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Generalized Riemann-Hilbert-Birkhoff decomposition and a new class of higher grading integrable hierarchies

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ABSTRACT

We propose a generalized Riemann-Hilbert-Birkhoff decomposition that expands the standard integrable hierarchy formalism in two fundamental ways: it allows for integer powers of Lax matrix components in the flow equations to be increased as compared to conventional models, and it incorporates constant non-zero vacuum (background) solutions.

Two additional parameters control these features. The first one defines the grade of a semisimple element that underpins the algebraic construction of the hierarchy, where a grade-one semi-simple element recovers known hierarchies such as mKdV and AKNS. The second parameter distinguishes between zero and non-zero constant background (vacuum) configurations.

Additionally, we introduce a third parameter associated with an ambiguity in the definition of the grade-zero component of the dressing matrices. While not affecting the decomposition itself, this parameter classifies different gauge realizations of the integrable equations (like for example, Kaup-Newell, Gerdjikov-Ivanov, Chen-Lee-Liu models).

For various values of these parameters, we construct and analyze corresponding integrable models in a unified universal manner demonstrating the broad applicability and generative power of the extended formalism.

1. Introduction

The algebraic formalism offers a powerful framework to construct and to classify integrable hierarchies. It rests on a zero curvature relations solved by Lie algebra valued gauge potentials (Lax operators) obtained via recursive considerations [1], [2], [3], [4], [5]. A direct consequence of these relations is the existence of infinite number of conservation laws that are responsible for presence and stability of soliton solutions. Soliton solutions are very special solutions that preserve their form and interact elastically with other solitons. In general, non-trivial solitons are constructed from a basic zero vacuum solution by gauge transformation (dressing transformation) [6], e.g. Korteweg-de Vries (KdV) and modified Korteweg-de Vries (mKdV) equations. Interestingly, there is also a class of non-linear equations, that only admits strictly non-zero constant vacuum solutions (see for instance [8]). This fact defines two distinct orbits, zero and non-zero constant vacuum, that will be embodied into a single formalism we shall propose.

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Table 1
Models obtained from g-RHB decomposition with $\hat{\mathcal{G}} = \hat{s}_2$
and principal gradation.

a	c	Hierarchy
$a = 1$	$0 \leq c \leq 1$	mKdV
$a = 2$	$c = 0$	GI
$a = 2$	$c = 1/2$	CLL
$a = 2$	$c = 1$	KN
$a = 3$	$0 \leq c \leq 1$	v_1, v_2, v_3 hierarchy of section 5

It is well-known that construction of commutative (positive and negative) flows satisfying zero curvature relations follows straightforwardly from the Riemann-Hilbert-Birkhoff (RHB) formalism [9,10]. It is based upon an affine Lie algebraic structure with decomposition into graded subspaces induced by a grading operator Q (see for instance [11]). In this setting we explore various ambiguities related to formulation of the underlying Heisenberg algebra in the graded environment. One ambiguity is connected with a freedom to define the decomposition formalism in terms of every second grade or every third grade and so on, rather than the conventional every neighboring grade (see discussion below equation (3.9) for explicit realization and discussion of this idea). Additionally there also exists an ambiguity that accounts for freedom of adding extra terms to the semisimple element generating the flows in the RHB formalism without a change to the Heisenberg algebra it satisfies but giving raise to the non-zero constant vacuum solution. Recognizing these ambiguities we here generalize the RHB decomposition by augmenting it with additional structures that are controlled by new a and b parameters. The parameter a enters directly the definition of a semi-simple element that is a starting block of the decomposition formalism and effectively increases the powers of fields entering the flow equations while b switches on and off the non-zero vacuum solution. The models with $a > 1$ will be referred to as higher grading hierarchies. We will refer to the generalized Riemann-Hilbert-Birkhoff decomposition by the abbreviation g-RHB.

In general, solitons are constructed by applying dressing transformation to the zero vacuum solution [6,7]. In practice this is accomplished algebraically by gauge transforming the two dimensional gauge potential from the vacuum configuration to some non-trivial configuration. The key point in our observation is that some hierarchies present more than one type of vacuum solution, i.e. zero or constant non-zero vacuum, e.g. mKdV, Chen-Lee-Liu model, etc. Each type of such solution is characterized by a Heisenberg algebra underlying the two dimensional gauge potential in vacuum, each having its own different Heisenberg algebra. The Heisenberg algebra, in turn, depends upon the vacuum parameters controlled here by a parameter b . The soliton solutions are therefore classified by orbits generated from vacuum solutions and characterized by their own different Heisenberg algebras. Related to the presence of the b parameter it is worthwhile to point out that in the reference [8] the negative even-graded sub-hierarchy of mKdV was proposed and its solitons were constructed from a strictly non-zero vacuum solution. This was later extended to the KdV hierarchy by Miura transformation and an interesting degeneracy of zero and non-zero vacuum solutions was pointed out in [12].

In addition, we also employ a new parameter c that, reflects ambiguity of the zero-grade components of the dressing matrices and realizes different gauge formulations of the underlying flow equations. Gauge equivalence via Miura transformations connecting different splitting of subalgebras associated with the zero grade ambiguity has been observed in the KP hierarchy treated by Adler-Kostant-Symes R -matrix scheme [13].

In this paper we show that Kaup-Newell, Gerdjikov-Ivanov and Chen-Lee-Liu hierarchies, respectively, can be derived directly from the g-RHB formalism for the value of the parameter a equal to $a = 2$ and for the appropriate values of the parameter c . Although their flow equations are related to each other through gauge ambiguity classified by the parameter c , remarkably the Chen-Lee-Liu hierarchy is the only one that admits non-zero constant vacuum solutions. In the related forthcoming publication [14] the phenomenon of strictly non-zero vacuum will be shown to appear within the Chen-Lee-Liu hierarchy. In particular in [14] it will be shown that the constant non-zero solitons solutions can be employed systematically in order to reduce the Chen-Lee-Liu hierarchy into the Burgers hierarchy.

The g-RHB formalism allows for a general construction of integrable hierarchies with both homogeneous and principal gradations but here we will focus on producing higher grading hierarchies models with principal gradation for which we are also able to study the new types of vacuum structures.

The paper is organized as follows.

The background material is presented in section 2, where we show how zero curvature relations are employed within algebraic approach to derive the commuting time evolution equations for both positive and negative flows. In section 3, the algebraic construction of section 2 is embedded in a formalism of g-RHB decomposition of an affine Lie algebra. This setting allows generalizations to “higher grading” integrable hierarchies in a unified framework that is controlled by the three parameters a, b and c , which we introduce to describe various models and their realizations. Section 4 presents integrable hierarchies obtained by construction presented in Section 3.

The models that are characterized by principal gradation and $a = 1, 2$ are presented in subsections 4.1 and 4.2. The higher gradation model with $a = 3$ is given in section 5. These results are summarized in Table 1 that lists principal gradation models obtained from the g-RHB decomposition formula (3.1) corresponding to the choice of algebra $\hat{\mathcal{G}}$ (here fixed to be $\hat{s}_2$), and the values of the parameters a and c .

In the above table we have used the following well-known integrable equations denoting them by their well-established abbreviations: modified Korteweg-de Vries (mKdV), Kaup-Newell (KN), Chen-Lee-Liu (CLL) and Gerdjikov-Ivanov (GI) that will be used consistently throughout the rest of this paper.

In addition, it is worthwhile to point out that the well-known homogeneous $\hat{sl}_2$ hierarchies Ablowitz-Kaup-Newell-Segur (AKNS) and its connection to Wadati-Konno-Ishikawa (WKI) hierarchy (see [12], [21]) corresponds to $a = 1$, $c = 0$ and $c = 1$ respectively, can be reproduced by g-RHB formalism as will be explained in a separate publication.

2. Algebraic background

The algebraic approach to studying integrable hierarchies forms a leading framework for systematically formulating hierarchies of time evolution (flow) equations, their infinite conservation laws, and for methodically deriving the corresponding soliton solutions. In order to develop such approach one starts with a decomposition of affine Lie algebra $\hat{\mathcal{G}}$ into graded subspaces $\hat{\mathcal{G}} = \bigoplus_{i \in \mathbb{Z}} \mathcal{G}_i$, with $\mathcal{G}_i$ satisfying

$$[Q, \mathcal{G}_i] = i\mathcal{G}_i, \quad [\mathcal{G}_i, \mathcal{G}_j] \subset \mathcal{G}_{i+j}, \quad (2.1)$$

where Q is a grading operator and $i, j \in \mathbb{Z}$.

For a semi-simple element of grade a , $E \equiv E^{(a)} \in \mathcal{G}_a$ an affine Lie algebra $\hat{\mathcal{G}}$ further decomposes into the kernel of E defined by

$$\mathcal{K} \equiv \{X \in \hat{\mathcal{G}} \mid [E, X] = 0\},$$

and its complement $\mathcal{M}$:

$$\hat{\mathcal{G}} = \mathcal{K} \oplus \mathcal{M}.$$

The integrable hierarchies are henceforth classified according to the decomposition of the affine algebra $\hat{\mathcal{G}}$ in terms of a grading operator Q and a constant semi-simple generator $E \in \hat{\mathcal{G}}$.

Within such algebraic structure it is convenient to introduce zero curvature relations in order to derive the commuting time evolution equations. A simple example of hierarchy defined in terms of a semisimple element of grade one, $E^{(1)} \in \mathcal{G}_1$ will illustrate basic features that will be easy to recognize in a variety of models presented below. The evolution flows defining such hierarchy are derived from the zero curvature relations in terms of a two dimensional gauge potentials, namely, A_x and A_{t_N} :

$$[D_x + A_x, D_{t_N} + A_{t_N}] = [D_x + E^{(1)} + A_0, D_{t_N} + D^{(N)} + D^{(N-1)} \dots + D^{(0)}] = 0, \quad (2.2)$$

$E^{(1)} \in \mathcal{G}_1$, $A_0 \in \mathcal{G}_0$, $N \in \mathbb{Z}_+^*$ and $D_{t_1} = D_x$. Decomposing equation (2.2) grade by grade according to (2.1), allows for obtaining the solution for $D^{(i)} \in \mathcal{G}_i$ (see for instance [8]). This is done recursively, starting from the highest grade projection of (2.2), namely

$$[E^{(1)}, D^{(N)}] = 0, \quad (2.3)$$

until we arrive at the zero grade component,

$$\partial_{t_N} A_0 - \partial_x D^{(0)} - [A_0, D^{(0)}] = 0, \quad (2.4)$$

which corresponds to the time evolution equation for the grade-zero field A_0 .

Analogously negative sub-hierarchy can be constructed by considering negative graded decomposition:

$$[D_x + A_x, D_{t_{-N}} + A_{t_{-N}}] = [D_x + E^{(1)} + A_0, D_{t_{-N}} + D^{(-N)} + D^{(-N+1)} \dots + D^{(-1)}] = 0, \quad (2.5)$$

$D^{(i)} \in \mathcal{G}_i$. Different gradings Q and/or different $E^{(1)}$ lead to different integrable hierarchies. The well-known examples of AKNS and mKdV hierarchies, closely follow such framework for $\hat{\mathcal{G}} = \hat{sl}_2$ algebra but with different homogeneous and principal gradations, respectively and consequently with different $E^{(1)}$ elements.

A universal framework unify positive and negative flows can be formulated using the RHB factorization formula and we will see below that in more general scheme presented in (3.1) it corresponds to a conventional choice of $a = 1$. In [11] this formula was introduced as:

$$\exp\left(-\sum_{N=1}^{\infty} E^{(N)} t_N\right) g \exp\left(\sum_{N=1}^{\infty} E^{(-N)} t_{-N}\right) = \Theta_-^{-1}(t) \Theta_+(t). \quad (2.6)$$

In the above formula $E^{(\pm N)} \in \mathcal{K}^{(\pm N)} \equiv \mathcal{K} \cap \mathcal{G}_{\pm N}$. For $N = 1$ we encounter the grade one semi-simple element $E^{(1)}$, a basic ingredient of a zero-curvature approach. The group element g denotes an arbitrary constant group element and $\Theta_- = \exp(-\sum_{i>0} \theta^{(-i)})$, $\Theta_+ = \exp(\sum_{i\geq 0} \theta^{(i)})$. In reference [11] equation (2.6) was used to derive the Lax pair A_x from a dressing factorization problem involving both the positive and negative flows labeled by $(t_{\pm 1}, t_{\pm 2}, \dots)$ within the same formulation. Both sub-hierarchies (2.2) and (2.5) were derived within such unified framework and the flows were shown to commute.

3. The generalized Riemann-Hilbert-Birkhoff (g-RHB) decomposition

In this paper we extend applicability of RHB decomposition formula by generalizing the construction of [11] in a three separate ways:

- (1) First, we consider the construction of *higher grading integrable hierarchies* by replacing the grade N semi-simple element $E^{(N)} \in \mathcal{G}_N$ in equation (2.6) by generalized Heisenberg generators (abelian) $\epsilon^{(aN)} \in \mathcal{G}_{aN}$.
- (2) Apart from carrying higher graded structure characterized by the integer a , $\epsilon^{(aN)}$ may incorporate information about non-zero vacuum induced by extra terms in $\epsilon^{(aN)}$ in addition to $E^{(aN)}$ signaled by non-zero value of parameter b in equation (3.2).
- (3) Incorporating explicitly a gauge ambiguity within the zero grade subspace that will be labeled by a parameter c such that $0 \leq c \leq 1$, $\tilde{B} = e^{-c\tilde{\theta}^{(0)}}$.

The models, like KN, CLL, GI models will emerge when a higher grading semi-simple element, e.g., $E = E^{(a)} \in \mathcal{G}_a$, $a = 2$ is introduced. We refer to the associated hierarchies as *higher grading integrable hierarchies*. To incorporate these models in an algebraic approach we propose g-RHB decomposition formula that generalizes the RHB decomposition formula as follows¹:

$$\Theta(t) = \exp \left(- \sum_{N=1}^{\infty} \epsilon^{(-aN)} t_{-N} + \epsilon^{(aN)} t_N \right) g \exp \left(\sum_{N=1}^{\infty} \epsilon^{(-aN)} t_{-N} + \epsilon^{(aN)} t_N \right) = \Theta_-^{-1}(t) \Theta_+(t), \quad (3.1)$$

where g again denotes an arbitrary group element, $a \in \mathbb{Z}_+^*$, $E^{(\pm aN)} \in \mathcal{K}^{(\pm aN)} \equiv \mathcal{K} \cap \mathcal{G}_{\pm aN}$, and the constant elements $\epsilon^{(\pm aN)}$:

$$\epsilon^{(aN)} = E^{(aN)} + b \sum_{i=0}^{aN-1} \epsilon_N^{(i)}, \quad (3.2)$$

$$\epsilon^{(-aN)} = E^{(-aN)} + b \sum_{i=1}^{aN-1} \epsilon_{-N}^{(-i)}, \quad \epsilon_{\pm N}^{(i)} \in \mathcal{G}_i, \quad (3.3)$$

describe two possible vacuum structures

- I. Type I hierarchies with *zero vacuum* solutions and $b = 0$, $\epsilon^{(\pm aN)} = E^{(\pm aN)}$.
- II. Type II hierarchies with *non-zero vacuum* solutions and $b = 1$ in relations (3.2) and (3.3).

In both cases the elements $\epsilon^{(\pm aN)}$ satisfy the center-less Heisenberg (abelian) subalgebra,

$$[\epsilon^{(\pm aN)}, \epsilon^{(\pm aM)}] = 0, \quad N, M \in \mathbb{Z}_+^*. \quad (3.4)$$

We now consider the g-RHB factorization (3.1) with $\Theta(t)_-$ and $\Theta_+(t)$ realized as:

$$\Theta_-(t) = \tilde{B} \Theta_-^{-1} = \tilde{B} \exp \left(- \sum_{k=1}^{\infty} \theta^{(-k)} \right), \quad \Theta_+(t) = \tilde{B} B \Theta_+ = \tilde{B} B \exp \left(\sum_{k=1}^{\infty} \theta^{(k)} \right), \quad (3.5)$$

for $\tilde{B}$ being a zero grade element and independent of $B = \exp \theta_0$. Also $\theta^{(-k)} \in \mathcal{G}_{-k}$ and $\theta^{(k)} \in \mathcal{G}_k$, where $k \in \mathbb{Z}_+$.

It is important to notice the obvious fact that the g-RHB decomposition formula (3.1) does not depend upon the $\tilde{B}$. The zero grade group element $\tilde{B}$ cancels out on the right hand side of equation (3.1) and we might as well gauge it away or alternatively set it to one: $\tilde{B} = 1$. Its presence, however turns to be important allowing definitions of different possible physical variables of the system. We will see below in (3.36), (3.37) that the addition of the $\tilde{B}$ in (3.5) reflects a freedom to gauge transform the model and therefore exhibits an ambiguity inherently present in the formalism.

In the remaining part of this section we will derive from a general decomposition of an affine Lie algebra the time evolution flow equations associated with algebra generators of positive and negative grade.

The connection with vacuum parameters of the theory given in (2.2) and (2.5) in vacuum configuration, is established through:

$$A_{t_N}^{vac} = E^{(aN)} + D_{vac}^{(aN-1)} + \dots D_{vac}^{(0)} \equiv \epsilon^{(aN)}, \quad \text{or} \quad (3.6)$$

$$A_{t_{-N}}^{vac} = E^{(-aN)} + D_{vac}^{(-aN+1)} + \dots D_{vac}^{(-1)} \equiv \epsilon^{(-aN)}. \quad (3.7)$$

$A_x^{vac} = A_{t_1}^{vac}$. Type I hierarchy is characterized by *zero-vacuum* configuration $D_{vac}^{(aN-1)} = \dots D_{vac}^{(0)} = 0$ or $D_{vac}^{(-aN+1)} = \dots D_{vac}^{(-1)} = 0$ and $\epsilon^{(\pm aN)} = E^{(\pm aN)}$, i.e., $b = 0$. For *type II hierarchy*, the *constant non-zero vacuum* configuration $D_{vac}^{(\pm aN \mp 1)} \neq 0$, i.e., $b = 1$. In both cases the zero curvature representation implies $[A_x^{vac}, A_{t_{\pm N}}^{vac}] = [\epsilon^{(a)}, \epsilon^{(\pm aN)}] = 0$ and from Jacobi identity we find that (3.4) follows straightforwardly. Accordingly we conclude that solution of the zero curvature equation for the vacuum configuration can be written, for both positive and negative sectors as

$$A_x^{vac} = -\partial_x \Psi_0 \Psi_0^{-1} = \epsilon^{(a)}, \quad A_{t_{\pm N}}^{vac} = -\partial_{t_{\pm N}} \Psi_0 \Psi_0^{-1} = \epsilon^{(\pm aN)}, \quad (3.8)$$

¹ In ref. [11] only negative/positive graded generators were considered in the left/right hand sided of (2.6). Here we consider both graded structure in each side of (2.6) without loss of generality since it corresponds to redefining Θ_+ and Θ_- .

where Ψ_0 is the Baker-Akhiezer function,

$$\Psi_0 = \exp \left(- \sum_{N=1}^{\infty} \epsilon^{(-aN)} t_{-N} + \epsilon^{(aN)} t_N \right), \quad x \equiv t_1. \quad (3.9)$$

In order to understand the commutativity of flows and the role of the parameter a it is instructive to define a submodel of the model defined by equation (3.1) with $a = 1$ by setting to zero the odd flows:

$$t_1 = t_3 = \dots = t_{2k+1} = 0,$$

on both sides of equation (3.1). Then such submodel of (3.1) with $a = 1$ agrees with the model obtained by setting $a = 2$ and $\epsilon^{(aN)} = E^{(aN)}$ in equation (3.1) after an appropriate redefinition of indices of the flows. Similar relations hold for the negative times. This, in principle, shows how to obtain the $a = 2, 3, \dots$ models as reductions of the $a = 1$ model and explains why their flows will commute.

Acting with derivatives with respect to $t_1 = x$ and $t_{\pm N}$ in equation (3.1) and using (3.8) we find,

$$\partial_x \Theta = [\Theta, \epsilon^{(a)}], \quad \partial_{t_{\pm N}} \Theta = [\Theta, \epsilon^{(\pm aN)}], \quad (3.10)$$

which leads from (3.6) and (3.7) to

$$\Theta_+ A_x^{vac} \Theta_+^{-1} - (\partial_x \Theta_+) \Theta_+^{-1} = \Theta_- A_x^{vac} \Theta_-^{-1} - (\partial_x \Theta_-) \Theta_-^{-1}, \quad (3.11)$$

$$\Theta_+ A_{t_{\pm N}}^{vac} \Theta_+^{-1} - (\partial_{t_{\pm N}} \Theta_+) \Theta_+^{-1} = \Theta_- A_{t_{\pm N}}^{vac} \Theta_-^{-1} - (\partial_{t_{\pm N}} \Theta_-) \Theta_-^{-1}. \quad (3.12)$$

It therefore follows that the generators $\Theta_{\pm}$ can be identified with the dressing operators mapping the vacuum into a non-trivial configuration by a gauge transformation, i.e.,

$$A_x \equiv \Theta_{\pm} A_x^{vac} \Theta_{\pm}^{-1} - (\partial_x \Theta_{\pm}) \Theta_{\pm}^{-1} = -\partial_x (\Theta_{\pm} \Psi_0) \Psi_0^{-1} \Theta_{\pm}^{-1}, \quad (3.13)$$

$$A_{t_{\pm N}} \equiv \Theta_{\pm} A_{t_{\pm N}}^{vac} \Theta_{\pm}^{-1} - (\partial_{t_{\pm N}} \Theta_{\pm}) \Theta_{\pm}^{-1} = -\partial_{t_{\pm N}} (\Theta_{\pm} \Psi_0) \Psi_0^{-1} \Theta_{\pm}^{-1}, \quad (3.14)$$

It is clear from relation (3.5) and (3.1) that the theory does not depend upon $\tilde{B}$. This means that *different choices of $\tilde{B}$ lead to the same hierarchy*. This fact is expressed at the zero curvature level from (3.13) and (3.14) by gauge invariance under transformation $\Theta_{\pm} \rightarrow \tilde{B} \Theta_{\pm}$, i.e.,

$$\tilde{A}_x = \tilde{B} A_x \tilde{B}^{-1} - \partial_x \tilde{B} \tilde{B}^{-1}, \quad (3.15)$$

$$\tilde{A}_{t_{\pm N}} = \tilde{B} A_{t_{\pm N}} \tilde{B}^{-1} - \partial_{t_{\pm N}} \tilde{B} \tilde{B}^{-1}. \quad (3.16)$$

From (3.11) and (3.12) we find (recall $N \in \mathbb{Z}_+^*$),

$$-\partial_{t_N} \Theta_- \Theta_-^{-1} + \partial_{t_N} \Theta_+ \Theta_+^{-1} = \Theta_+ \epsilon^{(aN)} \Theta_+ - \Theta_- \epsilon^{(aN)} \Theta_-^{-1}, \quad (3.17)$$

$$-\partial_{t_{-N}} \Theta_- \Theta_-^{-1} + \partial_{t_{-N}} \Theta_+ \Theta_+^{-1} = \Theta_+ \epsilon^{(-aN)} \Theta_+ - \Theta_- \epsilon^{(-aN)} \Theta_-^{-1}. \quad (3.18)$$

Considering the graded structure of $\Theta_{\pm}$ in (3.5) we find by projecting (3.17) and (3.18) into zero grade:

$$(\partial_{t_N} \tilde{B}) \tilde{B}^{-1} - (\partial_{t_N} \tilde{B} B) B^{-1} \tilde{B}^{-1} = -\tilde{B} (\partial_{t_N} B) B^{-1} \tilde{B}^{-1} = (\Theta_- \epsilon^{(aN)} \Theta_-^{-1})_0, \quad (3.19)$$

$$(\partial_{t_{-N}} \tilde{B}) \tilde{B}^{-1} - (\partial_{t_{-N}} \tilde{B} B) B^{-1} \tilde{B}^{-1} = -\tilde{B} (\partial_{t_{-N}} B) B^{-1} \tilde{B}^{-1} = -(\Theta_+ \epsilon^{(-aN)} \Theta_+^{-1})_0. \quad (3.20)$$

Acting with derivatives in $\Theta_{\pm}$ we obtain,

$$\begin{aligned} (\partial_{t_{\pm N}} \Theta_-) \Theta_-^{-1} &= \partial_{t_{\pm N}} (\tilde{B} \Theta_-^{-1}) \Theta_-^{-1} \\ &= (\partial_{t_{\pm N}} \tilde{B}) \tilde{B}^{-1} + \tilde{B} (\partial_{t_{\pm N}} \Theta_-^{-1}) \Theta_-^{-1}, \end{aligned} \quad (3.21)$$

$$\begin{aligned} (\partial_{t_{\pm N}} \Theta_+) \Theta_+^{-1} &= \partial_{t_{\pm N}} (\tilde{B} B \Theta_+) \Theta_+^{-1} \\ &= (\partial_{t_{\pm N}} \tilde{B}) \tilde{B}^{-1} + \tilde{B} (\partial_{t_{\pm N}} B) B^{-1} \tilde{B}^{-1} + \tilde{B} B (\partial_{t_{\pm N}} \Theta_+) \Theta_+^{-1} \tilde{B}^{-1}. \end{aligned} \quad (3.22)$$

From relations (3.17), (3.18), (3.21) and (3.22) we find

$$\partial_{t_N} B = B (\Theta_+ \epsilon^{(aN)} \Theta_+^{-1})_0 - (\Theta_-^{-1} \epsilon^{(aN)} \Theta_-)_0 B, \quad (3.23)$$

$$\partial_{t_{-N}} B = B (\Theta_+ \epsilon^{(-aN)} \Theta_+^{-1})_0, \quad (3.24)$$

$$\partial_{t_N} \Theta_+ = (\Theta_+ \epsilon^{(aN)} \Theta_+^{-1})_> \Theta_+ - B^{-1} (\Theta_-^{-1} \epsilon^{(aN)} \Theta_-)_> B \Theta_+, \quad (3.25)$$

$$\partial_{t_{-N}} \Theta_+ = (\Theta_+ \epsilon^{(-aN)} \Theta_+^{-1})_> \Theta_+, \quad (3.26)$$

$$\partial_{t_N} \Theta_{<} = -\Theta_{<} (\Theta_{<}^{-1} \epsilon^{(aN)} \Theta_{<})_{<}, \quad (3.27)$$

$$\partial_{t_{-N}} \Theta_{<} = \Theta_{>} B (\Theta_{>} \epsilon^{(-aN)} \Theta_{>}^{-1})_{<} B^{-1} \Theta_{<} - \Theta_{<} (\Theta_{<}^{-1} \epsilon^{(-aN)} \Theta_{<})_{<}. \quad (3.28)$$

As a consequence, from equations (3.21) and (3.22) we find

$$\partial_{t_N} \Theta_{+} = \left(\partial_{t_N} \tilde{B} \tilde{B}^{-1} + \Theta_{+} \epsilon^{(aN)} \Theta_{+}^{-1} - (\Theta_{-} \epsilon^{(aN)} \Theta_{-}^{-1})_{\geq} \right) \Theta_{+}, \quad (3.29)$$

$$\partial_{t_{-N}} \Theta_{+} = \left(\partial_{t_{-N}} \tilde{B} \tilde{B}^{-1} + \Theta_{+} \epsilon^{(-aN)} \Theta_{+}^{-1} - (\Theta_{+} \epsilon^{(-aN)} \Theta_{+}^{-1})_{<} \right) \Theta_{+}, \quad (3.30)$$

$$\partial_{t_N} \Theta_{-} = \left(\partial_{t_N} \tilde{B} \tilde{B}^{-1} + \Theta_{-} \epsilon^{(aN)} \Theta_{-}^{-1} - (\Theta_{-} \epsilon^{(aN)} \Theta_{-}^{-1})_{\geq} \right) \Theta_{-}, \quad (3.31)$$

$$\partial_{t_{-N}} \Theta_{-} = \left(\partial_{t_{-N}} \tilde{B} \tilde{B}^{-1} + \Theta_{-} \epsilon^{(-aN)} \Theta_{-}^{-1} - (\Theta_{+} \epsilon^{(-aN)} \Theta_{+}^{-1})_{<} \right) \Theta_{-}. \quad (3.32)$$

Introducing, $D_{t_N} \Theta_{-} = \partial_{t_N} \Theta_{-} + \Theta_{-} D_{t_N}$, we find from (3.31),

$$\Theta_{-} (D_{t_N} + \epsilon^{(aN)}) \Theta_{-}^{-1} = D_{t_N} + (\Theta_{-} \epsilon^{(aN)} \Theta_{-}^{-1})_{\geq} - (\partial_{t_N} \tilde{B}) \tilde{B}^{-1}. \quad (3.33)$$

From the abelian nature of $\mathcal{K}_e$, i.e., $[\epsilon^{(a)}, \epsilon^{(aN)}] = 0$ and denoting $t_1 = x$, we find

$$\Theta_{-} [D_x + \epsilon^{(a)}, D_{t_N} + \epsilon^{(aN)}] \Theta_{-}^{-1} = [\Theta_{-} (D_x + \epsilon^{(a)}) \Theta_{-}^{-1}, \Theta_{-} (D_{t_N} + \epsilon^{(aN)}) \Theta_{-}^{-1}] = 0. \quad (3.34)$$

It therefore follows that

$$[D_x + \tilde{A}_x, D_{t_N} + \tilde{A}_{t_N}] = 0, \quad (3.35)$$

where

$$\tilde{A}_x = \tilde{B} (\Theta_{<}^{-1} \epsilon^{(a)} \Theta_{<})_{\geq} \tilde{B}^{-1} - (\partial_x \tilde{B}) \tilde{B}^{-1} \equiv \tilde{B} E^{(a)} \tilde{B}^{-1} + \sum_{i=0}^{a-1} \tilde{A}_i, \quad (3.36)$$

$$\tilde{A}_{t_N} = \tilde{B} (\Theta_{<}^{-1} \epsilon^{(aN)} \Theta_{<})_{\geq} \tilde{B}^{-1} - (\partial_{t_N} \tilde{B}) \tilde{B}^{-1} \equiv \sum_{i=0}^{aN} \tilde{D}^{(i)}. \quad (3.37)$$

It thus follows from (3.15) and (3.16) that

$$A_x = (\Theta_{<}^{-1} \epsilon^{(a)} \Theta_{<})_{\geq} = E^{(a)} + A_{a-1} + \dots A_0, \quad (3.38)$$

$$A_{t_N} = (\Theta_{<}^{-1} \epsilon^{(aN)} \Theta_{<})_{\geq} = D^{(aN)} + D^{(aN-1)} + \dots D^{(0)}. \quad (3.39)$$

For the negative graded sector we find,

$$\Theta_{-} [D_x + \epsilon^{(a)}, D_{t_{-N}} + \epsilon^{(-aN)}] \Theta_{-}^{-1} = [\Theta_{-} (D_x + \epsilon^{(a)}) \Theta_{-}^{-1}, \Theta_{-} (D_{t_{-N}} + \epsilon^{(-aN)}) \Theta_{-}^{-1}] = 0, \quad (3.40)$$

leading to

$$[D_x + \tilde{A}_x, D_{t_{-N}} + \tilde{A}_{t_{-N}}] = 0, \quad (3.41)$$

where

$$\tilde{A}_{t_{-N}} = \tilde{B} B (\Theta_{>} \epsilon^{(-aN)} \Theta_{>}^{-1})_{<} B^{-1} \tilde{B}^{-1} - (\partial_{t_{-N}} \tilde{B}) \tilde{B}^{-1} \equiv \sum_{i=0}^{aN} \tilde{D}^{(-i)}. \quad (3.42)$$

For the quantity $A_{t_{-N}}$ from (3.16) we find,

$$A_{t_{-N}} = B (\Theta_{>} \epsilon^{(-aN)} \Theta_{>}^{-1})_{<} B^{-1} = D^{(-aN)} + D^{(-aN+1)} \dots D^{(-1)}. \quad (3.43)$$

Notice that (3.43) does not contain zero-grade terms as in (3.39) due to the asymmetric splitting of zero grade subgroup B in (3.5) when $\tilde{B} = 1$. The absence of such terms will be responsible for excluding the possibility of constant vacuum solutions for certain flows. This will explain why flows associated to $t_{-N} = t_{-1}$ only allow zero vacuum solutions in the explicit examples to be discussed in the next sections. We should also point out that the same result can be obtained using Θ_{+} from (3.29) and (3.30) instead of Θ_{-} . The group element $\tilde{B}$ act as a gauge freedom which can be chosen by convenience. In contrast, the zero grade group element B contains physical fields and satisfy (3.23) and (3.24). In particular for $N = 1$ in equation (3.23) we find

$$\partial_x B B^{-1} = b B \epsilon_1^{(0)} B^{-1} - (\Theta_{<}^{-1} \epsilon^{(a)} \Theta_{<})_0. \quad (3.44)$$

4. Integrable hierarchies obtained from g-RHB decomposition

Here we will illustrate our construction for the case of the affine algebra $\hat{\mathcal{G}} = \hat{sl}_2$. Neglecting for the moment the central terms, the nonzero commutators are²

$$\left[h^{(n)}, E_{\pm\alpha}^{(m)} \right] = \pm 2E_{\pm\alpha}^{(n+m)}, \quad \left[E_{\alpha}^{(n)}, E_{-\alpha}^{(m)} \right] = h^{(n+m)}, \quad \left[d, h^{(n)} \right] = nh^{(n)}, \quad \left[d, E_{\pm\alpha}^{(n)} \right] = nE_{\pm\alpha}^{(n)}, \quad (4.1)$$

where $n, m \in \mathbb{Z}$.

The structure of the two dimensional gauge potentials (3.36), (3.37) and (3.42) naturally decomposes into two sub-hierarchies namely, *positive and negative* according to the different gradings as we shall now discuss explicitly case by case.

We will work here exclusively with the principal gradation, with $Q = Q_p = 2d + \frac{1}{2}h$ such that $[Q_p, \mathcal{G}_i] = i\mathcal{G}_i$ where,

$$\mathcal{G}_{2n} = \{h^{(n)}\}, \quad \mathcal{G}_{2n+1} = \{E_{\alpha}^{(n)}, E_{-\alpha}^{(n+1)}\}, \quad (4.2)$$

and $\hat{\mathcal{G}} = \oplus_{i \in \mathbb{Z}} \mathcal{G}_i$.

4.1. The $a = 1$ mKdV hierarchy

For completeness we reproduce here the well-known mKdV hierarchy in the setting considered in this paper. Set $a = 1$ and $E \equiv E^{(1)} = \frac{1}{2}(E_{\alpha}^{(0)} + E_{-\alpha}^{(1)})$. It follows that the kernel $\mathcal{K}$ of E contains only *odd graded* generators and is given by

$$Ker_E = \mathcal{K} = \left\{ E^{(2n+1)} = \frac{1}{2} (E_{\alpha}^{(n)} + E_{-\alpha}^{(n+1)}) \in \mathcal{G}_{2n+1} \right\} \quad (4.3)$$

(see [8] for positive and negative flows).

In fact $E^{(1)}$ is a semi-simple element in the sense that $\hat{\mathcal{G}} = \mathcal{K} \oplus \mathcal{M}$ where $\mathcal{M}$ is an image of E . In this case $E^{(\pm N)}$ satisfy the centerless Heisenberg algebra,

$$[E^{(2n+1)}, E^{(2m+1)}] = 0. \quad (4.4)$$

An important second Heisenberg algebra satisfying (4.4) is generated by the combination³

$$\Sigma^{(2n+1)} = \frac{1}{2}(E_{\alpha}^{(n)} + E_{-\alpha}^{(n+1)}) + bv_0 h^{(n)}. \quad (4.5)$$

Denoting $\Sigma^{(1)} = \epsilon^{(1)} = E + bv_0 h^{(0)}$, $\theta^{(-1)} = \chi_{-1} E_{\alpha}^{(-1)} + \psi_{-1} E_{-\alpha}^{(0)}$ and $\tilde{B} = e^{-c\tilde{\phi}h^{(0)}}$, we obtain from (3.36):

$$\begin{aligned} \tilde{A}_x &= \tilde{B} (E^{(1)} + bv_0 h^{(0)} - [\theta^{(-1)}, E^{(1)}]) \tilde{B}^{-1} - \tilde{B} \partial_x \tilde{B}^{-1} \\ &= \frac{1}{2} \left(e^{-2c\tilde{\phi}(t)} E_{\alpha}^{(0)} + e^{2c\tilde{\phi}(t)} E_{-\alpha}^{(1)} \right) + \frac{1}{2} (-\chi_{-1} + \psi_{-1} + 2bv_0 + 2c\partial_x \tilde{\phi}) h^{(0)}. \end{aligned} \quad (4.6)$$

We set $A_0 = \frac{1}{2}(-\chi_{-1} + \psi_{-1} + 2bv_0)h^{(0)} = v(x, t_{\pm N})h^{(0)}$ and notice that $\tilde{B}A_0\tilde{B}^{-1} = A_0$ and henceforth the field $v(x, t_{\pm N})$ is *unchanged under gauge transformation* (3.15) and (3.16). As a consequence, although the gauge potentials A_x and $A_{t_{\pm N}}$ may depend upon $\tilde{B}$, the flow equations are independent of $\tilde{B}$ and we may take $\tilde{B} = 1$, or $c = 0$, to obtain:

$$A_x = E + A_0, \quad A_0 = \frac{1}{2}(-\chi_{-1} + \psi_{-1} + 2bv_0)h^{(0)} = v(x, t_{\pm N})h^{(0)}. \quad (4.7)$$

Equation (4.7) shows that the extra term in $\epsilon^{(1)}$, namely $\epsilon_1^{(0)} = v_0 h^{(0)}$ in (3.2) represents the non-zero vacuum value v_0 , for $b \neq 0$ characteristic for the type II hierarchy.

From the general structure of A_{t_N} from (3.37) we find the zero curvature,

$$\left[D_x + E^{(1)} + A_0, D_{t_N} + D^{(N)} + \sum_{i=0}^{N-1} D^{(i)} \right] = 0, \quad (4.8)$$

which decomposes grade by grade allowing to determine $D^{(i)} = D_{\mathcal{K}}^{(i)} + D_{\mathcal{M}}^{(i)}$. In particular the highest grade projection

$$[E^{(1)}, D^{(N)}] = 0,$$

leads to $D^{(N)} = D_{\mathcal{K}}^{(N)} \in \mathcal{K}$ and hence for the positive sub-hierarchy it holds that $N = 2k + 1$. The second highest grade projection,

$$[E^{(1)}, D^{(N-1)}] + [A_0, E^{(N)}] + \partial_x D^{(N)} = 0$$

² In centerless case (loop algebra) the affine generators are realized as $h^{(n)} = \zeta^n h$, $E_{\pm\alpha}^{(n)} = \zeta^n E_{\pm\alpha}$ and $d = \zeta \frac{\partial}{\partial \zeta}$.

³ Notice that Σ contain terms of different gradings. It does suggest that powers of v_0 could be incorporated into a generalized grading operator such that the two terms would have the same grade. The vacuum parameter v_0 would introduce an extra loop in the affine algebra (see for instance ref. [25]).

determines $D_{\mathcal{M}}^{(N-1)} \in \mathcal{M}, \dots$. This process continues until we reach the zero grade projection,

$$\partial_x D^{(0)} - \partial_{t_N} A_0 + [A_0, D^{(0)}] = 0,$$

which yields the flow equation for $v(x, t_N)$.

Solving (4.8) for t_3 and t_5 we find,

$$\begin{aligned} A_{t_3} = & \frac{1}{2} (E_a^{(1)} + E_{-a}^{(2)} + v h^{(1)} - (2v^3 - v_{xx}) h^{(0)} \\ & - (v^2 - v_x) E_a^{(0)} - (v^2 + v_x) E_{-a}^{(1)} \end{aligned} \quad (4.9)$$

and

$$\begin{aligned} A_{t_5} = & \frac{1}{2} (E_a^{(2)} + E_{-a}^{(3)} + v h^{(2)} + (v_x - v^2) E_a^{(1)} - (v_x + v^2) E_{-a}^{(2)} \\ & + (v_{xx} - 2v^3) h^{(1)} + [v_{3x} - 2vv_{xx} + v_x (v_x - 6v^2) + 3v^4] E_a^{(0)} \\ & - [v_{3x} + 2vv_{xx} - v_x (v_x + 6v^2) - 3v^4] E_{-a}^{(1)} \\ & + [v_{4x} - 2v (5vv_{xx} + 5v_x^2 - 3v^4)] h^{(0)} \end{aligned} \quad (4.10)$$

and arrive at the t_3 (mKdV) and t_5 flows:

$$\partial_{t_3} v = \partial_x^3 v - 6v^2 \partial_x v, \quad (4.11)$$

and

$$\partial_{t_5} v = \partial_x^5 v - 10v^2 \partial_x^3 v - 40v \partial_x v \partial_x^2 v - 10(\partial_x v)^3 + 30v^4 \partial_x v.$$

Higher projections of the zero curvature equations (4.8) lead to higher flow equations for t_N . It is clear that the *positive odd* sub-hierarchy admits *both, zero $v = 0$ and constant non-zero vacuum, $v = v_0 \neq 0$* structures. In vacuum configuration we find for t_3 and t_5 ,

$$\begin{aligned} A_{t_3}^{vac} = & \frac{1}{2} (E_a^{(1)} + E_{-a}^{(2)} + bv_0 h^{(1)} - 2bv_0^2 \left(\frac{1}{2} (E_a^{(0)} + E_{-a}^{(1)} + bv_0 h^{(0)}) \right) \\ & = \Sigma^{(3)} - 2bv_0^2 \Sigma^{(1)}, \end{aligned} \quad (4.12)$$

$$\begin{aligned} A_{t_5}^{vac} = & \frac{1}{2} (E_a^{(2)} + E_{-a}^{(3)} + bv_0 h^{(2)} - 2bv_0^2 \left(\frac{1}{2} (E_a^{(1)} + E_{-a}^{(2)} + bv_0 h^{(1)}) \right) \\ & + bv_0^4 \left(\frac{1}{2} (E_a^{(0)} + E_{-a}^{(1)} + bv_0 h^{(0)}) \right) \\ & = \Sigma^{(5)} - 2bv_0^2 \Sigma^{(3)} + 6bv_0^4 \Sigma^{(1)}. \end{aligned} \quad (4.13)$$

In general we expect the following structure

$$A_{t_{2k+1}}^{vac} = \sum_{j=0}^k c_j (bv_0)^{2k-2j} \Sigma^{(2j+1)}, \quad k = 0, 1, 2, \dots \quad (4.14)$$

for the positive sub-hierarchy, where c_j are constants independent of v_0 . It is interesting to observe that the sum the powers of v_0 and the grade of Σ always add up to $2k + 1$.

The very same argument works for negative flows, i.e. we can solve the zero curvature condition and determine a sub-class of negative hierarchy for $b = 0$. From (3.42),

$$\left[D_x + E^{(1)} + A_0, D_{t_{-2k-1}} + B E^{(-2k-1)} B^{-1} + \sum_{i=1}^{2k} D^{(-i)} \right] = 0. \quad (4.15)$$

The simplest case of $k = 0$ and $b = 0$ has,

$$A_{t_{-1}} = \frac{1}{2} \left(e^{-2\partial_x^{-1} v} E_a^{(-1)} + e^{2\partial_x^{-1} v} E_{-a}^{(0)} \right), \quad (4.16)$$

leading to the sinh-Gordon equation i.e.,⁴

$$\partial_{t_{-1}} v = \frac{1}{4} \left(e^{2\partial_x^{-1} v} - e^{-2\partial_x^{-1} v} \right) \longrightarrow \partial_{t_{-1}} \partial_x \phi = \frac{1}{4} (e^{2\phi} - e^{-2\phi}), \quad -\partial_x \phi = v \quad (4.17)$$

Next for t_{-3} we find,

⁴ Defining $B = e^{(\phi(t) + bv_0 x) h^{(0)}}$, the equation (3.44) implies $\partial_x \phi = -v$.

$$\partial_{t-3} \partial_x \phi = \frac{1}{8} e^{2\phi} \partial_x^{-1} (e^{-2\phi} \partial_x^{-1} (e^{2\phi} - e^{-2\phi})) + \frac{1}{8} e^{-2\phi} \partial_x^{-1} (e^{2\phi} \partial_x^{-1} (e^{2\phi} - e^{-2\phi})). \quad (4.18)$$

All the *odd graded* negative flows, are constructed from the *zero vacuum configuration*, of $b=0$, $v=0$, $\phi=0$ and the Lax potentials

$$A_x^{vac} = E^{(1)} \quad (4.19)$$

and

$$A_{t-2k-1}^{vac} = E^{(-2k-1)} = \frac{1}{2} (E_\alpha^{(-k-1)} + E_{-\alpha}^{(-k)}), \quad k=0, 1, 2, \dots \quad (4.20)$$

satisfying the Heisenberg subalgebra (4.4) as in (3.1) with $a=1$ and identified with *Type I hierarchy*.

Recall that as it was pointed out in [8] for the negative flows there is no restriction upon the values of N . There exists a second sub-hierarchy consisting of *negative even* grades (see [8], [20]). In this case we can find a second Heisenberg subalgebra by considering non-zero constant vacuum solutions e.g., $A_x^{vac} = \Sigma^{(1)} \equiv E^{(1)} + v_0 h^{(0)}$. In fact such class of integrable models admits strictly *non zero constant vacuum* solutions $v=v_0 \neq 0$, and is referred to as *Type II hierarchy*. Let us consider, apart from the positive odd flows, the negative even flows. The potential:

$$A_{t-2} = \frac{1}{2} h^{(-1)} + \frac{1}{2} e^{2\phi} \partial_x^{-1} e^{-2\phi} E_\alpha^{(-1)} - \frac{1}{2} e^{-2\phi} \partial_x^{-1} e^{2\phi} E_{-\alpha}^{(0)}, \quad (4.21)$$

leads to the t_{-2} flow,

$$\partial_{t-2} \partial_x \phi = \frac{1}{4} e^{2\phi} \partial_x^{-1} e^{-2\phi} + \frac{1}{4} e^{-2\phi} \partial_x^{-1} e^{2\phi}. \quad (4.22)$$

Similarly we find the t_{-4} flow,

$$\partial_{t-4} \partial_x \phi = \frac{1}{8} e^{2\phi} \partial_x^{-1} [e^{-2\phi} (\partial_x^{-1} e^{2\phi}) (\partial_x^{-1} e^{-2\phi})] + \frac{1}{8} e^{-2\phi} \partial_x^{-1} [e^{2\phi} (\partial_x^{-1} e^{2\phi}) (\partial_x^{-1} e^{-2\phi})]. \quad (4.23)$$

It is straightforward to verify that $\phi = -v_0 x$ indeed satisfies equations (4.22) and (4.23) for $v_0 \neq 0$. The vacuum structure of such a class of models implies a v_0 -deformation upon the algebraic structure. The vacuum structure is given by

$$\begin{aligned} A_{t-2}^{vac} &= \frac{1}{2v_0} \left(\frac{1}{2} (E_\alpha^{(-1)} + E_{-\alpha}^{(0)}) + v_0 h^{(-1)} \right) = \frac{1}{2v_0} \Sigma^{(-1)}, \\ A_{t-4}^{vac} &= \frac{1}{2v_0} \left(\frac{1}{2} (E_\alpha^{(-2)} + E_{-\alpha}^{(-1)}) + v_0 h^{(-2)} \right) = \frac{1}{2v_0} \Sigma^{(-3)}, \\ &\vdots \\ A_{t-2k-2}^{vac} &= \frac{1}{2v_0} \left(\frac{1}{2} (E_\alpha^{(-k-1)} + E_{-\alpha}^{(-k)}) + v_0 h^{(-k-1)} \right) = \frac{1}{2v_0} \Sigma^{(-2k-1)}, \quad k=0, 1, \dots \end{aligned} \quad (4.24)$$

One and two soliton solutions were constructed for the entire hierarchy in terms of deformed vertex operators in [8]. These are v_0 dependent eigenfunctions of $A_{t_{\pm N}}^{vac}$ $N=1, \dots$ and the soliton dispersion relations are given by their v_0 dependent eigenvalues.

4.2. Higher grading $a=2$ hierarchies - KN, GI and CLL hierarchies

Let us now set $a=2$ and, as a first example consider the GI hierarchy [18] generated from the semisimple element:

$$\epsilon^{(2)} = \frac{1}{2} h^{(1)} + br_0 E_\alpha^{(0)} + bs_0 E_{-\alpha}^{(1)} - br_0 s_0 h^{(0)}. \quad (4.25)$$

From equation (3.36) we find that $A_x = (\Theta_{<}^{-1} \epsilon^{(2)} \Theta_{<})_{\geq}$ is given by

$$A_x = \epsilon^{(2)} - [\theta^{(-1)}, E^{(2)}] - b[\theta^{(-1)}, r_0 E_\alpha^{(0)} + s_0 E_{-\alpha}^{(1)}] + \frac{1}{2} [\theta^{(-1)}, [\theta^{(-1)}, E^{(2)}]], \quad (4.26)$$

where $\theta^{(-1)} = \chi_{-1} E_\alpha^{(-1)} + \psi_{-1} E_{-\alpha}^{(0)}$, $\theta^{(-2)} = \phi_{-2} h^{(-1)}$. We therefore find

$$A_x = \frac{1}{2} h^{(1)} + r(t) E_\alpha^{(0)} + s(t) E_{-\alpha}^{(1)} - r(t) s(t) h^{(0)}, \quad (4.27)$$

where we have defined the field variables ($b^2 = b$)

$$r(x, t_{\pm N}) = br_0 + \chi_{-1}(x, t_{\pm N}), \quad s(x, t_{\pm N}) = bs_0 - \psi_{-1}(x, t_{\pm N}). \quad (4.28)$$

The flow equations are derived by solving the zero curvature relation:

$$\left[D_x + \frac{1}{2} h^{(1)} + r(t) E_\alpha^{(0)} + s(t) E_{-\alpha}^{(1)} - r(t) s(t) h^{(0)}, D_{t_N} + E^{(2N)} + \sum_{i=0}^{2N-1} D^{(i)} \right] = 0.$$

Solving for $N = 2$ we find⁵

$$A_{t_2} = \frac{1}{2}h^{(2)} + rE_{\alpha}^{(1)} + sE_{-\alpha}^{(2)} - rs h^{(1)} - r_x E_{\alpha}^{(0)} + s_x E_{-\alpha}^{(1)} - (r^2 s^2 - sr_x + rs_x - br_0^2 s_0^2) h^{(0)}. \quad (4.29)$$

Analogously for $N = 3$,

$$\begin{aligned} A_{t_3} = & \frac{1}{2}h^{(3)} + rE_{\alpha}^{(2)} + sE_{-\alpha}^{(3)} - rs h^{(2)} - r_x E_{\alpha}^{(1)} + s_x E_{-\alpha}^{(2)} - (rs_x - sr_x + r^2 s^2) h^{(1)} \\ & + (r_{xx} - 2r^2(s_x + rs^2)) E_{\alpha}^{(0)} + (s_{xx} + 2s^2(r_x - r^2 s)) E_{-\alpha}^{(1)} \\ & - (rs_{xx} + r_{xx}s - r_x s_x - 2r^3 s^3) h^{(0)}, \end{aligned} \quad (4.30)$$

leading to

$$\partial_{t_2} r = -\partial_x^2 r + 2r(r^2 s^2 - br_0^2 s_0^2) + 2r^2 s_x, \quad \partial_{t_2} s = \partial_x^2 s - 2s(r^2 s^2 - br_0^2 s_0^2) + 2s^2 r_x, \quad (4.31)$$

and

$$\partial_{t_3} r = \partial_x^3 r - 6rr_x(r s^2 + s_x), \quad \partial_{t_3} s = \partial_x^3 s - 6ss_x(r^2 s - r_x). \quad (4.32)$$

Other flow equations can be systematically derived from (3.37) for higher values of N and together they form the GI positive hierarchy. Equations (4.31) admit *zero vacuum* solution for $b = 0$ and *constant vacuum* for $b = 1$. Notice that explicit vacuum parameters appear explicitly in eq. (4.31) due to the fitting with of the b dependent terms from the Heisenberg operators $\epsilon^{(aN)}(b)$. Other equations, like (4.32) do not display explicit b dependent terms and admit both *zero* and *constant non-zero* vacuum solutions.

For the negative GI sub-hierarchy we solve the zero curvature equation

$$\left[D_x + \frac{1}{2}h^{(1)} + r(t) E_{\alpha}^{(0)} + s(t) E_{-\alpha}^{(1)} - r(t) s(t) h^{(0)}, \quad D_{t-N} + \sum_{i=1}^{2N} D^{(-i)} \right] = 0.$$

For the first few cases we find,

$$A_{t_{-1}} = \frac{1}{2}h^{(-1)} + e^{2\phi} \partial_x^{-1} (re^{-2\phi}) E_{\alpha}^{(-1)} - e^{-2\phi} \partial_x^{-1} (se^{2\phi}) E_{-\alpha}^{(0)}, \quad (4.33)$$

and

$$\begin{aligned} A_{t_{-2}} = & \frac{1}{2}h^{(-2)} + e^{2\phi} \partial_x^{-1} (re^{-2\phi}) E_{\alpha}^{(-2)} - e^{-2\phi} \partial_x^{-1} (se^{2\phi}) E_{-\alpha}^{(-1)} \\ & + \partial_x^{-1} (re^{-2\phi}) \partial_x^{-1} (se^{2\phi}) h^{(-1)} \\ & - e^{2\phi} \partial_x^{-1} [\partial_x^{-1} (re^{-2\phi}) - 2re^{-2\phi} \partial_x^{-1} (re^{-2\phi}) \partial_x^{-1} (se^{2\phi})] E_{\alpha}^{(-1)} \\ & - e^{-2\phi} \partial_x^{-1} [\partial_x^{-1} (se^{2\phi}) + 2se^{2\phi} \partial_x^{-1} (re^{-2\phi}) \partial_x^{-1} (se^{2\phi})] E_{-\alpha}^{(0)}, \end{aligned} \quad (4.34)$$

where $\partial_x \phi = rs$,⁶ leading to the following flow equations,

$$\partial_{t_{-1}} r = e^{2\phi} \partial_x^{-1} (re^{-2\phi}), \quad \partial_{t_{-1}} s = e^{-2\phi} \partial_x^{-1} (se^{2\phi}) \quad (4.35)$$

and

$$\partial_{t_{-2}} r = -e^{2\phi} \partial_x^{-1} [\partial_x^{-1} (re^{-2\phi}) - 2re^{-2\phi} \partial_x^{-1} (re^{-2\phi}) \partial_x^{-1} (se^{2\phi})], \quad (4.36)$$

$$\partial_{t_{-2}} s = e^{-2\phi} \partial_x^{-1} [\partial_x^{-1} (se^{2\phi}) + 2se^{2\phi} \partial_x^{-1} (re^{-2\phi}) \partial_x^{-1} (se^{2\phi})]. \quad (4.37)$$

Equations (4.36) and (4.37) both admit zero and constant non-zero vacuum solutions.

In order to discuss the vacuum structure of the GI hierarchy let us define the Heisenberg algebra generated by

$$\Sigma^{(2n)} = \frac{1}{2}h^{(n)} + br_0 E_{\alpha}^{(n-1)} + bs_0 E_{-\alpha}^{(n)} - br_0 s_0 h^{(n-1)}, \quad n = 0, \pm 1, \pm 2, \dots \quad (4.38)$$

For constant non-zero vacuum values we find for A_x^{vac} and $A_{t_N}^{vac}$,

$$A_x^{vac} = \Sigma^{(2)} = \frac{1}{2}h^{(1)} + r_0 E_{\alpha}^{(0)} + s_0 E_{-\alpha}^{(1)} - r_0 s_0 h^{(0)},$$

$$A_{t_2}^{vac} = \Sigma^{(4)},$$

$$A_{t_3}^{vac} = \Sigma^{(6)} + (r_0 s_0)^2 \Sigma^{(2)},$$

⁵ Where the constant term is fixed by a choice of integration constants.

⁶ Defining $B = e^{(\phi(t) - br_0 s_0 x) h^{(0)}}$, the equation (3.44) implies $\partial_x \phi = rs$.

$$\begin{aligned} \vdots &= \vdots, \\ A_{t_N}^{vac} &= \sum_{j=1}^N c_j(r_0 s_0)^{(N-j)\Sigma(2j)}, \quad N = 1, 2, 3, \dots \end{aligned} \quad (4.39)$$

For the negative grade sector we find for the zero vacuum, i.e., $b = 0$,

$$A_{t_{-N}}^{vac} = \frac{1}{2} h^{(-N)}, \quad N = 1, 2, \dots \quad (4.40)$$

and for the constant non-zero vacuum, i.e., $b = 1$, we find for t_{-2} ,

$$A_{t_{-2}}^{vac} = -\frac{1}{2r_0 s_0} \Sigma^{(-2)}. \quad (4.41)$$

We therefore expect the following structure for the vacuum configuration

$$A_{t_{-N}}^{vac} = \sum_{i=0}^N c_i(r_0 s_0)^{(-N+i)\Sigma(-2i)}, \quad N = 2, 3, \dots \quad (4.42)$$

Observe that from (3.43), the graded structure of $A_{t_{-N}} = D^{(-N)} + D^{(-N+1)} + \dots D^{(-1)}$ in vacuum cannot be fitted in terms of $\Sigma^{(2n)} \sim X^{(2n)} + X^{(2n-1)} + X^{(2n-2)}$, $X^{(i)} \in \mathcal{G}_i$ given in (4.38) for some values of n , e.g., $n = -1$. This explains the absence of t_{-1} flow for constant non-zero vacuum. This situation may be changed by the introduction of $\tilde{B}$ as we shall now discuss.

From equation (3.36) we find that $\tilde{A}_x = \tilde{B} A_x \tilde{B}^{-1} - \tilde{B} \partial_x \tilde{B}^{-1}$ where A_x is given by (4.26) and $\tilde{B} = e^{-c\tilde{\phi}h^{(0)}}$ with $\partial_x \tilde{\phi} = r(t)s(t)$.⁷ In terms of field variables r and s we find,

$$\tilde{A}_x = \frac{1}{2} h^{(1)} + r(t) e^{-2\phi} E_\alpha^{(0)} + s(t) e^{2\phi} E_{-\alpha}^{(1)} - r(t)s(t)(1-c)h^{(0)}.$$

It is convenient to define the c -dependent variables,

$$R_c(x, t_{\pm N}) = r(t) e^{-2c\tilde{\phi}(t)}, \quad S_c(x, t_{\pm N}) = s(t) e^{2c\tilde{\phi}(t)}. \quad (4.43)$$

In this notation we find

$$\tilde{A}_x = \frac{1}{2} h^{(1)} + R_c(t) E_\alpha^{(0)} + S_c(t) E_{-\alpha}^{(1)} - (1-c) R_c(t) S_c(t) h^{(0)}.$$

In order to simplify notation we shall drop for the moment the subscript c from variables R_c and S_c . Flow equations are derived by solving the zero curvature relation:

$$\left[D_x + \frac{1}{2} h^{(1)} - (1-c) R(t) S(t) h^{(0)} + R(t) E_\alpha^{(0)} + S(t) E_{-\alpha}^{(1)}, \quad D_{t_N} + E^{(2N)} + \sum_{i=0}^{2N-1} D^{(i)} \right] = 0.$$

For $N = 2$ we find,

$$\begin{aligned} \tilde{A}_{t_2} &= \frac{1}{2} h^{(2)} - R S h^{(1)} - (1-c) \left((1-4c) R^2 S^2 - (1-2c) b R_0^2 S_0^2 - S R_x + R S_x \right) h^{(0)} \\ &\quad + R E_\alpha^{(1)} + S E_{-\alpha}^{(2)} - (2c R^2 S + R_x) E_\alpha^{(0)} - (2c R S^2 - S_x) E_{-\alpha}^{(1)}, \end{aligned}$$

where R_0 and S_0 and the corresponding vacuum values of R and S while for $N = 3$,

$$\begin{aligned} \tilde{A}_{t_3} &= \frac{1}{2} h^{(3)} - R S h^{(2)} - \left((1-4c) R^2 S^2 - S R_x + R S_x \right) h^{(1)} \\ &\quad - (1-c) \left(-2(1-6c^2) R^3 S^3 + 4(1-2c) c b R_0^3 S_0^3 + 6c R S^2 R_x \right) h^{(0)} \\ &\quad - (1-c) \left(- (6c R^2 S + R_x) S_x + S R_{xx} + R S_{xx} \right) h^{(0)} \\ &\quad + R E_\alpha^{(2)} + S E_{-\alpha}^{(3)} - (2c R^2 S + R_x) E_\alpha^{(1)} - (2c R S^2 - S_x) E_{-\alpha}^{(2)} \\ &\quad + (2R \left(- (1-2c(1+c)) R^2 S^2 + 3c S R_x - (1-c) R S_x \right) + R_{xx}) E_\alpha^{(0)} \\ &\quad + (-2S \left((1-2c(1+c)) R^2 S^2 - (1-c) S R_x + 3c R S_x \right) + S_{xx}) E_{-\alpha}^{(1)} \end{aligned}$$

and the corresponding flows equations are:

$$\partial_{t_2} R = -\partial_x^2 R + 2(1-c)(1-2c) R (R^2 S^2 - b R_0^2 S_0^2) + 2(1-c) R^2 S_x - 2c \partial_x (R^2 S), \quad (4.44)$$

$$\partial_{t_2} S = \partial_x^2 S - 2(1-c)(1-2c) S (R^2 S^2 - b R_0^2 S_0^2) + 2(1-c) S^2 R_x - 2c \partial_x (R S^2), \quad (4.45)$$

⁷ Notice that unlike in the mKdV case, $\tilde{B} \neq 1$ changes the r and s into the c -dependent $R = R_c(x, t_{\pm N})$ and $S = S_c(x, t_{\pm N})$ variables.

$$\begin{aligned} \partial_{t_3} R &= \partial_x^3 R - 8c(1-c)(1-2c)R(R^3 S^3 - bR_0^3 S_0^3) - 6(1-c)(1-4c)R^2 S^2 R_x \\ &\quad - 6(1-c)RR_x S_x - 6c\partial_x [(1-2c)R^3 S^2 - RS_x], \end{aligned} \quad (4.46)$$

$$\begin{aligned} \partial_{t_3} S &= \partial_x^3 S + 8c(1-c)(1-2c)S(R^3 S^3 - bR_0^3 S_0^3) - 6(1-c)(1-4c)R^2 S^2 S_x \\ &\quad + 6(1-c)SR_x S_x - 6c\partial_x [(1-2c)R^2 S^3 + RS_x S_x]. \end{aligned} \quad (4.47)$$

Notice that in this case $\tilde{B}E^{(2)}\tilde{B}^{-1} = E^{(2)}$ and the fields $R(x, t_{\pm N})$ and $S(x, t_{\pm N})$ defined in (4.43), unlike in the mKdV case, change under the gauge transformation (3.15) and (3.16). They are actually c -dependent and so are the corresponding flow equations, e.g., (4.45)-(4.47), etc. Since the hierarchy is invariant under gauge transformation (3.15) and (3.16) and hence independent of c , three interesting cases of the *same hierarchy* naturally arise in terms of different gauge variant field variables defined in (4.43) for $c = 0$, $c = 1/2$ and $c = 1$. For $c = 0$ we find the GI hierarchy already discussed earlier. When $c = 1/2$ we get the CLL hierarchy [16], [17] while $c = 0$ corresponds to KN hierarchy [19]. This fact was already anticipated in ref. [15] in terms of a change of variables and in this paper we have established it on a more fundamental basis in terms of the g-RHB decomposition (3.1). Interestingly, we notice that the t_2 flows in (4.44), (4.45) satisfy the conservation law $(RS)_{t_2} = [(RS_x - S_x R_x) + (1-4c)R^2 S^2]_x$ that depends on the parameter c .

We should like to point out that the CLL hierarchy obtained with variables (4.43) for $c = 1/2$ is the only hierarchy that admits non-zero constant vacuum solutions, i.e.

$$R = R_0, \quad S = S_0.$$

It is straightforward to verify this since in this limit ($c = 1/2$) equations (4.44) and (4.45) become

$$\partial_{t_2} R + \partial_x^2 R + 2R S \partial_x R = 0, \quad \partial_{t_2} S - \partial_x^2 S + 2R S \partial_x S = 0, \quad (4.48)$$

and

$$\begin{aligned} \partial_{t_3} R - \partial_x^3 R - 3R^2 S^2 \partial_x R - 3SR_x^2 - 3RSR_{xx} &= 0, \\ \partial_{t_3} S + \partial_x^3 S - 3R^2 S^2 \partial_x S + 3RS_x^2 + 3RSS_{xx} &= 0. \end{aligned} \quad (4.49)$$

Next, we derive the negative sub-hierarchy. Consider

$$\tilde{A}_{t_{-1}} = \frac{1}{2}h^{(-1)} + \bar{R}e^{2(1-c)\partial_x^{-1}(RS)}E_{\alpha}^{(-1)} - \bar{S}e^{-2(1-c)\partial_x^{-1}(RS)}E_{-\alpha}^{(0)} + c\bar{R}\bar{S}h^{(0)}, \quad (4.50)$$

where

$$\bar{R} = \partial_x^{-1} \left(R e^{-2(1-c)\partial_x^{-1}(RS)} \right), \quad \bar{S} = \partial_x^{-1} \left(S e^{2(1-c)\partial_x^{-1}(RS)} \right). \quad (4.51)$$

It follows that

$$\partial_{t_{-1}} R = \bar{R}e^{2(1-c)\partial_x^{-1}(RS)} - 2cR\bar{R}\bar{S}, \quad \partial_{t_{-1}} S = \bar{S}e^{-2(1-c)\partial_x^{-1}(RS)} + 2cS\bar{R}\bar{S}. \quad (4.52)$$

For $c = 1$, the above equation can be put in a local form by defining new variables,

$$\bar{R} = \partial_x^{-1}(R) \equiv \Phi_R, \quad \bar{S} = \partial_x^{-1}(S) \equiv \Phi_S, \quad (4.53)$$

and equation (4.52) becomes [15],

$$\partial_{t_{-1}} \partial_x \Phi_R = \Phi_R - 2\partial_x \Phi_R (\Phi_R \Phi_S), \quad \partial_{t_{-1}} \partial_x \Phi_S = \Phi_S - 2\partial_x \Phi_S (\Phi_R \Phi_S). \quad (4.54)$$

A class of higher grading relativistic integrable models with similar structure has been considered in [22] in connection with reductions of WZWN model. It is clear that equation (4.54) only admits zero vacuum solution.

A condition for non-zero vacuum solution can be found by considering $R = R_0$, $S = S_0 = \text{constants}$ and obtaining from relation (4.51):

$$\bar{R}_{vac} = R_0 \int^x e^{-2(1-c)R_0 S_0 y} dy = \frac{R_0}{-2(1-c)R_0 S_0} e^{-2(1-c)R_0 S_0 x}, \quad (4.55)$$

$$\bar{S}_{vac} = S_0 \int^x e^{2(1-c)R_0 S_0 y} dy = \frac{S_0}{2(1-c)R_0 S_0} e^{2(1-c)R_0 S_0 x}. \quad (4.56)$$

Substituting equations (4.55) and (4.56) in both of equations (4.52) we find the following condition for c :

$$\frac{1-2c}{2(1-c)^2} = 0, \quad (4.57)$$

which implies that the only constant non-zero vacuum solution for t_{-1} -flow occurs for $c = 1/2$.

For the next negative flow for $t = t_{-2}$ we find

$$\begin{aligned}\partial_{t_{-2}} R &= -e^{2(1-c)\partial_x^{-1}(RS)} \partial_x^{-1} (\bar{R} - \bar{S} \partial_x \bar{R}^2) \\ &\quad + 2cR [\bar{R}^2 \bar{S}^2 - \bar{R} \partial_x^{-1} (\bar{S} + \bar{R} \partial_x \bar{S}^2) + \bar{S} \partial_x^{-1} (\bar{R} - \bar{S} \partial_x \bar{R}^2)], \\ \partial_{t_{-2}} S &= e^{-2(1-c)\partial_x^{-1}(RS)} \partial_x^{-1} (\bar{S} + \bar{R} \partial_x \bar{S}^2) \\ &\quad - 2cS [\bar{R}^2 \bar{S}^2 - \bar{R} \partial_x^{-1} (\bar{S} + \bar{R} \partial_x \bar{S}^2) + \bar{S} \partial_x^{-1} (\bar{R} - \bar{S} \partial_x \bar{R}^2)].\end{aligned}$$

Again, non-zero vacuum solution (4.55) and (4.56) implies equation (4.57) indicating that $c = 1/2$ is indeed a special value. We conclude that the CLL hierarchy is the only one of the $a = 2$ models that admits constant non-zero vacuum solution. In fact in reference [14] we have constructed explicit solutions for the CLL hierarchy in terms of vertex operators. These are classified in terms of “two flavor” vertices, namely $V_{\pm}(z)$. Non-trivial solutions for fields R and S are obtained from mixed powers of products of these two vertices. Another class of solutions, keeping one of the fields constant is obtained by considering powers of one of the vertices. The latter gives rise to Burgers hierarchy where one of the CLL fields is set to identity as we discuss in the next sub-section.

4.3. The Burgers hierarchy

Having settled the fact that the CLL hierarchy admits non-zero constant vacuum solutions, we are now in position to explore a particular set of solutions obtained by setting one of the fields (R or S) to a constant (normalized to one). This gives rise to the Burgers hierarchy as a sub-hierarchy with $R = 1$ and $S = w$ for $c = 1/2$.⁸ The first two flow equations are

$$\begin{aligned}\partial_{t_2} w &= \partial_x^2 w - 2w \partial_x w, \\ \partial_{t_3} w &= \partial_x^3 w + 3w^2 \partial_x w - 3\partial_x(w \partial_x w),\end{aligned}$$

and for the negative sector we find

$$\begin{aligned}\partial_{t_{-1}} w &= 1 + w e^{\partial_x^{-1} w} \partial_x^{-1} (e^{-\partial_x^{-1} w}), \\ \partial_{t_{-2}} w &= e^{\partial_x^{-1} w} \partial_x^{-1} (e^{-\partial_x^{-1} w}) + w e^{\partial_x^{-1} w} \partial_x^{-2} (e^{-\partial_x^{-1} w}), \\ \partial_{t_{-3}} w &= e^{\partial_x^{-1} w} \partial_x^{-2} (e^{-\partial_x^{-1} w}) + w e^{\partial_x^{-1} w} \partial_x^{-3} (e^{-\partial_x^{-1} w}).\end{aligned}$$

The positive and negative flows for the sub-hierarchies can be written in closed form as

$$\partial_{t_N} w = -\partial_x (e^{\partial_x^{-1} w} \partial_x^N (e^{-\partial_x^{-1} w})),$$

for $N = 1, 2, 3, \dots$. We observe that this reduction procedure applied on the CLL model reproduces a general closed formula for all positive flows of the Burgers hierarchy, see for example [24]. Here the reduction process yields

$$\partial_{t_{-N}} w = \partial_x (e^{\partial_x^{-1} w} \partial_x^{-N} (e^{-\partial_x^{-1} w})),$$

for $N = 1, 2, 3, \dots$, extending the definition of the Burgers hierarchy to include negative flows of the same structure.

5. A novel model. Higher grade hierarchy with $a = 3$ with principal gradation and $\hat{\mathcal{G}} = \hat{s}l_2$

Here we consider $\hat{\mathcal{G}} = \hat{s}l_2$ with the principal gradation, but set the parameter a to be equal to $a = 3$ (higher a then we have encountered up to now). Accordingly, the semi-simple element inducing the corresponding hierarchy is then equal to $e^{(3)} = E^{(3)} = \frac{1}{2}(E_{\alpha}^{(1)} + E_{-\alpha}^{(2)})$, i.e. $b = 0$. From equation (3.36) with $c = 0$, we parametrize $\theta^{(-3)} = \chi_{-3} E_{\alpha}^{(-2)} + \psi_{-3} E_{-\alpha}^{(-1)}$, $\theta^{(-2)} = \phi_{-2} h^{(-1)}$. $\theta^{(-1)} = \chi_{-1} E_{\alpha}^{(-1)} + \psi_{-1} E_{-\alpha}^{(0)}$ and obtain

$$\begin{aligned}A_x &= \frac{1}{2} (E_{\alpha}^{(1)} + E_{-\alpha}^{(2)}) + \frac{1}{2} (\psi_{-1} - \chi_{-1}) h^{(1)} + \\ &\quad - \left(\phi_{-2} + \frac{1}{2} \chi_{-1} (\psi_{-1} - \chi_{-1}) \right) E_{\alpha}^{(0)} + \\ &\quad + \left(\phi_{-2} - \frac{1}{2} \psi_{-1} (\psi_{-1} - \chi_{-1}) \right) E_{-\alpha}^{(1)} + \\ &\quad + \left(\frac{1}{3} \chi_{-1} \psi_{-1} (\psi_{-1} - \chi_{-1}) - \frac{1}{2} \phi_{-2} (\psi_{-1} + \chi_{-1}) \right) h^{(0)} + \\ &\quad + \frac{1}{2} (\psi_{-3} - \chi_{-3}) h^{(0)}.\end{aligned}\tag{5.1}$$

⁸ In ref. [14] we develop an explicit construction of such class of solutions using the dressing method and classify them by judicious choice of products of vertex operators.

Setting $v_1 = \frac{1}{2}(\psi_{-1} - \chi_{-1})$, $v_2 = \phi_{-2} - v_1\psi_{-1}$ and $v_3 = \frac{1}{2}(\psi_{-3} - \chi_{-3}) - v_2\psi_{-1} - \frac{1}{3}v_1(v_1 + \psi_{-1})$, where $v_1 = v_1(t)$, $v_2 = v_2(t)$ and $v_3 = v_3(t)$, results in

$$A_x = \frac{1}{2} (E_a^{(1)} + E_{-a}^{(2)}) + v_1 h^{(1)} - (2v_1^2 + v_2) E_a^{(0)} + v_2 E_{-a}^{(1)} + (v_1 v_2 + v_3) h^{(0)}.$$

We get for the lowest flow

$$v_{1,t_3} = v_{1,3x} - 6(v_1^2 + v_2) \left(v_{1,x}^2 + (v_2^2 - 2v_1 v_3)^2 \right) - 6\partial_x (v_1 v_3^2) + \\ - 6v_3 \partial_x (v_1^4 + 2v_1^2 v_2) + 2v_2^2 \partial_x (v_1^3 + 3v_3 + 3v_1 v_2), \quad (5.2)$$

$$v_{2,t_3} = v_{2,3x} + 6(v_1 v_2 - v_3) \left(v_{1,x}^2 + (v_2^2 - 2v_1 v_3)^2 \right) - 6v_2^2 v_{2,x} + \\ - 6\partial_x (v_1^2 v_3^2 + (v_2^2 - 2v_1 v_3) v_{1,x}) - 12v_1 v_2 \partial_x (v_2 v_3) + \\ + 4v_1^2 \partial_x (v_2^3 + 3v_3^2) + 6(v_{1,xx} + 4v_1^2 v_2 v_3) v_{1,x}, \quad (5.3)$$

$$v_{3,t_3} = v_{3,3x} - 6(v_2^2 + v_1 v_3) \left(v_{1,x}^2 - (v_2^2 - 2v_1 v_3)^2 \right) - 6(v_2^3 + v_3^2) v_{3,x} + \\ - 6\partial_x (-4v_1^3 v_3^2 + 3v_1^2 v_2^2 v_3 - 2v_1^2 v_3 v_{1,x} + (v_2^2 - 2v_1 v_3) v_{2,x}) + \\ + 6v_1 v_2 \partial_x (v_2^3 - v_2 v_{1,x}) - 6v_{1,x} (v_1 v_2 v_{2,x} - \partial_x (v_1 v_{1,x})) + \\ + 24v_1 v_3 (v_{1,x}^2 - v_1^2 v_{3,x}) + 3\partial_x (v_{1,x} v_{2,x}). \quad (5.4)$$

The above flows are the lowest evolution equations of a novel higher grading hierarchy characterized by $a = 3$. To reduce these equations to the $a = 1$ mKdV hierarchy one sets $v_1 = 0$, $v_2 = 0$ for which the flow equations (5.2) and (5.3) identically vanish. The remaining variable $v_3 = v$ satisfies the mKdV t_3 -flow (4.11) that agrees with equation (5.4) in this limit.

6. Conclusions and further developments

In this paper we have proposed a unified universal construction of a class of integrable hierarchies based upon a generalization of Riemann-Hilbert-Birkhoff decomposition, which incorporates higher graded structures and different types of vacuum. The key ingredient is the decomposition of the affine algebra $\hat{sl}_2$ into graded subspaces according to the principal gradation and a choice of semi-simple grade a constant generator $E = E^{(a)}$. The construction is systematic and generates a series of flow equations associated to positive and negative graded Lie algebraic elements. As a basic $a = 1$ example our construction generates the well-known mKdV hierarchy which naturally splits into two classes of solutions, those constructed from zero vacuum, for positive and negative odd graded flows and those build up from non-zero constant vacuum solutions associated to positive odd and negative even flows. The second example concerns a grade 2 construction yielding three known hierarchies namely, KN, CLL and GI hierarchies. In fact our construction indicates that they differ by a change of variables induced by an ambiguity in parameterizing the zero graded sector in g-RHB decomposition. Although these three hierarchies are equivalent, we show that the CLL hierarchy stands out in the sense that it is the only one admitting non-zero constant vacuum solution for all flows. Such peculiar property allows the reduction of the CLL to the Burgers hierarchy where one of the CLL fields is set to constant. In fact, the construction of soliton solutions for the CLL hierarchy [14] in terms of deformed vertex operators shows that this is indeed the case.

We should point out that the construction developed in this paper, apart from providing an advantage of unifying several known hierarchies within the same structure, also paves the way towards systematic construction of higher graded hierarchies as exemplified by t_3 -flow of the $a = 3$ model of section 5. Of course many new hierarchies may be constructed within the same universal framework considering the same $\hat{sl}_2$ affine algebra and principal gradation for $a = 4$ and higher values of a . An interesting study of such framework involving homogeneous gradation shows connection between AKNS, WKI and other hierarchies that will be developed elsewhere [23].

Generalizations of this construction to higher rank affine algebras, as well as, inclusion of other gradations may uncover interesting new and relationships among integrable hierarchies.

CRedit authorship contribution statement

H. Aratyn: Writing – review & editing. **C.P. Constantinidis:** Writing – original draft, Writing – review & editing. **J.F. Gomes:** Writing – review & editing. **T.C. Santiago:** Writing – original draft, Writing – review & editing. **A.H. Zimmerman:** Writing – original draft, Writing – review & editing.

Declaration of competing interest

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Data availability

No data was used for the research described in the article.

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