**IFT**

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# Application of open string field theory to the inflationary scenario

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*“Rir muito e com frequência; ganhar o respeito de pessoas inteligentes e o afeto das crianças; merecer a consideração de críticos honestos e suportar a traição de falsos amigos; apreciar a beleza, encontrar o melhor nos outros; deixar o mundo um pouco melhor, seja por uma saudável criança, um canteiro de jardim ou uma redimida condição social; saber que ao menos uma vida respirou mais fácil porque você viveu. Isso é ter tido sucesso.”* - Ralph Waldo Emerson

*"Bem-aventurados são aqueles que promovem o auto-conhecimento, que buscam entender o sentido de suas próprias vidas.*

*Felizes são aqueles que entendem que não fazem mal ao próximo, não por causa de alguma divindade externa, mas sim pelo reconhecimento de que o seu próprio eu é simplesmente incompatível com tais atos.*

*Bem-aventurados são aqueles que tiveram a oportunidade, ao seu tempo, de perceber e poder reconhecer o que significa uma amizade verdadeira.*

*Felizes são aqueles que possuem em seu círculo social pessoas respeitosas e que compreendem as diferenças dos outros, podendo assim, também, reconhecer esta bela atitude e respeitar o próximo.*

*Bem-aventurados são aqueles que não se deixam abalar pelas opressões conceituais sociais somente porque a maioria das pessoas inconscientemente consente.*

*Em prol da* religião *focada nos indivíduos, graças a dEUs."* - Patrice Camati

# Contents

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# Abstract

This thesis consists in a review of the String Field Theory framework, focusing in the classical properties of the open sector and its possible applications for inflation. Therefore, we intend to follow the Witten's prescription and build an action for the Open Bosonic String Field Theory. Then, recognizing that the theory has a tachyonic mode and motivated by the inflationary scenario, we calculate and consider the tachyonic potential in some order of approximation. As an application, we consider the tachyon field as a possible candidate for the inflaton. In order to work with this proposal, we first review the inflationary theory and study its modern approach, considering only its classical implications using the slow-roll approximation. Finally, we analyze the tachyonic potential as being the inflaton potential and explore its consequences. As a support, there are four appendices containing some aspects of String Theory, General Relativity, Cosmology and some relevant calculations that were omitted throughout the thesis.

# Resumo

Esta tese consiste em uma revisão da estrutura da Teoria de Campo na Corda, focando nas propriedades clássicas do setor da corda aberta e suas possíveis aplicações para inflação. Portanto, seguiremos a prescrição do Witten e construiremos uma ação para a Teoria de Campo na Corda Aberta Bosônica. Então, percebendo que a teoria tem uma modo taquiônico e motivado pelo cenário de inflação, calcularemos e consideraremos o potencial taquiônico em alguma ordem de aproximação. Como aplicação, tomaremos o campo taquiônico como um candidato possível para o *inflaton.* A fim de trabalhar com esta proposta, revisaremos primeiramente a teoria inflacionária e estudaremos sua abordagem moderna, considerando apenas suas implicações a nível clássico utilizando a aproximação *slow-roll.* Finalmente, analisaremos o potencial taquiônico como sendo o potencial do *inflaton* e exploramos suas consequências. Como suporte, há quatro apêndices contendo alguns aspectos de Teoria de Cordas, Relatividade Geral, Cosmologia e alguns cálculos relevantes que foram omitidos ao longo da tese.

# Objectives

The objective of this thesis is to provide that readers coming from cosmology can be able to understand string inflation through the tachyonic potential proposal from open bosonic string field theory, since it is expected that all the necessary elements are provided for this understanding. On the other hand, it is also expected that a string theorist can be able to understand the general aspects of inflation and a direct application coming from string theory only with the material here provided.

# 1. Introduction

## String Theory

In string theory, the basic assumption is that all the elementary particles are vibrations of a very small elastic string, generating a unification between the different kinds of elementary particles. The fact that a string has some extension also succeeds in eliminating infinites which come up when point particles approach each other too closely in Quantum Field Theory (QFT). These two properties together provide a good indication that String Theory could provide the correct framework to study a quantum gravity theory.

There are several types of string theory, depending essentially if we are considering bosons, fermions or both of them and what are the boundary conditions, resulting in open or closed strings. In this thesis, we will be concerned with the open bosonic sector.

Although we consider string-like objects, string theory is perturbative as QFT, that means, it is a formalism for calculating scattering amplitudes as a perturbation series over a small coupling constant. However, when one starts studying the spectrum of the open bosonic sector, the physical spectrum contains a tachyonic mode, indicating that a *non-perturbative* theory has to be developed.

## String Field Theory

The most successful non-perturbative theory developed so far was first introduced by Witten [3]. In fact, making use of the $bc$ ghosts introduced to fix the symmetries in the string action, which results in the presence of a $BRST$ symmetry, it was possible to construct a parallel with the already known Chern-Simons theory in $3-$dim, providing a covariant action for the bosonic open string field theory.

The construction, as we will see, is very elegant and based in very intuitive arguments, providing an algorithm to consider string interactions. Using this framework, it is possible to calculate the effective tachyonic potential of the open bosonic sector and analyze the tachyon condensation dynamics, that is, how the tachyon field decays and its implications.

## Cosmology and Inflation

It was known in the $70's$ that the Standard Cosmological Model by that time had some problems, related with the initial-conditions of the universe. As a matter of fact, the large isotropy and homogeneity of the universe in the Cosmic Background Radiation (CMB) were an indication of an extremely fine-tunning in the very early universe, which seemed very unnatural.

Then, in the beginning of the $80's$, some works [16, 17, 25] leading by Guth's paper [15] proposed a new scenario for the early universe, when a huge accelerated expansion of the universe would have happened. This was known as *inflation*. As we will see, it provides resolutions for several cosmological problems, even though it is more a framework than a theory itself.

## Strings and Cosmology

At first sight, string theory and cosmology, inflation in particular, could be thought as too "far away" to be related. However, recent developments of both fields have brought they together, especially concerning the very beginning of the universe.

In fact, cosmology and strings could complement each other in several ways. On one side, cosmology stills lacks an underlying theory to address the initial singularity problem and the origin of inflation. On the other side, string theory has not yet been confronted with experiments and this could be a good opportunity for it.

One way to put all the things together is considering the decaying of the tachyonic mode, having its dynamics (its potential) calculated in the open string field theory framework, as a candidate for the inflationary scenario. This will be the ultimate objective here, even though we might have to consider several approximations in order to keep the problem as simple as possible so that it remains pedagogic.

## Summary of the Thesis

In the second chapter we introduce briefly the Chern-Simons theory and explicit the relevant properties that will be important for the construction of the open string field action. Then, we follow Witten's prescription and provide the method to evaluate the action.

The third chapter presents the calculation of the tachyonic potential at the very first approximation (to be explained there). Some of the calculations are omitted, but they are developed in the appendices.

Inflation is introduced only in chapter 4 through a historical and qualitative description. We provide an easy analogy to understand the main problems that motivated the introduction of this scenario in the early universe cosmology.

Chapter 5 brings inflation to its modern approach, focusing only in the classical properties and its consequences. We treat some example models to demonstrate the slow-roll approximation and, finally, build the bridge between the tachyonic potential calculated in chapter 3 and the inflationary paradigm.

The appendices, besides presenting some useful calculations, are concerned about relevant aspects to the thesis, coming from String Theory up to General Relativity and Cosmology.

# 2. String Field Theory

The first construction of a gauge invariant string field theory appeared in [1] where the BRST approach was used. Soon after, in [2] a gauge invariant string field theory was constructed using a single string field and when tried to extend it to a minimal gauge invariant system, a BRST structure was seen to emerge. Even though these references will not be used any longer, they give a hint of the important task BRST symmetry shall play from now on.

This chapter is intend to review the construction of a consistent String Field Theory for bosonic open strings. The strategy is to interpret the interactions in this framework as defining a non-commutative and associative algebra which will have its elements identified with the string field and its derivative with the BRST operator. This algebra is analogous to the Chern-Simons theory, so we start providing a brief description of it.

Our prior references will be [3, 4, 5, 6] and others that will be cited when necessary.

## 2.1. Chern-Simons Theory

Let's start by reviewing the Chern-Simons theory in $2+1-$dimensions since the open string field action is inspired on the nonabelian version of it. A good reference for the classical and quantum aspects of this gauge theory can be found in [31].

The Chern-Simons theory is a gauge theory in $2+1$-dimensions[1] developed by Witten [32]. It has very interesting theoretical properties and practical application in condensed matter phenomena. Its Lagrangian is

$$\mathcal{L}_{CS} = \frac{\kappa}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho - A_\mu J^\mu \tag{2.1}$$

where $A_\mu$ is the gauge field in $2+1-$dim and $J^\mu$ is the matter current. Under the gauge transformation $A_\mu \to A_\mu + \partial_\mu\Lambda$ it transforms as

$$\delta\mathcal{L}_{CS} = \frac{\kappa}{2}\partial_\mu(\Lambda\epsilon^{\mu\nu\rho}\partial_\nu A_\rho). \tag{2.2}$$

Therefore, its action is gauge invariant if we can ignore boundary terms. The equation of motion is

$$\frac{\kappa}{2}\epsilon^{\mu\nu\rho}F_{\nu\rho} = J^\mu, \tag{2.3}$$

where $F_{\nu\rho} = \partial_\nu A_\rho - \partial_\rho A_\nu$ is the field strength. Note that if $J^\mu = 0$, we have $F_{\nu\rho} = 0$, which looks pretty trivial when compared with Maxwell theory, because there even the source-free Lagrangian provides interesting properties, as the plane-wave solutions. There are several ways to make the Chern-Simons theory more interesting[2] in $2+1$-dim, as considering the nonabelian version below.

The nonabelian Chern-Simons is given by

$$\mathcal{L}_{CS} = \kappa\epsilon^{\mu\nu\rho}tr\left(A_\mu\partial_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho\right), \tag{2.4}$$

[1] Actually, it is possible to construct a Chern-Simons gauge theory for all the odd dimensions. On the other hand, it is only in $3-$dim that it is quadratic in the gauge field.

[2] Other ways are: coupling to dynamical matter fields, to a Maxwell term or including gravity, for instance.

where $A_\mu = A_\mu^a T^a$, $A_\mu^a$ are the gauge fields and $T^a$ are the generators of a gauge Lie algebra, satisfying $[T^a, T^b] = f^{abc}T^c$; $f^{abc}$ are the structure constants of the algebra. A variation of $\delta A_\mu$ induces a variation in the Lagrangian,

$$\delta\mathcal{L}_{CS} = \kappa\epsilon^{\mu\nu\rho} tr(\delta A_\mu F_{\nu\rho}), \tag{2.5}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. The equation of motion has the same form as the abelian case, but now the field strength has an additional term.

In order to connect the nonabelian Chern-Simons theory with the open string field action more directly, we shall rewrite it using differential forms. Then, its action is

$$S(A) = \frac{1}{2}\int_M A \wedge dA + \frac{1}{3}\int_M A \wedge A \wedge A, \tag{2.6}$$

where $M$ is a $3-$manifold and $A = A_a T^a$ is a $1-$form. The integral symbol also includes a trace over the Lie-algebra generators. If we consider the following properties

1. $d$ is nilpotent: $d^2\omega = 0$ for all differential forms $\omega$;
2. $d$ is a derivation: $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{|\omega|}\omega \wedge d\eta$, where $|\omega|$ is the degree of $\omega$;
3. cyclic symmetry: $\int \omega \wedge \eta = (-1)^{|\omega||\eta|} \int \eta \wedge \omega$;
4. Stoke's theorem: $\int d\omega = 0$;
5. $\wedge$ is associative,

then (2.6) is invariant under the gauge transformation

$$\delta A = d\epsilon + A \wedge \epsilon - \epsilon \wedge A, \tag{2.7}$$

with $\epsilon$ being a $0-$form.

All the above properties, the gauge transformation and the Chern-Simons action will have analogs in Witten's open bosonic string field theory.

## 2.2. Generalization of Chern-Simons theory

We start by following the Witten's proposal to formulate the field theory of open string [3]. Its starting point is quite axiomatic: let's consider an associative non-commutative algebra $B$ with the defining properties:

1. a $Z_2$ grading, so that for every element $b \in B$ we have a degree[3] $(-1)^{k_b}$ which is $\pm 1$;
2. a multiplication law $\star$ defined as such: for any two elements $a$, $b \in B$, the degree of the product $a \star b$ is $(-1)^{k_a} \cdot (-1)^{k_b}$;
3. a linear map $Q$: for any $b \in B$, $Q(b) \in B$ which obeys $Q(a \star b) = (Q(a)) \star b + (-1)^{k_a} a \star Q(b)$. Besides, $Q$ is odd under the grading, which means that the degree of $Q(b)$ is $-(-1)^{k_b}$ and $Q$ is nilpotent, *i.e.*, $Q^2 = 0$, giving $Q(Q(b)) = 0$ for all $b \in B$. We will call this linear map a "derivation" of the algebra.

[3] The standard notation for the degree of an element $b \in B$ is $(-1)^b$.

4. a linear map $\int$: for any $b \in B$, $\int b \in \mathbb{C}$ which obeys $\int a \star b = (-1)^{k_a k_b} \int b \star a$, where $(-1)^{k_a k_b}$ is defined to be $(-1)$ if $a$ and $b$ are both odd and $+1$ otherwise. We also require that for any $b \in B$, $\int Q(b) = 0$. This map will be called "integration".[4]

Before we continue, let's call some attention for the above axioms in order to preview where we will get soon. The elements of our algebra will be identified with the cohomology of the BRST operator $Q_B$ with ghost number 1.

The above axioms are enough to generalize the Chern-Simons gauge theory. Let's consider a field $A$ with $k_A = 1$ (in the language of differential forms, this would means a 1-form) with the following gauge invariance

$$\delta A = Q(\epsilon) + A \star \epsilon - \epsilon \star A, \tag{2.8}$$

where $\epsilon$ is an arbitrary element of $B$ with $k_\epsilon = 0$ (a 0-form in the differential forms framework). Note that $B$ contains a subalgebra $B_0$ of elements with $k_{B_0} = 0$. We can now define a field strength to be

$$F = Q(A) + A \star A \tag{2.9}$$

which transforms under gauge transformations as

$$\delta F = F \star \epsilon - \epsilon \star A. \tag{2.10}$$

Now, we can imagine the most general action to be written for the field $A$ which preserves the above gauge invariance. It turns out that this action is given by the Chern-Simons three-form action,

$$S = \int \left( A \star Q\,(A) + \frac{2}{3} A \star A \star A \right). \tag{2.11}$$

Note that we do not consider higher Chern-Simons forms because they will have their integrals vanishing in the bosonic open string field framework (due to the precise ghost number associated with the correlators). We could have also considered the term $\int F \star F$, however it is easy to show it is a topological invariant since its variation under an arbitrary variation of $A$ is zero. Since this action pretty much constitutes the essence of this thesis, let's show that it is gauge invariant under (2.8):

$$\begin{aligned}
S &= \int (A \star Q\,(A) + \frac{2}{3} A \star A \star A) = \int (A \star F - \frac{1}{3} A \star A \star A) \\
\delta S &= \int \{\delta A \star F + A \star \delta F - \frac{1}{3}[\delta A \star A \star A + A \star \delta A \star A + A \star A \star \delta A]\} \\
&= \int \{[Q(\epsilon) + A \star \epsilon - \epsilon \star A] \star F + A \star (F \star \epsilon - \epsilon \star F) - \frac{1}{3}[Q(\epsilon) + A \star \epsilon - \epsilon \star A] \star A \star A + \\
&\quad + A \star [Q(\epsilon) + A \star \epsilon - \epsilon \star A] - \frac{1}{3} A \star A \star [Q(\epsilon) + A \star \epsilon - \epsilon \star A]\} \\
&= \int \{Q(\epsilon) \star F + \overbrace{A \star \epsilon \star F}^{3} - \epsilon \star A \star F + A \star F \star \epsilon - \overbrace{A \star \epsilon \star F}^{3} - \frac{1}{3}[Q(\epsilon) \star A \star A + A \star Q\,(\epsilon) \star A + \\
&\quad + A \star A \star Q(\epsilon)] - \frac{1}{3}[\overbrace{A \star \epsilon \star A \star A}^{1} + \overbrace{A \star A \star \epsilon \star A}^{2} + A \star A \star A \star \epsilon] + \frac{1}{3}[\epsilon \star A \star A \star A + \\
&\quad + \overbrace{A \star \epsilon \star A \star A}^{1} + \overbrace{A \star A \star \epsilon \star A}^{2}]\}.
\end{aligned} \tag{2.12}$$

[4] It is worth noting that each element or operation has its counterpart in the Chern-Simons theory above.

The terms with the same number cancel each other. Now, using axiom 4, we see that

$$\begin{aligned}
\int Q(\epsilon)\star A\star A &= (-1)^{k_A k_{Q(\epsilon)\star A}}\int A\star Q\,(\epsilon)\star A=\int A\star Q\,(\epsilon)\star A,\ \text{since } Q(\epsilon)\star A\text{is even};\\
\int A\star A\star Q(\epsilon) &= \int Q(\epsilon)\star A\star A,\ \text{since } A\star A\text{ is even};\\
\int A\star F\star\epsilon &= \int F\star\epsilon\star A=\int\epsilon\star A\star F,\ \text{since } F\star\epsilon\text{and } F\text{are even};\\
\int\epsilon\star A\star A\star A &= \int A\star A\star A\star\epsilon,\ \text{since } A\star A\text{ and } A\star A\star\epsilon\text{are even,}
\end{aligned}$$

so that (2.12) becomes

$$\begin{aligned}
\delta S &= \int\{Q(\epsilon)\star F-\overbrace{\epsilon\star A\star F}^{4}+\overbrace{\epsilon\star A\star F}^{4}-Q(\epsilon)\star A\star A\}\\
&= \int\{Q(\epsilon)\star F-Q(\epsilon)\star A\star A\}\\
&= \int\{Q(\epsilon)\star Q(A)+\overbrace{Q(\epsilon)\star A\star A}^{5}-\overbrace{Q(\epsilon)\star A\star A}^{5}\}\\
&= \int Q(\epsilon)\star Q(A)=\int Q[\epsilon\star Q(A)]=0.
\end{aligned}$$

## 2.3. Open String Field Theory

The first mode of the bosonic open string theory spectrum is a tachyon, which is a scalar field with an unstable potential. It is possible that this potential could be used to create an inflationary scenario. However, in order to study this possibility it is necessary to know how the potential associate with this tachyonic mode looks like. To calculate the potential, we have to go beyond the first-quantized framework, which is *onshell*. Hence we need an off-shell formulation of string theory, which is called *string field theory,* so that it is possible to obtain the potential. It is worth to mention that there are other reasons for the study of the off-shell formalism, between them the attempt to prove the Sen's conjectures [35] and find the correct vacuum of string theory.

This section will be based on [4, 5, 6].

### 2.3.1. The classical string field

As it can be reviewed in the appendix A, the Hilbert space $\mathcal{H}$ of the open string theory is constructed acting with the creation operators $\alpha^{\mu}_{-n}$, $b_{-m}$, $c_{-l}$, where $m$, $n$, $l>0$, and $c_0$ on the vacuum $|\Omega\rangle\equiv c_1|0\rangle$, defined by

$$\begin{aligned}
\alpha^{\mu}_{n}|\Omega\rangle &= 0 \qquad n>0\\
b_n|\Omega\rangle &= 0 \qquad n\geq 0\\
c_n|\Omega\rangle &= 0 \qquad n>0\\
p^{\mu}|\Omega\rangle\propto\alpha^{\mu}_{0}|\Omega\rangle &= 0. 
\end{aligned}\tag{2.13}$$

Note that regarding the state-operator mapping, it is expected that the vacuum of the theory should be mapped to the unit operator. This property is, indeed, satisfied by the state $|0\rangle=|0;0\rangle\otimes b_{-1}|\downarrow\rangle$. On the other hand, this is not really the ground state of the theory, which is given by $|0;0\rangle\otimes|\downarrow\rangle$

that is equal to $|\Omega\rangle$. So, due to the weights of $b$ and $c$, the unit operator in the vertex operator formalism is not mapped to the ghost vacuum in the state formalism. As one can check, $|\Omega\rangle$ is mapped to $c(0)$. Then, we have

$$|0\rangle \sim I \quad , \quad |\Omega\rangle \sim c(0). \tag{2.14}$$

A generic state is given by

$$\alpha^{\mu_1}_{-n_1} \dots \alpha^{\mu_i}_{-n_i} b_{-m_1} \dots b_{-m_j} c_{-l_1} \dots c_{-l_k} |\Omega\rangle, \tag{2.15}$$

where $n > 0$, $m > 0$, $l \geq 0$ and $i$, $j$, $k$ are arbitrary positive integers. As a result, any state $|\Phi\rangle \in \mathcal{H}$ can be expanded as

$$|\Phi\rangle = [\varphi(x) + A_\mu(x)\alpha^\mu_{-1} + B_{\mu\nu}(x)\alpha^\mu_{-1}\alpha^\nu_{-1} + \dots]c_1|0\rangle \equiv \Phi(z=0)|0\rangle. \tag{2.16}$$

Note that the coefficients multiplying the operators depend on the center-of-mass coordinate $x$ of the string. As you can imagine, we have infinitely many coefficient functions and they are identified with the spacetime particle fields, that is why we call $|\Phi\rangle$ *string field.* The corresponding vertex operator, $\Phi(0)$, has the same name.

The Hilbert space defined above is redundant. We have treated this problem in the appendix B considering the BRST quantization procedure. Now that we have established some aspects of the string field language, we can rewrite the physical conditions in this framework. In order to do this, let's consider the action below defined upon the ghost number +1 states[5]:

$$S_0 = \langle\Phi|Q_B|\Phi\rangle. \tag{2.17}$$

Using the reality condition for the string field, $\Phi[X^\mu(\pi - \sigma)] = \Phi^*[X^\mu(\sigma)]$, the equation of motion is

$$Q_B|\Phi\rangle = 0, \tag{2.18}$$

equal to the physical condition (B.15). Besides, it is easy to see that (2.17) is invariant under the gauge transformation

$$\delta|\Phi\rangle = Q_B|\chi\rangle, \tag{2.19}$$

where $|\chi\rangle$ has ghost number 0. This corresponds to the exact state defined in (B.18). From the above, we notice that an *on-shell state*, which is a solution to the equation of motion of $S_0$, is a physical state in string theory. In order to obtain the interacting field theory, we will have to include higher orders terms in $\Phi$ using the generalization we have exposed already in section 2.2.

## 2.4. Cubic String Field Theory Action

After studying the generalization of the Chern-Simons theory in section 2.2, we can construct our string field theory action considering the same axioms we have seen already with the identification of the BRST operator, $Q_B$, as the nilpotent derivative operator and string fields as the elements of the algebra B.

In order to understand better why this axiomatic approach makes sense to deal with open string field theory, we need to give a better look at the $\star$ product. First, let's call attention to what is pointed out in [3]: we do not need to work with reparametrization invariant strings once we have imposed the conservation of the BRST charge, which is natural giving our identifications. Hence, the following construction will not be reparametrization invariant as we will be choosing a preferred point of the string, the mid-point $\sigma = \pi/2$, remembering we are considering $\sigma \in [0, \pi]$. So, consider:

[5] All the physical states in the operator formalism have ghost number +1, as it is explained in the appendix .

1. The $\star-$product will be associative if we interpret this operation as gluing two half-strings together as in Figure 2.1 (a). We glue together the right hand piece of the string $U$, $(\pi/2 \leq \sigma \leq \pi)$, to the left hand piece of the string $V$, $(0 \leq \sigma \leq \pi/2)$, remaining a string-like object composed by the left hand piece of $U$ and right hand piece of $V$. The resulting string state on $U \star V$ is the string field $a \star b$ ;

2. Since we are considering a non-commutative operation, $U \star V$ and $V \star U$ are different. On the other hand, the integration is defined as $\int a \star b \; = (-1)^{k_a k_b} \int b \star a$, *i.e.,* it is "commutative" ignoring the sign difference. That suggests us that the integration procedures glues the remaining sides of $U$ and $V$. In $(c)$, that would mean to sew together the left- and right-hand ends of the string $U$ at its mid-point;

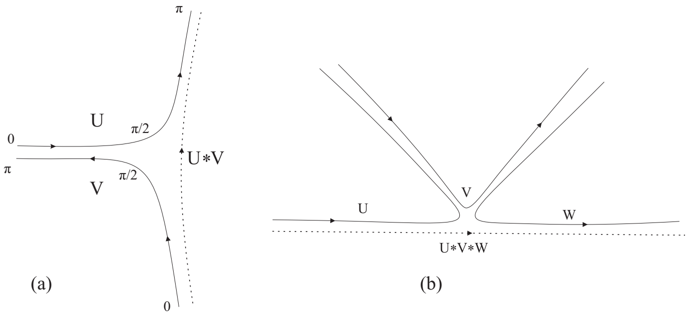


Figure 2.1.: (a) Gluing of two strings which have half of them coinciding in space-time. (b) Gluing three strings shows qualitatively why associativity follows. (c) The integration operation sews together both halves of the string at its mid-point. (d) Multiplication followed by an integration, $\int(U \star V)$.

Once the $\star$-product and the integration operation is understood as above, we can consider to write the so called *Witten vertex*,

$$\int \Phi_1 \star ... \star \Phi_n, \tag{2.20}$$

where $\Phi_i$ denotes a string field on the $i^{th}$ string Hilbert space. It is the 3-string vertex that matters for us, since it is for the cubic interaction term that we were able to generalize the gauge invariance. We represent this vertex below :

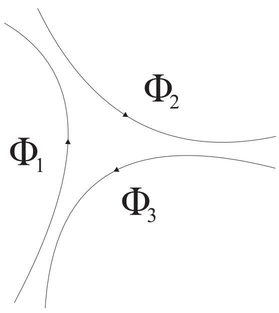


Figure 2.2.: 3-string vertex.

We have not finished with the identifications. Once the BRST charge $Q_B$ is qualified as the derivative operator, the algebra $Z$ grading comes from the ghost number. It was already explained that the string field has ghost number $+1$. Besides, we consider that $\#_{gh}(Q_B) = 1$ and $\#_{gh}(\star) = 0$.

Now, we can understand why only the quadratic term and the $3-$string vertex were considered from the beginning. We know that the ghost number counts the number of $c$ ghosts in the correlator minus the number of $b$ ghosts. On the other hand, according to the Riemann-Roch theorem, we must have for each correlator

$$\#c_{ghost} - \#b_{ghost} = 3\chi, \tag{2.21}$$

where $\chi$ is the Euler characteristic number for the surface considered. In the open string case, we are considering the disk[6], which gives $\chi = 1$. Hence, the ghost number for each correlator is 3, so that only the quadratic and cubic term were possible candidates from the beginning.

Finally, we can rewrite the action (2.11) for the open string interaction as

$$S = -\left(\int \Phi \star Q_B \Phi + \frac{2}{3} g_o \Phi \star \Phi \star \Phi\right), \tag{2.22}$$

which is gauge invariant under

$$\delta\Phi = Q_B\epsilon + g_o(\Phi \star \epsilon - \epsilon \star \Phi), \tag{2.23}$$

where $\epsilon$ is a gauge parameter with ghost number 0 and $g_o$ is the open string coupling constant. Indeed, the action with which we will be working comes from redefining the string field as $\Phi \to \frac{\alpha'}{g_o}\Phi$ and changing the overall normalization such that

$$S = -\frac{1}{g_o^2}\left(\frac{1}{2\alpha'}\int \Phi \star Q_B \Phi + \frac{1}{3}\int \Phi \star \Phi \star \Phi\right). \tag{2.24}$$

After considering the above construction, we can now state the correspondence between the nonabelian Chern-Simons theory in $2+1$- dim and the open string field theory in the table below:

| | Open String Field Theory | Chern-Simons theory in differential forms |
|---|---|---|
| element | string field | differential k-form |
| degree | $(-1)^{\#ghost}$ | $(-1)^k$ |
| multiplication | $\star-$product | $\wedge$ (wedge product) |
| derivation | $Q_B$ | exterior derivative $d$ |
| integration | $\int$ | $\int$ on a $k-$dimensional manifold |

## 2.5. Evaluation method

In the last section, we have justified why the relevant action to work the interactions in the string field theory framework is (2.24). However, we still do not know how to do calculations with it. Even though the quadratic term can be promptly associated with $\langle\Phi|Q_B|\Phi\rangle$, the cubic term does not have a ready translation.

There are more than one method to work out the interaction string field term. Here we will be considering a conformal field theory approach, in which conformal mappings and calculation of correlators in the disk will be relevant. The construction presented in [4] is very clear and we will follow it closely.

As it is explicit in appendix A, in the $z-$coordinate system an $i^{th}$ string evolving from $t = -\infty$, which corresponds to $z_i = 0$ $(P_i)$, propagates radially until it reaches an interaction point, which

[6] Because we will be considering only tree-level amplitudes.

we considered as $t = 0$, that is, $|z_i| = 1$. Note that in the CFT framework, these states have a corresponding vertex operator inserted at $P_i$. Since we are worried about the 3−string vertex, the idea is to map the three upper half-disks associated with the strings to one unique disk on a conformal plane. Graphically, we have:

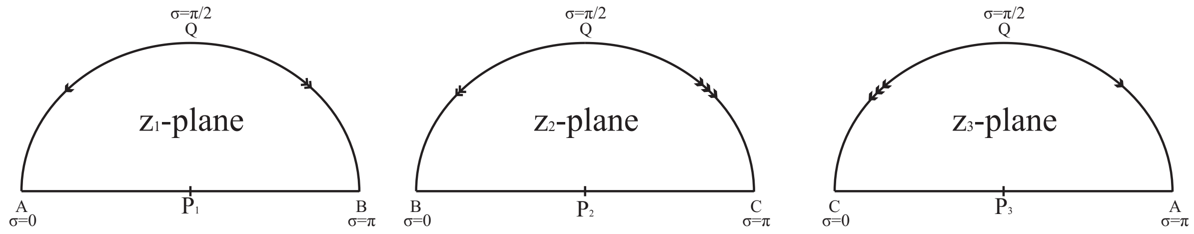


Figure 2.3.: As we learned in the last section, the gluing process will take half of each string and sew together to form a new string-like object.

In other words, we will be mapping three half-disks with their own local coordinates $z_i's$ to the interior of a unique disk with global coordinate $\zeta$. This map must satisfy these properties:

1. The common interaction point $Q$ is mapped to the center $\zeta = 0$ of the unit disk;
2. The real axes, which are the open string boundaries, are mapped to the boundary of the unit disk.

In order to implement this transformation, let's consider the first string. The mapping

$$z_1 \to \omega = h(z_1) = \frac{1 + iz_1}{1 - iz_1} \tag{2.25}$$

satisfy the above properties. On the other hand, we are trying to construct a unit disk from three half disks, so we need to consider yet another transformation over $\omega$ so that the the $z_1$−plane is mapped to a region with an angle of $120^o$. The second transformation is given by

$$\omega \to \zeta = \eta(\omega) = \omega^{2/3}. \tag{2.26}$$

Following the same ideas for the second and third strings, we end up with three wedges of $120^o$ angle that can be combined and form a unit disk. The last point to be made is to take care of the sewing, that is, we need to sew together the right-hand piece of the first string with the left-hand piece of the second string and so on, as it was establish in the last section. This can be achieved through the following transformations

$$\begin{aligned} g_1(z_1) &= e^{-\frac{2\pi i}{3}} \left(\frac{1 + iz_1}{1 - iz_1}\right)^{2/3} \\ \eta \circ h(z_2) = g_2(z_2) &= \left(\frac{1 + iz_2}{1 - iz_2}\right)^{2/3} \\ g_3(z_3) &= e^{\frac{2\pi i}{3}} \left(\frac{1 + iz_3}{1 - iz_3}\right)^{2/3}. \end{aligned} \tag{2.27}$$

These mappings are graphically represented as

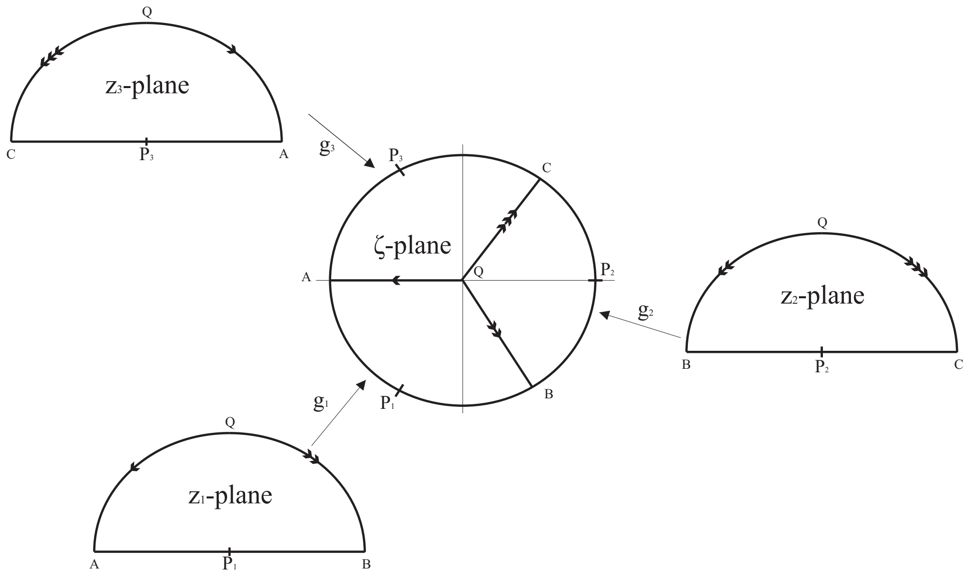


Figure 2.4.: 3−string vertex.

Therefore, using those mappings we can give a representation for the interaction term as a 3−point correlator function,

$$\int \Phi \star \Phi \star \Phi = \langle g_1 \circ \Phi(0)\, g_2 \circ \Phi(0)\, g_3 \circ \Phi(0)\rangle, \tag{2.28}$$

where it is defined on the global disk constructed above and evaluated in the combined matter and ghost CFT. Note that $g_i \circ \Phi(0)$ is just the conformal transformation of $\Phi(0)$ by $g_i$, *i.e.*

$$g_i \circ \Phi(0) = (g_i'(0))^h \Phi(g_i(0)), \tag{2.29}$$

if the $\Phi$ is a primary operator (A.31) with conformal weight $h$.

We can now go back to the upper half plane considering the inverse transformation of (2.25), that is,

$$z = h^{-1}(\zeta) = -i\frac{\zeta - 1}{\zeta + 1}, \tag{2.30}$$

so that the final expression for the 3−point vertex is given by

$$\begin{aligned} \int \Phi \star \Phi \star \Phi &= \langle \prod_{i=1}^{3} f_i \circ \Phi(0) \rangle & (2.31) \\ f_i(z_i) &= h^{-1} \circ g_i(z_i). & (2.32) \end{aligned}$$

The action of $h^{-1}$ is represented below

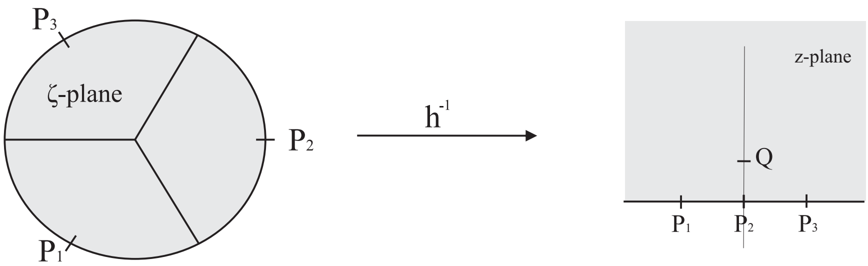


Figure 2.5.: Coming back to the upper half plane.

The above construction can be generalized for the $n-$point vertex, which becomes

$$\int \Phi \star \ldots \star \Phi = \langle f_1 \circ \Phi(0) \ldots f_n \circ \Phi(0) \rangle, \tag{2.33}$$

where

$$\begin{aligned} f_j(z_j) &= h^{-1} \circ g(z_j) \\ g_j(z_j) &= e^{\frac{2\pi i}{n}(j-1)} \left( \frac{1+iz_j}{1-iz_j} \right)^{\frac{2}{n}}, \; 1 \leq j \leq n. \end{aligned} \tag{2.34}$$

Hence, for the quadratic term in (2.24) $n = 2$, so that

$$f_1(z_1) = h^{-1}\left(\frac{1+iz_1}{1-iz_1}\right) = z_1 = id(z_1) \tag{2.35}$$

$$f_2(z_2) = h^{-1}\left(-\frac{1+iz_2}{1-iz_2}\right) = -\frac{1}{z_2} \equiv \mathcal{I}(z_2) \tag{2.36}$$

and

$$\int \Phi \star Q_B \Phi = \langle I \circ \Phi(0) Q_B \Phi(0) \rangle. \tag{2.37}$$

Finally, the string field theory action in terms of the CFT correlators is given by

$$S = -\frac{1}{g_0^2}\left[\frac{1}{2\alpha'}\langle I \circ \Phi(0) Q_B \Phi(0) \rangle + \frac{1}{3}\langle \prod_{i=1}^{3} f_i \circ \Phi(0) \rangle\right]. \tag{2.38}$$

# 3. The Tachyonic Potential

In the last chapter we have constructed an action for the string field considering the axiomatic proposal from Witten's generalization of the Chern-Simons theory. Besides, we have presented a CFT method to evaluate that action so that one is able to calculate, at least in principle, the full action for the spacetime fields.

Giving our interest in inflation, as we have already stated in the introduction, we will obtain an action for the tachyonic field in a perturbative way, which will be explained below. Then, we will consider its potential calculated in this framework in the last chapter as a candidate for producing an inflationary scenario.

Our basic references are again [3, 4, 5, 6].

## 3.1. The gauge choice

The action for the bosonic open string theory (2.24) is gauge invariant under the transformation (2.23). Therefore, in order to proceed with our calculations, we will be considering the so called Feynman-Siegel gauge in the state formalism[1],

$$b_0|\Phi\rangle = 0, \tag{3.1}$$

that is, there is no $c_0$ mode in the string field. This is a good gauge choice because:

1. *It can always be chosen, at least at the linearized level*: let's consider a state $|\Psi\rangle$ with $L_o^{tot}|\Psi\rangle = h|\Psi\rangle$ not obeying (3.1). We can define a new state $|\tilde{\Psi}\rangle$, which is a linearized gauge transformation of the original state, as

$$|\tilde{\Psi}\rangle = |\Psi\rangle - \frac{1}{h}Q_B|\Lambda\rangle$$

where $|\Lambda\rangle = b_0|\Psi\rangle$. The new state satisfies the above gauge condition:

$$\begin{aligned} b_0|\tilde{\Psi}\rangle &= b_0|\Psi\rangle - \frac{1}{h}b_0 Q_B b_0|\Psi\rangle \\ &= b_0|\Psi\rangle - \frac{1}{h}b_0\{Q_B, b_0\}|\Psi\rangle \\ &= b_0|\Psi\rangle - \frac{1}{h}b_0 L_0^{tot}|\Psi\rangle \\ &= 0, \end{aligned}$$

where we have used (B.26) in the third line. Note we have shown this property for $h \neq 0$ and that the $|\tilde{\Psi}\rangle$ and $|\Psi\rangle$ are physically equivalent because, as we have explained in the appendix B, $Q_B|\Lambda\rangle$ is a closed state;

[1] One of the motivations for this gauge choice comes from the fact that the string field theory action, when considered this gauge, has a global $Z_2$ symmetry, called the *twist symmetry,* which simplifies level calculations. We will not need to consider this symmetry for our calculations because they are the very first approximation.

2. *There are no residual gauge transformations preserving the gauge condition:* suppose we have two states satisfying the gauge condition, $b_0|\Psi_i\rangle$, $i = 1, 2$, and they are related by a gauge transformation: $|\Psi_1\rangle = |\Psi_2\rangle + Q_B|\xi\rangle$. Therefore, $Q_B|\xi\rangle$ is a residual gauge degree of freedom. Then,
$$h(Q_B|\xi\rangle) = L_0^{tot}(Q_B|\xi\rangle) = \{Q_B, b_0\}Q_B|\xi\rangle = Q_B b_0(Q_B|\xi\rangle) = 0,$$
where we have used the nilpotency of $Q_B$. Hence, since $h \neq 0$ from above, we see that $Q_B|\xi\rangle = 0$.

Besides, the usefulness of the Feynman-Siegel gauge can be appreciated in the quadratic piece of the action providing the canonical kinetic term for the states satisfying (3.1):

$$\begin{aligned}
\langle\Psi_1|Q_B|\Psi_2\rangle &= \langle\Psi_1|Q_B\{b_0, c_0\}|\Psi_2\rangle \\
&= \langle\Psi_1|Q_B b_0 c_0|\Psi_2\rangle \\
&= \langle\Psi_1|\{Q_B, b_0\}c_0|\Psi_2\rangle \\
&= \langle\Psi_1|L_0^{tot}c_0|\Psi_2\rangle \\
&= \langle\Psi_1|c_0 L_0^{tot}|\Psi_2\rangle,
\end{aligned}$$

where we used again (B.26) and $[L_0^{tot}, c_0] = 0$. As $L_0^{tot}$ is roughly $p^2 + m^2$, we have the familiar kinetic terms.

## 3.2. Level truncation scheme

As we have defined in the last chapter, the string field has infinite spacetime fields in it. Hence, in order to work out the action (2.24) we have to developed some kind of truncation.

Let's expand $|\Phi\rangle$ in the momentum basis using the state formalism, so

$$\begin{aligned}
|\Phi\rangle &= \int d^dk \Big(\varphi + A_\mu\alpha_{-1}^\mu + i\alpha b_{-1}c_0 + \frac{i}{\sqrt{2}}B_\mu\alpha_{-2}^\mu + \frac{1}{\sqrt{2}}B_{\mu\nu}\alpha_{-1}^\mu\alpha_{-1}^\nu \\
&\quad +\beta_0 b_{-2}c_0 + \beta_1 b_{-1}c_{-1} + i\kappa_\mu\alpha_{-1}^\mu b_{-1}c_0 + \ldots\Big)\, c_1|k\rangle, \qquad (3.2)
\end{aligned}$$

where we have not considered a gauge condition yet. The expansion coefficients might appear to be quite arbitrary, but they are adjusted so that the spacetime field terms recover their usual form in the end. Now, remembering from appendix A, we know that

$$L_0^{tot} = \alpha' p^2 + \sum_{n=1}^{\infty} \alpha_{-n}^\mu\alpha_{\mu n} + \sum_{n=-\infty}^{\infty} n \circ c_{-n}b_n \circ -1, \qquad (3.3)$$

where $\circ ... \circ$ denotes annihilation-creation normal ordering. Then, we define the *level* of a state as the sum of the level numbers $n$ of the creation operators acting on $c_1|k\rangle$, that is, the sum of the second and third terms above. Therefore, the zero momentum tachyonic mode has level 0.

We can also define the level of each term in the action, which is naturally defined to be as the sum of the level of each field involved. Therefore, we understand the truncation to level $N$ as keeping only those terms with level less than or equal to $N$. We shall denote *"level $(M, N)$ truncation"* when the string field has terms with $l \leq M$ while the action has ones with $l \leq N$.

Even though the truncation is well defined, it is just an approximation to deal with an infinite number of fields. There are some works trying to explore how good is this approximation through numerical analysis, in which they consider higher levels and see if the method is numerically convergent. Indeed, it seems to converge quite well, but we still lack a theoretical proof of this.

Another important point to be made is that when a particular level truncation is chosen the gauge invariance breaks down because the action was invariant under the full gauge transformations, which involves all the infinite fields. Hence, in order to proceed with calculations, we shall impose our gauge choice before use the level truncation.

## 3.3. The potential

In this section we intend to obtain a formula for the tachyonic potential using the methods developed above and results from appendix A so that we can consider it as a candidate for producing an inflationary scenario. Since our objective here is academic, we will keep the calculations as simple as we can using the very first approximation that the level truncation scheme provides us. As we have talked above, the string field has an infinite number of spacetime fields which can be classified according to their level. In order to do calculations, we use the level truncation scheme as an approximation. For our purposes here, let's consider the $|\Phi\rangle$ up to $l = 2$. Besides, we work in the Feynman-Siegel gauge, thus all the terms containing $c_0$ are dropped. Therefore, (3.2) becomes

$$|\Phi\rangle = \int d^dk \left[\varphi(k) + A_\mu(k)\alpha^\mu_{-1} + \frac{i}{\sqrt{2}}B_\mu(k)\alpha^\mu_{-2} + \frac{1}{\sqrt{2}}B_{\mu\nu}(k)\alpha^\mu_{-1}\alpha^\nu_{-1} + \beta_1(k)b_{-1}c_{-1}\right] c_1|k\rangle. \quad (3.4)$$

Using the vertex-operator map from section A.5, we can rewrite the string field in the operator formalism as

$$\begin{aligned} |\Phi\rangle \quad = \quad & \int d^dk \left[\varphi(k)c(0) + \frac{i}{\sqrt{2\alpha'}}A_\mu(k)c\partial X^\mu(0) - \frac{1}{2\sqrt{\alpha'}}B_\mu(k)c\partial^2 X^\mu(0)\right. \\ & \left. -\frac{1}{2\sqrt{2}\alpha'}B_{\mu\nu}(k)c\partial X^\mu\partial X^\nu(0) - \frac{1}{2}\beta_1(k)\partial^2 c(0)\right]|k\rangle, \end{aligned} \quad (3.5)$$

with $|k\rangle = e^{ik\cdot X(0)}|0\rangle$. As one can check, the vertex operators of the last three terms above are not primary operators (A.31) and they can complicate a lot the CFT method. Therefore, we will proceed with our calculations at the $(1, 3)$ truncation level.

The quadratic term in (2.24) is $\langle \mathcal{I} \circ \Phi(0)Q_B\Phi(0)\rangle$, then we have to consider the conformal transformations of the vertex operators in the string field under $\mathcal{I} = -z^{-1}$ given by (A.31), so that

1. $\mathcal{I} \circ \left[ce^{ik\cdot X}(\epsilon)\right] =$

$$\begin{aligned} &= \left[\partial_z\left(-\frac{1}{z}\right)\right]^{-1+\alpha' k^2}_{z\to\epsilon} ce^{ik\cdot X}\left(-\frac{1}{\epsilon}\right) \\ &= \left(\frac{1}{\epsilon^2}\right)^{-1+\alpha' k^2} ce^{ik\cdot X}\left(-\frac{1}{\epsilon}\right); \end{aligned} \quad (3.6)$$

2. $\mathcal{I} \circ \left[c\partial X^\mu e^{ik\cdot X}(\epsilon)\right] =$

$$\begin{aligned} &= \left[\partial_z\left(-\frac{1}{z}\right)\right]^{-1+\alpha' k^2+1}_{z\to\epsilon} c\partial X^\mu e^{ik\cdot X}\left(-\frac{1}{\epsilon}\right) \\ &= \left(\frac{1}{\epsilon^2}\right)^{\alpha' k^2} c\partial X^\mu e^{ik\cdot X}\left(-\frac{1}{\epsilon}\right). \end{aligned} \quad (3.7)$$

Note that we have to take the limit $\epsilon \to 0$ in the end. Then, using (B.24), the quadratic term becomes

$$\begin{aligned}\langle \mathcal{I}\circ\Phi(0)Q_B\Phi(0)\rangle &= \int d^dkd^dq\left\langle\left[\left(\frac{1}{\epsilon^2}\right)^{-1+\alpha'k^2} ce^{ik\cdot X}\left(-\frac{1}{\epsilon}\right)\varphi(k)+\right.\right.\\ &\quad \left.\frac{i}{\sqrt{2\alpha'}}A_\mu(k)\left(\frac{1}{\epsilon^2}\right)^{\alpha'k^2} c\partial X^\mu e^{ik\cdot X}\left(-\frac{1}{\epsilon}\right)\right]\oint\frac{dz}{2\pi i}\left[cT^m(z)+bc\partial c(z)\right]\\ &\quad \left.\times\left[\varphi(q)ce^{iq\cdot X}(\epsilon)+\frac{i}{\sqrt{2\alpha'}}A_\nu(q)c\partial X^\nu e^{iq\cdot X}(\epsilon)\right]\right\rangle_{\epsilon\to 0}\end{aligned}$$

$$\begin{aligned}&=\int d^dkd^dq\oint\frac{dz}{2\pi i}\left\langle\left(\frac{1}{\epsilon^2}\right)^{-1+\alpha'k^2}\varphi(k)\left[\varphi(q)ce^{ik\cdot X}\left(-\frac{1}{\epsilon}\right)cT^m(z)ce^{iq\cdot X}(\epsilon)\right.\right.\\ &+\frac{i}{\sqrt{2\alpha'}}A_\nu(q)ce^{ik\cdot X}\left(-\frac{1}{\epsilon}\right)cT^m(z)c\partial X^\nu e^{iq\cdot X}(\epsilon)+\varphi(q)ce^{ik\cdot X}\left(-\frac{1}{\epsilon}\right)bc\partial c(z)ce^{iq\cdot X}(\epsilon)\\ &\left.+\frac{i}{\sqrt{2\alpha'}}A_\nu(q)ce^{ik\cdot X}\left(-\frac{1}{\epsilon}\right)bc\partial c(z)c\partial X^\nu e^{iq\cdot X}(\epsilon)\right]+\frac{i}{\sqrt{2\alpha'}}A_\mu(k)\left(\frac{1}{\epsilon^2}\right)^{\alpha'k^2}\\ &\times\left[c\partial X^\mu e^{ik\cdot X}\left(-\frac{1}{\epsilon}\right)cT^m(z)ce^{iq\cdot X}(\epsilon)\varphi(q)+\frac{i}{\sqrt{2\alpha'}}A_\nu(q)c\partial X^\mu e^{ik\cdot X}\left(-\frac{1}{\epsilon}\right)cT^m c\partial X^\nu e^{iq\cdot X}(\epsilon)\right.\\ &\left.\left.+\varphi(q)c\partial X^\mu e^{ik\cdot X}\left(-\frac{1}{\epsilon}\right)bc\partial c(z)ce^{iq\cdot X}(\epsilon)+\frac{i}{\sqrt{2\alpha'}}A_\nu(q)c\partial X^\mu e^{ik\cdot X}\left(-\frac{1}{\epsilon}\right)bc\partial c(z)c\partial X^\nu e^{iq\cdot X}(\epsilon)\right]\right\rangle_{\epsilon\to 0}.\end{aligned}$$

We have eight different terms above, but we will keep our attention to the ones involving only the tachyonic field[2]. Therefore, we are left with

$$\begin{aligned}\langle \mathcal{I}\circ\Phi(0)Q_B\Phi(0)\rangle_\varphi &= \int d^dkd^dq\oint\frac{dz}{2\pi i}\left(\frac{1}{\epsilon^2}\right)^{-1+\alpha'k^2}\varphi(k)\varphi(q)\left\{\left\langle ce^{ik\cdot X}\left(-\frac{1}{\epsilon}\right)cT^m(z)ce^{iq\cdot X}(\epsilon)\right\rangle\right.\\ &\quad\left.+\left\langle ce^{ik\cdot X}\left(-\frac{1}{\epsilon}\right)bc\partial c(z)ce^{iq\cdot X}(\epsilon)\right\rangle\right\}.\end{aligned}\tag{3.8}$$

The detailed calculation is presented in appendix C. The final result is

$$\langle \mathcal{I}\circ\Phi(0)Q_B\Phi(0)\rangle_\varphi = (2\pi)^d\alpha'\int d^dk\left(k^2-\frac{1}{\alpha'}\right)\varphi(-k)\varphi(k).\tag{3.9}$$

Considering the Fourier-transformation to the position space,

$$\varphi(k)=\int\frac{d^dk}{(2\pi)^d}\varphi(x)e^{-ik\cdot x},\tag{3.10}$$

the quadratic part of the tachyonic field becomes

$$S^{(2)}_\varphi=\frac{1}{g_0^2}\int d^dx\left(-\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi+\frac{1}{2\alpha'}\varphi^2\right).\tag{3.11}$$

[2] Some of the other terms give no contribution at all or are responsible for the kinetic term of the vetor field. The curious reader can check the other terms in [4], page 30.

Now, let's calculate the cubic term. We will focus again on the terms of our interest, that is, involving only the scalar field. Even though just the main results are expressed below, we provide details in appendix C. From (2.31), we consider

$$\begin{aligned} f_i \circ \Phi(0) &= \int d^dk \left[\varphi(k) f_i \circ (ce^{ik\cdot X})(0) + \frac{i}{\sqrt{2\alpha'}} A_\mu(k) f_i \circ (c\partial X^\mu e^{ik\cdot X})(0)\right] \\ &= \int d^dk \left\{ [f_i'(0)]^{\alpha' k^2 - 1} \varphi(k) ce^{ik\cdot X}(f_i(0)) \right. \\ &\quad \left. + \frac{i}{\sqrt{2\alpha'}} [f_i'(0)]^{\alpha' k^2} A_\mu(k) c\partial X^\mu e^{ik\cdot X}(f'(0)) \right\}. \end{aligned}$$

Then, we need to calculate the $f_i's$ and their first derivatives. They are given by

$$\begin{aligned} w_1 &= -\sqrt{3} \\ w_2 &= 0 \\ w_3 &= \sqrt{3} \\ w_1' &= \frac{8}{3} \\ w_2' &= \frac{2}{3} \\ w_3' &= \frac{8}{3}, \end{aligned}$$

where $w_i \equiv f_i(0)$. The 3−point function has only one term involving only the tachyonic field, which is

$$\begin{aligned} \langle \prod_{i=1}^{3} f_i \circ \Phi(0) \rangle &= \int d^dk \int d^pk \int d^dq w_1'^{\alpha' k^2 - 1} w_2'^{\alpha' p^2 - 1} w_3'^{\alpha' q^2 - 1} \varphi(k)\varphi(p)\varphi(q) \times \\ &\quad \left\langle ce^{ik\cdot X}(w_1) ce^{ip\cdot X}(w_2) ce^{iq\cdot X}(w_3) \right\rangle \\ &= (2\pi)^d \int d^dk \int d^dp \int d^dq w_1'^{\alpha' k^2 - 1} w_2'^{\alpha' p^2 - 1} w_3'^{\alpha' q^2 - 1} \varphi(k)\varphi(p)\varphi(q) \times \\ &\quad |w_{12}|^{2\alpha' k\cdot p + 1} |w_{23}|^{2\alpha' p\cdot q + 1} |w_{13}|^{2\alpha' k\cdot q + 1} \\ &= 6\sqrt{3}\frac{3^3}{2^7}(2\pi)^d \int d^dk \int d^pk \int d^dq \delta^d(k+p+q)\varphi(k)\varphi(p)\varphi(q) F(k,p,q), \end{aligned} \tag{3.12}$$

where

$$F(k,p,q) = \exp\left[\alpha' \ln\left(\frac{4}{3\sqrt{3}}\right)(k^2+p^2+q^2)\right].$$

Thus, the $\varphi^3$ term in the action is given by

$$S_\varphi^{(3)} = -\frac{1}{3g_0^2}\left(\frac{3\sqrt{3}}{4}\right)^3 \int d^dx \tilde{\varphi}(x)^3, \tag{3.13}$$

where we have used the Fourier-transformation for the position space (3.10) and defined

$$\tilde{\varphi}(x) = \exp\left(-\alpha' \ln \frac{4}{3\sqrt{3}} \partial_\mu \partial^\mu\right) \varphi(x). \tag{3.14}$$

Therefore, the tachyonic action[3] is given by

$$S_\varphi = \frac{1}{g_0^2} \int d^dx \left[ -\frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi + \frac{1}{2\alpha'}\varphi^2 - \frac{1}{3}\left(\frac{3\sqrt{3}}{4}\right)^3 \tilde{\varphi}^3 \right]. \tag{3.15}$$

[3] The curious reader can check the full action at the level (1,3) truncated in [4], page 34.

There are higher-order calculations in the literature, up to level $(10, 20)$ [36]. In order to see how good is the approximations we have considered above, let's write the effective potential for the $(2, 4)-$level [4]:

$$\begin{aligned} V^{(2,4)} &= \frac{6\pi^2\varphi^2}{256(288+581\sqrt{3}\varphi)^2(432+786\sqrt{3}\varphi+97\varphi^2)^2} \times \\ & \times(-660451885056 - 4510794645504\sqrt{3}\varphi \\ & -32068942626816\varphi^2 - 25455338339328\sqrt{3}\varphi^3 + 27487773823968\varphi^4 \\ & +54206857131636\sqrt{3}\varphi^5 + 24845285906980\varphi^6 + 764722504035\sqrt{3}\varphi^7). \end{aligned} \tag{3.16}$$

The graphic of the potential in the level we calculated, which was $(0, 0)$ after all the approximations, and the level $(2, 4)$ is [4]

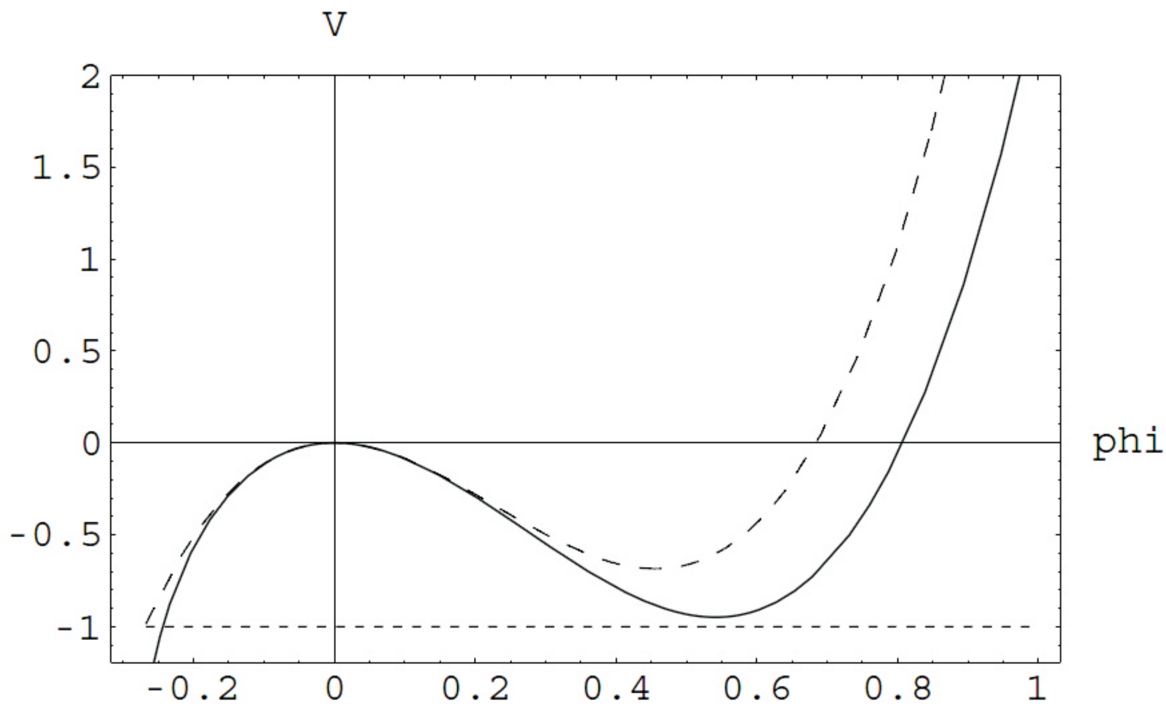


Figure 3.1.: The dashed line is the potential calculated here while the solid line is the potential for the $(2, 4)$ level.

so that we did not lose too much information after all.

# 4. Inflation: motivations, qualitative description and old approaches

Before we start to study inflation and its consequences, it is important to contextualize the *status quo* of cosmology by the end of $70's$, focusing in its properties and issues that preceded and motivated the inflationary scenario proposal. Some points of the discussion will be limited to qualitative aspects, hence the reader that is not familiar with General Relativity or the basic aspects of Cosmology should first address the appendix D.

## 4.1. Historical Context

The cosmological standard model before the inflationary proposal by [15, 16, 17, 23] had some arguable problems related with initial-conditions. After the observation of the CMB (Comic Microwave Background) radiation, which indicated that the universe was extremely homogeneous and isotropic at recombination[1], the best model at time to describe the primordial universe until the CMB emission was provided by the expanding radiation-dominated FRW metric[2].

On the other hand, we know that inhomogeneities are unstable gravitationally (if there is an accumulation of matter-energy at some point, this should grow with time because gravity is attractive). As a result, if the CMB pointed out a universe highly homogenous at the last-scattering surface, it was expected that the inhomogeneities were even smaller in the primordial universe. However, Rindler [18] had already noticed that an expanding radiation-dominated universe should have a particle horizon and, consequently, it would be composed of a lot of causally disconnected regions. Therefore, it is odd to imagine that these regions should present so similar physical properties without any dynamical reason. This is essentially the recipe for the initial-condition problems we will talk quantitatively below.

## 4.2. A simple analogy

It is generally better when we can have some intuition about a complicated problem using ordinary physics. Hence, let's consider the following situation: suppose we receive a photograph of a very soft material, showing the shape in the Figure 4.1.

[1] Footnote on page 67.
[2] Table D.1.

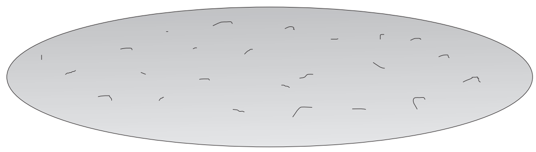

Figure 4.1.: Note that the shape presented is very flat and regular, with only some small "inhomogeneities" in its surface.

Knowing that it could be anything in the photo (it is a soft object after all), we could ask ourselves some natural questions: why such a regular object? Why is it so flat? Then, one may consider two different positions: the object was always just as above and that is it, or something might have happened to make the object as in the figure. In this very simple case, one who has chosen the second alternative might wonder that a racket have hit the object, Figure 4.2, letting it as above.

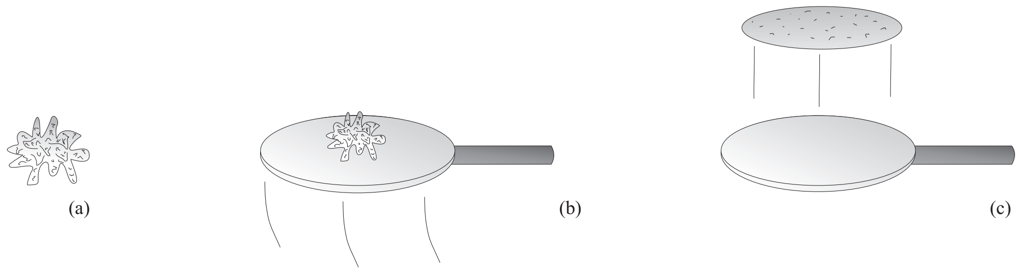


Figure 4.2.: (a) Very irregular object; (b) Ping-Pong racket hitting the material; (c) The material is deformed in a very regular form.

Well, what any of this has anything to do with the universe? Actually, it is pretty much the same physical intuition. Take the very irregular soft material as some initial state of the universe at a very early age, which was inhomogeneous. The photograph is representing the CMB, which tells us the universe was extremely homogeneous and isotropic, *i.e.*, the analogous to the object soon after being hit. Besides, the questions posed above can be translated as: why the universe was very homogeneous and isotropic? Why was it so flat? And the answer for both situations could be a very short period of acceleration, that is, a racket hit causing a stretching acceleration on the object and a very short period of accelerated expansion of the universe, also known as inflation.

Therefore, *inflation* will be understood as a stage of accelerated expansion of the universe in the very early ages. It will be argued quantitatively and qualitatively that it can address some initial-condition problems once one has assumed that Quantum Gravity is not relevant[3] after $t_{Pl} \sim 10^{-43} s$ and that inhomogeneities are not dissolved by expansion.

Before we continue, one should note that the problem to explain dynamically the initial-conditions of the universe, its kinematics, is essentially a philosophical one. When some of the others areas are

[3] So that it makes sense to use General Relativity and all its consequences when applied for this early age of the universe.

contemplated, coming from classical mechanics to quantum physics, we see that the dynamics is fundamentally concerned in predicting the future evolution of a system given the initial-conditions. Hence, it is not clear if cosmology should differ from this paradigm. On the other hand, once we have a dynamical theory in which these conditions are naturally produced, we shall explore it and try to infer some other results in order to corroborate this scenario.

## 4.3. The initial-condition issues

In this section, we will follow some calculations from [10] with minor changes and present some insights coming from different references which are cited in the text.

### 4.3.1. Homogeneity, isotropy (horizon) problem

The homogeneous and isotropic wedge of the universe at the last-scattering surface is at least as large as the horizon scale: $ct_{rec} \sim 10^{21}m$ (it is bigger if taking into account that it was expanding), where $t_{rec}$ is $380,000$ years.

Initially, this piece of the universe was much smaller, by a factor[4] of $\frac{a_i}{a_{rec}}$. Hence, the size of the isotropic homogeneous region from which the universe at the last-scattering originated at $t = t_i$ was of order

$$l_i \sim ct_{rec}\frac{a_i}{a_{rec}}. \tag{4.1}$$

On the other hand, the causal region at that time was of order $l_c \sim ct_i$ (in fact, it would be smaller if we considered the expansion). Comparing both regions,

$$\frac{l_i}{l_c} \sim \frac{t_{rec}}{t_i}\frac{a_i}{a_{rec}}. \tag{4.2}$$

In order to estimate this ratio, we assume that the primordial radiation dominates at $t_i \sim 10^2 \times t_{Pl}$ and use that $a(t) \propto t^{\frac{1}{2}}$ from Table D.1. Hence,

$$\frac{l_i}{l_c} \sim 10^{-1}\sqrt{\frac{t_{rec}}{t_{Pl}}} \sim 10^{-1}\sqrt{\frac{10^{13}}{10^{-43}}} \sim 10^{27}. \tag{4.3}$$

Thus, at $t_i$ the size of the homogeneous and isotropic universe exceeded the causality scale by 27 orders of magnitude! This means that in $10^{81}$ (volume $\sim l^3$) causally disconnected regions the energy density was distributed with a fractional variation not exceeding $\delta\rho/\rho \sim 10^{-4}$. This unnatural fine-tunning energy distribution cannot be explained by causal physical processes given that no signals propagate faster than light.

Note also that if $a \sim t^n$, then $\frac{a}{t} \sim \dot{a}$ and (4.2) becomes

$$\frac{l_i}{l_c} \sim \frac{\dot{a}_i}{\dot{a}_{rec}}, \tag{4.4}$$

which implies that the homogeneity scale was always larger than the causality one if gravity was always attractive (hence decelerating the expansion). That's why this problem is also called *horizon problem.*

An important point to be made that was emphasized by Guth [15] is that the horizon problem could be obviated by consideration of the full quantum gravitational theory if it has an unexpected behavior in the very early universe, which could compensate this huge scale difference between the causal connected regions and their homogeneity.

[4] Here, we denote the initial time using the index "$i$" and it refers to a point comfortably below the Planck time. Further comments about it will be given in Section 4.5.1.

### 4.3.2. Flatness problem

One of the Friedmann equations from General Relativity is given by (D.32)

$$H^2 + \frac{k}{a^2} = \frac{8\pi\rho}{3}, \tag{4.5}$$

where $H(t) = \frac{\dot{a}(t)}{a(t)}$ is the Hubble parameter, $k$ is associated with the spatial curvature and $\rho(t)$ is the energy density. We define the cosmological parameter, $\Omega(t)$, as

$$\Omega(t) = \frac{\rho(t)}{\rho^{cr}(t)}, \tag{4.6}$$

where $\rho^{cr}(t) = 3H^2/8\pi$ is the critical energy (it is often used to refer to the current value of the energy density). Then (4.5) can be rewritten as

$$\Omega(t) - 1 = \frac{k}{(Ha)^2}.$$

Thus,

$$\Omega_i - 1 = (\Omega_{rec} - 1)\frac{(Ha)^2_{rec}}{(Ha)^2_i} = (\Omega_{rec} - 1)\left(\frac{\dot{a}_{rec}}{\dot{a}_i}\right)^2 \leq 10^{-54}, \tag{4.7}$$

where we have used (4.4) and $\Omega_{rec} \approxeq 1$[5]. Hence, the above relation tells us that the cosmological parameter must be initially extremely close to unity, *i.e.*, we had a flat universe by that time. That's why we call this as the *flatness problem*. It was known at least since the end of $70's$ [19].

### 4.3.3. Initial perturbation problem

One might wonder how the primordial inhomogeneities that ended up forming the large structures of the universe were originated. An answer to this question is also expected and it turns out that inflation has something to say about it. In fact, inflation might not only be responsible for the large-scale homogeneity of the universe, but also for the small fluctuations in the primordial universe that were the seeds for the formation of the large-scale structures.

A qualitative understanding [12] is to imagine that the microscopic fluctuations in the energy density were created in the very early universe during a period of inflation. Then, they were stretched by the inflationary expansion to macroscopic scales, larger than the physical horizon at that time, leaving behind a perfectly homogeneous universe. These perturbations remained causally unaccessible until they re-enter the horizon at a later time during the FRW non-accelerated expansion, when the universe was about 100,000 years old, before recombination. Inside the horizon again, these perturbations could create the inhomogeneities we observe in the CMB spectrum that were the key for the large-structure formation.

This is all we will talk about cosmological perturbations. The interested reader can check [12, 10] for more informations.

## 4.4. Inflation: qualitative aspects

In the last section we saw that the initial-condition problems are related to the fact that $\dot{a}_i/\dot{a}_{rec} \gg 1$. This condition can be avoided only if during some period of expansion gravity acted as a repulsive

[5] In order to check this, we can use (4.7) with $t_i \to t_0 =$ today and $\Omega_0 = -0.001\begin{pmatrix} +0,0062 \\ -0,0065 \end{pmatrix}$ from [14]. The value of $\Omega_0$ comes from a combination of data from several sources as can be checked in the reference.

force, thus accelerating the expansion. In this situation, we can have $\dot{a}_i/\dot{a}_{rec} < 1$ and it becomes possible to have a large homogeneous universe coming from a single causally connected domain. This is a necessary but not suficient condition. Despite of general features of inflation that can be investigated, the specific model that provides this scenario will also play an important role, as we will be able to see studying the old, new and chaotic inflationary scenarios in the next sections and chapter.

Considering the above remarks, inflation is a stage of accelerated expansion of the universe when gravity acts as a repulsive force. The general picture of universe evolution in its early ages should be as the Figure 4.3,

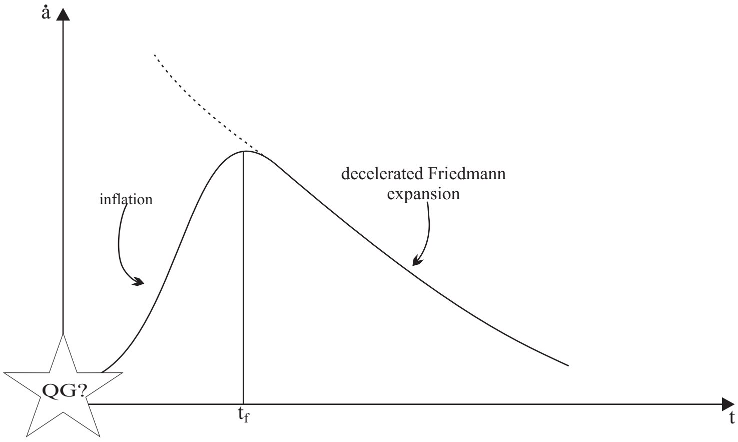


Figure 4.3.: Representation of what the expansion of the universe would look like in the first ages, where $t_f \sim 10^{-34} - 10^{-36}s$. $QG$ stands for Quantum Gravity.

where $t_f$ comes essentially from requiring the generation of primordial fluctuations. We also note that the curve is smooth at the connection between the end of inflation and the Friedmann expansion. This is important to guarantee that the homogeneity of the universe is not spoiled.

The idea of an accelerated expansion phase in the beginning of the universe is, in fact, better than stated above because it is even possible to relax the restriction of homogeneity on the initial conditions, *i.e.*, if starting with a strongly inhomogeneous causal domain, inflation can produce a large homogeneous universe (intuitively, it is possible to think of a very irregular rubber material being suddenly stretched). A quantitative argument is given at [10], page 232. This is one of the aspects regarding chaotic inflation, as we will see in chapter 5.

## 4.5. The old and new scenarios

Let's now consider the initial proposals of an inflationary universe, first Guth's and Sato's [15, 23], known as the *old inflation*, and then Linde's [16] and Steinhardt and Albrecht's [17], called *new inflation.*

### 4.5.1. The old inflation

In order to be pragmatic, we will follow the general proposal of Guth, with some comments coming from Sato's work.

Guth started recognizing that there is a singularity in the FRW metric at $t = 0$, what is called until today the Big Bang, so that the initial conditions could not be defined there. Besides, even after the singularity, we are in the Planck regime. Then, quantum gravity effects are expected to play a role and the Einstein's equation is meaningless at this point. Therefore, he established that within the scope of knowledge of the epoch, it only made sense to consider the FRW expansion at some energy scale comfortably below $E_P$. He proposed $10^{17}\,GeV$ as the limit because there was a lot of interest in *Grand Unification Theories* (GUTs) by that time, which were expected to be relevant in these energies scales. Thus, below this limit one should be able to describe the universe as a set of initial conditions and with the subsequent evolution given by the equations of motion coming from General Relativity.

Considering a thermodynamical approach, he noticed that the horizon and flatness problems were an indication that the expansion of the universe was not adiabatically, *i.e.,* the total entropy of the universe should not be a constant. Actually, there should be a huge increase of entropy at some moment of the early universe to prevent those problems. In order to create that moment, he proposed what he called an *inflationary universe*, described below.

Suppose that the equation of state for matter exhibits a phase transition at some critical temperature, $T_c$. As a result, when the universe reaches the temperature $T_c$ (remember it is cooling due to the expansion), bubbles of the low-temperature phase should start to nucleate and grow. A phase-transition should happen here. However, if the nucleation rate for this phase transition is low, the universe will continue to cool as it expands, supercooling in the high-temperature phase[6]. Suppose that this supercooling continues until some temperature $T_s$, where $T_s \ll T_c$. When the phase transition finally takes place, then the latent heat is released. Because the latent heat is characteristic of the energy scale of $T_c$, the universe is reheated to some temperature $T_\gamma \sim T_c$. Remembering from appendix that

$$a(t) \propto T^{-1},$$

the horizon and flatness problems are solved if $T_\gamma/T_s \sim 10^{27}$.

Even though the huge expansion was found, the dynamics remains to be determined. In other words, what is the expression for $a(t)$? The key point to answer this question is to notice that while the universe is supercooled in the high-temperature phase, its energy density must be modified. Until the system reaches the phase-transition, it is cooling not toward the true vacuum, but rather toward some metastable false vacuum with an energy density $\rho_0$ which is necessarily higher than that of the true vacuum. Hence, we end up with

$$\rho = \rho_0 + \rho_{matter} + \rho_{radiation}.$$

But, as it was deduced in the appendix, $\rho_{matter}$ and $\rho_{radiation}$ goes with $a^{-3}(t)$ and $a^{-4}(t)$, respectively. So, as soon as the temperature drops considerable, their contribution are negligible. Finally, using (D.32), we can obtain $a(t)$ :

$$a(t) \propto e^{\beta t},$$

where $\beta = \sqrt{\frac{8\pi\rho_0}{3}}$. Obviously, this behaviour is very different from what is expected from matter-radiation content, as it was showed in Table D.1. Actually, it reproduces the behavior of a cosmological constant. This is essentially the physics contained in the old inflation.

[6] Let's talk about this phenomenum in another context. If we have liquid water and start to drop the temperature passing the melting point, the water starts to freeze through a nucleation process. This is a *phase-transition* and it happens if there are others substances in the liquid water, possibiliting the water molecules to nucleate. These nucleations are the equivalent to the bubbles we are talking about. They are "transforming" the liquid- to the solid-phase. However, this nucleation process can be very slow if the water is pure. In fact, it is possible to drop the temperature until $-48.3^o\,C$ without freezing the water. Hence, if the nucleation process is very slow, we can keep the high-temperature phase even after crossing the critical temperature that characterizes the phase-transition. This is *supercooling.*

There are some interesting remarks already pointed by Guth in his paper:

1. We cannot use necessarily the FRW metric from the beginning, since it already assumes homogeneity and isotropy. The solution proposed was that even in the inhomogeneous universe, local homogeneous regions could appear due to thermalization of the particles distribution (and by local we mean smaller than the horizon length). Thus, assuming at least some region of the universe with a temperature higher than $T_c$, we should expect that, by the time the temperature in one of these regions falls to $T_c$, it will be locally homogeneous, isotropic and in thermal equilibrium. It will then be possible to describe this local region of the universe by a FRW metric, which will be accurate at small distance scales compared to the horizon scale. When the temperature falls off $T_c$, the inflationary scenario will take place. The result will be a huge isotropic and homogeneous regions. Depending on the final temperature after the latent heat is released, we can have a region bigger than our observed universe;

2. This scenario has some problems. The most relevant for us here is to notice that the model does not have an smooth end after the exponential expansion, which spoils the homogeneity of the universe. This happens because the bubbles of the low-temperature phase carry all this energy in their walls, as already pointed out in [28]. Hence, the bubbles walls have to collide so that this energy can be thermalized. However, the walls move almost at the speed of light [28] and when the collisions happen the bubbles are too large, generating a highly inhomogeneous and anisotropic universe.

### 4.5.2. The new inflation

Only one year after Guth's proposal, Linde and Albrecht with Steinhardt [16, 17] came up with an inflationary scenario in which the problems in the old inflation were essentially solved. They considered the phase-transition with the Coleman-Weinberg mechanism of symmetry breaking [20]. Let's explore the features of this model.

As we have seen above, during the exponential expansion phase, the energy density was given by $\rho_0$, implying that $\dot{H}_0 = 0$. Hence, through the Friedmann equations, it is easy to find that $p_0 = -\rho_0$. Now, if we consider that this vacuum energy density can be associated with a perfect fluid, the energy-momentum tensor is given by

$$T^{vac}_{\mu\nu} = \rho_0 g_{\mu\nu}. \tag{4.8}$$

If we suppose that the phase-transition is due to the Coleman-Weinberg mechanism of symmetry breaking, which is generated by a scalar field[7] $\varphi$, then we can use our results from appendix D and rewrite (4.8) as

$$T^{vac}_{\mu\nu} = V(0) g_{\mu\nu}, \tag{4.9}$$

where we have considered the potential at $\varphi = 0$ as the metastable vacuum. It is essentially the same phenomenon of supercooling above. However, the novelty here is to notice that [33] inflation can occur not only in this supercooled state, but also while the field is growing towards its true vacuum slowly enough, much larger than the cosmological time scale giving by $H^{-1}$. This condition

[7] It is interesting to note that at that time, the scalar field was thought to be the Higgs. Today, we know that the scalar field known as inflaton could not have been the Higgs if one consider that the potential coupling for the Higgs is constant. In fact, when the Higgs particle is considered as a candidate for the inflaton, the slow-roll approximation, as we will study below, implies that the Higgs potential is super-Planckian in the beginning of inflation. This happens because the potential coupling is not small enough. However, when we consider the renormalization and the running of this coupling, it might be possible that it gets lower enough for the scale in which inflation starts so that the Higgs potential could be responsible for inflation.

can be realized if the potential of the field is sufficiently flat around $\varphi = 0$, letting the universe inflates enough so that the field bubbles grow much more than the horizon scale, not damaging the homogeneity (as it happens in the old scenario). After reaching the true vacuum, the field executes damped oscilattions around it creating elementary particles.

The new scenario has problems also. There is a fine-tunning problem related to the period of time the field spends in the false vacuum to lead to sufficient amount of inflation. Besides, the justification for the initial conditions of the scalar field was the assumption that the universe was in a state of thermal equilibrium from its beginning, which is very unlikely.

The difference between the old and new scenarios can be essentially captured in the figure 4.4.

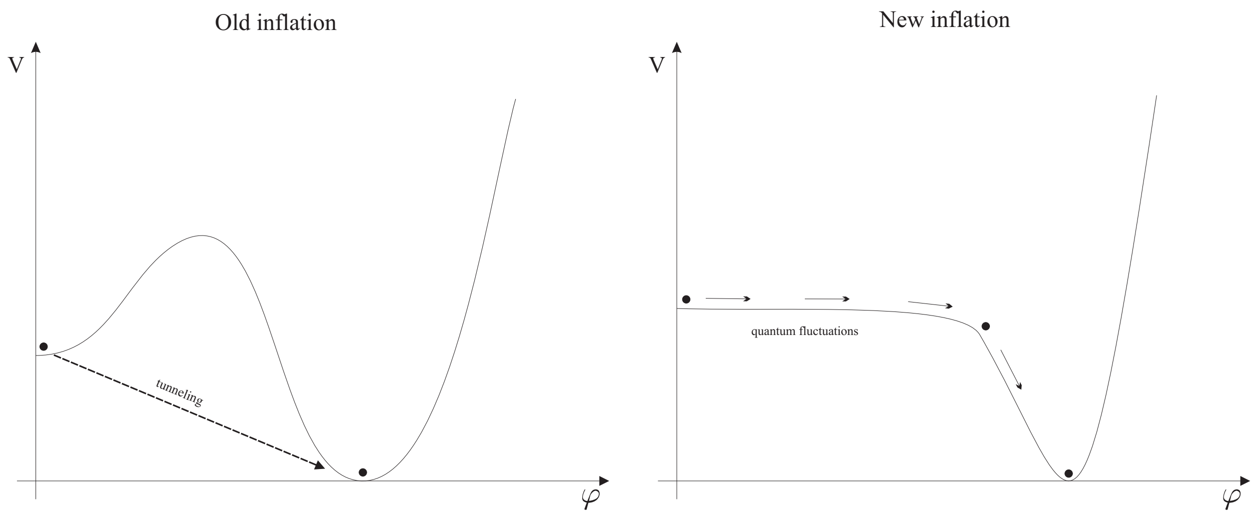


Figure 4.4.: In the old inflation scenario, the phase-transition begins with the formation of bubbles of the field $\varphi$, which is a tunneling process. So, it is intuitively clear why there is not a smooth transition from the inflationary phase to the FRW stage. On the other hand, in the new inflation, because the potential is very flat and has a maximum at $\varphi = 0$, the scalar field escapes from the maximum due to quantum fluctuations, rolling towards the global minimum [10].

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# 5. The chaotic inflation

In this chapter we will study a class of inflationary models called *chaotic inflation* considering the slow-roll limit. They were introduced by Linde [27] and are called chaotic due to the possibility of having almost arbitrary initial conditions for the scalar field. Then, the tachyonic potential calculated in chapter 3 will be considered as one of these models and we will analyze its classical consequences.

Even though the first article by Linde is pretty clear, we will take a modern approach and the main references to be considered are [10, 12, 21, 22].

## 5.1. Why chaotic?

Before start to study the slow-roll approximation and some examples, let's understand why the term *chaotic*. Briefly, one can show that [33] the unique exigency over the initial condition of the scalar field is that it must be larger than the Planckian value (that is why these models are called large field models, while the others are called small field ones). Once this is guaranteed, it is possible to drop the homogeneity on the initial conditions of the scalar field [10] (as it was the case for the old and new scenarios when thermal equilibrium of the universe in the pre-inflationary regime was assumed).

## 5.2. Slow-roll limit

As we have seen in the last chapter, a de Sitter evolution of the universe gives an approximately good description of how inflation solves the horizon and flatness problems of the standard cosmological model. On the other hand, we have not given a precise formulation of how this scenario takes place quantitatively yet. Then, let us start considering the action (D.18) for a scalar field minimally coupled to gravity[1]

$$S_\varphi = \int d^4x\sqrt{-g}\mathcal{L}_\varphi, \tag{5.1}$$

with

$$\mathcal{L}_\varphi = -\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi). \tag{5.2}$$

Considering flat spacetime in comoving coordinates with metric given by $g_{00} = -1$, $g_{ij} = a^2(t)\delta_{ij}$, $g_{oi} = 0$, the equation of motion (D.19) can be written as

$$\ddot{\varphi} - \frac{1}{a^2}\nabla^2\varphi + 3H\dot{\varphi} + \frac{dV}{d\varphi} = 0, \tag{5.3}$$

where an overdot indicates a derivative with respect to the time $t$ and $H \equiv \dot{a}/a$ is the Hubble parameter[2].

[1] There is no direct coupling between the field and the metric, which would be an interaction-like term that couples, for instance, a function of the Ricci scalar and the scalar field. In practice, we do not consider these kind of models because they can be reduced to the minimally coupled one with a field redefinition in general [12].

[2] The reader not familiar with the basics of cosmology should read the appendix D.

Instead of working with the above equation of motion for the scalar field, we will suppress the second term, reducing the above equation to

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{dV}{d\varphi} = 0. \tag{5.4}$$

That term was suppressed because after the expansion has started, it becomes negligible very rapidly. One could argued that our approach is assuming homogeneity *a priori,* otherwise we could not have used the FRW metric. In fact, this is true. However, since the scalar field has initial conditions very chaotic, there is a probability that a very small region is approximately homogeneous, permitting the FRW metric. Then, once this region starts to expand, the inhomogeneities are kicked out and the approximation only gets better (see page 231, [10]).

In the appendix D we have already calculated the energy-momentum tensor for the scalar field and rewritten it as a perfect fluid. Then, consider the De Sitter limit, $p \backsimeq -\rho$, which generates a quasi-exponential expansion as explicit at Table D.1. It is the same as to consider the limit

$$V(\varphi) \gg \dot{\varphi}^2, \tag{5.5}$$

coming from (D.24) and (D.25). This is known as *slow-roll* limit.

One could argue that this limit, which provides inflation to happen, is very restricted and not so common as an initial condition. However, in order to see this is reasonable, let us assume the kinetic energy is much bigger than the potential energy. Then, the resultant equation from the combination of the Friedmann equations,

$$\begin{aligned} \ddot{\varphi} + \dot{\varphi}\sqrt{24\pi}(\dot{\varphi}^2/2 + V)^{1/2} + \frac{dV}{d\varphi} &= 0 \\ \dot{\varphi}\frac{d\dot{\varphi}}{d\varphi} + \dot{\varphi}\sqrt{24\pi}(\dot{\varphi}^2/2 + V)^{1/2} + \frac{dV}{d\varphi} &= 0, \end{aligned} \tag{5.6}$$

can be approximated as

$$\frac{d\dot{\varphi}}{d\varphi} \simeq \sqrt{12\pi}\dot{\varphi}. \tag{5.7}$$

The solution for this is

$$\dot{\varphi} = C\exp(\sqrt{12\pi}\varphi). \tag{5.8}$$

Hence, the derivative of the scalar field decays exponentially faster than the scalar field itself. That means that the friction term damps the initial velocities and enforces a slow-roll regime in which the acceleration can be neglected compared to the friction term. This also can be seen as an argument about the chaotic status of this scenario, since it does not matter how is the initial time derivative of the field, it decays much faster than the field itself so that the slow-roll regime will take place sooner or later.

Let us define the *e-folds number*, $N$, given by

$$a(t) \propto \exp(\int H dt) \equiv e^{-N} \tag{5.9}$$

$$dN \equiv -H dt. \tag{5.10}$$

Note that $N$ is large in the past and it decreases as the scalar factor grows.

## 5.3. Slow-roll parameters

The Friedmann equations (D.32) and (D.31) for the scalar field are given by

$$\begin{aligned} H^2 &= \frac{8\pi}{3}\left[\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right] &(5.11)\\ \left(\frac{\ddot{a}}{a}\right) &= -\frac{8\pi}{3}(\dot{\varphi}^2 - V). &(5.12)\end{aligned}$$

The second equation can be rewritten as

$$\left(\frac{\ddot{a}}{a}\right) = H^2(1-\epsilon), \tag{5.13}$$

where we have defined the parameter $\epsilon$ as

$$\epsilon \equiv \frac{3}{2}\left(\frac{p}{\rho}+1\right) = 4\pi\left(\frac{\dot{\varphi}}{H}\right)^2. \tag{5.14}$$

The usefulness of defining this parameter is that it tells us that the universe is under accelerated expansion if $\epsilon < 1$. In the de Sitter limit, $\epsilon \to 0$, (5.11) becomes

$$H^2 \simeq \frac{8\pi}{3}V(\varphi). \tag{5.15}$$

If we also consider that the friction term in the equation of motion for the scalar field dominates[3], that is,

$$|\ddot{\varphi}| \ll 3H\dot{\varphi}, \tag{5.16}$$

we end up with

$$3H\dot{\varphi} + \frac{dV}{d\varphi} \simeq 0. \tag{5.17}$$

The equations (5.15) and (5.17) are referred to as the *slow-roll approximation*. Note that the dominance of the friction term can be expressed by another parameter, $\alpha$, defined as

$$\alpha \equiv -\frac{\ddot{\varphi}}{H\dot{\varphi}} = \epsilon + \frac{1}{2\epsilon}\frac{d\epsilon}{dN}. \tag{5.18}$$

This is the second slow roll parameter[4]. The *slow roll approximation* is the same as to consider $\epsilon$, $|\alpha| \ll 1$. It is worth noting that inflation happens independent of the $\alpha$ value. Thus, we could associated $\epsilon$ directly with if inflation is or not happening, while $\alpha$ define for us a whole class of potentials with some general characteristics (in fact, the smallness of the $\alpha$ parameter helps to ensure that inflation will occur by a sufficient period).

The slow-roll parameters written in the slow-roll approximation, that is, using (5.15) and (5.17), become

$$\begin{aligned} \epsilon &\simeq \frac{1}{16\pi}\left[\frac{V'(\varphi)}{V(\varphi)}\right]^2 \\ \alpha &\simeq \frac{1}{8\pi}\left\{\frac{V''(\varphi)}{V(\varphi)} - \frac{1}{2}\left[\frac{V'(\varphi)}{V(\varphi)}\right]^2\right\}. &(5.19)\end{aligned}$$

[3] Otherwise, $\dot{\varphi}$ starts growing too fast and the kinetic energy becomes bigger than the potential energy too early. Accelerated expansion will only be sustained for a sufficiently long period of time if the second time derivative is small enough.

[4] In the literature, this parameter is usually referred as $\eta$. However, in order to denote the proper time as $\tau$ in the appendix D, we have already used $\eta$ to denote the conformal time. That is the reason why we are considering an unusual notation.

Then, in order to have the parameters small, we see that the potential has to be flat $V'(\varphi) \ll V(\varphi)$ and its curvature, $V''(\varphi)$, small. The e-fold number in this approximation becomes

$$N = -\int H dt \tag{5.20}$$

$$\simeq 8\pi \int_{\varphi_e}^{\varphi} \frac{V(\varphi)}{V'(\varphi)} d\varphi = 2\sqrt{\pi} \int_{\varphi_e}^{\varphi} \frac{d\varphi}{\sqrt{\epsilon}}, \tag{5.21}$$

where $\varphi_e$ denotes the end of inflation. As we have already said, $N$ decreases as time goes on and represents the number of e-folds the expansion takes place between $\varphi$ and $\varphi_e$.

A natural question is how long inflation must go so that it solves the horizon and flatness problems. As we have seen in the last chapter, if the scale factor is $a_i$ at the beginning of inflation, then the scale factor at the end must be approximately $a_e \sim 10^{27} a_i$. Hence, the scale factor $a$ increases by a factor of $e^N$, where

$$N \sim 62, \tag{5.22}$$

given our calculations[5].

After the end of inflation the inflaton begins to oscillate around its true vacuum and the universe enters the stage of deceleration. In order to determine the effective equation of state for the scalar field, we go back to (5.4) suppressing the expansion term (which implies that the cosmological time is much bigger than the period of oscillation) and rewrite it as

$$\frac{d}{dt}(\varphi\dot{\varphi}) - \dot{\varphi}^2 + \varphi \frac{dV}{d\varphi} \simeq 0. \tag{5.23}$$

Averaging over a period the first term drops away and we get $\langle \dot{\varphi}^2 \rangle \simeq \langle \varphi \frac{dV}{d\varphi} \rangle$. Hence, the effective equation of state is

$$\omega = \frac{p}{\rho} \simeq \frac{\langle \varphi \frac{dV}{d\varphi} \rangle - \langle 2V \rangle}{\langle \varphi \frac{dV}{d\varphi} \rangle + \langle 2V \rangle}, \tag{5.24}$$

which simplifies to $\omega \simeq (n-2)/(n+2)$ for an $n-$monomial potential. Note that for $n = 2$ and $n = 4$, the oscillating field reproduces matter- or radiation-dominance, respectively.

In order to clarify the above considerations and calculations, let us consider some simple examples below.

### 5.3.1. Examples

**i)$\mathbf{V}(\varphi) = \lambda\varphi^{\mathbf{4}}$**

The second slow-roll equation (5.17) implies

$$\dot{\varphi} = -\frac{V'(\varphi)}{3H} = -\varphi\sqrt{\frac{2}{3\pi}}. \tag{5.25}$$

Then, we have $\dot{\varphi} \propto \varphi$ while $V(\varphi) \propto \varphi^4$. Therefore, the slow-roll limit is satisfied as the modulus of the field grows up, even though the potential itself not being properly flat graphically. The $\epsilon$ parameter is given by

$$\epsilon \simeq \frac{1}{16\pi} \left[\frac{V'(\varphi)}{V(\varphi)}\right]^2 = \frac{1}{\pi\varphi^2}, \tag{5.26}$$

[5] Even though we will not study the primordial density fluctuations here, they can constrain $N$ at the lower bound together with the so observed baryon asymmetry of the universe and the hot Big Bang scenario. In 2006 yet, the theoretical incertanties in the lower bound of the e-fold number was[34] $N_{min} \simeq [46, 60]$.

so that $\epsilon(\varphi_e) = 1$ implies

$$\varphi_e = \pm\frac{1}{\sqrt{\pi}}. \tag{5.27}$$

Hence, from the (5.13) we see that for $\varphi > \varphi_e$ the universe is inflating, while for $\varphi < \varphi_e$ it enters in a decelerated phase.

The e-fold number is given by

$$N = \pi\varphi^2 - 1. \tag{5.28}$$

Thus, in order to have $N \sim 62$, the field value with which inflation starts is

$$\varphi_{62} \sim \pm 4.48, \tag{5.29}$$

remembering we are working in Planck units. In fact, $\varphi_{62} \sim 4.48 m_{Pl}$, where $m_{Pl}$ is the Planck mass. Note that this simple potential provides an inflationary scenario, but the scalar field is assuming a huge value in the beginning of inflation in order to provide sufficient e-folds and one could argue that quantum gravity should take a role there. On the other hand, we see that the coupling constant of the potential has not appeared anywhere. Therefore, for a very tiny value[6] of $\lambda$, the energy density in the field, which is the physically important quantity, ends up being much less than the Planckian energy density. The effective equation of state tells us this potential imitates a radiation-dominant universe.

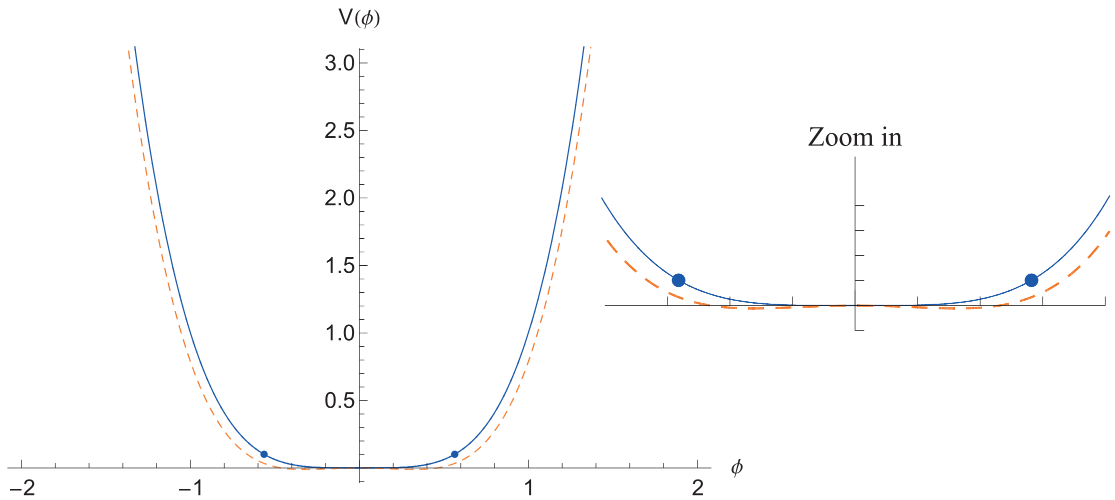


Figure 5.1.: We have plotted the potential in the blue full line; the orange dashed curve present $V(\varphi) - \dot{\varphi}^2$, so that the region in which this curve is above the $\varphi-$axis is where the potential starts dominates over the kinetic energy, that is, $\varphi < -0.460659$ or $\varphi > 0.460659$; the blue points indicate the field values for which $\epsilon = 1$, representing essentially the end of inflationary solution. In this plot, we are considering $\lambda = 1$, because the coupling value for a monomial potential does not affect the general behavior we are analysing.

[6] According to [25], $\lambda \simeq 10^{-13}$. This constraint comes from the fact that the couplings are important for the amplitudes of the density fluctuations, something we have observational access. Unfortunately, we are not considering here the quantum aspects of inflation, that is why we only provide the reference.

### ii) $\mathbf{V}(\phi) = \mathbf{m^2}\varphi^\mathbf{2}$

Following the detailed example above, we have

$$\dot{\varphi} = -m\sqrt{\frac{1}{12\pi}}. \tag{5.30}$$

Then, we have $\dot{\varphi} = const$ while $V(\varphi) \propto \varphi^2$. Therefore, the slow-roll limit is satisfied for all the field values in which $V(\varphi) \gg \dot{\varphi}^2$. The $\epsilon$ parameter is given by

$$\epsilon = \frac{1}{4\pi\varphi^2}, \tag{5.31}$$

so that $\epsilon(\varphi_e) = 1$ implies

$$\varphi_e = \pm\frac{1}{2\sqrt{\pi}}. \tag{5.32}$$

The e-fold number is given by

$$N = 2\pi\varphi^2 - 1. \tag{5.33}$$

Thus, in order to have $N \sim 62$, we field value with which inflation starts is

$$\varphi_{62} \sim \pm 3.06603. \tag{5.34}$$

We have $\omega_{eff} = 0$, imitating matter-dominance. As in the first example, the coupling is constrained by the amplitude of density perturbations. Data from COBE satellite [26] gives $m \simeq 10^{-6} m_{pl}$.

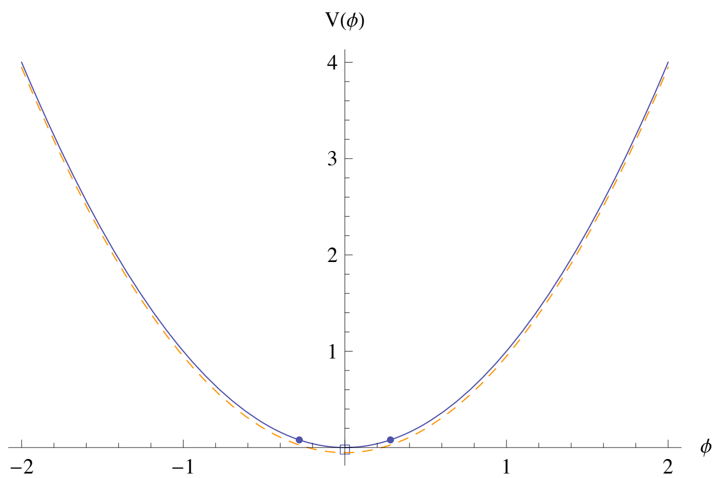


Figure 5.2.: For $V(\varphi) > \dot{\varphi}^2$, $\varphi > |0.230329|$; $\varphi_{\epsilon=1} = \pm 0.282095$.

### iii) Higgs-like potential: $\mathbf{V} = \mathbf{V_0}\left[\mathbf{1} - \left(\frac{\varphi}{\mu}\right)^\mathbf{2}\right]^\mathbf{2}$

The algorithm is the same as the above examples. The graphic alone can provide the general picture:

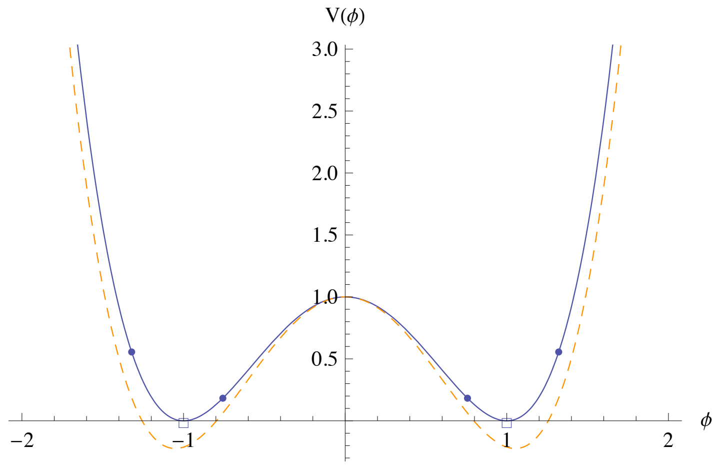


Figure 5.3.: For this plot, we have considered $\mu = 1$ for convenience and omitted $V_0$. The range of dominance of the potential over the kinetic energy is: $\varphi > |1.25651|$ or $\varphi < |0.795854|$; the $\epsilon-$parameter is $\epsilon = \frac{1}{\pi\mu^4}\frac{\varphi^2}{[1-(\varphi/\mu)^2]^2}$, so that $\varphi_{\epsilon=1} = \pm 1.32132, \pm 0.756932$ for $\mu = 1$. In order to calculate the $\varphi_{62,}$ we can consider, for instance, $\varphi_\epsilon = 0.756932$, so that $N = [\pi\varphi^2 - 2\pi\ln\varphi]\big|_{\varphi_\epsilon}^{\varphi_{62}} \to \varphi_{62} = 0.0000514141$, in Planck units as always.

**iv) Natural inflation:** $\mathbf{V}(\phi) = \mathbf{m^4}\left[\cos\left(\frac{\phi}{\mathbf{f}}\right) + \mathbf{1}\right]$

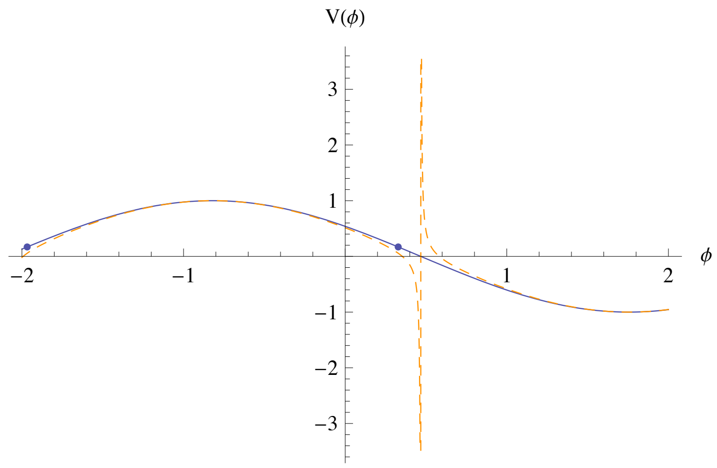


Figure 5.4.: From [22], we have $f \approx 10^{19} GeV$ and $m \approx 10^{16} GeV$, so that in Planck units we consider $f = \frac{50}{60}$ and $m = \frac{1}{1220}$. The $\epsilon-$parameter is $\epsilon = \frac{9\tan\left(1+\frac{6}{5}\varphi\right)}{100\pi}$, so that $\varphi_{\epsilon=1} = -2.00261, 0.33594$. In order to calculate the $\varphi_{62}$, we consider $\varphi_\epsilon = 0.33594$, so that $N = -\frac{50}{9}\pi\ln\left[\sin\left(1 + \frac{6}{5}\varphi\right)\right]\Big|_{\varphi_\epsilon}^{\varphi_{62}} \to \varphi_{62} = -0.809784$.

## 5.4. General picture in conformal diagrams

As we can see in appendix D, we can define the conformal time as

$$\eta = \int_0^t \frac{dt}{a(t)}, \tag{5.35}$$

so that light propagates at $\pm 45^o$ in the $\eta-\chi$ plane, being $\chi$ a comoving coordinate. Hence, during matter, radiation or cosmological constant domination the scale factor evolves as

$$a(\eta) \propto \begin{cases} \eta & \text{radiation} \\ \eta^2 & \text{matter} \\ -\frac{1}{H\eta} & \Lambda,\ \text{H=const} \end{cases}, \tag{5.36}$$

using the results from Table D.1. We see that for radiation and matter domination there is an initial singularity: $a(\eta_i \equiv 0) = 0$. Therefore, the standard cosmological model without inflation gives the conformal diagram as seen in Figure 5.5:

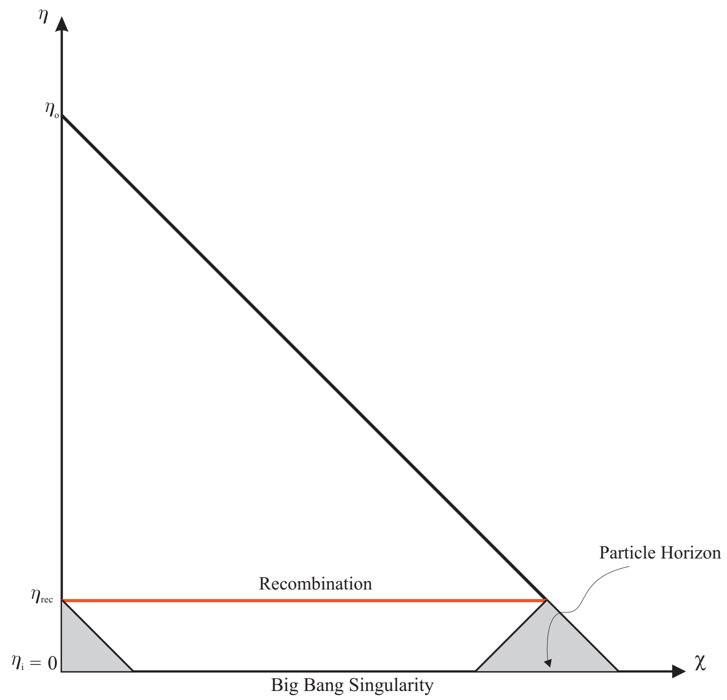


Figure 5.5.: There is a singularity at $\eta_i = 0$ where it is pretty clear the horizon problem, since the past light cones from different regions at the $\eta_{rec}$ consist of many causally disconnected regions.

On the other hand, if we have inflation in the very early universe, it behaves similar to a cosmological constant ($H \approx const$), so that the initial singularity is pushed to the infinite past, $\eta_i \to -\infty$. One could think that the scale factor would diverge at $\eta = 0$, but this would happen only for a de Sitter universe, which means an inflationary universe forever. We know that this is not true, because inflation ends at finite time and this approximation breaks down at the end o inflation. The new scenario can be observed in Figure 5.6:

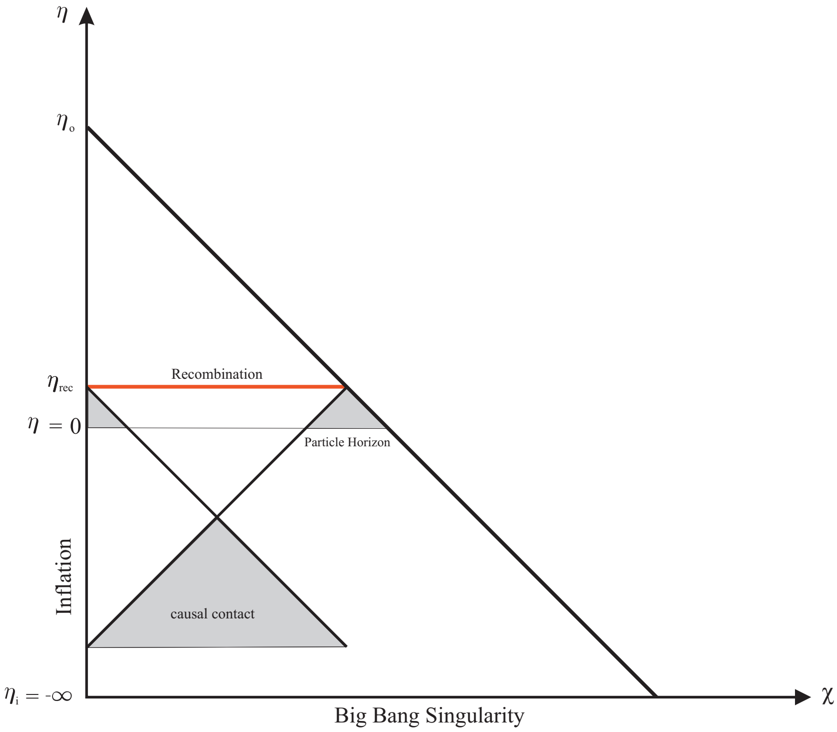


Figure 5.6.: Inflation "creates" additional conformal time, extending the diagram downwards and providing causal contact at past between regions that were causally disconnected in the standard cosmological model without inflation. There is no singularity at $\eta = 0$ any longer.

## 5.5. The tachyonic potential

The last model we analyze comes directly from calculations of chapter 3, where we have obtained the potential associated with the tachyonic mode of the open string spectrum using the String Field Theory framework. In fact, this is the last piece of the bridge connecting Strings and Cosmology in this thesis.

First, we will provide a naive analysis only for academic purposes, where we call attention for some problems that should be addressed so that the model could be really considered. Then, we use some already known results about the potential to carry a realistic analysis. We will not provide here the details of how the realistic model comes from the open string field theory.

### 5.5.1. A naive analysis

The tachyonic action obtained in Chapter 3 was

$$S_\varphi = \frac{1}{g_0^2}\int d^{26}x \left[-\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{2\alpha'}\varphi^2 - \frac{1}{3}\left(\frac{3\sqrt{3}}{4}\right)^3\tilde{\varphi}^3\right], \tag{5.37}$$

where we have already assumed 26 dimensions, as it is demanded by consistency of the bosonic string theory. However, in order to consider this as an academic exercise, we will be working with the following naive effective action

$$S_{\varphi}^{eff} = \frac{V^{(22)}}{g_0^2} \int d^4x \sqrt{-g^{(4)}} \left[ -\frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi + \frac{1}{2\alpha'}\varphi^2 - \frac{1}{3}\left(\frac{3\sqrt{3}}{4}\right)^3 \varphi^3 \right]. \tag{5.38}$$

There is a lot of issues in order to come from (5.37) to (5.38). We will explain qualitatively what are the assumptions in order to call the attention to some problems that usually appear when considering higher dimensional models. The considerations are:

1. There is no coupling with the 26−metric in (5.37). This is not a surprise, since we have started considering only open strings while the graviton appears as a closed string state. On the other hand, closed strings emerge in perturbative open string scattering amplitudes ([29]). Actually, it might be even possible that closed string states should also arise as asymptotic states of the quantum open string field theory, which has been under investigations and still lack of a final resolution [30]. Therefore, if one consider not only the classical theory, it makes sense to couple the Lagrangian from (5.37) with the 26−metric;

2. Once the above coupling is considered, the tachyonic action becomes

$$S_{\varphi}^{(1)} = \frac{1}{g_0^2} \int d^{26}x \sqrt{-g^{(26)}} \left[ -\frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi + \frac{1}{2\alpha'}\varphi^2 - \frac{1}{3}\left(\frac{3\sqrt{3}}{4}\right)^3 \tilde{\varphi}^3 \right].$$

   Therefore, we should compactify it in order to be more realistic. The net result from this matching has important consequences when the energy is of order of the radius used in the compactification, which is close to the Planckian length. On the other hand, from inflationary purposes, we do not want to consider very high energies, otherwise this would break down the whole approach from this and last chapter concerning the early universe. Therefore, in this low energy regime, the compactification of the additional dimensions gives effectively a volume factor, which it was represented as $V^{(22)}$;

3. So far, we have reduced the original action to

$$S_{\varphi}^{(2)} = \frac{V^{(22)}}{g_0^2} \int d^4x \sqrt{-g^{(4)}} \left[ -\frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi + \frac{1}{2\alpha'}\varphi^2 - \frac{1}{3}\left(\frac{3\sqrt{3}}{4}\right)^3 \tilde{\varphi}^3 \right].$$

   Now, remembering the slow-roll regime, we have seen that the first and second derivatives are small and they must continue like this in order to keep the approximation valid. Hence, higher derivatives must be even smaller, otherwise they would spoil the smallness of the first and second ones. This limit could be considered direct in the equation of motion, but we shall considered an effective action instead, that means, we consider

$$\tilde{\varphi} \approx \varphi - \alpha' \ln \frac{4}{3\sqrt{3}} \partial_\mu \partial^\mu \varphi.$$

   In fact, this looks like a perturbative expansion, because for $\alpha' = \frac{1}{2}$, we see that the constant in the exponential defining $\tilde{\varphi}$ is 0.130812035. Now, the action looks like

$$S_{\varphi}^{(3)} = \frac{V^{(22)}}{g_0^2} \int d^4x \sqrt{-g^{(4)}} \left[ \partial_\mu \varphi \partial^\mu \varphi \left( -\frac{1}{2} + \alpha' \ln \frac{4}{3\sqrt{3}} \varphi^2 \right) + \frac{1}{2\alpha'}\varphi^2 - \frac{1}{3}\left(\frac{3\sqrt{3}}{4}\right)^3 \varphi^3 \right].$$

4. The final step would be to redefine the field in order to recover the canonical kinetic term. In fact, there is a solution, given by

$$\varphi' = \varphi\sqrt{-\frac{\alpha'}{2}\ln\frac{4}{3\sqrt{3}}}\sqrt{1-2\alpha'\ln\frac{4}{3\sqrt{3}}\varphi}+\frac{1}{2}\ln\left(\sqrt{-\frac{\alpha'}{2}\ln\frac{4}{3\sqrt{3}}}\varphi+\sqrt{1-2\alpha'\ln\frac{4}{3\sqrt{3}}\varphi^2}\right).$$

   Then, we should invert this function in order to calculate the potential for $\varphi'$. However, the above function is too complicated and there is not an analytic inverse. Therefore, we will consider the approximation, where $\tilde{\varphi} \approx \varphi$. The reasons for this are two: i) we want to keep things simple as possible as an academic exercise so that it is possible to have contact with all the way coming from strings to inflation and ii) it can be checked that the full potential, considering even very high truncation levels, is reduced to a combination of second and third power law in the tachyonic field (graphic comparing levels $(0,0)$ and $(2,4)$), so that our effective action can provide all the basic intuition for a more involved analysis[7].

Once we have emphasized the above points, let's start our naive analysis. The potential for the tachyonic mode is given by[8]

$$V_{naive}(\varphi) = -\frac{1}{2\alpha'}\varphi^2+\frac{1}{3}\left(\frac{3\sqrt{3}}{4}\right)^3\varphi^3+\frac{2^{14}}{3^{10}}. \tag{5.39}$$

Note that we considered $\alpha' = \frac{1}{2}$, which is a common convention for the open strings. Besides, we have added a constant so that the energy value for $\varphi = 0$ is not zero. This comes from the tension of the $D-$brane [41], which makes the energy value in the local minimun to be zero. Then, the graphic of the potential is

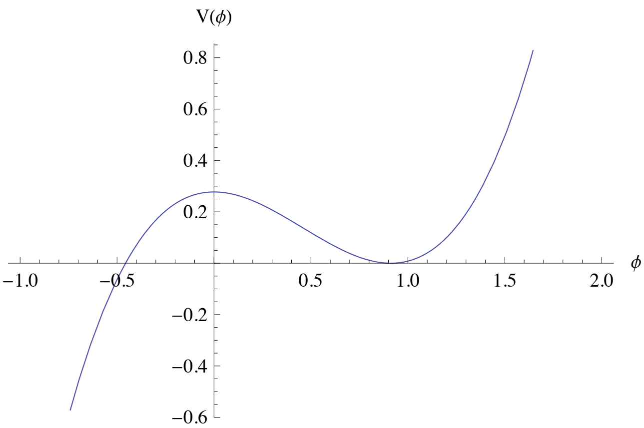


Figure 5.7.: The tachyonic potential (5.39).

The $\epsilon$ parameter is given by

$$\epsilon = \frac{\varphi^2}{16\pi}\left[\frac{81\sqrt{3}\varphi-128}{\frac{2^{20}}{3^{14}}-64\varphi^2-27\sqrt{3}\varphi^3}\right]^2, \tag{5.40}$$

[7] Also, we could argue that our analysis is restricted to the slow-roll approximation, in which the temporal second derivatives of the field are negligible and that we can consider the field homogeneously as an initial chaotic state at some region of the universe, as it was argued by Linde in [24].

[8] The common notation for the tachyonic field is $T(x)$. However, we will be using the same notation as we have been using for the inflaton in the last chapters.

so that we have four solutions for $\varphi$ that gives $\epsilon = 1$ (which indicates the end of inflation), they are $\varphi_{\epsilon=1} = -0.628827,\ 0.661862,\ -0.341043,\ 1.22036$, but only one is relevant for us because of our interest in the rolling tachyon phenomenum, *i.e.,* the decaying of the tachyon from the unstable equilibrium point to the minimum of the potential. Therefore, we use $\varphi_{\epsilon=1} = 0.661862$.

The e-fold number is given by

$$N(\varphi) = \int_{\varphi_{\epsilon=1}}^{\varphi} n(\varphi) d\varphi = \frac{4\pi}{3^9(128\sqrt{3} - 243\varphi)^2(64 + 81\sqrt{3}\varphi)}[81\varphi(-2^{27}\sqrt{3} + 2^{20}3^5\varphi + 2^{13}3^{10}\sqrt{3}\varphi^2 - 5 \times 2^6 3^{14}\varphi^3 + 3^{18}\sqrt{3}\varphi^4) - 2^{14}(2^{20} - 2^6 3^{10}\varphi^2 + 3^{13}\sqrt{3}\varphi^3)\ln\varphi]\Big|_{\varphi_{\epsilon=1}}^{\varphi} \tag{5.41}$$

If we plot the above function in order to have a picture of the behavior of the e-fold number, we obtain

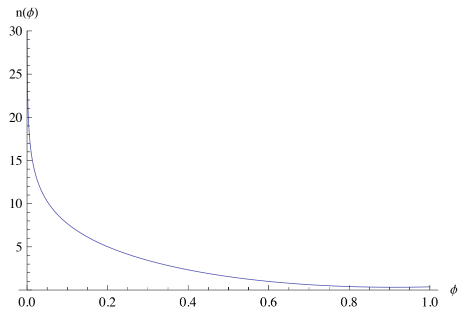


Figure 5.8.: Ignoring the contribution of $\varphi_{\epsilon=1}$ for the e-fold number (which is $-0,744494$), we see that it rapidly diverges. Therefore, we can produce any e-fold number that we want.

We have seen that the expected value for $N$ is close to 62. Therefore, we can obtain the initial value of the field so that enough inflation is produced, which is

$$\varphi_{initial} \approx 2,283 \times 10^{-8}. \tag{5.42}$$

Hence, our naive potential can, in fact, produce enough inflation. Should this surprise ourselves? Not really. If we remember the e-fold number formula, we had in the denominator the derivative of the potential. So, when the field approaches the value that gives the local maximum ($\varphi = 0$), the integrand diverges, so that we have field values for any e-fold number we want, including 62 as we saw above.

The conclusion here must be very conservative. Even though we have obtained a potential that produces enough inflation, this is the very first test for a model. The real tests come from the energy density fluctuation amplitudes that appear in the CMB, a subject we have not considered here. Therefore, this model shall be seen at this point only as a possible candidate for inflation, having passed in the very first possible classical test, which is if it can or cannot produce enough inflation.

# 6. Conclusion

One of the objectives of this thesis was the construction of a bridge connecting String Theory with inflationary cosmology. We have seen that the fundamental piece for that was the presence of a tachyonic mode in the open string spectrum, because this was an indication that its potential might have relevant properties to consider it as a possible candidate for inflation. However, in order to calculate the tachyonic potential, we had to introduce the framework of String Field Theory. Let's quickly review what we have done.

In chapter 2, we gave the basic intuition concerning String Field Theory from the nonabelian Chern-Simons theory in $2+1-$dim. Then, we investigated how real calculations could be made in that framework using a conformal field theory approach and established the tools for calculate the tachyonic potential.

Then, we obtained the tachyonic potential in chapter 3. However, it was necessary to define a scheme of truncation over the string field, because it is a combination of all the possible modes of the string, having as coefficients what were identified as the spacetime fields. Once this truncation was defined, the lowest order of the potential was obtained.

So far, those efforts were related only with one side of the bridge. Therefore, in chapter 4 we focused to introduce the inflationary scenario so that the other side of the problem could be established. This first part was worried in motivating the inflationary paradigm throughout problems cosmology was facing in the beginning of the $80's$.

Then, in chapter 5 we were able to consider the quantitative aspects of inflation and learned how we could check if a particular model for the inflaton potential could be useful in the context of the slow-roll approximation. In fact, we worked only the classical aspects related with this exponential expansion.

Once the two sides of the bridge were defined, the last piece was the analysis investigating if the tachyonic potential, which comes from calculations in String Field Theory, could be used to provide the necessary amount of inflation that is required by observations. In order to make the analysis of our model, we had to consider the very first approximation of the truncation method in the string field.

More than that, we had to consider some assumptions in order to let the model tractable pedagogically in this thesis. Those assumptions could be worked out better, indicating field for future work. First of all, one could work out higher levels of approximation in the truncation scheme, so that the tachyonic potential obtained would be more realistic. Secondly, one could, for instance, improve the calculations for the compactification of the model to $4-$dim, investigating different ways of compactify the extra dimensions. Besides, the higher-derivative terms that were not considered in the action could be included in the analysis, because those terms have proven to be relevant in other approaches. Another issue that can be improved in the calculations is the search for the canonical kinetic term in the action, since we only recovered it using approximations based in the slow-roll limit and justifying some steps as pedagogical. Probably, if one or more of these considerations were taken into account, analytical calculations would not be possible in several steps that we took. Therefore, it is hard to know how far one can go trying to let this model more realistic.

After the analysis of the tachyonic potential, we concluded that it could be responsible for the whole period of exponential acceleration in the early universe. We have estimated that the universe should expand around by a factor of $e^{62}$ during the inflationary period, which brought some conditions for the potential. However, for our particular naive potential, we have seen that it can produce how much inflation we want due to its particular form. Therefore, this indicates that a deeper analysis of the model is welcome, considering other classical tests and investigating its quantum aspects in comparison with the data. In particular, considering or not the above suggestions to make the model more realistic, one could work out the the scalar and tensorial fluctuations coming from the model with the data.

Finally, we emphasize that a lot of approximations were considered in order to arrive at some results and that we have worked here only with the open bosonic sector. Therefore, this same proposal could be carried out in the superstring theory yet.

# A. Elements of Bosonic Open String Theory

The objective of this appendix is to present the basic elements concerning Conformal Field Theory so that the thesis will be essentially self-contained. Besides, since we are considering here a transition between String Theory and Cosmology, more precisely inflation, it is expected that this can help the reader more familiarized with the latter than the former.

We will be reviewing below the string action in the conformal gauge, the Operator Product Expansion (OPE), the mode expansions of the scalar and *bc* theory and some results of the vertex-operator map and of tree-level amplitudes. Since we are concerned with only open strings throughout this work, we will be focusing in this subject below.

The reference for this analysis is [7].

## A.1. String Action in the conformal gauge

The fundamental action in String Theory is the Polyakov one, given by

$$S_P[X,\gamma] = -\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma(-\gamma)^{\frac{1}{2}}\gamma^{ab}\partial_a X^\mu \partial_b X_\mu, \tag{A.1}$$

where $M$ denotes the world-sheet, which is the generalization of the world-line for the point particle, that is, it is a two-dimensional surface embedded in the spacetime representing the string "trajectory"; $\tau$ and $\sigma$ are the parameters describing the world-sheet, being the former a temporal coordinate and the latter a spatial one limited in $[0,\pi]$ for the open string; $\gamma$ is the embedded metric[1]; $X^\mu$, with $\mu = 1, ..., 26$, are the spacetime coordinates of the string, being vectors in the spacetime but scalars in the world-sheet.

The above action has the following symmetries:

1. D-dimensional Poincaré invariance:

$$\begin{aligned} X'^\mu(\tau,\sigma) &= \Lambda^\mu{}_\nu X^\nu(\tau,\sigma) + a^\mu \\ \gamma'^{ab}(\tau,\sigma) &= \gamma^{ab}(\tau,\sigma); \end{aligned} \tag{A.2}$$

2. World-sheet diffeomorphism invariance:

$$\begin{aligned} X'^\mu(\tau',\sigma') &= X^\mu(\tau,\sigma) \\ \frac{\partial\sigma'^c}{\partial\sigma^a}\frac{\partial\sigma'^d}{\partial\sigma^b}\gamma'_{cd}(\tau',\sigma') &= \gamma_{ab}(\tau,\sigma), \end{aligned} \tag{A.3}$$

for new coordinates $\sigma'^a(\tau,\sigma)$;

3. Two-dimensional Weyl invariance:

$$\begin{aligned} X'^\mu(\tau,\sigma) &= X^\mu(\tau,\sigma) \\ \gamma'_{ab}(\tau,\sigma) &= \exp[2\omega(\tau,\sigma)]\gamma_{ab}(\tau,\sigma), \end{aligned}$$

for arbitrary $\omega(\tau,\sigma)$.

[1] $a, b = 1, 2$; The metric $\gamma$ has signature $-1$ and it is the analogous in two dimensions of the $\eta_{\mu\nu}$ from Special Relativity.

These symmetries are at the classical level. However, the attempt to keep them at the quantum level brings up amazing results concerning String Theory, as the reader can check in the reference we are following. For our purposes, we only need to work with the conformal properties regarding this action. Hence, we fix almost all of the above freedom considering the *unitary gauge,* given by

$$\gamma_{ab} = \delta_{ab}. \tag{A.4}$$

Then, we reduce (A.1) to

$$S_P[X,\gamma] = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma \partial_a X^\mu \partial^a X_\mu.$$

It is better to work with the action in an Euclidean world-sheet ($i\sigma^0 \to \sigma^2$), that means, we can consider the embedded metric with a $(+,+)$ signature. Hence, the action becomes

$$S_P[X,\gamma] = \frac{1}{4\pi\alpha'}\int d^2\sigma(\partial_1 X^\mu \partial_1 X_\mu + \partial_2 X^\mu \partial_2 X_\mu), \tag{A.5}$$

where the overall sign dropped out because we are following the same convention from [7]. It is worth to comment that our gauge fixing let a group of local symmetries not fixed, the so called *conformal transformations*, which is a combination of a diffeomorphism and a Weyl transformation.

In order to work out the conformal properties coming from (A.5), we will use the $z-$coordinate system that is convenient for the open string analysis:

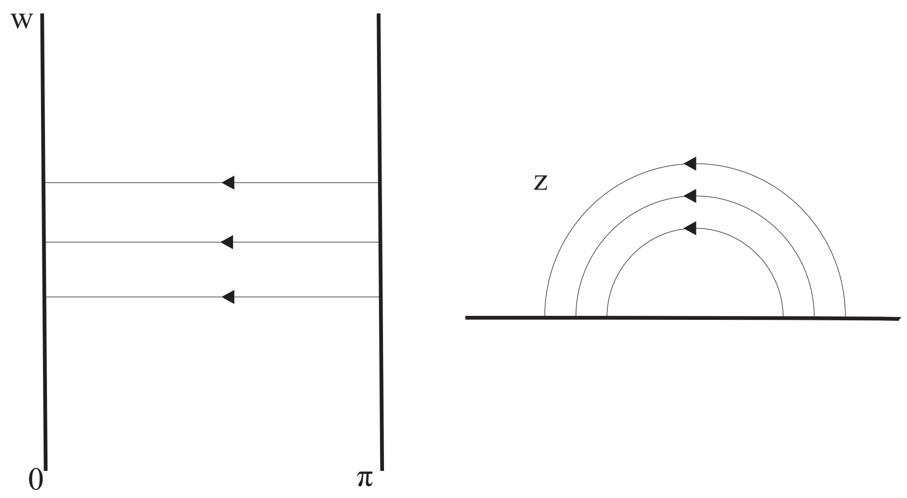


Figure A.1.: Open string coordinates. Note the equal time contours in both planes.

We present all the relevant information below for completeness:

| | |
|---|---|
| $w = \sigma^1 + i\sigma^2$ | |
| $z = -e^{-i\omega}$ | |
| $\partial_1 = ie^{-i\omega}(\partial_z - e^{2i\sigma^1}\partial_{\bar{z}}),$ | $\partial_2 = -e^{-i\omega}(\partial_z + e^{2i\sigma^1}\partial_{\bar{z}})$ |
| $d^2z = 2\exp(2\sigma^2)d\sigma^1 d\sigma^2$ | |
| $\int d^2z\delta^2(z,\bar{z}) = 1,$ | |

Then, the action with which we will work is given by

$$S = \frac{1}{2\pi\alpha'}\int d^2z \partial X^\mu \bar{\partial} X_\mu, \tag{A.6}$$

with equation of motion

$$\partial\bar{\partial}X^{\mu}(z,\bar{z})=0. \tag{A.7}$$

In order to fix the gauge, the Faddeev-Popov method[2] had to be used. Its straightforward application leads to the introduction of anticommuting ghost fields: a traceless symmetric 2-tensor $b_{ab}$ and a vector $c^a$ [3]. The action for the ghosts takes the form

$$S_{gh}=\frac{1}{2\pi}\int d^2z(b_{zz}\bar{\partial}c+b_{\bar{z}\bar{z}}\partial c^{\bar{z}}). \tag{A.8}$$

Considering the above coordinate transformations, the end points of the string are mapped to the real axis $Im\,z=0$, being the scalar field $X^{\mu}(z,\bar{z})$ only defined in the upper-half plane. In order to calculate the propagator, let us impose *Neumann* boundary conditions, in which the end of the open string move freely in spacetime:

$$\partial^{\sigma}X^{\mu}(\tau,0)=\partial^{\sigma}X^{\mu}(\tau,\pi)=0. \tag{A.9}$$

A convenient way to implement this boundary condition is to use the *doubling trick*: one can combine the holomorphic and anti-holomorphic fields $X^{\mu}(z)$ and $\tilde{X}(\tilde{z})$ on the upper half plane into a single holomorphic field defined on the whole complex plane by defining

$$X^{\mu}(z)=\tilde{X}^{\mu}(\bar{z})\ \text{ at } z=\bar{z}, \tag{A.10}$$

where

$$X^{\mu}(z,\bar{z})=X^{\mu}(z)+\tilde{X}^{\mu}(\bar{z}). \tag{A.11}$$

In this way, the Neumann boundary condition is automatically satisfied. So the CFT on the upper half plane can be reduced to the holomorphic sector of the CFT on the complex plane. Then, the propagator is defined as

$$\langle X^{\mu}(z,\bar{z})X^{\nu}(z',\bar{z}')\rangle=G^{\mu\nu}(z,\bar{z};z',\bar{z}'), \tag{A.12}$$

where $\partial^2G=-2\pi\alpha'\delta^2(z-z')$ subject to the boundary condition

$$0=\partial_1G^{\mu\nu}(z,\bar{z};z',\bar{z}')|_{\sigma^1=0,\pi}.$$

The solution can be found considering an "image charge" in the lower-half plane considering now $X^{\mu}(z,\bar{z})$ varying over the whole complex plane and it is given by

$$G^{\mu\nu}(z,\bar{z};z',\bar{z}')=-\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z-z'|^2-\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z-\bar{z}'|^2, \tag{A.13}$$

remembering that $\partial\bar{\partial}\ln|z|^2=2\pi\delta^2(z,\bar{z})$.

Quantizing the theory using the path integral formulation we can define expectation values as

$$\langle\mathcal{F}[X]\rangle=\int[dX]\exp(-S)\mathcal{F}[X], \tag{A.14}$$

where $\mathcal{F}[X]$ is any functional of $X$. Then, remembering that the path integral of a total derivative is zero, the classical equation of motion (A.7) becomes

$$\langle\partial\bar{\partial}X^{\mu}(z,\bar{z})\rangle=0. \tag{A.15}$$

[2] Check subsection (B.1.1).

[3] Actually, one should also introduce a ghost field for the Weyl transformations, which would couple with the trace part of $b_{ab}$. But this field turn out to be auxiliary, having no derivatives acting on them, so that it can be eliminated, imposing $b_{ab}$ to be traceless.

## A.2. Operator Product Expansion (OPE)

We have seen above that

$$X^\mu(z,\bar{z})X^\nu(z',\bar{z}') = -\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z-z'|^2 - \frac{\alpha'}{2}\eta^{\mu\nu}\ln|z-\bar{z}'|^2, \tag{A.16}$$

for the open string in the operator formalism. On the other hand, as it will be clear soon, when working only with open string interactions, due to a conformal transformation, the strings are identified with vertex operators on the disk boundary, where $Im\, z = 0$ and the above operatorial equation becomes

$$X^\mu(x_1)X^\nu(x_2) = -\alpha'\eta^{\mu\nu}\ln(x_1-x_2)^2. \tag{A.17}$$

Let us now define the *normal ordering:*

$$:X^\mu(z_1): \; = \; X^\mu(z_1) \tag{A.18}$$

$$:X^\mu(z_1)X^\nu(z_2): \; = \; X^\mu(z_1)X^\nu(z_2) + \frac{\alpha'}{2}\eta^{\mu\nu}\ln(|z_{12}|^2|z_1-\bar{z}_2|^2), \tag{A.19}$$

where $z_{12} = z_1 - z_2$. This definition has the following property

$$\partial_1^2 :X^\mu(z_1,\bar{z}_1)X^\nu(z_2,\bar{z}_2):= 0, \tag{A.20}$$

and satisfies the boundary equation.

From this definition, it is interesting to consider the normal ordering of a general functional of the field, $\mathcal{F}[X]$,

$$\begin{aligned} :\mathcal{F}: \; &= \; \exp\left[\frac{\alpha'}{4}\int d^2z_1 d^2z_2 \ln(|z_{12}|^2|z_1-\bar{z}_2|^2)\frac{\delta}{\delta X^\mu(z_1,\bar{z}_1)}\frac{\delta}{\delta X_\mu(z_2,\bar{z}_2)}\right]\mathcal{F} \\ &= \; \mathcal{F} + \sum subtractions, \end{aligned} \tag{A.21}$$

where the sum runs over all ways of choosing one, two or more pair of fields from the functional and replacing each pair with $\frac{\alpha'}{2}\eta^{\mu_i\mu_j}\ln(|z_{ij}|^2|z_i-z_j|^2)$.

A common object we will be considering is the expectation values of a product of local operators in the limit where two operators are taken to approach one another. The tool that gives a description of this limit is the *operator product expansion (OPE)*. It essentially states that given two pair of operators, $\mathcal{F}$ and $\mathcal{G}$, the OPE is given by

$$:\mathcal{F}::\mathcal{G}:=:\mathcal{F}\mathcal{G}: - \sum(cross\, subtractions). \tag{A.22}$$

The sum now runs over all ways of contracting pairs with one field in each functional. This can also be written as

$$:\mathcal{F}::\mathcal{G}:= \exp\left[-\frac{\alpha'}{2}\int d^2z_1 d^2z_2 \ln(|z_{12}|^2|z_1-\bar{z}_2|^2)\frac{\delta}{\delta X^\mu_{\mathcal{F}}(z_1,\bar{z}_1)}\frac{\delta}{\delta X_{\mathcal{G}\mu}(z_2,\bar{z}_2)}\right]:\mathcal{F}\mathcal{G}:, \tag{A.23}$$

where the functional indices indicates where the derivatives act (they will be indicated by (1) and (2) in the following).

Let us work out several examples that will be useful for us through the thesis[4]:

[4] We are using " $\sim$ " instead of " $=$ " when we are keeping only the singular terms.

1. $:\partial X^\rho(z,\bar{z})::\partial X^\sigma(y,\bar{y}):=$

$$
\begin{aligned}
&= \exp\left[-\frac{\alpha'}{2}\int d^2z_1 d^2z_2 \ln(|z_{12}|^2|z_1-\bar{z}_2|^2)\frac{\delta}{\delta X^\mu_{(1)}(z_1,\bar{z}_1)}\frac{\delta}{\delta X_{(2)\mu}(z_2,\bar{z}_2)}\right]:\partial X^\rho(z,\bar{z})\partial X^\sigma(y,\bar{y}): \\
&\sim -\frac{\alpha'}{2}\int d^2z_1 d^2z_2 \ln(|z_{12}|^2|z_1-\bar{z}_2|^2)\delta^\rho_\mu\partial_z\delta^2(z_1,z)\eta^{\sigma\alpha}\delta^\mu_\alpha\partial_y\delta^2(z_2,y) \\
&\sim -\frac{\alpha'}{2}\eta^{\sigma\rho}\partial_z\partial_y[\ln(z-y)+\ln(\bar{z}-\bar{y})+\ln(z-\bar{y})+\ln(\bar{z}-y)] \\
&\sim -\frac{\alpha'}{2}\eta^{\sigma\rho}\frac{1}{(z-y')^2};
\end{aligned}
\tag{A.24}
$$

2. $:T(z,\bar{z})::\partial X^\sigma(y,\bar{y}):=$

$$
\begin{aligned}
&\sim \eta^{\sigma\rho}\partial X_\rho(z)\partial_z\partial_y\ln(|z-y|^2|z-\bar{y}|^2) \\
&\sim \partial X^\sigma(z)\frac{1}{(z-y)^2} \\
&\sim \frac{1}{z-y}\partial^2 X^\sigma(y)+\frac{1}{(z-y)^2}\partial X^\sigma(y);
\end{aligned}
\tag{A.25}
$$

3. $:e^{iq\cdot X(z)}::e^{ik\cdot X(y)}:=$

$$
\begin{aligned}
&= \exp\left[\frac{\alpha'}{2}q\cdot k\ln(|z-y|^2|z-\bar{y}|^2)\right]:e^{iq\cdot X(z)}e^{ik\cdot X(y)}: \\
&= (|z-y||z-\bar{y}|)^{\alpha' q\cdot k}:e^{i(q+k)\cdot X(y)}[1+O(z,\bar{z})]:;
\end{aligned}
\tag{A.26}
$$

4. $:e^{ik\cdot X(z)}::\partial X^\rho(y):=$

$$
\begin{aligned}
&\sim -\frac{\alpha'}{2}ik^\rho:e^{ikX(z)}:\partial_y(\ln|z-y|^2|z-\bar{y}|^2) \\
&\sim \frac{\alpha'}{2}ik^\rho\left(\frac{1}{z-y}+\frac{1}{\bar{z}-y}\right):e^{ik\cdot X(y)}:;
\end{aligned}
\tag{A.27}
$$

5. $:T(z,\bar{z})::e^{ik\cdot X(y)}:=$

$$
\sim ik^\mu\left(\frac{1}{z-y}+\frac{1}{z-\bar{y}}\right):e^{ik\cdot X}\partial X_\mu(y):+\frac{\alpha'}{4}k^2\left(\frac{1}{z-y}+\frac{1}{z-\bar{y}}\right)^2 e^{ik\cdot X(y)};
\tag{A.28}
$$

6. $:T(z,\bar{z})::T(y,\bar{y}):=$

$$
\begin{aligned}
= \quad &\sim \underbrace{-\frac{2}{\alpha'}\int d^2z_1 d^2z_2 \ln(|z_{12}|^2|z_1-\bar{z}_2|^2)\delta^\alpha_\mu\partial_z\delta^2(z_1,z)\delta^\mu_\beta\partial_y\delta^2(z_2,y):\partial X_\alpha(z)\partial X^\beta(y):}_{-\frac{2}{\alpha'}:\partial X_\mu(z)\partial X^\mu(y):\frac{1}{(z-y)^2}} \\
&-\frac{2}{\alpha'}\frac{1}{(z-y)^2}\frac{1}{2}\left(-\frac{\alpha'}{2}\right)\int d^2z_1 d^2z_2 \ln(|z_{12}|^2|z_1-\bar{z}_2|^2)\eta^\mu_\mu\partial_z\delta^2(z_1,z)\partial_y\delta^2(z_2,y) \\
\sim \quad &\frac{\eta^\mu_\mu}{2}\frac{1}{(z-y)^4}+2T(y)\frac{1}{(z-y)^2}+\partial T(y)\frac{1}{(z-y)}.
\end{aligned}
\tag{A.29}
$$

In general, for a generic operator $\mathcal{A}(z', \bar{z}')$, the OPE with $T(z, \bar{z})$ has the following form

$$T(z)\mathcal{A}(z', \bar{z}') = ... + h\left(\frac{1}{z-z'} + \frac{1}{z-\bar{z}'}\right)^2 \mathcal{A}(z', \bar{z}') + \left(\frac{1}{z-z'} + \frac{1}{z-\bar{z}'}\right)\partial\mathcal{A}(z', \bar{z}') + ..., \quad (A.30)$$

where $h$, the constant multiplying the term with second order pole, is called *conformal weight.* As it can be seen above, sometimes there is not a term involving[5] $(z-\bar{z}')^{-1}$. If the OPE with the energy-momentum tensor has the higher order pole equals 2, then the operator is called a *tensor operator* or *primary field,* denoted $\mathcal{O}(z', \bar{z}')$. In this case, it transforms under a general conformal transformations $z \to z'(z)$ as

$$\mathcal{O}'(z', \bar{z}') = (\partial_z z')^{-h} (\partial_{\bar{z}} \bar{z}')^{-\tilde{h}} \mathcal{O}(z, \bar{z}). \quad (A.31)$$

## A.3. The $bc$ theory

We have seen that the Faddeev-Popov method of gauge fixing implied an additional ghost term in the Polyakov action, given by

$$S_{gh} = \frac{1}{2\pi}\int d^2 z b\bar{\partial}c, \quad (A.32)$$

where $b$ and $c$ are anticommuting. This action is conformally invariant for $b$ and $c$ transforming as tensors of weights $(\lambda, 0)$ and $(1-\lambda, 0)$, respectively. For our case, $\lambda = 2$ [6].

The operator equations of motion are

$$\begin{aligned} \bar{\partial}c(z) &= \bar{\partial}b(z) = 0 && (A.33)\\ \bar{\partial}b(z)c(0) &= 2\pi\delta^2(z, \bar{z}). && (A.34) \end{aligned}$$

The second equation gives that the normal ordered $bc$ product is

$$: b(z_1)c(z_2) := b(z_1)c(z_2) - \frac{1}{z_{12}}. \quad (A.35)$$

The energy-momentum tensor is

$$T(z) =: (\partial b)c - 2\partial(: bc :). \quad (A.36)$$

It is interesting to note that there is a *ghost number* symmetry coming from considering $\delta b = -i\epsilon b$, $\delta c = i\epsilon c$, with corresponding current

$$j = - : bc :, \quad (A.37)$$

which can be checked as not being a tensor.

## A.4. Mode Expansion

It is useful to consider the operators we will be working expanded as oscillators. Once we are working in the complex plane, we expand a holomorphic operator in a Laurent series as

$$\mathcal{O}(z) = \sum_{m=-\infty}^{\infty} \frac{\mathcal{O}_m}{z^{m+h}}, \quad (A.38)$$

where $h$ is the operator conformal weight. If the operator is antiholomorphic, the expansion would be over $\bar{z}$, with conformal weigh $\tilde{h}$.

[5] This kind of term does not appear when the two functionals have derivatives.

[6] This comes from consistency of the theory. In fact, the total central charge of the Virasoso algebra (see below) has to be zero so that the theory has no anomalies. The matter piece contributes with $c_m = 26$, while the ghost one gives $c_{gh} = -3(2\lambda - 1)^2 + 1$, so that for $\lambda = 2$ we have $c_{total} = 0$.

### A.4.1. Scalar field $X^\mu$

The scalar fields on the worldsheet satisfy

$$\partial\bar{\partial}X^\mu = 0, \tag{A.39}$$

with the additional boundary condition $\partial X^\mu = \bar{\partial}X^\mu$ on the real axis for the open string. Then, disregarding this boundary condition, we have two possible expansion

$$\partial X^\mu(z) = -i\left(\frac{\alpha'}{2}\right)^{1/2}\sum_{m=-\infty}^{\infty}\frac{\alpha_m^\mu}{z^{m+1}} \tag{A.40}$$

$$\bar{\partial}X(\bar{z}) = -i\left(\frac{\alpha'}{2}\right)^{1/2}\sum_{m=-\infty}^{\infty}\frac{\tilde{\alpha}_m^\mu}{\bar{z}^{m+1}}, \tag{A.41}$$

with

$$\alpha_m^\mu = \left(\frac{\alpha'}{2}\right)\oint\frac{dz}{2\pi}z^m\partial X^\mu(z), \tag{A.42}$$

and analogously for the $\tilde{\alpha}_m^\mu$ On the other hand, considering the boundary condition, we end up with $\alpha_m^\mu = \tilde{\alpha}_m^\mu$. Hence, the expansion for $X^\mu$ is given by

$$X^\mu(z,\bar{z}) = x^\mu - i\alpha' p^\mu \ln|z|^2 + i\left(\frac{\alpha'}{2}\right)^{1/2}\sum_{m=-\infty, m\neq 0}^{\infty}\frac{\alpha_m^\mu}{m}(z^{-m}+\bar{z}^{-m}),$$

where $x^\mu$ is the center of mass coordinate of the string and $p^u$ is the conserved charge associated with the spacetime translational current calculated by the Noether theorem, that is,

$$p^\mu = \frac{1}{2\pi i}\oint_C dz j^\mu = (2\alpha')^{-1/2}\alpha_0^\mu, \tag{A.43}$$

considering only the holomorphic current in the whole complex plane due to the doubling trick.

From the canonical commutator $[\pi_\mu, X^\nu] = -\delta_\mu^\nu$, we derive

$$[\alpha_m^\mu, \alpha_n^\nu] = m\delta_{m,-n}\eta^{\mu\nu} \tag{A.44}$$

$$[x^\mu, p^\nu] = i\eta^{\mu\nu}. \tag{A.45}$$

For the energy-momentum tensor, the expansion is given by

$$T = \sum_{m=-\infty}^{\infty}\frac{L_m}{z^{m+2}}, \tag{A.46}$$

where

$$L_m = \oint_C\frac{dz}{2\pi i z}z^{m+2}T, \tag{A.47}$$

with $C$ any contour encircling the origin counterclockwise. These $L_m$ coefficients are known as *Virasoro generators* and satisfy the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m,-n}, \tag{A.48}$$

where $c$ is the central charge. Expanding $T$ in modes of $X$, we find

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{\infty}\alpha_{m-n}^\nu\alpha_{\mu n},\ \ (m\neq 0) \tag{A.49}$$

$$L_0 = \alpha' p^2 + \sum_{n=1}^{\infty}\alpha_{-n}^\mu\alpha_{\mu n}. \tag{A.50}$$

Note also that from the OPE of the energy-momentum tensor with a primary operator, we have

$$[L_m, \mathcal{O}_n] = [(h-1)m - n]\mathcal{O}_{m+n}.$$

### A.4.2. $bc$ fields

The expansions of the $b$ and $c$ fields are given by

$$b(z) = \sum_{m=-\infty}^{\infty} \frac{b_m}{z^{m+2}} \tag{A.51}$$

$$c(z) = \sum_{m=-\infty}^{\infty} \frac{c_m}{z^{m-1}}, \tag{A.52}$$

with

$$b_m = \oint \frac{1}{2\pi i} dz z^{m-1} b(z) \tag{A.53}$$

$$c_m = \oint \frac{1}{2\pi i} dz z^{m-2} c(z), \tag{A.54}$$

where we are considering the specific case for which $\lambda = 2$. Besides, we are integrating over a whole path in the complex plane because we have implicitly considered the doubling trick so that the antiholomorphic fields in the upper half plane are written in terms of holomorphic fields in the whole plane, using the definition

$$c(z) \equiv \tilde{c}(\bar{z}'), \;\; b(z) \equiv \tilde{b}(\bar{z}'), \;\; Im(z) \leq 0, \, z' = \bar{z}. \tag{A.55}$$

The $bc$ OPE gives the anticommutators

$$\{b_m, c_n\} = \delta_{m,-n}. \tag{A.56}$$

The ghost vacuum will be denoted by $|\downarrow\rangle$ and defined by

$$b_0|\downarrow\rangle = 0 \tag{A.57}$$

$$c_0|\downarrow\rangle = |\uparrow\rangle \tag{A.58}$$

$$b_n|\downarrow\rangle = c_n|\downarrow\rangle = 0, \, n > 0. \tag{A.59}$$

The Virasoro generators are

$$L_m = \sum_{n=-\infty}^{\infty} (2m - n) \circ b_n c_{m-n} \circ -\delta_{m,0}, \tag{A.60}$$

where the operators between $\circ ... \circ$ are ordered with annihilation operators to the right and creation ones to the left.

For the ghost number current (A.37), the charge is

$$N^g = \sum_{n=1}^{\infty} (c_{-n} b_n - b_{-n} c_n) + c_0 b_0 - \frac{1}{2}, \tag{A.61}$$

satisfying

$$[N^g, b_m] = -b_m \quad [N^g, c_m] = c_m. \tag{A.62}$$

Note that the ground state has ghost number $-\frac{1}{2}$.

## A.5. Vertex Operators

In conformal field theory there is a map between the set of local operators and the space of states of the theory. For each state, there will be a local operator called *vertex operator* and they are identified through this symbol $\cong$ .

Let's provide the relevant identifications and then justify them. The map is

| States | Operators |
|---|---|
| $\|0;0\rangle_m$ | $1_m$ |
| $\alpha^\mu_{-m},\ m \geq 1$ | $i\left(\frac{2}{\alpha'}\right)^{1/2} \frac{1}{(m-1)!}\partial^m X^\mu(0)$ |
| $x^\mu_0$ | $X^\mu(0,0)$ |
| $\|0;k\rangle$ | $:e^{ik\cdot X(0,0)}:$ |
| $b_{-1}\|\downarrow\rangle$ | $1_g$ |
| $b_{-m},\ m \geq 2$ | $\frac{1}{(m-2)!}\partial^{m-2}b(0)$ |
| $c_{-m},\ m \geq -1$ | $\frac{1}{(m+1)!}\partial^{m+1}c(0)$ |

Note that the subscript $g$ and $m$ stand for *ghost* and *matter*. Now, we explain:

1. The states are identified with operators at $z = 0$ because an initial state in the $\sigma-$coordinate system has the temporal coordinate at $-\infty$. Hence, the behavior of the fields at the origin in the complex plane is equivalent to specifying the boundary condition for the path integral in the $\sigma-$coordinate, that is, the initial state;

2. The unity operators for matter, denoted $1_m$, is associated with the ground state because the matter oscillators for $m \geq 0$, defined by (A.42), multiplied by this unity operator have no poles at the origin and so vanish. Thus, we can think that $1_m$ is "annihilated" by these modes in the operator formalism. In the state formalism, this would be written as $\alpha^\mu_m|0;0\rangle = 0$ for $m \geq 0$. It is analogous for the ghost piece;

3. The fourth identification can be understood given the transformation the state and its corresponding operator suffers under a translation $X^\mu \to X^\mu + a^\mu$. The state transforms as $|0;k\rangle \to e^{ik\cdot a}|0;k\rangle$ from Quantum Mechanics while the operator transforms equally, since there is no contractions between the scalar field and a constant;

4. Note that for the open strings, the case of our interest, the origin is also the boundary of the strings. Therefore, when we will calculate OPE of our local operators for the open strings, we will be doing the calculations over the boundary and the OPE formulas will be simplified as we have already talked about in section A.2;

## A.6. Tree-level Amplitudes

We present here just some results of tree-level amplitudes calculations on the disk boundary:

$$1)\ \left\langle \prod_{i=1}^{n} e^{ik_i X(y_i)} \prod_{j=1}^{p} \partial X^{\mu_j}(y'_j) \right\rangle_{matter} = (2\pi)^d \delta^d\left(\sum_{i=1}^{n} k_i\right) \prod_{i,j=1;\, i<j}^{n} |y_i - y_j|^{2\alpha' k_i\cdot k_j} \times \left\langle \prod_{j=1}^{p} [v^{\mu_j}(y'_j) + q^{\mu_j}(y'_j)\rangle \right\rangle_{norm.}, \tag{A.63}$$

where

$$v^\mu(y) = -2i\alpha' \sum_{i=1}^{n} \frac{k_i^\mu}{y - y_i}, \quad \langle q^\mu(y) q^\nu(y') \rangle_{norm.} = -2\alpha' \frac{\eta^{\mu\nu}}{(y - y')^2}; \tag{A.64}$$

$$2)\ \langle c(z_1) c(z_2) c(z_3) \rangle = |z_{12} z_{23} z_{13}| \quad \longleftrightarrow \quad \langle 0 | c_{-1} c_0 c_1 | 0 \rangle. \tag{A.65}$$

The derivation of them are very simple but a little longer and they can be found in our reference.

# B. BRST Quantization

The BRST charge was fundamental in the construction of the open string field theory action, so that it is worth to dedicate an appendix to review this approach used to study the string spectrum of states.

We will be following closely [7].

## B.1. The String Spectrum

As we have seen above, in conformal gauge[1], the world-sheet fields are $X^\mu$ and the Faddeev-Popov ghosts $b_{ab}$ and $c^a$. This covariant gauge implies that the Hilbert space of states contains unphysical states and, therefore, is bigger than the physical spectrum of the string. These unphysical states either have negative norm, coming from the timelike oscillators and ghosts, or are null states, which are redundant states of a physical one.

In order to find out the set of physical states, one might consider the amplitude on the infinite cylinder for some initial state $|i\rangle$ propagating to a final state $|f\rangle$. Having fixed the metric to be $g_{ab}(\sigma)$ using the local symmetries, one would not expected that this amplitude would vary if a different gauge was considered: $g_{ab}(\sigma) + \delta g_{ab}(\sigma)$. Hence, the resulting variation on the amplitude should vanish,

$$0 = \delta\langle f|i\rangle \propto \int d^2\sigma g(\sigma)^{1/2}\delta g_{ab}(\sigma) < f|T^{ab}|i >, \tag{B.1}$$

where $T^{ab} = T^{ab}_X + T^{ab}_{ghost}$. Then, a necessary condition for arbitrary physical states is given by

$$\langle\psi|T^{ab}|\psi'\rangle = 0. \tag{B.2}$$

Note that before fixing the gauge we had the equation of motion for the metric telling us $T^{ab} = 0$. However, when we work with the gauge fixed theory, we do not have this equation of motion anymore since the metric is fixed. Hence, the above condition says that this missing equation must hold for matrix elements between physical states. Carrying (B.2) throughout quantization is called *Old Covariant Quantization.*

In this procedure, we have $(L_0^m - 1)|\psi\rangle = 0$ which really comes from $(L_0^m + L_0^g)|\psi, \downarrow\rangle = 0$. Hence, $N^g|\psi, \downarrow\rangle = -1/2|\psi, \downarrow\rangle$. So, the vertex operator associated with this physical state must have ghost number 1, since $Q^g = N^g + (\lambda - 1/2) = 1$. This difference between ghost number of the vertex operator and the physical state associated with it is due to the fact that the ghost number of states is conventionally defined by the cylindrical frame expression

$$N^g = -\frac{1}{2\pi i}\int_0^{2\pi} dw j_w = \sum_{n=1}^{\infty}(c_{-n}b_n - b_{-n}c_n) + c_0 b_0 - 1/2,$$

while the ghost number of the vertex operator to the radial frame

$$Q^g = \frac{1}{2\pi i}\oint dz j_z = N^g + q_0 \;\; (q_0 = 3/2 \;\text{ for } \lambda = 2).$$

[1] In the unitary gauge, we would have also the $B_A$ ghost associated with the Weyl transformation.

Summarizing, all the vertex operators associated with the physical states will have ghost number $+1$ once the states have ghost number $-1/2$.

As it should be clear, this is an *ad hoc* procedure and we intend to explore a more systematic method, the *BRST quantization*.

### B.1.1. BRST Quantization

In order to explore this quantization procedure, let's follow the very general approach presented by Polchinski to gauge fix an action through Faddeev-Popov procedure.

Let a path integral whose fields are denoted $\phi_i$, in our case they are $X^\mu(\sigma)$ and $g_{ab}(\sigma)$, having a local symmetry. The index $i$ labels the field and the coordinate $\sigma$. The gauge invariance is $\epsilon^\alpha\delta_\alpha$ with $\alpha$ also representing the coordinate and the parameters $\epsilon^\alpha$ being reals, since we can always separated a complex parameter into two real ones. The gauge transformations satisfy an algebra[2]

$$[\delta_\alpha, \delta_\beta] = f^\gamma_{\alpha\beta}\delta_\gamma. \tag{B.3}$$

We write the gauge fixing condition as

$$F^A(\phi) = 0, \tag{B.4}$$

where $A$ also includes the coordinate. Following Faddeev-Popov procedure, one should find

$$\int \frac{[d\phi]}{V_{gauge}} \exp(-S_1) \to \int [d\phi dB_A db_A dc^\alpha] \exp(-S_1 - S_2 - S_3), \tag{B.5}$$

where $S_1$ is the original action and $S_2$ and $S_3$ are given by

$$S_2 = -iB_A F^A(\phi) \tag{B.6}$$
$$S_3 = b_A c^\alpha \delta_\alpha F^A(\phi). \tag{B.7}$$

The $B_A$ was only introduced to create an integral representation of the gauge-fixing $\delta(F^A)$.

The total action coming from the Faddeev-Popov procedure turns out to have an additional symmetry. It is invariant under

$$\delta_B\phi_i = -i\epsilon c^\alpha\delta_\alpha\phi_i \tag{B.8}$$
$$\delta_B B_A = 0 \tag{B.9}$$
$$\delta_B b_A = \epsilon B_A \tag{B.10}$$
$$\delta_B c^\alpha = \frac{i}{2}\epsilon f^\alpha_{\beta\gamma}c^\beta c^\gamma. \tag{B.11}$$

Remember that $c^\alpha$ and $b_A$ are anticommuting objects; so is $\epsilon$. As we have seen in appendix A, there is a conserved ghost number, which is $+1$ for $c^\alpha$, $-1$ for $b_A$ and $\epsilon$, and 0 for all the other fields. The above symmetry is called *BRST symmetry.*

Note also that

$$\delta_B(b_A F^A) = i\epsilon(S_2 + S_3), \tag{B.12}$$

which implies that a small change $\delta F$ in the gauge-fixing condition at the amplitude level gives

$$\epsilon\delta\langle f|i\rangle = -\epsilon\langle f|\{Q_B, b_A\delta F^A\}|i\rangle. \tag{B.13}$$

[2] This algebra is not so general. In fact, we are considering the structure constants independent of the fields and we do not have additional terms proportional to the equations of motion in the right-hand side. If we had considered all this, the above BRST formalism would have to be generalized, resulting in what is called Batalin-Vilkovisky formalism.

where we have written the BRST variation as an anticommutator with the corresponding conserved charge $Q_B$, which will be called BRST operator from now on.

Therefore, as we have seen in the beginning of this section, physical states must satisfy

$$\langle\psi|\{Q_B, b_A\delta F^A\}|\psi'\rangle = 0. \tag{B.14}$$

For this to hold for arbitrary $\delta F$, we must have

$$Q_B|\psi\rangle = Q_B|\psi'\rangle = 0. \tag{B.15}$$

A physical state will satisfy this condition and it will be called a *BRST invariant state*[3] .

An important property of the BRST operator is its *nilpotency*, that is,

$$Q_B^2 = 0. \tag{B.16}$$

It is implicit the operator squared is acting in a state. A simple way to understand why this is true it is to remember that $Q_B$ is a charge, hence $[Q_B, H] = 0$, where $H$ is the Hamiltonian. While we move around in the space of gauge choices, $H \to H + \delta H$. On the other hand, $Q_B$ remains a charge. Therefore, we end up with the condition

$$\begin{aligned} 0 &= [Q_{B,}\delta H] \\ &= -[Q_{B,}\delta_B L] \\ &= [Q_{B,}\{Q_B, b_A\delta F^A\}] \\ &= [Q_B^2, b_A\delta F^A]. \end{aligned} \tag{B.17}$$

Because this is valid for any change of gauge, $Q_B$ is nilpotent[4]. A more direct way to check this property is to consider the action of the BRST operator twice in the fields, leaving them invariant. As an example, let's consider it on $\phi_i$ :

$$\delta_{B'}\delta_B\phi_i = -i\epsilon\delta_{B'}(c^\alpha\delta_\alpha\phi_i) = -i\epsilon\left[\frac{i}{2}\epsilon f^\alpha_{\beta\gamma}c^\beta c^\gamma\delta_\alpha\phi_i + c^\alpha(-i\epsilon c^\beta\delta_\beta)\delta_\alpha\phi_i\right] = 0.$$

Considering (B.15), the nilpotency of $Q_B$ has an important consequence. A state of the form

$$Q_B|\chi\rangle \tag{B.18}$$

is physical. On the other hand, it is orthogonal to all the other physical states including itself. Such a state is called *null state*. Note that if we have two physical states differing by a null state, as below,

$$|\psi'\rangle = |\psi\rangle + Q_B|\chi\rangle, \tag{B.19}$$

they will be equivalent because for all the practical calculations, such as physical amplitudes, we obtain the same results. A common terminology here is to call the physical states annihilated by $Q_B$ as *closed* and the ones with the form (B.18) as *exact.* When considering these redundant states, we end up with the physical Hilbert space given by

$$\mathcal{H}_{BRST} = \frac{\mathcal{H}_{closed}}{\mathcal{H}_{exact}}. \tag{B.20}$$

[3] We have assumed that $Q_B$ is hermitian.
[4] Note that $Q_B^2$ cannot be a constant, since it has ghost number 2.

## B.2. BRST quantization of the string

The BRST transformation for string theory is given by

$$\delta_B X^\mu = i\epsilon(c\partial + \tilde{c}\bar{\partial})X^\mu \tag{B.21}$$
$$\delta_B b = i\epsilon(T_m + T_g) \tag{B.22}$$
$$\delta_B c = i\epsilon c\partial c. \tag{B.23}$$

The BRST current, calculated using Noether's theorem, is

$$j_B = cT_m + :bc\partial c: + \frac{3}{2}\partial^2 c, \tag{B.24}$$

where the last term was added by hand in order to make the BRST current a tensor. Therefore, the BRST operator for the open string is

$$Q_B = \sum_{n=-\infty}^{\infty} cL^m_{-n} + \sum_{m,n=\infty}^{\infty} \frac{(m-n)}{2} \circ c_m c_n b_{-m-n} \circ - c_0. \tag{B.25}$$

Note that

$$\{Q_B, b_m\} = L_m^{matter} + L_m^{ghost}. \tag{B.26}$$

# C. Calculations from Chapter 3

## Quadratic action for the tachyonic field

We had the following calculation to be made

$$\begin{aligned}\langle \mathcal{I}\circ\Phi(0) Q_B \Phi(0)\rangle_\varphi &= \int d^dk d^dq \oint \frac{dz}{2\pi i}\left(\frac{1}{\epsilon^2}\right)^{-1+\alpha' k^2} \varphi(k)\varphi(q)\left\{\left\langle ce^{ik\cdot X}\left(-\frac{1}{\epsilon}\right) cT^m(z) ce^{iq\cdot X}(\epsilon)\right\rangle\right.\\ &\left. + \left\langle ce^{ik\cdot X}\left(-\frac{1}{\epsilon}\right) bc\partial c(z) ce^{iq\cdot X}(\epsilon)\right\rangle\right\},\end{aligned}$$

so let's consider each term individually:

1. We can rewrite the first term as

$$\begin{aligned}A &= \oint \frac{dz}{2\pi i}\left\langle ce^{ik\cdot X}\left(-\frac{1}{\epsilon}\right) cT^m(z) ce^{iq\cdot X}(\epsilon)\right\rangle = \oint \frac{dz}{2\pi i}\left\langle c\left(-\frac{1}{\epsilon}\right) c(z)c(\epsilon)\right\rangle_{gh} \times\\ &\left\langle e^{ik\cdot X\left(-\frac{1}{\epsilon}\right)} T^m(z) e^{iq\cdot X(\epsilon)}\right\rangle_m,\end{aligned}$$

focusing in the amplitudes for now. Then, considering the OPE's developed in appendix A, we have

$$\begin{aligned}: e^{ik\cdot X}\left(-\frac{1}{\epsilon}\right) :: T^m(z) :: e^{iq\cdot X}(\epsilon) : \;\sim\; & : e^{ik\cdot X\left(-\frac{1}{\epsilon}\right)} : \left[\frac{2}{z-\epsilon} : \partial_\epsilon e^{iq\cdot X(\epsilon)} : + \frac{\alpha' q^2}{2}\frac{1}{(z-\epsilon)^2} : e^{iq\cdot X(\epsilon)} :\right]\\ + & : e^{iq\cdot X(\epsilon)} : \left[\frac{2}{z+\frac{1}{\epsilon}} : \partial_{-\frac{1}{\epsilon}} e^{ik\cdot X\left(-\frac{1}{\epsilon}\right)} :\right.\\ & \left. + \frac{\alpha' k^2}{2}\frac{1}{(z+\frac{1}{\epsilon})^2} : e^{ik\cdot X\left(-\frac{1}{\epsilon}\right)} :\right]\\ + & : T^m(z) : \left|-\frac{1}{\epsilon}-\epsilon\right|^{2\alpha' q\cdot k} : e^{i(q+k)\cdot X(\epsilon)} : .\end{aligned}$$

Since we are integrating $z$ in a closed path around the operators, we have only one pole inside the path, which is $z=\epsilon$ ($z=-\frac{1}{\epsilon}$ is outside the region of integration because $\frac{1}{\epsilon}\gg 0$ - in fact, this argument is not even necessary if one remembers that the charge is an operator, therefore, only acting in the vertex operator in its right, the one over the point $\epsilon$). Therefore, the second and forth terms give no contributions. Besides, the fifth term also gives zero because the dependence on $z$ is holomorphic. The first term does not contribute neither because the dependence on $z$ is on the $c$ ghost, producing $c(\epsilon)^2=0$. Thus, we end up with

$$\begin{aligned}\oint \frac{dz}{2\pi i}\left\langle ce^{ik\cdot X}\left(-\frac{1}{\epsilon}\right) cT^m(z) ce^{iq\cdot X}(\epsilon)\right\rangle &= \frac{\alpha' q^2}{2}\left\langle e^{ik\cdot X\left(-\frac{1}{\epsilon}\right)} e^{iq\cdot X(\epsilon)}\right\rangle_m \left\langle c\left(-\frac{1}{\epsilon}\right)\partial cc(\epsilon)\right\rangle_{gh}\\ &= \frac{\alpha' q^2}{2}(2\pi)^d\delta^d(k+q)\left|-\frac{1}{\epsilon}-\epsilon\right|^{2\alpha' q\cdot k}\left(\epsilon+\frac{1}{\epsilon}\right)^2,\end{aligned}$$

using the calculations from section A.6. Finally, the final result is given by

$$\begin{aligned}
A &= \lim_{\epsilon\to 0}\int d^dk d^dq \left(\frac{1}{\epsilon^2}\right)^{-1+\alpha' k^2} \varphi(k)\varphi(q)\frac{\alpha' q^2}{2}(2\pi)^d\delta^d(k+q)\left|-\frac{1}{\epsilon}-\epsilon\right|^{2\alpha' q\cdot k}\left(\epsilon+\frac{1}{\epsilon}\right)^2 \\
&= \lim_{\epsilon\to 0}\int d^dk \left(\frac{1}{\epsilon^2}\right)^{-1+\alpha' k^2} \varphi(k)\varphi(-k)\frac{\alpha' k^2}{2}(2\pi)^d\left|-\frac{1}{\epsilon}-\epsilon\right|^{-2\alpha' k^2}\left(\epsilon+\frac{1}{\epsilon}\right)^2 \\
&= (2\pi)^d\int d^dk\frac{\alpha' k^2}{2}\varphi(k)\varphi(-k)\lim_{\epsilon\to 0}\left(\epsilon^2+1\right)^{2-2\alpha' k^2} \\
&= (2\pi)^2\alpha'\int d^dk\varphi(k)\varphi(-k)k^2,
\end{aligned}$$

which gives the canonical kinetic term for the tachyonic field after a Fourier-transformation for the position space.

2. The second term can be rewritten as

$$B = \int d^dk d^dq \oint\frac{dz}{2\pi i}\left(\frac{1}{\epsilon^2}\right)^{-1+\alpha' k^2}\varphi(k)\varphi(q)\left\langle c\left(-\frac{1}{\epsilon}\right)bc\partial c(z)c(\epsilon)\right\rangle_g\left\langle e^{ik\cdot X\left(-\frac{1}{\epsilon}\right)}e^{iq\cdot X(\epsilon)}\right\rangle_m .$$

Then, from the $bc$ theory, the ghost piece together with the $z$ integration gives

$$\begin{aligned}
\oint\frac{dz}{2\pi i}\left\langle c\left(-\frac{1}{\epsilon}\right)bc\partial c(z)c(\epsilon)\right\rangle_g &= \oint\frac{dz}{2\pi i}\left[: c\left(-\frac{1}{\epsilon}\right) :: c\partial c(\epsilon) : \frac{1}{z-\epsilon} - : c(\epsilon) : c\partial c\left(-\frac{1}{\epsilon}\right) : \frac{1}{z+\frac{1}{\epsilon}}\right] \\
&= -\left(\epsilon+\frac{1}{\epsilon}\right)^2,
\end{aligned}$$

using the amplitude results from section A.6. The last term does not contribute by the same reasons as in the above case. Therefore,

$$\begin{aligned}
B &= -\lim_{\epsilon\to 0}\int d^dk d^dq\left(\frac{1}{\epsilon^2}\right)^{-1+\alpha' k^2}\varphi(k)\varphi(q)\left(\epsilon+\frac{1}{\epsilon}\right)^2\left\langle e^{ik\cdot X\left(-\frac{1}{\epsilon}\right)}e^{iq\cdot X(\epsilon)}\right\rangle_m \\
&= -(2\pi)^d\lim_{\epsilon\to 0}\int d^dk d^dq\left(\frac{1}{\epsilon^2}\right)^{-1+\alpha' k^2}\left(\epsilon+\frac{1}{\epsilon}\right)^2\varphi(k)\varphi(q)\delta^d(k+q)\left|-\frac{1}{\epsilon}-\epsilon\right|^{2\alpha' q\cdot k} \\
&= -(2\pi)^d\lim_{\epsilon\to 0}\int d^dk\varphi(k)\varphi(-k)\left(\frac{1}{\epsilon^2}\right)^{-1+\alpha' k^2}\left(\epsilon+\frac{1}{\epsilon}\right)^2\left(\frac{1}{\epsilon}+\epsilon\right)^{-2\alpha' k^2} \\
&= -(2\pi)^d\int d^dk\varphi(k)\varphi(-k)\lim_{\epsilon\to 0}\left(\epsilon^2+1\right)^{2-2\alpha' k^2} \\
&= -(2\pi)^d\int d^dk\varphi(k)\varphi(-k),
\end{aligned}$$

giving the quadratic term on $\varphi$ in (3.11).

## Cubic action for the tachyonic field

→ Let's start calculating the $f_i'$s and their derivatives. Remembering first that

$$\begin{aligned}
h^{-1}(\zeta) &= -i\frac{\zeta-1}{\zeta+1} \\
f_i(z_i) &= h^{-1}\circ g_i(z_i),
\end{aligned}$$

then,

$$f_i'(z_i) = -\frac{2i g_i'(z_i)}{[g_i(z_i)+1]^2},$$

so that:

1. For $g_1(z_1) = e^{-2\pi i/3}\left(\frac{1+iz_1}{1-iz_1}\right)^{2/3}$, we have

$$\begin{aligned} g_1(0) &= e^{-2\pi i/3} \\ g_1'(0) &= \frac{4}{3} i e^{-2\pi i/3}. \end{aligned}$$

   Therefore, $f_1(0) = -\sqrt{3}$ and $f_1'(0) = 8/3$;

2. For $g_2(z_2) = \left(\frac{1+iz_2}{1-iz_2}\right)^{2/3}$, we end up with

$$\begin{aligned} g_2(0) &= 1 \\ g_2'(0) &= \frac{4i}{3}, \end{aligned}$$

   providing $f_2(0) = 0$ and $f_2'(0) = 2/3$;

3. For $g_3(z_3) = e^{2\pi i/3}\left(\frac{1+iz_3}{1-iz_3}\right)^{2/3}$, we get

$$\begin{aligned} g_3(0) &= -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\ g_3'(0) &= \frac{4}{3} i e^{2\pi i/3}, \end{aligned}$$

   giving $f_3(0) = \sqrt{3}$ and $f_3'(0) = 8/3$.

$\rightarrow$ Another important calculation that was omitted is the derivation of the $F(k,p,q)$ term. If one substitute the above calculations in (3.12), it will conclude that

$$F(k,p,q) = 2^{3\alpha'(k^2+q^2)+\alpha' p^2 + 2\alpha' q\cdot k} 3^{\alpha'(k\cdot p + k\cdot q + p\cdot q - k^2 - p^2 - q^2)}.$$

We can rewrite it as

$$\begin{aligned} F(k,p,q) &= \exp[\alpha' k\cdot p \ln 3 + \alpha' p\cdot q \ln 3 + \alpha' k\cdot q(\ln 3 + 2\ln 2) + \alpha' k^2(3\ln 2 - \ln 3) \\ &\quad + \alpha' p^2(\ln 2 - \ln 3) + \alpha' q^2(3\ln 2 - \ln 3) \end{aligned}$$

and using $q = -k - p$ from the delta function, we get

$$F(k,p,q) = \exp\left[\alpha' \ln\frac{4}{3\sqrt{3}}(k^2+p^2+q^2)\right].$$

# D. Elements of General Relativity and Cosmology

This appendix is intended to be a quick review on some aspects of General Relativity and Cosmology in order to fix the notation and to provide a short reference for the results we have been using so far. We will be concerned about our homogeneous and isotropic universe, its kinematics and dynamics and some particular examples that were important throughout the thesis. It is expected that this is enough to study the inflationary paradigm in Chapters 4 and 5.

The basic references for this short review are [10, 11, 12].

## D.1. Our Universe

The Cosmic Microwave Background (CMB) radiation, first observed by Penzias and Wilson [13], tells us that the universe was extremely homogeneous and isotropic in the past (around 380,000 years-old)[1]. In fact, we observe only small fluctuations of order $10^{-5}$ in the energy density distribution by that time, when the universe was $\approx 1100$ smaller than now. Together with the CMB observations, redshifts surveys, counts of radio sources and the isotropy of the X-ray and $\gamma$-ray background suggest that the universe is homogeneous and isotropic on large scales ($\gtrsim 100\,Mpc$), having inhomogeneous structures in small scales, *e.g.,* galaxies.

On theoretical grounds, in order to describe properly the universe we need to know the spacetime *metric* of the universe, $g_{\mu\nu}$. General Relativity provides the proper equations to describe the dynamics of the metric through the Einstein's equation:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} + \Lambda g_{\mu\nu}. \tag{D.1}$$

where $G = c = 1$; $\Lambda$ is a cosmological constant[2]; $T_{\mu\nu}$ is the *energy-momentum tensor*, responsible for the matter-energy content of the universe. The *Ricci tensor* and the *Ricci scalar* are defined, respectively, by

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\mu \Gamma^\alpha_{\nu\alpha} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha} \tag{D.2}$$

$$R = g_{\mu\nu} R^{\mu\nu}, \tag{D.3}$$

where the *Christoffel symbols* are

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}(\partial_\nu g_{\mu\beta} + \partial_\mu g_{\nu\beta} - \partial_\beta g_{\mu\nu}). \tag{D.4}$$

[1] Around $10^{-6}s$, the early universe was made up of high energy plasma composed by photons, electrons and baryons. As it expanded, the plasma temperature decreased until it became favorable for electrons to combine with protons, decoupling matter and radiation. This event, called *recombination*, happened when the temperature was approximately $3000\,K$. After the decoupling, those photons have remained in thermodynamical equilibrium and been affected only by the expansion of the universe, which *redshifted* them from infrared to the microwave ($\approx 2,73\,K$). We also refer to this event as the *last-scattering surface.*

[2] Its empirical value is $10^{-122}$ in Planck units and it will be considered zero from now on. This is justifiable because we will be worried here only with the early universe until recombination and the cosmological constant was not relevant for that epoch. In fact, the accelerated expansion promoted by $\Lambda$, as deduced below, only became dominant when the universe was $\approx 10^{10}$ years.

Given the above observational evidences, assuming homogeneity and isotropy[3] on large scales one is lead to the Friedmann-Robertson-Walker (FRW) metric

$$ds^2 = -dt^2 + a^2(t)[d\chi^2 + \Phi_K(\chi)(d\theta^2 + \sin^2\theta d\phi^2)] \tag{D.5}$$

$$\Phi_K(\chi) = \begin{cases} \sinh^2\chi, & k = -1 \\ \chi^2, & k = 0 \\ \sin^2\chi, & k = 1. \end{cases} \tag{D.6}$$

To understand the above elements, consider spacelike hypersurfaces $\Sigma$. So, the *scale factor*, $a(t)$, is associated with the relative size of these hypersurfaces at different times. Besides, if $\Sigma$ is positively curved, $k = 1$; if flat, $k = 0$; and if negatively curved, $k = -1$. Finally, it should be said we are using comoving coordinates, that means the coordinates $r$, $\theta$, $\phi$ are constant for all the observers/objects that are not under peculiar motion (this never happens, but it is a good approximation)[4]. Note that we have reduced all the dynamics of the homogeneous and isotropic universe to a unique function of time.

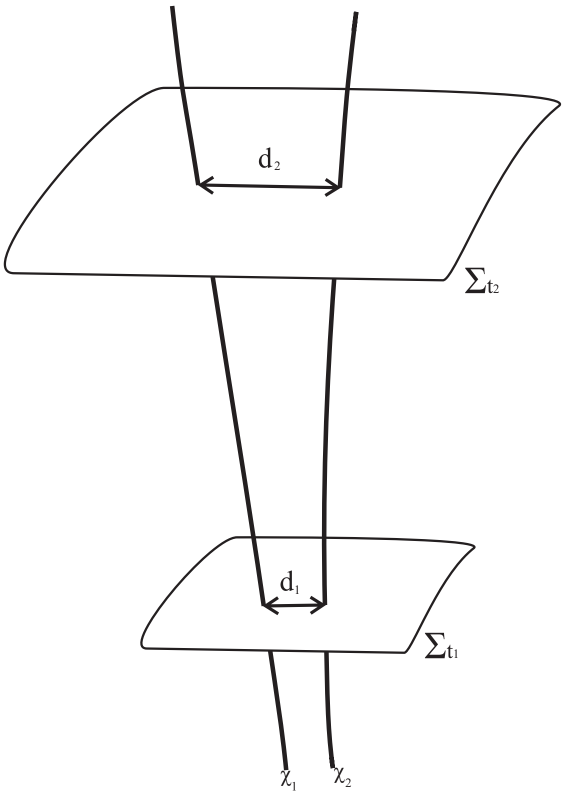


Figure D.1.: Representation of the spacelike hypersurfaces, comoving coordinates and the intuition of the scale factor. Note that $d_t = \int_{\Sigma_t} \sqrt{ds^2} = a(t)\Delta\chi$, having suppressed the other comoving coordinates.

[3] A homogeneous space is one which has translational invariance while an isotropic one has rotational invariance. For a formal definition, see [11], chapter 5. There one can also find a formal reduction of the generic metric to the FRW one.

[4] In fact, for objects far away, on the same scale where homogeneity and isotropy seems to be observable, the redshift due to peculiar motion is meaningless compared to the redshift coming from the expansion of the universe (considering peculiar velocities of order $100\,km/s$ and $H_0 = (67.4 \pm 1.4)\,km\,s^{-1}\,Mpc^{-1}$[14].

Considering the FRW-metric, now it is just a matter to insert it into the Einstein's equation to determine $a(t)$. An important quantity from now on is the *Hubble parameter,*

$$H = \frac{\dot{a}}{a}, \tag{D.7}$$

which is positive for an expanding universe and negative for a collapsing one. It sets the characteristic time- and length-scale of the universe: $t \sim H^{-1}$ and $d \sim H^{-1}$ ($c = 1$), respectively.

## D.2. Kinematics: causal structure

### D.2.1. Conformal Time

The causal structure of the universe is given by the propagation of light in the FRW metric. We know that light-rays follow *null geodesics*, $ds^2 = 0$, which look simple if, instead of the physical time $t$, we use the *conformal time*

$$\eta = \int \frac{dt}{a(t)}. \tag{D.8}$$

Then, the metric (D.5) in $\eta$, $\chi$ coordinates is

$$ds^2 = a(\eta)^2[-d\eta^2 + (d\chi^2 + \Phi_k(\chi)(d\theta^2 + \sin^2 d\phi^2))]. \tag{D.9}$$

Considering only the radial light geodesics of light (isotropic universe), their description are given by

$$\chi(\eta) = \pm\eta + \text{const.}, \tag{D.10}$$

corresponding to straight lines at angles $\pm 45^o$ in the $\eta - \chi$ plane.

Looking for (D.8), an intuitive way to understand the conformal time is to think it as a "clock" that slows down with the expansion of the universe.

### D.2.2. Particle Horizon

The comoving distance light can propagate between an emission time, $t_i$, and some latter time, $t$, is

$$\chi_p(\eta) = \eta - \eta_i = \int_{t_i}^{t} \frac{dt'}{a(t')}. \tag{D.11}$$

In a universe with an initial singularity, we define $t_i$ by $a(t_i) = 0$. Multiplying $\chi_p$ by the scale factor, we obtain the physical size of the *particle horizon*:

$$d_p(t) = a(t)\chi_p = a(t) \int_{t_i}^{t} \frac{dt'}{a(t')}. \tag{D.12}$$

The particle horizon is finite at any time in the past, limiting the distance of causal contact between different regions in the early universe.

### D.2.3. Event Horizon

The *event horizon* is the complement of the particle horizon. It defines the set of points from which signals sent at a given moment of time $\eta$ will never be received by an observer in the future. In comoving coordinates these points satisfy

$$\chi > \chi_e(\eta) = \int_{\eta}^{\eta_{\max}} d\eta = \eta_{max} - \eta. \tag{D.13}$$

Hence, the physical size of the event horizon at time $t$ is

$$d_e(t) = a(t)\int_t^{t_{max}} \frac{dt'}{a(t')},$$

where $\eta_{max}$ refers to the final moment of time, which may be finite or infinite.

### D.2.4. Redshift

Consider a source of radiation with comoving coordinate $\chi_{em}$, which emits a signal at conformal time $\eta_{em}$ with duration $\delta\eta$. The trajectory of the signal is $\chi(\eta) = \chi_{em} - (\eta - \eta_{em})$ and it is detected at $\chi_{obs} = 0$ at time $\eta_{obs} = \eta_{em} + \chi_{em}$. Even though $\delta\eta_{em} = \delta\eta_{obs} = \delta\eta$, the physical time intervals are different at the points of emission and detection and given by

$$\begin{aligned} \delta t_{em} &= a(\eta_{em})\delta\eta \\ \delta t_{obs} &= a(\eta_{obs})\delta\eta. \end{aligned}$$

If $\delta t$ is the period of the light wave, then its wavelength is $\lambda = \delta t$, so that

$$\frac{\lambda_{obs}}{\lambda_{em}} = \frac{a(\eta_{obs})}{a(\eta_{em})}. \tag{D.14}$$

We define the *redshift parameter* as

$$z = \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}}. \tag{D.15}$$

This formula can be rewritten using (D.14) as

$$1 + z = \frac{a_0}{a(t_{em})}, \tag{D.16}$$

where $a_o$ is the present value of the scale factor.

## D.3. Dynamics

### D.3.1. Energy, momentum and pressure

On the cosmic scales, each galaxy can be idealized as a "grain of dust". Besides, the peculiar velocities of the galaxies are small, so what would be an equivalent pressure, $p$, of this dust is negligible. Hence, the ordinary matter can be described as a perfect fluid with $p = 0$. However, there are others forms of energy content in the universe with non-zero pressure, *e.g.* radiation. Therefore, we shall consider the general perfect fluid form for $T_{\mu\nu}$, given by

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}, \tag{D.17}$$

where $\rho$ is the energy density and $u^\mu$ is the 4−velocity of the fluid. Note that $\rho$ and $p$ are defined in the fluid rest frame.

An important example of matter field that is of fundamental importance for us is the classical scalar field $\varphi$ with potential $V(\varphi)$ and action given by,

$$S_\varphi = \int d^4x\sqrt{-g}\left[-\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi)\right]. \tag{D.18}$$

The equation of motion is

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu\varphi) - \frac{dV}{d\varphi} = 0. \tag{D.19}$$

The energy-momentum tensor for the scalar field is[5]

$$T^{(\varphi)}_{\mu\nu} \equiv \frac{-2}{\sqrt{-g}}\frac{\delta S_\varphi}{\delta g^{\mu\nu}} = \partial_\mu\varphi\partial_\nu\varphi - g_{\mu\nu}\left(\frac{1}{2}\partial^\sigma\varphi\partial_\sigma\varphi + V(\varphi)\right). \tag{D.20}$$

If the kinetic term is negative ($\dot{\varphi}^2 > (\partial_i\varphi)^2$), then we can rewrite it in the form of a perfect fluid by defining

$$\rho \equiv -\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + V(\varphi) \tag{D.21}$$

$$p \equiv -\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi) \tag{D.22}$$

$$u^\mu \equiv \partial^\mu\varphi/\sqrt{-\partial_\alpha\varphi\partial^\alpha\varphi}. \tag{D.23}$$

In particular, if the field is homogeneous, *i.e.* $\partial_i\varphi = 0$, we have

$$\rho = \frac{1}{2}\dot{\varphi}^2 + V(\varphi) \tag{D.24}$$

$$p = \frac{1}{2}\dot{\varphi}^2 - V(\varphi). \tag{D.25}$$

If the potential has a local minimum at $\varphi_0$, then $\varphi(t) = \varphi_0$ is a solution of (D.19), for which

$$p = -\rho = -V(\varphi_0). \tag{D.26}$$

Back to the energy-momentum tensor, we find

$$T_{\mu\nu} = V(\varphi_0)g_{\mu\nu}, \tag{D.27}$$

which mimics a cosmological term with $\Lambda = 8\pi G V(\varphi_0)$. Remembering now that the cosmological constant produces an accelerated expansion, this example turns out to be a key point in inflation.

### D.3.2. Friedmann Equations

In the last section, we have seen that we can use the energy-momentum tensor of a perfect fluid to describe different energy-matter contents in the universe. Then, what remains to be done is to use it in the Einstein's equation with the FRW metric so that we can obtain the dynamics of the scale factor.

We can start saving ourselves some hand work noticing that, although (D.1) has 10 equations coming from the 10 independent components (remember we have two-index symmetric tensors), the symmetries in the FRW metric reduces the problem into only two independent equations. In order to understand it, span the space-time with four normalized 4-vectors, one of them the 4-velocity, which is timelike, and the three others spacelike, $s_\alpha$ (in fact, they are tangent to the homogeneous hypersurfaces, $\Sigma$, we talked before). In Cartesian coordinates, they are $\{\partial_t, \partial_i;\ i = x, y, z\}$. Now, imposing isotropy we must have[6]

$$G^\mu{}_\nu u^\nu \propto u^\mu,$$

[5] Remember that $\partial(\sqrt{-g}) = \frac{1}{2}\sqrt{-g}g^{\alpha\beta}\delta g_{\alpha\beta} = -\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta}$.

[6] We are ignoring a cosmological constant now.

otherwise, we would have a preferred direction in space. Hence, $G^{\mu}{}_{\nu}u^{\nu}s_{\mu} = 0$. As a result, the "time-space" components of the Einstein's equation are identically zero. Besides, the diagonal "space-space" components produce the same equation, as can be inferred by (D.17). Therefore, there remains two independent equations:

$$\begin{aligned} G_{\mu\nu}s^{\mu}s^{\nu} &= 8\pi T_{\mu\nu}s^{\mu}s^{\nu} \to G_{\star\star} = 8\pi p \qquad &(D.28)\\ G_{\mu\nu}u^{\mu}u^{\nu} &= 8\pi T_{\mu\nu}u^{\mu}u^{\nu} \to G_{tt} = 8\pi\rho, \qquad &(D.29)\end{aligned}$$

where we used Cartesian coordinates to keep it simple.

Now, it is just a matter of some tedious calculations to obtain the Ricci's scalar and tensor. The results are

$$\begin{cases} R_{00} &= -3\frac{\ddot{a}}{a} \\ R_{ij} &= \delta_{ij}\left[2\dot{a}^2 + a\ddot{a} + 2\frac{k}{a^2}\right] \\ R &= 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right]. \end{cases} \tag{D.30}$$

Using (D.30) into (D.28) and (D.29), we end up with the *Friedmann Equations:*

$$\begin{aligned} \frac{\ddot{a}}{a} &= \dot{H} + H^2 = -\frac{4\pi}{3}(\rho + 3p) \qquad &(D.31)\\ \left(\frac{\dot{a}}{a}\right)^2 &= H^2 = \frac{8\pi\rho}{3} - \frac{k}{a^2}. \qquad &(D.32)\end{aligned}$$

There are three important points to be made about them:

1. Combining (D.31) e (D.32), the continuity equation is obtained,

$$\frac{d\rho}{dt} + 3H(\rho + p) = 0; \tag{D.33}$$

2. The Friedmann equations are only two while we have to solve them for three variables. Hence, we need another relation, the *equation of state,* which relates the pressure and energy density. It is usually expressed as

$$\omega = \frac{p}{\rho}, \tag{D.34}$$

defining the parameter $\omega$. For radiation, matter and the cosmological constant, $w = \frac{1}{3}$, 0, $-1$, respectively;

3. We only have an accelerated expansion if the *strong energy condition,* $\rho + 3p > 0$, is violated. This happens for $\omega < -\frac{1}{3}$.

### D.3.3. General example

Let's solve in detail the equations for the scale factor and the energy density assuming an arbitrary $\omega$ for $k = 0$. From (D.33), we have

$$\begin{aligned} 0 &= \frac{1}{\rho}\frac{d\rho}{dt} + 3\frac{1}{a}\frac{da}{dt}(1 + \omega) \\ &= \frac{d\ln\rho}{d\ln a} + 3(1 + \omega). \end{aligned}$$

Integrating,

$$\rho(a) \propto a^{-3(1+\omega)}.$$

Together with (D.32), we find

$$\begin{aligned} \dot{a}^2 &\propto a^{-3\omega-1} \\ a^{\frac{3\omega+1}{2}} da &\propto dt. \end{aligned}$$

This leads to the time evolution of the scale factor,

$$a(t) \propto \begin{cases} t^{\frac{2}{3(1+\omega)}}, & w \neq -1, \\ \exp^{Ht}, & w = -1. \end{cases} \tag{D.35}$$

Then, we obtain

| | $\omega$ | $\rho(a)$ | $a(t)$ | $a(\eta)$ |
|---|---|---|---|---|
| matter | 0 | $a^{-3}$ | $t^{2/3}$ | $\eta^2$ |
| radiation | $\frac{1}{3}$ | $a^{-4}$ | $t^{1/2}$ | $\eta$ |
| $\Lambda$ | -1 | $a^0$ | $\exp^{Ht}$ | $-\frac{1}{\eta}$ |

Table D.1.: The scale factor evolution for different energy contents.

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