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Holography, BMN correspondence and deformations

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**HOLOGRAPHY, BMN CORRESPONDENCE
AND DEFORMATIONS**

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"If I have seen further, it is by standing on the shoulders of giants."

Sir Isaac Newton

Resumo

Desde à descoberta por Maldacena no final da década de 90, a correspondência conjecturada entre uma teoria da gravidade e uma teoria de campos cresceu tão rapidamente e de forma tão abrangente que se tornou uma área de pesquisa por si só. A gama de aplicabilidade dessa correspondência é hoje tão ampla que vai desde modelos cosmológicos, que descrevem a evolução do universo, até sistemas de matéria condensada não relativísticos, que modelam materiais fortemente correlacionados.

Entre todos os modelos conhecidos e úteis para a dualidade gravidade/gauge, particularmente interessantes são aqueles obtidos por meio de deformações de modelos já existentes, mapeando resultados do lado gravitacional para o lado de gauge ou vice-versa. Além disso, as deformações servem como ferramentas poderosas para compreender melhor como a dualidade se manifesta: tipicamente geradas por transformações no lado da supergravidade da correspondência, essas deformações estão intrinsecamente relacionadas a fluxos de energia, simetrias do modelo e integrabilidade, fornecendo, assim, resultados que nos ajudam a entender como tais propriedades aparecem na teoria.

Nesta dissertação abordamos esse problema estudando diferentes tipos de deformações, que em geral são geradas por técnicas similares no lado da supergravidade, frequentemente fazendo uso de transformações TsT, embora não necessariamente. Por exemplo, estudamos as deformações β, γ, $T\bar{T}$, η, λ, dentre outras, explicando suas propriedades, suas quantidades conservadas e como as deformações se manifestam em ambos lados da correspondência.

Fazemos ainda uso do mapeamento BMN/limite de Penrose e o generalizamos para modelos além de $\mathrm{AdS}_5/\mathrm{CFT}_4$. Esse procedimento consiste em mapear um setor particular da teoria de campos, contendo operadores massivos, para um limite de impulso no lado gravitacional, resultando em uma métrica que é geralmente do tipo onda plana (pp-wave), embora também possa ser uma generalização desse tipo.

Investigamos as propriedades não comutativas do limite de Penrose em relação às deformações geradas por transformações TsT no lado da supergravidade. Também realizamos uma análise do tipo "bootstrap" para deformações do tipo Leigh–Strassler, obtendo insights úteis para trabalhos futuros. Por fim, estudamos

como deformações TsT afetam modelos envolvendo branas e anti-branas que exibem características semelhantes às da QCD, como a presença de mésons e a quebra de simetria quiral.

Palavras Chaves: Deformações; Limit BMN; Correspondência de Penrose; Transformações TsT; AdS/CFT; Holografia.

Áreas do conhecimento: Física; Física Matemática; Física de Altas Energias; Holografia.

Abstract

Since the breakthrough paper by Maldacena in the late 90's, the correspondence conjectured between a gravity theory and a field theory has grown so rapidly and so extensively that it has become a research area in its own right. The range of applicability of such a correspondence is now so broad that it spans from cosmological models, describing the evolution of the universe, to non-relativistic condensed matter systems, modeling strongly correlated materials.

Among all the known models useful for gravity/gauge duality, particularly interesting are those obtained by deforming existing models, mapping the results from the gravity side to the gauge side or vice-versa. Moreover, deformations serve as powerful tools to better understand how the duality manifests: typically generated by transformations on the supergravity side of the correspondence, these deformations are intrinsically related to energy flows, symmetries of the model, and integrability, and therefore provide results that help us understand how such properties appear in the theory.

In this dissertation, we approach this problem by studying different types of deformations, which in general are generated by similar techniques on the supergravity side, often making use of TsT transformations, though not exclusively. For example, we study the β, γ, $T\bar{T}$, η, λ and other deformations, explaining their properties, their conserved quantities, and its gauge/gravity interpretation.

We further make use of the BMN mapping/Penrose limit and generalize it to models other than AdS_5/CFT_4. This procedure consists of mapping a particular sector of the field theory, containing massive operators, to a boosted limit on the gravity side, resulting in a metric that is generally of the pp-wave type, though it can also be a generalization thereof. Moreover, we also investigate the noncommutative properties of this Penrose limit in the presence of deformations.

We investigated the noncommutative properties of the Penrose limit in relation to deformations generated by TsT transformations on the supergravity side. We also carried out a bootstrap-type analysis of Leigh–Strassler deformations, obtaining useful insights for future work. Finally, we studied how TsT deformations affect models involving branes and anti-branes that display QCD-like features, such as the presence of mesons and chiral symmetry breaking.

Keywords: Deformations; BMN limit; Penrose correspondence; TsT transformations; AdS/CFT; Holography.

Knowledge Area: Physics; Mathematical Physics; High-energy Physics; Holography.

Content

Chapter 1

Introduction and Motivation

It is hardly arguable that Maldacena's paper [107] marked a watershed in high-energy physics. Since its proposal, string theory has achieved a new level of applicability. Indeed, while the majority of models and calculations take place in the supergravity limit [1], the correspondence is conjectured and expected to hold for all limits of string theory. Therefore, the most prominent version of the duality is really a correspondence between string theory and a pure gauge field theory. This relation, however, was not something that just popped out of nowhere, and we can give a brief summary of the events that preceded the breakthrough paper.

The connection between string theory (or gravity, or even "geometry" at least) and pure theoretical field theory has manifested itself in different ways throughout the literature, though Maldacena's work was the first to make the connection precise.

It was well known that open strings allow for gauge freedom by making use of Chan-Paton factors [135], which can be understood as charges at the endpoints of the string. Later, Kawai, Lewellen, and Tye [91] showed how closed string amplitudes, which we can understand as graviton ($g_{\mu\nu}$) amplitudes at the tree level, are mapped to a scattering of open strings consisting of, roughly speaking, gauge bosons (A_μ). This relation has influenced the BCJ relations [23], which are nowadays some of the great results in the perspective of connecting gravity and gauge theory.

This was not the only "hint" towards the AdS/CFT conjecture. In fact, the "topological nature" emerging from the 't Hooft limit [159] in gauge theory, which consists of taking the number of colors N to infinity, $N \to \infty$, supressing the subleading $1/N$ term in the Fierz relation between the generators of $\mathrm{SU}(N)$ and keeping the 't Hooft coupling $\lambda = N g_{\mathrm{YM}}^2$ coupling constant, where the amplitude scattering can be organized as a sum over topological objects $\sum N^\chi \lambda^{E-V}$. After identifying the Euler Characteristic $\chi = V - P + L$ (P for propagators, V for

[1] $l_s^2 = \alpha' \to 0$

vertices, and L for loops) with the characteristic of a surface $\chi = V - E + F = 2 - 2g$, this shows to be a series expansion in $\sum_g N^{2-2g} f(\lambda)$. The similarity with string amplitudes is remarkable. There, we have the scattering amplitude in closed string sector to admits a topological expansion on the genus of the resulting surface, $\mathcal{A} \sim \sum_g g_s^{2g-2} \mathcal{A}_g$.

Moreover, the DBI action [135, 133, 59], describing the corresponding generalization of the Nambu-Goto action for D-branes, also provides another useful hint towards the conjecture. Assuming a $U(1)$ gauge group for the field strength F, the DBI action is

$$S = \mu_D \int d^{d+1}\xi e^{-\phi} \sqrt{\det(G + B + 2\pi\alpha' F)} + \dots, \tag{1.1}$$

Where we have the pullback of the metric G, together with the "NS-NS" field B (which is one of the fields resulting from the massless states in the closed string, being one of the specific irreducible representation of it), ϕ is the pure trace irreducible part of this representation. F is the field strength of the gauge group in the world-volume of the D-branes, which are objects of supergravity (and string theory) models, as properly explained by Polchinski in another breakthrough paper [133]. Assuming F as a perturbation, the first-order (in α') term appearing in the DBI[2] is the gauge-field kinetic term, $-\frac{1}{4}F^2$, which makes clear there is a field theory on the world-volume.

Following that line of reasoning, it is fair to say that the main contributions that influenced Maldacena's idea were those by Klebanov and collaborators, presented in a sequence of papers suggesting a relation between the supergravity solution for D3-branes and $\mathcal{N} = 4$ SYM. More precisely, in these papers, the absorption cross sections calculated on the supergravity side for the scattering of dilatons and gravitons in the D3-brane background were compared with the corresponding calculations in $\mathcal{N} = 4$ SYM. In other papers, the authors computed the entropy of near-extremal black D3-branes and compared it with the analogous calculation for non-BPS states in $\mathcal{N} = 4$, finding agreement [71], [72].

All of these "hints" towards a possible connection between string theory and field theory, and naturally supergravity and field theory, do not, however, overshadow how insightful the conjecture made by Maldacena was. Indeed, the correspondence, as the name suggests, goes way beyond simple resemblances and similarities between both sides of the model, or connections between the ampli-

[2] By making use of the relation $\det A = \exp(\operatorname{tr} \ln A)$.

tudes. It is, in the most crude and literal sense, a correspondence between any information one can have on either side of the duality. Just to elucidate how powerful the correspondence is, we describe here some of its remarks: The isometries of the supergravity background translate to R-symmetry (to be explained later) and global symmetries of the field theory dual, the fields in the supergravity (or states, in terms of strings) and the operators in the field theory show a true duality between them, Green's functions and amplitudes can be mapped in both sides of the model, and all physical properties one expects from the field theory side, like confinement, glueball, chiral symmetry breaking or spontaneous symmetry breaking, central charge flows (for CFT), integrability, etc., can be described from the supergravity side (and vice-versa). Indeed, regarding the observables for example, Witten [170] extended the conjecture, which summarizes the correspondence pretty well by the following mapping formula,

$$W_{\text{field}}[O_i] = i \ln Z_{\text{supergravity}}[\phi_i], \tag{1.2}$$

with ϕ_i the asymptotic value of the field ϕ at the conformal boundary of AdS, acting as a source for O_i. We are going to into more details later, but since the purpose of this dissertation is not to describe the whole applications of gauge/gravity duality, we cite here useful guides for future students in the area: the books [121, 4, 124], the review from Polchinski [134], and the last chapters from Freedman's supergravity book [61] and Becker, Becker and Schwarz [13], though there are much more literature on the subject.

While the correspondence is clear when evaluated in the supergravity limit, as explained before, we expect this to hold and be true in any regime of energy and coupling, and therefore string theory is really a subject here. Some of the first connections between truly string states and gauge theory were made by Berenstein, Maldacena, and Nastase [20], where they showed there is a clear map between the states, their energies, and angular momentum, with the corresponding operators with a scaling dimension Δ and R-symmetry J, if we make a clever limit, which goes by the name of the Penrose limit [130] on the supergravity side, while from the field theory we can think of it as a BMN limit, sometimes also called pp-wave correspondence. Moreover, the Hamiltonian on both sides shows a matching as well. We are going into details in Chapter 5, where we also show how it can be done generically for any other model of gauge/gravity, not only AdS_5/CFT_4. The BMN analysis played a crucial role in uncovering the integrable structures of the correspondence, paving the way for the identification of the planar dilatation operator of $\mathcal{N} = 4$ SYM with the Hamiltonian of an XXX Heisenberg spin chain.

Maldacena's correspondence is remarkable not only because of the duality itself, but by the objects of the duality. Indeed, from the field theory point of view the $\mathcal{N} = 4$ SYM theory has been the subject of extensive study over the years, attaining a prominent position among field theories. Its remarkable properties, such as integrability, conformal symmetry [153], the non-perturbative $\mathrm{SL}(2, \mathbb{Z})$ Montonen-Olive duality, and maximal supersymmetry, make it a cornerstone for exploring phenomena in less constrained theories, such as the gauge theory of QCD in the Standard Model [31, 105].

The integrability that the theory exhibits has been thoroughly investigated over the past two decades [16], and it is widely believed that $\mathcal{N} = 4$ SYM is exactly integrable, at least in the planar limit (for deviations from the planar limit, see [101]). This is further supported by the AdS/CFT conjecture [107, 170], which proposes that $\mathcal{N} = 4$ SYM is equivalent to type IIB string theory with $Q_{F_5} \sim N$ in $\mathrm{AdS}_5 \times \mathrm{S}^5$. Since it has been shown that strings in the limit $R\alpha' \to 0$ and $g \to 0$ on the homogeneous spacetime $\mathrm{AdS}_5 \times \mathrm{S}^5$ are classically integrable [19, 101], it is also expected that $\mathcal{N} = 4$ SYM is integrable in the corresponding limit, $\lambda = N g_y^2 \gg 1$, $N \to \infty$. Other examples supporting the success of integrability in $\mathcal{N} = 4$ SYM include scattering amplitudes computed using integrable methods, which correctly predict the four-loop deviations observed in [15, 24], and the BMN limit, which accurately reproduces the matching of anomalous dimensions of operators with the string spectrum via spin-chain methods [20]. More generally, there appears to be an intrinsic connection between integrable field theories and spin-chain models, which are often integrable via the Bethe Ansatz [54, 118, 17], including the Hubbard model, XXX Heisenberg, and XXZ chains.

These highly successful results in planar $\mathcal{N} = 4$ SYM naturally raise the question: could other models exhibit those properties? This question has led to a whole subfield devoted to studying deformations of $\mathcal{N} = 4$ SYM, aiming for models that preserve some of the original theory's, mainly integrability. In this direction, numerous deformations have been investigated [106, 63, 46, 148],[151], and particularly interesting are those marginal deformations cataloged by Leigh and Strassler in [103], which include the β deformation. More recently, a different class of deformations has attracted considerable attention, namely the $\mathrm{T\bar{T}}$ deformation and its generalizations, including single-trace $\mathrm{T\bar{T}}$-type deformations defined on product manifolds. Unlike the Leigh-Strassler deformations, these are irrelevant deformations and explicitly break conformal invariance, but they retain remarkable solvability properties.

On the gravity side, the β deformation admits a clear realization via TsT transformations[3] of the supergravity background [106, 63]. Related geometric constructions have also been explored for certain generalized $T\bar{T}$-type deformations [68], as discussed earlier.

[3]See Appendix B for details.

Chapter 2

Field Theory

2.1 Introduction to Quantum Field Theory Fundamentals

Before discussing the correspondence, its deformations, or other more advanced concepts, it is useful to first review the basic ingredients that constitute a quantum field theory. We therefore begin by introducing key features of classical and quantum field theories, and then focus on the classes of models most relevant for our purposes, namely theories that exhibit conformal symmetry and/or supersymmetry.

To describe any model or phenomenon in modern physics, we typically require a Lagrangian $\mathcal{L}$[1]. The Lagrangian is a function of the fields and, generally, at most of their first derivatives, where if "good enough" it contains at least a quadratic term together with interaction terms, allowing to perform Gaussian integrations. The action S is defined as

$$S = \int d^{d+1}x \; \mathcal{L}(\phi_i, \partial\phi_i), \tag{2.1}$$

and the equation of motion is obtained by extremizing it. Basic aspects concerning the nature of particles, understood as representations of the Poincaré group, as well as the connection between field theory and group theory, including gauge symmetries, representations, and internal symmetries, are discussed in detail in Appendix A. We therefore recommend that readers who are not sufficiently familiar with these concepts consult that appendix.

[1]Not considering non-conservative models, like a potential V depending on velocity $\dot{x}$, for example.

2.2 RG flow

While in classical physics the notion of a constant of nature is clear and well defined in all situations, even at arbitrarily high energies, this picture breaks down when one attempts to describe physical systems at the quantum level. In quantum field theory, coupling constants, particle masses, and even the scaling dimensions are no longer strictly constant. Instead, they depend on the energy scale of the physical process under consideration.

This idea, that physical parameters depend on the energy scale, was made precise by Wilson through what is currently known as the Wilsonian effective action. The basic idea consists of integrating out degrees of freedom that are irrelevant at the energy scale of interest, by introducing an ultraviolet cutoff Λ. As the cutoff varies, the objects of the theory must be adjusted such that physical observables remain unchanged [146].

One may also understand this phenomenon from the perturbative point of view, by computing Feynman loop diagrams, which generically produce ultraviolet divergences. To make sense of these expressions, one follows a renormalization procedure, introducing a regulator such as a cutoff or heavy regulator fields. After regularization, the dependence of the objects on the regulator are absorbed into redefinitions of the couplings, masses, and fields through a procedure known as renormalization.

Once a renormalization is imposed by fixing the values of the couplings at a given energy scale μ, an explicit energy scale is introduced into the theory. As a consequence, the couplings acquire a scale dependence, leading to a set of differential equations describing their evolution with μ, $\mu\frac{dg_i}{d\mu} = \beta_i(g_1, g_2, \ldots)$, and these equations define the renormalization group.

If an operator is such that its effect becomes suppressed at low energies, then it is called *irrelevant*, a name that is justified once we realize that most physical predictions concern large distance scales. Otherwise, it is called *relevant*. If its coupling does not run with the energy scale at the classical level, it is called *marginal*. In general, the way the terms appear in the Lagrangian is enough to determine their classification classically. This follows from the leading scaling behavior of the operator, which is governed by

$$\mu\frac{dO}{d\mu} \sim (d-\Delta)O, \tag{2.2}$$

where μ is the energy scale, d is the spacetime dimension, and Δ is the scaling dimension of the operator. Thus, if $d = \Delta$ the operator is classically marginal, if $d > \Delta$ it is relevant, and it is irrelevant otherwise.

The previous discussion, however, was mainly focused on fundamental particles, represented by operators that are not composite, such as $\hat{\phi}(x)$ or $\hat{\psi}(x)$. It should be noted, however, that the same concepts also apply to composite operators, which are constructed from fundamental fields and local operators. In a Yang Mills theory, such composite operators must be gauge invariant. As a simple example, the operator

$$O(x) = \phi(x)^2 \tag{2.3}$$

is a local composite operator built from the fundamental bosonic field $\phi(x)$.

The concept of composite operators is necessary to describe composite particles, which are formed by combining, for example, quark fields q and $\bar{q}$, as in baryons or mesons. We note, as a side remark, that this is not the only generalization of operators that appears in a Yang Mills theory. For instance, one of the main objects of study in lattice gauge theory is the Wilson loop operator

$$W = \operatorname{tr} \mathcal{P} \exp\left(\int_C A^\mu dx_\mu \right), \tag{2.4}$$

which provides an example of a non local composite operator. In any case, composite operators can sometimes behave as simply as fundamental operators from the point of view of renormalization. In such cases, the renormalized operator and the bare operator are related by [4, 131]

$$O_0 = Z \cdot O. \tag{2.5}$$

This relation, nevertheless, is not the most general situation. To understand the subtleties associated with composite operators, one must take into account the quantum numbers of the fundamental fields. For example, in four dimensions the fields ϕ, A_μ, and ψ have mass dimensions and spins given by $(1,0)$, $(1,1)$, and $(\frac{3}{2}, \frac{1}{2})$, respectively. As a consequence, the bare fields can only be renormalized multiplicatively,

$$\phi_0 = Z_\phi \phi, \qquad \psi_0 = Z_\psi \psi, \qquad A^\mu_0 = Z_A A^\mu, \tag{2.6}$$

since the renormalization Z carries no quantum numbers.

This is no longer true for composite operators. Suppose, for instance, that we have $O_1 = \phi^4(x)$ and $O_2 = (\partial_\mu \phi)(\partial^\mu \phi)(x)$. These operators have the same classical dimensio and spin, therefore both can appear in the renormalization equation for any of them, leading to operator mixing and promoting the renormalization factor Z to a matrix with multiple indices.

To be more concrete, let $\{O_I\}$ be a set of operators with the same quantum numbers. Their bare and renormalized operators can then mix through a renormalization matrix Z_{IJ} according to

$$O_0^I = Z^I{}_J O^J, \tag{2.7}$$

see [131] for further details. We note that, in some cases, a constant term may be added to (2.7), referred to as additive renormalization [4].

2.3 Differential Forms

From now on, we will often use differential form notation, meaning that we will make use of the wedge product $\wedge$ and the exterior derivative d. In summary, a p form is a field carrying p indices that are totally antisymmetric,

$$F = \frac{1}{p!} F_{\mu_1 \ldots \mu_p} \, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}. \tag{2.8}$$

The wedge product is antisymmetric, satisfying $dx^a \wedge dx^b = -dx^b \wedge dx^a$, and therefore acts on a p form together with a q form to produce a $(p+q)$ form, which can be expressed as a map $\wedge : V^p \otimes V^q \to V^{p+q}$.

The exterior derivative increases the rank of a p form by one, mapping V^p into V^{p+1}, satisfying the important property of $d^2 = 0$, which follows from the commutativity of ordinary partial derivatives ∂_μ, as we now show,

$$d^2 V_p \sim \partial_\mu \partial_\nu V_{p\ldots} \, dx^\mu \wedge dx^\nu \wedge \cdots = 0. \tag{2.9}$$

This property allows one to define cohomology classes of differential forms. In particular, a p form is said to be closed if $dF = 0$, and exact if it can be written as $F = dA$. Closed forms that are not exact define nontrivial cohomology classes.

Finally, though perhaps it is not appropriate to define it in the present moment, since we haven't defined what a metric is, we nevertheless bypass this caveat and define another operator here, called the **Hodge star** $\star$, consisting of mapping p forms to $d-p$ forms

$$\star : \Lambda^p \to \Lambda^{d-p}. \tag{2.10}$$

For a more detailed treatment of differential forms, a good reference is Freedman's supergravity book [61], or [158].

2.4 Supersymmetry

We can finally define what a supersymmetric model is. In this section, we intend to introduce the mathematical apparatus for someone not familiar with that type of calculation.

We highlight some important points. To understand supersymmetry, it is sufficient to be familiar with the mathematical tools of quantum field theory or Yang-Mills theory. One should keep in mind the statistical nature of a field, that is, whether it is fermionic or bosonic, what is a symmetry and how they are related to current conservation through Noether's theorem. It is important to keep in mind what is an algebra and representation of it. We have touched on some of these points in the previous section, and recommend the reader the Appendix B. We just mention one last important fact: Coleman and Mandula developed a theorem showing that one cannot have an internal symmetry Q_1 together with a spacetime symmetry Q_2, such as the Poincaré algebra, and still obtain a nontrivial algebra,

$$[Q_1, Q_2] \neq 0, \tag{2.11}$$

where $[\,,\,]$ denotes the commutator, defined to be antisymmetric. For more details, see the first chapter of Wess's book [167]. As a consequence, for example, the representation of a particle or field under the Lorentz group cannot be nontrivially mixed with an internal symmetry, like a flavor symmetry.

While the theorem is true, it is only as true as its postulates. If we allow for a mixed algebra containing both commutators, $[,]$, and anticommutators, $\{,\}$, then the field theory is no longer necessarily trivial, and we can proceed with a consistent study in that direction. Indeed, this is essentially the core of supersymmetry: it is a fermionic symmetry, governed by anticommutators $\{,\}$, that maps bosonic degrees of freedom into fermionic degrees of freedom, and therefore acts as a symmetry relating bosons and fermions. The number of supersymmetry generators Q is denoted by $\mathcal{N}$. The specific sets of fields that transform into one another are organized into multiplets, and once these are identified, it is even possible to write a Lagrangian that is manifestly supersymmetric by introducing fermionic spacetime coordinates θ.

Since we believe it is easier to understand complex concepts by examples, we study the simplest case, which already contains all the information discussed in the previous paragraph. If the reader wishes to pursue a more advanced and detailed treatment, a suitable first course with proper details can be found in [82].

The Wess-Zumino model in $d = 4$ has the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\int d^4x \left((\partial A)^2 + (\partial B)^2 + \bar{\psi}\partial\!\!\!/\psi \right), \tag{2.12}$$

where A and B are real scalar fields with $[A] = 1$, and ψ is a Majorana fermion with $[\psi] = 3/2$. The first quantity one should compute in a supersymmetric theory is the number of degrees of freedom, which must necessarily match on shell. As discussed in Appendix B, a scalar field has one degree of freedom, while a spinor in $d = 4$ has $2 \times 2^2 \times \frac{1}{2} = 4$ degrees of freedom off shell, and $4 \times \frac{1}{2} = 2$ degrees of freedom on shell. Therefore, two scalar fields together with one fermion indeed carry the same number of degrees of freedom on-shell.

We need to guess a global supersymmetry transformation for this theory, and since $\mathcal{N} = 1$ here, we expect a fermionic parameter ϵ. As we saw, $[\partial A] \sim 1$, $[\bar{\psi}\partial\!\!\!/\psi] \sim 3$, and $[A] = 1$, $[\psi] = 3/2$. Therefore, if $[\epsilon] = -1/2$ such that $[\bar{\epsilon}\psi] \sim 1$, we can write

$$\delta A = \bar{\epsilon}\psi, \quad \delta B = i\,\bar{\epsilon}\gamma_5\psi, \quad \delta\psi = \partial\!\!\!/(A + i\gamma_5 B)\,\epsilon. \tag{2.13}$$

The transformations in (2.13) are of the form $\delta B(F) = \epsilon F(B)$, as required by supersymmetry transformations. Moreover, (2.13) closes on shell, and therefore the model is properly defined. As explained in Appendix B, a symmetry transformation is generated by a charge, the J^0 component of a conserved current, which in the case of supersymmetry is a fermionic generator Q. We can obtain the super Poincare algebra, which is

$$\begin{aligned} &\text{Poincare algebra} \quad + \\ \{Q^i_\alpha, \bar{Q}_{\dot{\alpha}j}\} = -2\,(\sigma^\mu)_{\alpha\dot{\alpha}}\,P_\mu\,\delta^i{}_j, \quad &\{Q^i_\alpha, Q^j_\beta\} = 0, \quad \{\bar{Q}_{\dot{\alpha}i}, \bar{Q}_{\dot{\beta}j}\} = 0, \\ [M_{\mu\nu}, Q_\alpha] = (\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta, \quad &[M_{\mu\nu}, \bar{Q}_{\dot{\alpha}}] = (\bar{\sigma}_{\mu\nu})_{\dot{\alpha}}{}^{\dot{\beta}}\bar{Q}_{\dot{\beta}}. \end{aligned} \tag{2.14}$$

We notice, however, that off shell the degrees of freedom do not match. While this is not necessarily a contradiction, one generally expects a theory to have its symmetries closed and consistent even off shell. To overcome this problem, one introduces *auxiliary fields*, defined as fields which have no dynamics and vanish on shell. In the present case, we introduce two auxiliary real scalar fields H and G,

$$S_f = S + \frac{1}{2}\int d^4x\,(H^2 + G^2). \tag{2.15}$$

Then, we have $4 = 4$ degrees of freedom. By including $\delta H = \bar{\epsilon}\partial\!\!\!/\psi$ and $\delta G = \bar{\epsilon}\gamma_5\partial\!\!\!/\psi$ in the algebra (2.13), and at the same time adding the terms $(H + i\gamma_5 G)\epsilon$ to the right hand side of $\delta\psi$, the algebra closes off shell. Moreover, we can rewrite everything as

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2}\int d^4x\left(|\partial\phi|^2 + \bar{\psi}\partial\!\!\!/\psi + |F|^2\right), \\ \delta\phi &= \bar{\epsilon}\psi, \quad \delta\psi = (\partial\!\!\!/\phi + F)\epsilon, \quad \delta F = \bar{\epsilon}\partial\!\!\!/\psi, \end{aligned} \tag{2.16}$$

where we used the properties of γ_5, $\gamma_5^\dagger = \gamma_5$ and $\gamma_5^2 = 1$, and defined $\phi = A + iB$ and $F = H + iG$.

This multiplet of a complex scalar together with a Majorana fermion forms what is called a chiral multiplet in $\mathcal{N} = 1$. This allows us to define a chiral superfield

$$\Phi = \phi(x) + \sqrt{2}\,\theta\psi + F\,\theta^2, \tag{2.17}$$

and to write the action in a superspace formalism, as described previously,

$$S = \int d^4x\, d^4\theta\ \Phi^\dagger\Phi, \tag{2.18}$$

where θ is a fermionic coordinate of the so-called superspace. In that way, the action is manifestly super Poincare invariant.

Chapter 3

(Super) Gravity

While quantum mechanics takes place before special relativity or general relativity, the latter remained essentially untouched, while the former has been subject to a large modification, by extending it to many body systems, relativistic models and the last generalization to quantum field theory. This general idea was true till Peter van Nieuwenhuizen, Ferrara, Freedman, and others extended Einstein's ideas to include supersymmetry in the 70s and 80s.

In this chapter, we intend to present the concepts of General Relativity (GR) as formulated by Einstein, and then the extension required to include supersymmetry.

3.1 Fundamentals of General Relativity

Einstein's General Relativity, just like Special Relativity, holds in the idea that physical laws should not depend on a specific reference, being universal. This has strong implications, illustrated by his wonderful mental experiments, "Gedankenexperiments", where we have a majestic connection involving simple elevators and a flashlight, with implications for light trajectory deviation due to the gravity of planets and stars.

General Relativity, therefore, must be invariant under any change of coordinates (the mathematical equivalent of "different observers"),

$$V^{\mu}(x) \to \frac{\partial x^{\mu}}{\partial x'^{\nu}} V^{\nu}(x'). \tag{3.1}$$

The transformation $x \to x'$ in (3.1) goes by the name of diffeomorphism, which is an isomorphism (one-to-one and onto) between differential manifolds. The simplest action that can be obtained by requiring invariance under diffeomorphism is the Einstein-Hilbert action,

$$S = \int d^{d+1}x \sqrt{-g} R, \tag{3.2}$$

where R is called the Ricci scalar, being invariant under the change of coordinates. $\sqrt{-g}$ is the square root of the determinant of the metric. This last, the metric, is the most important tensor in the subject of General Relativity, since it precisely defines the length of the spacetime, and all events in it. Indeed, the metric can be seen as a map from two vectors to a scalar g, $V \otimes V \to \mathbb{R}$

$$g_{\mu\nu} dx^{\mu} dx^{\nu} = ds^2, \tag{3.3}$$

where dx are infinitesimal distances, and ds^2 is the length element. For example, flat spacetime is simply the identity $g = I$, while the sphere S^2 in (θ, ϕ) coordinates is $g = \mathrm{diag}(1, \sin^2\theta)$. One can see that ds^2 is a scalar invariant by a straightforward calculation,

$$g_{\mu\nu} dx^{\mu} dx^{\nu} \to g_{\mu'\nu'} \frac{\partial x'^{\mu'}}{\partial x^{\mu}} \frac{\partial x'^{\nu'}}{\partial x^{\nu}} \frac{\partial x^{\mu}}{\partial x'^{\eta'}} \frac{\partial x^{\nu}}{\partial x'^{\chi'}} dx'^{\eta'} dx'^{\chi'} = g_{\mu'\nu'} dx'^{\mu'} dx'^{\nu'}. \tag{3.4}$$

Furthermore, to define the Ricci scalar of above, one needs to define the Riemann curvature, which is a quantity that measures how curved a spacetime is. It is a 4-index tensor $R^{\mu}_{\nu\rho\sigma}$ with some specific properties of the indices,

$$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma}, \quad R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho}, \quad R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}, \tag{3.5}$$

together with two Bianchi identities,

$$R_{\mu[\nu\rho\sigma]} = 0, \quad \nabla_{[\alpha} R_{|\mu\nu|\rho\sigma]} = 0. \tag{3.6}$$

We can say that Einstein's general relativity is the simplest spacetimes possible. The reason for that will be made clear, but for the moment we "define" the most general spacetime as being the one that is "well behaved" at all points, and that is equipped with a connection $\Gamma^{\mu}_{\nu\rho}$, which allows us to compare objects at one point X in the manifold M with a nearby point $X' = X + \delta X$. Then, we can characterize manifolds by three properties [128],

$$\text{Metricity postulate } \nabla_{\sigma} g_{\mu\nu} = 0, \quad \text{Torsionless } \Gamma^{\mu}_{[\nu\rho]} = 0, \quad \text{Flat } R_{\mu\nu\rho\sigma} = 0. \tag{3.7}$$

Einstein's General Relativity is the simplest because it satisfies both the metricity postulate and the torsionless postulate, while it can be flat in some specific cases. In that case, the connection goes by the name of Christoffel symbols. With those simplifications, we can express both the Christoffel symbols and the Riemann tensor in terms of the metric. For a detailed definition and derivation, see [164, 36, 119], here we quote the results,

$$\begin{aligned} \Gamma^{\mu}_{\nu\rho} &= \frac{1}{2} g^{\mu\alpha_1} (\partial_\nu g_{\alpha_1 \rho} + \partial_\rho g_{\alpha_1 \mu} - \partial_{\alpha_1} g_{\nu\rho}), \\ R^{\mu}{}_{\nu\rho\sigma} &= \partial_\rho \Gamma^{\mu}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}_{\nu\rho} + \Gamma^{\mu}_{\lambda\rho}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\mu}_{\lambda\sigma}\Gamma^{\lambda}_{\nu\rho}. \end{aligned} \tag{3.8}$$

Due to the equivalence principle explained before, the model is formulated such that locally, around any generic point, the metric reduces to the Minkowski metric. That is, $g_{\mu\nu}(p) = \eta_{\mu\nu}$, $\partial g = 0$. Indeed, this defines a Riemannian manifold, which in mathematical words is defined as a "smooth topological space" equipped with a metric g and an atlas of charts, such that around any point p in a small open set U, there exists a map ϕ diffeomorphic to $\mathbb{R}^d$, $\phi(p), U \to \phi(U) \subset \mathbb{R}^d$. A common example is the sphere S^2, where we can divide it into two patches, one containing only the north pole while the other contains only the south pole, and each patch has a straightforward chart to a plane, as can be seen by a stereographic projection, for example.

We stress, however, that Einstein-Hilbert action is in some sense incomplete. In fact, we can add a scalar term Λ, which goes by the name of cosmological constant, to the action and still remain invariant. We could also add more terms like

$$R^2, \quad R_{\mu\nu}R^{\mu\nu}, \quad R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, \dots, \tag{3.9}$$

to the action, though from the experimental point of view these terms do not contribute a value large enough to be perceptible in the laboratory. Moreover, (3.8) shows that the Riemann tensor is of second-order derivative of the metric $R \sim \partial^2 g$, and consequently, all the terms in (3.9) have higher-order derivatives. These types of extensions have been made in what is called "quadratic gravity" [145] and though it solves some problems of usual Einstein gravity, like renormalizability, it introduces new gravitons that are massive. Hořava gravity is an alternative approach to this problem, by foliating the spacetime with spacelike hypersurfaces at each specific time t, more details can be found in [83], with a review in [80].

One may ask why we are stressing this information. The reason we follow that direction is to touch the subject of renormalization of gravity since, in the end of the day, the most well motivated reason on why people are seeking for models beyond the Standard Model is that "quantum gravity", in the most crude sense of

Yang-Mill theory, of the Einstein-Hilbert action is non-renormalizable.

As explained in Section 2.2, the terms inside the Lagrangian of the gauge model can be characterized by being marginal, irrelevant, or relevant. For gravity, we have

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} R, \quad \Rightarrow \quad [\kappa] = -2. \tag{3.10}$$

After expanding R in terms of g and properly normalizing the kinetic term, the fact that the coupling $\kappa = \frac{8\pi G}{c^4}$ has a negative dimension[1] means the interactions regarding that coupling are irrelevant. This means, among other things, an uncontrollable increase of such couplings in the RG flow, which flows by definition from IR to UV, as properly explained by making use of the Wilsonian effective action (see [131]). Consequently, gravity is unrenormalizable, and the question of how to connect the "checked everyday by astronomers and cosmologists" theory of General Relativity and the "theory with an incredible precision in comparison with experiments"[2], Standard Model, has triggered a lot of research. String theory and supergravity, or more phenomenological models like $SU(5)$ or $SO(10)$ Grand Unified Theories (GUTs) [64], were all driven by that question.

3.2 The Vielbein Formalism and Local Lorentz Symmetry

We take one last section before we start to talk about Supergravity itself, to introduce the necessary language that is used there, which consists of using vielbeins and spin connection.

The vielbeins are 1-forms $e^a_\mu(x)$ which take one index μ in the curved spacetime $g_{\mu\nu}$, while the other index a is over the locally Minkowski spacetime. We stress one important possible misconception one can have: General Relativity *is not* invariant under Lorentz/Poincaré symmetry, which can be seen as the isometry group (that is, transformations in the spacetime coordinates that leave the metric invariant) of the Minkowski spacetime, but instead it is invariant under the diffeomorphism described earlier in (3.1). Therefore, the definition of particles as properly explained in Appendix A as transformations under the Lorentz group, is nonsense if we think globally in terms of general relativity.

[1] As always, $[m] = 1$.

[2] See, for example, the nowadays known as "muon g-2 experiment" performed at Fermilab, where the precision between the experimental value obtained for the anomalous dipole magnetic moment of the muon matched with the theoretical value of 0.24 parts-per-billion (ppb) [93].

One can, nevertheless, follow the insights realized by the creators of supergravity, where essentially the idea is to look at tiny regions around points where we know the metric is locally Minkowski, where the diffeomorphism reduces to the Lorentz transformations. With this in mind, we return to the vielbein concept and define it as a map from objects in the curved space at a point p to objects in the flat local region at that point p,

$$V^\mu \to V^a = e^a_{\ \mu} V^\mu. \tag{3.11}$$

More specifically, since the vielbein takes an index on curved and flat space, it serves as a map from the curved metric g to the flat η,

$$g_{\mu\nu} = e^a_\mu \eta_{ab} e^b_\nu. \tag{3.12}$$

Therefore, substituting (3.12) into (3.3) gives,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = e^a_\mu e^b_\nu dx^\mu dx^\nu \eta_{ab} = e^a \eta_{ab} e^b. \tag{3.13}$$

See [163, 122]. So, for example, in diagonal metrics which are the simplest ones, we have for S^2,

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2, \tag{3.14}$$

and the vielbeins are $e^1 = d\theta, e^2 = \sin\theta d\phi$, satisfying,

$$ds^2 = (e^1)^2 + (e^2)^2 = e^a e^b \delta_{ab}, \tag{3.15}$$

where δ_{ab} is the Euclidean/flat metric.

The vielbein, as an object itself, is quite interesting due to its structure involving indices of flat and curved spacetime. From the Lorentz point of view, it is a vector, and so infinitesimally $\delta_L e^a_\mu = \omega^a_b e^b_\mu$. For the curved spacetime, under the Lie derivative [3]) it transforms as

$$\delta_\xi e^a_\mu = \xi^p \partial_p e^a_\mu + (\partial_\mu \xi^p) e^a_p. \tag{3.16}$$

The similarities with the gauge group analysis of Appendix A will start to become clear, since the index a can be seen as a "local" index, though some caution needs to

[3]The essence of the Lie derivative is to compare a transformed object with the untransformed one, but at the same point of the spacetime. That is, we first transform the object, $O(x) \to O'(x')$, and then we move back to x, $O'(x') \to O'(x)$, and compare $O'(x)$ with $O(x)$.

be taken if we assume the analogy seriously. First, we remember that a Dirac spinor transforms as $\psi \to \Lambda_{1/2}\psi$, and therefore, following the rule $\partial \to D = \partial - \delta(A_\mu)$, we obtain that for the spinor ψ,

$$D_\mu \psi = \partial_\mu \psi - \omega_\mu^{ab} \frac{1}{4} \Sigma_{ab} \psi. \tag{3.17}$$

The object ω, defined exactly by the above expression, can be seen as the gauge field of the Lorentz symmetry. For the vector and scalars the rule is the same, but specifically for vectors the idea is to replace the parameter by the gauge field, and keep the generator. So, we know that

$$\delta \xi^a = \omega^a_{\ b} \xi^b = \frac{1}{2} \omega_{bc} (M^{bc})^a_{\ d} \xi^d, \tag{3.18}$$

where M is obtained by verifying that it satisfies the Lorentz algebra,

$$[M^{ab}, M^{cd}] = \eta^{ad} M^{bc} - \eta^{ac} M^{bd} - \eta^{bd} M^{ac} + \eta^{bc} M^{ad}, \tag{3.19}$$

which is obtained by imposing invariance of the Minkowski algebra under the $SO(3,1)$ transformations Λ. Then, we obtain that $(M^{ab})_{cd} = \eta_c{}^a \delta_d{}^b - \eta_d{}^a \delta_c{}^b$ as the generators of Lorentz symmetry for vector representation [131]. Finally, substituting this M in (3.18) together with the rule "parameter $\to$ gauge field", we obtain,

$$D_\mu \xi^a = \partial_\mu \xi^a - \omega_\mu^{ab} \xi^b. \tag{3.20}$$

The observation that ω really seems to play the role of the gauge field in the local Lorentz transformations has some caveats however (see Chapter 4 of [121] or [128]). This is because if we impose torsionless conditions, in the sense of Einstein-Cartan theory, where we have two equations, $\mathcal{R} = d\omega + \omega \wedge \omega$ and $T = de + \omega \wedge e$ for the curvature and the torsion, respectively, then $T = 0$ relates ω to e ($\omega = \omega(e)$). Therefore, it is not really a fundamental gauge field as in Yang-Mills, since it is defined by the vielbein. However, there is some connection once we allow torsion, as in the Palatini formalism (or first-order formalism), in which the "torsionless condition" is now understood as the equation of motion for ω, since it arises from extremizing the action, and is not imposed *a priori*.

3.3 From Gravity to Supergravity

Finally, once the appropriate language to describe supergravity has been introduced, we extend the discussion of the previous section to include fields with spin higher than one, namely the massless spin 3/2 Rarita–Schwinger field, or gravitino, and the spin 2 field, the graviton. Since the gravitino is more involved from the point of view of the gauge structure, we begin with the graviton.

First of all, it is useful to emphasize some general results obtained in the context of Yang–Mills theory. Weinberg and Witten[4] [166] showed that massless particles with spin greater than one cannot arise as composite states, nor can they possess a conserved, local, and gauge invariant stress energy tensor $T_{\mu\nu}$.

This result essentially rules out the existence of gravitons within an ordinary flat Minkowski spacetime quantum field theory. We can understand that by making an analogy: the classical electromagnetic field A_μ can be understood as a coherent state of a large number of photons, the $U(1)$ gauge bosons. One is therefore led to expect that the classical gravitational field $g_{\mu\nu}$ should similarly emerge as a coherent excitation associated with an underlying field with spin $s = 2$. From a group theoretical perspective, one may realize this $s = 2$ spin by considering the Lorentz algebra $\mathrm{so}(3,1)$, which decomposes as $\mathrm{su}(2) \oplus \mathrm{su}(2)$. Since the indices μ and ν of the metric transform independently, it can be understood as a tensor product of two vector representations, which can be decomposed as irreducible representations,

$$d \otimes d = \big(d \otimes_S d - 1\big) \oplus \big(d \otimes_A d\big) \oplus 1, \tag{3.21}$$

where d denotes the dimension of the spacetime dimension. The first term corresponds to the symmetric traceless representation, with dimension $\frac{d(d+1)}{2} - 1$, the second to the antisymmetric representation, with dimension $\frac{d(d-1)}{2}$, and the last term to the pure trace. The graviton is identified with the symmetric traceless representation, which can be shown, using Lorentz transformations, to describe a particle of spin $s = 2$.

Then, it may appear that Weinberg–Witten theorem renders the graviton description inconsistent, since the graviton does not admit a conserved, gauge invariant current. However, as with any no go theorem, its conclusions are limited by its assumptions. Supergravity evades this theorem by violating its initial hypotheses, most importantly the assumption of a globally flat background, since

[4] And also with the collaboration of Sidney Coleman, although he is not listed as an author of the paper.

supergravity necessarily involves a dynamical spacetime geometry. Moreover, even within General Relativity, the notion of an energy momentum tensor is more subtle than in flat space field theory. Gravity itself should contribute to the total energy momentum, but this contribution can only be defined in a precise way under special conditions, such as asymptotically flat spacetime within the ADM formalism [40]. In more general situations, a globally well defined local stress energy tensor does not exist [168].

The idea of Supergravity is [163] a generalization of supersymmetry that includes gravity. Supergravity, in the vielbein formalism explained previously, possesses a local Lorentz symmetry, and we also expect it to exhibit a local supersymmetry. This expectation is motivated by the general conjecture that consistent theories of gravity do not admit global symmetries, a statement that is widely believed to be true, although a fully general proof is still lacking. This feature appears repeatedly in holography, where a local symmetry in a supergravity theory is mapped to a global symmetry in the dual field theory, but never the other way around. Consequently, supersymmetry transformations in supergravity are parametrized by spacetime dependent fields $\epsilon(x)$, or more precisely spinors, since susy is (BOSE $\leftrightarrow$ FERMI) symmetry.

Being a local symmetry, which field plays the role of the gauge field for supersymmetry? To clarify this point, let us briefly recall the simplest $\mathrm{U}(1)$ example. Schematically, one has

$$\mathcal{L} = |\partial\phi|^2 - V(|\phi|^2). \tag{3.22}$$

Upon gauging the symmetry $\phi \to e^{i\alpha}\phi$, the derivative transforms as $\partial\phi \to e^{i\alpha(x)}\left(\partial\phi + i(\partial\alpha)\phi\right)$. Defining the covariant derivative $D = \partial + ieA$, and imposing the transformation rule $A \to A - \frac{1}{e}\partial\alpha$, the Lagrangian becomes locally $\mathrm{U}(1)$ invariant,

$$\mathcal{L} = |D\phi|^2 - V(|\phi|^2). \tag{3.23}$$

One then adds the field strength term to give A its own dynamics, and so on. The important point for our purposes is that the gauge field transforms as $\delta A \sim \partial\alpha$[5].

Translating this analogy to supersymmetry, where the parameter α is replaced by a spinor $\epsilon(x)$, the corresponding gauge field ψ must transform roughly as $\delta\psi_\mu \sim \partial_\mu\epsilon(x)$. This immediately implies that ψ carries both a spinor index and a vector index. In group language, assuming for simplicity that ϵ is a left handed

[5]For the abelian toy-model in question, since non-abelian groups contains $[A, \alpha]$ terms in the right-hand side of δA.

spinor, one has

$$(1/2, 1/2) \otimes (1/2, 0) = (1, 1/2) \oplus (0, 1/2) = \psi_\mu. \tag{3.24}$$

Equation (3.24) expresses the fact that the vector ∂_μ acting on the spinor ϵ produces a spin $1 + 1/2 = 3/2$ field together with an additional pure spin $1/2$ component. To eliminate this unwanted spinor part, one contracts ψ_μ with the Clifford generator γ^μ, which removes all vector information in the representation (by contracting with ψ_μ). This corresponds precisely to the $(0, 1/2)$ representation, and imposing the constraint $\gamma^\mu \psi_\mu = 0$ ensures that this component vanishes.

Therefore, the gravitino corresponds to

$$\text{Gravitino} \quad = \quad (1/2, 1) \oplus (1, 1/2) \quad \text{of } \mathrm{su}(2) \oplus \mathrm{su}(2), \tag{3.25}$$

and the graviton,

$$\text{Graviton} \quad = \quad (1/2, 1/2) \otimes_S (1/2, 1/2) \; - \; \text{trace} \quad \text{of } \mathrm{su}(2) \oplus \mathrm{su}(2). \tag{3.26}$$

We have carried out the analysis in $d = 4$, but it can of course be generalized to higher dimensions. However, the simplification of representing spins in terms of su factors is not necessarily always possible. This can already be anticipated by comparing the dimension of $\mathrm{so}(d-1, 1)$, namely $\frac{d(d-1)}{2}$, with that of $\mathrm{su}(N)$, which is $N^2 - 1$, which are equal only for $2 \times N = 2$ and $d = 4$.

Finally, we have all tools to describe a supergravity model, stressing how some of them are remarkably simple. In fact, writing a Lagrangian that is invariant under local supersymmetry is not necessarily the hardest part, but rather that the hardest part is to verify whether the supersymmetry algebra itself is closed or not [122].

Among all possible supergravity models, those that stand out for their importance in the historical development of supergravity are

- The original $d = 4$, $\mathcal{N} = 1$ theory, introduced in the seminal paper by the creators of supergravity [60].
- The $d = 11$, $\mathcal{N} = 1$ theory [41], which is the only possible supersymmetry in that dimension. This strongly constrains the allowed supergravity models, since one cannot have $\mathcal{N} > 1$ in eleven dimensions, as this would lead, upon compactification, to multiplets containing fields with spin higher than 2,

see [61]. Moreover, this theory is of central importance from the perspective of string theory, since it can be understood, roughly speaking, as the low energy limit of Witten's M theory [169]. Witten established a deep connection between this eleven dimensional theory and the five consistent string theories through dualities and compactifications, see [13] for details.

- Finally, the $d = 10, \mathcal{N} = 1$ theory [32], which is closely related to type I and heterotic string theories.

There are many other important models, however, such as the $d = 10, \mathcal{N} = 2$ theories, and we will encounter several additional supergravity models in the holographic setups discussed later. Most of these will be treated from a classical point of view, where one typically has $\langle \text{fermions} \rangle = 0$, making the calculations technically simpler. We also emphasize that not all supergravity models necessarily arise as low energy limits of string theory or can be obtained via consistent truncations. Those that do not belong to this class fall into what is known as the swampland of supergravity theories, namely models that do not admit a consistent ultraviolet completion within string theory and therefore lie outside the string theory landscape [161].

3.3.1 The 11D Supergravity Model

The $d = 11, \mathcal{N} = 1$ model of Cremmer, Julia, and Scherk [41] has a remarkably simple form, with a supersymmetry algebra that closes, and therefore we discuss it here as a useful pedagogical example. We naturally have a graviton, and consequently a gravitino, almost by definition, as some of the fields in a multiplet in this dimension.

Before mentioning the other fields, it is useful to step back and count the degrees of freedom. In fact, supergravity is supersymmetry with gravity, and if supersymmetry is to be taken seriously, it must at least satisfy its most general principle, namely a mapping between bosons and fermions.

To simplify, we can think of the graviton as a symmetric matrix, but with d diffeomorphisms as a gauge symmetry that we can exploit, and therefore off shell we have $\frac{d(d+1)}{2}$ components. Moreover, the equations of motion can be obtained by linearizing Einstein's equations for a perturbation around the flat metric η, writing $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, which leads to a free wave equation $\Box \tilde{h}_{\mu\nu} = 0$, with $\tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{h}{2}\eta_{\mu\nu}$. Due to the structure of $h_{\mu\nu}$, even after imposing a gauge condition such as $\partial_\mu \tilde{h}^{\mu\nu} = 0$, we can still shift $h_{\mu\nu} \to h_{\mu\nu} - \partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu$ without changing the equations of motion, provided the parameters ϵ_μ satisfy $\Box\epsilon_\mu = 0$. These are d additional conditions that can be used to remove further freedom. We are then left with $\frac{d(d-3)}{2}$ on shell,

$$\text{D.O.F of } e_\mu^a \quad = \quad \frac{d(d-3)}{2}. \tag{3.27}$$

For the gravitino, the analysis is simpler. As can be seen from its representation in (3.25), it is a spinor valued vector, minus the degrees of freedom corresponding to one spinor due to the constraint $\gamma^\mu \psi_\mu = 0$. The number of degrees of freedom is therefore

$$\text{D.O.F of } \psi^\mu \quad = \quad \frac{d(d-3)}{2} 2^{[\frac{d}{2}]}. \tag{3.28}$$

In $d = 11$, there is a mismatch between these two contributions, leaving 84 excess bosonic degrees of freedom. In higher dimensions, however, we are allowed to include additional p-form fields, generalizing the familiar two forms of electromagnetism in $d = 4$. By definition, a field strength of degree $p+2$ is given by

$$F_{p+2} = dA_{p+1}, \tag{3.29}$$

where A_{p+1} couples naturally to a brane with p spatial dimensions. This can be understood by noting that a p brane sweeps out a $(p+1)$ dimensional spacetime volume, and the corresponding coupling term takes the form $A \wedge \star J$, with $\star J$ a $(d-p)$ form.

We have mentioned this fact because, as we are going to show, 84 is precisely the number of degrees of freedom of a three form in $d = 11$. A three form, by definition, is an antisymmetric tensor A_{ijk}, and therefore it has $\binom{11}{3}$ independent components. Furthermore, it possesses a gauge symmetry that leaves F_4 invariant, namely $A_3 \to A_3 + d\Lambda_2$, which removes $\binom{11}{2}$ components. However, part of this freedom is not completely fixed, since $\Lambda_2 \to \Lambda_2 + d\Lambda_1$, so we must add back $\binom{11}{1}$ components. The same logic continues once more, adding $\binom{11}{0}$. Off shell, the total number of degrees of freedom is therefore

$$\binom{11}{3} - \binom{11}{2} + \binom{11}{1} - \binom{11}{0} = \binom{d-1}{p} = \binom{10}{3}. \tag{3.30}$$

On shell, the equations of motion impose an additional constraint of $\binom{d-2}{p-1}$ components, leading to a total number of physical degrees of freedom

$$\binom{d-1}{p} - \binom{d-2}{p-1} = \binom{d-1}{p-1}, \tag{3.31}$$

or, equivalently, $\binom{d-2}{p}$. Substituting $d = 11$ and $p = 3$, we obtain precisely 84.

Then, it is possible to demonstrate that the $\mathcal{N} = 1$ supergravity model in $d = 11$ is given by [122, 61, 41]

$$\begin{gathered} L = -e\frac{1}{2k^2}R(e,\omega) - \frac{e}{2}\bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\rho(\omega+\hat{\omega})\psi_\nu - \frac{e}{48}F_{\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma} \\ -\frac{\sqrt{2}\kappa}{384}e\left(\bar{\psi}_\mu\gamma^{\mu\alpha\beta\gamma\delta\nu}\psi_\nu F_{\alpha\beta\gamma\delta} + 12\bar{\psi}^\alpha\gamma^{\beta\gamma}\psi^\delta F_{\alpha\beta\gamma\delta}\right) + \frac{\sqrt{2}\kappa}{3456}\epsilon^{\mu_1\ldots\mu_{11}}F_{\mu_1\ldots\mu_4}F_{\mu_5\ldots\mu_8}A_{\mu_9\mu_{10}\mu_{11}}. \end{gathered} \tag{3.32}$$

The corresponding supersymmetry transformations are

$$\begin{aligned} \delta e^a_\mu &= \frac{k}{2}\bar{\epsilon}\gamma^a\psi_\mu, \\ \delta\psi_\mu &= \frac{1}{k}D_\mu(\hat{\omega})\epsilon + \frac{\sqrt{2}}{288}\left(\gamma_\mu{}^{\alpha\beta\gamma\delta} - 8\delta^\alpha_\mu\gamma^{\beta\gamma\delta}\right)\epsilon\hat{F}_{\alpha\beta\gamma\delta}, \\ \delta A_{\mu\nu\rho} &= -\frac{\sqrt{2}}{8}\bar{\epsilon}\gamma_{\mu\nu}\psi_\rho. \end{aligned} \tag{3.33}$$

Note that the numerical coefficients denoted by D, E, and C in the original literature are fixed by supersymmetry invariance.

Chapter 4

AdS/CFT

4.1 (A)dS

The previous two chapters presented the basic ideas one should know in order to understand Maldacena's conjecture [107]. Once these ideas are firmly kept in mind, understanding the correspondence becomes easier. The last two important concepts to keep in mind are specific examples of spacetime and a field theory, namely Anti de-Sitter spacetime and $\mathcal{N} = 4$.

As explained in Chapter 3, beyond the Einstein–Hilbert term, which is invariant under diffeomorphisms, $\sim \sqrt{|g|}R$, one can add a simple term to the action that is essentially a constant times the volume, $\int -\Lambda\sqrt{|g|}\, d^{d+1}x$. We obtain the following action,

$$S = \int d^{d+1}x\sqrt{-g}\,(R - 2\Lambda)\,. \tag{4.1}$$

Out of curiosity, Einstein was already aware of this term and even included it in his model for a static spacetime solution, which required Λ to be positive, together with a matter content in the action, characterized by a pressure p and a matter density ρ. Although this model contained a positive and constant radius and curvature, it was later shown by Eddington that the solution is unstable [152]. This construction is now known as the Einstein static universe.

Later, a similar model was proposed by de-Sitter, but this time without matter, so the action is simply (4.1). It is a maximally symmetric model, meaning that it possesses the maximal possible number of Killing vectors[1], which in d dimensions is $\frac{d(d+1)}{2}$ [164], [36]. This is easy to understand once we realize that the most symmetric possible space, the flat manifold, possesses this amount of symmetry: d translations plus $\frac{d(d-1)}{2}$ rotations. This de-Sitter[2] model has the simple equation

[1] Killing vectors are the same as isometry generators. For example, in a simple metric such as $ds^2 = dx^2 - dy^2$, we could have Killing vectors $\xi_{x,y} = \partial_{x,y}$ and $\chi = y\partial_x + x\partial_y$, etc.

[2] who was a Dutch physicist

of motion

$$G_{\mu\nu} = -\Lambda g_{\mu\nu}, \tag{4.2}$$

where $G_{\mu\nu} \sim \int d^{d+1}x \frac{\delta}{\delta g^{\mu\nu}}(R\sqrt{-g})$. It is, moreover, stable and exhibits an exponential expansion

$$ds^2 = -dt^2 + e^{2Ht}(dx^2 + dy^2 + dz^2), \tag{4.3}$$

where H is the Hubble constant [165]. This constant characterizes the evolution of the scale factor $a(t)$ through the relation $H = \frac{\dot{a}}{a}$. This scale factor is a standard quantity defined in the standard metric of cosmology, the FLRW metric, which describes how the universe evolves in time, $ds^2 = -dt^2 + a^2(t)\gamma_{ij}dx^i dx^j$. It is no coincidence that the de-Sitter metric is naturally a type of FLRW metric, with H related to the cosmological constant by $H = \sqrt{\frac{\Lambda}{3}}$.

Moreover, we mentioned that the spacetime is maximally symmetric, but the form presented in (4.3) does not make this information immediately clear. Indeed, maximally symmetric spacetimes admit several equivalent definitions and possess interesting properties, and here we mention one of them.

A maximally spacetime admits that the Riemann tensor with all indices lowered can be expressed purely in terms of the Ricci scalar R and the metric,

$$R_{\mu\nu\rho\sigma} = \frac{R}{d(d-1)}\left(g_{\nu\sigma}g_{\mu\rho} - g_{\nu\rho}g_{\mu\sigma}\right). \tag{4.4}$$

Another property relates maximally symmetric spacetimes to coset spaces. As defined in [76], a manifold M is a homogeneous space if a group G acts transitively on it. If it is also maximally symmetric, it is homogeneous and isomorphic to a coset space, defined as G_1/G_2,

$$G_1/G_2 \;=\; \{\, g \in G_1 \mid g' \sim g \iff \exists\, g_2 \in G_2 \text{ such that } g' = g_2 g \,\}. \tag{4.5}$$

In particular,

$$\begin{aligned} \text{Minkowski} \;&\cong\; ISO(1,3)/SO(1,3), \\ \text{de-Sitter} \;&\cong\; SO(1,4)/SO(1,3), \\ \text{anti-de-Sitter} \;&\cong\; SO(2,3)/SO(1,3). \end{aligned} \tag{4.6}$$

These correspond to $\Lambda = 0, +1, -1$, respectively. The isometry content becomes particularly clear if the spacetime is embedded in a higher dimensional flat space, which can indeed be done. For example, de-Sitter spacetime can be embedded as a hyperboloid with one negative coordinate,

$$\mathrm{dS}_n = \left\{ X \in \mathbb{R}^{1,n} \,\middle|\, -(X^0)^2 + (X^1)^2 + \cdots + (X^n)^2 = \alpha^2 \right\}. \tag{4.7}$$

Equation (4.7) makes explicit the isometry content of (4.6) and, consequently, the corresponding number of Killing vectors. Consequently, de-Sitter spacetime satisfies all of them.

Finally, we can discuss Anti de-Sitter (AdS) spacetime. Since de-Sitter (dS) spacetime was properly introduced, it is straightforward to describe AdS. We can think of it simply as an analytic continuation of de-Sitter spacetime, (4.8),

$$\alpha^2 \to -\alpha^2, \quad X^1 \to iX^1. \tag{4.8}$$

Thus,

$$\mathrm{AdS}_n = \left\{ X \in \mathbb{R}^{1,n} \,\middle|\, -(X^0)^2 - (X^1)^2 + \cdots + (X^n)^2 = -\alpha^2 \right\}. \tag{4.9}$$

The analytic continuation shown in (4.8) has proven useful in understanding extensions of the AdS/CFT correspondence to the positive curvature case, often referred to as dS/CFT [108], [157].

Anti de-Sitter spacetime, however, presents intrinsic differences from de-Sitter spacetime when one considers its boundary and horizon structure. To illustrate this, we can use static patch coordinates to describe both spacetimes. One finds

$$\begin{aligned} ds^2_{\mathrm{dS}} &= -\left(1 - H^2 r^2\right) dt^2 + \frac{dr^2}{1 - H^2 r^2} + r^2 d\Omega^2_{d-1}, \\ ds^2_{\mathrm{AdS}} &= -\left(1 + \frac{r^2}{L^2}\right) dt^2 + \frac{dr^2}{1 + \frac{r^2}{L^2}} + r^2 d\Omega^2_{d-1}. \end{aligned} \tag{4.10}$$

We immediately observe a qualitative difference. De Sitter spacetime naturally possesses a cosmological horizon at $r = 1/H$, whereas Anti de-Sitter spacetime can be extended from $r = 0$ to infinity. The the horizon at $r = 1/H$ in the de-Sitter spacetime can be used to define an entropy and a temperature, derived from the Hawking Bekenstein prescription [78], extended to cosmological horizons [65],

$$S = \frac{A}{4G}, \qquad T = \kappa, \tag{4.11}$$

where S is the entropy, A is the area of the horizon of the cosmological horizon, κ is the surface gravity, defined for asymptotically flat spacetime as $\kappa k^\mu = k^\nu \nabla_\nu k^\mu$ [4], and T is the temperature. This is in contrast with Anti de-Sitter, where temperature is not defined unless black hole is inserted on it or we compactify the time direction [79].

To understand the asymptotic behavior of both spacetimes, we use global coordinates, which can be obtained by solving the embedding constraints in (4.9) and (4.7),

$$\begin{aligned} ds^2_{\text{dS}} &= -d\tau^2 + H^{-2}\cosh^2(H\tau)\, d\Omega_d^2, \\ ds^2_{\text{AdS}} &= L^2\left[-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\Omega^2_{d-1}\right]. \end{aligned} \tag{4.12}$$

We first notice that spatial slices ($d\tau = 0$) are spherical for de-Sitter, while they are hyperbolic for Anti de-Sitter. To analyze the boundary behavior of (4.12), we take the limits $\tau \to \infty$ and $\rho \to \infty$, and then introduce the coordinate changes $\eta = e^{H\tau}$ and $u = \frac{e^\rho}{2}$. This yields the asymptotic forms

$$\begin{aligned} ds^2_{\text{dS}} &\to -\frac{1}{H^2}\frac{d\eta^2}{\eta^2} + \frac{\eta^2}{H^2} d\Omega_d^2, \\ ds^2_{\text{AdS}} &\to L^2\left[-u^2dt^2 + \frac{du^2}{u^2} + u^2 d\Omega^2_{d-1}\right]. \end{aligned} \tag{4.13}$$

Neglecting subleading terms and the overall factor of L^2 (with $L^2_{\text{dS}} = 1/H^2$), we obtain

$$\begin{aligned} ds^2_{\text{dS}} &\to \eta^2 d\Omega_d^2, \\ ds^2_{\text{AdS}} &\to u^2\left[-dt^2 + d\Omega^2_{d-1}\right]. \end{aligned} \tag{4.14}$$

Equation (4.14) implies the following. The boundary of de-Sitter spacetime is spacelike, with the isometry of a sphere S_d, since the induced metric has positive signature $(++\cdots+)$. This is one of the main difficulties of the dS/CFT conjecture: if a quantum field theory is supposed to live on the boundary, the absence of a timelike direction implies the absence of time evolution and, consequently, of a Hamiltonian. This issue does not arise for Anti de-Sitter spacetime, since its boundary is conformal to $\mathbb{R} \times S_{d-1}$, and ∂_t generates time translations, allowing a quantum field theory to be defined on the boundary[3].

[3] From my understanding, this does not mean that dS/CFT is impossible, but rather that the construction is more subtle than in the AdS case, which naturally satisfies the required properties.

4.2 CFT

Once the groundwork for supergravity has been prepared, we can prepare the field theory side by discussing what a Conformal Field Theory (CFT) is. This subject is not restricted only to the AdS/CFT conjecture, although it has certainly been advanced to another level since this connection was established. Indeed, there are many models known to be conformal. The most famous example is the $d = 2$ Ising model, one of the most important models ever proposed in physics for describing ferromagnetism, which is also known to be integrable [28]. The most well known reference on CFT, sometimes referred to as the "big yellow book" (in analogy with Wald's "big black book" in gravity [164], or Green, Schwarz, and Witten's "big green book" in string theory), is Di Francesco [49], but we also recommend [28] for a more concise approach.

The introduction to CFT can be separated according to the spacetime dimension, $d = 2$ or higher dimensions ($d \neq 2$). Both cases are important not only for generic models but also for superstrings, where $d = 2$ corresponds to the string worldsheet. For example, requiring the condition $\beta = 0$ for exact conformal symmetry in $d = 2$ at one loop order leads to the equations of motion in $d = 26$ for the bosonic string or $d = 10$ for the superstring. We first study the $d \neq 2$ case, which is more compact, and then turn to the $d = 2$ case.

4.2.1 $d \neq 2$

From the gravity point of view, a conformal transformation is the set of transformations (for a **flat metric** only) that transform the metric into itself up to a conformal factor, $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega(x)\, g_{\mu\nu}$, where the conformal factor $\Omega(x)$ can be local. It can also be defined as a transformation that preserves the angle between any two vectors, that is, $\vec{v}_1 \cdot \vec{v}_2$ is invariant. It is easy to see that the two definitions are equivalent by defining the angle θ as $\theta = \arccos \frac{\langle v_1, v_2 \rangle}{|v_1|\,|v_2|}$, where $\langle\, ,\, \rangle$ denotes the inner product, $\langle x, y \rangle = g_{ij} x^i y^j$.

Using the Lie derivative defined in Chapter 3, we know that the metric transforms infinitesimally as $\tilde{g}_{\mu\nu} = g_{\mu\nu} + \epsilon^p \partial_p g_{\mu\nu} + g_{\mu p} \partial_\nu \epsilon^p + g_{\nu p} \partial_\mu \epsilon^p$. By requiring infinitesimally that $\Omega = 1 + \xi$, and assuming $\partial g = \partial \eta = 0$, we obtain

$$\eta_{\mu p} \partial_\nu \epsilon^p + \eta_{\nu p} \partial_\mu \epsilon^p = \xi\, \eta_{\mu\nu}. \tag{4.15}$$

Contracting both sides of (4.15), we find $\xi = \frac{2\,\partial_p \epsilon^p}{d}$, which leads to the equation for the vector field ϵ,

$$\partial_{(\mu}\epsilon_{\nu)} - \frac{\partial_p \epsilon^p}{d}\eta_{\mu\nu} = 0. \tag{4.16}$$

Equation (4.16) defines the vector fields that generate conformal symmetry and is known as the conformal Killing equation [61]. Furthermore, a feature that is not immediately apparent from (4.16) is the special case $d = 2$. Applying another partial derivative to both sides of (4.16), we obtain

$$\Box\epsilon_\nu = \frac{2-d}{d}\partial_\nu(\partial_p\epsilon^p). \tag{4.17}$$

For $d = 2$, this implies that ϵ can be obtained by solving the equation of motion $\Box\epsilon_\nu = 0$, which always admits solutions and, in fact, infinitely many vector solutions. This is easily seen, for example, by Fourier transforming the equation, which is possible since the background metric is flat. In contrast, when $d \neq 2$, the equation is more restrictive, and the number of independent conformal Killing vectors is finite.

Focusing on the $d \neq 2$ case, we can show that for a d-dimensional flat metric we have $\frac{(d+1)(d+2)}{2}$ conformal vectors, d translations and $\frac{d(d-1)}{2}$ rotations (evidently), plus the new transformations, dilatations ($x \to \lambda x$, which immediately preserve the relative angle between vectors) and special conformal transformations (SCT), which can be understood as an inversion, a translation, and then an inversion again (SCT $= I \circ T \circ I$). This results in the finite coordinate transformation $x^\mu \to \frac{x^\mu - b^\mu x^2}{1-2b\cdot x+b^2x^2}$, giving d more transformations and the final number of transformations as $d + \binom{d}{2} + 1 + d = \binom{d+2}{2}$ [28].

To understand the implications of these transformations from the field theory point of view, we need to take a step back. Analogously, recall that we classify the fields ϕ_I present in a field theory by a non-compact (and thus non unitary) Poincare group, see Appendix B. We can boost to a frame where the little group for massive particles is SO(3), and classify the particles by the two Casimir invariants of the group algebra, the eigenvalue of $-P^2$ (the mass) and the eigenvalue of the Pauli Lubanski operator [4] W^2, with $W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}J^{\nu\rho}P^\sigma$. These two eigenvalues label each field in a catalog of representations, and the states are $|p^\mu, s, s_3\rangle$ for massive fields. For massless fields, however, the little group is ISO$(1,2)$, with the non compact generators trivialized, while the compact generator J_{12} gives the helicity of the massless particle, $|p^\mu, \lambda\rangle$. The same classification applies to conformal transformations. Here we focus on the $d = 4$ case, which is generally more familiar due to $\mathcal{N} = 4$ Super Yang Mills.

We have $SO(4,2)$ as the isometry group of a CFT in $d = 4$, which can be viewed as containing $ISO(3,1) \times SO(1,1)$. The labels describing the eigenvalues of the operators are (j_1, j_2), corresponding to the spin as for ordinary fields, together with the eigenvalue of the $SO(1,1)$ subgroup, the dilatation charge Δ,

$$(j_1, j_2, \Delta). \tag{4.18}$$

These three labels together characterize the states in a CFT in $d = 4$. The label Δ specifies how the field transforms under the dilatation operator D,

$$\delta_D \phi(0) = \Delta \phi(0). \tag{4.19}$$

Of course, the field also transforms under Poincare transformations, $\delta^L_{\mu\nu}\phi(0) = -\mathcal{J}_{\mu\nu}\phi(0)$. We can obtain the transformation of $\phi(x)$ by using the translation operator $T(x)$ acting on ϕ, $\phi(x) = T\phi(0)T^{-1}$, which leads to the commutation relations [4]

$$\begin{aligned}
[P_\mu, \phi(x)] &= -i\,\partial_\mu \phi(x), \\
[D, \phi(x)] &= -i\,\Delta\,\phi(x) - i x^\mu \partial_\mu \phi(x), \\
[J_{\mu\nu}, \phi(x)] &= -\mathcal{J}_{\mu\nu}\phi(x) + i(x_\mu \partial_\nu - x_\nu \partial_\mu)\phi(x), \\
[K_\mu, \phi(x)] &= i\Big(-x^2\partial_\mu + 2x_\mu x^\rho \partial_\rho + 2x_\mu \Delta\Big)\phi(x) - 2x^\nu J_{\mu\nu}\phi(x).
\end{aligned} \tag{4.20}$$

These commutation relations follow from the standard quantum mechanics prescription $\delta \to i[\,,\,]$ [144]. It is important to emphasize that these relations are derived for operators that are eigenvectors of the dilatation operator. Such operators are known as conformal primary operators. From the algebra (4.20), an equivalent defining property is to impose $[K_\mu, \phi(0)] = 0$.

Moreover, conformal symmetry requires a traceless energy-momentum tensor ($T_{\mu\nu}$). This can be shown in two ways. First, we can construct the conserved current associated with the dilatation operator using Noether's theorem, which states that for each continuous symmetry transformation there exists a conserved current [69]. The current associated with D is $J^\nu_d = x_\mu T^{\mu\nu}$. Applying the conservation equation and recalling that $T_{\mu\nu}$ itself is conserved, we obtain

$$\partial_\nu J^\nu_d = \eta_{\nu\mu} T^{\mu\nu} = \mathrm{tr}(T) = 0. \tag{4.21}$$

The second way is to recall that the energy-momentum tensor can be defined as

$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}S$[4]. For scale transformations, the metric transforms as $\delta g^{\mu\nu} = \xi g^{\mu\nu}$, so we have

$$\delta g^{\mu\nu} T_{\mu\nu} = \xi \operatorname{Tr}(T) = -\frac{2}{\sqrt{-g}}\delta_\xi S. \tag{4.22}$$

If $\delta_\xi S = 0$, meaning that the action is invariant under the transformation, then $\operatorname{Tr}(T) = 0$.

For the $\mathcal{N} = 4$ case, one adds two more fermionic generators, S and Q, turning the above algebra into a superalgebra.

Before proceeding to the $d = 2$ case, we mention a subtlety regarding the precise definition of the energy-momentum tensor in quantum and gravitational theories. There are distinct approaches to defining the stress tensor $T_{\mu\nu}$, each leading to a slightly different expression. The only property shared by all of them, as we will see, is conservation, $\nabla_\mu T^{\mu\nu} = 0$ (on shell in some cases, as explained later).

One of the earliest definitions of the energy-momentum tensor follows from Noether's theorem applied to translation symmetry, yielding what we denote by $T^{\mathrm{N}}_{\mu\nu}$. This tensor is conserved by construction, but other properties, such as symmetry under the exchange of indices, are not guaranteed. However, one can always render $T^{\mathrm{N}}_{\mu\nu}$ symmetric by adding a conserved term $N_{\mu\nu}$. This procedure defines the Belinfante tensor $T^{B}_{\mu\nu}$, which is symmetric by construction[5].

In contrast, the Hilbert energy-momentum tensor, defined by varying the action with respect to the metric, $T_{\mu\nu} = \frac{2}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}}$, is automatically symmetric, but its conservation generally holds only when the fields satisfy the equations of motion.

Finally, one can define an improved energy-momentum tensor $T^{I}_{\mu\nu}$ such that $\operatorname{tr} T^{I} = 0$. This is possible in classically conformal models and, as is often the case, the improved tensor is traceless on shell. Moreover, if the theory possesses Weyl invariance, a local scaling symmetry in curved spacetime, the improved tensor can be made traceless even off shell.

4.2.2 $d = 2$

As we saw, in $d = 2$ the conformal Killing vectors are much less constrained. Indeed, the equation of motion $\Box\epsilon_\nu = 0$ can be solved if we impose $\partial_0\epsilon_1 = -\partial_1\epsilon_0$ and $\partial_0\epsilon_0 = \partial_1\epsilon_1$. These are, not coincidentally, the Cauchy–Riemann equations in the complex plane with coordinates $z, \bar{z}$. Therefore, if we change coordinates to

[4] see [164].

[5] The explicit form of $N_{\mu\nu}$ is not needed here.

$z, \bar{z} = x_0 \pm ix_1$, the new fields $\epsilon, \bar{\epsilon}$ are holomorphic and anti-holomorphic. We can then Laurent expand both fields,

$$\epsilon = \sum_n \frac{\epsilon_n}{z^{n+1}}, \quad \bar{\epsilon} = \sum_n \frac{\bar{\epsilon}_n}{\bar{z}^{n+1}}. \tag{4.23}$$

The generators of these transformations are $l_n, \bar{l}_n$, such that $(1 + l_n) f(z) = f(z - \epsilon_n z^{-(n+1)}) = f(z) - \epsilon_n z^{-(n+1)} \partial f(z)$. This leads to

$$l_n = -z^{n+1}\partial_z, \quad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}}. \tag{4.24}$$

The algebra satisfied by the subset of generators $\{l_{-1}, l_0, l_1, \bar{l}_{-1}, \bar{l}_0, \bar{l}_1\}$ is $\mathrm{sl}(2, \mathbb{C}) \times \mathrm{sl}(2, \mathbb{C})$, but the full algebra is infinite,

$$[l_m, l_n] = (m - n) l_{m+n}. \tag{4.25}$$

The same applies to the conjugate generators, and $[l_m, \bar{l}_n] = 0$. This algebra is known as the de Witt algebra. However, upon quantization it becomes the Virasoro algebra and includes a central term of the form $\frac{c}{12}(m^3 - m)\delta_{m+n,0}$, where c is the central charge of the theory. The central charge measures the anomaly of conformal symmetry, that is, how conformal symmetry is broken at the quantum level, and it encodes information about the matter content of the model.

We can also define primary operators, although we also have the presence of quasi-primary operators or fields. In contrast to the $d = 4$ case, where we focused on the algebra, here we use an analogy closer to a tensor definition. Let a field transform under a group G as

$$\phi(z, \bar{z})' = (\partial_z f(z))^h (\partial_{\bar{z}} f(\bar{z}))^{\bar{h}} \phi(f(z), f(\bar{z})). \tag{4.26}$$

If G is the group of global transformations $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ mentioned before, which can be understood as $z \to \frac{az+b}{cz+d}$ and similarly for the complex conjugate, then the field is called a quasi-primary operator [28]. If G is the full group of conformal generators in $d = 2$, the field is called a primary operator. The exponents h and $\bar{h}$ are called the conformal dimensions $(h, \bar{h})$ of the field $\phi(z, \bar{z})$ and describe how the field transforms under a dilatation operation.

Finally, an important property of $d = 2$ conformal fields is related to their energy-momentum tensor. It is also traceless and therefore, in the off-diagonal metric $\eta_{z\bar{z}}$, it only has the components $T_{zz} = T$ and $T_{\bar{z}\bar{z}} = \bar{T}$. Moreover, it is possible to show, by evaluating the conservation equation, that T and $\bar{T}$ are holomorphic and anti-holomorphic, respectively.

4.3 AdS/CFT

Before talking about AdS/CFT itself, we mention that a large number of models have been proposed based on this idea, sometimes generalizing to the most universal idea of gravity/gauge-theory duality. To cite only some of them, we have

Polchinski-Strassler [135], Maldacena-Nuñez [109], Klebanov-Strassler [96],
Domain-wall models [4], Matrix models[6], Klebanov-Witten [97], JT/SYK [95], (4.27)
Non-relativistic Lifshitz geometry [87], probe-flavor [88], ABJM [1].

The models listed in (4.27) are not even the full set of holographic constructions available today [4], [121]. There are models with black branes to simulate finite temperature, with black holes to further introduce phase transitions, as well as holographic cosmology, holographic superconductors, and many others. We hope this convinces the reader of our initial statement in the dissertation: holography, which began with Maldacena, has gained its own life and can now be regarded as a distinct area of research in theoretical physics.

Some of the models in (4.27) are dual to conformal field theories, while others are not. Some reproduce renormalization-group flow equations through gravitational analogues, among other features. In general, beyond the wide range of models that can be studied, many physical phenomena can be understood from the gravitational point of view, such as deformations and energy-dependent couplings.

With this introduction completed, we may finally explain what the AdS/CFT conjecture is. In order to do so in a more didactic way, we focus on the most studied and best understood example, $\mathrm{AdS}_5/\mathrm{CFT}_4$. We emphasize once more that Maldacena's conjecture applies to other dualities as well, such as $\mathrm{AdS}_7 \times S^4$ and $\mathrm{AdS}_4 \times S^7$. From now on, however, we will often use the term AdS/CFT to specifically refer to the $\mathrm{AdS}_5/\mathrm{CFT}_4$ example, so as to avoid carrying the indices 5 and 4 throughout the text.

Type IIB string theory admits odd p-branes, while Type IIA admits even p-branes. One of the possible solutions of the Type IIB model involves the presence

[6] see [171] for details.

of several stacked 3-branes. It is worth noting that membranes in string theory are analogous to particles in quantum field theory, since they possess their own Lagrangian and interact with external fields. In particular, a p-brane couples to a $(p+1)$-form potential, described by a field strength $F_{\mu_1 \ldots \mu_{p+2}}$, which is a $(p+2)$-form. Thus, in our case, a 3-brane is accompanied by a field strength F_5.

Another peculiarity is that Type IIB string theory is intrinsically more subtle than the other cases. One cannot write a conventional covariant Lagrangian because of the constraint $F_5 = \star F_5$, where $\star$ denotes the Hodge star. The field F_5 is self-dual, meaning that it is equal to its own dual. This property could have been anticipated, since the Hodge star maps a p-form into a $(D-p)$-form. In $D = 10$, the only possibility for a self-dual field strength is $p + 2 = D - (p+2)$, which implies $p + 2 = 5$ and therefore $p = 3$.

This peculiarity turns out to be useful in the present context. The field F_5 produces a flux through S^5 proportional to N, the number of branes. Instead of a single brane, we may therefore consider N D3-branes stacked on top of each other. The N D3-branes generate an effective theory on their interior, the worldvolume, which has dimension $d = 3 + 1$. This effective theory resembles a Yang-Mills theory with gauge group $SU(N)$, essentially due to the presence of Chan-Paton factors at the endpoints of open strings.

We must also pay attention to supersymmetry. A string background with D-branes is not automatically supersymmetric, and this must be analyzed by studying solutions of the equations of motion. Without going into technical details, the idea is to obtain the Lagrangian of the model and verify whether it is invariant under any supersymmetry transformation, schematically $\delta_\epsilon B \sim F$ and $\delta_\epsilon F \sim B$, where B denotes bosonic fields and F fermionic fields. Another useful approach is to evaluate the existence of Killing spinors for the corresponding supergravity solution.

Alternatively, one can recall that the Type IIB theory is $\mathcal{N} = 2$ supersymmetric, meaning that it possesses two supersymmetry generators, corresponding to a total number of supercharges

$$Q = \mathcal{N} \times \frac{2^{[d/2]} \times 2}{2^{W+M}}. \tag{4.28}$$

Here $[d/2]$ denotes the integer part of $d/2$. The factor of 2 appears because fermions are complex, and the factor 2^{W+M} refers to the spinor representation, as explained in Appendix A. In this case there are 32 supercharges. When a D3-brane is introduced, one can project half of the fermions along the brane directions and the other half orthogonally. As a result, half of the supersymmetry is broken, giving $Q = 32/2 = 16$. Indeed, D-branes are $\frac{1}{2}$ BPS solitons.

For consistency, the effective theory living on the D3-brane must therefore possess 16 supersymmetries. In $d = 4$, with $W = 1$ and $M = 0$, this implies $Q = 16$ and hence $\mathcal{N} = 4$. Moreover, supersymmetric theories admit an automorphism acting on their supercharges, known as an R-symmetry. For $d = 4$ and $\mathcal{N} = 4$, this symmetry group is $SU(4)$.

Having established the effective theory on the D3-branes, we now analyze the exterior region. Outside the stack of branes, we look for the gravitational solution generated by these objects. The metric describing N coincident D3-branes in Type IIB supergravity is well known and is given by

$$ds^2 = H(r)^{-1/2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + H(r)^{1/2}\left(dr^2 + r^2 d\Omega_5^2\right), \tag{4.29}$$

where $H(r) = 1 + \dfrac{L^4}{r^4}$ and L is a constant related to the flux of F_5, and therefore to N. The coordinates (t, x_1, x_2, x_3) parametrize the world volume directions, while r and the angles of S^5 parametrize the transverse space.

This metric can be rewritten to emphasize its behavior in different regions. For large r, one has $H(r) \approx 1$, and the metric approaches ten-dimensional Minkowski spacetime,

$$ds^2 \approx -dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dr^2 + r^2 d\Omega_5^2. \tag{4.30}$$

For small r, corresponding to the near-horizon region, the function $H(r)$ approaches L^4/r^4, and the metric becomes

$$ds^2 = \frac{r^2}{L^2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + \frac{L^2}{r^2}dr^2 + L^2 d\Omega_5^2. \tag{4.31}$$

This geometry is precisely $\mathrm{AdS}_5 \times \mathrm{S}^5$ with common radius L. Explicitly, the AdS part is

$$ds^2_{\mathrm{AdS}_5} = \frac{r^2}{L^2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + \frac{L^2}{r^2}dr^2, \tag{4.32}$$

while the spherical part is

$$ds^2_{\mathrm{S}^5} = L^2 d\Omega_5^2. \tag{4.33}$$

Thus, the presence of the N D3-branes produces a spacetime that interpolates between flat space at infinity and $\mathrm{AdS}_5 \times \mathrm{S}^5$ near the horizon. Since L is related to the string coupling g_s and the number of branes by

$$L^4 = 4\pi g_s N \alpha'^2, \tag{4.34}$$

the curvature is small, and the supergravity description is valid, when $g_s N \gg 1$.

We therefore have two descriptions of the same physical system. From the perspective of open strings ending on the branes, we obtain $\mathcal{N} = 4$ Super Yang-Mills theory with gauge group $SU(N)$ living on the worldvolume. From the perspective of closed strings propagating outside the branes, we have Type IIB supergravity in the background generated by the branes, whose near-horizon limit is $\mathrm{AdS}_5 \times \mathrm{S}^5$.

Maldacena's idea is that these two descriptions are not merely approximations, but are in fact exactly dual to each other. Taking the low-energy limit on the open-string side removes massive string modes and gravity, leaving only the four-dimensional $\mathcal{N} = 4$ theory. On the closed-string side, taking the near-horizon limit removes the asymptotically flat region, leaving only the $\mathrm{AdS}_5 \times \mathrm{S}^5$ throat. In other words,

$$\begin{array}{rcl} \text{Low-energy open strings on D3-branes} & \Longleftrightarrow & \mathcal{N} = 4\ SU(N)\ \text{SYM in } d = 4 \\ \text{Near-horizon closed strings around D3-branes} & \Longleftrightarrow & IIB\ \text{string theory on } \mathrm{AdS}_5 \times \mathrm{S}^5 \end{array} \tag{4.35}$$

Thus, the conjecture states that

$$\boxed{\text{IIB string theory on } \mathrm{AdS}_5 \times \mathrm{S}^5 \ \cong\ \mathcal{N} = 4\ SU(N)\ \text{SYM in } d = 4} \tag{4.36}$$

This is the celebrated AdS/CFT correspondence. The conformal symmetry of the gauge theory is $SO(4,2)$, which is exactly the isometry group of AdS_5. The $SU(4)$ R-symmetry coincides with the isometry group of the internal sphere S^5. This matching of symmetries is compelling evidence for the duality. Furthermore, the dictionary between parameters relates the string coupling and curvature to the Yang-Mills coupling, $g_{\mathrm{YM}}^2 = 4\pi g_s$, and the 't Hooft coupling is $\lambda = g_{\mathrm{YM}}^2 \mathcal{N} = 4\pi g_s N = \frac{L^4}{\alpha'^2}$. Thus, strong curvature (small λ) corresponds to weak coupling in the gauge theory, while weak curvature (large λ) corresponds to strong coupling in the gauge theory. This remarkable strong–weak relation makes the duality extremely powerful for studying non-perturbative physics.

An essential feature of the correspondence is the limit in which each description becomes classical. On the gravity side, Type IIB supergravity is valid when both the string coupling and the spacetime curvature are small. The conditions are $g_s \ll 1$, $\frac{\alpha'}{L^2} \ll 1$. Using the relations already established, these inequalities translate into conditions on the gauge theory parameters, $g_s = \frac{g_{\text{YM}}^2}{4\pi} \ll 1, \frac{\alpha'}{L^2} = \lambda^{-1/2} \ll 1$. In other words, the gravity description is classical when $N \gg 1, \lambda \gg 1$. Thus, large-N and strong coupling on the gauge theory side correspond to weakly curved classical gravity on the string side. Conversely, small λ means high curvature, bringing stringy corrections into play. This provides the first indication that AdS/CFT is a strong–weak duality: perturbative Yang-Mills correspond to highly quantum gravity, while classical gravity corresponds to non-perturbative Yang-Mills.

A second key consequence arises from the fact that conformal field theories admit no intrinsic energy scale. In a CFT, physical observables depend only on dimensionless ratios. On the gravity side, this scale invariance manifests geometrically as a radial symmetry of AdS_5. Indeed, consider the coordinate dilatation

$$x^\mu \to \alpha x^\mu, \quad r \to \frac{r}{\alpha} \tag{4.37}$$

applied to the AdS metric,

$$ds^2_{\text{AdS}_5} = \frac{r^2}{L^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{r^2}dr^2, \tag{4.38}$$

one verifies that it is invariant under such a scaling. This is precisely the same symmetry as the dilatations of the conformal group $SO(4,2)$. The radial coordinate r therefore acquires a physical interpretation; it corresponds to the energy scale of the dual gauge theory. Large r corresponds to short distances and high energies, while small r corresponds to long distances and low energies. This geometric renormalization-group interpretation is one of the most celebrated aspects of AdS/CFT. From the duality, observables in the gauge theory must correspond to fields propagating in the bulk of $\text{AdS}_5 \times \text{S}^5$.

The precise statement of the duality, as formulated by Witten [170], is that

$$Z_{\text{string}}\left[\Phi \to \phi_0\right] = \left\langle \exp\left(\int d^4x\, \phi_0(x)\mathcal{O}(x)\right)\right\rangle_{CFT}, \tag{4.39}$$

where ϕ_0 is the boundary asymptotic value of $\Phi(x)$. In the classical supergravity limit, the left-hand side reduces to $\exp(-S_{cl})$, where S_{cl} is the classical action evaluated on the solution with boundary condition $\phi_0(x)$. Thus,

$$S_{cl}[\phi_0] = -\ln\left\langle \exp\left(\int d^4x\, \phi_0(x)\mathcal{O}(x)\right)\right\rangle. \tag{4.40}$$

This relation provides a practical method for computing strongly coupled gauge theory correlators by evaluating classical gravitational actions. Of particular relevance is the case of fluctuations around the AdS background. Consider a scalar field Φ of mass m in AdS_5. Its behavior close to the boundary is governed by the differential equation

$$r^2\partial_r^2\Phi - 3r\partial_r\Phi + r^2\Box_x\Phi - m^2L^2\Phi = 0, \tag{4.41}$$

which leads to the asymptotic scaling

$$\Delta(\Delta - 4) = m^2L^2. \tag{4.42}$$

Thus, the conformal dimension (Δ) of the dual operator is determined by the mass (m) of the bulk field. This is one of the central entries of the AdS/CFT dictionary.

Evidently, despite all the similarities we have observed, including the aspects of symmetry, there has been no attempt at a proof in this report (and none exists in the literature so far). Therefore, we call this duality the AdS/CFT conjecture. Assuming that this conjecture is indeed correct, we may,

- Compare (enumerate and compare the spectrum of) operators of the $\mathcal{N} = 4$ theory with solutions of the IIB supergravity equation of motion (i.e., compare spectra).
- Compare observables (Witten, $Z_{\mathrm{CFT}} = Z_{\mathrm{AdS}}$), 2- and 3-point functions, Wilson operators.
- Compare anomalies.

Chapter 5

pp-wave and BMN correspondence

5.1 Penrose limit on gravity

Perhaps the best path to follow in order to understand the meaning of the BMN correspondence is by analyzing the sequence of facts chronologically. Indeed, the history in that case provides a good logical sequence that culminated in one of the most well-known concepts in holography and integrability in general, a precise map between states and operators, between the (potential part of) Hamiltonian of both fields, and an intrinsic connection with spin-chains [20].

To properly explain and motivate the BMN correspondence, we can start in 1976 with Penrose, in his paper [130]. There, he explains how any spacetime, when boosted to go near a null-light trajectory, reduces to a metric with a specific form. The way this is shown is simple. Consider a metric in coordinates such that $V = Y^i = 0$. We remember a trajectory is described by one free parameter, which we call U here. Then, the metric is defined, as explained in Section 3, by the line element ds^2, but in the specific coordinate patch we choose, it can be written as the following: Define the spacetime by the coordinates (V, U, Y), and call $x^\mu = (V, Y)$, $x^i = (Y)$. Then,

$$ds^2 = g_{vv}dV^2 + g_{vi}dVdY^i + g_{ij}dY^idY^j + g_{uu}dU^2 + g_{u\mu}dUdx^\mu. \tag{5.1}$$

The metric in the form (5.1) is the most generic possible, but if we now impose that $V = Y^i = 0$ is the null-light trajectory, then necessarily $g_{uu} = 0$, as can be easily checked. We have enough degree of freedom to make $g_{ui} = 0$ and $g_{uv} = 1$, reducing the neighborhood of a null geodesic, after calling $g_{vi} = \beta_i$, to

$$ds^2 = g_{vv}dV^2 + \sum_i \beta_i dVdY^i + dUdV + \sum_{i,j} g_{ij}dY^idY^j. \tag{5.2}$$

We stress that V and U are the null-coordinates describing the trajectory itself, while Y are the transverse coordinates, "localizing" the geodesic on the space.

Therefore, we boost along a mixing direction of U, V by a scaling of the type

$$U \to U, \quad Y^i \to \lambda^{-1} Y^i, \quad V \to \lambda^{-2} V \tag{5.3}$$

where, more precisely, $\lambda = e^\eta$, with η being the rapidity [164]. Then, the metric (5.1) goes, keeping only the leading order term, to

$$ds^2 = g_{ij} dY^i dY^j - 2dUdV + \beta_i dV dY^i \tag{5.4}$$

What follows from this depends on the specific metric you are working on: to be a pp-wave, g_{ij} need to be the flat metric [1], and if g_{iv} satisfies some conditions we are touching on now, then it is the most generic pp-wave metric possible. Before we go onto specific metric, however, we mention that generally we can identify the scale λ with an overall factor R of the metric,

$$U = u, \quad V = \frac{v}{R^2}, \quad Y^i = \frac{y^i}{R} \tag{5.5}$$

taking the limit of $R \to \infty$ and then absorbing the overall factor left in ds^2. This is how things are generally done in usual supergravity backgrounds, though the boost interpretation is valid to give an intuition of the meaning of that scaling, and we can call either of the limits the famous **Penrose limit**.

Going into details later, we call the metric in (5.4), when $g_{ij} = g_{ij}(U)$ only, as the *pp-wave metric in the Rosen coordinates*, though probably the most recognized form is in the Brinkmann coordinates,

$$ds^2 = 2dx^+ dx^- + H(x^+, x^0)(dx^+)^2 + d\vec{x}^2. \tag{5.6}$$

We intend to spend the rest of this section classifying some spacetimes that are pp-wave, though not of the type used for example in the BMN limit [20], which is the simplest and most important pp-wave in the context of supergravity (more on why this is true later). More precisely, the metric

$$ds^2 = -2dudv - F(v, Y^i)dv^2 + 2A_i(v, Y^i)dvdY^i + g_{ij}(u, x^k)dx^i dx^j \tag{5.7}$$

is called "Bargmann metric" [50].

The metric in question is, of course, an Einstein solution, but its applicability

[1] Since pp means parallel and **plane**.

for supergravity is somewhat still obscure, as we are not aware of it being applied in the literature[2], and the terms A_j, F and g_{ij} are constrained, either by the Einstein equations of motion or by the supergravity equations of motion. A more constrained type is the special type of the Bargmann structure, which can be obtained by imposing $A_j = 0, g_{ij} = \delta_{ij}$ in (5.7) [140],

$$ds^2 = -2dudv - F(v, Y^i)dv^2 + d\vec{x}^2 \tag{5.8}$$

(5.8) becomes a plane-wave, in the definitions of [140], if F can be written as a 2-degree homogeneous polynomial of x, $F = f_{ij}(u)x^i x^j$. In that case, we can find a specific change of coordinates from the metric (5.4) to (5.8). Finally, the two most important cases are a special type of this homogeneous F function: the most general case of f_{ij} being a constant, and the second one is that condition with the imposition of it being diagonal, $f_{ij} = \mu_i \delta_{ij}$.

We have focused only on the metric, though supergravity solutions have more fields than this, like the other NSNS fields together with the RR-fields. Depending on the specific form of these fields, the model is doomed to break all of the supersymmetry, but for specific cases known as *parallelizable pp-wave*, in which the parameters μ_j explained above are present both in H_3 as in F_3, we can have preserved supersymmetry [140]. These types of backgrounds can preserve $16, 18, 20, 22$ or 24 supercharges if in type IIA, or $16, 20, 24, 28$ in type IIB, depending on how the parameters in the metric are related to the parameters in the fields [140]. Later, we are going to apply these properties in one of the results of the current master's degree [12], where we apply the Penrose Limit for an I-brane, and compare the commutation of this limit with deforming the model, though we leave the details for later.

[2]To the best of my knowledge.

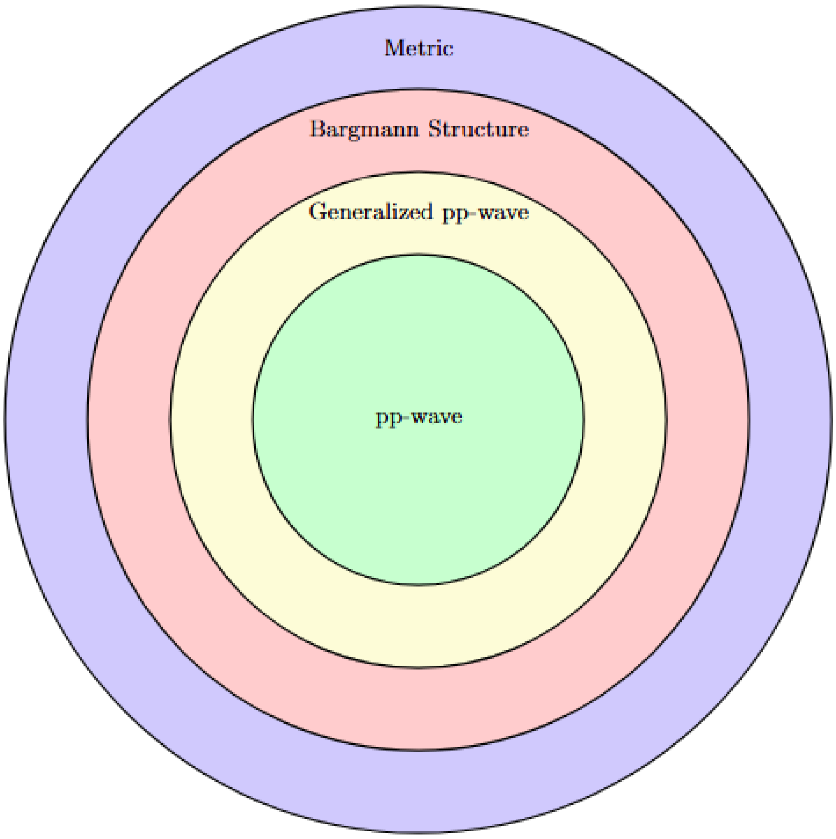

Figure 5.1: Spacetime classification under "pp-wave" metric, where generalized are the parallelizable cases.

We mention, out of curiosity, the historical facts of the year 2002, with a burst of papers involving the concept of Penrose Limit and pp-waves. In January 2002, a paper by Blau, O'Farril, Hull and Papadopoulos [27] showed that the maximally supersymmetric pp-wave solution in type IIB superstring could be obtained as the Penrose limit of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background. Only a month later, in February, came the publication of the BMN correspondence proposal, by Berenstein, Maldacena, and Nastase. Some months later, O'Farril and Papadopoulos [57] obtained that possible spacetimes in $d = 10$ and $d = 11$ with maximally preserved supersymmetry, 32 supercharges, are either flat, or of $AdS_p \times S^q$ (sometimes called Freund-Rubin type due to a study about compactification on this spacetime [62]), or are of a homogeneous pp-wave type, with the latter being obtained by taking the Penrose limit of the Freund-Rubin background [3].

Finally, since the focus here is to explain precisely the meaning of the BMN correspondence, which is done in the usual $\mathrm{AdS}_5/\mathrm{CFT}^4$ background, we study the

[3] The Kowalski-Gilkman theorem proves these are the only maximally supersymmetric spacetimes possible in those dimensions.

Penrose limit of the $AdS_5 \times S^5$ supergravity solution. For more detailed reviews of the Penrose limit, we recommend [141, 26]. To take the Penrose limit of $AdS_5 \times S^5$ is quite simple in global coordinates,

$$ds^2 = R^2\left(-\cosh^2\rho\, d\tau^2 + d\rho^2 + \sinh^2\rho\, d\Omega_3^2\right) + R^2\left(\cos^2\theta\, d\psi^2 + d\theta^2 + \sin^2\theta\, d\Omega_3'^2\right). \tag{5.9}$$

The null geodesic is located in the middle of AdS_5, at $\rho = 0$, and moves along an equator of S^5, at $\theta = 0$, with motion in τ, ψ. Then, near the geodesic we find

$$ds^2 \simeq R^2\left[-(1+\rho^2)\, d\tau^2 + d\rho^2 + \rho^2 d\Omega_3^2\right] + R^2\left[(1-\theta^2)\, d\psi^2 + d\theta^2 + \theta^2 d\Omega_3'^2\right]. \tag{5.10}$$

Now, we define the null coordinates to put the metric in the Penrose form. $\tilde{x}^\pm = \frac{\tau\pm\psi}{\sqrt{2}}$ and then rescale to perform the Penrose limit,

$$\tilde{x}^+ = x^+, \quad \tilde{x}^- = \frac{x^-}{R^2}, \quad \rho = \frac{r}{R}, \quad \theta = \frac{y}{R}. \tag{5.11}$$

We can simply rescale the null coordinates to obtain the metric in the usual form

$$ds^2 = -2\, dx^+ dx^- - \mu^2(\vec{r}^{\,2} + \vec{y}^{\,2})(dx^+)^2 + d\vec{y}^{\,2} + d\vec{r}^{\,2}. \tag{5.12}$$

together with the 5-form field, $F_{+1\ldots4} = F_{+5\ldots8} = 4\mu$.

5.2 Penrose limit on field theory

Now, how does this limit reflect in the $\mathcal{N} = 4$ SYM theory? Which sector are we referring to? As we saw, the pp-wave null coordinates are related to those of $AdS_5 \times S^5$ by $\tilde{x}^\pm = \frac{\tau\pm\psi}{\sqrt{2}}$, where τ is the global AdS time coordinate, and ψ is the equatorial angle of S^5. The momentum in the pp-wave and in AdS is, consequently, related. We can obtain that

$$p^- \sim (\partial_\tau + \partial_\psi) \to \Delta - J, \quad p^+ \sim (\partial_\tau - \partial_\psi) \to \frac{\Delta + J}{R^2}. \tag{5.13}$$

Thus, the energy E (time generator) and angular momentum J (from ψ rotations) are conserved in supergravity. It is possible to construct configurations with up to 6 conserved charges. On the dual $\mathcal{N} = 4$ side, we identify the energy E with the conformal dimension Δ, and the charge J with one of the R-symmetry charges, $J = (J_1, 0, 0)$.

We do not want p^+ to diverge or vanish. To avoid a nonphysical spectrum

($p^{\pm} \to \infty$) we take, using $R^2 = \sqrt{\lambda}\alpha'$,

$$p^+ \Rightarrow \Delta + J \sim \sqrt{\lambda}, \quad p^- \Rightarrow \Delta \sim J, \quad \Delta \sim J \sim \sqrt{\lambda}. \tag{5.14}$$

The last condition comes from the fact that $p^- = H_{lc}$, that is, p^- is the light-cone Hamiltonian (by definition, technically), and if $\Delta \gg J$ or vice versa, in the large N limit, the Hamiltonian would diverge. Then, we require $\Delta \sim J$, and obtain the results of (5.14).

5.3 BMN correspondence

The line of reasoning in equation (5.14) is the heart of the BMN correspondence, and can already be considered a result of it. The Penrose limit of AdS/CFT maps type IIB string states in the pp-wave metric to a sector of $\mathcal{N} = 4$ whose operators satisfy $\Delta \sim J \sim R^2$, i.e., a large-R sector.

One of the most interesting points, not yet mentioned, is that string theory can be quantized in the pp-wave background by analyzing the worldsheet via the Polyakov action. We consider only the bosonic part of the Green-Schwarz (GS) action

$$\begin{aligned} S_B &= \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-h}h^{\alpha\beta}G_{MN}\partial_\alpha X^M\partial_\beta X^N \\ &= \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-h}h^{\alpha\beta}\left(-2\partial_\alpha X^+\partial_\beta X^- + \partial_\alpha X^I\partial_\beta X^I - \mu^2(X^I)^2\partial_\alpha X^I\partial_\beta X^I\right), \end{aligned} \tag{5.15}$$

where we substitute the pp-wave metric.

Now, fixing the diffeomorphism and Weyl symmetries by choosing $\sqrt{-h}h^{\alpha\beta} = \eta^{\alpha\beta}, \eta_{\tau\tau} = -1, \eta_{\sigma\sigma} = 1$ and by imposing $X^+ = \alpha'\tilde{p}^+\tau$, we may express $X^-(\tau,\sigma)$ in terms of the X^I. Moreover, the action for the bosonic sector becomes quadratic in the X^I,

$$S_B = \frac{1}{4\pi\alpha'}\int d\tau \frac{1}{2\pi\alpha'\tilde{p}^+}\int_0^{2\pi\alpha'\tilde{p}^+} d\sigma\left(\partial_\tau X^I\partial_\tau X^I - \partial_\sigma X^I\partial_\sigma X^I - \mu^2 X_I^2\right). \tag{5.16}$$

This is an action for eight scalars X^I with mass μ, with the massive wave equations of motion for the fields X,

$$(\partial_\tau^2 - \partial_\sigma^2 - \mu^2)X^I(\tau,\sigma) = 0. \tag{5.17}$$

with a solution of the form

$$X^I(\tau,\sigma) = \cos(\mu\tau)\frac{x_0^I}{\mu} + \sin(\mu\tau)\frac{p_0^I}{\mu} + \sum_{n\neq 0}\frac{i}{\sqrt{2\omega_n}}\left(\alpha_n^I e^{-i(\omega_n\tau - k_n\sigma)} + \tilde{\alpha}_n^I e^{-i(\omega_n\tau + k_n\sigma)}\right), \tag{5.18}$$

where $\omega_n = \text{sign}(n)\sqrt{k_n^2+\mu^2}, k_n = \frac{n}{\alpha'\tilde{p}^+}$. Then, since we have found a solution in terms of creation and annihilation operators (the coefficients of the Fourier expansion), it is possible to quantize the Hamiltonian. For now, we write the classical Hamiltonian,

$$H_{\text{lc}} = \frac{1}{p^+}\int d\sigma\left[(P_I)^2 + (\partial_\sigma X^I)^2 + \mu^2(X^I)^2\right]. \tag{5.19}$$

Introducing the zero modes,

$$\alpha_0^I = \frac{1}{\sqrt{2}\mu}\left(p_0^I + i\mu x_0^I\right), \tag{5.20}$$

and imposing the usual commutation relation, we obtain the Hamiltonian after substituting the solution of the equations of motion, written in terms of oscillator modes as

$$H_{\text{lc}} = \mu\left(\alpha_0^{I\dagger}\alpha_0^I\right) + \frac{1}{\alpha'\tilde{p}^+}\sum_{n=1}^{\infty}\sqrt{n^2 + (\alpha'\tilde{p}^+\mu)^2}\left(\alpha_{-n}^I\alpha_n^I + \tilde{\alpha}_{-n}^I\tilde{\alpha}_n^I\right), \tag{5.21}$$

the resulting spectrum is therefore

$$E_{\text{lc}} = \mu N_0 + \mu\sum_{n=1}^{\infty}(N_n)\sqrt{1+\frac{n^2}{(\alpha'\tilde{p}^+\mu)^2}}, \tag{5.22}$$

where $N_n = \alpha_{-n}^I\alpha_n^I$.

This allows us to map string states to operators in $\mathcal{N}=4$. For example, the vacuum in string theory is given by $E_{pp} = p^- = 0, \Rightarrow \Delta = J, \quad p^+ = \Delta$. Thus $\Delta = J$, and we can obtain a mapping between the "vacuum" of both sides of the duality by a line of reasoning analogous, in some sense, to the mapping logic of flavor holography, that is, we seek a gauge-invariant operator that has $\Delta = J$. Since we know Δ is, at tree level, equal to the mass dimension of operators, and knowing that $\phi_1 + i\phi_2$ has a charge under the $U(1)$ rotation over the equatorial S^5, together with a mass $\Delta = 1$, the simplest operator we can obtain is $\text{Tr}Z^J$, which is precisely the case for the mapping,

$$|0, p^+\rangle \sim \mathrm{tr}(Z^J). \tag{5.23}$$

We notice we can rewrite the 6 scalar bosons as real and complex fields, stressing that $|0, p^+\rangle$ is the string vacuum ($N = 0$) with momentum p^+. The next states are those with $p^- = 1$, corresponding to operators with $\Delta - J = 1$:

$$a^\dagger_{0,r} b^\dagger_{0,b} |0, p^+\rangle \sim \sum_1^J \mathrm{tr}[\phi^r Z^I \psi^b_{J=1/2} Z^{J-I}] \tag{5.24}$$

To make it clear, the operator on the left-hand side of (5.24) is there because both ψ, which has $\Delta = 3/2$ and $J = 1/2$, and ϕ with $\Delta = 1$, satisfy the condition of $\Delta - J = 1$. Therefore, we sum over all possible ways to combine all of them in a covariant way, where of course the overall Z^J is there because it is the "vacuum operator". Moreover, one notices that the states we studied are for $n = 0$ energy. If, on the other hand, we decide to study states obtained by applying creation operators with higher levels, then the level "N", say, on the left-hand side of the matching (5.23), (5.24), should manifest itself on the right-hand side as well. This is done by remembering the principle of quantum mechanics, where all of these "quantization" started, this comes from the fact that wavefunctions in a closed/compact space need to be consistent with the boundary conditions, so it is either quantized or periodic [144], and then naturally we could guess that what happens on the other side is the presence of a wave-like function $\sim e^{ikx} = e^{i\frac{2n\pi}{L}}$[4]. This is precisely the case [121].

The operator/states correspondence in the BMN context properly introduced, we return to the energy (5.22) and take the Penrose limit[5],

$$E_{pp} = 2\sqrt{1 + \frac{n^2 R^4}{\alpha'^2 J^2}} \tag{5.25}$$

Again, calling $R^2 = \alpha' \sqrt{\lambda}$, the BMN conjecture predicts (recall $E_{pp} = p^- = \Delta - J$),

$$\Delta - J \sim 2 + n^2 \frac{\lambda}{J^2} \tag{5.26}$$

[4] 2π because it is periodic. Indeed, we can think of it as a chain! This is just a hint towards the whole subject involving spin-chains, integrability, BMN [16], etc.

[5] Though, in general, in the context of BMN we can call it the BMN limit also

We wrote (5.26) in terms of gauge-field parameters because the calculation was done on the string (gravity) side, but since it needs to be compared with the results on the field theory, we already convert it to the usual parameters. If the BMN correspondence is correct, at least to such order, if we do the same calculations on the field side we should obtain exactly the relation of (5.26). We can do this, and it has been done and matched precisely, showing to be one of the most prominent matches in the holography correspondence.

Then, to calculate it we can follow two approaches: the first one is more intuitive and derives an astonishing result, with a match between the results but also an interaction term in the Hamiltonian of $\mathcal{N} = 4$, the potential say, when acting on BMN operators (those operators are, $\mathrm{tr}(Z^J)$, etc) showing to be exactly equal to the discrete action of the string theory in the pp-wave background, and the result (5.26) is obtained as the eigenvalue of that interaction term. Or we can do a laborious calculation by remembering that Δ is, in the end of the day, the dimension of the operator, which is the term that appears in the correlation functions of conformal operators in $\mathcal{N} = 4$, and then we can calculate the first loop order contribution for the anomalous dimensions in the **planar limit** by making use of Feynman diagrams, define something called a dilaton operator, which is of the form of the Heisenberg XXX spin-chain, and then obtain the spectrum of this Hamiltonian, which matches with (5.26). At this point of the dissertation, we still did not mention it, but to restrict the $SU(4)$ "rotations" only to one angle, is like collecting a specific sector of the field theory made of operators with labels that define such a sector. The sector in question is the $SU(2)$, the simplest sector of $\mathcal{N} = 4$. For more about this last discussion, see [118].

We decided to follow the second option. To obtain the correction for the anomalous dimension Δ, the calculations are literally the same as those done in any standard QFT book [131], of course, since in the end of the day $\mathcal{N} = 4$ is only an extremely symmetric field theory. So, in that sense, we can either make use of Feynman diagrams, which highlights some diagrammatical aspects of double-line calculations and the 't Hooft limit, or directly calculate the deviations of the 2-point functions by using Wick contractions, which is an algebraic approach and consequently less intuitive, though at the same time it is more direct (we don't need to think about all the possible Feynman diagrams allowed, they naturally appear. Moreover, the symmetrical factors we generally need to include and negative signs are also automatically present in the Taylor expansion). We follow the last approach, and focus on the first loop correction to the anomalous dimension.

We can consider generic operators of the form $O_{i_1 i_2 \ldots i_l}(x) \sim \mathrm{Tr}\left[\phi_{i_1}(x)\phi_{i_2}(x)\cdots\phi_{i_l}(x)\right]$, ignoring the normalization for the moment. The operator is only required to be gauge invariant. We can calculate the two-point function by making use of straightforward Wick techniques $\langle O_{\ldots}(x)O_{\ldots}(y)\rangle$, consisting of essentially contracting all fields from the first operator $O(x)$ with the fields of the operators $O(y)$. The general result is of the form $\langle O_{\ldots}(x)O_{\ldots}(y)\rangle = \mathrm{color}(G) \times \mathrm{kinematics}(x,y)$, with the color term containing information about the contractions of the different representations the fields are in the group G, while the kinematic term contains the part of the calculation regarding propagators and interactions, which generally need to be regularized, etc.

Then, for example, for the operator O of the type we defined above, with ϕ being precisely the scalar fields, we obtain

$$\left\langle O_{i_1\ldots i_l}(x)\,\overline{O}_{j_1\ldots j_l}(y)\right\rangle_{\mathrm{tree}} \sim \left(\delta_{i_1}^{j_1}\delta_{i_2}^{j_2}\cdots\delta_{i_l}^{j_l} + \text{cyclic perm.}\right)\frac{1}{(x-y)^{2l}}. \qquad (5.27)$$

To obtain the first order correction for l, we use the 4-point function interaction of the scalar fields in $\mathcal{N}=4$,

$$I = \int d^4z \mathrm{Tr}([\phi_i,\phi_j][\phi_i,\phi_j]) = 2\int d^4z \mathrm{Tr}(\phi_i^2\phi_j^2 - (\phi_i\phi_j)^2), \qquad (5.28)$$

then, inserting I inside the 2-point function of O_{ii+1}, we can obtain the correction for one of those terms present in (5.27), $\langle \phi_{i_k}\phi_{i_{k+1}}(x) I \phi_{j_{k+1}}\phi_{j_k}(y)\rangle$,

$$\langle \phi_{i_k}\phi_{i_{k+1}}(x) I \phi_{j_{k+1}}\phi_{j_k}(y)\rangle \sim \lambda\left(\delta_{i_k}^{j_k}\delta_{i_{k+1}}^{j_{k+1}} + \delta_{i_{k+1}}^{i_k}\delta_{j_{k+1}}^{j_k} - 2\delta_{i_k}^{j_{k+1}}\delta_{i_{k+1}}^{j_k}\right)\int d^4z\,\frac{1}{(z-x)^4(z-y)^4}, \qquad (5.29)$$

and to regularize this last term one notices that $f(z) = \frac{1}{(z-x)^4(z-y)^4}$ can be interpreted as two symmetric peaks, and then the integration (5.29) is roughly two times the integration from the first peak, located at x, to the second peak, located at y (two because we would need to consider the region to the left of the first peak, and to the right of the second peak).

Assuming $|x| > |y|$, then

$$\int_0^\infty d^4z\frac{1}{(z-x)^4(z-y)^4} \sim 2\int_y^x \frac{d^4z}{(z-x)^4(z-y)^4}. \qquad (5.30)$$

Changing coordinates, and calling Λ^{-1} the cutoff regulator, and then Wick rotate,

$$\int_0^\infty d^4z \frac{1}{(z-x)^4(z-y)^4} \sim \frac{2\Omega_3}{|x-y|^4} \int_{\Lambda^{-1}}^{|x-y|} \frac{d\epsilon}{\epsilon} = \frac{2\pi^2}{|x-y|^4} \ln \Lambda^2 |x-y|^2. \quad (5.31)$$

Then, considering the whole operator O instead of only a pair inside of it, we can generalize the above to obtain the final expression. Defining, beforehand, the operators P and K,

$$\begin{aligned} P_{k,k+1}\, \delta^{j_1}_{i_1} \cdots \delta^{j_l}_{i_l} &= \delta^{j_1}_{i_1} \cdots \delta^{j_{k+1}}_{i_k} \delta^{j_k}_{i_{k+1}} \cdots \delta^{j_l}_{i_l}, \\ K_{k,k+1}\, \delta^{j_1}_{i_1} \cdots \delta^{j_l}_{i_l} &= \delta^{j_1}_{i_1} \cdots \delta^{i_k}_{i_{k+1}} \delta^{j_k}_{j_{k+1}} \cdots \delta^{j_l}_{i_l}, \end{aligned} \quad (5.32)$$

as the permutation and the contraction operator, we obtain the first loop 2-point function,

$$\langle O_{i_1 \ldots i_l}(x)\, \overline{O}_{j_1 \ldots j_l}(y) \rangle = \frac{1}{(x-y)^{2l}} \left[1 - \ln\!\left(\mu^2 (x-y)^2\right) \hat{D}_{\text{1-l}} \right] \left(\delta^{j_1}_{i_1} \delta^{j_2}_{i_2} \cdots \delta^{j_l}_{i_l} + \text{cyclic perm.} \right). \quad (5.33)$$

We defined in (5.33) an operator, $\hat{D}_{\text{1-loop}}$, that contains P and K. By its form, the spectrum defined by it gives precisely the anomalous dimension correction for generic gauge-invariant operators of the theory. More precisely, it is $\hat{D}_{\text{1-loop}} = \frac{\lambda}{16\pi^2} \sum_{k=1}^{l} \left(1 - C - 2P_{k,k+1} + K_{k,k+1}\right)$, where C is a constant that we can fix by requiring chiral primary operators (see Section of CFT in Chapter 2) to have 0 eigenvalue over it, implying $C = -1$.

The case of BMN, however, the $\hat{D}$ operator is simpler. This is because K vanishes there, since it only involves the holomorphic fields Z, W and not their Hermitian conjugates. Moreover, we do not need to worry about the presence of other fields of $\mathcal{N} = 4$ contributing to the anomalous dimensions, for example coming from terms $\lambda \phi \bar{\lambda} \subset \mathcal{L}_{\mathcal{N}=4}$ in the Lagrangian, and this is a consequence of the sector being studied in the BMN case (remember, as explained in the Sections above, restricting the geodesic to the equator, etc.). The $SU(2)$ sector is closed, and no mixing phenomena occurs. The $\hat{D}$ operator in the $SU(2)$ sector is to first loop order simply,

$$\hat{D}^{SU(2)}_{\text{1-loop}} = \frac{\lambda}{8\pi^2} \sum_{k=1}^{l} \left(1 - P_{k,k+1}\right). \quad (5.34)$$

The fact that the sector is $SU(2)$ permits us to interpret the fields present in it as representations. We can, therefore, interpret Z as the upper spin $|\uparrow\rangle$ and W as the down spin $|\downarrow\rangle$, and we can at the same time write the operators in a basis where the elements are the Pauli-spin matrices. It is easy to see that $1 - P_{k,k+1}$ does the same in W, Z as $\frac{1}{2} - 2\vec{S}_k \cdot \vec{S}_{k+1}$ in $|\uparrow,\downarrow\rangle$, and then

$$\hat{D}^{SU(2)}_{\text{1-loop}} = \frac{\lambda}{8\pi^2}\sum_{k=1}^{l}\left(\frac{1}{2} - 2\vec{S}_k \cdot \vec{S}_{k+1}\right), \tag{5.35}$$

it just turns out that (5.35) is simply the Hamiltonian of the Heisenberg XXX spin-chain. A spin-chain is generally made of a term representing interactions between the spins of the particles in the chain, plus a term representing interactions of the spin with an external magnetic field. For the present case, it is easier to consider local interactions[6], in the sense that interactions only occur for neighboring particles, and a global magnetic field which is constant throughout the spin chain

$$\mathcal{H} = \sum_{i=1}^{N}\sum_{j=1}^{d}\left(\kappa_j\, \sigma^j_i \sigma^j_{i+1} + H_j \sigma^j_i\right), \tag{5.36}$$

where κ_j is the coupling describing the neighbor interactions, depending on the spatial direction the spins are pointed, and H_j are the components of the magnetic field H. Moreover, we have a periodic condition, $\sigma_{N+1} = \sigma_1$. Perhaps the most famous example of a spin-chain of this type is the $d = 1$ Ising model, an exactly solvable spin-chain [18] without a phase transition at finite temperature, but when generalized to $d = 2$ with $H = 0$, though also exactly solvable (by Onsager), it does present a phase transition with a critical temperature T_c related to κ_3.

Coming back to the main point here, the Heisenberg XXX is the $d = 1$ spin-chain (particles arranged in a row, but with spins pointing in the 3-dimensional direction x, y, z), with neighborhood coupling $\kappa_1 = \kappa_2 = \kappa_3$, without external magnetic field, $H = 0$. We see that the second term in (5.35) is precisely of that type, and then the problem of calculating the anomalous dimensions of non-chiral primary operators at first order is equivalently interpreted as the problem of obtaining the spectrum of a spin-chain Hamiltonian.

The problem was solved by Hans Bethe by making use of an insightful ansatz, which goes by its name "Bethe Ansatz" [25]. Nowadays, this ansatz, which we are not going to focus so much on, has been refined by making use of modern techniques, like the monodromy matrix, transfer matrix, R matrix, etc., and then the literature distinguishes between two different approaches to solving for the

[6] Some cases of "long-range" spin-chains are also studied in the holography context, though it can be really thought of as a long chain of neighborhood interactions, and generally correlated to "wrapping effects" emerging when the loop order in question is higher than, or equal to, the range of the spin-chain [14].

spectrum [150], the coordinate Bethe ansatz [104], based on the original method proposed by Bethe that solved the problem, or the algebraic Bethe ansatz, which makes use of modern techniques. See [137] or the cited papers in this paragraph to see this subject in detail in the context of integrability, with the techniques being applied not only for the Heisenberg spin-chain but more general integrable systems.

Since the calculations are too extensive, and the focus of the present dissertation is not to go through the complex details of integrability and so on, we simply state here the deformed Δ to 1-loop solution, and one not so interested in integrability etc. can see the derivation straightforwardly in [4],

$$\Delta_{\text{1-loop}} = \frac{\lambda}{\pi^2}\sin^2\left(\frac{\pi n}{\ell - 1}\right), \tag{5.37}$$

(5.37) in the large l limit, $l \sim J \to \infty$, goes to $\Delta = n^2 \frac{\lambda}{J^2}$, which is precisely the deviation obtained in (5.26). We see, therefore, how powerful the BMN correspondence is, being one of the few examples where we can verify a matching in the gauge/gravity duality beyond BPS conditions.

Chapter 6

Deformations

As the name suggests, the idea behind deformations is to modify a model. This immediately raises the following question: why deform it? Some possible reasons include:

- **Preservation of symmetries.** Suppose we have a d dimensional theory with $\mathcal{N}$ supersymmetry. This model possesses its own symmetries and features, such as R symmetry, global symmetry, integrability, gauge symmetry, among others. We may look for deformations of this model that preserve specific symmetries, and it is easier to deform because finding solutions to the supergravity equation of motion is a non-trivial work, especially solutions preserving some amount of supersymmetry. Thus, starting from a background that already satisfies the equations of motion makes it substantially easier to obtain new solutions and if the original solution preserves some symmetry, the deformed one can preserve a residue of it. This provides, moreover, useful tools to study holography to any extension.

- **Duality.** Similarly, one may want to find a gravitational dual of the gauge model being deformed, or vice-versa. Suppose, for example, that we start with the model from the previous item and find a deformation in the field-theory side that breaks supersymmetry from $\mathcal{N} = 4$ down to $\mathcal{N} = 1$. By identifying a gravitational analogue of this deformation, we obtain a new gravitational model dual to an $\mathcal{N} = 1$ theory, with some deformation of the original spacetime, and moreover it can be a useful tool to study the gauge/gravity duality.

- **RG flow.** Introducing operators in a Yang Mills theory generally affects the renormalization group equations. In particular, adding irrelevant operators may drive the theory toward the ultraviolet (UV), while relevant operators may drive it toward the infrared (IR). It is quite interesting to see those limits manifest in the supergravity background, which can sometimes show to be non-trivial.

The following sections present some well known deformations from the literature. We emphasize that these deformations can be interpreted as those that are more transparent from the field theory point of view, while their implications on the gravity side still need to be carefully considered. This should be contrasted with some holographic examples. For instance, the Klebanov Strassler solution can be thought of as a deformation of a model, but the supergravity side is more straightforward to understand, since we approach it directly from the gravity perspective, than the field theory side. Then, we can interpolate between both points of view, with the most clear side depending on the deformation.

6.1 β

The beta deformation, originally proposed by Lunin and Maldacena [106], is an exactly marginal deformation in the planar limit, and deviations from this limit are known to be exact at first loop order. In fact, Rossi [138, 139] and Mauri [113] showed the exactness of the finiteness constraint in the planar limit, $N \to \infty$, of the β deformation, suggesting that integrability is preserved by this deformation, at least in the planar limit. This means, among other things, that the deformation does not drive the theory toward either the UV or the IR. Moreover, it preserves integrability and, since it is a deformation of $\mathcal{N} = 4$ SYM, this is equivalent to saying that the deformation preserves the conformal symmetry of the model. Highlighting these statements is important, since they play a crucial role in the search for a gravitational dual of the deformation. Preserving $SO(4,2)$ essentially means, from the gravity point of view, preserving the AdS$_5$ space. Thus, regardless of the final gauge model, the gravitational dual must still be of the form AdS$_5 \times Z^5$.

It also breaks part of the supersymmetry, reducing $\mathcal{N} = 4$ to $\mathcal{N} = 1$. This was demonstrated by Leigh and Strassler in their paper on deformations of $\mathcal{N} = 4$, with the real beta deformation[1] being a special case [103]. These deformations preserve part of the supersymmetry, resulting in a wide class of $\mathcal{N} = 1$ theories, while also maintaining conformal symmetry, since they are marginal deformations. This class of deformations is fully characterized by four parameters (g, q, h, κ), corresponding to the undeformed coupling, the parameter deforming the algebra,

[1] From now on, real beta deformation and beta deformation are going to be used as synonymous, unless otherwise stated

the one responsible for a new term in the superpotential, and the deformed coupling, respectively. The deformation acts on the superpotential, and it was also shown in [103] that two special symmetries are preserved, the cyclic property among the scalar fields ($X_i \to X_{i+1}$) and invariance under $X_i \to X_i\omega^{i+1}$. These symmetries, together with conformal exactness, imply a finiteness constraint $\mathcal{F}(g, q, h, \kappa)$ relating the four parameters. Among this family of models, some, such as the present β deformation [107], correspond to special points for which $\mathcal{F}$ does not receive quantum corrections at any higher loop order [113] in the planar limit, and they admit a clear interpretation in terms of transformations of the supergravity background [63]. Other deformations have also been shown to be exactly conformal and related to twists [111], although these cases are less well understood from the point of view of gauge gravity duality.

Furthermore, we obtain an interesting solution that preserves part of the supersymmetry while maintaining conformal symmetry, which is always desirable. From the gravitational perspective, this has a clear meaning. Breaking $\mathcal{N} = 4$ implies breaking the $SU(4)$ R symmetry and, consequently, $SO(6)$, thereby deforming the sphere S^5. In the end, we are left with a $U(1)$ R symmetry, that is, $SO(2) \sim U(1)$ from the gravitational point of view. What does this imply? Do we obtain a torus or a circle? Not necessarily. With the information provided so far, this simply indicates that whatever the final space Z^5 is, it must possess a $U(1)$ isometry. In other words, there exists an angle ϕ such that Z^5 is invariant under $\phi \to \phi + \epsilon$, corresponding to a rotation.

Given this information, this is all that can be deduced,

$$\mathcal{N} = 4 \underbrace{\longrightarrow}_{\text{marginal}} \mathcal{N} = 1 \quad \cong \quad \mathrm{AdS}_5 \times \mathrm{S}^5 \;\longrightarrow\; \mathrm{AdS}_5 \times Z^5\Big|_{U(1)\subset Z^5} . \tag{6.1}$$

To define the deformed Lagrangian, we make use of the fact that the $\mathcal{N} = 4$ vector multiplets can be decomposed into a sum of two different $\mathcal{N} = 1$ multiplets: one $\mathcal{N} = 1$ vector multiplet and three chiral multiplets Φ^i.

We may therefore rewrite the $\mathcal{N} = 4$ model in terms of $\mathcal{N} = 1$ chiral and vector superfields. Concerning the superpotential, which is the relevant part for the deformation, we first write its most general form

$$\mathcal{W}_{\mathrm{LG}} = \kappa \left(\mathrm{Tr}(\Phi_1\Phi_2\Phi_3 - q\Phi_1\Phi_3\Phi_2) + h' \,\mathrm{Tr}(\Phi_1^3 + \Phi_2^3 + \Phi_3^3) \right) . \tag{6.2}$$

The real beta deformation is such that $h' = 0$ and $|q|^2 = 1$. We can therefore rewrite the above superpotential as

$$\mathrm{Tr}\Big(e^{i\pi\beta}\,\phi_1\phi_2\phi_3 - e^{-i\pi\beta}\,\phi_1\phi_3\phi_2\Big)\,. \tag{6.3}$$

We can now better understand the structure of the spacetime Z^5. It possesses not only a $U(1)$ isometry, but in fact a $U(1) \times U(1) \times U(1)$ isometry. Why is this interesting, or more precisely, what is more informative than simply preserving $U(1)_R$? Note that $U(1)_R$ was not necessarily manifest in the original S^5 geometry. The symmetry group may break in such a way that X $(= SO, SU, Sp, \dots) \rightarrow U(1)_R$. Thus, while $U(1)_R$ must be present in Z^5, this fact alone does not determine how S^5 is deformed. In contrast, the fact that the beta deformation preserves $U(1)_1 \times U(1)_2$ is meaningful, since these symmetries were present before and must remain present afterwards. With this observation, the central point is that the gravitational dual is determined by a transformation acting on the complex structure modulus τ of a T^2 geometry,

$$\tau = B + i\sqrt{g_{\mathrm{torus}}} \;\longrightarrow\; \tau_\beta = \frac{\tau}{1+\beta\,\tau}. \tag{6.4}$$

In the original model, having a $U(1) \times U(1)$ symmetry implies that, intrinsically, the geometry contains a product of two tori, $T^1 \times T^1$, hidden inside it in a certain sense. Using the rules of Appendix B for the TsT transformation over two isometric angles of $\mathrm{AdS}_5 \times \mathrm{S}^5$, and omitting the RR and NSNS fields, we obtain

$$ds^2_\beta = R^2\left[ds^2_{\mathrm{AdS}_5} + \sum_i (d\mu_i^2 + G\,\mu_i^2 d\phi_i^2) + \hat{\gamma}^2 G\,\mu_1^2\mu_2^2\mu_3^2\left(\sum_i d\phi_i\right)^2\right] \tag{6.5}$$

where the three $U(1)$ symmetries are manifest, since μ_i, G, and $\hat{\gamma}$ do not depend on the angles $\phi_{i=1,2,3}$.

Moreover, regarding integrability, this can also be verified more directly by using the $\mathbf{R}-\mathbf{matrix}$ and the Yang-Baxter equation for the quantum R-matrix, (6.6) [150],

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{6.6}$$

We then borrow the results already derived in [111] for the R matrix of general Leigh Strassler type deformations. We rewrite them here for the reader's convenience,

$$R = \frac{1}{|q|^2+|h|^2+1} \begin{pmatrix} 1+q\bar{q}-h\bar{h} & 0 & 0 & 0 & 0 & -2\bar{h} & 0 & 2h\bar{q} & 0 \\ 0 & 2\bar{q} & 0 & 1-q\bar{q}+h\bar{h} & 0 & 0 & 0 & 0 & 2h\bar{q} \\ 0 & 0 & 2q & 0 & -2h & 0 & q\bar{q}+h\bar{h}-1 & 0 & 0 \\ 0 & q\bar{q}+h\bar{h}-1 & 0 & 2q & 0 & 0 & 0 & 0 & -2h \\ 0 & 0 & 2h\bar{q} & 0 & 1+q\bar{q}-h\bar{h} & 0 & -2\bar{h} & 0 & 0 \\ 2h\bar{q} & 0 & 0 & 0 & 0 & 2\bar{q} & 0 & 1-q\bar{q}+h\bar{h} & 0 \\ 0 & 0 & 1-q\bar{q}+h\bar{h} & 0 & 2h\bar{q} & 0 & 2\bar{q} & 0 & 0 \\ -2h & 0 & 0 & 0 & 0 & q\bar{q}+h\bar{h}-1 & 0 & 2q & 0 \\ 0 & -2\bar{h} & 0 & 2h\bar{q} & 0 & 0 & 0 & 0 & 1+q\bar{q}-h\bar{h} \end{pmatrix} \tag{6.7}$$

Equation (6.7) represents a quantum R matrix in a general form. We should note that we are slightly abusing the notation, since the matrix above satisfies the Yang-Baxter equation only for specific pairs (q, h). It would therefore be more appropriate to refer to it as an R matrix only for these pairs. In any case, when $h = 0$ and $|q| = 1$, equation (6.7) satisfies (6.6), and therefore the beta deformation is shown to be integrable.

6.2 γ

The idea behind the gamma deformation is similar to that of the beta deformation, and thus we can borrow concepts from Section 6.1. Originally obtained by Frolov [63], he was able to show how to construct the gravitational dual without explicitly referring to tori or to the $U(1)$ symmetries. One simply applies three TsT transformations, that is, T duality, shift, and T duality. Since S^5 has $SO(6)$ symmetry, it possesses three isometric angles when the metric is written in spherical coordinates. Writing the metric as in the beta deformation case, making the isometric angles explicit, we can apply three TsT operations, for instance (ϕ_1, ϕ_2), (ϕ_2, ϕ_3), and (ϕ_1, ϕ_3). In this way, the metric becomes very similar to (6.5),

$$ds^2_{\text{str}} = R^2 \left[ds^2_{\text{AdS}_5} + \sum_i (d\mu_i^2 + G\, \mu_i^2 d\phi_i^2) + G\, \mu_1^2\mu_2^2\mu_3^2 \left(\sum_i \gamma_i\, d\phi_i \right)^2 \right]. \tag{6.8}$$

Note that the $U(1)^3$ symmetry is still present. However, this spacetime is not a supersymmetric solution of supergravity. Therefore, we expect that the dual of this spacetime will likewise have no supersymmetry, that is, the deformation breaks all the supersymmetry of the model. Why is this deformation still relevant? Because, as is evident from the resulting metric, the AdS part of the spacetime is preserved and, therefore, the dual theory is expected to remain invariant under conformal transformations. Moreover, integrability is preserved. Before and after the deformation, there still exist infinitely many Lax pairs, with $[A, B] = 0$.

Indeed, the deformed potential, obtained by making use of the F terms and discarding the D terms, is $V = \sum_i |\partial_{\phi_i} \mathcal{W}|^2$,

$$V = \mathrm{Tr}\Big[|\Phi_1\Phi_2 - e^{-2i\pi\gamma_3}\,\Phi_2\Phi_1|^2 + |\Phi_2\Phi_3 - e^{-2i\pi\gamma_1}\,\Phi_3\Phi_2|^2 + |\Phi_3\Phi_1 - e^{-2i\pi\gamma_2}\,\Phi_1\Phi_3|^2\Big] \\ + \mathrm{Tr}\Big[\big([\Phi_1, \bar{\Phi}_1] + [\phi_2, \bar{\phi}_2] + [\phi_3, \bar{\phi}_3]\big)^2\Big], \tag{6.9}$$

As explained in [63], the existence of Lax pairs implies that the model is integrable[2].

6.3 Yang-Baxter deformations

Another important class of deformations in the literature are known as Yang-Baxter (YB) deformations. They are qualitatively different from the previous ones because, while those were mainly highlighted through their supergravity solutions and field theory duals, YB deformations can be understood as deformations at the level of the string worldsheet. We therefore take the worldsheet action as the main object of study, while the supergravity solutions and the field theory are left for later analysis. Consequently, since the worldsheet is a $D = 2$ model, we can expect topological terms such as the Wess-Zumino (WZ) term[3], and it is natural to use the language of chiral models, where the action can in general be described in terms of currents.

One can understand these deformations within string theory by making use of an analogy with simpler models. Consider a generic $d = 2$ model with classical action

$$S = -\frac{1}{2}\int d^2x\, \eta^{\mu\nu}\, \mathrm{tr}(J_\mu J_\nu), \tag{6.10}$$

where the current J takes values in the Lie algebra $\mathfrak{g}$ of a group $G(x)$, and the trace ensures invariance. The current J_μ is sometimes referred to as a left invariant one form, since it is given by

$$J_\mu = g^{-1}\partial_\mu g, \tag{6.11}$$

with $g(x)$ taking values in G. The equation of motion of the action (6.10) is obtained by varying (6.11), and after discarding boundary terms it reduces to $\partial_\mu J^\mu = 0$, which expresses current conservation [172].

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[2] at least classically.

[3] or WZW, where the level, now in the mathematical sense, is quantized as required for conformal theories [94]

The action has a $G_L \times G_R$ global symmetry of the form $g(x) \to g_L \cdot g(x)$ and $g(x) \to g(x) g_R$. By Noether's theorem, these symmetries imply the existence of conserved currents, namely the right invariant current I and the left invariant current J introduced above.

Another important concept is that these currents satisfy the flatness condition[4],

$$\partial_{[\mu} j_{\nu]} + \frac{1}{2}[j_\mu, j_\nu] = 0, \tag{6.12}$$

where in (6.12) we use the convention $A_{[\mu\nu]} = \frac{A_{\mu\nu} - A_{\nu\mu}}{2}$. Equation (6.12) is extremely important in the context of integrability. Classically, it underlies the chain of implications stating that the Lax condition leads to flatness, which implies that the monodromy matrix depends only on the endpoints, and therefore guarantees the existence of a generating function producing conserved quantities for any value of the spectral parameter u.

With this information, Yang-Baxter deformations become straightforward to understand. At the classical level, one first considers the classical analogue of the quantum R-matrix equation. Starting from (6.6) and using $R \sim e^{ir}$, one obtains

$$[r_{12}, r_{13}] + [r_{13}, r_{23}] + [r_{12}, r_{23}] = 0, \tag{6.13}$$

where $r = \sum_i a_i \otimes b_i$ is the classical r matrix taking values in $\mathfrak{g} \otimes \mathfrak{g}$, and for instance $r_{12} = r \otimes I$. Equation (6.13) can be modified by adding a non zero term on the right hand side, $\omega[c_{12}, c_{13}]$, where c denotes the Casimir operators of the algebra [172]. The parameter ω controls the deformation of the Yang-Baxter equation and can take three distinct values,

$$\begin{aligned} \omega = 0, &\quad \text{homogeneous CYBE} \\ \omega = -1, &\quad \text{split type CYBE} \\ \omega = 1, &\quad \text{non split type CYBE.} \end{aligned} \tag{6.14}$$

In some conventions one writes $\omega = -\theta^2$, in which case (6.14) becomes $0, 1, i$ [81].

Yang-Baxter deformations focus on the modified case, with $\omega \neq 0$ in (6.14), and were originally studied in the context of the principal chiral model in [98, 99].

[4] In analogy with the expression for curvature given by the second Cartan structure equation, $\rho = d\omega + \omega \wedge \omega$

In its simplest form, the deformation modifies the Lagrangian as

$$S(\chi) = -\frac{1}{2}\left(\eta^{\mu\nu} - \epsilon^{\mu\nu}\right) \int d^2x \, \mathrm{Tr}\left(J_\mu \frac{1}{1-\chi R} J_\nu\right), \tag{6.15}$$

where χ is the deformation parameter. The global symmetry of the undeformed model is reduced to a single copy, G_L, but the deformed action still admits Lax pairs satisfying both the equations of motion and the flatness condition. As a result, the model remains classically integrable, provided the deformed R matrix satisfies the modified Yang-Baxter equation [172].

The discussion above can be extended to symmetric cosets, where $G_L = G_R$, as shown in [47]. One year later, the same authors applied these ideas to the string worldsheet on $\mathrm{AdS}_5 \times \mathrm{S}^5$, which can be written as the coset space $\frac{PSU(2,2|4)}{SO(4,1)\times SO(5)}$ [46]. In that work, fermionic symmetries were also included, although there are subtleties related to the treatment of gauge degrees of freedom that lie beyond the scope of the present discussion.

We therefore simply state that, among the Yang-Baxter deformed models describing strings propagating on the supersymmetric background $\mathrm{AdS}_5 \times \mathrm{S}^5$, two are particularly well known in the literature: the λ and the η deformations [122, 172]. An important point, which we now emphasize, is that Yang-Baxter deformations differ from the previous classes not only in their construction, but also in the fact that they do not necessarily lead to a supergravity solution.

In particular, the λ deformation is a special case that satisfies the homogeneous classical Yang-Baxter equation, and whose R matrix is nilpotent. As shown in the original work [90], this deformation does generate a supergravity solution. The η deformation, on the other hand, satisfies a generalized supergravity equation, and only when its r matrix obeys the unimodularity condition does it correspond to a standard supergravity solution [46].

6.4 $\mathrm{T}\bar{\mathrm{T}}$

Some of the deformations presented in the last Sections were shown to be marginal, preserving the conformality of the model, but of course, not all types of deformations must belong to this class. The $\mathrm{T}\bar{\mathrm{T}}$ deformation is essentially different from the others not only by its nature in driving the theory, but also in the way it deforms the model. Taking some analogy, the $\mathrm{T}\bar{\mathrm{T}}$ could be thought of as an "explicit deformation", borrowing the concept from field theory when talking about chiral symmetry, where we can break it simply by adding small massive terms to the Lagrangian. Indeed, while the γ/β deformation, purely from the point of view of the terms in the Lagrangian, could be thought as

$$[,] \subset \mathcal{L} \to [,]_q \subset \mathcal{L}_{\text{deformed}}, \tag{6.16}$$

the T$\bar{\text{T}}$ explicitly deforms the action by adding a new term to it, $\int d^d x \lambda O$. In this way, depending on the dimensionality of λ, the deformation may be marginal, irrelevant, or relevant. For the present case, it is an irrelevant one with striking properties generally not expected for irrelevant interactions.

While the previous deformations typically deformed some specific type of supersymmetric Yang–Mills models, the T$\bar{\text{T}}$ deformation applies in general to $d = 2$ models, and while it is not necessarily of conformal type, the models are way more controlled when conformal symmetry is present, and moreover, the spectrum is easily obtained, only depending on the space (cylinder, torus) we study the field theory.

To be more precise, we deform the Lagrangian as

$$\mathcal{L} \to \mathcal{L} + \int d^2 x \lambda O(x), \tag{6.17}$$

but since we are working in $d = 2$ we can use complex coordinates,

$$\mathcal{L} \to \mathcal{L} + \int dz\, d\bar{z}\, \lambda O(z, \bar{z}). \tag{6.18}$$

The operator $O(z, \bar{z}) = T(z)\bar{T}(z') - \Theta(z)\Theta(z')$, where T denotes the stress–energy tensor of the model in complex coordinates, so $T = T_{zz}$, $\bar{T} = T_{\bar{z}\bar{z}}$, and $\theta = T_{z\bar{z}}$, up to overall factors.

Note that the stress–energy tensor has units $[T] = d$, since the 0-component of it integrated over the spatial volume is the energy and then $[O] = 2d$. Finally, $[\lambda] = -d$, in this case -2.

Since that operator deforms mainly conformal models, as explained before, it automatically becomes an interesting object in the context of gauge/gravity duality. Let us show some interesting properties by following Gross et. al [70]. There, the authors showed that the spectrum of the deformed model is purely a function of the undeformed spectrum E_0 and of the deformation parameter λ. Indeed, let

$$S = S_0 + \lambda \int d^2 x\, O_{\text{T}\bar{\text{T}}}, \tag{6.19}$$

then, the energy can be obtained as $T^{00} = -2\frac{\delta S}{\delta \eta^{00}}$, so

$$E = -2\int \frac{\delta S}{\delta \eta^{00}} d^2x = -2\int \frac{\delta(S_0 + \lambda \int d^2x' O_{\mathrm{T\bar{T}}})}{\delta \eta^{00}}\, d^2x = E_0 - 2\lambda \int\int \frac{\delta O_{\mathrm{T\bar{T}}}}{\delta \eta^{00}}\, d^2x\, d^2x', \tag{6.20}$$

but $O_{\mathrm{T\bar{T}}} = \mathrm{T\bar{T}} - \theta^2 = T_{zz}T_{\bar{z}\bar{z}} - T_{z\bar{z}}^2 \sim -(T_{00}T_{11} - T_{01}^2)$, where "$\sim$" means we are ignoring irrelevant constant factors, which can be absorbed into λ anyway. We can rewrite $O_{\mathrm{T\bar{T}}}$ as

$$O_{\mathrm{T\bar{T}}} = -\left(T^{00}(\eta_{00})^2 T_{11} - (\eta_{00}\eta_{11}T^{01})^2\right), \tag{6.21}$$

and therefore, the deformed spectrum is

$$\begin{aligned} E = E_0 - 2\lambda \int\int (\eta^{00}T_{00}(x)T_{11}(x') - \eta_{00}(\eta_{11}T^{01})^2(x,x'))\, d^2x d^2x' = \\ E_0 - 2\lambda \int\int (-T_{00}(x)T_{11}(x') + T_{01}^2(x,x'))\, d^2x d^2x', \end{aligned} \tag{6.22}$$

where $T_{01}(x,x')^2 = T_{01}(x)T_{01}(x')$. Finally, defining $\int T_{01}(x)d^2x = J$ and using the traceless property $T_{00} - T_{11} = 0$ due to conformal invariance, allow us to write the energy as

$$E = E_0 - 2\lambda(-E^2 + J^2) \tag{6.23}$$

and solving for E, we obtain

$$E = \frac{1}{4\lambda}\left(1 - \sqrt{1 - 8E_0\lambda + 16J^2\lambda^2}\right). \tag{6.24}$$

Note that this matches the result obtained in [70]. One interesting point is the resemblance between this energy and that of a $d = 2$ black hole with a radial cutoff r_c, allowing the identification of $\lambda \sim 1/r_c^2$.

This deformation was shown to be integrable in the seminal paper of Smirnov and Zamolodchikov [151]. The derivation of that is quite simple, which perhaps explains the success of the work, and then here we follow a similar line of reasoning, using a more familiar notation consistent with this dissertation. Moreover, we emphasize that integrability here literally means classical integrability: of infinite conserved quantities which are in involution (that is $[X_i, X_j] = 0$) among themselves, and then the concept of R-matrix or Yang-Baxter is not mentioned anywhere.

Holographically, the $\mathrm{T\bar{T}}$ operator was proposed to be equivalent to bringing the boundary to a finite radius $\rho = \rho_c$ [116], though some caveats and problems with this interpretation were later realized. We will touch upon these points later. Now, to understand the motivation behind this interpretation, we first show how the deformation operator modifies the field on the gravity side. To explain that, let's define the following: S^λ is a deformed Lagrangian with a deformation term $\delta S^\lambda_{TT} = \lambda \int d^2x\ \mathrm{T\bar{T}}$, and since it is linear on λ, we have the following relation between the actions of the deformed model:

$$S^{\lambda+\Delta\lambda} = S^\lambda + \Delta\lambda\ \delta S_{TT}. \tag{6.25}$$

But since $T\bar{T} = \det T$, we can write δS as

$$(\delta S)_{TT} = -\frac{1}{2}\int d^2x\sqrt{\gamma}(T^{ab}T_{ab} - T^2). \tag{6.26}$$

We remember that

$$\delta S^\lambda = -\frac{1}{2}\int d^2x T^{ab}{}_\lambda \delta\gamma_{ab}\sqrt{\gamma}, \tag{6.27}$$

and then, we can vary both sides of (6.25) and substitute (6.26) to obtain

$$\frac{1}{2}\int d^2x\left(\sqrt{\gamma}T^{ab}\delta\gamma_{ab}\right)_{\lambda+\Delta\lambda} - \frac{1}{2}\int d^2x\left(\sqrt{\gamma}T^{ab}\delta\gamma_{ab}\right)_\lambda = -\frac{\Delta\lambda}{2}\delta\left(\int d^2x\sqrt{\gamma}(T^{ab}T_{ab} - T^2)\right), \tag{6.28}$$

which can be written as a differential equation,

$$\partial_\lambda\left(\int d^2x\left(\sqrt{\gamma}T^{ab}\delta\gamma_{ab}\right)\right) = \delta\left(\int d^2x\sqrt{\gamma}(T^{ab}T_{ab} - T^2)\right). \tag{6.29}$$

This differential equation was obtained and solved in [74], where they expressed the deformed operators in terms of the undeformed,

$$\gamma^\lambda_{ab} = \gamma^0_{ab} - 2\lambda\hat{T}^0_{ab} + \lambda^2\hat{T}^0_{ap}\hat{T}^0_{\sigma\beta}\gamma^{0p\sigma}, \quad \hat{T}^\lambda_{ab} = \hat{T}^0_{ab} - \lambda\hat{T}^0_{ap}\hat{T}^0_{\sigma\beta}\gamma^{0p\sigma}, \tag{6.30}$$

where $T = \hat{T} - \gamma\mathrm{tr}\hat{T}$. To fully express γ in terms of the undeformed metric, we first express the general metric in Fefferman-Graham coordinates [56],

$$ds^2 = \frac{dp^2}{4p^2} + g_{ab}dx^a dx^b. \tag{6.31}$$

Then, the perturbation terms in (6.31) are of the form, $g_{ab} = \frac{g^0_{ab}}{p} + g^2_{ab} + \ldots$, and there is a well-known matching between the perturbation g^2_{ab} and the expectation value of the energy-momentum tensor, $g^2_{ab} = 8\pi G T^0_{ab}$ [45], which allows us to write (6.30) as

$$\gamma^{\lambda}_{ab} = g^0_{ab} - \frac{1}{4\pi G}\lambda g^2_{ab} + \frac{\lambda^2}{(8\pi G)^2} g^2_{\alpha\gamma} g^{0\gamma\delta} g^2_{\delta\beta}, \quad \hat{T}^{\lambda}_{ab} = \frac{g^2_{ab}}{8\pi G} - \frac{\lambda}{(8\pi G)^2} g^2_{\alpha\gamma} g^{0\gamma\delta} g^2_{\delta\beta}. \quad (6.32)$$

A useful result to understand the cutoff idea was obtained in [149], where the authors showed that for a pure gravity solution in $d = 3$, the Fefferman expansion truncates at the third term, and then the metric has the exact form

$$\gamma^{\lambda}_{ab} = g^{(0)}_{\alpha\beta} + \rho\, g^{(2)}_{\alpha\beta} + \frac{\rho^2}{4}\, g^{(2)}_{\alpha\gamma}\, g^{(0)\,\gamma\delta}\, g^{(2)}_{\delta\beta}. \quad (6.33)$$

Comparing (6.32) with (6.33), if we assume a cutoff at $\rho_c = -\frac{\lambda}{4\pi G}$, we see there is an exact matching between the deformed metric driven by the $T\bar{T}$ operator and the $d = 3$ asymptotically AdS_3 metric (for $\lambda < 0$). Moreover, the stress tensor in (6.32) can also be compared with those expected for a theory with a cutoff/boundary. As is well known in the literature, a gravitational theory which has a boundary needs to modify the definition of the energy-momentum tensor by adding boundary terms. More specifically, a gravitational model with a boundary requires a gravitational boundary term known as the Gibbons-Hawking (GH)

$$S_{\partial M} = -\frac{1}{8\pi G}\int_{\partial M} d^{d-1}x\sqrt{\gamma}K, \quad (6.34)$$

where γ is the induced metric on the boundary, and K is the trace of the extrinsic curvature, which is a curvature depending on the embedding space the surface is on. The GH term (6.34) can be understood as a counterterm in the action, to balance the variation of the usual Einstein-Hilbert action S_{EH} since we cannot assume boundary terms vanish (as is usually done, considering them at infinity and neglecting the contribution). To calculate K we can either use the induced metric γ together with a unit vector normal to the boundary n, $K = \gamma^{\mu\nu}\nabla_\mu n_\nu$ or we can directly use a tangent vector to the boundary as well, $K = t^a n_b \nabla_a t^b$. Then, the energy-momentum tensor with such a term is known as the Brown-York stress tensor, and has the expression in $d = 2$ [73],

$$T_{\alpha\beta} = -\frac{1}{8\pi G}\left(K_{\alpha\beta} - g_{\alpha\beta}K + g_{\alpha\beta}\right). \quad (6.35)$$

We can then compute it for the pure gravity solution. By substituting the cutoff $\rho_c = -\frac{\lambda}{4\pi G}$ into (6.32), both energy momentum tensors match precisely [116].

Although remarkable, one should be concerned about some apparently contradictory consequences of introducing a cutoff. Bringing the boundary to a finite radial position is dual to integrating out modes with energies higher than some scale E_c. This seems to be in tension with the UV completeness of the $\mathrm{T\bar{T}}$ deformation. Moreover, as argued in [74], the cutoff matching breaks down once matter fields are included, which should not pose a problem within the holographic context.

Motivated by these observations, several attempts were made to generalize the $\mathrm{T\bar{T}}$ deformation in order to study more interesting dual models. From now on, we refer to these as double trace deformations, with $O_{\mathrm{T\bar{T}}} \sim \det = \lambda_1 \lambda_2$, while $\mathrm{Tr} = \lambda_1 + \lambda_2$. Thus, one schematically has $O_{\mathrm{T\bar{T}}} \sim \mathrm{Tr}(O_{\mathrm{T\bar{T}}}) \times \mathrm{Tr}(O_{\mathrm{T\bar{T}}}) + O(\lambda^2)$. One of the most interesting analogous constructions that emerged from this line of reasoning is the so called single trace version of the $\mathrm{T\bar{T}}$ deformation.

We would like to identify the gravitational background appropriate for describing a conformal theory in $d = 2$. From the equivalence between spacetime isometries and global symmetries, this naturally requires an AdS_3 factor. Furthermore, after performing a Kaluza Klein reduction, the remaining space X_7 must be compact. We are therefore led to a background of the form $\mathrm{AdS}_3 \times X_7$.

It can be shown that solutions of supergravity sourced by p branes typically approach, in the near horizon limit, a geometry of the form $\mathrm{AdS}_{p+2} \times \mathrm{S}^{8-p}$, up to a conformal factor. Consequently, we expect our model to involve one branes, more specifically fundamental strings,

$$\text{F1 brane} \sim \mathrm{AdS}_3 \times X^7. \tag{6.36}$$

As discussed earlier in the context of dualities, type IIB string theory admits a Hodge star duality. Therefore, although not strictly required, it is natural to expect that in addition to F1 branes there are also NS5 branes, which are magnetically dual to the F1 branes.

The NS5 brane solution is given by

$$ds^2 = dx^\mu dx_\mu + \left(1 + \frac{N_5 \alpha'}{r^2}\right)\left(dr^2 + r^2 d\Omega_3^2\right), \quad e^{2\Phi} = g_s^2 \left(1 + \frac{N_5 \alpha'}{r^2}\right), \quad H = \star_4 d\Phi. \tag{6.37}$$

In this case, the decoupling limit requires keeping $u = r/g_s$ fixed as $g_s \to 0$, which forces the presence of NS5 branes.

From this solution, we see that while the region $r = \infty$ is regular, the limit $r \to 0$ leads to a divergent dilaton. Thus, even though the NS5 brane background is a valid supergravity solution, an additional condition must be imposed near $r = 0$ to regulate this divergence.

One possible resolution is to add F1 branes parallel to the NS5 branes, since perpendicular configurations would break supersymmetry. This configuration is illustrated in Figure 7.4.

There remains, however, a further issue, since there exist trajectories that reach $r = 0$ without intersecting the F1 branes. A way to resolve this is to compactify the remaining NS5 brane directions on a T^4. The resulting background takes the form

$$ds^2 = \frac{-dt^2 + dx_1^2}{f_1} + N_5\alpha' \left(\frac{dr^2}{r^2} + d\Omega_3^2\right) + ds^2_{T^4}, \quad e^{2\Phi} = \frac{N_5\alpha'}{r^2 f_1}, \quad f_1 = 1 + \frac{r_1^2}{r^2}, \quad r_1^2 = \frac{N_1\alpha'}{v}. \tag{6.38}$$

A central and interesting feature of this solution is that it can be obtained relatively straightforwardly from the near horizon limit of the NS5 F1 configuration by applying a transformation already mentioned earlier, namely a TsT transformation. In this case, however, the transformation involves time rather than only spatial isometries. It is noteworthy how frequently, and sometimes unexpectedly, TsT transformations appear in the context of deformations of gauge theories and their gravitational duals.

The gravitational dual of this deformation was proposed by Kutasov et al. [68]. In their work, they argue that the dual conformal field theory is described by a symmetric orbifold. An orbifold is defined as the quotient of a space by a discrete group, $\mathcal{M}/X$. In the present case, the proposal is

$$\mathrm{CFT} \sim \frac{(\mathcal{M}_{6N_5})^{N_1}}{S_{N_1}}, \tag{6.39}$$

where $\mathcal{M}_{6N_5}$ is a conformal theory with central charge $c = 6N_5$, so that $[L_m, L_n] \sim L_{m+n} + c(\ldots)$, with L_n the Virasoro generators. The duality can be tested by comparing the volume of the deformed spectrum, $Z_{\mathrm{T\bar{T}}}$, with its gravitational counterpart. Furthermore, $Z_{\mathrm{T\bar{T}}}$ can be computed directly from the undeformed partition function Z.

Given the $\mathrm{T\bar{T}}$ deformation, in particular its single trace version, and its proposed gravitational dual, several open questions remain. Is this construction restricted to $d = 2$? If not, how can the $\mathrm{T\bar{T}}$ deformation be generalized to higher dimensions, and what would the corresponding gravitational duals be? Several authors, including Marika Taylor [160] and Bonelli et al. [29], have attempted to address these questions by searching for deformations that retain T$\bar{\text{T}}$ like properties.

For example, since $O_{\text{T}\bar{\text{T}}} \sim \det T_{\mu\nu}$, and $\det T \sim \frac{1}{n!}\epsilon^{\mu_1\ldots\mu_n}\epsilon^{\nu_1\ldots\nu_n}T_{\mu_1\nu_1}\cdots T_{\mu_n\nu_n}$, it is tempting to generalize the T$\bar{\text{T}}$ deformation as the determinant of the stress energy tensor, possibly supplemented by additional potentials or constants. However, a fully satisfactory analogue for dimensions greater than two is still not known.

Chapter 7

Current progress and ongoing work

7.1 Penrose limit and TsT for fibered I-brane

Related work by the author

Available on arXiv: https://arxiv.org/abs/2510.24406 [12]

Introduction

In that work we analyze the single $T\bar{T}$ deformation, explained in Section 6.4 defined by a TsT transformation, of the fibered I-brane solution from [126]. We use the Penrose limit to understand it, and we consider both the TsT followed by the Penrose limit, as well as the Penrose limit followed by TsT. We describe the spin chains obtained in field theory. In the first case we find, indeed, that the TsT transformation preserves solvability in a simple way, as in the standard "$T\bar{T}$" case. In the second case, we have several options, but none is simple enough to be conclusive, however, one case gives us an asymptotically free and IR nontrivial field theory sector, and another a new parallelizable pp wave.

Review of the background

The background describes a $(1+1)$-dimensional theory obtained when two stacks of $D5$-branes intersect along two spacetime directions, the "I-brane" configuration. Effectively, the theory acquires an extra worldvolume direction, as explained in [86], thus being described by an $(2+1)$-dimensional theory.

In [126], an IR completion for this theory was introduced by a twisted compactification on a shrinking cycle, obtaining a "fibered I-brane".

The supergravity background contains the metric $G_{\mu\nu}$, a dilaton Φ, and an RR

field C_2. In the string frame they can expressed as

$$
\begin{aligned}
ds^2_{st} r &\left\{ -dt^2 + dx^2 + f_s(r) d\varphi^2 + \frac{4}{r^2 f_s(r)} dr^2 + \frac{2}{e_A^2}\left[\hat{\omega}_1^2 + \hat{\omega}_2^2 + (\hat{\omega}_3 - e_A Q_A \zeta(r) d\varphi)^2\right]\right. \\
&\left. + \frac{2}{e_B^2}\left[\tilde{\omega}_1^2 + \tilde{\omega}_2^2 + (\tilde{\omega}_3 - e_B Q_B \zeta(r) d\varphi)^2\right]\right\}, \\
C_2 \psi_A &\left(\frac{2Q_A}{e_A}\zeta'(r) dr \wedge d\varphi - \frac{2}{e_A^2}\sin\theta_A d\theta_A \wedge d\phi_A\right) + \frac{2}{e_A}\cos\theta_A Q_A \zeta(r) d\varphi \wedge d\phi_A \\
+\psi_B &\left(\frac{2Q_B}{e_B}\zeta'(r) dr \wedge d\varphi - \frac{2}{e_B^2}\sin\theta_B d\theta_B \wedge d\phi_B\right) + \frac{2}{e_B}\cos\theta_B Q_B \zeta(r) d\varphi \wedge d\phi_B, \\
&\Phi = \log r.
\end{aligned}
\tag{7.1}
$$

The functions $f_s(r)$ and $\zeta(r)$ are given by

$$
\begin{aligned}
f_s(r) &= \frac{e_A^2 + e_B^2}{2} - \frac{m}{r^2} - \frac{2\left(Q_A^2 + Q_B^2\right)}{r^4} \equiv \frac{e_A^2 + e_B^2}{2r^4}\left(r^2 - r_+^2\right)\left(r^2 - r_-^2\right), \\
\zeta(r) &= \frac{1}{r^2} - \frac{1}{r_+^2}, \quad r_\pm^2 = \frac{m \pm \sqrt{m^2 + 4\left(Q_A^2 + Q_B^2\right)\left(e_A^2 + e_B^2\right)}}{e_A^2 + e_B^2}.
\end{aligned}
\tag{7.2}
$$

Here $\hat{\omega}_i, \tilde{\omega}_i$ are the Maurer-Cartan forms for su(2), given by

$$
\begin{aligned}
\hat{\omega}_1 &= \cos\psi_A d\theta_A + \sin\psi_A \sin\theta_A d\phi_A, & \tilde{\omega}_1 &= \cos\psi_B d\theta_B + \sin\psi_B \sin\theta_B d\phi_B, \\
\hat{\omega}_2 &= -\sin\psi_A d\theta_A + \cos\psi_A \sin\theta_A d\phi_A, & \tilde{\omega}_2 &= -\sin\psi_B d\theta_B + \cos\psi_B \sin\theta_B d\phi_B, \\
\hat{\omega}_3 &= d\psi_A + \cos\theta_A d\phi_A, & \tilde{\omega}_3 &= d\psi_B + \cos\theta_B d\phi_B.
\end{aligned}
$$

The supersymmetry of this background was studied in [126], where it was found that it preserves four supersymmetries when the parameters $e_{A,B}, m, Q_{A,B}$ satisfy $m = 0,\ e_A Q_B = \pm e_B Q_A$.

Tst+Penrose

Then, we deform it by a TsT, see Appendix B for details, The NSNS part of the background is

$$
\begin{aligned}
ds^2_{st} &= r\left\{ f_s(r) d\varphi^2 + \frac{2}{e_A^2}\left[\hat{\omega}_1^2 + \hat{\omega}_2^2 + (\hat{\omega}_3 - e_A Q_A \zeta(r) d\varphi)^2\right]\right. \\
&\left. + \frac{4}{r^2 f_s(r)} dr^2 + \frac{2}{e_B^2}\left[\tilde{\omega}_1^2 + \tilde{\omega}_2^2 + (\tilde{\omega}_3 - e_B Q_B \zeta(r) d\varphi)^2\right]\right\} + \left(\frac{r}{1-(r\eta)^2}\right)\left(-dt^2 + dx^2\right), \\
B &= -\left(\frac{r}{1-(r\eta)^2}\eta r\right) dx \wedge dt, \quad \Phi = \frac{1}{2}\ln\left(\frac{r^2}{1-(\eta r)^2}\right).
\end{aligned}
\tag{7.3}
$$

The RR fields are also transformed under TsT, and we get a new RR field C_4, with the same C_2,

$$
\begin{aligned}
C_2 = \psi_A \left(\frac{2Q_A}{e_A} \zeta'(r) dr \wedge d\varphi - \frac{2}{e_A^2} \sin\theta_A d\theta_A \wedge d\phi_A \right) + \frac{2}{e_A} \cos\theta_A Q_A \zeta(r) d\varphi \wedge d\phi_A \\
+\psi_B \left(\frac{2Q_B}{e_B} \zeta'(r) dr \wedge d\varphi - \frac{2}{e_B^2} \sin\theta_B d\theta_B \wedge d\phi_B \right) + \frac{2}{e_B} \cos\theta_B Q_B \zeta(r) d\varphi \wedge d\phi_B, \\
C_4 = B \wedge C_2.
\end{aligned} \tag{7.4}
$$

Summarizing, the resulting configuration in the target space consists of two stacks of D5-branes wrapping distinct three-cycles, $(\psi_{A,B}, \theta_{A,B}, \phi_{A,B})$, each carrying positive charge. Additionally, there are D3-branes extended over (x, r, φ) with flux over the two five-cycles $(\psi_{A,B}, \theta_{A,B}, \phi_{A,B}, \phi_{B,A}, \theta_{B,A})$. We also have F1 strings, extended along the x-direction, and located above ψ_A and ψ_B, as represented in Table 7.1. The resulting configuration in the target space breaks all the supersymmetry that was left after twisting the background I of [126] to obtain the background II we considered here, which is expected, since our TsT deformation does not correspond to a susy-preserving deformation.

Table 7.1: Brane configuration in the given coordinate system

	t	x	r	φ	θ_a	ϕ_a	ψ_a	θ_b	ϕ_b	ψ_b
$F1$	○	○	×	×	×	×	×	×	×	×
$D3$	○	○	○	○	×	×	×	×	×	×
$D5_1$	○	○	×	○	○	○	○	×	×	×
$D5_2$	○	○	×	○	×	×	×	○	○	○

For the analysis of the dual field theory, the charges we have obtained, (D5, D3, F1), should give us information about the result. A simplification arises if we consider the S-dual of the theory, which in the Field theory just amounts to a description exchanging weak and strong coupling. The S-dual maps the charges to (NS5, D3, D1) charges, which were studied in the literature [68], [2]. The theory, initially in 1+1 dimensions, but effectively, as we said, extended to 2+1 dimensions, then still contains (as in [11]) the gauge group (with Chern-Simons levels) $SU(N_A)_{N_B} \times SU(N_B)_{N_A}$, with Yang-Mills and Chern-Simons terms, and fermion bifundamentals under both groups, but now also with Yang-Mills groups for $SU(N_1) \times SU(N_3)$, due to the D1 and D3 charges, and still has a global Lorentz $SO(1,1)$ symmetry, not broken by the $B_{tx} \propto \epsilon_{t,x}$. The R-symmetry, as seen from the solution (7.3), is unchanged from the one in [11], namely $SO(4) \times SO(4)$, for the two S^3's, since TsT acted only on x, t. The R-symmetry corresponds in field theory to a global symmetry rotating the scalar fields (and the fermions).

Then, we take the Penrose Limit. To identify a suitable null geodesic in the background described by (7.3), solving the geodesic equations of motion, we consider the path defined by

$$r = r_+, \quad x = 0, \quad \theta_A = \theta_B = \frac{\pi}{2}, \quad \phi_A = \phi_B = \theta = 0 \tag{7.5}$$

where r_+ is the larger of the two zeroes of $f_s(r)$. We can easily check that this solves the (null) geodesic equations of motion.

Along this trajectory, the string Lagrangian simplifies to

$$\mathcal{L} = \frac{2r_+ d\psi_A^2}{e_A^2} + \frac{2r_+ d\psi_B^2}{e_B^2} - \frac{r_+ dt^2}{1 - \eta^2 r_+^2}. \tag{7.6}$$

Introducing new angular coordinates $\psi_A = e_A(\theta + \psi)$, $\psi_B = e_B(\theta - \psi)$, and setting $\theta = 0$, so that we have a motion on the equators of both S^3 spheres, but in opposite directions, which is of the type also considered in [11], the Lagrangian becomes

$$\mathcal{L} = r_+ \left(4d\psi^2 - \frac{dt^2}{1 - \eta^2 r_+^2} \right). \tag{7.7}$$

This expression can be recast into light-cone coordinates via the transformation

$$du = 2d\psi - \frac{dt}{\sqrt{1 - \eta^2 r_+^2}}, \quad dv = 2d\psi + \frac{dt}{\sqrt{1 - \eta^2 r_+^2}}, \tag{7.8}$$

leading to the simplified form

$$\mathcal{L} = r_+ du dv. \tag{7.9}$$

To fully define the null geodesic, we further impose $u = 0$. Then the metric finally is

$$\begin{aligned} ds^2 = -2\,du\,dV + dz^2 + dy^2 + dV \bigg(&\frac{1}{e_B} (\Phi_B\, d\Theta_B - \Theta_B\, d\Phi_B) + \frac{1}{e_A} (\Theta_A\, d\Phi_A - \Phi_A\, d\Theta_A) \\ &- \frac{(Q_B - Q_A)\, \zeta'(r_+)}{2} (z\, dy - y\, dz) \bigg). \end{aligned} \tag{7.10}$$

From this metric, we obtain for F_3

$$F_3 = \frac{1}{e_A e_B}\left(-\frac{1}{r_+}e_B d\Phi_A \wedge d\Theta_A \wedge dV + \frac{1}{r_+}e_A d\Phi_B \wedge d\Theta_B \wedge dV \right. \\ \left. -\frac{1}{2r_+}e_A e_B (Q_A - Q_B)\zeta'(r_+) dz \wedge dV \wedge dy\right) , \tag{7.11}$$

Since the background has a constant dilaton, and it is S-dual of the parallelizable pp wave background, we know that it is a supersymmetric solution preserving, if $Q_A = Q_B$ (zero coefficient of $dVdy$ and $dVdz$ in the metric) and $e_A = e_B$ (equal contributions for the remaining transverse coordinates in the metric), 24 supersymmetries. Note that since $e^2_{A,B} = 8/N_{B,A}$ (from the background before TsT), $e_A = e_B$ means $N_A = N_B$.

In this case, since, as shown in [126], the original solution (before TsT), was supersymmetric if $e_A Q_B = \pm e_B Q_A$, or $Q_B\sqrt{N_A} = \pm Q_A\sqrt{N_B}$, it means that in our case, the pp wave with 24 supersymmetries corresponds to an original solution, so undeformed field theory as well, that was supersymmetric, before TsT.

Moreover, we see that the condition $Q_A = Q_B$ makes N_{D3} exactly 0. This is interesting, though not so surprising: It is hard to find a complex configuration involving a large number of different branes that preserves supersymmetry, and therefore we should expect that we need to get rid of some of the branes in order to maintain something.

The momenta p_u and p_v become, in gauge theory notation,

$$p^- \equiv p^u = -p_V = \frac{\sqrt{1-\eta^2 r_+^2}}{2}\Delta - \frac{J_\psi}{4}, \quad p^+ \equiv p^V = -p_u = -\frac{\frac{\sqrt{1-\eta^2 r_+^2}}{2}\Delta + \frac{J_\psi}{4}}{L^2} , \tag{7.12}$$

where Δ, J are eigenvalues of the energy H, corresponding to conformal dimension, and momentum $p_\psi = \frac{1}{2}(e_A p_{\psi_A} + e_B p_{\psi_B})$, corresponding to a $U(1)$ (global) R-charge, respectively. The negative sign here is a result of the choice in doing the change of coordinates in (7.8), and does not affect the results. Therefore, the Penrose limit corresponds to a specific sector of the gauge theory in which the surviving operators (à la BMN [20]) have large charge J_ψ, approximately equal to a large conformal dimension Δ,

$$J_\psi \simeq 2\sqrt{1-(\eta r_+)^2}\Delta \propto L^2 . \tag{7.13}$$

Here $J_\psi = \frac{1}{2}(e_A J_{\psi_A} + e_B J_{\psi_B})$,[1] and therefore we can have operators with a large J_{ψ_A} charge, large J_{ψ_B} charge, or a mixture of both.

The naive correspondence between the string oscillators, coming from pp wave coordinates, and the field theory oscillators, would be

$$\begin{aligned} (x_1, x_2) = (\rho, \varphi) \to D_i \,, \quad (x_7, x_8) = (x, \Theta) \Rightarrow W \,, \\ (x_3, x_4) = (\theta_A, \phi_A) \to W_2 \,, \quad (x_5, x_6) = (\theta_B, \phi_B) \to W_3 \,, \end{aligned} \tag{7.14}$$

but, as explained also in [11], this does not match, as expected, since the field theory is effectively $(2+1)-$dimensional.

The radial coordinates

$$\bar{u} = \sqrt{\sum_i \phi^i_{(1)} \phi^i_{(1)}} \,, \quad \bar{v} = \sqrt{\sum_j \phi^j_{(2)} \phi^j_{(2)}} \,, \tag{7.15}$$

transverse to each of the D5-branes, are joined at $\bar{u} = \bar{v} = 0$, obtaining the extra spatial coordinate called w, and replacing the original spacetime coordinate x, and still corresponding to the x_7 oscillator.

In 2+1 dimensions, a scalar has the *classical* (conformal) dimension 1/2, so we must define the unit of J_ψ to be 1, so that $E = p^- = 0$ for it. But, of course, the implicit assumption is that the TsT deformation by a parameter η is such that $\Delta_\eta \sqrt{1-(\eta r_+)^2} = 1$, or $\Delta_\eta = 1/\sqrt{1-(\eta r_+)^2}$. We redefine the zero of the energy by an $E_0 = 1$, and then we find that, in order to get the correct spectrum, we must rescale H by $\mu = -2$. Summarizing,

field	Z	W	$\bar{Z}$	$\bar{W}$	$W_2, \bar{W}_2$	$W_3, \bar{W}_3$	D_{x_i}	D_w
Δ	1/2	1/2	1/2	1/2	1/2	1/2	1	1
J_ψ	-1	-1	1	1	0	0	0	0
$\Delta - J_\psi/2$	1	1	0	0	1/2	1/2	1	1
$H/\mu = \Delta - J_\psi/2 - E_0$	0	0	-1	-1	-1/2	-1/2	0	0
oscillator	-	x_8	-	-	x_3, x_4	x_5, x_6	x_1, x_2	x_7

Then the vacuum is acted upon by the $U(2)$ part of the R-symmetry, rotating Z and W, so that we have

$$|0_a, p^+\rangle = \frac{1}{\sqrt{J_\psi N^{J_\psi/2}}} Tr[Z_a^{J_\psi}] \,, \tag{7.16}$$

[1]Note that we have chosen the relative sign, which amounts to a choice of the relative positive R-charge in field theory, which we can always do.

where $Z_a = (Z, W)$, so either one can be inserted. Then we insert fields for oscillators from the table above inside the trace, in order to obtain the string oscillator states.

Penrose+TsT

We can now study the other way around of the chain limits, and then first take the Penrose to after take the TsT transformations. The Penrose Limit is trivial, just like the case before but without the deformation parameter η. Then, we TsT deform it, specially for the v, φ pair of coordinates, but also for more general cases (v, X), (v, Θ), (u, φ), (X, φ), (Θ, φ), (v, u).

As explained in [63], the TsT transformation is done by picking two different coordinates that possess killing vectors ∂_{x_i} of translations. Evaluating the background we notice we have several possibilities to do the Penrose Limit, with the most prominent being u, v, Θ, φ, X. Moreover, we can evaluate the TsT transformation directly on the supergravity background, or we can try to deform the string world-sheet (the Polyakov action [136]) in a specific way.

The TsT transformation that we did in the last section, before doing the Penrose limit, was over the coordinates t and x, and since we want to compare what are the effects of doing a TsT after the Penrose limit, we naturally need to deform in these same coordinates. However, the metric we obtained is in terms of the more useful coordinates for the Penrose limit u, v, and since these coordinates are related to t via $t = \frac{v}{2} - \frac{u}{2L^2}$, and for large L the one that prevails is v, we do a TsT over v and $X \propto x$.

After TsT transforming with parameter η, the metric gets a correction of the order of η^2, and therefore the full metric can be written as $\tilde{ds}^2 = ds^2 + \eta^2 \delta ds^2$, with δds^2 being

$$\delta ds^2 = -\frac{r_+^3}{4e_A^2 e_B^2}\left(-e_B \Theta_A d\Phi_A + e_A \Theta_B d\Phi_B + e_A e_A du\right)^2 , \tag{7.17}$$

and for the NS-NS fields, we have the intriguing result that the dilaton is constant, and the field strength $H = dB$ is constant,

$$\phi = \log r_+, \quad B = \frac{\eta r_+^2}{2e_A e_B}\left(-e_B \Theta_A d\Phi_A + e_A \Theta_B d\Phi_B + e_A e_B du\right) \wedge dX. \tag{7.18}$$

The compact form of the deformation of the metric, (7.17), together with the field strength $H = dB$ being constant, may make one ask oneself if we can't transform the coordinates in the metric to reduce it to a parallelizable pp wave metric. However, we now show it is not possible to find a coordinate transformation that satisfies this. We do that by showing that invariant objects in both metrics, like the Ricci scalar R, can't be adjusted to match, that is $R_{\text{parallelizable}} \neq R_{\text{deformed}}$, and therefore both metrics can't be connected by a transformation, and therefore they aren't the same spacetime.

First, from the metric above, we ignore the terms that are not relevant to our analysis, Θ, ρ, φ and X, since the terms of the metric involving them weren't affected by the deformation. Then we are left with, if $Q_A = Q_B$,

$$\begin{aligned}\tilde{ds}^2 = r_+ \, du \, dv + \frac{2r_+}{e_A^2} \, d\Phi_A^2 + \frac{2r_+}{e_A^2} \, d\Theta_A^2 + \frac{2r_+}{e_B^2} \, d\Phi_B^2 + \frac{2r_+}{e_B^2} \, d\Theta_B^2 \\ + dv \left(\frac{r_+ \Theta_B}{e_B} \, d\Phi_B - \frac{r_+ \Theta_A}{e_A} \, d\Phi_A \right) + \eta^2 \delta ds^2.\end{aligned} \tag{7.19}$$

Using a Mathematica notebook, we have calculated the Ricci form, which takes the simple form

$$R = \frac{1}{32} \left(e_A^2 + e_B^2 \right) r_+ \eta^2. \tag{7.20}$$

The whole spacetime/metric is then of the form $\mathcal{M} = \mathbb{R}^4 \times M_6$, where $\mathbb{R}^4$ is Euclidean, and M_6 is the metric above. Since M_6 is curved, and $\mathbb{R}^4$ is flat, the whole spacetime has a total Ricci curvature $\neq 0$ (This wouldn't be true if, for example, instead of $\mathbb{R}^4$ we had another spacetime that was not flat. In fact, for example the $\mathrm{AdS}_5 \times \mathrm{S}^5$ metric has a zero net Ricci scalar).

To prove that we cannot turn M_6 into a parallelizable pp wave metric form, we assume that we can, with the general parallelizable wave metric (here we use 10 coordinates), see equation (5.8) of Chapter 5. The Ricci scalar of a generalized pp wave metric was calculated in [50], and it was showed to be $R = 0$, as a general property of any pp wave metric. Since (7.20) is only 0 for $\eta = 0$, which is the undeformed background, both Ricci scalars are different, and then the metric for a TsT transformation over x, v (we are going to show later that we can obtain parallelizable wave, but for other pairs of coordinates) is not of parallelizable type.

We could also have guessed that the metric can't reduce to a parallelizable pp-wave just by looking at the $\sum_i g_{ij} dx^i dx^j$ part of the metric, where the pp wave metric occurs when $g_{ij} = \delta_{ij}$, but that was not an invariant statement, the Ricci scalar calculation is. Otherwise, perhaps we could at least have a Bargmann structure (5.7), so we expand δds^2

$$
\begin{aligned}
ds^2 = \frac{1}{4}\Bigg(& -\frac{r_+\left(-8+\eta^2 r^2 \Theta_A^2\right)}{e_A^2}\, d\Phi_A^2 - \frac{r_+\left(-8+\eta^2 r^2 \Theta_B^2\right)}{e_B^2}\, d\Phi_B^2 + 16 r_+\, d\Theta^2 \\
& + \frac{8 r_+}{e_A^2}\, d\Theta_A^2 + \frac{8 r_+}{e_B^2}\, d\Theta_B^2 - \eta^2 r^3\, du^2 - \frac{2 r_+ \Theta_B}{e_B}\, d\Phi_B \left(\eta^2 r_+^2 du - 2 dv\right) \\
& + 4 r_+\, du\, dv + \frac{2 r_+ \Theta_A}{e_A e_B}\, d\Phi_A \left(\eta^2 r_+^2 \Theta_B\, d\Phi_B + e_B \eta^2 r_+^2 du - 2 e_B dv\right) \\
& + 4 r_+\, dX^2 + \frac{64}{r_+ f_s'}\, d\rho^2 + 4 \rho^2 r_+ f_s'\, d\varphi^2 \Bigg).
\end{aligned}
\tag{7.21}
$$

Changing coordinates by

$$
\begin{aligned}
& \Theta \to \frac{\Theta}{2\sqrt{r_+}}, \Phi_A \to \frac{e_A\, \Phi_A}{\sqrt{2 r_+}}, \Phi_B \to \frac{e_B\, \Phi_B}{\sqrt{2 r_+}}, X \to \frac{X}{\sqrt{r_+}}, \rho \to \frac{\sqrt{r_+ f_s'}\, \rho}{4}, \\
& \varphi \to \frac{4\, \varphi}{r_+ f_s'}, \Theta_A \to \frac{e_A\, \Theta_A}{\sqrt{2 r_+}}, \Theta_B \to \frac{e_B\, \theta_B}{\sqrt{2 r_+}}, u \to \frac{2(v-u)}{\eta r_+^{3/2}}, v \to -\eta \sqrt{r}\, u\ ,
\end{aligned}
\tag{7.22}
$$

we obtain

$$
\begin{aligned}
ds^2 &= d\rho^2 + \rho^2 d\varphi^2 + dX^2 + d\Theta^2 \\
&+ \left(1 - \frac{1}{8}\eta^2 r_+^2 \Theta_A^2\right) d\Phi_A^2 + \left(1 - \frac{1}{8}\eta^2 r_+^2 \Theta_B^2\right) d\Phi_B^2 \\
&+ \frac{1}{4}\eta^2 r_+^2 \Theta_A \Theta_B\, d\Phi_A\, d\Phi_B + d\Theta_A^2 + d\Theta_B^2 \\
&- dv^2 + du^2 + \frac{\eta r_+ \Theta_A\, d\Phi_A\, dv}{\sqrt{2}} - \frac{\eta r_+ \Theta_B\, d\Phi_B\, dv}{\sqrt{2}}.
\end{aligned}
\tag{7.23}
$$

We see that the third line of the metric mixes the coordinates $\Phi_{A,B}, \Theta_{A,B}$ in a nontrivial way, and we can't reduce to a trivial Euclidean part $\sum_i x_i^2$ on the metric. In fact, even the most general metric we cited here, the Bargmann metric, is hard to obtain. Due to the nature of this metric it would be very hard to obtain the spectrum of the state directly from the usual quantization of the Polyakov action. Yet, some assumptions can be made, in order to make a connection between the supergravity solution and a deformed field theory.

The Penrose limit selects a large charge operator sector of the field theory, and the TsT deformation should deform this sector. The presence of a non-constant B-field indicates, in general, either a dipole-type, or a noncommutative deformation of the field theory, so in this case, the product algebra of the large R-charge operators should be so deformed. However, it would be very hard to define this, so we will leave it for further work.

We proceed to consider also deforming in different pairs of isometric directions. In fact, we will find that some of the deformation pairs present similar behaviors from the field point of view, like being dual to irrelevant deformations, or involving dipole or noncommutative deformations.

Starting with the pair (v, φ), and to simplify the calculation, we can define an M matrix element M for the coordinates (v, φ) (see Appendix B for the definitions of the terms and their applications), we obtain the deformed metric

$$\begin{aligned} ds^2 = \frac{2r_+}{e_A^2} d\phi_A^2 + \frac{2r_+}{e_B^2} d\phi_B^2 + \frac{2r_+}{e_A^2} d\theta_A^2 + \frac{2r_+}{e_B^2} d\theta_B^2 + 4r_+ \, d\theta^2 + r_+ \, dX^2 + \frac{16}{r_+ f_s'(r_+)} d\rho^2 \\ - \frac{\eta^2\rho^2 r_+^3 f_s'(r_+)}{Y} du^2 + \frac{4\rho^2 r_+ f_s'(r_+)}{Y} d\varphi^2 \\ + \frac{2r_+ Y - \eta^2\rho^2 r_+^3 \theta_A^2 f_s'(r_+)}{e_a^2 Y} d\phi_A^2 + \frac{2r_+ Y - \eta^2\rho^2 r_+^3 \theta_B^2 f_s'(r_+)}{e_b^2 Y} d\phi_B^2 \\ + dv \left(\frac{4\rho^2 (Q_B - Q_A) r_+ \zeta'(r_+)}{Y} d\varphi + \frac{4r_+ \Theta_B}{e_b Y} d\Phi_B - \frac{4r_+ \Theta_A}{e_a Y} d\Phi_A \right) \\ + du \left(\frac{2\eta^2\rho^2 r_+^3 \Theta_A f_s'(r_+)}{e_a Y} d\Phi_A - \frac{2\eta^2\rho^2 r_+^3 \Theta_B f_s'(r_+)}{e_b Y} d\Phi_B \right) \\ + \frac{2\eta^2\rho^2 r_+^3 \Theta_A \Theta_B f_s'(r_+)}{e_a e_b Y} d\Phi_A d\Phi_B + \frac{4r_+}{Y} du\, dv \, , \end{aligned} \tag{7.24}$$

where $Y = 4 - \eta^2\rho^4 (Q_A - Q_B)^2 r_+^2 \zeta'(r_+)^2$, and there are also nonzero B and $\phi(\rho)$ fields present. It is important to emphasize that the dilaton ϕ becomes radial dependent, implying an energy scale dependence in the field theory. Again, the C_2 field remains the same, and we have a C_4 just like in the previous cases. The form of the metric suggest again that the TsT deformation is dual to an irrelevant operator, because as $\rho \to 0$, it reduces again to the undeformed pp wave metric. The general structure of the solution does not easily suggest a field theory interpretation, since it is a complex combination of coordinates with a not so obvious meaning from the dual side. However, we can try to evaluate the deformation of the special case $Q_A = Q_B$ of the pp wave, where (if also $e_A = e_B$) 24 supersymmetries were preserved. Under this condition, the metric reduces to the simple form

$$\begin{aligned} ds^2 = r_+ \, du \, dv + \frac{2r_+}{e_A^2} \, d\Phi_A^2 + \frac{2r_+}{e_A^2} \, d\Theta_A^2 + \frac{2r_+}{e_B^2} \, d\Phi_B^2 + \frac{2r_+}{e_B^2} \, d\Theta_B^2 + 4r_+ \, d\Theta^2 + \frac{16}{r_+ f_s'(r_+)} \, d\rho^2 \\ + \rho^2 r_+ f_s'(r_+) \, d\varphi^2 + r_+ \, dX^2 + dv \left(\frac{r_+ \Theta_B}{e_B} \, d\Phi_B - \frac{r_+ \Theta_A}{e_A} \, d\Phi_A \right) \\ - \frac{\eta^2 \rho^2 r_+^3 f_s'(r_+)}{4 e_A^2 e_B^2} \left(e_A e_B \, du + e_A \Theta_B \, d\Phi_B - e_B \Theta_A \, d\Phi_A \right)^2 \, , \end{aligned} \tag{7.25}$$

where the last line represents the effect of the TsT transformation in the metric. In this case it is easier to notice that as $\rho \to 0$, the metric reduces to the initial case, but we will talk more about this later. Also, the dilaton field ϕ under this supersymmetry condition reduces to the initial dilaton value, $\phi = \log r_+$, a constant scalar. Finally, the B-field reduces to the simple form

$$B = \frac{\eta \rho^2 r_+^2 f_s'(r_+)}{2e_A e_B} \left(e_B \Theta_A d\Phi_A \wedge d\varphi - e_A \Theta_B d\Phi_B \wedge d\varphi - e_A e_B du \wedge d\varphi \right) \;, \tag{7.26}$$

with the new F_5 also having a ρ^2 dependence. Note that, even in the susy $e_A = e_B$ case, the $H = dB$ field is not constant anymore, since then we have

$$B = \text{constant} \times \rho^2 (\Theta_A d\Phi_A - \Theta_B \Phi_B - e_A u) \wedge d\varphi. \tag{7.27}$$

Moreover, the metric (7.25) presents the same problem as the one calculated for v, X, but even worse, since now the deformation is dependent on the coordinate ρ as well. This makes the analysis very difficult, since the Hamiltonian is not so easily diagonalized as it is for pp waves (even less than for parallelizable pp waves), see [123].

Moving on to a TsT transformation over V, u, the lightcone coordinate for pp wave, using the notation of Appendix B, we obtain for $\mathcal{M} = (1-\eta^2)^{-1}$, and then the deformed background,

$$\begin{aligned} ds^2 = \frac{1}{\eta^2 - 1} dV^2 \left(\frac{e_B^2}{16} r_+ |\omega_1|^2 + \frac{e_A^2}{16} r_+ |\omega_2|^2 + \frac{(Q_B - Q_A)^2 \zeta'(r_+)^2}{16} |\omega_3|^2 \right) \\ - \frac{2}{\eta^2 - 1} du dV + |d\omega_1|^2 + |d\omega_2|^2 + |d\omega_3|^2 + \dots \\ B_2 = \frac{\eta}{\eta^2 - 1} du \wedge dv, \quad \Phi = \log r_+ - \frac{1}{2} \log(1 - \eta^2). \end{aligned} \tag{7.28}$$

As before, we also find nontrivial B-field and dilaton ϕ fields. Moreover, we also have the C_4 field, which as we have seen is simply related to the B and C_2 field, the last being intact under the TsT transformation,

$$\begin{aligned} C_4 = B_2 \wedge C_2 \\ = \frac{\eta V}{2(1-\eta^2) r_+^2 f_s'(r_+)} (-e_B f_s'(r_+) d\Phi_B \wedge d\Theta_B + e_A f_s'(r_+) d\Phi_A \wedge d\Theta_A \\ + 4(Q_B - Q_A) \sqrt{r_+ f_s'(r_+)} \zeta'(r_+) \wedge dy \wedge dz) \wedge du \wedge dV. \end{aligned} \tag{7.29}$$

We notice that now there is a similarity between the Penrose limit after TsT transformation over x, t and the current case, the TsT transformation over u, v after the

Penrose limit, even though it is was not obvious that they should be similar. In fact, the deformed metric obtained above is still a parallelizable pp wave type, and since the B and dilaton ϕ fields are still constants, the analysis of the field theory dual sector (spin chain for operator sector) is simple, and similar to the cases done before. In fact, the BMN mapping is essentially the same (up to some constant factors).

Instead of investigating case by case, it is better to summarize the most important specific cases and find the differences and similarities between the TsT transformations over distinct pairs of coordinates. Also, by using the example of the last subsection, where we saw that the effect of the TsT transformation in the general background is not so illuminating, we focus here only on the susy condition, $Q_A = Q_B$ (and $e_A = e_B$) for a clearer analysis.

We first notice that doing a TsT transformation over v and any other coordinates that is not u produces a rather similar metric. Calling $\delta ds^2 = ds^2_{\mathrm{TsT}} - ds^2$, we explicitly write the case for each other coordinate pair involving v,

Table 7.2: TsT deformations and induced fields

TsT pair	δds^2	B	ϕ
(v, X)	$-\dfrac{\eta^2 r_+^3}{4e_A^2 e_B^2}\,\chi^2$	$\dfrac{\eta r_+^2}{2e_A e_B}\,\chi \wedge dX$	$\log r_+$
(v, φ)	$-\dfrac{\eta^2\rho^2 r_+^3 f_s'(r_+)}{4e_A^2 e_B^2}\,\chi^2$	$\dfrac{\eta\rho^2 r_+^2}{2e_A e_B}\, f_s'(r_+)\chi \wedge d\varphi$	$\log r_+$
(v, Θ)	$-\dfrac{\eta^2 r_+^3}{e_A^2 e_B^2}\,\chi^2$	$\dfrac{\eta r_+^2}{2e_A e_B}\,\chi \wedge d\Theta$	$\log r_+$

where χ is a 1-form, $\chi = -e_B\Theta_A d\Phi_A + e_A\Theta_B d\Phi_B + e_A e_B dv$. Beside the remarkable similarity between the metrics, we notice that the (v, φ) case is a radially dependent one, and moreover, as $\rho \to 0$ (which means $r \to r_+$) the deformed background reduces to the original one. This will be explored later, where we will show that all deformations over φ have this structure. For now, we emphasize that this coincidence may suggest that there are some relations between the deformed gauge theories.

In fact, since ρ ranges over $(0, \infty)$, from the formulas above for the deformations, we see that for some specific value of ρ (which from the gauge side means for some energy) the (v, φ) deformed gauge theory is equivalent to the deformed (v, X) and the deformed (v, Θ), both of which are marginal-like deformations of the field theory. So, the (v, φ) gauge field passes over the other two deformed fields theories, as the energy increases.

Moreover, the dilaton field is equal in the three cases above, and constant. The more notable difference between these deformations is how the B-field transforms, where we see that one leg of B is in the X, φ, Θ, corresponding to the TsT coordinate.

This implies that, while the deformed dual gauge theory may be similar, there is still a difference between them, defined by the B field, responsible for either a dipole-like deformation, or non-commutativity. Indeed, Seiberg and Witten [147] have shown that configurations involving constant Kalb-Ramond field B, and open strings, can be interpreted as a non-commutative field theory involving operators and a modified product, the Moyal star product. In [106] it was generalized to a non-constant (though still associative) B field. Furthermore, we expect that TsT deformations and noncommutative field theory are related, as explained in [106], with star product equals, at first order, to

$$A(x) \cdot B(x) \to A(x) \star B(x) = A(x) \cdot B(x) + \frac{i}{2}\theta^{ij}\partial_i A \partial_j B + ... \tag{7.30}$$

The Moyal star product in the constant B, therefore constant θ, case is

$$A(x) * B(x) = \lim_{x \to x'} A(x) e^{i\theta^{ij}\partial_i \partial'_j} B(x'). \tag{7.31}$$

The noncommutativity θ^{ij}, the open string metric G_{ij} and the open string coupling constant g_o are related to the closed string supergravity background by

$$\begin{aligned} G_{ij} = -(2\pi\alpha')^2 (Bg^{-1}B)^{ij}, \quad \theta^{ij} = (B^{-1})^{ij} \Rightarrow (G + 2\pi\alpha'\theta)^{ij} = (g + 2\pi\alpha' B)^{-1}_{ij}, \\ g_o = g_s \left(\frac{\det G}{\det g}\right)^{1/4} \end{aligned} \tag{7.32}$$

The closed string metrics of the three deformations above have the same structure. The (v, X) and (v, Θ) ones only differ by a constant factor, and there is an energy factor $\sim \rho^2 f'_s(r_+)$ relating (v, ϕ) to the other two. The B field differs only by the direction of the noncommutativity, but the general form (since $|B|^2$ are also essentially the same) is the same, again with the (v, φ) having *the same* $\rho^2 f'_s(r+)$ dependence. In the open string metric and θ^{ij}, the latter will translate into $1/\rho^2 f'_s(r_+)$, but that is as expected.

Moving on, there is another interesting relation among the TsT deformations, when the pair of coordinates involves the φ one. As we saw before, the TsT over

(v, φ) generates a deformation in the gauge side analogous to what would be the deformation done by an irrelevant operator. We now show that this is not a coincidence: any TsT over φ generates dual gauge theories deformed by an irrelevant operator (since we already studied (v, φ), we will ignore it here),

Table 7.3: TsT deformations involving φ

TsT pair	δds^2	B	ϕ
$(u, \varphi)^\star$	$-\frac{\eta^2 r_+^3 f_s'(r_+)}{4}\rho^2 dV^2$	$-\frac{\eta\rho^2 r_+^2}{2} f_s'(r_+) d\varphi \wedge dv$	$\log r_+$
(X, φ)	$-\frac{\eta^2 r_+^3 f_s'(r_+)}{1+\eta^2\rho^2 r_+^2 f_s'(r_+)}\rho^2(dX^2 + \rho^2 f_s'(r_+) d\varphi^2)$	$-\frac{\eta\rho^2 r_+^2 f_s'(r_+)}{1+\eta^2\rho^2 r_+^2 f_s'(r_+)} d\varphi \wedge dX$	$\log r_+ - \log\sqrt{1+\eta^2 r_+^2 f_s'(r_+)\rho^2}$
(Θ, φ)	$-\frac{4\eta^2 r_+^3 f_s'(r_+)}{1+4\eta^2\rho^2 r_+^2 f_s'(r_+)}\rho^2(4d\Theta^2 + \rho^2 f_s'(r_+) d\varphi^2)$	$-\frac{4\eta\rho^2 r_+^2 f_s'(r_+)}{1+4\eta^2\rho^2 r_+^2 f_s'(r_+)} d\varphi \wedge d\Theta$	$\log r_+ - \log\sqrt{1+4\eta^2 r_+^2 f_s'(r_+)\rho^2}$

Therefore, we see that every metric deformed by a TsT transformation that involves φ has a structure of the type $\delta ds^2 \sim \rho^2 \ldots$, and reproduces in the supergravity side the behavior of *an irrelevant operator* in the gauge side.

Besides that, we see two interesting facts regarding these deformations: while the (Θ, φ) and (X, φ) cases vanish for $\rho \to 0$, and the dilaton goes to a constant, for $\rho \to \infty$ the B field goes to a constant (and therefore, can be gauged away, though one could still have noncommutativity, in principle), and the dilaton goes to $\phi \to -\log\rho$, which means that the coupling constant $g_s = e^\phi \sim \frac{1}{\rho} \to 0$, meaning that there is no noncommutativity, and therefore the deformed gauge side is free, and perturbative at high energies, with operators interacting via the usual product. This is a remarkable result, since this suggests that we have an *asymptotically free gauge theory*, at least in some large R-charge sector.

The (u, φ) case is different from the others, and that's why we put a star over it in the table, while its gauge dual also can be described as an irrelevant operator, there is no asymptotic freedom, and the B field increases as we go to higher energies. However, there is a still more striking observation about the metric of this deformed model: it is still a parallelizable pp wave! In fact, we will analyze it in more detail next.

Doing the change of coordinates in that specific pair-deformed metric

$$\rho \to R\frac{\sqrt{r_+ f_s'(r_+)}}{4}, \quad \Phi \to \Phi\frac{4}{r_+ f_s'(r_+)}, \quad \Phi_{A,B} \to \Phi_{A,B}\frac{e_{A,B}}{\sqrt{2r_+}}, \quad \Theta_{A,B} \to \Theta_{A,B}\frac{e_{A,B}}{\sqrt{2r_+}} \tag{7.33}$$

obtaining

$$
\begin{aligned}
ds^2 = r_+\, du\, dv + d\Phi_A^2 + d\Theta_A^2 + d\Phi_B^2 + d\Theta_B^2 + 4r_+\, d\Theta^2 + dR^2 + R^2 d\Theta^2 \\
+r_+\, dX^2 \\
+dv\left(\frac{e_B\Theta_B}{2}d\Phi_B - \frac{e_A\Theta_A}{2}d\Phi_A + \frac{R^2(Q_B-Q_A)r_+\zeta'(r_+)}{4}d\Phi\right) \\
-\frac{\eta^2R^2r_+^4f_s'(r_+)^2}{64}\, dv^2.
\end{aligned}
\tag{7.34}
$$

Ignoring for a while the angular and X terms, since they are equal, we focus on the expression involving R, Φ, v. We first change coordinates to $R^2 = z^2 + y^2$, and then redefine z, y in terms of a complex coordinate $z_1 = z + iy$. We get

$$
ds^2 = dz_1 d\bar{z}_1 - \frac{i}{2}dv\left(\frac{(Q_B-Q_A)r_+\zeta'(r_+)}{4}(\bar{z}_1 dz_1 - z_1 d\bar{z}_1)\right) - \frac{\eta^2 r_+^4 f_s'(r_+)^2}{64L^4}z\bar{z}dv^2. \tag{7.35}
$$

We next do the rescaling $z_1 \to \omega e^{-\frac{iva}{2}}$, where $a = \frac{Q_B-Q_A}{4}r_+\zeta'(r_+)$, and $v \to \frac{-2V}{r_+}$, so we get

$$
ds^2 = d\omega d\bar{\omega} - \left(\frac{a^2}{r_+^2} + \frac{\eta^2 r_+^2 f_s'(r_+)^2}{16}\right)|\omega|^2 dV^2. \tag{7.36}
$$

This shows that we have a parallelizable metric, since it is just like the previous case for the Penrose limit after TsT transformation, but with a new term,

$$
\begin{aligned}
ds^2 &= -2\, du\, dV - L^4\, dV^2\left(r_+\frac{e_b^2}{16}|\omega_1|^2 + r_+\frac{e_a^2}{16}|\omega_2|^2 \right.\\
&\left. + \left(\frac{\eta^2 r_+^2 f_s'(r_+)^2}{16} + \frac{(Q_B-Q_A)^2\zeta'(r_+)^2}{16}\right)|\omega_3|^2\right) \\
&\quad + |d\omega_1|^2 + |d\omega_2|^2 + |d\omega_3|^2 \\
B_2 &= i\frac{\eta r_+ f_s'(r_+)}{2L^2}(\omega_3 d\bar{\omega}_3 - \bar{\omega}_3 d\omega_3)\wedge dV \\
\Phi &= \log r_+
\end{aligned}
\tag{7.37}
$$

This is more interesting than the (v, u) case, since we have a true physical modification, not just a rescaling of coordinates. In fact, we could obtain a spin-chain Hamiltonian just as we did before, and the fact that the two Hamiltonians (gauge and strings) match before we do the TsT transformations means that the deformed Hamiltonian must be the same for both cases.

For the string action, using the gauge $X^V = \tau$, the new terms add

$$S \supset \int d\sigma\, d\tau \left[|\partial\omega_3|^2 - \left(\frac{\eta^2 r_+^2 f_s'(r_+)^2}{16} + \frac{(Q_B - Q_A)^2 \zeta'(r_+)^2}{16} \right) |\omega_3|^2 \right. \tag{7.38}$$

$$\left. + \frac{i\,\eta r_+ f_s'(r_+)}{2} \left(\omega_3\, \partial_\sigma \bar{\omega}_3 - \bar{\omega}_3\, \partial_\sigma \omega_3 \right) \right]. \tag{7.39}$$

In the AdS_3/CFT_2 case with NS-NS fields, corresponding to a symmetric product CFT, it was shown in [68] that a "single trace" deformation, by $\sum_i T_i \bar{T}_i$ (instead of the standard $\sum_i T_i \sum_j \bar{T}_j$) is dual to a deformed holographic dual, which later, in [5], it was shown to be a TsT transformation, and has very similar properties to the standard $T\bar{T}$ deformation. Therefore, in [123], it was proposed that in other other cases and in other dimensions, like for instance in AdS_5/CFT_4, (perhaps several) TsT transformations correspond to some "generalized single-trace" $T\bar{T}$ deformations, that have similar properties to the $T\bar{T}$ deformation. Here, as we saw, we are still in 3 dimensions in gravity, with CFT defined along the x, t coordinates, though an extra coordinate appears effectively.

Therefore, even if the "single-trace" interpretation is not directly relevant anymore, we believe that the same properties that make it similar to the standard $T\bar{T}$ as in [68], in particular solvability, should be still valid. This is indeed what we see in the TsT followed by Penrose limit case, that we analyzed first, where we saw that we just have a modification of Δ_η and J_ψ, as well as g^2_{YM}, by the same factor $1/\sqrt{1-\eta^2 r_+^2}$, but otherwise the system is as solvable as before the TsT deformation.

In the case of a TsT deformation after the Penrose limit, things are less simple, since we saw that in the most relevant case, of deformation over (X, v), the resulting string Hamiltonian is hard to analyze, as are most of the other cases. The cases of TsT deformations over (v, X), (v, Θ) and (u, φ) hold most promise to preserve solvability in a simple way, but we have not yet fully analyzed them.

7.2 Bootstrapping the Leigh-Strassler deformations and the hidden symmetries of the model

Related work by the author

Available on arXiv: https://arxiv.org/abs/2601.02692 [156]

Introduction

As explained in Chapter 6, there are a large class of deformations of $\mathcal{N} = 4$ SYM defined by Leigh and Strassler [103]. These deformations preserve part of the supersymmetry, resulting in a wide set of $\mathcal{N} = 1$ theories, while also maintaining conformal symmetry, being marginal deformations. This class of deformations is fully characterized by four parameters (g, q, h, κ): the undeformed coupling, the parameter deforming the algebra, the one responsible for a new term in the superpotential, and the deformed coupling, respectively. The deformation acts on the superpotential, and it was also shown in [103] that two special symmetries are preserved: the cyclic property between the scalar fields ($X_i \to X_{i+1}$) and invariance under $X_i \to X_i \omega^{i+1}$. These symmetries, together with conformal exactness, imply a "finiteness constraint" $\mathcal{F}(g, q, h, \kappa)$ that relates the four parameters. Among this family of models, some, such as the β-deformation [107], are special points since $\mathcal{F}$ does not receive quantum corrections at any higher loop order [113] in the planar limit, and they admit a clear interpretation in terms of transformations in the supergravity background [63]. Other deformations have also been shown to be exactly conformal and related to twists [111], though these cases are less well understood in terms of gauge/gravity duality.

Rossi [138, 139] and Mauri [113] showed the exactness of the finiteness constraint for the planar limit, $N \to \infty$, for the real β-deformation, suggesting that integrability is preserved by this deformation, at least in the planar limit. Later, Mansson [111, 110] and Bundzik [34] constructed several different "integrable" deformations[2], for which the finiteness constraint does not receive quantum corrections and which possess an R-matrix satisfying the Yang-Baxter (YB) equation of motion. These deformations are controlled by the pair of parameters (q, h), which are related by

$$(q, h) = (0, \frac{1}{\bar{h}}), \quad ((1+p)e^{in}, pe^{im}), \quad (e^{in}, e^{im}). \tag{7.40}$$

We should mention that in [113, 30, 92], the authors identified a "symmetry" involving the q parameter that deforms the algebra in the amplitudes and in the expansion of the $\mathcal{F}$ constraint, which will be shown to be crucial in the present case, although their main results were not based on this symmetry. Moreover, [113] for example focused on β-deformation, while here we generalized it to any LS-type,

[2]As always, in the planar limit.

while the authors of [30] expanded over what we are going to call “old-variables” for the constraint function $\mathcal{F}$, (g, κ, h, q), which we are going to show to be “over counting” coefficients, by showing the new symmetry constrained them. Here, this symmetry is extended/generalized and we refer to as the “q-symmetry”, to guide the structure of the calculations, generalizing it to the other three parameters of the model, without focusing on Feynman diagrams (though we will, of course, later use a result obtained in the literature via Feynman diagrams). For instance, while [92] performed an expansion in g, here we focus primarily on this symmetry, and the expansion series for $\mathcal{F}$ are taken with respect to objects that are invariant under the “q-symmetry”[3], presenting how this strongly constrains the form of the finiteness condition.

Leigh-Strassler deformed model and discussion of the results

We are looking for a constraint relation between the four global parameters q, κ, h, and g of the deformed $\mathcal{N} = 4$ theory introduced in [103]. The first parameter, q, deforms the algebra of the fields in $\mathcal{N} = 4$ into a q algebra of the form $[X, Y]_q = XY - qYX$. The second parameter, κ, is the deformed coupling, which reduces to the undeformed coupling g in the original theory, and the third parameter, h, multiplies a new term in the superpotential of the deformed theory. In this section we use the $\mathcal{N} = 1$ supersymmetry description of $\mathcal{N} = 4$, and the deformed superpotential is [103]

$$\mathcal{W} = \kappa \text{ tr } (X_1[X_2, X_3]_q + \frac{h}{3}(\sum_i X_i^3)) + \text{h.c.} \tag{7.41}$$

where X_i are the chiral superfields in the superspace formalism. The constraint we are seeking to describe can be expressed as a function of the four parameters

$$\mathcal{F}(g, q, h, \kappa) = 0, \tag{7.42}$$

resulting from the $\beta = 0$ conditions for all the parameters, together with the symmetries LS-theory presents.

In the *real case*, $\mathcal{F}$ assumes a general homogeneous form

$$\mathcal{F}(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = \sum_{i,j}^{\infty} \sum_{p=0}^{j} d_{[ij]\ p(j-p)} \gamma_2^p \gamma_3^{j-p} \gamma_1^i \gamma_4^j, \quad d_{00} = 0, \tag{7.43}$$

[3] For real parameters at first, later generalizing it.

where γ_i are parameters defined in terms of the old ones. (7.43) together with further conditions as explained in next sections, suggest the first correction in the planar limit should be of order g^8, in agreement with results reported in the literature [3]. The coefficients $d_{...}$ are not fully determined by the constraints imposed on $\mathcal{F}$, which is expected: ultimately, these parameters depend on the numerical values obtained by summing over all possible Feynman diagrams at all loop orders.

Nevertheless, we can obtain non-trivial results solely from the relation between $\mathcal{F}$ and the parameters. In the Section that follows, we derive a general expression that can be directly compared with results available in the literature. We show that to first order in g^2,

$$\lambda = N\kappa^2 \left[\frac{\eta_1}{N^2} q + \left(\frac{1}{2} - \frac{\eta_1}{2N^2} \right) q^2 + \left(\frac{1}{2} - \frac{\eta_1}{2N^2} + \Theta_4 h^2 \right) \right], \tag{7.44}$$

where η_1 and Θ_4 are the only free coefficients that must be determined from Feynman loop calculations. This result, obtained solely from intrinsic conditions, can be directly compared with results in the literature [3], assuming real variables,

$$\lambda = N\kappa^2 \left[\frac{2}{N^2} q + \left(\frac{1}{2} - \frac{1}{N^2} \right) q^2 + \left(\frac{1}{2} - \frac{1}{N^2} \right) + \left(\frac{1}{2} - \frac{2}{N^2} \right) h^2 \right]. \tag{7.45}$$

Following two different heuristic approaches in which we analyze the planar limit of the finiteness constraint, we obtain interesting characteristic features for each of them. In the first approach, where we make an ansatz for the coefficients in the series, we correctly predict the conformal and integrable pairs of (7.40), although some caveats remain. In the second approach, where we proceed in the opposite direction and instead impose the vanishing condition at (7.40) together with another ansatz, and we find that the first correction to $\mathcal{F}$ appears at order κ^8, consistent with the literature [129, 138, 34].

For complex parameters, the analysis becomes more involved, but we can still obtain a constrained function $\mathcal{F}$. We find that it cannot be expressed in terms of the so-called good variables γ_i, as was possible for the real parameters in (7.43). Instead, we introduce a systematic method for constructing the structure function in the complex case. We also show that this function is significantly more constrained than one might expect, since a general expansion would allow for three additional parameters $(\bar{q}, \bar{\kappa}, \bar{h})$. To test its predictive power, we compare it with a previous result from the literature, demonstrating that it can be correctly fitted.

We also find a relation between the R matrix of the transformed complex pair $(\tilde{q} = q^{-1}, \tilde{h} = -\frac{h}{q})$ and that of the original pair (q, h), showing that it is exactly equal to the inverse of the initial R matrix

$$R_{\tilde{q},\tilde{h}} = R^{-1}_{q,h}, \tag{7.46}$$

and consequently it satisfies the Yang-Baxter equation whenever the original pair does, providing a systematic method for generating integrable pairs from the standard one. We further investigate the properties of this transformation, including its implications for deformed algebras, such as the affine Lie algebra associated with the integrable XXZ spin chain, and demonstrate that the transformed R matrix remains integrable.

The last section is devoted to examining this transformation from the perspective of AdS/CFT [107], identifying the parameter k in TsT transformations, which on the gravity side maps to the parameter q, as anticipated in [106, 63]. We obtain, as the simplest solution, a linear relation between the logarithm of q and k, $k \sim \log q$, where for the special case of real beta, we obtain a linear relation between k and β. This provides additional evidence, consistent with results in the literature, that our approach is well founded.

Finiteness constraint

Real parameters

Mathematical conditions on the structure of $\mathcal{F}$

To obtain the finiteness constraint, we follow an approach analogous to that used in bootstrap analyses [132], but applied to the finiteness $\mathcal{F}$, which is defined as the condition for the vanishing of the beta function. We assume a set of mathematical axioms and physical principles, and by imposing them we progressively restrict the allowed shape of $\mathcal{F}$.

In principle, $\mathcal{F}$ is intended to describe infinitely many different theories, depending on the relations among the parameters, and our approach highlights this powerful generalization property without specifying any particular quadruple, as illustrated in Figure 7.1.

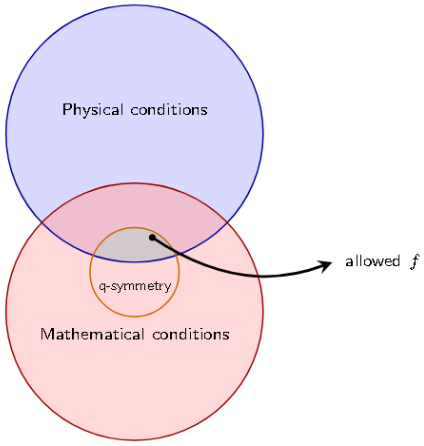

Figure 7.1: A Venn diagram illustrating the bootstrapping approach. The regions representing mathematical and physical conditions (e.g., loop effects and consistency) can grow with higher-loop orders in g. The allowed region of $\mathcal{F}$ expands with these corrections, but remains far more constrained than generally expected, a "smaller infinity" in a sense.

First, we demonstrate a discrete symmetry satisfied by the superpotential $\mathcal{W}(g, q, h, \kappa, X_i)$. Consider two models, 1 and $1'$, both deformed from $\mathcal{N} = 4$ SYM and within the LS classification [103]. Each model has a generic deformed superpotential $\mathcal{W}$ and the same matter content. In the unprimed model, we denote the chiral fields as (X_1, X_2, X_3), while in the primed model they are denoted (X_1', X_2', X_3'), with the superpotential being

$$\mathcal{W} = \kappa \text{ tr } (X_1[X_2, X_3]_q + \frac{h}{3}(\sum_i X_i^3)), \quad \mathcal{W}' = \kappa' \text{ tr } (X_1'[X_3', X_2']_q + \frac{h'}{3}(\sum_i X_i'^3)). \tag{7.47}$$

Although not identical in principle, the theories 1 and $1'$, being subjected to the same type of interactions with identical field content and arranged in exactly the same way, should behave identically under loop corrections. Consequently, the constraint relation among the parameters in 1, $\mathcal{F}(g, q, h, \kappa)$, and those in $1'$, $\mathcal{F}'(g', q', h', \kappa')$, must be identical,

$$\mathcal{F}(g, q, h, \kappa) = \mathcal{F}'(g', q', h', \kappa') = \mathcal{F}(g', q', h', \kappa'). \tag{7.48}$$

Now, we perform the following relabeling in the $1'$ theory. Since this is a simple relabeling rather than a transformation, all terms in the action remain unchanged,

$$g' = g, \quad \kappa' = -\kappa q, \quad q' = \frac{1}{q}, \quad h' = -\frac{h}{q}, \quad X_1' \to x_1, \quad X_2' \to x_2, \quad X_3' \to x_3, \tag{7.49}$$

then, the superpotential of $1'$ becomes (the other terms of the deformed LS action, which are identical to those of the original $\mathcal{N} = 4$ action, remain unaffected by this relabeling),

$$\begin{aligned} \mathcal{W}' &= (-\kappa q) \;\operatorname{tr}\left[x_1\left(x_3x_2 - \frac{1}{q}x_2x_3\right) - \frac{h}{3q}(\sum_i x_i^3)\right] \\ &= \kappa \;\operatorname{tr}\left[x_1\left(x_2x_3 - qx_3x_2\right) + \frac{h}{3}(\sum_i x_i^3)\right] \\ &= \kappa \;\operatorname{tr}\left[x_1[x_2, x_3]_q + \frac{h}{3}(\sum_i x_i^3)\right]. \end{aligned} \tag{7.50}$$

Therefore, the deformed superpotential of $1'$ is identical to that of 1, (7.47), under the simple relabeling $X_i' = X_i$, and hence the Lagrangians of both theories are the same. What is more interesting, and precisely what we are seeking, is the implication of the non trivial transformations (7.49) for the equality of the constraints (7.48). By applying (7.49) to (7.48), we find that $\mathcal{F}$ undergoes a non trivial transformation,

• 1. $$\mathcal{F}(g,q,h,\kappa) = \mathcal{F}\left(g, \tfrac{1}{q}, -\tfrac{h}{q}, -\kappa q\right) \qquad \text{“q-symmetry”}. \tag{7.51}$$

We refer to this as the “q-symmetry”, since it appears to be induced by the parameter q. This provides a strong condition that severely constrains the form of $\mathcal{F}$. Then, $\mathcal{F}$ is constrained to be invariant under such symmetry, and following that line of reasoning we will later seek for new variables that are automatically invariant under all four transformations (one trivial and three non-trivial). For the moment, however, we continue to work with the original variables.

The second and third assumptions state that we can simply change the sign of both couplings, $-\kappa$ and $-g$. This is valid because all terms in the action, aside from those in the superpotential, are quadratic in the fields X_i. Therefore, we can perform the transformation $X_i \to -X_i$ for all three complex fields, while simultaneously taking $\kappa \to -\kappa$, or $g \to -g$ in the undeformed case, where the only appearance of g is inside τ, leaving the superpotential invariant. The implication of this transformation for the constraint $\mathcal{F}$ is

$$\begin{array}{lll} \bullet\, 2. & \mathcal{F}(g,q,h,\kappa) = \mathcal{F}(g,q,h,-\kappa) & \text{sign 1,} \\ \bullet\, 3. & \mathcal{F}(g,q,h,\kappa) = \mathcal{F}(-g,q,h,\kappa) & \text{sign 2.} \end{array} \tag{7.52}$$

Next, we perform a change of variables, $q_i \to \gamma_i$, which are automatically invariant under q-symmetry, as well as under the sign flip transformations 1 and 2, (7.51) and (7.52). They are

$$\gamma_1 = g^2, \quad \gamma_2 = q + \frac{1}{q}, \quad \gamma_3 = \frac{h^2}{q}, \quad \gamma_4 = \kappa^2 q, \tag{7.53}$$

and then, we can write $\mathcal{F}$ as $\mathcal{F} = \mathcal{F}(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = \mathcal{F}(\gamma_i)$.

Further, if $g = 0$, then $\kappa = 0$, and none of the analysis carried out here would make sense. Otherwise, we would be computing a loop correction for a model that has no propagators. This implies

$$\bullet\, 4. \qquad \mathcal{F}(0, \gamma_2, \gamma_3, 0) = 0 \qquad \text{free theory.} \tag{7.54}$$

Looking back at the superpotential (7.41), we see that nothing prevents us from choosing $q = 0$. Therefore, $q = 0$ lies within the range of possible values for the deformation parameters and consequently corresponds to a specific constraint $\mathcal{F}(\gamma_i)$ among the constants. In other words, the general form of $\mathcal{F}$ can't contain terms that diverge as $q \to 0$. We therefore impose analyticity,

$$\bullet\, 5. \qquad \mathcal{F}(\gamma_1, \gamma_2, \gamma_3, \gamma_4) \text{ is analytic} \qquad \text{A.1.} \tag{7.55}$$

To see how to avoid such singular terms, we assign an order number $\mathcal{O}$ to each of the γ_i. This order number is related to the strongest power of q as $q \to 0$, and we find

$$\mathcal{O}_{\gamma_1} = 0, \quad \mathcal{O}_{\gamma_2} = -1, \quad \mathcal{O}_{\gamma_3} = -1, \quad \mathcal{O}_{\gamma_4} = 1. \tag{7.56}$$

Since $\mathcal{O}_{\gamma_1^i \gamma_4^j} = j$, we need $\mathcal{O}_{c_{ij}}$ to be at most $-j$ in order to keep a non negative order number. The final expression for the complete structure is therefore obtained without invoking any additional physical input beyond the assumptions stated above,

$$\mathcal{F}(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = \sum_{i,j}^{\infty} \sum_{p=0}^{j} d_{[ij]\ p(j-p)} \gamma_2^p \gamma_3^{j-p} \gamma_1^i \gamma_4^j, \quad d_{00} = 0 \tag{7.57}$$

Although the general expression is best suited for illustrating the overall structure, for practical purposes we will use the form given with the explicit form with $\gamma_{2,3}$ is hidden.

We still have one last conditions $\mathcal{F}$ need to satisfies,

- 6. $$\mathcal{F}(\gamma, 2, 0, \gamma) = 0 \qquad \text{CC.1.} \tag{7.58}$$

This condition essentially means that $\mathcal{F}$ should reproduce a feature of $\mathcal{N} = 4$, where the theory is exactly conformal to all orders. Since the deformed model reduces to the undeformed $\mathcal{N} = 4$ case when $q = 1$ and $h = 0$, this implies $g \to \kappa$.

If we impose all the above conditions at first order in κ^2, we obtain that

$$\mathcal{F} = (\alpha_2 + \alpha_3\gamma_2 + \alpha_4\gamma_3)\gamma_4 - (\alpha_2 + 2\alpha_3)\gamma_1 = 0. \tag{7.59}$$

Or, by rewriting (7.59) in a clearer way,

$$\mathcal{F} = (-2\alpha_3 + \alpha_0 + \alpha_3\gamma_2 + \alpha_4\gamma_3)\,\gamma_4 - \alpha_0\gamma_1 = 0. \tag{7.60}$$

We can also change the constants and open the new variables γ in terms of the older to organize (7.60) as

$$g^2 = \kappa^2 \left[\Theta_3 q + \left(\frac{1}{2} - \frac{\Theta_3}{2}\right) q^2 + \left(\frac{1}{2} - \frac{\Theta_3}{2} + \Theta_4 h^2\right)\right]. \tag{7.61}$$

This is the further we can go by imposing only the mathematical conditions we discussed.

However, we can go further by making use of the planar limit of the theory and a back-of-the-envelope calculation. The potential V of the model is given by summing the F terms, ignoring the D terms of the Lagrangian,

$$V = \sum_i \mathrm{tr}(|\partial_i W|^2) = \sum_i \mathrm{tr}(|[\phi_i, \phi_{i+1}]_q + h\phi_{i+2}^2|^2), \tag{7.62}$$

and we can use only one term of the sum, since it is related to the others by cyclic permutations. Considering $\mathrm{tr}(|[\phi_1, \phi_2]_q + h\phi_3^2|^2)$, we collect the q, $|h|^2$, and q^2 terms to obtain

$$\mathrm{tr}\Big(|q|^2(\phi_2\phi_1\phi_1^\dagger\phi_2^\dagger) - q(\phi_2\phi_1\phi_2^\dagger\phi_1^\dagger) + |h|^2\phi_3\phi_3\phi_3^\dagger\phi_3^\dagger\Big) \subset \mathcal{L}, \tag{7.63}$$

where the term involving $\bar{q}$ is analogous to the one involving q, and we do not consider it separately. We see that, under Wick's theorem, $|h|^2$ and $|q|^2$ stands out as they contain neighborhood contractions $\phi_1\phi_1^\dagger, \phi_3\phi_3^\dagger$, in opposite to q

The handwaving check above has a practical consequence for us. Since the first-loop calculations of $\mathcal{F}$ shouldn't be identically zero, should reduce to $\mathcal{F} = \lambda - Ng^2$ when $q = 1, h = 0$ and q is in a smaller order in comparative with $|h|^2$ and $q|^2$, then necessarily the correct leading power of the terms differs by a factor of $1/N^2$. Then, the finiteness becomes

$$\lambda = N\kappa^2 \left[\frac{\eta_1}{N^2}q + \left(\frac{1}{2} - \frac{\eta_1}{2N^2}\right)q^2 + \left(\frac{1}{2} - \frac{\eta_1}{2N^2}\right) + \Theta_4 h^2\right] + O(N^{-3}). \tag{7.64}$$

A second information we can extract from (7.63) is that, at least in the planar limit, the term involving $|h|^2$ and the term involving $|q|^2$ should be related. In fact, this is expected, since they contribute identically to (7.63), as is clear if we write the $|q|^2$ and $|h|^2$ term as

$$\operatorname{tr}\left(T_a T_b T_c T_d\right)\left(|q|^2\phi_2^a\phi_1^b\phi_1^{\dagger c}\phi_2^{\dagger d} + |h|^2\phi_3^a\phi_3^b\phi_3^{\dagger c}\phi_3^{\dagger d}\right). \tag{7.65}$$

From (7.65) we can conclude that, to first order and in the planar limit, $|h|^2$ and $|q|^2$ contributes the same for $\mathcal{F}$, this is because it doesn't matter if ϕ_3 are the same in the second term, as in the planar limit the only thing that is important is the neighborhood. However, this is not true for the first correction to the planar order: In that case, we allow to ϕ_3^a pass over ϕ_3^b, and then we can have twice contractions from the second term, $\delta^{bc}\delta^{ad} + \delta^{ac}\delta^{bd}$, in comparative with the first term, where we only have $\delta^{bc}\delta^{ad}$.

Then, on the finiteness constraint, this implies that

$$\lambda = N\kappa^2 \left[\frac{\eta_1}{N^2}q + \left(\frac{1}{2} - \frac{\eta_1}{2N^2}\right)q^2 + \left(\frac{1}{2} - \frac{\eta_1}{2N^2}\right) + \left(\frac{1}{2} - \frac{2\eta_1}{2N^2}\right)h^2\right]. \tag{7.66}$$

We stress that although the calculations here were carried out for real parameters, we can similarly do it for complex parameters.

For N going to infinite, (7.66) reduces to

$$\lambda = \frac{N\kappa^2}{2}\left[1 + |q|^2 + |h|^2\right]. \tag{7.67}$$

And then, we were able to obtain exactly the planar limit, to first order, of $\mathcal{F}$.

This remarkable result, obtained solely by imposing intrinsic conditions together with some hand waving calculations, can be compared with results from the literature. [3] obtained an expression that gives exactly the same large N limit at first order, but the matching between our result and theirs is not restricted to this large N limit. There, the authors used the superpotential

$$\mathcal{W} = i\frac{\eta\sqrt{2}}{3!}\epsilon_{ijk}\ \mathrm{tr}\ (\phi^i[\phi^j,\phi^k]) + \frac{h_{ijk}}{3!}\ \mathrm{tr}\ (\phi^i\{\phi^j,\phi^k\}), \tag{7.68}$$

and by assuming that the only non zero components of h are $h_{111} = h_{222} = h_{333} = h\kappa$ and h_{123}, together with the rescaling $\eta = \frac{\kappa(1+q)}{i2\sqrt{2}}$ and $h_{123} = \frac{\kappa(1-q)}{2}$, the superpotential reduces to our expression (7.41). The authors obtained a loop correction of the form $g^2 = \epsilon^2 + \frac{\alpha^2}{2}\left(\frac{-1}{2} + \frac{2}{N^2}\right)$, but we should note that their coupling g does not match our convention. This can be seen, for instance, from their NSVZ correction [125], which agrees with the standard literature only after the rescaling $g = i\frac{g}{\sqrt{2}}$. Substituting this into their correction expression, together with the expressions for ϵ, h_{123}, and h, their result expressed in terms of our parameters, if we assume $q = \bar{q}$, is

$$\lambda = N\kappa^2\left[\frac{2}{N^2}q + \left(\frac{1}{2} - \frac{1}{N^2}\right)q^2 + \left(\frac{1}{2} - \frac{1}{N^2}\right) + \left(\frac{1}{2} - \frac{2}{N^2}\right)h^2\right], \tag{7.69}$$

to be compared with our result (7.66). We see that the Feynman diagrams only have the real parameters $\eta_1 = 2$. The fact that our highly constrained function $\mathcal{F}$ is nevertheless able to reproduce the results found in the literature, which in our analogy is similar to fitting a theoretical prediction to experimental data, strongly suggests that our method is correct.

So, returning to the general discussion for $\mathcal{F}$ to any order, we can split the series defining it into three series, those with powers of κ higher than two, those with only powers of g higher than two, and the expression obtained previously in (7.59), which we denote here by $\mathrm{F}_{g^2,\kappa^2}$,

$$\mathcal{F}(\gamma_i) = \mathrm{F}_{g^2,\kappa^2} + \sum_i c_1^i(\gamma_{2,3})\gamma_1^i + \sum_{jk} c_2^{jk}(\gamma_{2,3})\gamma_1^j\gamma_4^k. \tag{7.70}$$

Imposing CC.1 on (7.70), $\mathcal{F}$ should reduce to the simple constraint of $\mathcal{N} = 4$, namely $g^2 = |\kappa|^2$. As we have seen, $\mathrm{F}_{g^2,\kappa^2}$ already reproduces this relation, and the expression above therefore reduces to

$$0 = \sum_i c_1^i(\gamma_{2,3})\gamma_1^i + \sum_{jk} c_2(\gamma_{2,3})^{jk}\gamma_1^j\gamma_4^k. \tag{7.71}$$

Now, for this equality to be satisfied, we observe the following. While $c_1(\gamma_{2,3})$ is, a priori, a function of γ_2 and γ_3, analyticity together with the order counting discussed above implies that the terms in the pure γ_1 series cannot depend on either γ_2 or γ_3. Therefore, to satisfy the equality above, the coefficients c_2 need to satisfy $c_2(\gamma_2 = 2, \gamma_3 = 0)$ and $c_1^i = 0$. Therefore, (7.70) can be written as

$$\mathcal{F}(\gamma_i) = \mathrm{F}_{g^2,\kappa^2} + \sum_{jk} c_2(\gamma_{2,3})\gamma_1^j\gamma_4^k. \tag{7.72}$$

Higher order terms

After all the discussion, one might worry that the q-symmetry is exact only at tree level, that is, at the classical level, and that loop effects start to break it. We have demonstrated that this is not the case, and that the symmetry is genuinely a quantum symmetry, since it acts on complex scalar fields and is a symmetry of the superpotential $\mathcal{W}$, which by itself is non renormalizable. Nevertheless, we can still compare with results from the literature to test whether our expansion of $\mathcal{F}$ can be fitted to known expressions, since this would indicate that our ansatz is correct.

In [138], the authors calculated a higher order correction to the propagator of a pair of fields. In that analysis, the authors assumed $h = 0$, and therefore our expression for the coefficients of $\mathcal{F}$ simplifies drastically to an expansion in γ_2. The correction they obtained, in our notation and assuming real variables, was

$$\kappa^2\Big|_6 = \kappa^6 g^6 (q-1)^2 \left(((q^2+q+1)^2 - 9q^2)N^2 + 5(q-1)^4 \right) \frac{(N^2-4)}{N^3}. \tag{7.73}$$

For $\kappa^6 g^6 = \frac{1}{q^3}\gamma_4^3\gamma_1^3$, the corresponding component in our framework, with $\gamma_3 = 0$, is

$$(a_1 + a_2\gamma_2 + a_3\gamma_2^2 + a_4\gamma_2^3)\gamma_4^3\gamma_1^3. \tag{7.74}$$

Imposing CC.1 gives $a_4 = -\frac{a_1+2a_2+4a_3}{8}$. We then find a solution,

$$\alpha_1 \to \frac{8\left(20 - 13N^2 + 2N^4\right)}{N^3}, \quad \alpha_2 \to -\frac{12\left(20 - 9N^2 + N^4\right)}{N^3}, \quad \alpha_3 \to -\frac{30\left(-4 + N^2\right)}{N^3}. \tag{7.75}$$

However, as argued in [138], the constants can be adjusted so that the g^6 correction becomes a g^8 correction. We do not pursue this point further here, since the previous analysis was restricted to real parameters. We emphasize a potential source of confusion. The terms obtained in $\mathcal{F}$ are also the most general terms that arise from loop calculations after summing all amplitudes in the same order. In this sense, each term in $\mathcal{F}$ must be consistent with standard Feynman diagram computations. Nevertheless, the expression (7.73) is not the final form that should appear in the full sum defining $\mathcal{F}$. This can be seen, for example, by noting that the planar limit of (7.73) does not vanish for the β deformation.

Complex parameters

Since we did not impose any restriction on whether the parameters take values in $\mathbb{R}$ or $\mathbb{C}$ when demonstrating the invariance of the superpotential $\mathcal{W}$ (7.41), the q-symmetry should, and indeed does, hold for both real and complex values. However, in the analysis that followed this demonstration, we restricted the parameters to lie in the real field $\mathbb{R}$. We now justify this choice by showing how the calculations become considerably more involved over $\mathbb{C}$, and afterwards we suggest a less direct approach to the function $\mathcal{F}$.

Following the analysis of Section 7.2, one might naively attempt to construct variables γ_i that are invariant under q-symmetry, which is the same symmetry as before but now extended to complex values. Before doing so, however, we observe that there is an additional requirement on the structure of $\mathcal{F}$, namely that it must be real. Suppose we have a finiteness condition of the general form

$$\mathcal{F} = \sum_i c_i g^{2i} = 0, \tag{7.76}$$

with c_i depending on the complex variables $(q, \bar{q}, \kappa, \bar{\kappa}, h, \bar{h})$. If (7.76) holds, then so does its complex conjugate, recalling that g is the only real variable by definition,

$$\bar{\mathcal{F}} = \sum_i \bar{c}_i g^{2i} = 0. \tag{7.77}$$

Subtracting (7.76) and (7.77), we find that either $\bar{c}_i - c_i = 0$ for all i, or $g = 0$. We of course select the first option, and therefore the coefficients c_i are real, which implies that $\mathcal{F}$ is real. We refer to this property as Reality R.1,

- 7. $$\mathcal{F}(g, \alpha_i) \text{ is real} \qquad \text{R.1.} \tag{7.78}$$

We now try to identify good variables in which to expand $\mathcal{F}$, paying particular attention to conditions R.1 and q-symmetry, which are the strongest constraints in this case. The simplest variables one might guess are

$$\begin{gathered} \Gamma_1 = g^2, \quad \Gamma_2 = \frac{1}{2}\left(c_1(q+q^{-1}) + \bar{c}_1(\bar{q}+\bar{q}^{-1})\right) + c_2, \\ \Gamma_3 = \frac{h\bar{h}}{2}\left(c_3 q^{-1} + \bar{c}_3\bar{q}^{-1}\right), \quad \Gamma_4 = \frac{\kappa\bar{\kappa}}{2}\left(c_4 q + \bar{c}_4\bar{q}\right). \end{gathered} \tag{7.79}$$

The difficulty with complex variables arises when imposing analyticity A.1. By treating each complex variable as independent, it is not possible to obtain a simple expression that is simultaneously non zero and non singular in $(q, \bar{q})$. For example, one might expect an expression of the following form at lowest order,

$$\mathcal{F} = \gamma_1\alpha + \Gamma_4(a_1 + a_2\Gamma_2 + a_3\Gamma_3), \tag{7.80}$$

which, when expanded in terms of the old variables, becomes

$$\begin{aligned} g^2\alpha + \frac{|\kappa|^2}{2}\left(a_1c_4q + a_2c_4\left(\frac{c_2(q^2+1)+\bar{c}_2(|q|^2+q\bar{q}^{-1})}{2}\right) + a_3c_4\left(\frac{|h|^2}{2}\left(c_3+\bar{c}_3q\bar{q}^{-1}\right)\right)\right) \\ + \frac{|\kappa|^2}{2}\left(a_1\bar{c}_4\bar{q} + a_2\bar{c}_4\left(\frac{c_2(|q|^2+\bar{q}q^{-1})+\bar{c}_2(\bar{q}^2+1)}{2}\right) + a_3\bar{c}_4\left(\frac{|h|^2}{2}\left(c_3\bar{q}q^{-1}+\bar{c}_3\right)\right)\right). \end{aligned} \tag{7.81}$$

To eliminate the singular terms, and since the c_i are constants, we must impose $c_2 = \bar{c}_2 = c_3 = \bar{c}_3 = 0$, which contradicts the implications of the reductionism condition R.2. This singular behavior persists at any loop order. The origin of the problem can be understood by noticing that the variables are, in analogy with complex analysis or group theoretic language, of degree $(i, 0) + (0, i)$. Consequently, any attempt to use Γ_4 to get rid of the singularity-arising due to the symmetries works only for certain terms, but not for all of them. One might try to redefine the variable Γ_4 to have degree (i, i), for instance as $(q + \frac{1}{q})(\bar{q} + \frac{1}{\bar{q}})$, or to modify the other variables accordingly, however, this approach is incorrect for two reasons. First, it does not reduce to the real variables γ_i, thus violating the reductionism condition. First, it significantly increases the power of the variables, leading to higher order corrections already at lowest order, which is not expected. Second, and most importantly, it fails to match previous results from the literature, which at the bootstrap level play a role analogous to experimental validation.

Our situation might appear hopeless, since we do not have good variables beyond the original ones with which to perform the expansion properly. Even though this seems to be the case, we show that it is still possible to obtain a constrained finiteness function $\mathcal{F}$ by following a simple prescription.

Before proceeding, we observe a symmetry under which the superpotential $\mathcal{W}$, together with its conjugate $\bar{\mathcal{W}}$, is invariant. This symmetry was not apparent in the real case because it is intrinsically tied to the complex nature of the variables. We refer to it as the Hermitian condition H.1, and now demonstrate its origin. As is well known, the Leigh Strassler action can be written as

$$S_{\mathcal{N}=4} + \underbrace{\int d^2\theta\, \mathcal{W}(g_i, X_i) + \int d^2\bar{\theta}\, \bar{\mathcal{W}}(g_i, \bar{X}_i)}_{S_W} = S_{\text{LS}}. \tag{7.82}$$

The action is Hermitian, as in almost all physical models, and therefore it is self conjugate, as is the Lagrangian $\mathcal{L} = \mathcal{L}^\dagger$. This has direct implications for the path integral, since $Z[g_i] = \int D\phi D\bar{\phi}\, e^{-S[\phi,\bar{\phi},g]}$ is mapped to $\overline{Z[g_i]} = \int D\phi D\bar{\phi}\, e^{-S[\phi,\bar{\phi},\bar{g}]} = Z[\bar{g}_i]$. In terms of observables, this implies $\langle O\rangle_g = \overline{\langle O\rangle}_{\bar{g}}$. Since observables, or expectation values, are themselves subject to Reality R.1, this further implies $\langle O\rangle_g = \langle O\rangle_{\bar{g}_i}$, and therefore

- 8. $$\mathcal{F}(g, \alpha_i) = \mathcal{F}(g, \bar{\alpha}_i) \qquad \text{H.1.} \tag{7.83}$$

Procedure for complex variables

The method to obtain the finiteness constraint is simple. First, we treat all variables as real variables, so that we can expand $\mathcal{F}$ in terms of the γ_i and impose all the real conditions on its structure. We then expand the resulting expression in terms of the original variables q_i, still treating them as real. Second, we allow the original variables to become complex in the following way,

$$\kappa^{2n} \to (\kappa\bar{\kappa})^n, \quad h^{2n} \to (h\bar{h})^n. \tag{7.84}$$

This replacement is always valid, since in the real expansion only even powers of h and κ appear. In fact, this prescription automatically satisfies the Hermitian condition H.1 discussed in the previous section. The most laborious step, though not excessively so, concerns the parameter q and its coefficients. We first decompose q^j into all possible combinations of the form $\sum_{i+k=j} c_{ik} q^i \bar{q}^k$, where each term is, in principle, multiplied by an independent coefficient. We then require analyticity in the complex variables and allow only non singular terms. Recalling that the expression must also be invariant under q-symmetry, the coefficients $c_{i_1 i_2}$ are related, with the precise relations depending on the factor of $(\kappa\bar{\kappa})^n$ multiplying each term. For example,

$$\kappa^6 a_1 q^5 \to (\kappa\bar{\kappa})^3 \big(c_1 q^3 \bar{q}^2 + c_1 q^2 \bar{q}^3\big) \subset \mathcal{F}. \tag{7.85}$$

After imposing this requirement on all terms in the expansion of $\mathcal{F}$, the analysis is complete. As an illustration, we now repeat the same exercise carried out before and verify that the results agree.

This can also be used to check with the calculations in [138], this time not restricted to real parameters. In fact, the exact result obtained there is

$$\begin{aligned} \frac{(-4+N^2)\,(-1+q)\,(-1+\bar{q})}{N^3} \Big(& 5\,(-1+q)^2(-1+\bar{q})^2 + \\ & N^2\left(-9\,q\bar{q} + (1+q+q^2)(1+\bar{q}+\bar{q}^2)\right) \Big)(\kappa\bar{\kappa})^3 g^6. \end{aligned} \tag{7.86}$$

We now follow the prescription described above. First, we write the expression for $\mathcal{F}$ expected in the real case, which is the same as obtained previously and which we reproduce here for the reader's convenience,

$$\left(a_1 + a_2\gamma_2 + a_3\gamma_2^2 + a_4\gamma_2^3\right)\gamma_4^3, \tag{7.87}$$

we then expand the invariant variables in terms of the original ones and impose Reality R.1, analyticity A.1, the Hermitian condition H.1, and q-symmetry, obtaining

$$\begin{aligned} \propto\; a_4 + \frac{a_3}{2}(q+\bar{q}) + (c_1 q^2 + c\, q\bar{q} + c_1\bar{q}^2) + (c_2 q^3 + c_3 q^2\bar{q} + c_3 q\bar{q}^2 + c_2\bar{q}^3) \\ +(c_1 q^3\bar{q} + c\, q^2\bar{q}^2 + c_1 q\bar{q}^3) + \frac{a_3}{2}(q^3\bar{q}^2 + q^2\bar{q}^3) + a_4 q^3\bar{q}^3. \end{aligned} \tag{7.88}$$

We thus see that the number of real parameters increases from (a_1, a_2, a_3, a_4) to $(a_4, a_3, c, c_1, c_2, c_3)$, that is, only two additional constants compared with the real case. It is possible to adjust (7.88) to fit (7.86) exactly, with the solution

$$c_1 = -\frac{a_3}{2}, \quad c_2 = -a_4, \quad c_3 = -c, \tag{7.89}$$

and one can verify it fits well, as expected.

Integrability

The Yang-Baxter equation for the quantum R-matrix was presented in (6.6) [150],

We then borrow the results already derived in [111] for the R-matrix of general deformations of LS-type. We rewrite it here for the reader's convenience,

$$R = \frac{1}{|q|^2+|h|^2+1}\begin{pmatrix} 1+q\bar{q}-h\bar{h} & 0 & 0 & 0 & 0 & -2\bar{h} & 0 & 2h\bar{q} & 0 \\ 0 & 2\bar{q} & 0 & 1-q\bar{q}+h\bar{h} & 0 & 0 & 0 & 0 & 2h\bar{q} \\ 0 & 0 & 2q & 0 & -2h & 0 & q\bar{q}+h\bar{h}-1 & 0 & 0 \\ 0 & q\bar{q}+h\bar{h}-1 & 0 & 2q & 0 & 0 & 0 & 0 & -2h \\ 0 & 0 & 2h\bar{q} & 0 & 1+q\bar{q}-h\bar{h} & 0 & -2\bar{h} & 0 & 0 \\ 2h\bar{q} & 0 & 0 & 0 & 0 & 2\bar{q} & 0 & 1-q\bar{q}+h\bar{h} & 0 \\ 0 & 0 & 1-q\bar{q}+h\bar{h} & 0 & 2h\bar{q} & 0 & 2\bar{q} & 0 & 0 \\ -2h & 0 & 0 & 0 & 0 & q\bar{q}+h\bar{h}-1 & 0 & 2q & 0 \\ 0 & -2\bar{h} & 0 & 2h\bar{q} & 0 & 0 & 0 & 0 & 1+q\bar{q}-h\bar{h} \end{pmatrix}. \tag{7.90}$$

Equation (6.7) represents a quantum R-matrix in a general form. We should note that we are slightly abusing the notation, since the matrix above satisfies the Yang-Baxter (YB) equation only for specific pairs (q, h), and it would therefore be more appropriate to refer to it as the R-matrix only for these pairs. In any case, if $(\tilde{q}, \tilde{h})$ satisfies the YB equation, then the model is quantum integrable. The deformed R-matrix is, in terms of the old one,

$$\tilde{R}\left(\frac{1}{\tilde{q}}, -\frac{\tilde{h}}{\tilde{q}}\right) = R(\tilde{q}, \tilde{h})^{-1}. \tag{7.91}$$

This property, namely that the deformed R-matrix can be expressed in terms of the undeformed R-matrix, also holds in the complex case. It is straightforward to see that R^{-1} also satisfies the YB equation (6.6) by recalling the definition of R_{ij} [51],

$$\begin{aligned} R_{12} &= R \otimes I, & R_{12}^{-1} &= R^{-1} \otimes I, \\ R_{13} &= P_{23}R_{12}P_{23}, \Rightarrow & R_{13}^{-1} &= P_{23}R_{12}^{-1}P_{23},. \\ R_{23} &= I \otimes R & R_{23}^{-1} &= I \otimes R^{-1} \end{aligned} \tag{7.92}$$

Thus, if (6.6) is satisfied by (7.92), then, by taking the inverse of both sides and using $P_{ij}^{-1} = P_{ij}$, we also obtain a solution.

In fact, this can be seen explicitly by making use of the relations in (7.40) and verifying that they are mapped either to each other or to themselves. We illustrate this for the second pair, which is the most natural one to consider under inversion,

$$\left((1+\rho)e^{in}, \rho e^{im}\right) \to \left(\frac{e^{-in}}{1+\rho}, -\frac{\rho}{1+\rho}e^{i(m-n)}\right) \underbrace{=}_{\substack{m-n=M \\ n=-N \\ \frac{1}{1+\rho}=1+x}} \left((1+x)e^{iN}, xe^{iM}\right). \tag{7.93}$$

Heuristic proposals for (real) $\mathcal{F}$

As we saw, the bootstrap+back-of-the-envelope calculations are quite powerful, but still not enough to understand the general behavior $\mathcal{F}$ should have, and one asks if we can't go further. Motivated by those questions we remember that perturbations of the first order contribution to $\mathcal{F}$ have further roots (7.40), but only in the planar limit. Then, in the present Section we follow two heuristic approach for $\mathcal{F}$:

- We first use a simple ansatz for the coefficients of the planar-limit of $\mathcal{F}$, and see if it predicts some of the roots.
- We use an ansatz again, but now also bootstrap the planar-limit of the corrections to F by imposing it to vanish on (7.40).

To approach that problem, we recall the superpotential $\mathcal{W}$ in the form given in [3], (7.68). We saw there are three cubic interaction vertices, one antisymmetric, ϵf_{abc}, and two symmetric ones, $h_{ijk}d_{abc}$ and $\frac{h}{3}d_{abc}$. The terms in the constraint $\mathcal{F}$ that contain κ (that is, all terms, since we have seen that $\sum_i g^i$ must vanish in comparative with $\sum_i \kappa^i$) necessarily originate from interactions involving these different vertices.

Then, we stress the total contribution to higher order can be written, strategically motivated, as

$$\mathcal{F} = \underbrace{g^2 - \kappa^2\left(\frac{1+q^2+h^2}{2}\right)}_{\mathrm{F}_{g^2, N\to\infty}} - \sum_{i=2}^{\infty} \mathcal{A}_{2i} + O(1/N), \tag{7.94}$$

where $\mathcal{A}$ is planar perturbation to $\mathcal{F}$, which we called F as in the previous Sections. $\mathcal{A}$, therefore, starts at order κ^4. Moreover, $\mathcal{A}$ will be expanded in [3] variables, $(1+q)$, $(1-q)$ and h.

Since the condition $q = 1, h = 0$ make the $\mathcal{A}$ vanishes, These requirements imply that, for any even n,

$$\mathcal{A}_n = \kappa^n \sum_j \sum_{2k=0}^{n} \sum_{2i=2}^{n-k} h^k C_{i,j,k}^{(n)} (1+q)^j (1-q)^i \delta_{j,n-i-k}. \tag{7.95}$$

Order by order

The coefficients $C_{i,j,k}$ are not necessarily correlated, but we make the following assumptions. As we are going to study the large N limit, we may expect in this regime the Feynman diagram and loop calculations simplify considerably, because not only are the relevant contributions of planar order, but also the contractions between d and f become essentially the same[4], as one can see for instance in [44], where the authors manipulate different types of contractions ($\mathrm{tr}(F)^2$, $\mathrm{tr}(F^2)$, etc.) up to sixth order, or by following the appendix of [75]. Furthermore, the possible interaction vertices are

$$f_{abc}\epsilon_{ijk}\phi^i\phi^j\phi^k, \quad d_{abc}h_{ijk}\phi^i\phi^j\phi^k. \tag{7.96}$$

As we saw, the first factor corresponds to the $(1+q)f_{abc}$ interactions, while the last factor splits into $(1-q)d_{abc}$ and $\frac{h}{3}d_{abc}$, where the factor of $1/3$ is compensated, when counting the Feynman diagrams, by acting on the three terms in $\mathrm{tr}(\phi_i^3)$. We can therefore expect that the coefficients in $\mathcal{A}_n$ coming from $(1-q^2)$ and $(1+q)h$ are, at least to some extent, correlated apart from symmetric factors, since from the point of view of Feynman diagrams the interactions are the same, this was explored and correctly verified in the first order correction to $\mathcal{F}$.

Since the interactions $(1-q)(1+q)$ and $(1+q)h$ are the same, one can propose an ansatz for c to investigate the consequences in $\mathcal{A}$. Of course, the ansatz must satisfy certain conditions, and moreover, we should expect to obtain at least a general formula valid for any order n. Otherwise, $\mathcal{A}$ would be out of control and we would have no final result.

[4]The contractions involving an odd number of terms are the same up to a factor of the complex unit i. However, as we saw, q-symmetry eliminates such terms, and we are left only with even contractions, which are indeed the same.

Naive ansatz

We can examine how $\mathcal{A}$ behaves after assigning to C different functions of i, j, k. Of course, the function we assume for c cannot give a zero value in the sum for any q, κ. We can simulate generic functions to see how (7.95) behaves for real parameters. To illustrate, for $n = 6$ we have,

$$\begin{aligned}\mathcal{A}_6 = \kappa^6 \left(C^{(6)}_{2,4,0}(1+q)^4(1-q)^2 + C^{(6)}_{4,2,0}(1-q)^4(1+q)^2 + C^{(6)}_{6,0,0}(1-q)^6 \right) \\ + h^4 \left(C^{(6)}_{0,2,4}(1+q)^2 + C^{(6)}_{2,0,4}(1-q)^2 \right) \\ + h^2 \left(C^{(6)}_{0,4,2}(1+q)^4 + C^{(6)}_{2,2,2}(1+q)^2(1-q)^2 + C^{(6)}_{4,0,2}(1-q)^4 \right).\end{aligned} \tag{7.97}$$

As explained, one part of the present ansatz will be to require C to be symmetric in h and $(1-q)$, so

$$C^{(n)}_{k,i,j} = C^{(n)}_{j,i,k}. \tag{7.98}$$

We assume a splitting property, such that $C^{(n)}_{x_1,x_2,x_3} = a(x_2)\, d(x_1, x_3)$, with b symmetric in its indices. The different types of b include exponential functions, power functions, and constants, combined in various ways,

$$d(i,n) = \begin{array}{c} a_0 \left(b_0^i + b_0^{(n-i)} \right), \quad \underbrace{a_0 \left(i^N + (n-i)^N \right)}_{N=3,2,1,0}, \quad a_0 \binom{(n+i)/2}{i/2}, \quad a_0 i(n-i), \\ a_0 \binom{n/2}{i}, \end{array} \tag{7.99}$$

and for $a(x_2)$, we can assume,

$$a(x_2) = \underbrace{c_0 x_2^N}_{N=2,1,0}, \quad c_0 (x_2!)^{\pm 1}, \quad c_0 2^{\pm x_2}. \tag{7.100}$$

It is hard to compare how each of them behaves just by looking at the results themselves, since some of them involve a messy combination of parameters, the simplest case being the constant one. However, we can compute a contour plot for each after assigning numerical values $a_0 = c_0 = 1$ and $b_0 = 2$, and then compare the different plots to observe similarities between them. To investigate this, we simulated values of κ varying in a given range, and values of κ and q over a range with integer steps, summing $\mathcal{F}$ up to the 20th order in κ. We collected the points for which the modulus of $\mathcal{F}$ is less than a specific numerical value, depending on the marks, as we explicitly show in the graphic that follows. In total, we performed 49 simulations, but there are particular plots that stand out among the others. This happens when we have $c(k)$ of linear order or higher, with $d(i,n)$ being the first binomial case in (7.99).

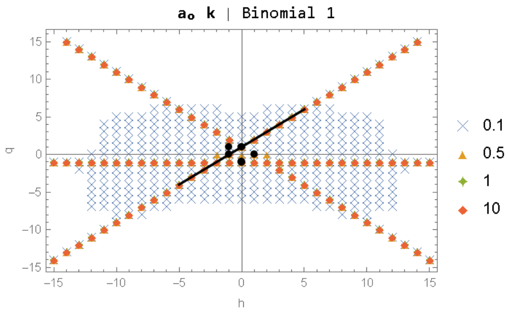

Figure 7.2: The black line represents the pairs in (7.40), while the marks in the graph represent the possible roots. The main marks is X, where the function is nearly 0.

The points and lines in the graphs represent the one-planar finiteness conditions of (7.51): $(q = -1, h = 1)$, $(q = 1, h = 0)$, $(q = -1, h = 0)$, $(q = 0, h = 1)$, $(q = 0, h = -1)$, $q = 1 + h$, together with another line $q = 1 - h$. The fittings show, to some extent, a prediction for these one-planar finiteness roots.

As we explained, from all the plots we produced, those that stand out are the results from a combination of power functions in $c(k)$ with the first combinatorial case in $d(k)$. We should mention, however, that a similar behavior occurs for any power function of $c(k)$, excluding the constant case (power $= 0$), if combined with this specific combinatorial choice. These combinations capture all the discrete roots, along with the relation $q = 1 + h$ for any order in κ, but also suggest that $(q, h) = (-1, h)$ is truly finite for any h, a consequence of the choice for c. Moreover, the pair $(q = h, 1 - h)$ was verified to not be integrable, by making use of the same techniques in [111] later, which resembles the limitation of the ansatz. In any case, the others roots were described before, and the coincidence seem to suggest that the asymptotic behavior of the coefficients should be described by the formula $\mathrm{b}_{i,n}\mathrm{c} \sim \exp(k)\binom{\frac{n+i}{2}}{\frac{i}{2}}$, as this includes all powers while discarding the constant term.

Bootstrap + "more refined" ansatz

Even though the previous approach was suggestive, in that depending on different combinations of functions we could expect an asymptotic behavior for c, the oversimplification was perhaps too trivial. Of course, the amplitudes coming from h and $(1 - q)$ are not completely equal. In that direction, one could try to make a better ansatz by noticing that h comes from a vertex with all three fields equal, ϕ_I^3, and a better fitting happens when, for each loop resulting from h^{2n}, we include a 2^n combinatorial factor compared to $(1-q)^n$, as occurs, for example, in the first-order correction

$$\frac{\kappa^2}{2}\left(c_1(2h^2+(1-q)^2)+\frac{1}{2}(1+q)^2\right), \tag{7.101}$$

which reduces to the usual expression when $c_1 = 1/2$.

This new approach, together with the ansatz, will be essentially different from the previous one. Instead of guessing a function and selecting those that are more appropriate, we make use of the finiteness pairs in (7.51) to further restrict the coefficients by imposing that each of them should vanish when (q, h) takes any of the values in (7.51), at each loop order.

We then start with these pair conditions. By imposing the 2^n factor explained above together with the splitting property, and also with the imposition of (7.40) to be roots of $\mathcal{A}$, and by assuming the same splitting property as before, we obtain an interesting relations between the coefficients d, while the a coefficients are independent of each other.

The d coefficients has to satisfy two relations between them

$$\begin{gathered} d_{2n,n} = -\sum_{i=0}^{n-2} d_{2n-2-2i,2n+2+2i}\alpha_i, \quad d_{2n,2n+2} = -\sum_{i=0}^{n-2} d_{2n-2-2i,2n+2i+4}\beta_i, \\ d_{0,k} = 0 \quad d_{2,2} = d_{2,4} = 0, \end{gathered} \tag{7.102}$$

where α and β are recursive relations,

$$\alpha_i = 2^{1+i} + 2^{-(1+i)}, \quad \beta_i = \frac{1}{6}(2^{3+n} + 2^{-n}). \tag{7.103}$$

These sequences are not new to the literature. Indeed, the first terms of the recurrence are

$$\alpha = \frac{3}{2}, \frac{11}{4}, \frac{43}{8}, \frac{171}{16}, \ldots, \quad \beta = \frac{5}{2}, \frac{17}{4}, \frac{65}{8}, \frac{257}{16}, \ldots. \tag{7.104}$$

The terms in the numerator of α are tabulated in [127], A007583, and are known to be a type of Lucas sequence, while β corresponds to the A092896 sequence. We can make use of a technique known as the binomial transform, which allows us to find an expression for d if we know the inverse sequences of α and β, which are known to exist. To summarize, suppose we have a sequence α_i that is related to β by

$$\alpha_j = \sum_{i=0}^{j} \binom{j}{i} \beta_i, \tag{7.105}$$

then α and β are related by a binomial transform, and we can straightforwardly obtain an expression for β in terms of the α_j,

$$\beta_j = \sum_{i=0}^{j} \binom{j}{i} (-1)^{j-i} \alpha_i. \tag{7.106}$$

The fact is that the sequence α, A007583, has a binomial transform, which is A025192, while the transform of A092896 is A092897. The sequences in (7.104) do not have the initial terms of the cited sequences, and they are also shifted by factors of 2, but we can straightforwardly fix this by imposing $c_0 = -1$ and by noticing that multiplying all $d_{a,b}$ by 2^{-a} (calling this new sequence u) does not modify the relations in (7.102) above, since the overall factor of powers of 2 on both sides will be the same. By doing this, (7.104) reduces to the sequences $3, 11, 43, \ldots$ and $5, 17, 65, \ldots$. The fact that we are left without the first terms of the cited sequences is not a problem, and we can find a solution for u in terms of combinatorial factors. First, we define the rule for the sequence transform of α, namely A025192,

$$k_2(x) = \begin{cases} 1, & x = 0, \\ 2 \cdot 3^{x-1}, & x \neq 0\,, \end{cases} \tag{7.107}$$

and for A092897,

$$k_1(x) = \frac{3^x + 4 \cdot 0^x - (-1)^x + 4x(-1)^x}{4}, \tag{7.108}$$

then, b is given completely by four relations,

$$b(M,N)=\begin{cases} k_1\left(\frac{M}{2}\right)+(-1)^{\frac{M+2}{2}-1}\left(\frac{M}{2}-1\right), & M=N,\\ k_2\left(\frac{M}{2}-1\right)-(-1)^{N/2}, & N=M+2,\\ -(-1)^i\binom{1+n_0}{i}, & 2+2i=M,\ 8+4n_0-2i=N,\\ -(-1)^i\binom{(M+N)/4}{i}, & 2+2i=M,\ 6+4n_0-2i=N. \end{cases} \tag{7.109}$$

For example, $\mathcal{A}$ up to 12 order is

$$\begin{aligned} h^2(1+h-q)(1+2h-q)(-1+q)^2(-1+h+q)(-1+2h+q)\,\kappa^8\bigg((1+q)^2\kappa^2\Big(c_2 \\ +\big(c_2(2h^2+(-1+q)^2)+c_4(1+q)^2\big)\kappa^2\Big) \\ +c_0\Big(-1+(2h^2+(-1+q)^2)\kappa^2+(4h^4+2h^2(-1+q)^2+(-1+q)^4)\kappa^4\Big)\bigg), \end{aligned} \tag{7.110}$$

with the c_i arbitrary.

We see that a first perturbation of *any* pair (q,h) in the *planar limit* is, at minimum, of eight order in κ. This is expected from the literature, where it is known that the g^4, g^6 terms vanish [129, 138, 34].

General

In general, only the condition $q=1+p$, $h=p$ needs to be imposed at each order. This can be justified by noticing that it is a function of the type $q(h)$, which is more complex than just discrete points like $(-1,1)$ or $(1,0)$. For example, suppose we do not impose vanishing at all orders. Then we have three possibilities. The first is that, at a specific order n, the correction is simply zero. The second and third options mimic what happened before: perhaps it is enough to set $q=1+p$ or $h=p$ to make it vanish, but this is nonsensical since p is a parameter, and rewriting q and h in terms of p is just a relabeling of parameters. In the end, we would obtain a term that depends neither on h nor on q, and therefore is a constant, which necessarily must be zero.

So, either the contribution at that order vanishes, or it vanishes in the specific combination $q=1+h$. We adopt the latter approach because, in any case, we can impose the constants, such as a and c before, to vanish. Moreover, if we wanted to impose finiteness up to the third order, we could also solve for the specific cases $\kappa^4(\dots) = \kappa^6(\dots) = 0$. The fact is that, while this can be done, the constraint is much weaker. The first correction, of order κ^8, already allows for six different solutions, with the coefficients not necessarily correlated among themselves, and therefore the approach tends to be too broad.

We could, nevertheless, make use of a more precise analysis using the Wick contraction, which is the correct approach, to seek a relation between amplitudes involving $(1-q)$ and h, but this is not done here and is left for later investigation. Indeed, the previous subsections show that the ansatz method can be iteratively refined, becoming increasingly rigorous, and that, when combined with bootstrap techniques, it could perhaps guide us toward a general form of $\mathcal{F}$, at least in the planar limit.

q-symmetry on states and gauge/gravity

AdS/CFT interpretation

We can also show the invariance under the q-transformation from the supergravity point of view by a back-of-the-envelope calculation. For the specific case considered here, we are going to impose $h = 0$, since this case is more obscure from the gravity side. Frolov [63], and Maldacena and Lunin [106], have shown that deformations such as the β deformation, or more general γ deformations involving three parameters (three β deformations), can be interpreted as TsT transformations of the background. These consist of a T-duality applied to a coordinate, followed by a shift along an isometric coordinate $x_1 \to x_1 + \gamma x_2$, and then another T-duality along the first coordinate. These transformations, consisting of abelian T-duality and translations, map supergravity solutions to other supergravity solutions, and [63] showed that they are dual to the LS deformation we are studying here (again, with $h = 0$ in this specific case). Moreover, TsT transformations have been exhaustively applied to several models, with an astonishing level of success in being dual to general deformations, in some cases integrable [116], [68].

Type IIA and type IIB supergravity models have an action of the form, ignoring all other fields except the metric and working in the string frame,

$$S = \frac{1}{2\kappa^2}\int e^{-2\phi}\sqrt{-g}R + \dots \tag{7.111}$$

Moreover, the Yang-Mills coupling (obtained from the relation $e^{\phi} = g_s = g^2_{\text{YM}}$, up to overall factors) transforms as

$$g^4_{\text{YM}} \underbrace{\rightarrow}_{\text{TsT}} \tilde{g}^4_{\text{YM}} = \frac{g^4_{\text{YM}}}{1 + (k^2 \mathcal{M})} \tag{7.112}$$

where $\mathcal{M}$ is a determinant of the metric involving the components affected by the TsT transformation, its precise form is not important for the present discussion.

To connect this with the LS parameter, we assume $h = 0$ and rewrite the superpotential in a more symmetric form

$$\mathcal{W} = \tilde{\kappa}\text{tr}\left(\phi_1\left(\sqrt{q}\phi_2\phi_3 - \phi_3\phi_2\sqrt{q}^{-1}\right)\right) = \tilde{\kappa}\text{tr}\left(\phi_1\left(\tilde{q}\phi_2\phi_3 - \phi_3\phi_2\frac{1}{\tilde{q}}\right)\right) \tag{7.113}$$

The new parameters $\tilde{q}$ and $\tilde{\kappa}$ are related to κ and q by $\tilde{\kappa} = \kappa\sqrt{q}$ and $\tilde{q} = \sqrt{q}$, and they transform under the q-symmetry as

$$\tilde{\kappa} \underbrace{\rightarrow}_{q} -\tilde{\kappa}, \quad \tilde{q} \underbrace{\rightarrow}_{q} \frac{1}{\tilde{q}} \tag{7.114}$$

(note that $\tilde{\kappa}^2 = \gamma_2$). In this notation, the q-symmetry splits into two independent transformations, with $\tilde{\kappa}$ transforming independently of $\tilde{q}$, and vice versa. This therefore provides a more natural notation for writing the objects of the theory, properly highlighting the q-symmetry (in the same sense as writing irreducible representations of a group).

We now have four parameters, $\tilde{\kappa}$, $\tilde{q}$, k, and $\tilde{g}_s$, with the first two coming from the field-theory side and the last two from the supergravity side. By the AdS/CFT conjecture, we should expect relations among them. The first relation that is immediately apparent is

$$\tilde{\kappa}^2 = \tilde{g}^2_{\text{YM}} = 2\pi\tilde{g}_s \tag{7.115}$$

We are then left with a relation of the form $k(\tilde{q}, \tilde{\kappa})$. There is no precise derivation of this relation, as the duality for the general Leigh-Strassler deformation is not fully understood, but we can make some contact in this direction by using the transformations (7.112) and (7.114). Under (7.114), $\tilde{\kappa}^2$ is invariant, and therefore $\tilde{g}^4_{\text{YM}}$ in (7.112) remains unchanged. Nevertheless, the parameter k in the context of the TsT transformation is independent of the coupling[5], and therefore one may

[5]More precisely, k is an independent parameter that affects the coupling. If $k = k(g_s)$, then the transformed coupling $\tilde{\phi} = \phi_0 + \frac{1}{2}\log(\dots)$ would become an equation for $\phi(g_s)$, which would

expect $k = k(\tilde{q})$. Following this logic, under the q-symmetry (7.114), k should transform as $k(\tilde{q}) \to k(\tilde{q}^{-1})$, but constrained in such a way that $\tilde{g}_{\rm YM}$ in (7.112) remains invariant. Consequently, we require k^2 to be invariant, leading to two possible transformations,

$$k(\tilde{q}) \to k(\tilde{q}^{-1}) = -k(\tilde{q}), \quad k(\tilde{q}) \to k(\tilde{q}^{-1}) = k(\tilde{q}) \tag{7.116}$$

We expect the function $k(\tilde{q})$ to have only a simple root, since the undeformed case $k = 0$ corresponds to $\mathcal{N} = 4$, that is, $\tilde{q} = 1$, as implied by the above relations. The antisymmetric case under inversion is harder to constrain, while the second case is simpler, although a more general function is not known and would require a series expansion, as in the previous case, and the simplest function that satisfies the required properties is

$$k \propto \ln \tilde{q} \tag{7.117}$$

in which case we choose the first transformation in (7.116). Of course, from this solution one could obtain infinitely many functions with higher transcendentality (such as $\ln^3 \tilde{q}$, etc.), but we choose the simplest closed-form solution rather than a series expansion.

We note that, for the real β deformation, $\tilde{q} = \exp\left(\frac{i\pi}{2}\beta\right)$, and (7.117) implies a linear relation between the parameters β and k,

$$\beta \propto k \tag{7.118}$$

Remarkably, this relation is the same as the one expected from the literature [63], [106], where the proportionality becomes an equality, $\beta = k$.

7.3 From AdS_5 to AdS_3: TsT deformations, Magnetic fields and Holographic RG Flows.

Related work by the author

Available on arXiv: https://arxiv.org/abs/2512.24267 [155]

contradict this assumption.

Introduction

Since the proposal of the AdS/CFT correspondence by Maldacena [107], a lot of work has been done in trying to generalize this gauge/gravity duality to a variety of other models, each of them with particular properties. Yet, we hope that, at the end of the day, these models can somehow be useful to approximate results from our Standard Model. In that direction, people have proposed new models and obtained results, generally from the gravity side of the correspondence, to compare with results known to hold, either numerically or analytically, in the Standard Model (SM), or more specifically in Quantum Chromodynamics (QCD). Even if QCD is a theory with fewer symmetries than general SYM models, being neither supersymmetric nor conformal (though QCD is approximately conformal in the UV, where quark masses m are negligible), nor integrable (although in some limits, and by studying amplitudes and form factors, the model shows deep connections with integrability, see [31], [55], [105]), we can still compare certain results with those obtained from gauge/gravity duality. Indeed, while it is an obvious fact that our Standard Model is not $\mathcal{N} = 4$ SYM, calculations done in AdS/CFT settings can still produce values close to real results. A famous example is the ratio η/s of shear viscosity to entropy density, whose value obtained from holographic calculations in AdS is numerically close to that extracted for the QGP, see [100].

But a lot of things still need to be implemented, and certainly the supergravity background obtained from N D3 branes alone [107] is not enough to describe all properties we expect, in principle, from a QCD-like gauge theory, namely chiral symmetry breaking, confinement, asymptotic freedom, nontrivial phase transitions at $T \neq 0$ and $\mu \neq 0$, fundamental and antifundamental matter fields, and so on. Each of the models that follow the AdS/CFT correspondence in the direction of reproducing QCD is, in some sense, a deformation of the original model, since the field theory we want to obtain is four dimensional in the end, so we certainly expect a D3 brane configuration, although with more ingredients added to it. Even so, different configurations can provide us with insight into the final gravity background.

For example, [67] and [66] proposed a way to break conformal symmetry on the gauge side and restore it in the ultraviolet, while at the same time obtaining a confinement-like behavior, [154], [67]. It is easy to see the relation between these effects by noting that the Wilson loop is of the area-law type, $W \sim \exp[A]$, which necessarily signals confinement and, consequently, the absence of conformal symmetry. This can be achieved by introducing, from the gravity point of view, a hard wall in the model or by allowing a dilaton-like flow, see [39]. Regarding mass spectra, several models have made considerable advances in showing how to add flavors, [88], by studying probe branes with $N_c \gg N_f$ in a supergravity background, or how to obtain the glueball mass spectrum, beyond the fact that conformal symmetry is also broken through the introduction of a cutoff, either soft or hard [89], [42]. As another example, if we want chiral symmetry breaking, it is necessarily true that supersymmetry must be broken, and the simplest way to do that while generating a quark condensate is to introduce a constant magnetic field into an already established background with probe branes, for example a D7 brane as a probe in the D3 brane geometry. Finally, models simulating chiral symmetry breaking by a condensate have also been proposed in the literature, by introducing black holes and probes [142], [143], or a constant magnetic field [58], or even an RG-flow-sensitive dilaton that breaks supersymmetry and conformal symmetry, while presenting a confining-like behavior and, by introducing probe branes, chiral symmetry breaking [39]. We also mention that these problems have been studied from the bottom-up point of view by constructing supergravity or pure gravity models that display the properties we want in order to mimic QCD. This approach goes by the name of AdS/QCD, see [52].

In that work we are interested in deforming the model of [58], which is a simple setup consisting of a D3 brane background with D7 brane probes, inspired by [88]. The model includes a constant Kalb-Ramond field B, which, as we are going to explain later, can be equivalently interpreted as a constant magnetic field B, and, being in the probe approximation, the model breaks supersymmetry. The magnetic field is responsible for both chiral symmetry breaking and a Zeeman-like effect in the spectrum. We are interested in knowing whether we can weaken the constancy of B, and if so, what the effects of a non constant magnetic field are on the phase structure and on the meson spectrum. To approach this problem, we deform the model by applying TsT transformations in the directions parallel to the constant magnetic field on the supergravity side.

We briefly mention that a magnetic field can be consistently incorporated on the field-theory side without appealing to the probe approximation or interpreting it as a Kalb-Ramond field. This can be achieved by considering a dyonic black hole in the bulk. The AdS_4 construction, dual to a $d = 3$ condensed-matter system, was presented in [77], and this approach was later extended to AdS_5 in [48].

Review of constant B background

This section follows essentially the second section of [58], so familiar reader with the procedure can skip it.

The near-horizon of the D3-brane background can be shown [107] to consist of a $AdS_5 \times S^5$ metric, with a self-dual F_5 field strength, with a flux over the sphere S^5 proportional to N_c, and a constant dilaton ϕ,

$$ds^2 = \frac{u^2}{L^2}(-dx_0^2 + \sum_{i=1}^{3} dx_i^2) + \frac{l^2}{u^2}(du^2 + u^2 d\Omega_5^2), \; F_5 = dC_4 + \ldots, \; \phi = \ln g_s, \qquad L^4 = 4\pi g_s N \alpha'^2, \tag{7.119}$$

where ... are terms imposing the $F_5 = \star F_5$ condition, L is the radius of the space and ϕ is the dilaton. Throughout this Section we will, however, adopt $L = 1$ unless otherwise stated. Also, C_4 has the form

$$C_4 = u^4 dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3. \tag{7.120}$$

We now embed the D7-brane in the background of the D3-branes. To do that, first we rewrite the radial+sphere part of the metric

$$ds^2 = \left(\rho^2 + l^2\right)(-dx_0^2 + \sum_{i=1}^{3} dx_i^2) + \frac{1}{\rho^2 + l^2}(d\rho^2 + \rho^2 d\Omega_3^2 + dl^2 + l^2 d\Phi^2), \tag{7.121}$$

assuming $x_0, x_1, x_2, x_3, \rho, d\Omega_3$ to be part of the world-volume of the D7-brane, the embedding is $l = l(\rho)$, $\Phi =$ constant. Therefore, the "pull-back" of the metric is given by

$$ds^2 = \left(\rho^2 + l^2\right)(-dx_0^2 + \sum_{i=1}^{3} dx_i^2) + \frac{1}{\rho^2 + l^2}(\left(1 + l'^2\right) d\rho^2 + \rho^2 d\Omega_3^2). \tag{7.122}$$

Moreover, as described in [58], we have to include a constant magnetic field B with components along x_2, x_3,

$$B = \mathrm{H} dx_2 \wedge dx_3. \tag{7.123}$$

We can add such a magnetic field without worrying about whether it is a supergravity solution, because since H is constant, B is a pure gauge field and therefore the field strength vanishes, $dB = 0$. Since only the field strength appears in the type IIB supergravity equations of motion, this does not affect the solution. We can now substitute the fields above into the DBI action, also assuming $2\pi = 1$. To understand why we can purposely confuse the B field with F, it is clear if we pay attention to the NSNS part of the action, given by

$$S_{NS} = -\mu \int_{M_8} d^8\xi \sqrt{-\det(G + B + \alpha' F)}, \tag{7.124}$$

as one can see, $B + F$ appears as an antisymmetric combination, and this shows to be true even in the Wess-Zumino term (more on this later). Consequently, if we only know $A_{23} = B_{23} + F_{23}$, we can say $F_{23} = A_{23}$ is equivalently described by $B_{23} = A_{23}$, with the last being the interpretation of Clifford et. al [58], where if A is constant, then a constant magnetic field $B_1 = \epsilon_{123} F_{23}$ is equivalent to a constant Kalb-Ramond field B_{23}. Then, we can expand to first order in α' by rewriting the above as

$$\sqrt{-\det(G + B + \alpha' F)} = \sqrt{-\det(E + \alpha' F)} = \sqrt{-\det E}\sqrt{\det(1 + \alpha' E^{-1} F)} \tag{7.125}$$

where $E = G + B$. Making use of the mathematical relation $\det e^A = e^{\operatorname{tr} \ln A}$, we approximate $\sqrt{\det(1 + \alpha' c_1)} \approx 1 + \frac{1}{2}\alpha' \operatorname{tr} c_1$,

$$S_{NS} = -\mu \int d^8\xi \sqrt{-\det E} - \frac{\mu\alpha'}{2} \int \sqrt{-\det E} \operatorname{tr}(E^{-1} F). \tag{7.126}$$

Moreover, the presence of F, B and C_4 requires Wess-Zumino (WZ) terms in the action [120]. These terms take the form (ignoring the pullback symbol)

$$S_{WZ} = \sum_i \int C_i \wedge e^{B + 2\pi\alpha' F}, \tag{7.127}$$

where C denotes the formal sum of all R–R potentials, and the Wess–Zumino term contains only the combinations that remain linear in C after expanding the exponential. The antisymmetric fields relevant for calculation are

$$C_4, \quad \tilde{C}_6, \quad B_2, \quad F_2, \tag{7.128}$$

here $\tilde{C}_6$ is an induced charge on D7-brane due to D3, necessary to show consistency (see [58] for details). In their embedding one has

$$B_2 \wedge B_2 = 0, \qquad B_2 \wedge C_4 = 0, \tag{7.129}$$

so the Wess–Zumino term is simplified. Keeping terms up to first order in α' we obtain

$$S_{\mathrm{WZ}} = \mu_7 \int \tilde{C}_6 \wedge B_2 + \alpha' \mu_7 \int \tilde{C}_6 \wedge F_2, \tag{7.130}$$

and as we are going to show later (similar case in [8]), $\tilde{C}_4$, the magnetic dual of C_4, is of order α', and therefore is not present in the action. So, to first order in α' is

$$\begin{aligned} S = -\mu \int d^8\xi \sqrt{-\det E} + \mu \int C_6 \wedge B - \frac{\mu}{4} \int d^{10}x \sqrt{-G} |dC_6|^2 \\ -\frac{\mu\alpha'}{2} \int \sqrt{-\det E}\, \mathrm{tr}(E^{-1}F) + \mu\alpha' \int C_6 \wedge F_2, \end{aligned} \tag{7.131}$$

with the third term being the kinetic term for C_6 in the (super)gravity model. As shown in detail in [58], the equation of motion for the world-volume gauge field A (with $F = dA$), obtained from the α' terms in the action, induces a constraint on the C_6 field. Consequently, the solution of the equation of motion for C_6 must satisfy this constraint. The resulting solution is shown to take the form $C_6 = f(\rho, L_0, \psi)$, where ψ is an angle of the S_5 sphere.

Once consistency is established for the R–R equation of motion, we can focus on the leading part of the action coming from the NSNS sector, which is, roughly speaking, proportional to $\sqrt{\det E}$. The asymptotic solution for $l(\rho)$, obtained from its own equation of motion, takes the form

$$l(\rho) \sim m + \frac{c}{\rho^2}. \tag{7.132}$$

As is standard in the holographic literature [53], [9], [107], the non–normalizable part of a bulk field near the boundary corresponds to the source of the dual operator, while the normalizable part corresponds to its vacuum expectation value. In our case, m plays the role of the quark mass and c is identified with the chiral condensate,

$$m \sim M, \; c \sim \langle \bar{\psi}\psi \rangle. \tag{7.133}$$

[58] showed the relation between m and c, and how one can have $m = 0$ while $c \neq 0$, which represents a chiral symmetry breaking by non-perturbative effects (and not by explicitly breaking it by adding a small mass parameter) [102],

$$\langle \bar{\psi}\psi \rangle = -\tilde{c}H^2 = \frac{-H^2}{4m}, \;\; \tilde{c} = \frac{1}{4m}. \tag{7.134}$$

It is possible to obtain the meson spectrum of the model by studying the fluctuations of the fields in the supergravity model. These fields are the vector potential A, that represents the gauge field inside the world-volume of the brane, and the scalars l, Φ, representing the perpendicular coordinates to the D7-brane. Moreover, scalar meson spectra are obtained by fluctuations of the scalar fields and fluctuations of the vector perpendicular to the D3-brane. However, just like was done in [58], we will study the mixing of scalar field fluctuations with the vector part of the vector fluctuation (along the D3-brane).

In [58] the spectrum was obtained for the weak field, $l(p) \sim m + O(H^2)$, from the equation of motion of $A_{0,1,2,3}$ and Φ ($\Phi = 0 + \Phi(x_0, x_1, \rho)$). They obtained a quantization of the mass to leading order

$$\tilde{M}_0 = 2\sqrt{(n+1)(n+2)}, \tag{7.135}$$

and a splitting by the magnetic field, just like the Zeeman effect,

$$M_\pm = M_0 \pm \frac{H}{m}, \tag{7.136}$$

The spectrum was also obtained for strong magnetic field, but as we are going to show later, this is the same for the TsT deformed.

Deformed D3 background

Applying the rules of Appendix B, we can obtain the TsT-transformed NSNS and RR fields.

NSNS fields

We decided to perform a TsT over x_2, x_3, the coordinates of the "original" B field. For details, see Appendix B, we obtain that

$$\begin{aligned} G_{22,33} &\to \frac{l^2+\rho^2}{(1+Hk)^2+k^2(l^2+\rho^2)^2}, \; B_{23} \to \frac{H+kH^2+k(l^2+\rho^2)^2}{(1+Hk)^2+k^2(l^2+\rho^2)^2}, \\ \phi &\to \phi_0 + \frac{1}{2}\log\left[\frac{1}{(1+Hk)^2+k^2(l^2+\rho^2)^2}\right]. \end{aligned} \tag{7.137}$$

RR fields

The previous section focused primarily on evaluating the effects of the TsT deformation on the NSNS sector of the model. However, we still need to consider the RR part of the action. In fact, this sector can introduce constraints, as discussed earlier. Although these terms turn out to be irrelevant for obtaining chiral symmetry breaking, they are important for determining the meson spectrum.

As we know, the D3 brane admits a self-dual F_5 form [107], which, together with the B field, gives rise to a non trivial transformation of the RR fields under TsT and can even generate new fields. Following [84] and [117] for the transformation rules, we obtain

$$\mathcal{F}_3 = k f^5_{[x_2 x_3]}, \; F_5 = f_5 + k[f_5 \wedge b]_{x_2 x_3} - k[f_5]_{[x_2 x_3]} \wedge B, \tag{7.138}$$

where F_5, F_3, and B are the new, or deformed, fields generated by the transformation, with B as obtained previously. The notation $[A]_{[x_i x_j]}$ simply indicates a contraction over the coordinates x_i and x_j. For a more detailed explanation of (7.138), see [84].

Before proceeding, it is useful to present the explicit form of the RR fields after the TsT transformation, as they will be needed when studying fluctuations of the action up to order α'^2. The explicit form of the RR fields C can be obtained through an analysis similar to the one described above, and we again refer to [84] for a more detailed discussion. Using the formulas provided there, one finds, since $C_2 = k(c_4)_{2,3}$,

$$C_4 = c_4 - k\,(c_4)_{[x_2 x_3]} \wedge B, \tag{7.139}$$

where B is the transformed Kalb Ramond field. Thus,

$$\begin{gathered} C_4 = \left((l^2+\rho^2)^2 - k(l^2+\rho^2)^2 B_{23}\right) dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3, \\ \tilde{C}_4 = \frac{(1+Hk)\, l^2\, (l^2+2\rho^2)}{2(l^2+\rho^2)^2} \sin(2\psi) d\psi \wedge d\beta \wedge d\alpha \wedge d\Phi, \quad C_2 = k(l^2+\rho^2)^2 dx_0 \wedge dx_1. \end{gathered} \tag{7.140}$$

To obtain $\tilde{C}_4$, with $\star \tilde{f}_5 = d\tilde{C}_4$, one must manipulate the algebra carefully, since the formulas in [84] do not directly apply to the magnetic duals. The procedure is as follows. Given the C_2 form, one has $F_5 = dC_4 + \mathrm{H} \wedge C_2 + \dots$, where the ellipsis indicates the imposition of the self duality condition on F_5.

Finally, one can verify from (7.140) that, in the limit $k \to 0$, the correct values of C_4 and $\tilde{C}_4$ are recovered, see [58]. It is worth emphasizing how remarkable this result is in simplifying calculations. Terms involving C_4 do not affect the equations of motion, while terms involving $\tilde{C}_4$ do. Even more strikingly, the result for $\tilde{C}_4$ is simply the undeformed value $\tilde{c}_4$ multiplied by a very simple factor $(1 + \mathrm{H}k)$, that is,

$$\tilde{C}_4 = \tilde{c}_4(1 + \mathrm{H}k). \tag{7.141}$$

To conclude, we compute F_5 from (7.140), since if the construction is correct it should at least be self dual. One can verify that this condition is satisfied,

$$F_5 = f_5 + k[f_5 \wedge b]_{x_2 x_3} - k[f_5]_{[x_2 x_3]} \wedge B = dC_4 + \mathrm{H} \wedge C_2 + d\tilde{C}_4 =$$
$$2(1+Hk)\left(\frac{2u^3\, du \wedge dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3}{(1+Hk)^2 + k^2u^4} - \cos\theta\, \sin^3\theta\, \sin(2\psi)\, d\alpha \wedge d\beta \wedge d\theta \wedge d\Phi \wedge d\psi\right), \tag{7.142}$$

and it is straightforward to check that $\mathcal{F}_5 = \star\mathcal{F}_5$ using the volume form. Moreover, F_5 reduces to the D3 brane flux f_5 in the limit $k \to 0$.

Chiral Symmetry Breaking

To study the effects of deforming the model on chiral symmetry breaking, we follow the standard procedure from the gravity side. We solve for the embedding of the probe brane in the background and then examine its asymptotic behavior, as done in [58] and reviewed above. We work at first order in α', so we only consider the $\sqrt{\det E}$ part of the action, together with the dilaton factor $e^{-\phi}$. As a consequence of the embedding, we assume $l = l(\rho)$ when solving the equations of motion. More explicitly, the action is

$$S = -\mu \int d^8\xi\, e^{-\phi(\rho)} \sqrt{-\det E}, \tag{7.143}$$

where G, B, and ϕ were obtained in (7.137). The matrix E, apart from the usual terms present in the undeformed case, is given by

$$E = \begin{pmatrix} \ddots & 0 & 0 & 0 \\ 0 & \frac{(l^2+\rho^2)}{(1+Hk)^2+k^2(l^2+\rho^2)^2} & \frac{\mathrm{H}+kH^2+k(l^2+\rho^2)^2}{(1+Hk)^2+k^2(l^2+\rho^2)^2} & 0 \\ 0 & -\frac{\mathrm{H}+kH^2+k(l^2+\rho^2)^2}{(1+Hk)^2+k^2(l^2+\rho^2)^2} & \frac{(l^2+\rho^2)}{(1+Hk)^2+k^2(l^2+\rho^2)^2} & 0 \\ 0 & 0 & 0 & \ddots \end{pmatrix}. \tag{7.144}$$

Rather than writing the equation of motion explicitly, it is more instructive to analyze the quantities $\sqrt{-\det E}$ and $e^{-\phi}$ separately. They are given by

$$\begin{aligned} \sqrt{-\det E} &= \frac{\rho^3\sqrt{H^2 + (\rho^2 + l(\rho)^2)^2}\sqrt{1 + (l'(\rho))^2}}{(\rho^2 + l(\rho)^2)\sqrt{(1+\mathrm{H}k)^2 + k^2(\rho^2 + l(\rho)^2)^2}}, \\ e^{-\phi} &= \sqrt{(1+\mathrm{H}k)^2 + k^2(\rho^2 + l(\rho)^2)^2}, \end{aligned} \tag{7.145}$$

and, as one can explicitly verify, the factor involving k in the determinant exactly cancels against the factor coming from the dilaton. In other words, the deformed action, or equivalently the deformed Lagrangian $\mathcal{L}_{\text{def}}$, is equal to the undeformed one $\mathcal{L}$ to first order in α'. This implies, in particular, that the undeformed and deformed models share the same equation of motion.

This result is remarkable, because the quantities that determine chiral symmetry breaking, namely the coefficients m and c, are fixed by the asymptotic behavior of the solution $l(\rho)$ to the equation of motion. Since the equations of motion are identical, the solutions and their asymptotic expansions coincide as well. Therefore, the results are the same as those obtained in (7.132), with the relation between m and c given in (7.134). Although this outcome may appear surprising at first, it admits a natural interpretation from the field theory point of view, which we postpone for later Sections.

Meson spectrum

To obtain the mesonic spectrum, we need to study the fluctuation solutions of the equations of motion. This requires allowing terms of order α'^2 in the action. However, the calculation is rather lengthy, so for the interested reader we refer to Appendix C, while the main algebraic steps are presented in Appendix D.

Weak magnetic field

We can now study the meson spectrum. In order to analyze the effect of the TsT deformation, we compare our results with those previously obtained for the undeformed case. As in [58], we focus on fluctuations of Φ to extract the spectrum. As follows from (D.37) and (D.43), the corresponding equations must first be decoupled before the spectrum can be analyzed.

A natural strategy would be the following. To study fluctuations perpendicular to the magnetic field plane, $x_{0,1}$, one could first expand the equations to first order in H and k, and then extract the leading correction to the mass spectrum. To access the regime of stronger fields, one would instead need to analyze the spectrum arising from fluctuations along the parallel plane, $x_{2,3}$, since the differential equation governing fluctuations in the $x_{0,1}$ directions becomes highly nontrivial when higher orders are retained.

Before proceeding we emphasize that fluctuations along the $x_{0,1}$ directions are in fact problematic. This is because a term appearing in the corresponding equation of motion diverges in the deformed case as the radial coordinate $u^2 = \rho^2 + L^2 \to 0$, whereas in the undeformed background this divergence is absent. As a consequence, the spectrum associated with these fluctuations is ill-defined and cannot be reliably obtained. This behavior can be understood by noting that deforming the magnetic field through a TsT transformation, for a generic value of the parameter k, effectively introduces a contribution to the background that is independent of the original constant magnetic field. Indeed, from previous results one sees that it is possible to take the limit $H \to 0$ while keeping $B \neq 0$. Intuitively, this can be viewed as adding a background with a non constant B field on top of the original constant B field background. Such non constant magnetic backgrounds are, in general, not suitable for defining a well behaved meson spectrum.[6]

To make this issue explicit, we adopt the weak magnetic field approximation, retaining only terms linear in k and H, while neglecting terms of order H^2, k^2, kH, and higher.

- $x_{0,1}$

From (D.43), we can manipulate the equation and rewrite it in terms of F_{01}, obtaining

$$\frac{1}{\sqrt{\det E}} e^{\phi_0} \partial_\rho \left(\frac{\sqrt{\det E}}{G_1 G_3} e^{-\phi_0} \partial_\rho F_{01} \right) + \frac{1}{G_1 G_4} \Delta_{\Omega_3} F_{01} + \frac{1}{G_1^2} \Delta_{0,1} F_{01} + \frac{G_2}{G_1 (G_2^2 + B^2)} \Delta_{2,3} F_{01} - \frac{e^{\phi_0}}{\sqrt{\det E}} \partial_\rho (K(\rho) B_{23}) (\partial_0^2 - \partial_1^2) \Phi = 0, \tag{7.146}$$

This is only one of the equations we can obtain, since the other equation vanishes as a consequence of the $A_{p,\sigma,\psi,\ldots}$ equation (D.38). For the scalar Φ, we find

$$\frac{e^{\phi_0}}{\sqrt{\det E}\tilde{C}^{(\partial\phi)^2}} \partial_\rho \left(e^{-\phi_0} \tilde{C}^{(\partial\phi)^2} \frac{\sqrt{\det E}}{G_3} \partial_\rho \Phi \right) + G_1^{-1}(\rho) \Delta_{0,1} \Phi + \frac{G_2}{G_2^2 + B^2} \Delta_{2,3} \Phi + G_4^{-1}(\rho) \Delta_\Omega \Phi - \frac{e^{\phi_0}}{\sqrt{\det E}\tilde{C}^{(\partial\phi)^2}} \partial_\rho (K(\rho) B_{23}) F_{01} = 0. \tag{7.147}$$

We now have two parameters to work with, k and H, where k is the deformation parameter and H is the original magnetic field of the undeformed background. Our goal is to understand how the deformation parameter k modifies the equations.

[6] Of course, the dynamics are nonlinear, so this should not be interpreted as a literal superposition, but the analogy is nevertheless useful.

We therefore retain only terms linear in k and H in the Kalb–Ramond field,

$$B = \mathrm{H} + k(\rho^2 + L_0^2)^2. \tag{7.148}$$

Moreover, to first order we have $K(\rho) = (1 + Hk)\tilde{K}(\rho) \approx \tilde{K}(\rho)$, with

$$\tilde{K}(\rho) = -R^4 L_0^2 \frac{2\rho^2 + L_0^2}{(\rho^2 + L_0^2)^2}. \tag{7.149}$$

The contribution from e^{ϕ_0} enters at second order, of order $O(kH)$, and can therefore be neglected at this stage. We also assume $L_0 = m + O(H^2)$, which leads to

$$\begin{gathered} G_1 = \rho^2 + m^2, \quad G_2 = \rho^2 + m^2, \quad G_3 = \frac{1}{m^2 + \rho^2}, \quad G_4 = \frac{\rho^2}{m^2 + \rho^2}, \\ C = \frac{m^2}{m^2 + \rho^2}, \quad \sqrt{\det E} = \rho^3. \end{gathered} \tag{7.150}$$

With these simplifications, the vector equation reduces to

$$\begin{aligned} \frac{1}{\rho^3}\partial_\rho(\rho^3 \partial_\rho F_{01}) + \frac{1}{\rho^2}\Delta_{\Omega_3} F_{01} + \frac{1}{(\rho^2 + m^2)^2}\Delta_{0,1} F_{01} + \frac{1}{(\rho^2 + m^2)^2}\Delta_{2,3} F_{01} \\ - \frac{1}{\rho^3}\partial_\rho(K(\rho) B_{23})(\partial_0^2 - \partial_1^2)\Phi = 0, \end{aligned} \tag{7.151}$$

while the scalar equation becomes

$$\begin{aligned} \frac{1}{\rho^3}\partial_\rho(\rho^3 \partial_\rho \Phi) + \frac{1}{(\rho^2 + m^2)^2}\Delta_{0,1}\Phi + \frac{1}{(\rho^2 + m^2)^2}\Delta_{2,3}\Phi \\ + \frac{1}{\rho^2}\Delta_\Omega \Phi - \frac{1}{m^2 \rho^3}\partial_\rho(K(\rho) B_{23}) F_{01} = 0. \end{aligned} \tag{7.152}$$

We can now decouple the system by introducing the combinations

$$\phi_\pm = F_{01} \pm m\mathcal{P}\Phi, \tag{7.153}$$

where $\mathcal{P} = (-\partial_0^2 + \partial_1^2)^{1/2}$. This leads to

$$\left(\frac{1}{\rho^3}\partial_\rho(\rho^3 \partial_\rho) + \frac{1}{(\rho^2 + m^2)^2}\Delta_{0,1} + \frac{1}{(\rho^2 + m^2)^2}\Delta_{2,3} + \frac{1}{\rho^2}\Delta_\Omega \mp \frac{1}{\rho^3}\partial_\rho(K(\rho) B_{23})\mathcal{P} \right)\phi^\pm = 0. \tag{7.154}$$

Denoting by $O_1^{k=0}$ the differential operator in (7.154) evaluated at $k = 0$, the equation can be written as

$$\left(O_1^{k=0} \pm \frac{4km}{\rho^2}\mathcal{P}\right)\phi^{\pm} = 0. \tag{7.155}$$

To solve this equation we follow exactly the same ansatz used in [58]. Remarkably, the same ansatz remains valid in the present linearized case,

$$\phi_{\pm} = \eta_{\pm}(\rho)e^{-iq_0x_0+iq_1x_1}. \tag{7.156}$$

The equation then reduces to

$$\frac{1}{\rho^3}\partial_\rho(\rho^3\partial_\rho\eta_\pm) + \frac{1}{(\rho^2+m^2)^2}M_\pm^2\eta_\pm \mp \frac{4\mathrm{H}m}{(\rho^2+m^2)^3}M_\pm\eta_\pm \pm \frac{4km}{\rho^2}M_\pm\eta_\pm = 0. \tag{7.157}$$

By splitting $\eta_\pm = \eta_0 \pm \mathrm{H}\eta_1 \pm k\eta_2$ and $M_\pm = M_0 \pm \mathrm{H}M_1 \pm kM_2$, and then adding and subtracting the two differential equations obtained previously, we arrive at three independent differential equations, corresponding to the orders 1, H, and k,

$$\begin{aligned}
\frac{1}{\rho^3}\partial_\rho(\rho^3\partial_\rho\eta_0) + \frac{1}{(\rho^2+m^2)^2}M_0^2\eta_0 &= 0,\\
\frac{H}{\rho^3}\partial_\rho(\rho^3\partial_\rho\eta_1) + \frac{1}{(\rho^2+m^2)^2}\left(M_0^2\mathrm{H}\eta_1 + 2\mathrm{H}M_0M_1\eta_0\right) - \frac{4\mathrm{H}m}{(\rho^2+m^2)^3}M_0\eta_0 &= 0,\\
\frac{k}{\rho^3}\partial_\rho(\rho^3\partial_\rho\eta_2) + \frac{1}{(\rho^2+m^2)^2}\left(M_0^2k\eta_2 + 2kM_0M_2\eta_0\right) + \frac{4km}{\rho^2}M_0\eta_0 &= 0.
\end{aligned} \tag{7.158}$$

We solve these equations in the order presented. The first equation is standard, and its solution is given in terms of a Gauss hypergeometric function, exactly as expected from [58]. Imposing normalizability at infinity leads to a quantization condition on the parameter M_0,

$$\begin{aligned}
\eta_0 = m^{-1+\frac{\sqrt{m^2+M_0^2}}{m}}\left(m^2+\rho^2\right)^{\frac{m-\sqrt{m^2+M_0^2}}{2m}} &\times\\
{}_2F_1\left(\frac{m-\sqrt{m^2+M_0^2}}{2m}, \frac{3}{2}-\frac{\sqrt{m^2+M_0^2}}{2m}, 1-\frac{\sqrt{m^2+M_0^2}}{m}, -1-\frac{\rho^2}{m^2}\right)&,\\
\tilde{M}_0 = 2\sqrt{(n+1)(n+2)}&.
\end{aligned} \tag{7.159}$$

To solve the second equation in (7.158), we substitute the solution for η_0, specializing in the simplest case $n = 0$. Requiring regularity at $\rho = 0$ yields a perturbative solution for M_1, which is well behaved,

$$M_1 = \frac{1}{m}. \tag{7.160}$$

As a result, the mass spectrum becomes $M_\pm = M_0 \pm \frac{H}{m} + \ldots$, reproducing the Zeeman splitting discussed in [58].

The difficulty arises in the third equation of (7.158). Substituting the values of M_0 and M_1, again for $n = 0$, the full solution is highly nontrivial. Instead, we focus directly on the behavior of the equation near $\rho = 0$. In this limit, the asymptotic solution of the differential equation takes the form

$$\eta_2 \propto -\frac{2\sqrt{2}\,m^{7/2}\left(m^{3/2} + \sqrt{2m + \sqrt{2}M_2}\,\pi\rho Y_1\left(\frac{2\sqrt{2m+\sqrt{2}M_2}\rho}{m^{3/2}}\right)\right)}{(2m + \sqrt{2}M_2)\rho^2}, \tag{7.161}$$

which is ill defined and diverges as $\rho \to 0$. Even if one substitutes $M_2 = -\frac{2m}{\sqrt{2}}$ directly into the differential equation, a divergent contribution remains and cannot be eliminated. This unavoidable divergence originates from the $\sim k/\rho^3$ term in the equation of motion, which persists at any order in H and k.

As a consequence, the meson spectrum associated with fluctuations along the $x_{0,1}$ directions is not well defined, since no finite and normalizable solution exists. As anticipated at the beginning of this section, we therefore turn to the analysis of fluctuations along the plane parallel to the magnetic field, where the effects of the TsT deformation turn out to be trivial.

Strong magnetic field

To evaluate the effects of the deformation in the strong magnetic field regime, we follow the approach of [58] and study fluctuations not in the x_0, x_1 plane, but instead along the x_2, x_3 directions. This case might be expected to be more interesting, since it is precisely this plane that undergoes the TsT deformation, and one might therefore anticipate nontrivial effects. However, as will become clear from the following analysis, we find a rather surprising result: the TsT deformation has no effect on the meson spectrum associated with fluctuations in the plane parallel to the deformation, even in the strong magnetic field regime. In fact, this statement holds for arbitrary values of H and k, since, as we will show, no expansion is required.

- $x_{2,3}$

Since the TsT deformation along $dx_2^2 + dx_3^2$ preserves the isometries associated with these coordinates, we can Fourier expand the fluctuation Φ into a radial function, a plane wave, and a spherical harmonic on the S^3, following [58],

$$\Phi(\rho) = h(\rho)e^{iqx}Y_l(S_3). \tag{7.162}$$

Substituting this ansatz into (7.147), we obtain

$$\begin{aligned}\frac{e^{\phi_0}}{\sqrt{\det E}C^{(\partial\phi)^2}}\partial_\rho\left(\tilde{C}^{(\partial\phi)^2}\frac{\sqrt{\det E}}{G_3}\partial_\rho h\right) + \frac{G_2}{G_2^2+B^2}M^2h \\ +G_4^{-1}l(l+1)h - \frac{e^{\phi_0}}{\sqrt{\det E}}\partial_\rho(K(\rho)B_{23})F_{01} = 0,\end{aligned} \tag{7.163}$$

while from (7.146) we obtain

$$\begin{aligned}\frac{1}{\sqrt{\det E}}e^{\phi_0}\partial_\rho\left(\frac{\sqrt{\det E}}{G_1G_3}e^{-\phi_0}\partial_\rho F_{01}\right) + \frac{1}{G_1G_4}\Delta_{\Omega_3}F_{01} + \frac{1}{G_1^2}\Delta_{0,1}F_{01} \\ + \frac{G_2}{G_1(G_2^2+B^2)}\Delta_{2,3}F_{01} = 0.\end{aligned} \tag{7.164}$$

We immediately see that $F_{01} = 0$ is a solution of this sourceless equation. As a result, the remaining equation reduces to a single differential equation for a scalar function depending only on the radial coordinate,

$$\frac{e^{\phi_0}}{\sqrt{\det E}C^{(\partial\phi)^2}}\partial_\rho\left(e^{-\phi_0}C^{(\partial\phi)^2}\frac{\sqrt{\det E}}{G_3}\partial_\rho h\right) + \frac{G_2}{G_2^2+B^2}M^2h + G_4^{-1}l(l+2)h = 0. \tag{7.165}$$

The terms G_i are

$$\begin{aligned}G_1 = \rho^2 + l^2, \quad G_2 = \frac{\rho^2+l^2}{(1+\mathrm{H}k)^2 + k^2(\rho^2+l^2)^2}, \quad G_3 = \frac{1+l'^2}{\rho^2+l^2}, \\ G_4 = \frac{\rho^2}{\rho^2+l^2}, \quad C^{(\partial\phi)^2} = \frac{l^2}{l^2+\rho^2}.\end{aligned} \tag{7.166}$$

After substituting these expressions into (7.165), similarly as in the chiral symmetry case, $e^{-\phi}\sqrt{\det E}$ cancels the k contribution. Moreover, upon algebraic manipulation, each term in the equation becomes independent of the deformation parameter k. Consequently, the final differential equation is exactly identical to the one obtained in [58]. This cancellation arises because the TsT deformation enters the metric, the dilaton, and the open string data through the same algebraic combination. As a result, the solution is not affected by the deformation, and the corresponding meson spectrum remains unchanged, and the mass quantization is the one obtained in [58],

$$M = 2m\sqrt{(n+l+1)(n+l+2)} \tag{7.167}$$

Field-theory interpretation

We summarize here the main results of the effect of TsT deforming the model, which has two major implications for the spectrum of the mesonic fluctuations,

- The fluctuations over $x_{0,1}$ are sensitive to the deformation, and the TsT completely removes these massive operators.
- The fluctuations over $x_{2,3}$ generate the same spectrum at any order, and are insensitive to the TsT deformation.

As discussed in the subsection above on chiral symmetry breaking, the TsT effect near the horizon of AdS$_5$ is negligible, in the sense that it reduces to the original model as we take the limit $l^2+\rho^2 \to 0$, and moreover after a shift $\phi \to \phi + \mathrm{H}k$, which is always allowed. Let's make this more explicit. In the $\rho^2+l^2 \to 0$ limit, the NSNS fields of the background reduces to, after assuming k as a perturbation and H small (which is not necessary, but it is easier to understand the dual field behavior),

$$\begin{aligned}
ds &= \frac{(l^2+\rho^2)}{(1+Hk)^2+k^2(l^2+\rho^2)^2}(dx_2^2+dx_3^2) \longrightarrow (l^2+\rho^2)(dx_2^2+dx_3^2)+\ldots, \\
B &= \frac{\left(H+kH^2+k(l^2+\rho^2)^2\right)dx_2\wedge dx_3}{(1+Hk)^2+k^2(l^2+\rho^2)^2} \longrightarrow (H+k(l^2+\rho^2)^2)dx_2\wedge dx_3+\ldots, \\
\phi &= \phi_0+\frac{1}{2}\log\left[\frac{1}{(1+Hk)^2+k^2(l^2+\rho^2)^2}\right] \longrightarrow \phi_0-\frac{1}{2}k^2(l^2+\rho^2)^2+\ldots.
\end{aligned} \tag{7.168}$$

For the UV limit $l^2+\rho^2 \to \infty$ the situation is less immediate. The NSNS fields behave as Near the boundary, the same approximation yields

$$\begin{aligned}
ds &\longrightarrow \frac{1}{k^2(l^2+\rho^2)}(dx_2^2+dx_3^2)+\dots,\\
B &= \frac{\left(H+kH^2+k(l^2+\rho^2)^2\right)dx_2\wedge dx_3}{(1+Hk)^2+k^2(l^2+\rho^2)^2} \quad\longrightarrow\quad \left(\frac{1}{\mathrm{k}}-\frac{1}{k^3}\frac{1}{u^4}\right)dx_2\wedge dx_3+\dots,\\
\phi &= \phi_0+\frac{1}{2}\log\left[\frac{1}{(1+Hk)^2+k^2(l^2+\rho^2)^2}\right] \quad\longrightarrow\quad \phi_0+\frac{1}{2}\log\left[\frac{1}{k^2(l^2+\rho^2)^2}\right]+\dots,
\end{aligned} \tag{7.169}$$

where $u^2 = l^2+\rho^2$.

The fact that the near-horizon region of the deformed background coincides with the undeformed background explains why chiral symmetry breaking in the dual field theory is preserved and independent of the deformation parameter k. This is consistent with the expectations from field theory calculations, since chiral symmetry breaking is an infrared phenomenon.

When moving away from the horizon, the small H and small k regime shows that the first nontrivial perturbation appears only in the B field, which ceases to be constant. Consequently, its interpretation as a magnetic field becomes less clear, and one may reconsider the regime in which B can be consistently interpreted as a constant magnetic field. As shown above, both near the boundary and near the horizon the B field approaches a constant value. Therefore, in these asymptotic limits the model is expected to describe a constant magnetic field rather than a dynamical Kalb-Ramond field.

Since the metric is already written in Fefferman-Graham coordinates [56], we can immediately see that the boundary metric is degenerate. Indeed, the metric takes the form (with $v^{-2} = l^2+\rho^2$)

$$\mathrm{ds}^2 = \frac{1}{z^2}\left(dz^2+g_{ij}(x,z)\,dx^i dx^j\right), \tag{7.170}$$

where, to zeroth order in z, we have $g_{22}=g_{33}=0$. Therefore, the boundary metric g^0 is degenerate,

$$g^0_{ij}(x) = \mathrm{diag}(1,1,0,0). \tag{7.171}$$

Furthermore, the metric written above resembles a non-isotropic Lifshitz-like geometry [87], though strictly speaking it is not a Lifshitz, since the time x_0 scales as the coordinates coordinate 1, which are the asymptotically the coordinates of the boundary field theory. This can be seen from the invariance of the metric under the anisotropic scaling

$$v \to \lambda v, \quad x_{0,1} \to \lambda x_{0,1}, \quad x_{2,3} \to \lambda^{-1} x_{2,3}. \tag{7.172}$$

In this case, the dynamical critical exponent is $z = 1$[7] but the spatial directions $x_{2,3}$ scale differently from $x_{0,1}$. This behavior reveals a hierarchy of symmetry breaking in the model: Lorentz invariance is preserved only in the (x_0, x_1) subspace, while scale invariance is broken in an anisotropic way. This chain of broken symmetries is summarized schematically in Fig.7.3.

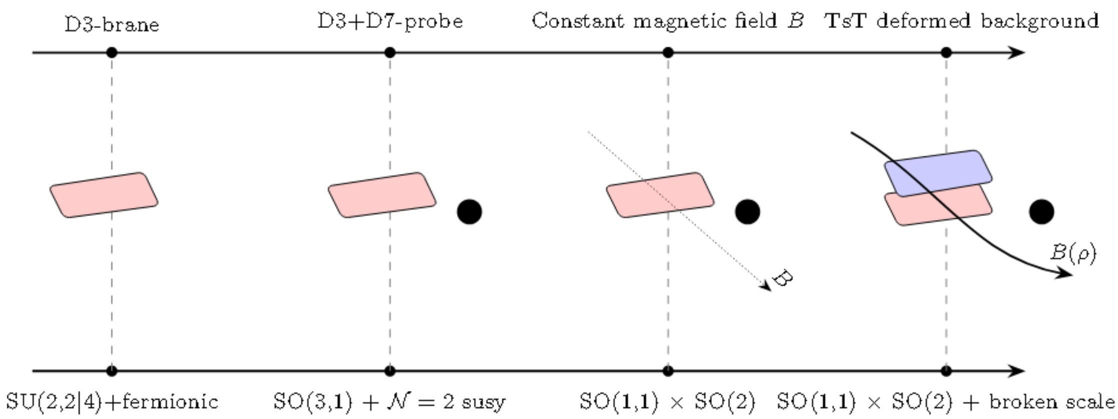

Figure 7.3: Sequence of deformations of the original model and the corresponding loss of symmetries.

The breaking of the $SO(4)$ symmetry of the D3-brane ($SO(3,1)$ after Wick rotation) down to $SO(2) \times SO(2)$ could, a priori, be attributed to the undeformed model, with B_{23} being responsible for the breaking of the isometry, as illustrated in Fig.7.3 and as explained in [58]. Here, however, this isometry breaking acts together with a breaking of scale invariance. As a consequence, while Lorentz symmetry is preserved on the boundary spanned by $x_{0,1}$, with the usual scaling $u, x_{0,1} \to \lambda u, \lambda x_{0,1}$, the fluctuations of the metric along $x_{2,3}$ is invariant under the anisotropic scaling (7.172). The paper [37] has made progress in studying generalizations of this anisotropic behavior as solutions of Einstein-Proca theory, that is, gravity coupled to a massive spin-1 vector field, although the present case does not fit directly into the class of solutions studied there.

The B field breaks the global spacetime symmetry $SO(3,1)$ rather than an

[7] z is the parameter that determines how the time coordinate scales relative to the spatial coordinates in Lifshitz geometries, where $z \neq 1$.

internal symmetry. Combined with the anisotropic structure of the background, this makes it difficult to construct an effective UV action that reproduces all expected operators and symmetries. Nevertheless, one may still infer certain general features that such terms must possess from the point of view of the remaining global symmetries. Moreover, the difficulty is not only due to this explicit breaking of spacetime symmetries: the presence of a non-pure gauge and ρ-dependent B field suggests, as conjectured in [147] and [106], that the supergravity background may be dual to a non-commutative field theory, as previously explained in the first work of this Section, 7.2.

Indeed, [147] showed that configurations involving a constant Kalb-Ramond field B, and hence open strings ending on D-branes, can be interpreted in terms of a non-commutative field theory, where operators are multiplied using a deformed product. See also [106] and [63] for generalizations to non-constant B fields and TsT-deformed supergravity backgrounds. In this framework, the usual commutative product of operators is promoted to the non-commutative $\star$-product [38],

$$A(x) \cdot B(x) \;\to\; A(x) \star B(x) \;\sim\; A(x) \cdot B(x) + \frac{i}{2}\,\theta^{ij}\,\partial_i A\,\partial_j B, \tag{7.173}$$

and the relevant quantities on the non-commutative gauge theory side are the non-commutativity parameter θ, the open-string metric G, and the open-string coupling g_o. These are related to the supergravity background through [147], [7], [6]

$$G_{ij} = -(2\pi\alpha')^2 (Bg^{-1}B)^{ij}, \quad \theta^{ij} = (B^{-1})^{ij}, \quad g_o = g_s \left(\frac{\det G}{\det g}\right)^{1/4}. \tag{7.174}$$

The relation between TsT deformations (or, more generally, deformations of $\mathcal{N} = 4$ SYM) and non-commutative field theories has been known for a long time; see, for example, [22] and [43]. Moreover, the fact that the boundary metric is degenerate, and therefore exhibits a nontrivial relation to dipole non-commutative field theories, has also been discussed in a similar context in [21]. This provides further support for the interpretation presented here, although a more detailed analysis is left for future work.

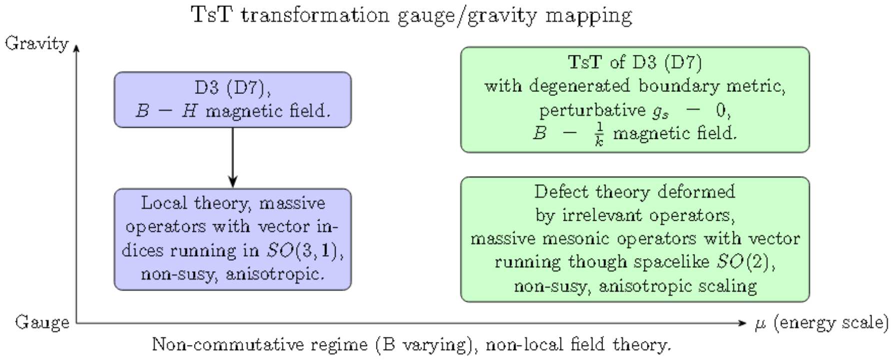

Figure 7.4: The TsT deforming of the D3-D7 + H constant [58] model in different regime of energies.

We stress one important fact: As discussed, near the horizon the supergravity background reduces to the undeformed case, and this automatically translates to the IR of the field theory being intact by the deformation. Moreover, what is perhaps obvious from the just given information, though is going to be useful later, is that the near-horizon region of this model is not degenerate. Then, the deformed model shares an irrelevant behavior from the RG flow point of view, it is non-local and present defects terms on the Lagrangian since the boundary goes like $\sim \left(\frac{1}{u^2}d\vec{x}_{2,3} + u^2 d\vec{x}_{0,1}\right)$, and therefore the preserved *isometric*[8] subgroup of the conformal symmetry $\mathrm{SO}(4,2)$ is $\mathrm{SO}(2,2)$.

Moreover, the coupling is constant asymptotically in both limits, though with different constant values, and $C_{2,4}$ are only known from their constant charge, which translate in the dimension of the gauge group N_1, N_3.

N_3 is trivially obtained by integrating $\star F_5$, which in the present background gives

$$|N_3| = (1 + Hk)\Omega 5, \tag{7.175}$$

the N_1, nevertheless, is less trivial. The equation of motion in type IIB supergravity give us the correct conserve Flux to be integrate,

$$d\left(F_7 + B \wedge F_2\right) = 0, \tag{7.176}$$

which can be integrated over the specific cycles to obtain Q_{N_1},

$$|N_1| = 4HA_{23}\Omega_5. \tag{7.177}$$

[8]A symmetry of the metric only, since $\phi(u)$ breaks the scale invariance of $x_{0,1,u}$ at large u.

The conserved combination of flux involving H_3 is given by $d(e^{-2\phi}H_3 + C_2 \wedge F_5) = 0$. Remarkably, in our background this specific combination of form fields vanishes identically. As a consequence, the associated flux is zero, and the presence of $H = dB$ in the dual field theory manifests itself purely through the non locality of operator interactions.

Therefore, the present supergravity model is dual to a field theory with anisotropic scaling, which coincides with $\mathcal{N} = 4$ in the IR. The theory is deformed both by the explicit insertion of defect operators and by deforming existing terms in $\mathcal{L}_{\mathcal{N}=4}$ through their promotion to non local interactions, and it flows asymptotically to a weakly coupled, $g_s \to 0$, AdS$_3 \times S^3$ regime. From the energy flow point of view, the gauge group is always $SU(N_1) \times SU(N_3)$, and the internal symmetry $SU(4)$ is preserved in the flow, but supersymmetry is broken and the spacetime symmetry of the model is reduced from $SO(4,2)$ to the isometry group $SO(2,2)$. Even though the $SO(2,2)$ group isometry is suggesting a possible conformal symmetry, the dilaton $\phi(u)$ breaks the scale invariance of the near boundary AdS$_3$, that would otherwise be present in the IR.

Finally, we emphasize that the IR limit of the field theory is precisely that of the $d = 4$, $\mathcal{N} = 4$ super Yang Mills theory; this flow appears to be driven by defect operators which are irrelevant.

D1 + magnetic field background

Throughout the whole analysis of the last Section, we have neglected the fact that the theory points to a special value for the parameter k, and therefore we address this issue in the present Section. Indeed, (7.140), (7.137), and even the explicit form of $K(\rho)$ make it clear that there exists a specific value for the deformation parameter as a function of the magnetic field magnitude,

$$\mathrm{k} = -\frac{1}{\mathrm{H}}. \tag{7.178}$$

Equation (7.178) is far from a generic or meaningless result. In fact, one immediately notices that, if (7.178) holds, the function $K(\rho)$ vanishes identically. As a consequence, the ill-defined term that appears in the equations of motion for fluctuations perpendicular to the magnetic field disappears. This makes it possible to study these fluctuations consistently again, and potentially to obtain a well-defined meson spectrum in this sector. Moreover, equation (7.178) has several additional implications, but for the moment we focus only on the two main physical aspects, namely chiral symmetry breaking and the hadronic spectrum.

Chiral symmetry breaking

The chiral symmetry breaking calculated before can be straightforwardly repeated here, since we can simply substitute (7.178) into the calculations performed there, and the implications are the same as before: the model exhibits chiral symmetry breaking. This time, however, one can directly relate it to the TsT deformation parameter,

$$\langle \bar{\psi}\psi \rangle \sim H^2 = \frac{1}{k^2}. \tag{7.179}$$

Then, there is a double interpretation in the present model, since we may interpret the magnetic field as $H = -\frac{1}{\mathrm{k}}$, or instead the TsT parameter as $\mathrm{k} = -\frac{1}{\mathrm{H}}$. Both points of view are valid, although, following the logic of the calculation, the latter interpretation is more appropriate.

The transformation of the NSNS fields, for the special value (7.178), is

$$G_{22,33} \to \frac{H^2}{(l^2+\rho^2)}, \quad B_{23} \to -H, \quad \phi \to \phi_0 + \log\left[\frac{H}{(l^2+\rho^2)}\right], \tag{7.180}$$

and therefore $\det E$ and e^{ϕ} becomes

$$\sqrt{-\det E} = \mathrm{H}\frac{p^3\sqrt{(H^2+(\rho^2+l(p)^2)^2)}\sqrt{1+(l'(p))^2}}{(p^2+l(p)^2)^2}, \quad e^{-\phi} = \frac{e^{-\phi_0}}{H}(l^2+\rho^2). \tag{7.181}$$

However, one should worry about the divergence of $g_s = e^{\phi}$ in the near horizon scale, as quantum corrections are non-suppressed. Therefore, a classical approach can not be the appropriate regime to be used near the horizon. While this is true, we postpone the discussion for later Section, and for the moment we just assume this solution is valid even under quantum corrections, in order to study its features.

Meson spectrum

The meson spectrum resulting from fluctuations over $x_{2,3}$ is the same as in the case of Strong magnetic field, since the overall effects of the TsT deformation cancel among themselves, and therefore we obtain the same result as before.

More interestingly, the fluctuations over the perpendicular plane deserve closer attention. The differential equation for the scalar mesons is the same as in the previous Section, but we rewrite it here for the reader's convenience,

$$\begin{aligned} &\frac{e^{\phi_0}}{\sqrt{\det E}\tilde{C}^{(\partial\phi)^2}}\partial_\rho\left(e^{-\phi_0}\tilde{C}^{(\partial\phi)^2}\frac{\sqrt{\det E}}{G_3}\partial_\rho\Phi\right) \\ +G_1^{-1}(\rho)\Delta_{0,1}\Phi - &\frac{e^{\phi_0}}{\sqrt{\det E}\tilde{C}^{(\partial\phi)^2}}\partial_\rho(K(\rho)B_{23})F_{01} = 0. \end{aligned} \tag{7.182}$$

We recall that

$$K(\rho) = (1 + Hk)\,\tilde{K}(\rho), \tag{7.183}$$

and therefore (7.178) implies $K(\rho) = 0$.

We are going to assume the weak magnetic field limit, which in the present case, one should interpret as a large k TsT parameter.

The meson spectrum associated with these fluctuations is straightforward to understand by looking at the differential equation. The linear term in H appearing in (7.182) vanishes, and therefore the leading correction to the mass spectrum arises only at order H^2. As discussed in the previous Sections, however, this order is technically difficult to analyze, since the embedding function $l(\rho)$ is no longer constant. At this stage, the only robust conclusion we can draw is that, for $\mathrm{k} = -\frac{1}{\mathrm{H}}$, the effects of the deformation on the mass spectrum arising from fluctuations along the $x_{0,1}$ directions are increased to order H^2.

Then, the simply conclusion from the discussion is that, for the mass spectrum over x_{01},

$$M_{01} = 2\sqrt{(n+1)(n+2)} + O(H^2). \tag{7.184}$$

Field theory

To understand the field theory dual to the present supergravity background it is useful to go back to the configuration involving only the D3-brane together with a constant magnetic field. In this case, using the usual coordinates $u^2 = l^2 + \rho^2$, the deformed background takes the form

$$\begin{aligned} \mathrm{ds}^2 = u^2(-dx_0^2 + dx_1^2) + \frac{1}{u^2}(\frac{1}{k^2}\left(dx_2^2 + dx_3^2\right) + du^2 + u^2 d\Omega_5^2), \quad \phi = \phi_0 + \log\frac{H}{u^2}, \\ B = -H, \quad C_{(2)} = -\frac{u^4}{H}, \end{aligned} \tag{7.185}$$

where both C_4 and its magnetic dual vanish, as can be seen directly from the deformed expressions in (7.57), and consequently F_5 also vanishes. This represents a surprising simplification of the configuration. The vanishing of C_4 and F_5 means that there is no D3-brane charge present. Therefore, the specific value of the TsT parameter k in (7.178) is special precisely because it removes the original D3-brane content of the background.

One can understand this perspective by evaluating the Wess-Zumino part of the brane action, as we did in the previous sections for the general k value. Indeed, since [58] one can "properly confuses itself" to attribute the value of $\mathcal{F}$ either to F_{ab} or to B_{ab}, we can assume $F_{ab} = 0$ (not being too strict about the equation of motion for F, as we did before) as B is constant, and then the WZ terms are

$$\int_{D_3} C_4 + C_2 \wedge B + \frac{1}{2} C_0 B \wedge B \subset S_{\mathrm{WZ}}. \tag{7.186}$$

The last term vanishes, since $B^2 = 0$ for the present B field. Then, we see the magnetic field induces a C_2 charge in D_3, which can be effectively described as a D_1 brane when $B = -k^{-1}$.

It is also straightforward to see that the resulting supergravity background does not preserve supersymmetry. Indeed, the dilatino variation reads

$$\delta\lambda = \frac{1}{2}\left(\gamma^u \partial_u \phi + \frac{e^\phi}{2}\gamma^{u01} F_{u01}\sigma_1\right)\epsilon \sim \frac{1}{2}\left(\gamma^u u^{-1} - \gamma^{u01} u \sigma_1\right)\epsilon, \tag{7.187}$$

which does not vanish for any nontrivial Killing spinor ϵ, since u is spacelike and the two terms don't have the same power of u, not allowing for a cancellation of the coefficients. Consequently, this special TsT deformation breaks all the supersymmetry of the original D3-brane background.

The fact that the configuration now contains only a B_2 field and a C_2 field suggests that the spacetime is sourced by a nontrivial bound state involving D1, F1. To identify the relevant charges more precisely, we can compute the fluxes of the RR fields. The three-form field strength is

$$F_3 = -\frac{4u^3}{H}\, dx^0 \wedge dx^1 \wedge du, \tag{7.188}$$

with Hodge dual

$$F_7 = - \star F_3 = 4H \, \sin^4\psi \, \sin^3\beta \, \sin\theta \, \cos\theta \, dx^2 \wedge dx^3 \wedge d\psi \wedge d\beta \wedge d\alpha \wedge d\phi \wedge d\theta. \tag{7.189}$$

The flux of F_7, namely $\int \star F_7$, is independent of the radial coordinate and is therefore not useful for charge quantization in this case. On the other hand, F_3 leads to a simple, radially independent flux $\int \star F_3$, which can be straightforwardly quantized. We stress, before calculating the flux, that the existence of different definitions for brane charges, some quantized but not gauge-invariant while others the other way around [112], is not something we need to worry in the present background, as $H_3 = 0, F_5 = 0$ and the charges turns out to be the same, $Q_{\text{Max}} = Q_{\text{Page}}$[9].

The D1-brane charge is

$$Q_{D_1} = -\int \star F_3 = 4\,\mathrm{H}\,A_{23}\,\Omega_5, \tag{7.190}$$

where A_{23} is the area of $x_{2,3}$ plane with flux through it, and Ω_5 denotes the volume of the five-sphere. We see, moreover, the quantity of N_1 is the same as for a generic value of k.

In the end, we are left with a D1-brane background together with a constant Kalb-Ramond field, which can again be interpreted as a magnetic field. In this sense, the construction closes consistently, recovering a configuration with a magnetic field but with a completely different origin.

Finally, it is worth emphasizing a subtle but important point about the deformed background. In general, a deformation necessarily preserves some information about the original background, since it is constructed from it. At first sight, it may seem contradictory that the final configuration no longer resembles the original D3-brane geometry. The resolution of this apparent paradox becomes clear once one reuses dimensionful coordinates in the background, where information of the original D3-brane scale reappears implicitly through the deformation parameters.

We recall that we set $L_3 = 1$, but restoring dimensions the background can be written as

$$\begin{aligned} \mathrm{d}s^2 &= \frac{u^2}{L_1^2}(-dx_0^2 + dx_1^2) + \frac{L_1^2}{k^2u^2}(dx_2^2 + dx_3^2) + \frac{L_1^2}{u^2}du^2 + L_1^2 d\Omega_5^2, \\ \phi &= \phi_0 + \log\frac{HL_1^2}{u^2}, \quad B = \frac{1}{\mathrm{k}}, \quad C_{(2)} = k\frac{u^4}{L_1^4}, \end{aligned} \tag{7.191}$$

where L_1 is the radius of the deformed background, with dimensions $[L_1] = -1$.

[9]We thank prof Carlos Núñez for pointing that.

We can determine L_1 by relating it to the number of D1-branes, obtaining

$$L_1^4 = \left|\frac{N_1}{4H}\right|\frac{2\kappa_{10}^2}{A_{23}\Omega_5} = \left|\frac{N_1}{4H}\right|\frac{(2\pi)^6 g_s^{(0)}\alpha'^3}{A_{23}\Omega_5}, \tag{7.192}$$

where we are ignoring the volume factor resulting from integrating S^5, etc. We also recall that the S^5 part of the metric is undeformed, and by consistency we must have

$$L_1 = L_3, \tag{7.193}$$

and we know that,

$$L_3^4 = 4\pi g_s^{(0)}\alpha'^2 |N_3|^{(\mathrm{k}=0)}, \tag{7.194}$$

where $\mathrm{k}=0$ is to remind the reader that the flux of F_5 was computed before the TsT deformation together with the condition $\mathrm{k} = -\frac{1}{\mathrm{H}}$, and should not be confused with an actual flux in the present background. Combining (7.192), (7.193), and (7.194), we obtain

$$|N_1| = \left|\frac{16\pi g_s^{(0)} H A_{23}\Omega_5 \alpha'^2 N_3^{(k=0)}}{(2\pi)^6 g_s^{(0)}\alpha'^3}\right| = \frac{16\pi^4 H A_{23} N_3}{64\pi^6\alpha'} = \frac{1}{4\pi^2\alpha'} H A_{23} |N_3^{(k=0)}| \Rightarrow$$
$$\frac{N_1}{N_3} = \frac{A_{23}}{4\pi^2}\frac{H}{\alpha'}. \tag{7.195}$$

Therefore, the information from the initial background is not completely lost. Equation (7.195) is interesting by itself, since it shows that the number of D1-branes is proportional to the number of D3-branes in the original background. One notice an interesting point, that we are going to discuss more about later: if we compactified A_{23} over a torus with radius $R_1 = R_2 = \sqrt{\alpha'}$, then (7.195) reduces to a simple relation without any numerical factor, $\frac{N_1}{N_3} = H$.

It is interesting to study the different limits of the supergravity background (7.185), but with caution, since limits in general do not commute.

Near boundary

Near the boundary, or equivalently in the UV of the field theory, as $u \to \infty$, and writing $g_s = e^{\phi}$, the background behaves as

$$\mathrm{d}s^2 \sim u^2(-dx_0^2 + dx_1^2) + \frac{du^2}{u^2} + d\Omega_5^2, \quad g_s \to 0, \quad B = \frac{1}{\mathrm{k}}, \quad C_2 \to \infty. \tag{7.196}$$

The metric is effectively eight-dimensional because we have discarded the degenerate directions at the boundary. More precisely, the behavior is similar to what was found in the previous Section, where the boundary metric is degenerate. In the present case, however, since B is constant, the dual field theory remains local, and there is no need to worry about non-locality associated with non-commutative effects. The degeneracy of the boundary indicates instead that we are deforming the $\mathcal{N} = 4$ theory by introducing defect operators. The boundary geometry is $\mathrm{AdS}_3 \times S^5$.

Since the $SO(6)$ symmetry is preserved, while the dilaton exhibits a nontrivial flow, one can point out similarities between the present background and Janus-like configurations [10], as well as with the Constable–Myers background [120].

The fact that the theory is asymptotically AdS_3, together with the vanishing of the coupling, suggests the following UV interpretation:

$$\boxed{\begin{array}{r} d = 2 \text{ effective field theory, non-supersymmetric,} \\ \text{with } SO(2,2) \text{ global symmetry, gauge group } SU(N_1), \text{ and internal symmetry } SU(4). \end{array}} \tag{7.197}$$

Near horizon

To study the IR behavior, we move toward the horizon $u \to 0$, where the background becomes

$$\mathrm{ds}^2 = \frac{1}{u^2}\Big(\frac{1}{k^2}\left(dx_2^2 + dx_3^2\right) + du^2 + u^2 d\Omega_5^2\Big), \quad g_s \to \infty, \quad B = \frac{1}{\mathrm{k}}, \quad C_2 \to 0. \tag{7.198}$$

After rescaling the coordinates $x_{2,3}$, the metric can be brought to a more familiar form. The Kalb-Ramond field is rescaled accordingly but remains constant. In this limit, the coupling diverges as we explained in the beginning of the section.

Near the horizon, the geometry approaches $\mathrm{EAdS}_3 \times S^5$, where EAdS denotes Euclidean AdS, or hyperbolic space. Consequently, the IR interpretation of the field theory is

$$\boxed{\begin{array}{r} d = 2 \text{ effective field theory, non-supersymmetric,} \\ \text{with global } SO(3,1), \text{ gauge group } SU(N_1), \text{ and internal symmetry } SU(4). \end{array}} \tag{7.199}$$

It is interesting how the gauge group and the internal symmetry is preserved in the energy flow.

What is perhaps more surprising is that, as we explained before, the TsT deformation is in most cases dual to an irrelevant operator. This can be easily seen by taking the limit $k \to 0$ in the deformed supergravity background without imposing $\mathrm{k} = -\frac{1}{\mathrm{H}}$, which results in the undeformed $\mathrm{AdS}_5 \times S^5$, as we have shown. However, if one proceeds in the opposite order and chooses the special value before taking any limit, the deformed contribution to the $x_{2,3}$ part of the metric,

$$ds^{\mathrm{TsT}\,2}_{23} = \frac{1}{k^2}\frac{1}{u^2}(dx_2^2 + dx_3^2), \tag{7.200}$$

is non-perturbative, that is, we can't write $ds^2_{TsT} \approx \mathrm{ds}^2 + k^2\delta\mathrm{ds}^2$, which is generally possible by taking the $k \to 0$ or $u \to 0$ limit. Therefore, the field theory is perturbed by operators whose quantum effects show to be non-perturbative in the IR and in the k parameter. This is why the IR of the field is effectively $x_{0,1}$, since the contributions form the deformation blows up at that energy. At the same time, at UV the theory is effectively $x_{-2,3}$, and not CFT_4. Consequently, it is hard to describe the effective field theory completely in terms of deformed operators in the $\mathcal{N} = 4$, but rather it is better to understand it as really a non-isotropic flow.

We emphasize that supersymmetry is not recovered in any asymptotic limit. We recall that the original model was $\mathrm{AdS}_5 \times S^5 + B$, and therefore the deformation acts as a defect in the dual field theory. This happens because the original model lives effectively in $d = 10$, while the deformed model becomes degenerate and effectively $d = 8$ in certain limits. We stress that even though the two opposite limits can be described by a $d = 8$ model, the region interpolating between those limits is still genuinely $d = 10$. Moreover, the effective $d = 8$ description involves two distinct sets of non-spherical coordinates, one timelike, (u, x_0, x_1), and the other spacelike, (u, x_2, x_3), but only the first case is truly understandable classically.

Analytical continuation perspective

Finally, we notice a dual interpretation for the model if we Analytic continuated $x_{2,3}$ to complex values, or alternatively Wick-rotate $x_{0,1}$, both limits of the background appear to be $\mathrm{AdS}_3 \times S^5$. Then it is easier to understand how the field theory behaves as energy flows. Although B and C_2 are not the same in these cases, and therefore the backgrounds are different, we observe that B can always be gauged away and that C_2 is known, from the field-theory point of view, only through its color factor. As a result, we can have two possible symmetry interpretations at the asymptotic limits, which do not interpolate as in the previous case,

$$\begin{aligned} SU(N_1) &\text{ gauge group with isometry } \mathrm{AdS}_3 \times S^5, \\ SU(iN_1) &\text{ gauge group with isometry } \mathrm{EAdS}_3 \times S^5, \end{aligned} \tag{7.201}$$

since C_2 has components over $x_{0,1}$, and the gauge group comes from integrating $\star dC_2$. We can obtain this as a direct consequence of the first case in (7.201), which is obtained by analytically continuing $x_2 \to ix_2$, while the last is for the continuation $x_0 \to ix_0$. The first case is better to understand the gauge field, while the second is easier to understand from the supergravity side, as we are going to show.

The appearance of EAdS and continuous groups with ill defined rank, such as a negative or complex value, are concepts present in the domain-wall/cosmology correspondence [115], [114], where amplitudes from a pseudo-QFT (the reason for pseudo is obvious, once you realize the complex rank for the group) in a 3d field theory is obtained by analytically continuation an Euclidean field theory in a Domain-Wal. However, we should mention the analytical continuation in the works cited involves also the radial coordinate, where it is analytically continued to $r \to it$, such that the RG flow can be understood as a time evolution. Then, to be more precise, at IR we would have after changing $du/u = -dz_1$,

$$\mathrm{d}s^2 = L_1^2 \left(dz_1^2 + \frac{e^{2z_1}}{k^2}(dx_2^2 + dx_3^2) \right) \tag{7.202}$$

while at UV, changing instead $du/u = dz_2$,

$$\mathrm{d}s^2 = L_1^2 \left(dz_2^2 + e^{2z_2}\frac{1}{L_1^4}(dx_0^2 + dx_1^2) \right). \tag{7.203}$$

We have a non-isotropic flow, running between two perpendicular domain wall, one over $x_{2,3}$ in the IR, and the other over $x_{0,1}$ in the UV, with the domain-wall being defects on the field theory point of view (which is hard to understand, due to the complexity nature of the gauge group), see Figure 7.5.

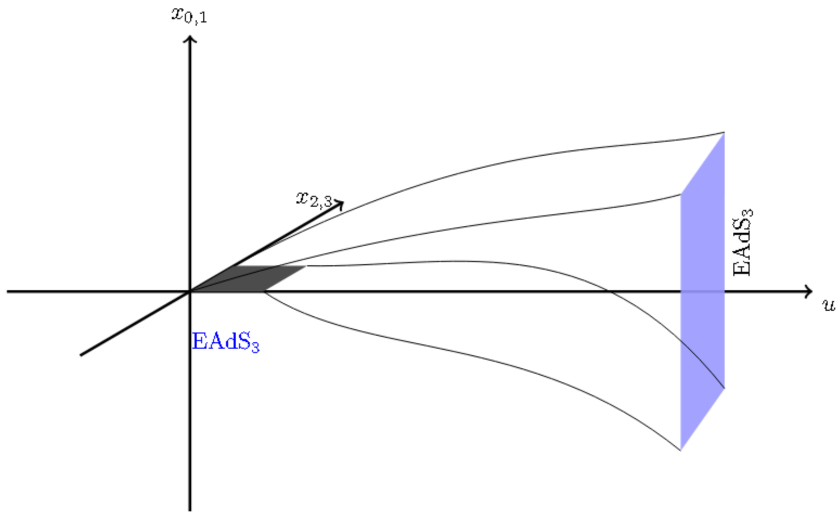

Figure 7.5: The background as the radius coordinates increases, the planes representing the spacelike domain-walls.

It is easier to understand the model from the field theory point of view by analyzing the first case of (7.201), where the QFT is well defined since the group rank is real. In this choice, the asymptotic theories along the RG flow are $SO(2,2)$, $d = 2$ and $SO(2,2)$, $d = 2$, with the dilaton running breaking the conformal symmetry in both limits, and the flow anisotropic.

Supergravity approximation

Despite the calculations from last sections, one should take the divergence involving ϕ more seriously. Indeed, while a divergence of the type $\phi \to -\infty$ is not a problem, as it is equivalent to $g_s \to 0$, a divergence $\phi \to \infty$ is more dangerous, as in that case we are in a strong coupling supergravity, $g_s \to \infty$, regime or even a string theory itself limit.

To verify this, we calculate terms relevant to the deviation of the first order supergravity action. For example, R, $R_{\mu\nu}R^{\mu\nu}$ and $R^{\mu}{}_{\nu\rho\sigma}R_{\mu}{}^{\nu\rho\sigma}$. However, the frame we are working on is the string frame, while those curvature quantities are properly defined in the Einstein frame.

We follow the rules and conventions defined in the nicely written paper [162] to go to the Einstein frame. The transformations are quite simple, since all fields (NS–NS and RR) vanish except for F_3 and ϕ. The Einstein metric can be obtained as

$$ds_E^2 = e^{-\frac{\phi-\phi_0}{2}} ds_S^2 = \frac{u}{\sqrt{H}L}\left(\frac{u^2}{L_1^2}(-dx_0^2 + dx_1^2) + \frac{H^2L_1^2}{u^2}(dx_2^2 + dx_3^2) + \frac{L_1^2}{u^2}du^2 + L_1^2 d\Omega_5^2\right), \tag{7.204}$$

and since $F_5 = F_1 = 0$, we just need to worry about the transformation of the axion-dilaton τ and the G_3 field, where for $H_3 = dB = 0$ is simply $\tau = ie^{-\phi}, G_3 = F_3$. From [162], the rules for these fields involves only ϕ_0[10], which can always be chosen to be 0, and then the mapping is trivial for the present specific case.

In the Einstein frame the Ricci scalar calculated is $R_E = -\frac{2\sqrt{H}}{Lu}$, but this is expected since it is now a mixing of R_S and ϕ_S. Indeed, when we combine it with α'_E expression, which is [162] $\alpha'_E = e^{-\frac{\phi-\phi_0}{2}}\alpha'_S$, we obtain for the dimensionless scalar involving the curvature the following expressions,

$$(R_{\mu\nu}\alpha'_E)^2 = 116\frac{(\alpha'_S)^2}{L^4}, \quad (R^{\mu}_{\nu\rho\sigma})^2(\alpha'_E)^2 = 97\frac{(\alpha'_S)^2}{L^4}, \quad (R\alpha'_E)^2 = 4\frac{(\alpha'_S)^2}{L^4}. \tag{7.205}$$

Then, if we assume the number of D1-branes N_1 (or equivalently, N_3) as large enough to suppress the α'^3 corrections, the supergravity limit of superstring is still valid, if we properly tune the parameters.

Nevertheless, the dilaton divergence can't be controlled in that way, and therefore the near-horizon regime of that classical solution is not completely trustable, as it receives quantum corrections in that regime. Yet, not everything in the near-horizon regime is lost. As is usual in homogeneous dilaton behavior, one can introduce a cutoff in the theory to avoid the quantum regime. Consequently, we would need to introduce a Λ_{IR} cutoff near $u \to 0$, integrating out the horizon and promoting the supergravity solution to be in a box-like spacetime with a mass gap, $r \in [\frac{1}{\Lambda_{IR}}, \infty]$.

Another alternative to introducing an IR cutoff, by taking advantage of IIB supergravity, is to perform an S-duality as we flow from large r to the horizon. The S-duality is an isomorphic mapping between two equivalent type IIB supergravity solution, where for the present case we can write the $\mathbb{Z}_2$ transformation as

$$\phi \to -\phi, \quad B_2 \to C_2, \quad C_2 \to -B_2. \tag{7.206}$$

Consequently, for $F_3 \neq 0$, we now have a non-zero H_3 flux and, as a result, instead of a D1-brane we obtain an F1 string. This new solution admits a classical

[10] As explained by the authors of the paper, ϕ_0 is the vacuum of ϕ where the string and Einstein dilaton coincides, then we doesn't need to worry which frame we are for that term.

supergravity regime near the horizon, $u \to 0$, since this limit is now equivalent to $\phi \to -\infty$ and $g_s \to 0$. However, the price we pay is a reversal of the non-classical behavior with respect to the radial coordinate u: the problematic regime is now near the boundary, i.e. in the UV. From the field-theory point of view, this way of bypassing quantum corrections is difficult to understand purely in terms of RG flow, as one must interpret it as a flow between two distinct theories. We do not explore this issue further, leaving a more detailed discussion for future work.

D1/D5 resonances

Despite these complexities, the model is interesting in its own right. It appears to realize a mixing of different holographic models present in the literature. These include a confining behavior in the IR driven by a dilaton flow [120], a defect field theory description [10], the presence of a constant magnetic field as in [58], together with group properties and supergravity solutions like those obtained in holographic cosmology [115], [114].

To finish, we discuss one more interesting characteristic of this supergravity solution. As discussed, the metric is

$$\mathrm{ds}^2 = u^2(-dx_0^2 + dx_1^2) + \frac{1}{u^2}\Big(\frac{1}{k^2}\left(dx_2^2 + dx_3^2\right) + du^2 + u^2 d\Omega_5^2\Big). \tag{7.207}$$

Since the S^5 part of the metric is not affected, we can rewrite the round five sphere metric as a three sphere S^3 foliated over S^1,

$$d\Omega_5^2 = \cos^2\theta\, d\Omega_3^2 + \sin^2\theta\, d\phi^2 + d\theta^2, \tag{7.208}$$

and then, if we approach $\theta \approx 0$ and keep only the leading terms in the metric, together with a change of coordinates in that limit of (7.208) $d\theta^2 + \theta^2 d\phi = dx_4^2 + dx_5^2$, the metric can be written as

$$\mathrm{ds}^2 \approx u^2(-dx_0^2 + dx_1^2) + \frac{du^2}{u^2} + d\Omega_3^2 + e^{\phi(u)}\widetilde{\mathrm{ds}^2_{M_4}}(u), \tag{7.209}$$

where $\widetilde{\mathrm{ds}^2_{M_4}}$ is a 4-dimensional made of the local flat metric of the compact S^2 ($\mathbb{R}^2$) + (rescaled) non-compact ($x_{2,3}$) and dependent on u.

This should be compared with the D1/D5 brane background [33], which is a supersymmetric model with gauge group $U(N_1) \times U(N_5)$ and constant coupling, dual to a two dimensional CFT with central charge $c \propto N_1 N_5$. The near horizon geometry of that background is $\mathrm{AdS}_3 \times S^3 \times M^4$,

$$\mathrm{d}s^2 = r^2(dt^2 + dx^2) + \frac{dr^2}{r^2} + d\Omega_3^2 + \left(\frac{Q_1}{Q_5}\right)^{1/2} \mathrm{d}s^2_{M_4}, \tag{7.210}$$

where M_4 is a compact internal manifold, and

$$F_3 = 2Q_5\, S^3, \qquad e^{2\phi} = \frac{Q_1}{Q_5}, \qquad Q_5 = N_5. \tag{7.211}$$

The metric (7.210), using (7.211), can be cast as

$$\mathrm{d}s^2 = r^2(dt^2 + dx^2) + \frac{dr^2}{r^2} + d\Omega_3^2 + e^{\phi}\mathrm{d}s^2_{M_4}. \tag{7.212}$$

Despite the similarity between the deformed D1 and D1/D5 background, evidently an exact matching is not possible. The radius L, which we set to one in the above calculations, depends on $Q_{1,5}$, which are constant in (7.210) in opposite to our model. Such a dependence would be required by the equations of motion here, and it would be inconsistent to have $L(Q_{1,5})$. Moreover, this limit should be interpreted with some caution, since keeping only the leading terms of S^5 near $\theta = 0$ shows to be an inconsistent truncation, since it is not a solution of the type IIB supergravity equations of motion.

Then, perhaps the most appropriate alternative for properly classifying the D1 plus magnetic field background is to compactify the $x_{2,3}$ coordinates by imposing periodicity, so that the metric can be consistently understood as a compact S^5 times a warped geometry,

$$\mathrm{d}s^2 = \left(u^2(-dx_0^2 + dx_1^2) + \frac{du^2}{u^2}\right) + e^{\phi(u)}\left(dx_2^2 + dx_3^2\right) + d\Omega_5^2, \tag{7.213}$$

together with the flux

$$C_2 = \frac{u^4 k}{L_1^4} dx_0 \wedge dx_1. \tag{7.214}$$

It is necessary to impose consistently the T^2 periodic identifications on the fields. For the NSNS sector it is trivial, since the dilaton ϕ is independent of $x_{2,3}$ and the Kalb Ramond field B is constant. Moreover, the modulus of the RR potential C_3 is also independent of these coordinates, so periodicity can be imposed in this sector as well. Therefore, the metric becomes

$$\mathrm{ds}^{2\ \mathrm{TT}}_{\mathrm{k}=-\frac{1}{\mathrm{H}}} = \underbrace{\mathrm{AdS}_3(u) \times e^{2\phi(u)}\ T^2}_{\text{Warped Geometry}} \times S^5, \tag{7.215}$$

where TT in ds means we compactified $x_{2,3}$ in a 2-torus, T^2. (7.215) should be compared with the near-horizon D1/D5 background, $\mathrm{AdS}_3(u) \times M^4 \times S^3$.

An appealing feature in favor of the compactification follows from recalling how N_1 is obtained, namely from the flux of the dual field F_7, which is proportional to the volume of S^5 times the area $A_{2,3}$. In the compactified case, $A_{2,3} = \kappa R_2 R_3$, where R_2 is the radius of one torus and R_3 is the radius of the other, with $\kappa = 4\pi^2$.

Chapter 8

Conclusion and Perspectives

The main goal of this dissertation was to present a coherent and accessible overview of advanced topics in string theory and holography from the perspective of a master student. Much of the literature in this area is highly technical, and even texts intended to be pedagogical can be difficult for readers who have only recently completed their undergraduate studies. With this in mind, the guiding principle of this work was to make the subject as clear as possible, with an effort to motivate the physical ideas and place them in a broader conceptual framework.

Within high energy physics and holography, the unifying theme of this dissertation was the study of deformations. We emphasized how deformations provide a concrete way to navigate between the two sides of the gauge gravity duality. While different holographic models are often proposed to capture specific physical phenomena, such as confinement or renormalization group flows with flavors, deformations tend to connect several of these aspects simultaneously. This characteristic makes them a powerful object of study to explore further the correspondence.

The three works discussed in Section 7 illustrate this from a complementary perspective. We investigate the behavior of deformations in different sectors by means of the BMN limit, analyze how deformations affect magnetic fields and QCD like phenomena in setups with flavor branes, and show that a bootstrap analysis of general deformed Leigh Strassler models can already predict a variety of nontrivial shared properties. Taken together, these results highlight the universality of deformations as a tool in holography and point toward several possible directions for future work.

A Basic of group, representations and particles

Particle Classification and Lorentz Group

Being more specific now, the Lagrangians we are going to study are generally relativistic, and therefore invariant under the Lorentz group $SO(d, 1)$. This is the cornerstone of modern physics and the Standard Model, since all known particles belong to some representation of this group. Indeed, focusing now on $d = 4$, the more intuitive case, we know there exist different types of particles, scalar (Higgs particle or mesons), fermionic/spinorial particles (electrons, positrons, quarks), or vector particles (the gauge bosons, like the photon γ particle). The terms scalar, vector, and spinor are due to representations of the $SO(3, 1)$ Lorentz group[1].

It can be shown that the Lie algebra resulting from the Lorentz group, $\mathrm{so}(3, 1)$, is isomorphic to $\mathrm{su}(2) \oplus \mathrm{su}(2)$. Therefore, we are going to use the Cartan labels [64] of both $\mathrm{su}(2)$ groups, j_1 and j_2, to describe the representations of $\mathrm{so}(3, 1)$.

Scalars

Scalars are those particles whose fields do not transform when we perform a Lorentz transformation, hence, they are invariant particles. They belong, consequently, to the $(0, 0)$ classification of the Lie algebra $\mathrm{su}(2) \oplus \mathrm{su}(2)$,

$$\text{Scalar} \quad = \quad (0,0) \text{ of } \mathrm{su}(2) \oplus \mathrm{su}(2) \tag{A.1}$$

Fermions (Spinors)

Fermionic particles, which in the Standard Model are restricted to fermions, are particles that transform under Lorentz transformations as [131],

$$\delta\psi = \frac{i}{4}\Sigma_{\mu\nu}\psi \tag{A.2}$$

[1]To be more specific, to truly classify the particles we need to study representations of the Poincaré group, which is the Lorentz group plus translations. This introduces a new label related to the mass of the particles, and makes use of two Casimir operators, P^2 and the Pauli-Lubanski vector, to classify the particles.

where $\Sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$, with γ being the gamma matrices, which are the generators of the anticommutator Clifford algebra,

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu} \tag{A.3}$$

with η being the metric of the spacetime, in our case simply the Minkowski spacetime, $(-1, 1, 1, 1)$ in high-energy conventions[2].

The transformation of the spinor field, however, is reducible. Spinors have some particularities when compared to vectors or scalars, mainly due to chirality. Generally, we differentiate between,

- Dirac spinors (never irreducible), which transform under the standard Lorentz action.
- Majorana spinors (generally called real spinors, though not in the literal sense $\psi^* = \psi$ [61, 163]), which are more restricted, only existing for specific dimensions $d = 2, 3$ and 4 mod 8.
- Weyl spinors, which depend on the eigenvalues of γ_5 (or γ_d),

  $$\gamma^5\psi = \pm\psi \tag{A.4}$$

  Depending on the sign in (A.4), we have left or right fermions. Weyl spinors only exist in even spacetime, $d = 2, 4, 6, \ldots$, where the general matrix γ_d can be defined with the property,

  $$\{\gamma_d, \gamma_\mu\} = 0, \quad \gamma_d^2 = \pm 1, \quad \gamma_d^\dagger = \gamma_d \tag{A.5}$$

The possibility of having a Majorana-Weyl spinor (simultaneously satisfying both conditions) is algebraically possible only if the signature of the spacetime, calling (d_1, d_2) the number of negative and positive signs in the metric, respectively, satisfies,

$$|d_1 - d_2| = 2 \mod 8 \tag{A.6}$$

Equation (A.6) shows that we can have Majorana-Weyl fermions in $(1, 1)$ ($d = 2$) or $(9, 1)$ ($d = 10$) spacetimes. It is no coincidence that these spacetimes arise in string theory as the string worldsheet dimension and the target spacetime itself, respectively.

[2]This convention is better for Wick rotations.

Returning to the classification, Dirac spinors belong to the $(1/2, 0) \oplus (0, 1/2)$ representation,

$$\text{Dirac fermion} \quad = \quad (1/2, 0) \oplus (0, 1/2) \text{ of } \mathrm{su}(2) \oplus \mathrm{su}(2) \tag{A.7}$$

which is reducible, as the algebra itself suggests. We can split the Dirac fermion into two Weyl fermions, left and right (A.4), which themselves are irreducible representations and transform in a similar way to (A.2), with transformation matrices $L(\lambda)$ and $\tilde{L}(\lambda)$ built from the Pauli matrices $\sigma_\mu = (I, \sigma_i)$ and $\tilde{\sigma}_\mu = (I, -\sigma_i)$, where the Pauli matrices are the generators of the su(2) algebra,

$$[\sigma_i, \sigma_j] = i\epsilon_{ijk}\sigma_k \tag{A.8}$$

Therefore,

$$\text{Weyl fermions} \quad = \quad (1/2, 0) \text{ and } (0, 1/2) \text{ of } \mathrm{su}(2) \oplus \mathrm{su}(2) \tag{A.9}$$

Vectors

Vector representations are, paraphrasing Carroll's definition [36], "things that transform like vectors",

$$V^\mu \to \Lambda^\mu_\nu V^\nu \tag{A.10}$$

Ignoring the orbital part (the x dependence), they belong to the representation,

$$\text{Vector} \quad = \quad (1/2, 1/2) \text{ of } \mathrm{su}(2) \oplus \mathrm{su}(2) \tag{A.11}$$

(Note, The vector representation is commonly denoted by $(1/2, 1/2)$, as $j_1 + j_2$ must sum to 1 for a four-vector, not $(1, 1)$ as you previously wrote). This finishes the group theoretical approach for the classification of the particles in the Standard Model. Later, when we talk about supergravity, we will include two more important representations, the gravitino particle with spin 3/2 and the graviton particle, with spin 2.

Internal Symmetries and Gauge Theory

The last general property almost all field theory models share, and that is also correlated to group theory, is the representations of the particles under internal groups. Global and local groups arising from specific internal symmetries are almost always present in a model, following the analogy made clear by Glashow, Abdus Salam, Weinberg, and other physicists for the description of forces in the Standard Model and group representations. In general, when talking about internal symmetries, we restrict ourselves to compact groups because they allow for unitary representations [64].

Fundamental and Adjoint Representations

Defining representations, a field ψ^i in the fundamental representation of a group G and Lie algebra $\mathfrak{g}$ transforms as [76],

$$\psi^i \to U^i_j \psi^j \tag{A.12}$$

with U being an element of the group G. Infinitesimally, this is the same as

$$\delta\psi^i = i\epsilon_a T^{ai}_{\ j}\psi^j \tag{A.13}$$

These are generally matter fields, like quarks or leptons. The field strength, $F_{\mu\nu} = \partial_{[\mu}A_{\nu]} + [A_\mu, A_\nu]$, transforms as the adjoint representation,

$$F^{ij}_{\mu\nu} \to U^{ii'}F_{\mu\nu\, i'j'}U^{j'j} = U^{-1}F_{\mu\nu}U \tag{A.14}$$

or, infinitesimally in terms of generators of the Lie algebra (f_{abc} are the structure constants),

$$\delta F_{\mu\nu a} = iF_{\mu\nu}^{\ \ b} f_{abc}\epsilon^c \tag{A.15}$$

The transformation for A itself is obtained by promoting the global group symmetry of $\mathcal{L}$ to a local symmetry (meaning the group transformation and parameters are now coordinate-dependent, $U(x), \epsilon(x)$). This is also called a gauge transformation, and a field model with these elements for a group SU(n) is generally called Yang-Mills theory. Moreover, the field strength F satisfies a Bianchi identity in the vacuum,

$$D_{[\mu}F_{\nu\rho]} = 0 \tag{A.16}$$

B TsT transformation rules

The derivation of the resulting background after doing a TsT transformation is easier after knowing how the fields transform under T-duality. In fact, this is well known from the literature. Regarding the NS-NS field, the transformations (under T-duality) are known as Buscher's rules, and were derived in [35]. We simply reproduce them here.

Doing a T-duality over a coordinate we call "1", the rules for the NSNS-fields are

$$\begin{gathered}\tilde{G}_{11}=\frac{1}{G_{11}},\quad \tilde{G}_{1i}=\frac{B_{1i}}{G_{11}},\quad \tilde{G}_{ij}=G_{ij}-\frac{G_{1i}G_{1j}-B_{1i}B_{1j}}{G_{11}},\quad \tilde{B}_{1i}=\frac{G_{1i}}{G_{11}},\\ \tilde{B}_{ij}=B_{ij}+\frac{G_{1i}B_{1j}-B_{1i}G_{1j}}{G_{11}},\quad \tilde{\Phi}=\Phi-\frac{1}{2}\log G_{11}\end{gathered}\tag{B.1}$$

And for the RR-fields [117], the rules consist of mappings among fields in type IIB to type IIA or vice-versa:

$$\tilde{C}^{(n)}_{\mu\nu\alpha}=C^{(n-1)}_{\mu\nu\alpha}-(n-1)\frac{C^{(n-1)}_{\mu\nu\beta}G_{\alpha\gamma}}{G_{11}},\quad \tilde{C}^{(n)}_{\mu\nu\beta}=C^{(n+1)}_{\mu\nu\gamma}+nC^{(n-1)}_{\mu\nu\alpha}B_{\gamma\beta}+n(n-1)\frac{C^{(n-1)}_{\mu\nu\alpha}B_{\alpha\beta}G_{\gamma\beta}}{G_{11}}.\tag{B.2}$$

Then, the TsT transformation consists of doing the T-duality, performing a shift over another (isometric) coordinate, and then applying the T-duality rules again. This is a possible way to do that, but the algebra takes too much time, so we decided to follow the general rules for a TsT transformation derived in [84], [85]. There, an element $\mathcal{M}$ is defined as follows: Let T-dualize over 2, and shift 3. Then,

$$\mathcal{M}=\left\{1-\gamma\left(e_{\alpha 2\alpha 3}-e_{\alpha 3\alpha 2}\right)+\gamma^2\det\begin{pmatrix}e_{\alpha 2\alpha 2} & e_{\alpha 2\alpha 3}\\ e_{\alpha 3\alpha 2} & e_{\alpha 3\alpha 3}\end{pmatrix}\right\}^{-1}\tag{B.3}$$

Using this element, the NSNS fields transformations under the TsT, defining $e=B+G$, are

$$\begin{gathered}\tilde{e}_{MN}=\mathcal{M}\left\{e_{MN}-\gamma\left[\det\begin{pmatrix}e_{\alpha 2\alpha 3} & e_{\alpha N}\\ e_{M\alpha 3} & e_{MN}\end{pmatrix}\right]-\det\begin{pmatrix}e_{\alpha 3\alpha 2} & e_{\alpha N}\\ e_{M\alpha 2} & e_{MN}\end{pmatrix}+\gamma^2\det\begin{pmatrix}e_{\alpha 3\alpha 2} & e_{\alpha 2\alpha 3} & e_{\alpha 2N}\\ e_{\alpha 2\alpha 3} & e_{\alpha 2\alpha 3} & e_{\alpha 3N}\\ e_{\alpha 3\alpha 3} & e_{M\alpha 3} & e_{MN}\end{pmatrix}\right\}\\ e^{2\tilde{\phi}}=\mathcal{M}e^{2\phi}\end{gathered}\tag{B.4}$$

and the RR-fields are

$$\begin{aligned}F_1&=f_1+\gamma\left[f_3+f_1\wedge B\right]_{2,3}\\ F_3+F_1\wedge B&=f_3+f_1\wedge B+\gamma\left[f_5+f_3\wedge B+\frac{1}{2}f_1\wedge B\wedge B\right]_{2,3}\\ F_5+F_3\wedge B+\frac{1}{2}F_1\wedge B\wedge B&=f_5+f_3\wedge B+\frac{1}{2}f_1\wedge B\wedge B+\gamma\left[f_7+f_5\wedge B+\frac{1}{2}f_3\wedge B\wedge B+\frac{1}{6}f_1\wedge B\wedge B\wedge B\right]_{2,3}\end{aligned}\tag{B.5}$$

where the indices with brackets simply mean contraction.

C Terms of the action and equation of motion

The present appendix is dedicated to give the terms of the action relevant to the equation of motion, and are used to derive the equation of motion for the fluctuations, up to α'^2, in the work 7.3.

Terms of the action

We give the entries value of each matrix here (notice that $\text{tr}(E) = \text{tr}(G+B) = \text{tr}(G)$). Explicitly, δ_2 is

$$\begin{aligned}\delta^2_{mn} = \frac{1}{2}\frac{\partial^2 E_{mn}}{\partial l^2}\Big|_{\chi=0} \chi^2 + \frac{1}{\rho^2+l_0^2}((\partial_m\chi)(\partial_n\chi) + l^2\partial_m\Phi\partial_n\Phi) \\ -\frac{2l_0 l_0'}{(\rho^2+l_0^2)^2}\left(\delta_{mp}\delta_{ni}+\delta_{np}\delta_{mi}\right)\chi\partial_i\chi.\end{aligned} \tag{C.1}$$

Then,

$$\begin{aligned}\text{tr}(\delta_2) = \left(\frac{1}{2}\frac{\partial^2 \text{tr}(G)}{\partial l_0^2}\right)\chi^2 + \frac{1}{\rho^2+l_0^2}\Big((\partial_i\chi)^2 + l_0^2(\partial_i\Phi)^2\Big) - \frac{4l_0 l_0'}{(\rho^2+l_0^2)^2}\chi\partial_\rho\chi, \\ S = S_{mn}, J = J_{mn}, \mathcal{F} = F_{mn} + \delta^1_A = F_{mn} + \frac{\partial B_{mn}}{\partial l_0}\chi, \\ \delta^1_S = \left(\frac{\partial G_{nm}}{\partial l_0}\right)\chi + \frac{1}{\rho^2+l_0^2}\Big(l_0'\left(\delta_{mp}\delta_{ni}+\delta_{np}\delta_{mi}\right)\partial_i\chi\Big),\end{aligned} \tag{C.2}$$

with $l_0' = \partial_\rho l_0$. Let's take each term of (D.12) separately, ignoring the $\sqrt{\det E}$ factor,

$$\frac{\partial^2 e^{-\phi_0}}{\partial l_0^2}\chi^2 \tag{C.3}$$

and

$$\begin{aligned}\frac{\partial e^{-\phi_0}}{\partial l_0}(\text{tr}(\delta_1^S S + J\mathcal{F}))\chi = \frac{\partial e^{-\phi_0}}{\partial l_0}\Bigg(\Bigg(\left(\frac{\partial G_{mn}}{\partial l_0}\right)\chi + \frac{1}{\rho^2+l_0^2}\Big(l_0'\left(\delta_{mp}\delta_{ni}+\delta_{np}\delta_{mi}\right)\partial_i\chi\Big)\Bigg)S_{nm} \\ + J_{nm}(F_{mn} + \frac{\partial B_{mn}}{\partial l_0}\chi)\Bigg)\chi,\end{aligned} \tag{C.4}$$

$$\text{Tr}(J\,\mathcal{F})^2 = J_{nm}\Big(F_{mn} + \frac{\partial B_{mn}}{\partial l_0}\chi\Big)J_{jr}\Big(F_{rj} + \frac{\partial B_{rj}}{\partial l_0}\chi\Big), \tag{C.5}$$

$$2\mathrm{tr}(S\delta_1^S)\mathrm{tr}(JF) = 2S_{mn}\Big(\left(\frac{\partial G_{nm}}{\partial l_0}\right)\chi + \frac{1}{\rho^2+l_0^2}\Big(l_0'\left(\delta_{mp}\delta_{ni}+\delta_{np}\delta_{mi}\right)\partial_i\chi\Big)\Big)J_{jr}\Big(F_{rj}+\frac{\partial B_{rj}}{\partial l_0}\chi\Big), \quad (C.6)$$

$$\begin{aligned}\mathrm{tr}(S\delta_1^S)^2 = S_{mn}&\Big(\left(\frac{\partial G_{nm}}{\partial l_0}\right)\chi + \frac{1}{\rho^2+l_0^2}\Big(l_0'\left(\delta_{mp}\delta_{ni}+\delta_{np}\delta_{mi}\right)\partial_i\chi\Big)\Big)\\ &\times S_{jr}\Big(\left(\frac{\partial G_{rj}}{\partial l_0}\right)\chi + \frac{1}{\rho^2+l_0^2}\Big(l_0'\left(\delta_{rp}\delta_{jk}+\delta_{jp}\delta_{rk}\right)\partial_k\chi\Big)\Big),\end{aligned} \quad (C.7)$$

$$\mathrm{tr}(J\mathcal{F}J\mathcal{F}) = J_{rm}\Big(F_{mn}+\frac{\partial B_{mn}}{\partial l_0}\chi\Big)J_{nj}\Big(F_{jr}+\frac{\partial B_{jr}}{\partial l_0}\chi\Big), \quad (C.8)$$

$$4\mathrm{tr}(J\delta_1^S S\mathcal{F}) = 4J_{rn}\Big(\left(\frac{\partial G_{nm}}{\partial l_0}\right)\chi + \frac{1}{\rho^2+l_0^2}\Big(l_0'\left(\delta_{mp}\delta_{ni}+\delta_{np}\delta_{mi}\right)\partial_i\chi\Big)\Big)S_{mj}\Big(F_{jr}+\frac{\partial B_{jr}}{\partial l_0}\chi\Big), \quad (C.9)$$

$$\begin{aligned}\mathrm{tr}(J\delta_1^S J\delta_1^S) = J_{rn}&\Big(\left(\frac{\partial G_{nm}}{\partial l_0}\right)\chi + \frac{1}{\rho^2+l_0^2}\Big(l_0'\left(\delta_{mp}\delta_{ni}+\delta_{np}\delta_{mi}\right)\partial_i\chi\Big)\Big)\\ &\times J_{mj}\Big(\left(\frac{\partial G_{jr}}{\partial l_0}\right)\chi + \frac{1}{\rho^2+l_0^2}\Big(l_0'\left(\delta_{jp}\delta_{rk}+\delta_{rp}\delta_{jk}\right)\partial_k\chi\Big)\Big),\end{aligned} \quad (C.10)$$

$$\begin{aligned}\mathrm{tr}(S\delta_1^S S\delta_1^S) = S_{rn}&\Big(\left(\frac{\partial G_{nm}}{\partial l_0}\right)\chi + \frac{1}{\rho^2+l_0^2}\Big(l_0'\left(\delta_{mp}\delta_{ni}+\delta_{np}\delta_{mi}\right)\partial_i\chi\Big)\Big)\\ &\times S_{mj}\Big(\left(\frac{\partial G_{jr}}{\partial l_0}\right)\chi + \frac{1}{\rho^2+l_0^2}\Big(l_0'\left(\delta_{jp}\delta_{rk}+\delta_{rp}\delta_{jk}\right)\partial_k\chi\Big)\Big),\end{aligned} \quad (C.11)$$

$$\mathrm{tr}(S\mathcal{F}S\mathcal{F}) = S_{rm}\Big(F_{mn}+\frac{\partial B_{mn}}{\partial l_0}\chi\Big)S_{nj}\Big(F_{jr}+\frac{\partial B_{jr}}{\partial l_0}\chi\Big), \quad (C.12)$$

$$4\mathrm{tr}(\delta_2) = 2\left(\frac{\partial^2 \mathrm{tr}(G)}{\partial l_0^2}\right)\chi^2 + \frac{4}{\rho^2+l_0^2}\Big((\partial_i\chi)^2 + l_0^2(\partial_i\Phi)^2\Big) - \frac{l_0 l_0'}{(\rho^2+l_0^2)^2}\chi\partial_\rho\chi. \quad (C.13)$$

Finally, collecting term by term, we obtain

• $\chi\partial\chi$

$$\begin{aligned}\Bigg(\frac{\partial e^{-\phi_0}}{\partial l_0}\frac{1}{\rho^2+l_0^2}l_0' 2S_{pi}+\frac{e^{-\phi_0}}{4}\Bigg(&-\frac{8l_0l_0'}{(\rho^2+l_0^2)^2}\delta_{ip}+2\frac{l_0'}{\rho^2+l_0^2}\,2S_{pi}\,J_{jr}\,\frac{\partial B_{rj}}{\partial l_0}\\
&+\frac{l_0'}{\rho^2+l_0^2}\,2S_{pi}\,S_{jr}\,\frac{\partial G_{rj}}{\partial l_0}+\frac{l_0'}{\rho^2+l_0^2}\,S_{mn}\,\frac{\partial G_{nm}}{\partial l_0}2S_{pi}\\
&-8\frac{l_0'}{\rho^2+l_0^2}\,(S_{pj}J_{ri}+S_{ij}J_{rp})\,\frac{\partial B_{jr}}{\partial l_0}-2\Big(\frac{l_0'}{\rho^2+l_0^2}\,(J_{jp}J_{ri}+J_{pi}J_{ij})\,\frac{\partial G_{jr}}{\partial l_0}\\
&+\frac{l_0'}{\rho^2+l_0^2}\,\frac{\partial G_{nm}}{\partial l_0}\,(J_{mp}J_{in}+J_{pn}J_{mi})\Big)\\
-2\Big(\frac{l_0'}{\rho^2+l_0^2}\,(S_{ri}S_{pj}+S_{ij}S_{rp})\,\frac{\partial G_{jr}}{\partial l_0}&+\frac{l_0'}{\rho^2+l_0^2}\,\frac{\partial G_{nm}}{\partial l_0}\,(S_{in}S_{mp}+S_{pn}S_{mi})\Big)\Bigg)\Bigg)\chi\partial_i\chi.\end{aligned} \tag{C.14}$$

Or, since S and G is diagonal and $\partial_\rho\chi = 0$, and B and J has only $2,3$ non-zero components,

$$\begin{aligned}\Bigg(\frac{\partial e^{-\phi_0}}{\partial l_0}\frac{1}{\rho^2+l_0^2}l_0' 2S_{pp}+\frac{e^{-\phi_0}}{4}\Bigg(&-\frac{8l_0l_0'}{(\rho^2+l_0^2)^2}+4\frac{l_0'}{\rho^2+l_0^2}\,S_{pp}\,J_{jr}\,\frac{\partial B_{rj}}{\partial l_0}\\
&+\frac{l_0'}{\rho^2+l_0^2}\,4S_{pp}\,S_{jr}\,\frac{\partial G_{rj}}{\partial l_0}\\
&-8\Big(\frac{l_0'}{\rho^2+l_0^2}\,S_{pp}S_{pp}\,\frac{\partial G_{pp}}{\partial l_0}\Big)\Bigg)\Bigg)\chi\partial_\rho\chi.\end{aligned} \tag{C.15}$$

Therefore, only $\chi\partial_\rho\chi$ is non-zero and present. We can, then call $\chi\partial_\rho\chi = 2^{-1}\partial_\rho\chi^2$, integrate by parts and then add the term to the χ^2 coefficient term, reducing to

$$\chi\partial_i\chi = 0. \tag{C.16}$$

We collect equally each term,

- χ^2

$$\begin{aligned}&\Big(\frac{\partial^2 e^{-\phi_0}}{\partial l_0^2}+\frac{\partial e^{-\phi_0}}{\partial l_0}\Big(\frac{\partial G_{mn}}{\partial l_0}S_{nm}+J_{nm}\frac{\partial B_{mn}}{\partial l_0}\Big)\\&+\frac{e^{-\phi_0}}{4}\Big[J_{nm}\frac{\partial B_{mn}}{\partial l_0}J_{jr}\frac{\partial B_{rj}}{\partial l_0}+2\,S_{mn}\frac{\partial G_{nm}}{\partial l_0}J_{jr}\frac{\partial B_{rj}}{\partial l_0}+S_{mn}\frac{\partial G_{nm}}{\partial l_0}S_{jr}\frac{\partial G_{rj}}{\partial l_0}+2\,\frac{\partial^2 G_{mm}}{\partial l_0^2}\\&-2\Big(J_{rm}\frac{\partial B_{mn}}{\partial l_0}J_{nj}\frac{\partial B_{jr}}{\partial l_0}+4\,J_{rn}\frac{\partial G_{nm}}{\partial l_0}S_{mj}\frac{\partial B_{jr}}{\partial l_0}+J_{rn}\frac{\partial G_{nm}}{\partial l_0}J_{mj}\frac{\partial G_{jr}}{\partial l_0}+S_{rn}\frac{\partial G_{nm}}{\partial l_0}S_{mj}\frac{\partial G_{jr}}{\partial l_0}\\&+S_{rm}\frac{\partial B_{mn}}{\partial l_0}S_{nj}\frac{\partial B_{jr}}{\partial l_0}\Big)\Big]\Big)\chi^2,\end{aligned} \tag{C.17}$$

- F^2

$$\frac{e^{-\phi_0}}{4}\Big(J_{nm}\,J_{rj}-\Big(J_{rm}J_{nj}-J_{jm}J_{nr}+S_{rm}S_{nj}-S_{jm}S_{nr}\Big)\Big)F_{mn}F_{jr}, \tag{C.18}$$

- $(\partial\chi)^2$

$$e^{-\phi_0}\Big(\frac{1}{\rho^2+l_0^2}\,\delta_{ik}+\frac{l_0'^2}{(\rho^2+l_0^2)^2}S_{pp}S_{ki}\Big)(\partial_i\chi)(\partial_k\chi), \tag{C.19}$$

- $F\partial\chi$

$$\begin{aligned}&\frac{e^{-\phi_0}}{4}\Big(2\frac{l_0'}{\rho^2+l_0^2}\,S_{mn}(\delta_{mp}\delta_{ni}+\delta_{np}\delta_{mi})\,J_{jr}-4\frac{l_0'}{\rho^2+l_0^2}\,(\delta_{mp}\delta_{ni}+\delta_{np}\delta_{mi})\,(S_{mr}J_{jn}-S_{mj}J_{rn})\Big)\,F_{rj}\,\partial_i\chi\\&=\frac{e^{-\phi_0}}{4}\frac{2l_0'}{\rho^2+l_0^2}\Big(S_{pi}J_{jr}+S_{ip}J_{jr}-2(S_{pr}J_{ji}-S_{pj}J_{ri}+S_{ir}J_{jp}-S_{ij}J_{rp}\Big)\Big)F_{rj}\,\partial_i\chi\\&=\frac{e^{-\phi_0}}{4}\frac{4l_0'}{\rho^2+l_0^2}\Big(S_{pi}J_{jr}-(S_{pr}J_{ji}-S_{pj}J_{ri}+S_{ir}J_{jp}-S_{ij}J_{rp}\Big)\Big)F_{rj}\,\partial_i\chi\\&=\frac{e^{-\phi_0}}{4}\frac{4l_0'}{\rho^2+l_0^2}\Big(S_{pi}J_{jr}-S_{pr}J_{ji}+S_{pj}J_{ri}\Big)F_{rj}\,\partial_i\chi\\&=\frac{e^{-\phi_0}}{4}\frac{4l_0'}{\rho^2+l_0^2}\Big(S_{pi}J_{jr}+S_{pr}J_{ij}+S_{pj}J_{ri}\Big)F_{rj}\,\partial_i\chi,\end{aligned} \tag{C.20}$$

where we used that $J_{p\ldots} = 0$. The term in parenthesis is cyclic in i, j, r, so we can partially integrate and obtain

$$-\frac{e^{-\phi_0} l_0'}{\rho^2 + l_0^2} \partial_i \Big(S_{pi} J_{jr} + S_{pr} J_{ij} + S_{pj} J_{ri} \Big) F_{rj}\, \chi. \tag{C.21}$$

Therefore, this is really a $F\chi$ term. So we can also say $C^{F\partial\chi} = 0$

- χF

$$\begin{gathered} \Big(\frac{\partial e^{-\phi_0}}{\partial l_0} J_{jr} + \frac{e^{-\phi_0}}{2} \Big(J_{nm} \left(\frac{\partial B_{mn}}{\partial l_0} \right) J_{rj} + S_{mn} \left(\frac{\partial G_{nm}}{\partial l_0} \right) J_{rj} \\ -\Big(\left(\frac{\partial B_{mn}}{\partial l_0} \right) (J_{rm} J_{nj} - J_{jm} J_{nr}) + 2 \left(\frac{\partial G_{nm}}{\partial l_0} \right) (J_{rn} S_{mj} - J_{jn} S_{mr}) + \left(\frac{\partial B_{mn}}{\partial l_0} \right) (S_{nj} S_{rm} - S_{nr} S_{jm}) \Big) \Big) \Big) \chi F_{jr}. \end{gathered} \tag{C.22}$$

Then, we can add (C.21)

$$\begin{aligned} &\Big(\frac{\partial e^{-\phi_0}}{\partial l_0} J_{jr} + \frac{e^{-\phi_0}}{2} \Big(J_{nm} \left(\frac{\partial B_{mn}}{\partial l_0} \right) J_{rj} + S_{mn} \left(\frac{\partial G_{nm}}{\partial l_0} \right) J_{rj} \\ &-\Big(\left(\frac{\partial B_{mn}}{\partial l_0} \right) (J_{rm} J_{nj} - J_{jm} J_{nr}) + 2 \left(\frac{\partial G_{nm}}{\partial l_0} \right) (J_{rn} S_{mj} - J_{jn} S_{mr}) \\ &+ \left(\frac{\partial B_{mn}}{\partial l_0} \right) (S_{nj} S_{rm} - S_{nr} S_{jm}) \Big) + \frac{2 l_0'}{\rho^2 + l_0^2} \partial_i \Big(S_{pi} J_{jr} + S_{pr} J_{ij} + S_{pj} J_{ri} \Big) \Big) \Big) F_{jr}\, \chi, \end{aligned} \tag{C.23}$$

simplifying further,

$$\begin{aligned} &\Big(\frac{\partial e^{-\phi_0}}{\partial l_0} J_{jr} + \frac{e^{-\phi_0}}{2} \Big(J_{nm} \left(\frac{\partial B_{mn}}{\partial l_0} \right) J_{rj} + S_{mn} \left(\frac{\partial G_{nm}}{\partial l_0} \right) J_{rj} \\ &-\Big(2 \left(\frac{\partial B_{mn}}{\partial l_0} \right) J_{rm} J_{nj} + 2 \left(\frac{\partial G_{nm}}{\partial l_0} \right) (J_{rn} S_{mj} - J_{jn} S_{mr}) + 2 \left(\frac{\partial B_{mn}}{\partial l_0} \right) S_{nj} S_{rm} \Big) \\ &+ \frac{2 l_0'}{\rho^2 + l_0^2} \partial_i \Big(S_{pi} J_{jr} + S_{pr} J_{ij} + S_{pj} J_{ri} \Big) \Big) \Big) F_{jr}\, \chi. \end{aligned} \tag{C.24}$$

- $(\partial\Phi)^2$

$$\frac{e^{-\phi_0}}{4} \frac{4}{\rho^2 + l_0^2} l_0^2 (\partial_i \Phi)^2. \tag{C.25}$$

Equation of motion

• ϕ

$$\frac{1}{4}\partial_i\left(C^{(\partial\phi)^2}\,\partial_i\phi\right)-\frac{\sin 2\psi}{4}\partial_\rho(K(p)B_{23})\delta_{0a}\delta_{1b}F_{ab}=0 \tag{C.26}$$

or, rewritten more appropriately,

$$\partial_i(\sqrt{\det E}\tilde{C}^{(\partial\phi)^2}G^{ij}\partial_j\Phi)-\sin 2\psi\partial_\rho(K(p)B_{23})F_{01}=0. \tag{C.27}$$

Now, $C^{(\partial\phi)^2}$ only depends on ρ. We can split the first term as following,

$$\begin{aligned}\partial_i(\sqrt{\det E}\tilde{C}^{(\partial\phi)^2}G^{ij}\partial_j\Phi)=\partial_\rho(\sqrt{\det E}\tilde{C}^{(\partial\phi)^2}G^{pp}\partial_\rho\Phi)+\partial_{x_i}(\sqrt{\det E}\tilde{C}^{(\partial\phi)^2}G^{x_ix_j}\partial_{x_j}\Phi)\\+\partial_\mu(\sqrt{\det E}\tilde{C}^{(\partial\phi)^2}G^{\mu\nu}\partial_\nu\Phi)+\partial_{\alpha_i}(\sqrt{\det E}\tilde{C}^{(\partial\phi)^2}G^{\alpha_i\alpha_j}\partial_{\alpha_j}\Phi).\end{aligned} \tag{C.28}$$

Using (D.14), (D.15) and (D.16),

$$\begin{aligned}\partial_i(\sqrt{\det E}\tilde{C}^{(\partial\phi)^2}G^{ij}\partial_j\Phi)=\partial_\rho(\sqrt{\det E}\tilde{C}^{(\partial\phi)^2}G^{pp}\partial_\rho\Phi)\\+\sqrt{\det E_i}\tilde{C}^{(\partial\phi)^2}G_1^{-1}(p)\partial_{x_i}(\sqrt{\det(i)}\eta^{x_ix_j}\partial_{x_j}\Phi)\\+\sqrt{\det E_\mu}\tilde{C}^{(\partial\phi)^2}\frac{G_2(p)}{G_2(p)^2-G_1(p)^2}\partial_\mu(\sqrt{\det(\mu)}\delta^{\mu\nu}\partial_\nu\Phi)\\+\sqrt{\det E_\Omega}\tilde{C}^{(\partial\phi)^2}G_4(p)^{-1}\partial_{\alpha_i}(\sqrt{\det(\Omega)}G^{\alpha_i\alpha_j}\partial_{\alpha_j}\Phi),\end{aligned} \tag{C.29}$$

which can be rewritten as

$$\begin{aligned}\partial_i(\sqrt{\det E}\tilde{C}^{(\partial\phi)^2}G^{ij}\partial_j\Phi)=\partial_\rho(\sqrt{\det E}\tilde{C}^{(\partial\phi)^2}G^{pp}\partial_\rho\Phi)+\sqrt{\det E}\tilde{C}^{(\partial\phi)^2}\Delta_i\Phi+\sqrt{\det E}\tilde{C}^{(\partial\phi)^2}\Delta_\mu\Phi\\+\sqrt{\det E}\tilde{C}^{(\partial\phi)^2}\Delta_\Omega\Phi).\end{aligned} \tag{C.30}$$

Substituting (D.16),

$$\begin{aligned}\partial_i(\sqrt{\det E}\tilde{C}^{(\partial\phi)^2}G^{ij}\partial_j\Phi)=\partial_\rho(\sqrt{\det E}\tilde{C}^{(\partial\phi)^2}G^{pp}\partial_\rho\Phi)\\+\sqrt{\det E}\tilde{C}^{(\partial\phi)^2}\left(G_1^{-1}(p)\Delta_\eta\phi+\frac{G_2(p)}{G_2(p)^2-G_1(p)^2}\Delta_\delta\phi+G_4(p)^{-1}\Delta_\Omega\Phi\right),\end{aligned} \tag{C.31}$$

and therefore, the equation of motion is

$$\partial_\rho(\sqrt{\det E}\tilde{C}^{(\partial\phi)^2}G_3\partial_\rho\Phi) + \sqrt{\det E}\tilde{C}^{(\partial\phi)^2}\left(G_1^{-1}(p)\Delta_\eta\phi + \frac{G_2(p)}{G_2(p)^2+B_{23}^2}\Delta_\delta\phi + G_4(p)^{-1}\Delta_\Omega\Phi\right) - \sin 2\psi\partial_\rho(K(p)B_{23})F_{01} = 0. \tag{C.32}$$

- $A_{5,6,7,8}$

  Now, $\delta_{0,s}\delta_{1,s} = 0$, and also $C^{\chi F}_{rs} = C^{\partial\chi F}_{rsi} = 0$ for $s \neq 2, 3$, so

$$\frac{1}{4}\partial_r\left[2\Big(C^{FF}_{rs\,mn} + \frac{1}{\sqrt{\det E}}(\rho^2+l_0^2)^2\epsilon_{rsmn}\Big)F_{mn}\right] = 0. \tag{C.33}$$

Substituting $C = \sqrt{\det E}\tilde{C}$, with $\tilde{C}$ calculated in C, we obtain the equation of motion,

$$\partial_r\left[\Big(C^{FF}_{rp\,mn} + \frac{1}{\sqrt{\det E}}(\rho^2+l_0^2)^2\epsilon_{rsmn}\Big)F_{mn}\right] = 0. \tag{C.34}$$

But under these constraints, the second term vanishes (since mnr can only take values on the s above, but $F_{s1s2} = 0$). So

$$\partial_r(C^{FF}_{rsmn}F_{mn}) = 0. \tag{C.35}$$

Finally, C is

$$\begin{aligned} C_{rsmn} &= \sqrt{\det E}\frac{e^{-\phi_0}}{4}\Big(J_{sr}\,J_{nm} - \Big(J_{nr}J_{sm} - J_{mr}J_{sn} + S_{nr}S_{sm} - S_{mr}S_{sn}\Big)\Big) \\ &= \sqrt{\det E}\frac{e^{-\phi_0}}{4}\Big(S_{mr}S_{sn} - S_{nr}S_{sm}\Big)\Big), \end{aligned} \tag{C.36}$$

then

$$\partial_r(\sqrt{\det E}e^{-\phi_0}S_{mr}S_{sn}F_{mn}) = 0, \tag{C.37}$$

but S is diagonal

$$\partial_r(\sqrt{\det E}e^{-\phi_0}S_{rr}S_{ss}F_{rs}) = 0, \tag{C.38}$$

and $F_{rs} = \partial_r A_s - \partial_s A_r = -\partial_s A_r$,

$$\partial_r(\det\sqrt{A}e^{-\phi_0}S_{rr}S_{ss}\partial_s A_r) = 0. \tag{C.39}$$

Now, nothing depends on r (it goes from 0 to 3), so we can discard terms multiplying the whole sum, and obtain

$$S^{rr}\partial_s\partial_r A_r = 0. \tag{C.40}$$

So we can still impose the same conditions of eq 52 in [58],

$$\partial_1 A_1 - \partial_0 A_0 = \partial_2 A_2 + \partial_3 A_3 = 0, \tag{C.41}$$

because $S_{00} = -S_{11}, S_{22} = S_{33}$.

- $A_{2,3}$

$$\partial_r \left[2\left(C^{F}_{rs\,mn} \right) F_{mn} + C^{\chi F}_{rs} \chi \right] = 0 \tag{C.42}$$

where, actually, we really have $\partial_r (C^{FF}_{ksmn} G^{sr} F_{mn})$. So we can separate r in three subset

$$r = \rho, \quad r = \alpha_i \in \Omega_3, \quad r = x_0, \ldots, x_3 \in D_3, \tag{C.43}$$

and the first term of (C.42),

$$2(\partial_\rho (G^{pp} C_{psmn} F_{mn}) + \partial_{\alpha_i} (G^{\alpha_i \alpha_j} C_{\alpha_j smn} F_{mn}) + \partial_{x_i} (G^{x_i x_j} C_{x_j smn} F_{mn})). \tag{C.44}$$

We are going now to obtain for each possible value of r, with the same notation of (D.14), (D.16) and, (D.15),

- $r = p$,

$$C_{psmn} = \sqrt{\det E} \frac{e^{-\phi_0}}{4} \left(S_{mp} S_{sn} - S_{np} S_{sm} \right), \tag{C.45}$$

$$(\partial_\rho (G^{pp} C_{psmn} F_{mn}) = \partial_\rho (G^{pp} \sqrt{\det E} \frac{e^{-\phi_0}}{2} S_{pp} S_{ss} F_{ps}). \tag{C.46}$$

- $r = \alpha_i$

$$\partial_{\alpha_i} (G_{\alpha_i \alpha_j} C^{\alpha_j smn} F_{mn}) = \partial_{\alpha_i} (G^{\alpha_i \alpha_j} \sqrt{\det E} \frac{e^{-\phi_0}}{2} S_{\alpha_j \alpha_j} S_{ss} F_{\alpha_j s}), \tag{C.47}$$

$$\partial_{\alpha_j} (C^{\alpha_j smn} F_{mn}), \tag{C.48}$$

$$C_{rsmn} = \sqrt{\det E} \frac{e^{-\phi_0}}{4} \left(S_{mr} S_{sn} - S_{nr} S_{sm} \right), \tag{C.49}$$

so,

$$\begin{aligned}
&\partial_{\alpha_i}(G_{\alpha_i\alpha_j}\sqrt{\det E}\frac{e^{-\phi_0}}{2}S_{\alpha_j\alpha_j}S_{ss}F_{\alpha_j s}) = \partial_{\alpha_i}(G_{\alpha_i\alpha_j}\sqrt{\det E}\frac{e^{-\phi_0}}{2}\frac{1}{G_{\alpha_j\alpha_j}}\frac{G_2}{G_2^2+B^2}F_{\alpha_j s}) \\
&= \partial_{\alpha_i}(G^{\alpha_i\alpha_j}\sqrt{\det E}\frac{e^{-\phi_0}}{2}\frac{1}{G_{\alpha_j\alpha_j}}\frac{G_{22}}{G_2^2+B^2}\partial_{\alpha_j}A_s) = \partial_{\alpha_j}(G^{\alpha_j\alpha_j}\sqrt{\det(\Omega)\times\kappa_G}\frac{e^{-\phi_0}}{2G_{\alpha_j\alpha_j}}\frac{G_2}{G_2^2+B^2}\partial_{\alpha_j}A_s) \\
&= \sqrt{\kappa_G}\frac{e^{-\phi_0}}{2}\frac{G_2}{G_2^2+B^2}\partial_{\alpha_i}(\sqrt{\det\Omega}\partial_{\alpha_i}A_s) = \sqrt{\kappa_G}\frac{e^{-\phi_0}}{2}\frac{G_2}{G_2^2+B^2}\partial_{\alpha_i}(\sqrt{\det G_{\Omega_3}}G^{ij}\partial_{\alpha_j}A_s).
\end{aligned} \tag{C.50}$$

Since A_s is a component of a vector, and $G^{ij} = G_4^{-1}G_\Omega^{ij}$

$$\sqrt{\det E}\frac{e^{-\phi_0}}{2}\frac{G_2G_4^{-1}}{G_2^2+B^2}\Delta_{\Omega_3}A_s. \tag{C.51}$$

- $r = x_0, x_1$

$$\begin{aligned}
&\partial_i(G_{ij}\sqrt{\det E}\frac{e^{-\phi_0}}{2}S_{jj}S_{ss}F_{\alpha_j s}) = \partial_i(G_{ij}\sqrt{\det E}\frac{e^{-\phi_0}}{2}\frac{1}{G_{jj}}\frac{G_2}{G_2^2+B^2}F_{\alpha_j s}) \\
&= \partial_{\alpha_i}(G^{\alpha_i\alpha_j}\sqrt{\det E}\frac{e^{-\phi_0}}{2}\frac{1}{G_{\alpha_j\alpha_j}}\frac{G_{22}}{G_2^2+B^2}\partial_{\alpha_j}A_s) = \partial_{\alpha_j}(G^{\alpha_j\alpha_j}\sqrt{\det(\Omega)\times\kappa_G}\frac{e^{-\phi_0}}{2G_{\alpha_j\alpha_j}}\frac{G_2}{G_2^2+B^2}\partial_{\alpha_j}A_s) \\
&= \sqrt{\kappa_G}\frac{e^{-\phi_0}}{2}\frac{G_2}{G_2^2+B^2}\partial_{\alpha_i}(\sqrt{\det\Omega}\partial_{\alpha_i}A_s) = \sqrt{\kappa_G}\frac{e^{-\phi_0}}{2}\frac{G_2}{G_2^2+B^2}\partial_{\alpha_i}(\sqrt{\det G_{\Omega_3}}G^{ij}\partial_{\alpha_j}A_s),
\end{aligned} \tag{C.52}$$

and so,

$$\partial_{x_i}(G^{x_ix_j}C_{x_jsmn}F_{mn}) = \sqrt{\det E}\frac{e^{-\phi_0}}{2}\frac{G_2G_1^{-1}}{G_2^2+B^2}\Delta_{0,1}A_s. \tag{C.53}$$

But for x_2, x_3,

$$\begin{aligned}
&\partial_{x_i}(G^{x_jx_j}\sqrt{\det E}\frac{e^{-\phi_0}}{2}S_{x_jx_j}S_{ss}F_{x_js}) = \partial_{x_i}(G_{x_jx_j}\sqrt{\det E}\frac{e^{-\phi_0}}{2}\frac{G_{x_jx_j}}{G_{x_jx_j}^2+B^2}\frac{G_2}{G_2^2+B^2}F_{x_js}) \\
&= \partial_{x_j}(\sqrt{\det E}\frac{e^{-\phi_0}}{2}\frac{G_{x_jx_j}^2}{G_{x_jx_j}^2+B^2}\frac{G_2}{G_2^2+B^2}\partial_{x_j}A_s) = \sqrt{\det E}\frac{e^{-\phi_0}}{2}\frac{G_2}{G_2^2+B^2}\Big(\frac{G_2}{G_2+B^2}\Delta_{1,2}A_s\Big).
\end{aligned} \tag{C.54}$$

Now, we add all the terms,

$$\begin{aligned}
&\partial_\rho(G^{pp}\sqrt{\det E}e^{-\phi_0}S_{pp}S_{ss}\partial_\rho A_s) + \sqrt{\det E}e^{-\phi_0}\frac{G_2G_4^{-1}}{G_2^2+B^2}\Delta_{\Omega_3}A_s + \sqrt{\det E}e^{-\phi_0}\frac{G_2G_1^{-1}}{G_2^2+B^2}\Delta_{0,1}A_s \\
&+\sqrt{\det E}e^{-\phi_0}\frac{G_2}{G_2^2+B^2}\Big(\frac{G_2}{G_2+B^2}\Delta_{1,2}A_s\Big) + \partial_r(C_{rs}^{\chi F}\chi) = 0
\end{aligned} \tag{C.55}$$

- $A_{0,1}$

$$\frac{1}{4}\partial_r\left[2\left(C^F_{rs\,mn}\right)F_{mn}+\frac{1}{\sqrt{\det E}}\left(\partial_\rho(K(p)B_{23})(\delta_{0r}\delta_{1s}-\delta_{0s}\delta_{1r})\sin 2\psi\right)\Phi\right]=0. \tag{C.56}$$

Where, actually, we really have $\partial_r(C^{FF}_{ksmn}G^{sr}F_{mn})$. So we can separate r in three subset again

$$r=\rho, \quad r=\alpha_i\in\Omega_3, \quad r=x_0,\ldots,x_3\in D_3, \tag{C.57}$$

obtaining

$$2(\partial_\rho(G^{pp}C_{psmn}F_{mn})+\partial_{\alpha_i}(G^{\alpha_i\alpha_j}C_{\alpha_j smn}F_{mn})+\partial_{x_i}(G^{x_ix_j}C_{x_jsmn}F_{mn})). \tag{C.58}$$

So, for

- $r=p$,

$$C_{psmn}=\sqrt{\det E}\frac{e^{-\phi_0}}{4}\left(S_{mp}S_{sn}-S_{np}S_{sm}\right), \tag{C.59}$$

so

$$(\partial_\rho(G^{pp}C_{psmn}F_{mn})=\partial_\rho(G^{pp}\sqrt{\det E}\frac{e^{-\phi_0}}{2}S_{pp}S_{ss}F_{ps}). \tag{C.60}$$

- $r=\alpha_i$

$$\partial_{\alpha_i}(G_{\alpha_i\alpha_j}C^{\alpha_j smn}F_{mn})=\partial_{\alpha_i}(G^{\alpha_i\alpha_j}\sqrt{\det E}\frac{e^{-\phi_0}}{2}S_{\alpha_j\alpha_j}S_{ss}F_{\alpha_j s}), \tag{C.61}$$

so

$$\partial_{\alpha_i}(G^{\alpha_j\alpha_j}\sqrt{\det E}\frac{e^{-\phi_0}}{2}S_{\alpha_j\alpha_j}S_{ss}F_{\alpha_j s})=\sqrt{\kappa_G}\frac{e^{-\phi_0}}{2}\frac{1}{G_1}\partial_{\alpha_i}(\sqrt{\det G_{\Omega_3}}G^{ij}\partial_{\alpha_j}A_s), \tag{C.62}$$

but since A_s is a component of a vector,

$$\sqrt{\det E}\frac{e^{-\phi_0}}{2}\frac{1}{G_1G_4}\Delta_{\Omega_3}A_s. \tag{C.63}$$

- $r=x_0,x_1$

$$\partial_{x_i}(G^{x_ix_j}C_{x_jsmn}F_{mn})=\sqrt{\det E}\frac{e^{-\phi_0}}{2}\frac{1}{G_1^2}\Delta_{0,1}A_s, \tag{C.64}$$

but for x_2, x_3

$$\partial_{x_i}(G^{x_j x_j}\sqrt{\det E}\frac{e^{-\phi_0}}{2}S_{x_j x_j}S_{ss}F_{x_j s}) = \sqrt{\det E}\frac{e^{-\phi_0}}{2}\frac{1}{G_1}\Big(\frac{G_2^2}{G_2^2+B^2}\Delta_{1,2}A_s\Big). \tag{C.65}$$

Now, we add to the equation of motion, and use $A_p = 0$,

$$\partial_\rho(G^{pp}\sqrt{\det E}\frac{e^{-\phi_0}}{2}S_{pp}S_{ss}\partial_\rho A_s) + \sqrt{\det E}e^{-\phi_0}\frac{1}{G_1 G_4}\Delta_{\Omega_3}A_s + \sqrt{\det E}e^{-\phi_0}\frac{1}{G_1^2}\Delta_{0,1}A_s$$
$$+\sqrt{\det E}e^{-\phi_0}\frac{1}{G_1}\Big(\frac{G_2^2}{G_2^2+B^2}\Delta_{3,4}A_s\Big) + \partial_r\Bigg(\Bigg(\partial_\rho(K(p)B_{23})(\delta_{0r}\delta_{1s}-\delta_{0s}\delta_{1r})\sin 2\psi\Bigg)\Phi\Bigg) = 0. \tag{C.66}$$

D Action for fluctuations

While the previous appendix was dedicated to organizing the terms in the action, the explicit derivation of them is made in the present appendix, complemeneting C on obtaining the equation of motion for the fluctuations.

Action for fluctuation terms around the classical solution

We now allow for fluctuations over the $l(p)$ and Φ (the coordinate, not dilaton) classical solution of the type,

$$l = l_0 + \alpha'\chi,\ \Phi = \alpha'\Phi. \tag{D.1}$$

First we consider the fluctuations (D.1) on E

$$E \to E_0 + \alpha'\delta_1 + \alpha'^2\delta_2. \tag{D.2}$$

Ignoring the explicit form of $\delta_{1,2}$ for while, the NS-NS part of the DBI action is

$$\sqrt{\det(E+\alpha' F)} = \sqrt{\det(E_0+\alpha'(F+\delta_1)+\alpha'^2\delta_2)} = \sqrt{\det E_0}\sqrt{\det(1+E_0^{-1}(\alpha'(F+\delta_1)+\alpha'^2\delta_2)}). \tag{D.3}$$

However, we also need to consider fluctuations of the dilaton ϕ, since it is no longer constant after the TsT deformation. We can write the expansion as

$$e^{-\phi} = \beta_0 + \beta_1\alpha'\chi + \beta_2\alpha'^2\chi^2, \tag{D.4}$$

where $\beta_0 = e^{-\phi}\Big|_{\chi=0} = e^{-\phi_0}, \beta_1 = \partial_{l_0} e^{-\phi_0}, \beta_2 = \frac{1}{2}\frac{\partial^2 e^{-\phi_0}}{\partial l_0^2}$. To calculate $\sqrt{\det(1 + c_1\alpha' + c_2\alpha'^2)}$ perturbatively, which is the expression we have in D.3, we obtain

$$\sqrt{\det(1+\alpha'\, c_1 + \alpha'^2\, c_2)} \approx 1 + \frac{1}{2}\alpha'\, \mathrm{tr}(c_1) + \alpha'^2 \left(\frac{1}{2}\mathrm{tr}(c_2) - \frac{1}{4}\mathrm{tr}(c_1^2) + \frac{1}{8}(\mathrm{tr} c_1)^2 \right), \tag{D.5}$$

and combining with (D.4), the NSNS part of the DBI action becomes

$$\begin{aligned} &\int \sqrt{\det E_0}\left(\beta_0 + \beta_1\alpha'\chi + \beta_2\alpha'^2\chi^2\right)\left(1 + \frac{\alpha'}{2}\mathrm{tr}(E_0^{-1}(F+\delta_1)) + \right. \\ &\left. \frac{\alpha'^2}{8}\left[4\mathrm{tr}(\delta_2) - 2\mathrm{tr}(E_0^{-1}(F+\delta_1)E_0^{-1}(F+\delta_1)) + (\mathrm{tr}(E_0^{-1}(F+\delta_1))^2\right]\right). \end{aligned} \tag{D.6}$$

The WZ term of the action is, on the other hand,

$$\int C_6 \wedge B + \alpha' \left(C_6 \wedge F_2 + \tilde{C}_4 \wedge B_2 \wedge F_2\right) + \frac{\alpha'^2}{2}\left(C_4 \wedge F_2 \wedge F_2 + C_2 \wedge F_2 \wedge F_2 \wedge B_2\right). \tag{D.7}$$

Then, by collecting terms of the full action by α' order,

$$\begin{aligned} &\int \\ &\beta_0\sqrt{\det E_0} + C_6 \wedge B \\ &+\alpha'\left(C_6 \wedge F_2 + \sqrt{\det E_0}\beta_1\chi + \frac{\beta_0}{2}\sqrt{\det E_0}\mathrm{tr}(E^{-1}(F+\delta_1))\right) \\ &+\alpha'\tilde{C}_4 \wedge B_2 \wedge F_2 + \alpha'^2\frac{1}{2}\left(C_4 \wedge F_2 \wedge F_2 + C_2 \wedge F_2 \wedge F_2 \wedge B_2\right) \\ \alpha'^2\Big[\beta_0 &\left(\sqrt{\det E_0}\frac{1}{8}\left[4\mathrm{tr}(\delta_2) - 2\mathrm{tr}(E_0^{-1}(F+\delta_1)E_0^{-1}(F+\delta_1)) + (\mathrm{tr}E_0^{-1}(F+\delta_1))^2\right]\right) \\ &+\frac{\beta_1\chi}{2}\sqrt{\det E_0}\mathrm{tr}(E_0^{-1}(F+\delta_1)) + \beta_2\chi^2\sqrt{\det E_0}\Big]. \end{aligned} \tag{D.8}$$

The zeroth order terms were useful to obtain the chiral symmetry breaking and the equation of motion, but we do not use them to obtain the fluctuations, and therefore they are not required from now on. Linear terms in the perturbation around the classical solution either vanishes by the stationary action principle or arises as a constraint (first order formalism in F).

We recall that we must check whether this is a consistent background for embedding our D7-brane. This consistency is verified mainly through the equations of motion of the DBI+WZ action, which are essentially the non-abelian gauge field equations of motion for F_2, acting as a constraint, together with the C_6 equation of motion, as explained in [58]. Since we again have a D7 probe, the situation is analogous to the undeformed case, with the only linear term acting on F being C_6. We still have the α' constraint,

$$\epsilon^{abm_1\ldots m_6}\frac{\partial_a C_{m_1\ldots m_6}}{6!} = -\partial_a(\beta_0\sqrt{E}E^{[ab]}). \tag{D.9}$$

The right-hand side vanishes for our deformed ansatz because nothing depends on $x_{2,3}$, and therefore we have $dP[C_6] = 0$.

At the same time we have the equation of motion resulting from $S_{\text{sugra}} + S_{\text{WZ}}$ with solution near the boundary of the brane

$$\partial_L(\sqrt{-G}dC_6^{L01\psi\alpha\beta}) = -\frac{\mu_7\kappa_0^2}{\pi}B(\rho)\delta(L-L_0). \tag{D.10}$$

Since $\partial_x\Theta(x) = \delta(x)$, we can obtain a solution analogous to the one in [58], by calling $C^6_{01L\psi\alpha\beta} \propto f(\rho)$ such that

$$f_7^{L01\rho\psi\gamma\beta} = -\frac{1}{\sqrt{-G}}\frac{\mu_7^2\kappa_0^2}{\pi}\beta_0 B(p)\Theta(l-l_0), \tag{D.11}$$

where we called $f_7 = dC_6$. We could use the metric to down the indices of f_7, and by substituting in the end $k = 0$, see the result boils down to the one in [58].

The rest of the terms at order α' involves only the fields, and its fluctuations, in the zeroth-order equation of motion, and therefore they are the terms that vanish in the extreme principle of Action. Consequently, we can ignore the $O(\alpha')$ contribution for the action. Therefore, the relevant part of the action for fluctuations are

$$\begin{aligned}S = \int \alpha'^2\sqrt{\det E}(\beta_2\chi^2 + \beta_1\frac{\chi}{2}\text{tr}(E^{-1}(F+\delta_1)) + \beta_0\frac{1}{8}\Big[4\text{tr}(\delta_2) - 2\text{tr}\Big(E^{-1}(F+\delta_1)E^{-1}(F+\delta_1)\Big)\\ +(\text{tr}E^{-1}(F+\delta_1))^2\Big]) + +\alpha'\tilde{C}_4\wedge B_2\wedge F_2 + \frac{\alpha'^2}{2}\left(C_4\wedge F_2\wedge F_2 + C_2\wedge F_2\wedge F_2\wedge B_2\right).\end{aligned} \tag{D.12}$$

- **NSNS part**

Explicitly, the metric and the B field we have is

$$\begin{aligned}G_{\text{diagonal}} &= (\rho^2+l^2)\Big(-dt^2+dx_1^2+\frac{1}{(1+Hk)^2+k^2(\rho^2+l^2)^2}\left(dx_2^2+dx_3^2\right)\Big)+\frac{R^2}{\rho^2+l^2}((1+l'^2)d\rho^2+\rho^2 d\Omega_3^2))\\ B_{23} &= \frac{H+H^2k+k(\rho^2+l^2)^2}{((1+Hk)^2+k^2(\rho^2+l^2)^2)}.\end{aligned} \tag{D.13}$$

From now on, however, to simplify the presentation of the calculations we adopt the following notation,

$$G = G_1(p)(-dt^2 + dx_3^2) + G_2(p)(dx_1^2 + dx_2^2) + G_3(p)d\rho^2 + G_4(p)d\Omega_3^2, \quad \text{(D.14)}$$

and therefore, the determinant of $E = G + B$ is

$$\det E = G_1^2 G_2^2 G_4^3 \left(G_3 \det(i) \det(\mu) \det(\Omega) + G_3 B_{23}^2 \det(\mu) \det(\Omega) \right), \quad \text{(D.15)}$$

while the inverse metric is

$$G^{-1} = G_1(p)^{-1}(-dt^2 + dx_3^2) + \frac{G_2(p)}{G_2(p)^2 + B_{23}^2}(dx_1^2 + dx_2^2) + G_3(p)^{-1}d\rho^2 + G_4(p)^{-1}d\Omega_3^2. \quad \text{(D.16)}$$

Proceeding, since G is diagonal and B is antisymmetric with only one non-zero component, B_{23}, it can be shown [8] that the inverse of E also satisfies this property and can be decomposed into a diagonal part plus an antisymmetric term with only one non-zero component. That is,

$$E_0^{-1} = S + J, \quad \text{(D.17)}$$

where S is diagonal and J is antisymmetric, and therefore

$$(\text{tr} E_0^{-1}(F + \delta_1)) = \text{tr} S\delta_1 + \text{tr} JF. \quad \text{(D.18)}$$

However, since B is not constant, the analysis is a bit different from [58] because δ_1 is not totally symmetric anymore (because $\partial_l B \neq 0$). In that case, we can split it as

$$\delta_1 = \delta_1^S + \delta_1^A, \quad \text{(D.19)}$$

or explicitly

$$\delta_{nm}^1 = \underbrace{\left(\frac{\partial G_{nm}}{\partial l}\Big|_{\chi=0} \right) \chi + \frac{1}{\rho^2 + l_0^2}\left(\partial_\rho l_0 \left(\delta_{mp}\delta_{ni} + \delta_{np}\delta_{mi}\right) \partial_i \chi\right)}_{\delta_1^S} + \underbrace{\left(\frac{\partial B_{nm}}{\partial l}\Big|_{\chi=0} \right) \chi}_{\delta_1^A}. \quad \text{(D.20)}$$

We can then combine $\delta_1^A + F = \mathcal{F}$ to reduce the term (D.18) to a purely symmetric part plus the product of two antisymmetric matrices

$$\text{tr} E_0^{-1}(F + \delta_1) = \text{tr}(S\delta_1^S) + \text{tr}(J\mathcal{F}). \quad \text{(D.21)}$$

Now, for $\mathrm{tr}(E_0^{-1}(F+\delta_1)E_0^{-1}(F+\delta_1))$, we have

$$\mathrm{tr}(E_0^{-1}(F+\delta_1)E_0^{-1}(F+\delta_1)) = \mathrm{tr}(S\delta_1^S S\delta_1^S) + \mathrm{tr}(S\mathcal{F}S\mathcal{F}) + \mathrm{tr}(J\delta_1^S J\delta_1^S) + 4\mathrm{tr}(J\delta_S^1 S\mathcal{F}) + \mathrm{tr}(J\mathcal{F}J\mathcal{F}), \tag{D.22}$$

and therefore, substituting for β in the action,

$$\begin{aligned} S = \int \frac{\sqrt{\det E_0}}{2} \Bigg(\frac{\partial^2 e^{-\phi_0}}{\partial l_0^2}\chi^2 + \frac{\partial e^{-\phi_0}}{\partial l_0}(\mathrm{tr}(\delta_1^S S + J\mathcal{F}))\chi + \\ \frac{e^{-\phi_0}}{4}\Bigg[4\,\mathrm{Tr}(\delta_2) - 2\Big(\mathrm{Tr}(S\,\delta_1^S\,S\,\delta_1^S) + \mathrm{Tr}(S\,\mathcal{F}\,S\,\mathcal{F}) + \mathrm{Tr}(J\,\delta_1^S\,J\,\delta_1^S) \\ +4\,\mathrm{Tr}(J\,\delta_1^S\,S\,\mathcal{F}) + \mathrm{Tr}(J\,\mathcal{F}\,J\,\mathcal{F})\Big) + \Big(\mathrm{Tr}(S\,\delta_1^S)^2 + 2\mathrm{Tr}(S\,\delta_1^S)\mathrm{Tr}(J\,\mathcal{F}) + \mathrm{Tr}(J\,\mathcal{F})^2\Big)\Bigg]\Bigg). \end{aligned} \tag{D.23}$$

We can calculate each term explicitly, and this is done in Appendices C. There, we show the "surviving terms" are the coefficients of the kinetic terms F^2, $(\partial\chi)^2$, $(\partial\Phi)^2$, the interaction term χF, and the "mass term" for χ, χ^2. Under these conditions, the NSNS action can therefore be rewritten as

$$S = \int \frac{1}{8}(C_{\chi^2}\chi^2 + C^F_{mnjr}F_{mn}F_{jr} + C^{(\partial x)^2}(\partial\chi)^2 + C^{\chi F}_{nm}\chi F_{nm} + C^{(\partial\phi)^2}(\partial\phi)^2), \tag{D.24}$$

where $\det E$ has been absorbed in the coefficients of the Lagrangian, $C = \sqrt{\det E}\tilde{C}$.

• RR part

The action must now be supplemented by the equations arising from the R-R fields, which, as we have seen, at order α'^2 are

$$S = \alpha'\tilde{C}_4 \wedge B_2 \wedge F_2 + \frac{\alpha'^2}{2}\left(C_4 \wedge F_2 \wedge F_2 + C_2 \wedge F_2 \wedge F_2 \wedge B_2\right), \tag{D.25}$$

since the new fields C_2 is only present at α'^2, and D_6, C_6 is only present at α', the results of the calculations are the same as in [58] but with a term $(1+Hk)$ multiplying it, as explained in Section 7.3,

$$\tilde{C}^4 - \frac{\alpha'}{g_s}\frac{\sin 2\psi}{2}K(\rho)\,\partial_a\Phi\, d\psi \wedge d\gamma \wedge d\beta \wedge dx^a \tag{D.26}$$

where,

$$K(\rho) = -R^4 L_0^2 \frac{2\rho^2 + L_0^2}{(\rho^2 + L_0^2)^2}(1+Hk). \tag{D.27}$$

In [58] they used the Bianchi identity $dF = 0$ together with the fact that $\partial B = 0$. The last is no longer true, however. But this is not a problem if we keep $K(\rho)B$ together, as we are going to show. The first term of (D.25), for $F = F_{mn}\, dx^m \wedge dx^n$, is

$$
\begin{aligned}
&- \int \frac{\alpha'^2}{g_s} \frac{\sin 2\psi}{2} K(\rho)\, \partial_a \Phi B_{23} F_{mn}\, d\psi \wedge d\alpha \wedge d\beta \wedge dx^2 \wedge dx^3 \wedge dx^a \wedge dx^m \wedge dx^n = \\
&- \int \frac{\alpha'^2}{g_s} \frac{\sin 2\psi}{2}\, \Phi \Big(\partial_a (K(\rho) B_{23}) F_{mn} + K(\rho) B_{23} \partial_a F_{mn} \Big)\, d\psi \wedge d\alpha \wedge d\beta \wedge dx^2 \wedge dx^3 \wedge dx^a \wedge dx^m \wedge dx^n,
\end{aligned} \tag{D.28}
$$

and then, using the Bianchi identity for F in the last term, $\partial_{[a} F_{mn]} = 0$, and that k and B only depends on ρ, we obtain

$$
\begin{aligned}
- \frac{\alpha'^2}{g_s} \frac{\sin 2\psi}{2}\, \Phi \partial_\rho (K(\rho) B_{23}) F_{01}\, d\psi \wedge d\alpha \wedge d\beta \wedge dx^2 \wedge dx^3 \wedge d\rho \wedge dx^0 \wedge dx^1 = \\
\int d^8 \xi \frac{\alpha'^2}{g_s} \frac{\sin 2\psi}{2}\, \Phi \partial_\rho (K(\rho) B_{23}) F_{01}.
\end{aligned} \tag{D.29}
$$

The second and third term from (D.25) can be treated together since they are of order F^2. Moreover, because C_4 and $C_2 \wedge B_2$ have components along the D3-brane world-volume, the structures involving these terms are identical, and we can write it as

$$
\frac{\alpha'^2}{8 g_s} \int d^8 \Xi \chi(p) F_{ab} F_{cd} \epsilon^{abcd}, \tag{D.30}
$$

where we defined $\Xi(p) = C^4_{0123}(p) + C_{01}(p) B_{23}(p)$. Also, the indices in (D.30) are over transverse coordinates to D3-brane, $\gamma, \psi, \gamma, \phi$. The action is

$$
S_{RR} = \frac{\alpha'^2}{8 g_s} \int d^8 \xi \left[\chi(p) F_{ab} F_{cd} \epsilon^{abcd} - 2 \sin 2\psi \Phi \partial_\rho (K(p) B_{23}) \delta_{0a} \delta_{1b} F_{ab} \right], \tag{D.31}
$$

which contributes for the Φ equation and for dA. Notice that the first term above only contributes if index $a = \rho, \psi, \beta, \gamma$ and the second only if $a = 0, 1$.

Equation of motion of the fluctuations

We now calculate the equation of motion for the fluctuations. The interested reader can look at the appendices D, where all calculations are open and made explicitly.

- χ

The χ equation of motion from (D.24) is

$$2C_{\chi^2}\chi + C^{\chi F}_{nm}F_{nm} - \partial_i\left(2C^{(\partial\chi)^2}\,\partial_i\chi\right) = 0 \tag{D.32}$$

with

$$C = \sqrt{(\det E)}\tilde{C}, \tag{D.33}$$

and $\tilde{C}$ was obtained in Appendix C.

- Φ

The equation of motion is derived in detail in the appendices C. There, we start with

$$\frac{1}{4}\partial_i\left(C^{(\partial\Phi)^2}\,\partial_i\phi\right) - \frac{\sin 2\psi}{4}\partial_\rho(K(p)B_{23})\delta_{0a}\delta_{1b}F_{ab} = 0, \tag{D.34}$$

and, after some manipulation, we can write is more appropriately,

$$\begin{aligned} &\partial_\rho(\sqrt{\det E}\tilde{C}^{(\partial\phi)^2}G_3^{-1}\partial_\rho\Phi) \\ +\sqrt{\det E}\tilde{C}^{(\partial\phi)^2}&\left(G_1^{-1}(p)\Delta_\eta\Phi + \frac{G_2(p)}{G_2(p)^2+B_{23}^2}\Delta_\delta\Phi + G_4(p)^{-1}\Delta_\Omega\Phi\right) - \\ &\sin 2\psi\partial_\rho(K(p)B_{23})F_{01} = 0. \end{aligned} \tag{D.35}$$

Notice that the equations are almost the same as in the original paper from [58], but there $\sqrt{\det E}$ is called g, and B appears always outside $\partial(KB_{23})$. Also, the Δ_{M_j} is the d'Alambertian/Laplacian in the M_j space, so

$$\begin{aligned} \Delta_\eta = -\partial_0^2 + \partial_1^2,\ \ \Delta_\delta = \partial_2^2 + \partial_3^2, \\ \Delta_{\Omega_3} = \quad \text{Laplacian in spherical harmonic.} \end{aligned} \tag{D.36}$$

Finally, we rewrite it as,

$$\frac{1}{\sqrt{\det E}}\partial_\rho(\sqrt{\det E}\tilde{C}^{(\partial\phi)^2}G_3^{-1}\partial_\rho\Phi) + \tilde{C}^{(\partial\phi)^2}\left(G_1^{-1}(p)\Delta_\eta\Phi + \frac{G_2(p)}{G_2(p)^2+B_{23}^2}\Delta_\delta\Phi + G_4(p)^{-1}\Delta_\Omega\Phi\right) - \frac{\sin 2\psi}{\sqrt{\det E}}\partial_\rho(K(p)B_{23})F_{01} = 0. \tag{D.37}$$

- A_s

First, we want to impose the constraints $A_\mu = 0$, where μ is for coordinates not into the D3-brane world-volume (so $\mu = \psi, \beta, \gamma, \rho$) and i is an index for the whole spacetime. Moreover, we gauge fix $\partial \cdot A = 0$.

- $s \in \psi, \beta, \gamma, \rho$

  The details of the derivation are in C. The equation of motion for the perpendicular coordinates of the vector to the brane are

$$S^{rr}\partial_s\partial_r A_r = 0, \tag{D.38}$$

which can be seen as imposing the following constraints in the vector components along the world-volume of the D3-brane,

$$\partial_1 A_1 - \partial_0 A_0 = \partial_2 A_2 + \partial_3 A_3 = 0. \tag{D.39}$$

These are exactly the same conditions as in [58].

- $s \in 2, 3$

  Again, the details are in C. We obtain

$$\begin{aligned}\partial_\rho(G^{pp}\sqrt{\det E}e^{-\phi_0}S_{pp}S_{ss}\partial_\rho A_s) + \sqrt{\det E}e^{-\phi_0}\frac{G_2G_4^{-1}}{G_2^2+B^2}\Delta_{\Omega_3}A_s + \sqrt{\det E}e^{-\phi_0}\frac{G_2G_1^{-1}}{G_2^2+B^2}\Delta_{0,1}A_s \\ +\sqrt{\det E}e^{-\phi_0}\frac{G_2}{G_2^2+B^2}\left(\frac{G_2}{G_2+B^2}\Delta_{1,2}A_s\right) + \partial_r(C^{\chi F}_{rs}\chi) = 0,\end{aligned} \tag{D.40}$$

but since $G_{pp} = S^{-1}_{pp}$,

$$\begin{aligned}\frac{1}{\sqrt{\det E}}\partial_\rho\left(e^{-\phi_0}\frac{G_2}{G_2^2+B^2}\partial_\rho A_s\right) + e^{-\phi_0}\frac{G_2G_4^{-1}}{G_2^2+B^2}\Delta_{\Omega_3}A_s + e^{-\phi_0}\frac{G_2G_1^{-1}}{G_2^2+B^2}\Delta_{0,1}A_s \\ +e^{-\phi_0}\frac{G_2}{G_2^2+B^2}\left(\frac{G_2}{G_2+B^2}\Delta_{1,2}A_s\right) + \frac{1}{\sqrt{\det E}}\partial_r(C^{\chi F}_{rs}\chi) = 0.\end{aligned} \tag{D.41}$$

- $s \in 0, 1$

  The details are in section C,

$$\begin{aligned}\partial_\rho(G^{pp}\sqrt{\det E}e^{-\phi_0}S_{pp}S_{ss}\partial_\rho A_s) + \sqrt{\det E}e^{-\phi_0}\frac{1}{G_1G_4}\Delta_{\Omega_3}A_s + \sqrt{\det E}e^{-\phi_0}\frac{1}{G_1^2}\Delta_{0,1}A_s \\ +\sqrt{\det E}e^{-\phi_0}\frac{1}{G_1}\left(\frac{G_2^2}{G_2^2+B^2}\Delta_{1,2}A_s\right) + \partial_r\left(\left(\partial_\rho(K(p)B_{23})(\delta_{0r}\delta_{1s} - \delta_{0s}\delta_{1r})\sin 2\psi\right)\Phi\right) = 0.\end{aligned} \tag{D.42}$$

So we have the same case as before,

$$\frac{1}{\sqrt{\det E}}\partial_\rho(e^{-\phi_0}\frac{1}{G_1}\partial_\rho A_s) + e^{-\phi_0}\frac{1}{G_1 G_4}\Delta_{\Omega_3}A_s + e^{-\phi_0}\frac{1}{G_1^2}\Delta_{0,1}A_s$$
$$+e^{-\phi_0}\frac{1}{G_1}\left(\frac{G_2^2}{G_2^2+B^2}\Delta_{1,2}A_s\right) + \frac{1}{\sqrt{\det E}}\partial_r\left(\partial_\rho(K(p)B_{23})(\delta_{0r}\delta_{1s}-\delta_{0s}\delta_{1r})\sin 2\psi\Phi\right) = 0 \tag{D.43}$$

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