f + \mu_w \bar{\beta}_f)}{\hat{\beta}} + \frac{\bar{\beta}_w \mu_f \ln(Q_w)}{\sigma} + \frac{M^-}{2\eta} (\mu_w \bar{\beta}_f - \mu_f \bar{\beta}_w) - \frac{\bar{\beta}_w \bar{\beta}_f \ln(Q_w)}{\sigma} \frac{\mu_h N_h}{\gamma_h + \mu_h}.$$

Collecting the constant terms and the terms in $\bar{\beta}_w$, we rewrite b as

$$b = \mu_w \bar{\beta}_f \left(\frac{\mu_h + \nu_h}{\hat{\beta}} + \frac{M^-}{2\eta} \right) + \left[\mu_f \left(\frac{\mu_h + \nu_h}{\hat{\beta}} + \frac{\ln(Q_w)}{\sigma} - \frac{M^-}{2\eta} \right) - \frac{\mu_h N_h}{\gamma_h + \mu_h} \bar{\beta}_f \frac{\ln(Q_w)}{\sigma} \right] \bar{\beta}_w.$$

Assuming

$$\mu_f \left(\frac{\mu_h + \nu_h}{\hat{\beta}} + \frac{\ln(Q_w)}{\sigma} - \frac{M^-}{2\eta} \right) - \frac{\mu_h N_h}{\gamma_h + \mu_h} \bar{\beta}_f \frac{\ln(Q_w)}{\sigma} \neq 0,$$

solving $b = 0$ for $\bar{\beta}_w$ defines the threshold value ψ_b , namely

$$\psi_b = \frac{\mu_w \bar{\beta}_f \sigma (\gamma_h + \mu_h) [2\eta (\mu_h + \nu_h) + \hat{\beta} M^-]}{2\eta \hat{\beta} \bar{\beta}_f \mu_h N_h \ln(Q_w) + \mu_f (\gamma_h + \mu_h) [\hat{\beta} \sigma M^- - 2\eta \sigma (\mu_h + \nu_h) - 2\eta \hat{\beta} \ln(Q_w)]}.$$

Since b depends linearly on $\bar{\beta}_w$, with slope given by

$$\mu_f \left(\frac{\mu_h + \nu_h}{\hat{\beta}} + \frac{\ln(Q_w)}{\sigma} - \frac{M^-}{2\eta} \right) - \frac{\mu_h N_h}{\gamma_h + \mu_h} \bar{\beta}_f \frac{\ln(Q_w)}{\sigma},$$

the sign regions are determined by the following two cases:

$$\text{If } \mu_f \left(\frac{\mu_h + \nu_h}{\hat{\beta}} + \frac{\ln(Q_w)}{\sigma} - \frac{M^-}{2\eta} \right) - \frac{\mu_h N_h}{\gamma_h + \mu_h} \bar{\beta}_f \frac{\ln(Q_w)}{\sigma} > 0,$$

$$\bar{\beta}_w > \psi_b \Rightarrow b > 0, \quad \bar{\beta}_w = \psi_b \Rightarrow b = 0, \quad \bar{\beta}_w < \psi_b \Rightarrow b < 0,$$

$$\text{If } \mu_f \left(\frac{\mu_h + \nu_h}{\hat{\beta}} + \frac{\ln(Q_w)}{\sigma} - \frac{M^-}{2\eta} \right) - \frac{\mu_h N_h}{\gamma_h + \mu_h} \bar{\beta}_f \frac{\ln(Q_w)}{\sigma} < 0,$$

$$\bar{\beta}_w > \psi_b \Rightarrow b < 0, \quad \bar{\beta}_w = \psi_b \Rightarrow b = 0, \quad \bar{\beta}_w < \psi_b \Rightarrow b > 0.$$

Finally, we characterize the positivity regions for I_h^c . By Descartes' rule of signs, the number of positive roots of the quadratic equation for I_h^c is determined by the signs of b and c . Accordingly:

- (I) $c < 0 \Rightarrow$ two real roots with opposite signs, hence exactly one positive root, for any b ;
- (II) $c = 0 \Rightarrow$ roots are 0 and $-b/a$, thus exactly one positive root if, and only if, $b < 0$;
- (III) $c > 0 \Rightarrow$ two positive roots if, and only if, $b < 0$ and $\Delta = b^2 - 4ac \geq 0$.

Therefore, once the signs of the coefficients b and c are determined, considering the thresholds ψ_b and ψ_c , the number of positive roots of I_h^c follows from the three regions defined above. $\square$

Proposition A.2.9. For the biologically relevant parameter sets, present on Table 3, representing distinct varieties of *Wolbachia* bacterium, the discriminant satisfies $\Delta(\bar{\beta}_w) > 0$. Moreover, the system lies in region (I), and therefore the quadratic equation defining I_h^c admits exactly one positive root.

Proof A.2.9. First, we write the discriminant of the quadratic expression I_h^c as a function of the human-*Wolbachia* mosquito transmission rate, $\Delta(\bar{\beta}_w)$. For brevity, define the quantities

$$k := \frac{M^-}{2\eta}, \quad \hat{\gamma} := \frac{\mu_h N_h}{\gamma_h + \mu_h}, \quad \ell := \frac{\ln(Q_w)}{\sigma}.$$

Then we rewrite the coefficients a , b , and c as functions of the transmission parameter $\bar{\beta}_w$. First,

$$a(\bar{\beta}_w) = \bar{\beta}_w \bar{\beta}_f \left(\ell + \frac{\mu_h + \nu_h}{\hat{\beta}} \right).$$

Moreover, collecting the terms in $\bar{\beta}_w$, we write

$$b(\bar{\beta}_w) = b_0 + b_1 \bar{\beta}_w, \quad c(\bar{\beta}_w) = c_0 + c_1 \bar{\beta}_w,$$

where

$$b_0 = \mu_w \bar{\beta}_f \left(\frac{\mu_h + \nu_h}{\hat{\beta}} + k \right), \quad b_1 = \mu_f \left(\frac{\mu_h + \nu_h}{\hat{\beta}} + \ell - k \right) - \hat{\gamma} \bar{\beta}_f \ell,$$

and

$$c_0 = \frac{(\mu_h + \nu_h) \mu_w \mu_f}{\hat{\beta}} - \hat{\gamma} k \mu_w \bar{\beta}_f, \quad c_1 = \hat{\gamma} \mu_f (k - \ell).$$

The discriminant is defined as

$$\Delta(\bar{\beta}_w) := b(\bar{\beta}_w)^2 - 4a(\bar{\beta}_w) c(\bar{\beta}_w).$$

Substituting the expressions for a , b , and c , we obtain

$$\begin{aligned}\Delta(\bar{\beta}_w) &= (b_0 + b_1 \bar{\beta}_w)^2 - 4 \left[\bar{\beta}_w \bar{\beta}_f \left(\ell + \frac{\mu_h + \nu_h}{\hat{\beta}} \right) \right] (c_0 + c_1 \bar{\beta}_w) \\ &= A_2 \bar{\beta}_w^2 + A_1 \bar{\beta}_w + A_0,\end{aligned}$$

with coefficients

$$\begin{aligned}A_0 &= b_0^2 = \mu_w^2 \bar{\beta}_f^2 \left(\frac{\mu_h + \nu_h}{\hat{\beta}} + \frac{M^-}{2\eta} \right)^2, \\ A_1 &= 2b_0 b_1 - 4 \bar{\beta}_f \left(\frac{\ln(Q_w)}{\sigma} + \frac{\mu_h + \nu_h}{\hat{\beta}} \right) c_0, \\ A_2 &= b_1^2 - 4 \bar{\beta}_f \left(\frac{\ln(Q_w)}{\sigma} + \frac{\mu_h + \nu_h}{\hat{\beta}} \right) c_1.\end{aligned}$$

That is,

$$\begin{aligned}A_1 &= 2 \mu_w \bar{\beta}_f \left(\frac{\mu_h + \nu_h}{\hat{\beta}} + \frac{M^-}{2\eta} \right) \left[\mu_f \left(\frac{\mu_h + \nu_h}{\hat{\beta}} + \frac{\ln(Q_w)}{\sigma} - \frac{M^-}{2\eta} \right) - \frac{\mu_h N_h}{\gamma_h + \mu_h} \bar{\beta}_f \frac{\ln(Q_w)}{\sigma} \right] \\ &\quad - 4 \bar{\beta}_f \left(\frac{\ln(Q_w)}{\sigma} + \frac{\mu_h + \nu_h}{\hat{\beta}} \right) \left[\frac{(\mu_h + \nu_h) \mu_w \mu_f}{\hat{\beta}} - \frac{\mu_h N_h}{\gamma_h + \mu_h} \frac{M^-}{2\eta} \mu_w \bar{\beta}_f \right],\end{aligned}$$

and

$$\begin{aligned}A_2 &= \left[\mu_f \left(\frac{\mu_h + \nu_h}{\hat{\beta}} + \frac{\ln(Q_w)}{\sigma} - \frac{M^-}{2\eta} \right) - \frac{\mu_h N_h}{\gamma_h + \mu_h} \bar{\beta}_f \frac{\ln(Q_w)}{\sigma} \right]^2 \\ &\quad - 4 \bar{\beta}_f \left(\frac{\ln(Q_w)}{\sigma} + \frac{\mu_h + \nu_h}{\hat{\beta}} \right) \left[\frac{\mu_h N_h}{\gamma_h + \mu_h} \mu_f \left(\frac{M^-}{2\eta} - \frac{\ln(Q_w)}{\sigma} \right) \right].\end{aligned}$$

One can observe that the discriminant $\Delta(\bar{\beta}_w)$ is written as quadratic polynomial, given by

$$\Delta(\bar{\beta}_w) = A_2 \bar{\beta}_w^2 + A_1 \bar{\beta}_w + A_0.$$

Considering the parameter set representing the biological features of the wMel *Wolbachia* strain, presented on Table 3, we obtain

$$A_0 = 2.1521 \times 10^{-06} > 0 \implies A_0 = b_0^2 > 0, \quad A_1 = 0.0696 > 0, \quad A_2 = 571.9730 > 0,$$

and

$$\frac{\ln(Q_w)}{\sigma} = 941.1207 > \frac{M^-}{2\eta} = 146.2536.$$

Highlighting the admissible domain for the transmission parameter, $\bar{\beta}_w \in [0, \infty)$, we present a direct demonstration that the discriminant is strictly positive, that is, $\Delta(\bar{\beta}_w) > 0$. Recall that

$$\Delta(\bar{\beta}_w) = A_2 \bar{\beta}_w^2 + A_1 \bar{\beta}_w + A_0, \quad A_0 = b_0^2 = \left[\mu_w \bar{\beta}_f \left(\frac{\mu_h + \nu_h}{\hat{\beta}} + \frac{M^-}{2\eta} \right) \right]^2 > 0.$$

Since $A_2 > 0$ and $A_1 \geq 0$, for all $\bar{\beta}_w \geq 0$ we have

$$\frac{d}{d\bar{\beta}_w} [\Delta(\bar{\beta}_w)] = 2A_2 \bar{\beta}_w + A_1 \geq A_1 \geq 0,$$

so Δ is nondecreasing on $[0, \infty)$. Consequently,

$$\Delta(\bar{\beta}_w) \geq \Delta(0) = A_0 = b_0^2 > 0, \quad \forall \bar{\beta}_w \geq 0.$$

Therefore,

$$\Delta(\bar{\beta}_w) > 0 \quad \text{for every admissible } \bar{\beta}_w \geq 0,$$

and the quadratic equilibrium expression I_h^c admits two real and distinct roots for all admissible values of the human–*Wolbachia* transmission rate $\bar{\beta}_w$.

Moreover, for the same parameter set,

$$\bar{\beta}_w = \frac{\bar{\beta}_w}{N_h} = 10^{-4} > \psi_c = 3.3041 \times 10^{-05} \implies c < 0,$$

showing that the system lies in region (I), and the quadratic expression in I_h^c has exactly one positive root.

Thus, we have proved that the quadratic equation for I_h^c has two real roots of opposite signs, and therefore admits exactly one positive root. The same pattern was observed when considering the biologically relevant parameter sets presented on Table 3, representing distinct variety of *Wolbachia* bacterium.

□

Proposition A.2.10. The human population equilibrium under mosquito coexistence conditions is locally asymptotically stable if $R^c > 1$, and unstable if $R^c < 1$.

Proof A.2.10. The following subsystem corresponds to human population under mosquito coexistence regime.

$$\begin{cases} \frac{dS_h}{dt} = \mu_h N_h - \hat{\beta} I_h G(I_h) S_h - (\mu_h + \nu_h) S_h, \\ \frac{dI_h}{dt} = \hat{\beta} I_h G(I_h) S_h - (\gamma_h + \mu_h) I_h, \end{cases}$$

$$\text{with } G(I_h) = \frac{\bar{\beta}_w \ln(Q_w)}{\sigma(\mu_w + \bar{\beta}_w I_h)} + \frac{\bar{M}(\mu_w \bar{\beta}_f - \mu_f \bar{\beta}_w)}{2\eta(\mu_w + \bar{\beta}_w I_h)(\mu_f + \bar{\beta}_f I_h)}.$$

$$S_h^{DFE} = \frac{\mu_h N_h}{\mu_h + \nu_h}, \quad I_h^{DFE} = 0.$$

$$F(S_h, I_h) = \hat{\beta} S_h I_h G(I_h), \quad V(I_h) = (\gamma_h + \mu_h) I_h.$$

$$\left. \frac{\partial F}{\partial I_h} \right|_{\text{DFE}} = \hat{\beta} S_h^{DFE} G(0), \quad \left. \frac{\partial V}{\partial I_h} \right|_{\text{DFE}} = \gamma_h + \mu_h.$$

$$G(0) = \frac{\bar{\beta}_w \ln(Q_w)}{\sigma \mu_w} + \frac{\bar{M}(\mu_w \bar{\beta}_f - \mu_f \bar{\beta}_w)}{2\eta \mu_w \mu_f}.$$

$$F = \left. \frac{\partial F}{\partial I_h} \right|_{\text{DFE}}, \quad V = \left. \frac{\partial V}{\partial I_h} \right|_{\text{DFE}}, \quad K = F V^{-1}.$$

$$R_0 = \rho(K) = \frac{\hat{\beta} S_h^{DFE} G(0)}{\gamma_h + \mu_h} = \frac{\hat{\beta} \mu_h N_h}{(\mu_h + \nu_h)(\gamma_h + \mu_h)} \left[\frac{\bar{\beta}_w \ln(Q_w)}{\sigma \mu_w} + \frac{\bar{M}(\mu_w \bar{\beta}_f - \mu_f \bar{\beta}_w)}{2\eta \mu_w \mu_f} \right].$$

Therefore, Disease-free Equilibrium is LAS $\iff R_0 < 1$, and conversely, DFE is unstable $\iff R_0 > 1$ and henceforth the human population equilibrium under mosquito coexistence conditions is locally asymptotically stable if $R^c > 1$, and unstable if $R^c < 1$. Moreover, we can conclude that E_- equilibrium point is LAS whenever $R^c > \max \{1, R^f\}$.

□