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Kernel polynomials from L-orthogonal polynomials [☆]

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ABSTRACT

A positive measure ψ defined on $[a,b]$ such that its moments $\mu_n = \int_a^b t^n\, d\psi(t)$ exist for $n = 0, \pm 1, \pm 2, \ldots$, is called a strong positive measure on $[a,b]$. If $0 \leqslant a < b \leqslant \infty$ then the sequence of (monic) polynomials $\{Q_n\}$, defined by $\int_a^b t^{-n+s} Q_n(t)\, d\psi(t) = 0$, $s = 0, 1, \ldots, n-1$, is known to exist. We refer to these polynomials as the L-orthogonal polynomials with respect to the strong positive measure ψ. The purpose of this manuscript is to consider some properties of the kernel polynomials associated with these L-orthogonal polynomials. As applications, we consider the quadrature rules associated with these kernel polynomials. Associated eigenvalue problems and numerical evaluation of the nodes and weights of such quadrature rules are also considered.

1. Introduction

Let $0 \leqslant a < b \leqslant \infty$ and let ψ be a bounded, non-decreasing function on $[a,b]$ with infinitely many points of increase in $[a,b]$ and such that all the moments $\mu_n = \int_a^b t^n\, d\psi(t)$, $n = 0, \pm 1, \pm 2, \ldots$, exist. We refer to ψ as a strong positive measure on $[a,b]$ and consider the sequence of monic polynomials $\{Q_n\}_{n=0}^{\infty}$ defined by

$$\int_a^b t^{-n+s} Q_n(t)\, d\psi(t) = 0, \quad s = 0, 1, \ldots, n-1. \tag{1.1}$$

Such polynomials were introduced in [12] in order to study the strong Stieltjes moment problem.

The sequence of Laurent polynomials or L-polynomials $\{t^{-\lfloor (n+1)/2 \rfloor} Q_n(t)\}$ form a sequence of orthogonal functions with respect to the measure ψ (see [10]). Thus for convenience, we refer to $\{Q_n\}$ as a sequence of L-orthogonal polynomials.

It is known that these polynomials satisfy the three term recurrence relation

$$Q_{n+1}(z) = (z - \beta_{n+1}) Q_n(z) - \alpha_{n+1} z Q_{n-1}(z), \quad n \geqslant 1, \tag{1.2}$$

with $Q_0(z) = 1$ and $Q_1(z) = z - \beta_1$, where $\beta_1 = \sigma_{0,0}/\sigma_{0,-1} = \mu_0/\mu_{-1}$,

$$\beta_{n+1} = -\alpha_{n+1} \frac{\sigma_{n-1,-1}}{\sigma_{n,-1}} \quad \text{and} \quad \alpha_{n+1} = \frac{\sigma_{n,n}}{\sigma_{n-1,n-1}}, \quad n \geqslant 1. \tag{1.3}$$

The numbers $\sigma_{n,s} = \int_a^b t^{-n+s} Q_n(t)\, d\psi(t)$, which by (1.1) must satisfy $\sigma_{n,s} = 0$ for $0 \leqslant s \leqslant n-1$, $n \geqslant 1$, also satisfy

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$$\sigma_{n,n} > 0 \quad \text{and} \quad (-1)^n \sigma_{n,-1} > 0, \quad n \geqslant 0. \tag{1.4}$$

The above inequalities are consequences of the linear system obtained from (1.1) using the positiveness of the associated Hankel determinants as verified from the quadratic forms $\int_a^b t^m p^2(t)\, d\psi(t)$ for any non-zero polynomial p and for any $m = 0, \pm 1, \pm 2, \ldots$. Thus, from (1.3) one can observe that $\beta_n > 0$, $\alpha_{n+1} > 0$, $n \geqslant 1$, and consequently, from (1.2), $(-1)^n Q_n(0) = \beta_1 \beta_2 \cdots \beta_n > 0$, $n \geqslant 1$.

Knowing the coefficients $\{\alpha_n, \beta_n\}$ in the three term recurrence relation (1.2) is very useful in many contexts such as the numerical generation of the values of the polynomials Q_n and their zeros (see Section 6), etc.

Studies of polynomials satisfying three term recurrence relations of the type (1.2) have appeared prior to [12] in the theory of continued fractions and two-point Padé approximants (see, for example, [11] and [14]). For some very recent contributions regarding the polynomials satisfying such three term recurrence relations see, for example, [13] and [17].

It is known that the zeros $z_{n,i}$, $i = 1, 2, \ldots, n$, of Q_n are all positive, distinct and lie within (a, b). The zeros of Q_n also interlace with the zeros of Q_{n-1}. Moreover, if $\lambda_{n,i} = \int_a^b [Q_n'(z_{n,i})(t - z_{n,i})]^{-1} Q_n(t)\, d\psi(t)$, $i = 1, 2, \ldots, n$, then

$$\int_a^b f(t)\, d\psi(t) = \sum_{i=1}^{n} \lambda_{n,i} f(z_{n,i}) \quad \text{for } t^n f(t) \in \mathbb{P}_{2n-1}. \tag{1.5}$$

This is the quadrature rule of highest algebraic degree of precision associated with the L-orthogonal polynomial Q_n (see, for example, [2] and [15]), which is analogous to the Gaussian rule associated with the n-th degree orthogonal polynomial on the real line.

The objective in the present manuscript is to look at the properties of kernel polynomials defined from L-orthogonal polynomials, associated quadrature rules and also numerical generation of the nodes and weights of these quadrature rules.

2. Kernel polynomials

Most of the studies in this section and also in Section 3 were inspired by results found in Freud [5] regarding ordinary orthogonal polynomials on the real line.

With w real, let

$$G_{n+1}(z; w) = Q_n(w) Q_{n+1}(z) - Q_{n+1}(w) Q_n(z), \quad n \geqslant 0.$$

Clearly, $G_{n+1}(z; w)$ is a polynomial in z and if we denote its degree by $\tilde{n} + 1$ then

$$\tilde{n} = \begin{cases} n & \text{if } Q_n(w) \neq 0, \\ n-1 & \text{if } Q_n(w) = 0, \end{cases} \quad n \geqslant 0.$$

From (1.1) the L-orthogonality property

$$\int_a^b t^{-n+s} G_{n+1}(t; w)\, d\psi(t) = 0, \quad 0 \leqslant s \leqslant n-1, \tag{2.1}$$

of $G_{n+1}(z; w)$ can be verified for $n \geqslant 1$.

Theorem 2.1.

$$\frac{G_{n+1}(z; w)}{z - w} = Q_n(w) Q_n(z) + \alpha_{n+1} \beta_n Q_{n-1}(w) Q_{n-1}(z) + \alpha_{n+1} \alpha_n w z \frac{G_{n-1}(z; w)}{z - w}, \tag{2.2}$$

for $n \geqslant 2$, *with* $\frac{G_1(z;w)}{z-w} = 1$ *and* $\frac{G_2(z;w)}{z-w} = Q_1(w) Q_1(z) + \alpha_2 \beta_1 Q_0(w) Q_0(z)$. *Consequently, for* $G_{n+1}'(z; w) = Q_n(w) Q_{n+1}'(z) - Q_{n+1}(w) Q_n'(z)$, *then*

$$G_{n+1}'(w; w) = Q_n^2(w) + \alpha_{n+1} \beta_n Q_{n-1}^2(w) + \alpha_{n+1} \alpha_n w^2 G_{n-2}'(w; w), \quad n \geqslant 2,$$

with $G_1'(w; w) = 1$ *and* $G_2'(w; w) = Q_1^2(w) + \alpha_2 \beta_1 Q_0^2(w)$.

Proof. Applying the three term recurrence relation (1.2) for $Q_{n+1}(z)$ and $Q_{n+1}(w)$ gives

$$G_{n+1}(z, w) = (z - w) Q_n(w) Q_n(z) + \alpha_n \big[w Q_{n-1}(w) Q_n(z) - z Q_{n-1}(z) Q_n(w) \big].$$

Again applying the three term recurrence relation for $Q_n(z)$ and $Q_n(w)$ in the second term on the right leads to (2.2). The initial conditions for (2.2) are also easily verified.

To obtain the remaining results of the theorem, one only needs to consider the limits in (2.2) as $z \to w$. □

The above theorem also means that $G_{n+1}'(w; w) > 0$ for all real w and all $n \geqslant 0$.

Theorem 2.2. *For any $w \in (-\infty, \infty)$ all the zeros of $G_{n+1}(z; w)$ are real and distinct, and at least n of these zeros are in (a, b). If $z_{n,1}(w), z_{n,2}(w), \ldots, z_{n,\tilde{n}}(w)$ and w are the zeros of $G_{n+1}(z; w)$, then*

$$G_{n+1}(z; w) = \chi_{n,i}(w) G_{n+1}\big(z; z_{n,i}(w)\big), \quad i = 1, 2, \ldots, \tilde{n}, \tag{2.3}$$

where $\chi_{n,i}(w) = Q_{n+1}(w)/Q_{n+1}(z_{n,i}(w))$ if $Q_n(w) = 0$ and $\chi_{n,i}(w) = Q_n(w)/Q_n(z_{n,i}(w))$ otherwise.

Proof. Since $(a, b) \subseteq (0, \infty)$, using the L-orthogonality property (2.1) one can show that $G_{n+1}(z; w)$ changes sign at least n times in (a, b) and thus leading to the initial results of the theorem.

To obtain (2.3), first assuming $Q_n(w) \neq 0$ we have from $G_{n+1}(z_{n,i}(w); w) = 0$,

$$Q_{n+1}\big(z_{n,i}(w)\big) = \frac{Q_{n+1}(w)}{Q_n(w)} Q_n\big(z_{n,i}(w)\big).$$

Since $Q_{n+1}(z)$ and $Q_n(z)$ do not have common zeros, $Q_n(z_{n,i}(w)) \neq 0$ and

$$\frac{Q_{n+1}(z_{n,i}(w))}{Q_n(z_{n,i}(w))} = \frac{Q_{n+1}(w)}{Q_n(w)}.$$

Thus, from $\frac{G_{n+1}(z;w)}{Q_n(w)} = Q_{n+1}(z) - \frac{Q_{n+1}(w)}{Q_n(w)} Q_n(z)$,

$$\frac{G_{n+1}(z; w)}{Q_n(w)} = Q_{n+1}(z) - \frac{Q_{n+1}(z_{n,i}(w))}{Q_n(z_{n,i}(w))} Q_n(z) = \frac{G_{n+1}(z; z_{n,i}(w))}{Q_n(z_{n,i}(w))},$$

which is the same as (2.3) when $Q_n(w) \neq 0$.

When $Q_n(w) = 0$ then $G_{n+1}(z; w) = -Q_{n+1}(w) Q_n(z)$ and $z_{n,i}(w)$ are the zeros of $Q_n(z)$. Hence, $z_{n,i}(w)$ is not a zero of $Q_{n+1}(z)$ and (2.3) follows immediately. □

Now let $\kappa_n(z; w)$ and $\Lambda_n(w)$ be defined by

$$\kappa_n(z; w) = \frac{1}{G'_{n+1}(w; w)} \frac{G_{n+1}(z; w)}{z - w}, \qquad \Lambda_n(w) = \int_a^b t^{-n} \kappa_n(t; w)\, d\psi(t), \quad n \geqslant 0. \tag{2.4}$$

Clearly, $\kappa_n(z; w)$ is a polynomial in z of degree $\tilde{n}$ such that its zeros are the zeros $z_{n,1}(w), z_{n,2}(w), \ldots, z_{n,\tilde{n}}(w)$ of $G_{n+1}(z; w)$ as defined in Theorem 2.2. Moreover,

$$(z - w)\kappa_n(z; w) = \hat{\chi}_{n,i}(w)\big(z - z_{n,i}(w)\big)\kappa_n\big(z; z_{n,i}(w)\big), \quad i = 1, 2, \ldots, \tilde{n},$$

where $\hat{\chi}_{n,i}(w) = \chi_{n,i}(w) G'_{n+1}(z_{n,i}(w); z_{n,i}(w))/G'_{n+1}(w; w)$ and $\chi_{n,i}(w)$ are as in Theorem 2.2.

The function $\Lambda_n(w)$ is a continuous function of w in $(-\infty, \infty)$ and the following theorem (with its demonstration) shows its importance in the studies of quadrature rules.

Theorem 2.3. *The functions $\Lambda_n(w)$, $n \geqslant 0$, are such that $\Lambda_n(w) > 0$ for $w \in (-\infty, \infty)$.*

Proof. Let $F(z) = z^n f(z) \in \mathbb{P}_{n+\tilde{n}}$. With w any finite real number, let $z_{n,i}(w)$, $i = 1, 2, \ldots, \tilde{n}$, and $z_{n,\tilde{n}+1}(w) = w$ be the zeros of $G_{n+1}(z; w)$. Hence, using the interpolation formula on the zeros of $G_{n+1}(z; w)$,

$$\begin{aligned} F(z) = {} & \sum_{i=1}^{\tilde{n}+1} \frac{G_{n+1}(z; w)}{G'_{n+1}(z_{n,i}(w); w)(z - z_{n,i}(w))} \big(z_{n,i}(w)\big)^n f\big(z_{n,i}(w)\big) \\ & + F\big[z_{n,1}(w), \ldots, z_{n,\tilde{n}}(w), z_{n,\tilde{n}+1}(w), z\big] G_{n+1}(z; w), \end{aligned}$$

where the interpolation polynomial is given by the Lagrange formula and the error term is given by the Newton divided difference formula. Since $F(z) \in \mathbb{P}_{n+\tilde{n}}$, we must have $F[z_{n,1}(w), \ldots, z_{n,\tilde{n}}(w), z_{n,\tilde{n}+1}(w), z] \in \mathbb{P}_{n-1}$. Hence, from (2.1),

$$\int_a^b f(t)\, d\psi(t) = \int_a^b t^{-n} F(t)\, d\psi(t) = \sum_{i=1}^{\tilde{n}+1} c_{n,i} \big(z_{n,i}(w)\big)^n f\big(z_{n,i}(w)\big),$$

where $c_{n,i} = \int_a^b \frac{t^{-n} G_{n+1}(t; w)}{G'_{n+1}(z_{n,i}(w); w)(t - z_{n,i}(w))}\, d\psi(t)$, $i = 1, 2, \ldots, \tilde{n} + 1$. Thus, from (2.3) and (2.4),

$$c_{n,i} = \int_a^b \frac{t^{-n} G_{n+1}(t; z_{n,i}(w))}{G'_{n+1}(z_{n,i}(w); z_{n,i}(w))(t - z_{n,i}(w))}\, d\psi(t) = \Lambda_n\big(z_{n,i}(w)\big), \quad i = 1, 2, \ldots, \tilde{n} + 1.$$

Consequently, we have the quadrature rule

$$\int_a^b f(t)\,d\psi(t) = \sum_{i=1}^{\tilde{n}+1} \big(z_{n,i}(w)\big)^n \Lambda_n\big(z_{n,i}(w)\big) f\big(z_{n,i}(w)\big), \quad n \geqslant 1, \tag{2.5}$$

which holds for any f such that $z^n f(z) \in \mathbb{P}_{n+\tilde{n}}$.

Using

$$f(z) = z^{-n}\left(\frac{G_{n+1}(z, z_{n,k}(w))}{z - z_{n,k}(w)}\right)^2$$

in (2.5) it follows that

$$\Lambda_n\big(z_{n,k}(w)\big) = \big(G'_{n+1}\big(z_{n,k}(w); z_{n,k}(w)\big)\big)^{-2} \int_a^b t^{-n}\left(\frac{G_{n+1}(t, z_{n,k}(w))}{t - z_{n,k}(w)}\right)^2 d\psi(t) > 0,$$

for $k = 1, 2, \ldots, \tilde{n}+1$. Since $z_{n,\tilde{n}+1}(w) = w$ is chosen to be any finite real number, we conclude that $\Lambda_n(w) > 0$ for $w \in (-\infty, \infty)$. □

Theorem 2.4. *The polynomial $\kappa_n(z; w)$ is a reproducing kernel for all $p \in \mathbb{P}_n$ in the sense*

$$\frac{1}{\Lambda_n(w)} \int_a^b p(t) t^{-n} \kappa_n(t; w)\,d\psi(t) = p(w).$$

Proof. Note that $\kappa_n(z_{n,\tilde{n}+1}(w); w) = \kappa_n(w; w) = 1$ and $\kappa_n(z_{n,i}(w); w) = 0$ for $i = 1, 2, \ldots, \tilde{n}$. Since $z^n[p(z)z^{-n}\kappa_n(z; w)] \in \mathbb{P}_{n+\tilde{n}}$ whenever $p(z) \in \mathbb{P}_n$, then from the quadrature rule (2.5),

$$\int_a^b p(t) t^{-n} \kappa_n(t; w)\,d\psi(t) = \sum_{i=1}^{\tilde{n}+1} \Lambda_n\big(z_{n,i}(w)\big)\kappa_n\big(z_{n,i}(w); w\big) p\big(z_{n,i}(w)\big) = \Lambda_n(w) p(w)$$

and thus concluding the proof of the theorem. □

Because of the reproducing property given by Theorem 2.4, we will refer to $\kappa_n(z; w)$ as the kernel polynomials associated with the L-orthogonal polynomials Q_n. See also the nomenclature used on p. 35 of Chihara [3].

Theorem 2.4 also means that for all $n \geqslant 1$,

$$w^s \Lambda_n(w) = \int_a^b t^{-n+s} \kappa_n(t; w)\,d\psi(t), \quad 0 \leqslant s \leqslant n. \tag{2.6}$$

Hence from Theorem 2.1, by considering $w\Lambda_n(w) = \int_a^b t^{-n+1}\kappa_n(t; w)\,d\psi(t)$, we obtain $\Lambda_n(w)G'_{n+1}(w; w) = \alpha_{n+1}\alpha_n \Lambda_{n-2}(w) G'_{n-1}(w; w)$, $n \geqslant 2$. Consequently,

$$\Lambda_n(w)^{-1} = \sigma_{n,n}^{-1} G'_{n+1}(w; w), \quad n \geqslant 1. \tag{2.7}$$

3. More on the zeros and quadrature rules

The following theorem gives further information regarding the positions of the zeros of $G_{n+1}(z; w)$ with respect to w.

Theorem 3.1. *Let the zeros $z_{n,i}$ of Q_n and $z_{n+1,i}$ of Q_{n+1} be such that*

$$z_{n+1,1} < z_{n,1} < z_{n+1,2} < z_{n,2} < \cdots < z_{n+1,n} < z_{n,n} < z_{n+1,n+1}.$$

Also let $z_{n,0} = -\infty$ and $z_{n,n+1} = \infty$. Then, assuming that the zero w of $G_{n+1}(z; w)$ is not a zero of Q_n and assuming also that the remaining $\tilde{n} = n$ zeros of $G_{n+1}(z; w)$ are such that $z_{n,1}(w) < z_{n,2}(w) < \cdots < z_{n,n}(w)$, we have the following:

(1) *For any k, $k = 0, 1, \ldots, n$, as w increases on $(z_{n,k}, z_{n+1,k+1}]$ then, for $i = 1, 2, \ldots, n$,*

the zero $z_{n,i_1+i_2}(w)$ monotonically increases on $(z_{n,i_1}, z_{n+1,i_1+1}]$,

where $i_1 = i + k$ mod $n+1$ and $i_2 = 1$ if $i_1 < k$, $i_2 = 0$ otherwise.

(2) *For any k, $k = 0, 1, \ldots, n$, as w increases on $[z_{n+1,k+1}, z_{n,k+1})$ then for $i = 1, 2, \ldots, n$,*

$$\textit{the zero } z_{n,i_1+i_2}(w) \textit{ monotonically increases on } [z_{n+1,i_1+1}, z_{n,i_1+1}),$$

where $i_1 = i + k \bmod n + 1$ and $i_2 = 1$ if $i_1 < k$, $i_2 = 0$ otherwise.

Proof. Note that

$$\frac{d}{dw}\frac{Q_{n+1}(w)}{Q_n(w)} = \frac{G'_{n+1}(w; w)}{[Q_n(w)]^2} > 0 \quad \text{for all real } w.$$

Moreover, $Q_{n+1}(w)/Q_n(w) < 0$ if $w < z_{n+1,1}$ and $Q_{n+1}(w)/Q_n(w) > 0$ if $w > z_{n+1,n+1}$. Hence the theorem follows from Theorem 2.2 and from

$$G_{n+1}(z; w) = Q_n(w)\left[Q_{n+1}(z) + \frac{Q_{n+1}(w)}{Q_n(w)}Q_n(z)\right],$$

with the use of [1, Lemma 2]. □

From Theorem 3.1 we can also make the following observations.

(A) If w is chosen inside $(a, z_{n,1}(b))$ or inside $(z_{n,n}(a), b)$ or inside any one of the intervals $(z_{n,k}(a), z_{n,k+1}(b))$, $k = 1, 2, \ldots, n-1$, then all $n+1$ zeros of $G_{n+1}(z; w)$ are within (a, b).
(B) If the range of values for w is taken to be $(-\infty, z_{n+1,1}) \cup [z_{n+1,n+1}, \infty)$ then the range of values covered by all the zeros of $G_{n+1}(z; w)$ is $(-\infty, \infty) \setminus \{z_{n,1}, z_{n,2}, \ldots, z_{n,n}\}$.
(C) $\lim_{w\to\pm\infty} z_{n,i}(w) = z_{n,i}$ for $i = 1, 2, \ldots, n$, and if, for example, the range of values for w is assumed to be $(-\infty, z_{n+1,1}) \cup [z_{n+1,n+1}, \infty]$ then we can assume that the range of values covered by all the zeros of $G_{n+1}(z; w)$ is the extended real line.

Theorem 3.2. *Let $w \in (-\infty, z_{n+1,1}) \cup [z_{n+1,n+1}, \infty)$ and let $z_{n,i}(w)$, $i = 1, 2, \ldots, n$, and $z_{n,n+1}(w) = w$ be the zeros of $G_{n+1}(z; w)$. Also let*

$$\lambda_{n,i}(w) = \big(z_{n,i}(w)\big)^n \Lambda_n\big(z_{n,i}(w)\big), \quad i = 1, 2, \ldots, n+1.$$

Then for any $n \geqslant 1$,

$$\int_a^b f(t)\,d\psi(t) = \sum_{i=1}^{n+1} \lambda_{n,i}(w) f\big(z_{n,i}(w)\big), \tag{3.1}$$

which holds for $z^n f(z) \in \mathbb{P}_{2n}$. Moreover, when $z^n f(z) \in \mathbb{P}_{2n-1}$ by letting $w \to \infty$,

$$\int_a^b f(t)\,d\psi(t) = \sum_{i=1}^{n} \lambda_{n,i} f(z_{n,i}), \quad n \geqslant 1.$$

Here, $z_{n,i} = \lim_{w\to\infty} z_{n,i}(w)$ and $\lambda_{n,i} = \lim_{w\to\infty} \lambda_{n,i}(w)$, $i = 1, 2, \ldots, n$, are as in (1.5).

Proof. When $w \in (-\infty, z_{n+1,1}) \cup [z_{n+1,n+1}, \infty)$ we must have $\tilde{n} = n$. Therefore, the first quadrature rule is a restatement of (2.5) for such values of w. This quadrature rule is an $n+1$ point quadrature rule of almost highest algebraic degree of precision with the prescribed node $z_{n,n+1}(w) = w \in (-\infty, z_{n+1,1}) \cup [z_{n+1,n+1}, \infty)$.

The latter quadrature rule in the theorem is the quadrature rule (1.5) of highest algebraic degree of precision associated with the L-orthogonal polynomial Q_n. This quadrature rule is the same as the quadrature rule (2.5) when $\tilde{n} = n - 1$ holds. Note that we also get this quadrature rule from (3.1) by substituting n by $n-1$ and w by $z_{n,n}$.

From (2.4) and (2.6),

$$\lim_{w\to\infty} \lambda_{n,i}(w) = \lim_{w\to\infty} \int_a^b \frac{G_{n+1}(t; z_{n,i}(w))}{G'_{n+1}(z_{n,i}(w); z_{n,i}(w))(t - z_{n,i}(w))}\,d\psi(t), \quad i = 1, 2, \ldots, n.$$

Since $\lim_{w\to\infty} z_{n,i}(w) = z_{n,i}$, $i = 1, 2, \ldots, n$, then

$$\lim_{w\to\infty} \lambda_{n,i}(w) = \int_a^b \frac{Q_n(t)}{Q'_n(z_{n,i})(t - z_{n,i})}\,d\psi(t) = \lambda_{n,i}, \quad i = 1, 2, \ldots, n.$$

Moreover, from (2.7) one has $\lim_{w\to\infty} w^{2n}\Lambda_n(w) = \sigma_{n,n}$, $n \geqslant 1$. This also means

$$\lim_{w\to\infty} \lambda_{n,n+1}(w) f(w) = \lim_{w\to\infty} w^n \Lambda_n(w) f(w) = 0 \quad \text{if } z^n f(z) \in \mathbb{P}_{2n-1}.$$

Hence, the second quadrature rule follows as a limit of the first quadrature rule when $z^n f(z) \in \mathbb{P}_{2n-1}$. $\square$

The L-orthogonality of $\{Q_n\}$ also means that the set $\{z^{n-l}Q_l(z)\}_{l=0}^{n}$ forms a basis for $\mathbb{P}_n$. This can be verified as in the proof of Theorem 4.1. Hence, if we write $\kappa_n(z;w) = \sum_{l=0}^{n} \gamma_l z^{n-l} Q_l(z)$, then from (1.1),

$$\int_a^b Q_k(t) t^{-n} \kappa_n(t;w)\, d\psi(t) = \sum_{l=0}^{n} \gamma_l \int_a^b t^{-l} Q_l(t) Q_k(t)\, d\psi(t) = \sigma_{k,k} \sum_{l=0}^{k} \gamma_l,$$

for $0 \leqslant k \leqslant n$, where $\sigma_{k,k} = \int_a^b Q_k(t)\, d\psi(t) = \mu_0 \alpha_2 \ldots \alpha_{k+1}$. On the other hand from Theorem 2.4 that $\int_a^b Q_k(t) t^{-n} \times \kappa_n(t;w)\, d\psi(t) = Q_k(w)\Lambda_n(w)$. Consequently,

$$\frac{\gamma_0}{\Lambda_n(w)} = \frac{Q_0(w)}{\sigma_{0,0}} \quad \text{and} \quad \frac{\gamma_k}{\Lambda_n(w)} = \left[\frac{Q_k(w)}{\sigma_{k,k}} - \frac{Q_{k-1}(w)}{\sigma_{k-1,k-1}}\right], \quad k = 1, 2, \ldots, n.$$

Therefore, $\kappa_n(z;w) = \sum_{l=0}^{n} \gamma_l z^{n-l} Q_l(z)$ can be written as

$$\kappa_n(z;w) = \Lambda_n(w) \frac{Q_0(w)}{\sigma_{0,0}} z^n Q_0(z) + \sum_{l=1}^{n} \Lambda_n(w) \left[\frac{Q_l(w)}{\sigma_{l,l}} - \frac{Q_{l-1}(w)}{\sigma_{l-1,l-1}}\right] z^{n-l} Q_l(z). \tag{3.2}$$

Using matrix notation, this can also be written as

$$\frac{1}{\Lambda_n(w)} \kappa_n(z;w) = \mathbf{x}_{n+1}^{\mathfrak{l}}(w)^T \mathbf{x}_{n+1}^{\mathfrak{r}}(z),$$

where

$$\mathbf{x}_{n+1}^{\mathfrak{l}}(w) = \left[\frac{Q_0(w)}{\sigma_{0,0}}, \frac{Q_1(w)}{\sigma_{1,1}} - \frac{Q_0(w)}{\sigma_{0,0}}, \ldots, \frac{Q_n(w)}{\sigma_{n,n}} - \frac{Q_{n-1}(w)}{\sigma_{n-1,n-1}}\right]^T$$

and

$$\mathbf{x}_{n+1}^{\mathfrak{r}}(z) = \left[z^n Q_0(z), z^{n-1} Q_1(z), \ldots, Q_n(z)\right]^T.$$

This means, in particular,

$$\mathbf{x}_{n+1}^{\mathfrak{l}}(w)^T \mathbf{x}_{n+1}^{\mathfrak{r}}\big(z_{n,i}(w)\big) = \delta_{\tilde{n}+1,i} \frac{1}{\Lambda_n(w)}, \quad 1 \leqslant i \leqslant \tilde{n}+1.$$

Considering also the polynomials $\kappa_n(z; z_{n,j}(w))$, $j = 1, 2, \ldots, \tilde{n}$, we can then conclude that

$$\mathbf{x}_{n+1}^{\mathfrak{l}}\big(z_{n,j}(w)\big)^T \mathbf{x}_{n+1}^{\mathfrak{r}}\big(z_{n,i}(w)\big) = \delta_{j,i} \frac{1}{\Lambda_n(z_{n,j}(w))}, \quad 1 \leqslant i, j \leqslant \tilde{n}+1,$$

which means the two sets of vectors

$$\begin{gathered}\big\{\mathbf{x}_{n+1}^{\mathfrak{l}}\big(z_{n,1}(w)\big), \ldots, \mathbf{x}_{n+1}^{\mathfrak{l}}\big(z_{n,\tilde{n}}(w)\big), \mathbf{x}_{n+1}^{\mathfrak{l}}\big(z_{n,\tilde{n}+1}(w)\big)\big\}, \\ \big\{\mathbf{x}_{n+1}^{\mathfrak{r}}\big(z_{n,1}(w)\big), \ldots, \mathbf{x}_{n+1}^{\mathfrak{r}}\big(z_{n,\tilde{n}}(w)\big), \mathbf{x}_{n+1}^{\mathfrak{r}}\big(z_{n,\tilde{n}+1}(w)\big)\big\}\end{gathered}$$

are biorthogonal.

We can also rearrange (3.2) to be

$$\kappa_n(z;w) = \sum_{l=0}^{n-1} \Lambda_n(w)\big[z^{n-l} Q_l(z) - z^{n-l-1} Q_{l+1}(z)\big] \frac{Q_l(w)}{\sigma_{l,l}} + \Lambda_n(w) Q_n(z) \frac{Q_n(w)}{\sigma_{n,n}}$$

and obtain the two sets of biorthogonal vectors

$$\begin{gathered}\big\{\mathbf{y}_{n+1}^{\mathfrak{l}}\big(z_{n,1}(w)\big), \ldots, \mathbf{y}_{n+1}^{\mathfrak{l}}\big(z_{n,\tilde{n}}(w)\big), \mathbf{y}_{n+1}^{\mathfrak{l}}\big(z_{n,\tilde{n}+1}(w)\big)\big\}, \\ \big\{\mathbf{y}_{n+1}^{\mathfrak{r}}\big(z_{n,1}(w)\big), \ldots, \mathbf{y}_{n+1}^{\mathfrak{r}}\big(z_{n,\tilde{n}}(w)\big), \mathbf{y}_{n+1}^{\mathfrak{r}}\big(z_{n,\tilde{n}+1}(w)\big)\big\},\end{gathered}$$

such that

$$\mathbf{y}_{n+1}^{\mathfrak{l}}\big(z_{n,j}(w)\big)^T\mathbf{y}_{n+1}^{\mathfrak{r}}\big(z_{n,i}(w)\big)=\delta_{j,i}\frac{1}{\Lambda_n(z_{n,j}(w))},\quad 1\leqslant i,j\leqslant \tilde{n}+1, \tag{3.3}$$

where

$$\mathbf{y}_{n+1}^{\mathfrak{l}}(z)=\begin{bmatrix} z^nQ_0(z)-z^{n-1}Q_1(z)\\ z^{n-1}Q_1(z)-z^{n-2}Q_2(z)\\ \vdots\\ z^1Q_{n-1}(z)-Q_n(z)\\ Q_n(z)\end{bmatrix}\quad\text{and}\quad \mathbf{y}_{n+1}^{\mathfrak{r}}(w)=\begin{bmatrix}\sigma_{0,0}^{-1}Q_0(w)\\ \sigma_{1,1}^{-1}Q_1(w)\\ \vdots\\ \sigma_{n-1,n-1}^{-1}Q_{n-1}(w)\\ \sigma_{n,n}^{-1}Q_n(w)\end{bmatrix}.$$

Using (3.3) we also have that the functions $\Lambda_n(w)$ for $n\geqslant 1$ and for any real w satisfy

$$\Lambda_n(w)^{-1}=\sum_{j=0}^{n-1}\big[w^{n-j}Q_j(w)-w^{n-j-1}Q_{j+1}(w)\big]\frac{Q_j(w)}{\sigma_{j,j}}+Q_n(w)\frac{Q_n(w)}{\sigma_{n,n}}. \tag{3.4}$$

4. Kernel polynomials as L-orthogonal polynomials

Let $w\in(-\infty,\infty)\setminus(a,b)$. Hence the degree $\tilde{n}$ of $\kappa_n(z;w)$ satisfies $\tilde{n}=n$ for $n\geqslant 0$. Let $\{K_n(z;w)\}_{n=0}^N$ be the sequence of monic kernel polynomials given by

$$K_n(z;w)=\frac{G'_{n+1}(w;w)}{Q_n(w)}\kappa_n(z;w)=\frac{G_{n+1}(z;w)}{Q_n(w)\,(z-w)},\quad n\geqslant 0,$$

and let $\sigma_{n,s}(w)=\int_a^b t^{-n+s}K_n(t;w)\,d\psi(t;w)$, where

$$d\psi(t;w)=|t-w|\,d\psi(t).$$

Hence from (2.1), since $|t-w|=(t-w)\mathrm{sgn}(t-w)$ in (a,b),

$$\sigma_{n,s}(w)=0,\quad 0\leqslant s\leqslant n-1,$$
$$\sigma_{n,n}(w)=\int_a^b t^{-n}\big[K_n(t;w)\big]^2|t-w|\,d\psi(t)>0,$$

for $n\geqslant 1$. Also $\sigma_{0,0}(w)=\int_a^b|t-w|\,d\psi(t)>0$. Hence, $\{K_n(z;w)\}_{n=0}^\infty$ is the sequence of monic L-orthogonal polynomials with respect to the strong positive measure $\psi(t;w)$.

Moreover, by denoting

$$\nu_{n+1}(w)=\frac{Q_{n+1}(w)}{Q_n(w)},\quad n\geqslant 0,$$

then from (1.4) and (2.1) $\sigma_{n,-1}(w)=|\nu_{n+1}(w)|\sigma_{n,-1}$ for $n>0$.

Theorem 4.1. *The polynomials* $\{K_n(z;w)\}$ *satisfy the three term recurrence relation*

$$K_{n+1}(z;w)=\big(z-\beta_{n+1}(w)\big)K_n(z;w)-\alpha_{n+1}(w)zK_{n-1}(z;w),\quad n=1,2,\dots,N-1,$$

with $K_0(z;w)=1$ *and* $K_1(z;w)=z-\beta_1(w)$, *where*

$$\beta_1(w)=\frac{\sigma_{0,0}(w)}{\sigma_{0,-1}(w)}=\beta_1\left(1-\frac{\alpha_2}{\nu_1(w)}\right),$$
$$\beta_{n+1}(w)=-\alpha_{n+1}(w)\frac{\sigma_{n-1,-1}(w)}{\sigma_{n,-1}(w)}\quad\textit{and}\quad \alpha_{n+1}(w)=\frac{\sigma_{n,n}(w)}{\sigma_{n-1,n-1}(w)},\quad n\geqslant 1.$$

Proof. If $\sum_{j=0}^n\gamma_jt^{n-j}K_j(t;w)=0$, then multiplying by t^{-n+m} and integrating with respect to $\psi(t;w)$ we obtain the triangular system $\sum_{j=0}^m\gamma_j\sigma_{j,m}(w)=0$, $m=0,1,\dots,n$. Since $\sigma_{m,m}(w)\neq 0$, one must have $\gamma_m=0$ and thus the $n+1$ polynomials $t^{n-j}K_j(t;w)$, $j=0,1,\dots,n$, are linearly independent and form a basis for $\mathbb{P}_n$.

Since $K_{n+1}(z;w)-zK_n(z;w)\in\mathbb{P}_n$, thus we can write

$$K_{n+1}(t; w) - tK_n(t; w) = \sum_{j=0}^{n} \gamma_j t^{n-j} K_j(t; w).$$

Multiplying by t^{-n+s} and integrating with respect to $\psi(t; w)$ we obtain that $\sum_{j=0}^{s} \gamma_j \sigma_{j,s} = 0$, $0 \leqslant s \leqslant n-2$. Since, $\sigma_{s,s} \neq 0$, it follows that $\gamma_s = 0$, $s = 0, 1, \ldots, n-2$. Moreover, taking $s = n-1$ and $s = -1$ gives

$$\gamma_{n-2} = -\frac{\sigma_{n,n}(w)}{\sigma_{n-1,n-1}(w)} = -\alpha_{n+1}(w) \quad \text{and} \quad \gamma_{n-1} = \gamma_{n-2}\frac{\sigma_{n-1,-1}(w)}{\sigma_{n,-1}(w)} = -\beta_{n+1}(w).$$

This completes the proof of the theorem. □

Together with $(z-w)K_n(z; w) = Q_{n+1}(z) - \nu_{n+1}(w)Q_n(z)$, $n \geqslant 0$, we have from the three term recurrence relation for $\{Q_n(z)\}$,

$$(z-w)K_n(z; w) = \big(z - \beta_{n+1} - \nu_{n+1}(w)\big)Q_n(z) - \alpha_{n+1} z Q_{n-1}(z), \quad n \geqslant 1.$$

Substitutions of these in the three term recurrence relation for $\{K_n(z; w)\}$ leads to the three term recurrence relation

$$\begin{aligned} Q_{n+1}(z) = {} & \left(\frac{-\alpha_{n+2} + \alpha_{n+1}(w) + \nu_{n+1}(w)}{\beta_{n+2} - \beta_{n+1}(w) + \nu_{n+2}(w)} z - \frac{\beta_{n+1}(w)\nu_{n+1}(w)}{\beta_{n+2} - \beta_{n+1}(w) + \nu_{n+2}(w)}\right) Q_n(z) \\ & - \frac{\alpha_{n+1}(w)\,\nu_n(w)}{\beta_{n+2} - \beta_{n+1}(w) + \nu_{n+2}(w)} z Q_{n-1}(z), \quad n \geqslant 1. \end{aligned}$$

Hence, from the uniqueness of the three term recurrence relation for $\{Q_n\}$, we obtain

$$\nu_{n+2}(w) + \beta_{n+2} - \beta_{n+1}(w) = \nu_{n+1}(w) - \alpha_{n+2} + \alpha_{n+1}(w), \quad n \geqslant 1,$$

and also the following theorem which is useful in the sense of numerical generation.

Theorem 4.2. *Let $w \in (-\infty, \infty) \setminus (a, b)$. Then the coefficients in the three term recurrence relation for $\{K_n(z; w)\}$ satisfy*

$$\begin{aligned} \beta_1(w) &= \frac{\nu_2(w) + \beta_2}{w}\beta_1 = \frac{\nu_1(w) - \alpha_2}{\nu_1(w)}\beta_1, \\ \beta_{n+1}(w) &= \frac{\nu_{n+2}(w) + \beta_{n+2}}{\nu_{n+1}(w) + \beta_{n+1}}\beta_{n+1} = \frac{\nu_n(w)}{\nu_{n+1}(w)}\,\frac{\nu_{n+1}(w) - \alpha_{n+2}}{\nu_n(w) - \alpha_{n+1}}\beta_{n+1}, \\ \alpha_{n+1}(w) &= \frac{\nu_{n+1}(w) - \alpha_{n+2}}{\nu_n(w) - \alpha_{n+1}}\alpha_{n+1}, \quad n \geqslant 1, \end{aligned}$$

where the values of $\nu_n(w)$ can be generated by

$$\nu_1(w) = w - \beta_1 \quad \textit{and} \quad \nu_{n+1}(w) = w - \beta_{n+1} - \frac{w\alpha_{n+1}}{\nu_n(w)}, \quad n \geqslant 1.$$

The above recursive relation for $\{\nu_n(w)\}$ follows from the three term recurrence relation for $\{K_n(z; w)\}$.
Since $t^n f(t) \in \mathbb{P}_{2n-1}$ implies $(t-w)t^n f(t) \in \mathbb{P}_{2n}$, from the quadrature rule given by (3.1),

$$\int_a^b f(t)|t-w|\,d\psi(t) = \sum_{i=1}^{n+1} \big|z_{n,i}(w) - w\big|\lambda_{n,i}(w) f\big(z_{n,i}(w)\big),$$

where $z_{n,i}(w)$ are the zeros of $K_n(z; w)$ and $z_{n,n+1}(w) = w$. Thus, we have

Theorem 4.3. *The quadrature rule of highest algebraic degree of precision associated with the L-orthogonal polynomial $K_n(z; w)$ is*

$$\int_a^b f(t)\,d\psi(t; w) = \sum_{i=1}^{n} \hat{\lambda}_{n,i}(w) f\big(z_{n,i}(w)\big), \tag{4.1}$$

for $t^n f(t) \in \mathbb{P}_{2n-1}$, where $\hat{\lambda}_{n,i}(w) = |z_{n,i}(w) - w|\,\lambda_{n,i}(w)$, $i = 1, 2, \ldots, n$.

5. A special case

One of the nicest examples of L-orthogonal polynomials and quadrature rules is the one associated with the measure given by $d\psi(t) = (b-t)^{-1/2}(t-a)^{-1/2}\,dt$, where $0 < a < b < \infty$. It was shown in [15] that the associated L-orthogonal polynomials are

$$Q_n(z) = \{\gamma_1(z)\}^n + \{\gamma_2(z)\}^n, \quad n \geqslant 1,$$

with $2\gamma_1(z) = (z-\beta) + \sqrt{(z-\beta)^2 - 4\alpha z}$ and $2\gamma_2(z) = (z-\beta) - \sqrt{(z-\beta)^2 - 4\alpha z}$, and in the associated three term recurrence relation (1.2)

$$\alpha_2 = 2\alpha, \qquad \beta_n = \beta, \qquad \alpha_{n+2} = \alpha, \quad n \geqslant 1,$$

where $\beta = \sqrt{ab}$, $\alpha = (\sqrt{b} - \sqrt{a})^2/4$. Moreover, in the associated quadrature rule (1.5) (or equivalently in the quadrature rule (3.1) with n replaced by $n-1$ and w by $z_{n,n}$)

$$\lambda_{n,i} = \frac{2\pi}{n}\frac{z_{n,i}}{z_{n,i}+\beta}, \quad i = 1, 2, \ldots, n,$$

$$z_{n,n+1-i} = (\beta + \alpha x_{n,i}) + \sqrt{(\beta + \alpha x_{n,i})^2 - \beta^2}, \qquad z_{n,i} = \frac{\beta^2}{z_{n,n+1-i}}, \quad i = 1, 2, \ldots, \lfloor (n+1)/2 \rfloor,$$

where $x_{n,i} = 2(\cos\frac{2i-1}{2n}\pi)^2$. In [15], the above expression for $\lambda_{n,i}$ was obtained by considering the rational functions

$$\frac{P_n(z)}{Q_n(z)} = \sum_{i=1}^{n} \frac{\lambda_{n,i}}{z - z_{n,i}}, \quad n \geqslant 1,$$

where

$$P_n(z) = \frac{\pi}{\sqrt{(z-\beta)^2 - 4\alpha z}}\{\gamma_1(z)\}^n - \frac{\pi}{\sqrt{(z-\beta)^2 - 4\alpha z}}\{\gamma_2(z)\}^n, \quad n \geqslant 1.$$

These polynomials have been shown to satisfy

$$Q_n'(z) = \frac{n(z+\beta)}{2z\pi}P_n(z) + \frac{n}{2z}Q_n(z), \quad n \geqslant 1.$$

Now for $G_{n+1}'(w; w)$ we obtain

$$\begin{aligned} G_{n+1}'(w, w) &= \frac{(n+1)(w+\beta)}{2\pi w}\sigma_{n,n} w^n + \frac{1}{n}Q_{n+1}(w)Q_n'(w) \\ &= \frac{n(w+\beta)}{2\pi w}\sigma_{n,n} w^n + \frac{1}{n+1}Q_n(w)Q_{n+1}'(w). \end{aligned}$$

Here, $\sigma_{n,n} = \pi\alpha^n$. Hence from (2.7), for $\lambda_{n,i}(w)$ and $\hat{\lambda}_{n,i}(w)$ in the associated quadrature rules (3.1) and (4.1),

$$\frac{|z_{n,i}(w) - w|}{\hat{\lambda}_{n,i}(w)} = \frac{1}{\lambda_{n,i}(w)} = \frac{Q_{n+1}(z_{n,i}(w))Q_n'(z_{n,i}(w))}{(z_{n,i}(w))^n \sigma_{n,n} n} + \frac{(n+1)(z_{n,i}(w)+\beta)}{2\pi z_{n,i}(w)}, \quad i = 1, 2, \ldots, n,$$

and

$$\frac{1}{\lambda_{n,n+1}(w)} = \frac{Q_{n+1}(w)Q_n'(w)}{w^n \sigma_{n,n} n} + \frac{(n+1)(w+\beta)}{2\pi w}.$$

In particular, if we choose $w = 0$ then

$$\frac{z_{n,i}(0)}{\hat{\lambda}_{n,i}(0)} = \frac{1}{\lambda_{n,i}(0)} = \frac{Q_{n+1}(z_{n,i}(0))Q_n'(z_{n,i}(0))}{(z_{n,i}(0))^n \sigma_{n,n} n} + \frac{(n+1)(z_{n,i}(0)+\beta)}{2\pi z_{n,i}(0)}, \quad i = 1, 2, \ldots, n,$$

and $\lambda_{n,n+1}(0) = 0$.

Since $\nu_n(0) = -\beta$, $n \geqslant 1$, from Theorem 4.2

$$\beta_1(0) = \beta + 2\alpha, \qquad \beta_2(0) = \frac{\beta+\alpha}{\beta+2\alpha}\beta, \qquad \alpha_2(0) = \frac{\beta+\alpha}{\beta+2\alpha}\alpha,$$

$$\beta_n(0) = \beta, \qquad \alpha_n(0) = \alpha, \quad n \geqslant 3,$$

for the coefficients of the three term recurrence relation associated with the L-orthogonal polynomials $\{K_n(z; 0)\}$.

Now if we choose $w = -\beta$ then

$$\frac{(z_{n,i}(-\beta)+\beta)}{\hat{\lambda}_{n,i}(-\beta)} = \frac{1}{\lambda_{n,i}(-\beta)} = \frac{Q_{n+1}(z_{n,i}(-\beta))\,Q_n'(z_{n,i}(-\beta))}{(z_{n,i}(-\beta))^n \sigma_{n,n} n} + \frac{(n+1)(z_{n,i}(-\beta)+\beta)}{2\pi z_{n,i}(-\beta)}, \quad i = 1, 2, \ldots, n,$$

and

$$\frac{1}{\lambda_{n,n+1}(-\beta)} = \frac{Q_{n+1}(-\beta)\,Q_n'(-\beta)}{(-\beta)^n \sigma_{n,n} n} = \frac{Q_{n+1}(-\beta)\,Q_n(-\beta)}{(-\beta)^{n+1} 2\sigma_{n,n}}.$$

Since $Q_n(-\beta) = (-\beta)^n[(1+\sqrt{1+\alpha/\beta})^n + (1-\sqrt{1+\alpha/\beta})^n]$, $n \geqslant 1$, we also easily obtain that $\nu_1(-\beta) = -2\beta$, $\nu_1(-\beta) - \alpha_2 = -2\beta(1+\alpha/\beta)$,

$$\nu_{n+1}(-\beta) = -\beta \frac{[(1+\sqrt{1+\alpha/\beta})^{n+1} + (1-\sqrt{1+\alpha/\beta})^{n+1}]}{[(1+\sqrt{1+\alpha/\beta})^n + (1-\sqrt{1+\alpha/\beta})^n]}, \quad n \geqslant 1,$$

and

$$\nu_{n+1}(-\beta) - \alpha_{n+2} = -\beta\sqrt{1+\alpha/\beta}\,\frac{[(1+\sqrt{1+\alpha/\beta})^{n+1} - (1-\sqrt{1+\alpha/\beta})^{n+1}]}{[(1+\sqrt{1+\alpha/\beta})^n + (1-\sqrt{1+\alpha/\beta})^n]}, \quad n \geqslant 1,$$

which permits one to obtain from Theorem 4.2 the explicit representations of the coefficients $\beta_n(-\beta)$, $\alpha_{n+1}(-\beta)$, $n \geqslant 1$, of the three term recurrence relation associated with the L-orthogonal polynomials $\{K_n(z; -\beta)\}$.

6. Eigenvalue problems and numerical generation

Given the sequences of real numbers $\{\alpha_m^{(0)}\}_{m=2}^N$ and $\{\beta_m^{(0)}\}_{m=1}^N$, where $\alpha_{m+1}^{(0)} > 0$, $\beta_m^{(0)} > 0$, $m = 1, 2, \ldots, N-1$, and $\beta_N^{(0)}$ not necessarily positive, let the sequence of polynomials $\{R_m\}_{m=0}^N$ be given by the recurrence relation

$$R_{m+1}(z) = \bigl(z - \beta_{m+1}^{(0)}\bigr)R_m(z) - \alpha_{m+1}^{(0)} z R_{m-1}(z), \quad m = 1, 2, \ldots, N-1,$$

with $R_0(z) = 1$ and $R_1(z) = z - \beta_1^{(0)}$. Then, as in Theorem 2.1, from the positiveness of $R_{m-1}(z)R_m'(z) - R_m(z)R_{m-1}'(z)$ for any real z and for $m = 1, 2, \ldots, N$, one can verify that the zeros of R_m are positive and distinct for $m = 1, 2, \ldots, N-1$, and also that the zeros of R_m interlace with the zeros of R_{m-1} for $m = 2, 3, \ldots, N$. Consequently, the zeros $\xi_{N,i}$, $i = 1, 2, \ldots, N$, of R_N are real and distinct and at least $N-1$ of them are positive. If we restrict $\beta_N^{(0)}$ also to be positive then all N zeros of R_N are positive.

With $\tilde{R}_m(z) = z^{N-1-m}R_n(z)$ and $\hat{R}_m(z) = \varsigma_m^{-1}R_m(z)$, $m = 0, 1, \ldots, N-1$, where $\varsigma_0 \neq 0$ is arbitrary and $\varsigma_m = \alpha_{m+1}^{(0)}\varsigma_{m-1}$, $m = 1, 2, \ldots, N-1$, the above recurrence relation can be given in the following two different forms (see [4] and [16]):

$$\begin{aligned}
z\bigl[\hat{R}_0(z)\bigr] &= -\beta_1^{(0)}\hat{R}_0(z) + \alpha_2\hat{R}_1(z),\\
z\bigl[-\hat{R}_{m-1}(z) + \hat{R}_m(z)\bigr] &= -\beta_{m+1}^{(0)}\hat{R}_m(z) + \alpha_{m+2}^{(0)}\hat{R}_{m+1}(z), \quad 1 \leqslant m \leqslant N-2,\\
z\bigl[-\hat{R}_{N-2}(z) + \hat{R}_{N-1}(z)\bigr] &= -\beta_N^{(0)}\hat{R}_{N-1}(z) + \varsigma_{N-1}^{-1}R_N(z)
\end{aligned}$$

and

$$\begin{aligned}
z\bigl[\tilde{R}_0(z) - \tilde{R}_1(z)\bigr] &= -\beta_1^{(0)}\tilde{R}_0(z),\\
z\bigl[\tilde{R}_m(z) - \tilde{R}_{m+1}(z)\bigr] &= \alpha_{m+1}^{(0)}\tilde{R}_{m-1}(z) - \beta_{m+1}^{(0)}\tilde{R}_m(z), \quad 1 \leqslant m \leqslant N-2,\\
z\bigl[\tilde{R}_{N-1}(z)\bigr] &= \alpha_N^{(0)}\tilde{R}_{N-1}(z) - \beta_N^{(0)}\tilde{R}_{N-1}(z) + R_N(z).
\end{aligned}$$

In matrix representation these are

$$z\mathbf{A}_N\hat{\mathbf{b}}_N(z) = \mathbf{B}_N^{(0)}\hat{\mathbf{b}}_N(z) + \varsigma_{N-1}^{-1}R_N(z)\mathbf{e}_N \tag{6.1}$$

and

$$z\tilde{\mathbf{b}}_N(z)^T\mathbf{A}_N = \tilde{\mathbf{b}}_N(z)^T\mathbf{B}_N^{(0)} + R_N(z)\mathbf{e}_N^T, \tag{6.2}$$

respectively, where $\mathbf{e}_N$ is the Nth column of the $N \times N$ identity matrix,

$$\hat{\mathbf{b}}_N(z) = \bigl[\hat{R}_0(z), \hat{R}_1(z), \ldots, \hat{R}_{N-1}(z)\bigr]^T, \qquad \tilde{\mathbf{b}}_N(z) = \bigl[\tilde{R}_0(z), \tilde{R}_1(z), \ldots, \tilde{R}_{N-1}(z)\bigr]^T$$

and the $N \times N$ matrices $\mathbf{A}_N$ and $\mathbf{B}_N^{(0)}$ are

$$\mathbf{A}_N = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix}, \qquad \mathbf{B}_N^{(0)} = \begin{pmatrix} \beta_1^{(0)} & \alpha_2^{(0)} & 0 & \cdots & 0 & 0 \\ 0 & \beta_2^{(0)} & \alpha_3^{(0)} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{N-1}^{(0)} & 0 \\ 0 & 0 & 0 & \cdots & \beta_{N-1}^{(0)} & \alpha_N^{(0)} \\ 0 & 0 & 0 & \cdots & 0 & \beta_N^{(0)} \end{pmatrix}.$$

From the matrix representation (6.1), $\hat{\mathbf{b}}_N(\xi_{N,i})$ is a right eigenvector of the lower Hessenberg matrix

$$\mathbf{H}_N^{(0)} = \mathbf{A}_N^{-1}\mathbf{B}_N^{(0)} = \begin{pmatrix} \check{\gamma}_1^{(0)} & \alpha_2^{(0)} & 0 & \cdots & 0 & 0 \\ \check{\gamma}_1^{(0)} & \check{\gamma}_2^{(0)} & \alpha_3^{(0)} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \check{\gamma}_1^{(0)} & \check{\gamma}_2^{(0)} & \check{\gamma}_3^{(0)} & \cdots & \alpha_{N-1}^{(0)} & 0 \\ \check{\gamma}_1^{(0)} & \check{\gamma}_2^{(0)} & \check{\gamma}_3^{(0)} & \cdots & \check{\gamma}_{N-1}^{(0)} & \alpha_N^{(0)} \\ \check{\gamma}_1^{(0)} & \check{\gamma}_2^{(0)} & \check{\gamma}_3^{(0)} & \cdots & \check{\gamma}_{N-1}^{(0)} & \check{\gamma}_N^{(0)} \end{pmatrix},$$

associated with the eigenvalue $\xi_{N,i}$. Here, $\check{\gamma}_m^{(0)} = \alpha_m^{(0)} + \beta_m^{(0)}$, $m = 1, 2, \ldots, N$, with $\alpha_1^{(0)} = 0$. Moreover, since $\xi_{N,i}$, $i = 1, 2, \ldots, N$, are distinct, the corresponding right eigenvectors $\hat{\mathbf{b}}_N(\xi_{N,i})$, $i = 1, 2, \ldots, N$, are all linearly independent.

Likewise, from the matrix representation (6.2), the vector $\mathbf{A}_N^T\tilde{\mathbf{b}}_N(\xi_{N,i})$ is a left eigenvector of $\mathbf{H}_N^{(0)}$ associated with the eigenvalue $\xi_{N,i}$. Again, there are N linearly independent left eigenvectors $\mathbf{A}_n^T\tilde{\mathbf{b}}_N(\xi_{N,i})$, $i = 1, 2, \ldots, N$, associated with the N distinct $\xi_{N,i}$.

We remark that the vector $\tilde{\mathbf{b}}_N(z)$ defined here is a scalar multiple of the vector $\tilde{\mathbf{b}}(z)$ considered in [4]. With the present definition, we have $\tilde{\mathbf{b}}_N(0) \neq \mathbf{0}$ and so origin being an eigenvalue, which is the case when $\beta_N^{(0)} = 0$, can be dealt without any problem.

We can identify our eigensystem with results obtained in Sections 2 and 3. We have $R_n(z) = Q_n(z)$, $n = 0, 1, \ldots, N-1$, and the vectors given by $\mathbf{A}_N^T\tilde{\mathbf{b}}_N(z)$ and $\hat{\mathbf{b}}_N(z)$ are respectively the vectors $\mathbf{y}_N^{\mathfrak{l}}(z)$ and $\mathbf{y}_N^{\mathfrak{r}}(z)$ given in (3.3), provided that $\alpha_{m+1}^{(0)} = \alpha_{m+1}$, $\beta_m^{(0)} = \beta_m$, $m = 1, 2, \ldots, N-1$, and $\varsigma_0 = \mu_0$. Here we use the knowledge that $\sigma_{0,0} = \mu_0 = \varsigma_0$ and $\sigma_{m,m} = \mu_0\alpha_2 \cdots \alpha_{m+1} = \varsigma_m$, $m = 1, 2, \ldots, N-1$.

Moreover, if $\beta_N^{(0)} = \beta_N$ then we obtain the eigensystem associated the polynomial Q_N defined in Section 1 as the Nth degree L-orthogonal polynomial with respect to the strong positive measure ψ. Thus, the eigenvalues $\xi_{N,i}$ are the zeros of $R_N(z) = Q_N(z)$.

Since we can write

$$(z - w)K_{N-1}(z; w) = \bigl(z - \beta_N - \nu_N(w)\bigr)Q_{N-1}(z) - \alpha_N z Q_{N-2}(z),$$

where $\nu_N(w) = Q_N(w)/Q_{N-1}(w)$, if we choose $\beta_N^{(0)} = \beta_N + \nu_N(w)$ then the eigensystem is the one associated with the polynomial $(z - w)K_{N-1}(z; w)$ (also $G_N(z; w)$) defined in Sections 2 and 3. Thus, the eigenvalues $\xi_{N,i}$ are the zeros $z_{N-1,i}(w)$ associated with the polynomial $R_n(z) = (z - w)K_{N-1}(z; w)$. Here we have assumed that $z_{N-1,N}(w) = w$. Hence, from the definition of $G_N(z; w)$, taking w to be equal to $z_{N,N}$, the largest zero of Q_N, also gives the above eigenvalue problem associated with the choice $\beta_N^{(0)} = \beta_N$.

We obtain from (3.3) and from the left and right eigenvectors $\mathbf{y}_N^{\mathfrak{l}}(z_{N-1,i}(w)) = \mathbf{A}_N^T\tilde{\mathbf{b}}_N(z_{N-1,i}(w))$ and $\mathbf{y}_N^{\mathfrak{r}}(z_{N-1,i}(w)) = \hat{\mathbf{b}}_N(z_{N-1,i}(w))$ that

$$\bigl[\Lambda_{N-1}\bigl(z_{N-1,i}(w)\bigr)\bigr]^{-1} = \mathbf{y}_N^{\mathfrak{l}}\bigl(z_{N-1,i}(w)\bigr)^T\mathbf{y}_N^{\mathfrak{r}}\bigl(z_{N-1,i}(w)\bigr), \quad 1 \leqslant i \leqslant N,$$

where the values $\Lambda_N(z_{N-1,i}(w))$, together with the eigenvalues $z_{n,i}(w)$, give us the weights of the quadrature rules given by (3.1) and (4.1) with $n = N-1$. The orthogonality of the left and right eigenvectors associated with different eigenvalues are also confirmed by (3.3).

If the $N \times N$ matrices $\mathbf{Y}_N^{\mathfrak{l}}$, $\mathbf{Y}_N^{\mathfrak{r}}$, $\mathbf{D}_N$ and $\mathbf{Z}_N$ are defined by

$$\begin{aligned} \mathbf{Y}_N^{\mathfrak{l}} &= \bigl[\mathbf{y}_N^{\mathfrak{l}}\bigl(z_{N-1,1}(w)\bigr)^T, \mathbf{y}_N^{\mathfrak{l}}\bigl(z_{N-1,2}(w)\bigr)^T, \ldots, \mathbf{y}_N^{\mathfrak{l}}\bigl(z_{N-1,N}(w)\bigr)^T\bigr]^T, \\ \mathbf{Y}_N^{\mathfrak{r}} &= \bigl[\mathbf{y}_N^{\mathfrak{r}}\bigl(z_{N-1,1}(w)\bigr), \mathbf{y}_N^{\mathfrak{r}}\bigl(z_{N-1,2}(w)\bigr), \ldots, \mathbf{y}_N^{\mathfrak{r}}\bigl(z_{N-1,N}(w)\bigr)\bigr], \end{aligned}$$

$$\mathbf{D}_N^{-1} = \begin{pmatrix} \Lambda_{N-1}(z_{N-1,1}(w)) & 0 & \cdots & 0 \\ 0 & \Lambda_{N-1}(z_{N-1,2}(w)) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Lambda_{N-1}(z_{N-1,N}(w)) \end{pmatrix}$$

and

$$\mathbf{Z}_N = \begin{pmatrix} z_{N-1,1}(w) & 0 & \cdots & 0 \\ 0 & z_{N-1,2}(w) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_{N-1,N}(w) \end{pmatrix},$$

then

$$\mathbf{Y}_N^{\mathfrak{l}}\mathbf{Y}_N^{\mathfrak{r}} = \mathbf{D}_N, \qquad \mathbf{H}_N^{(0)}\mathbf{Y}_N^{\mathfrak{r}} = \mathbf{Y}_N^{\mathfrak{r}}\mathbf{Z}_N \quad \text{and} \quad \mathbf{Y}_N^{\mathfrak{l}}\mathbf{H}_N^{(0)} = \mathbf{Z}_N\mathbf{Y}_N^{\mathfrak{l}}.$$

The nodes and weights of the quadrature rule (4.1), with $n = N$, can also be obtained directly from the three term recurrence relation given by Theorem 4.1 by considering the eigenvalue problem with

$$\varsigma_0 = \mu_0(w), \qquad \alpha_{m+1}^{(0)} = \alpha_{m+1}(w), \quad 1 \leqslant m \leqslant N-1, \quad \text{and}$$
$$\beta_m^{(0)} = \beta_m(w), \quad 1 \leqslant m \leqslant N.$$

In this case $R_n(z)$ are the polynomials $K_n(z; w)$ for $n = 1, 2, \ldots, N$.

Now we see how the eigenvalue problem associated with our special Hessenberg matrix $H_N^{(0)}$, with $\varsigma_0 = \mu_0$, $\alpha_{m+1}^{(0)} = \alpha_{m+1}$, $\beta_m^{(0)} = \beta_m$, $m = 1, 2, \ldots, N-1$, and $\beta_N^{(0)} = \beta_N + \nu_N(w)$, can be used to derive numerically the nodes and weights of the quadrature rules (3.1) and (4.1). The technique presented below can be considered as an extension of an idea used for determining the nodes and weights of Gaussian quadrature rules (see Gautschi [6]).

First we observe that if an eigenvalue $\xi_{N,i}$ and the associated right eigenvector $\hat{\mathbf{y}}_N^{\mathfrak{r}}(\xi_{N,i}) = \mu_0\mathbf{y}_N^{\mathfrak{r}}(\xi_{N,i})$ with its leading element equal to 1 are known then the associated left eigenvector $\mathbf{y}_N^{\mathfrak{l}}(\xi_{N,i})$ can be obtained from the relation

$$\mathbf{y}_N^{\mathfrak{l}}(z) = z^{N-1}\mathbf{A}_N^T\mathbf{C}_N(z)\hat{\mathbf{y}}_N^{\mathfrak{r}}(z),$$

where the $N \times N$ diagonal matrix $\mathbf{C}_N(z)$ is

$$\mathbf{C}_N(z) = \begin{pmatrix} \varsigma_0(z) & 0 & \cdots & 0 & 0 \\ 0 & \varsigma_1(z) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \varsigma_{N-2}(z) & 0 \\ 0 & 0 & \cdots & 0 & \varsigma_{N-1}(z) \end{pmatrix},$$

with $\varsigma_0(z) = 1$ and $\varsigma_i(z) = \alpha_{i+1}^{(0)}z^{-1}\varsigma_{i-1}(z)$, $i = 1, 2, \ldots, N-1$. This also means

$$\begin{aligned}\left[\Lambda_{N-1}(z)\right]^{-1} &= \sum_{j=0}^{N-2}\left[z^{N-1-j}Q_j(z) - z^{N-j-2}Q_{j+1}(z)\right]\frac{Q_j(z)}{\varsigma_j} + Q_{N-1}(z)\frac{Q_{N-1}(z)}{\varsigma_{N-1}} = \mathbf{y}_N^{\mathfrak{l}}(z)^T\mathbf{y}_N^{\mathfrak{r}}(z) \\ &= z^{N-1}\mu_0^{-1}\hat{\mathbf{y}}_N^{\mathfrak{r}}(z)^T\mathbf{C}_N(z)\mathbf{A}_N\hat{\mathbf{y}}_N^{\mathfrak{r}}(z).\end{aligned}$$

Since one needs to determine the right eigenvectors such that the leading element in them is equal to 1, they can also be easily derived from the system $\mathbf{H}_N^{(0)}\hat{\mathbf{y}}_N^{\mathfrak{r}}(\xi_{N,i}) = \xi_{N,i}\hat{\mathbf{y}}_N^{\mathfrak{r}}(\xi_{N,i})$, once the eigenvalues $\xi_{N,i}$ are known.

Algorithm 1. Given the eigenvalues $\xi_{N,i} = z_{N-1,i}(w)$ of $\mathbf{H}_N^{(0)}$, to obtain the corresponding right eigenvectors $\hat{\mathbf{y}}_N^{\mathfrak{r}}(\xi_{N,i}) = [\varepsilon_{i,1}, \varepsilon_{i,2}, \ldots, \varepsilon_{i,N}]^T$ and, by performing the operations within [ ], to obtain also the corresponding quadrature weights $\lambda_{N-1,i}(w) = (z_{N-1,i}(w))^{N-1}\Lambda_{N-1}(z_{N-1,i}(w))$.

$$\left\{\begin{array}{l}
\text{For } i = 1, 2, \ldots, N, \\
\quad \varepsilon_{i,1} = 1, \quad S_1 = \beta_1^{(0)} \times \varepsilon_{i,1}, \quad \varepsilon_{i,2} = (\varepsilon_{i,1} \times \xi_{N,i} - S_1)/\alpha_2^{(0)} \\
\quad \big[\varsigma = \alpha_2^{(0)}/\xi_{N,i}, \quad S_2 = 1 + (\varepsilon_{i,2} - \varepsilon_{i,1}) \times \varepsilon_{i,2} \times \varsigma\big] \\
\quad \left\{\begin{array}{l}
\text{For } j = 2, 3, \ldots, N-1, \\
\quad S_1 = S_1 + \check{\gamma}_j^{(0)} \times \varepsilon_{i,j}, \quad \varepsilon_{i,j+1} = (\varepsilon_{i,j} \times \xi_{N,i} - S_1)/\alpha_{j+1}^{(0)} \\
\quad \big[\varsigma = \varsigma \times \alpha_{j+1}^{(0)}/\xi_{N,i}, \quad S_2 = S_2 + (\varepsilon_{i,j+1} - \varepsilon_{i,j}) \times \varepsilon_{i,j+1} \times \varsigma\big]
\end{array}\right. \\
\big[\lambda_{N-1,i}(w) = \mu_0 \times S_2^{-1}\big]
\end{array}\right.$$

Here, $\check{\gamma}_j^{(0)} = \alpha_j^{(0)} + \beta_j^{(0)}$, $j = 2, 3, \ldots, N$.

This algorithm is essentially the inverse power method for generating the eigenvectors and hence it is expected to be very efficient. Note that, even though the matrix involved is of Hessenberg type, because of the structure the number of arithmetic operations involved is smaller in the order of a three diagonal matrix.

Observe that when $i = N$ and $w = 0$ the operations within [ ] should be avoided. In this case we can simply assign $\lambda_{N-1,N}(0) = 0$.

Since the eigenvalues of $\mathbf{H}_N^{(0)}$ are real and distinct, the QR algorithm (see [8,18]) developed independently by John G.F. Francis and Vera Kublanovskaya (see [7]) can be effectively applied to obtain the eigenvalues and also, for example, the right eigenvectors of this matrix.

However, there is also another procedure to determine the eigenvalues which is very simple to implement.

The matrix $\mathbf{H}_N^{(0)}$ is a product of the matrices $\mathbf{A}_N^{-1}$ and $\mathbf{B}_N^{(0)}$. We consider the decomposition $\mathbf{H}_N^{(0)} = \mathbf{F}_N^{(0)} \mathbf{G}_N^{(0)}$, where

$$\mathbf{F}_N^{(0)} = \begin{pmatrix}
\hat{\gamma}_1^{(0)} & 0 & 0 & \cdots & 0 & 0 \\
\hat{\gamma}_1^{(0)} & \hat{\gamma}_2^{(0)} & 0 & \cdots & 0 & 0 \\
\hat{\gamma}_1^{(0)} & \hat{\gamma}_2^{(0)} & \hat{\gamma}_3^{(0)} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\hat{\gamma}_1^{(0)} & \hat{\gamma}_2^{(0)} & \hat{\gamma}_3^{(0)} & \cdots & \hat{\gamma}_{N-1}^{(0)} & 0 \\
\hat{\gamma}_1^{(0)} & \hat{\gamma}_2^{(0)} & \hat{\gamma}_3^{(0)} & \cdots & \hat{\gamma}_{N-1}^{(0)} & \hat{\gamma}_N^{(0)}
\end{pmatrix}$$

and

$$\mathbf{G}_N^{(0)} = \begin{pmatrix}
\beta_1^{(0)}/\hat{\gamma}_1^{(0)} & \alpha_2^{(0)}/\hat{\gamma}_1^{(0)} & 0 & \cdots & 0 & 0 \\
0 & \beta_2^{(0)}/\hat{\gamma}_2^{(0)} & \alpha_3^{(0)}/\hat{\gamma}_2^{(0)} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \beta_{N-1}^{(0)}/\hat{\gamma}_{N-2}^{(0)} & 0 \\
0 & 0 & 0 & \cdots & \beta_{N-1}^{(0)}/\hat{\gamma}_{N-1}^{(0)} & \alpha_N^{(0)}/\hat{\gamma}_{N-1}^{(0)} \\
0 & 0 & 0 & \cdots & 0 & \beta_N^{(0)}/\hat{\gamma}_N^{(0)}
\end{pmatrix}.$$

Here, $\hat{\gamma}_j^{(0)} = \alpha_{j+1}^{(0)} + \beta_j^{(0)}$, $j = 2, 3, \ldots, N$, with $\alpha_{N+1}^{(0)} = 0$. What is interesting about the above decomposition is that the matrix $\mathbf{G}_N^{(0)} \mathbf{F}_N^{(0)} = \mathbf{H}_N^{(1)}$, apart from being a similar matrix to $\mathbf{H}_N^{(0)}$, has exactly the same structure as $\mathbf{H}_N^{(0)}$. We have,

$$\mathbf{H}_N^{(1)} = \begin{pmatrix}
\check{\gamma}_1^{(1)} & \alpha_2^{(1)} & 0 & \cdots & 0 & 0 \\
\check{\gamma}_1^{(1)} & \check{\gamma}_2^{(1)} & \alpha_3^{(1)} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\check{\gamma}_1^{(1)} & \check{\gamma}_2^{(1)} & \check{\gamma}_3^{(1)} & \cdots & \alpha_{N-1}^{(1)} & 0 \\
\check{\gamma}_1^{(1)} & \check{\gamma}_2^{(1)} & \check{\gamma}_3^{(1)} & \cdots & \check{\gamma}_{N-1}^{(1)} & \alpha_N^{(1)} \\
\check{\gamma}_1^{(1)} & \check{\gamma}_2^{(1)} & \check{\gamma}_3^{(1)} & \cdots & \check{\gamma}_{N-1}^{(1)} & \check{\gamma}_N^{(1)}
\end{pmatrix},$$

where

Table 1
Number of iterations and approximate execution time required to generate the eigenvalues of $\mathbf{H}_N^{(0)}$ accurate to 8 significant digits.

	Iterations QR algorithm	Execution time QR algorithm	Iterations Algorithm 2	Execution time Algorithm 2
$N=8,\ w=0.0$	50	0.0017	130	0.0035
$N=8,\ w=-1.0$	51	0.0017	124	0.0032
$N=16,\ w=0.0$	102	0.0052	497	0.0135
$N=16,\ w=-1.0$	100	0.0050	485	0.0126
$N=24,\ w=0.0$	152	0.0120	1070	0.0269
$N=24,\ w=-1.0$	146	0.0115	1052	0.0264

$$\check{\gamma}_j^{(1)} = \hat{\gamma}_j^{(0)}, \qquad \alpha_{j+1}^{(1)} = \frac{\hat{\gamma}_{j+1}^{(0)}}{\hat{\gamma}_j^{(0)}}\alpha_{j+1}^{(0)}, \quad j = 1, 2, \ldots, N.$$

By successive application of this idea we then obtain the matrices

$$\mathbf{H}_N^{(k)} = \begin{pmatrix} \check{\gamma}_1^{(k)} & \alpha_2^{(k)} & 0 & \cdots & 0 & 0 \\ \check{\gamma}_1^{(k)} & \check{\gamma}_2^{(k)} & \alpha_3^{(k)} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \check{\gamma}_1^{(k)} & \check{\gamma}_2^{(k)} & \check{\gamma}_3^{(k)} & \cdots & \alpha_{N-1}^{(k)} & 0 \\ \check{\gamma}_1^{(k)} & \check{\gamma}_2^{(k)} & \check{\gamma}_3^{(k)} & \cdots & \check{\gamma}_{N-1}^{(k)} & \alpha_N^{(k)} \\ \check{\gamma}_1^{(k)} & \check{\gamma}_2^{(k)} & \check{\gamma}_3^{(k)} & \cdots & \check{\gamma}_{N-1}^{(k)} & \check{\gamma}_N^{(k)} \end{pmatrix},$$

where $\check{\gamma}_j^{(k)}$ and $\alpha_j^{(k)}$, $j = 1, 2, \ldots N$, can be recursively generated by

$$\check{\gamma}_j^{(k)} = \hat{\gamma}_j^{(k-1)}, \qquad \alpha_{j+1}^{(k)} = \frac{\hat{\gamma}_{j+1}^{(k-1)}}{\hat{\gamma}_j^{(k-1)}}\alpha_{j+1}^{(k-1)}, \quad j = 1, 2, \ldots, N.$$

Here, for any $k \geqslant 0$, $\check{\gamma}_j^{(k)} = \alpha_j^{(k)} + \beta_j^{(k)}$ and $\hat{\gamma}_j^{(k)} = \alpha_{j+1}^{(k)} + \beta_j^{(k)}$ for $j = 1, 2, \ldots, N$, with $\alpha_1^{(k)} = \alpha_{N+1}^{(k)} = 0$.

Clearly, because of their structure, $\mathbf{H}_N^{(k)}$ as $k \to \infty$ cannot converge to a diagonal matrix with the eigenvalues $\xi_{N,i}$ as diagonal entries. However, it turns out that, since the eigenvalues are distinct and positive, $\lim_{k\to\infty} \alpha_j^{(k)} = 0$, $j = 2, 3, \ldots, N$, and the matrix $\mathbf{H}_N^{(k)}$ as $k \to \infty$ converge to a triangular matrix with the eigenvalues $\xi_{N,i}$ as diagonal entries. Consequently, we can state the following algorithm, which is the same algorithm obtained by Jones and Magnus [9] for the computation of poles of two-point Padé approximants.

Algorithm 2. A simple procedure to obtain the eigenvalues of $\mathbf{H}_N^{(0)}$.

$$\begin{cases} \text{For } k = 1, 2, 3, \ldots, \\ \alpha_{N+1}^{(k-1)} = 0, \qquad \hat{\gamma}_j^{(k-1)} = \alpha_{j+1}^{(k-1)} + \beta_j^{(k-1)}, \quad j = 1, 2, \ldots, N, \\ \beta_1^{(k)} = \hat{\gamma}_1^{(k-1)}, \\ \beta_j^{(k)} = \beta_{j-1}^{(k-1)} \dfrac{\hat{\gamma}_j^{(k-1)}}{\hat{\gamma}_{j-1}^{(k-1)}}, \qquad \alpha_j^{(k)} = \alpha_j^{(k-1)} \dfrac{\hat{\gamma}_j^{(k-1)}}{\hat{\gamma}_{j-1}^{(k-1)}}, \quad j = 2, 3, \ldots, N. \end{cases}$$

As $k \to \infty$, we obtain from $\beta_j^{(k)}$, $j = 1, 2, \ldots, N$, the required eigenvalues.

The convergence of the algorithm follows from [9] and, since all the elements involved are positive, we can also expect reasonable stability.

We apply both QR algorithm with shifts and Algorithm 2 to determine the eigenvalues of the matrix $\mathbf{H}_N^{(0)}$ corresponding to the particular example treated in Section 5. The eigenvalues are the zeros of the polynomial $(z - w)K_{N-1}(z; w)$ when

$$\alpha_2 = 2\alpha, \qquad \beta_n = \beta, \qquad \alpha_{n+2} = \alpha, \quad n \geqslant 1.$$

To obtain the results given in Table 1 we take $\alpha = \beta = 1$. To compare the algorithms, we observe the number of iterations and approximate execution time required to obtain the eigenvalues of the matrices $\mathbf{H}_8^{(0)}$, $\mathbf{H}_{16}^{(0)}$ $\mathbf{H}_{24}^{(0)}$, for $w = 0$ and $w = -1$.

Clearly, from the results presented in Table 1, the QR algorithm shows a superior convergence behavior than Algorithm 2, especially in terms of the number of iterations. However, the execution time involved in Algorithm 2 is only about twice as much as the QR algorithm. Hence, because of the simplicity in programming Algorithm 2 we consider it to be an excellent choice when the value of N is not too large.

Having obtained the eigenvalues, i.e. the nodes of the quadrature rule (3.1), Algorithm 1 can be effectively used to obtain the associated weights of this quadrature rule.

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