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Polynomials generated by a three term recurrence relation: bounds for complex zeros ☆

A.P. da Silva [a], A. Sri Ranga [b,*]

[a] *Ciências de Computação e Matemática Computacional, ICMC, USP – Universidade de São Paulo, Av. do Trabalhador São-Carlense 400, 13560-970 São Carlos, SP, Brazil*
[b] *Departamento de Ciências de Computação e Estatística, IBILCE, UNESP – Universidade Estadual Paulista, Rua Cristóvão Colombo 2265, 15054-000 São José do Rio Preto, SP, Brazil*

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Abstract

This paper deals with the zeros of polynomials generated by a certain three term recurrence relation. The main objective is to find bounds, in terms of the coefficients of the recurrence relation, for the regions where the zeros are located. In most part, the zeros are explored through an Eigenvalue representation associated with a corresponding Hessenberg matrix. Applications to Szegö polynomials, para-orthogonal polynomials and polynomials with non-zero complex coefficients are considered.

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* Corresponding author.
E-mail address: ranga@dcce.ibilce.unesp.br (A. Sri Ranga).

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1. Introduction

We consider the sequence of polynomials $\{Q_m\}$ generated by the three term recurrence relation

$$Q_{m+1}(z) = (z + \beta_{m+1})Q_m(z) - \alpha_{m+1} z Q_{m-1}(z), \quad m = 1, 2, \ldots, \tag{1.1}$$

with $Q_0 = 1$ and $Q_1(z) = z + \beta_1$, where the complex numbers α_m and β_m are such that $\alpha_m \neq 0$, $m = 2, 3, \ldots$ and $\beta_m \neq 0$, $m = 1, 2, 3, \ldots$

This recurrence relation has been thoroughly studied when

$$\alpha_{m+1} > 0 \quad \text{and} \quad \beta_m < 0 \quad \text{for } m \geqslant 1. \tag{1.2}$$

We cite the paper [11] of Jones, Thron and Waadeland which initiated the motivation for the study of this special case of the recurrence relation (1.1) and the recent paper [15] for some further references. When (1.2) holds, it was shown in [11] that the corresponding polynomials satisfy the orthogonality property, or more precisely, the Laurent Orthogonality (L-orthogonality) property,

$$\int t^{-m+s} Q_m(t) \mathrm{d}\phi(t) = 0, \quad s = 0, 1, \ldots, m-1. \tag{1.3}$$

Here $\mathrm{d}\phi$ is a strong positive measure defined on the positive real axis. The word strong is used to indicate that the moments $\mu_m = \int t^m \mathrm{d}\phi(t)$ exist also for all negative values of m.

In this paper we study the behavior of the zeros of the polynomials $\{Q_m\}$ under other restrictions on the coefficients α_m and β_m. As special cases of this study we consider the Szegö polynomials $\{S_m(z)\}$ and also of certain so called para orthogonal polynomials $\{S_m(z) + S_m^*(z)\}$ and $\{S_m(z) - S_m^*(z)\}$, especially when the reflection coefficients are real.

As another special case, the zeros of the polynomial $P_n(z) = \sum_{m=0}^{n} b_m z^m$, where the complex coefficients $b_m \neq 0$ for $m = 0, 1, \ldots, n$, are also considered.

2. Zeros and their eigenvalue representations

From the three term recurrence relation (1.1), clearly $Q_m(0) = \beta_1 \beta_2 \cdots \beta_m \neq 0$, $m \geqslant 1$. Furthermore, since (1.1) can be written in the form

$$\begin{aligned} \alpha_2 z &= (z + \beta_2) Q_1(z) - Q_2(z), \\ \alpha_{m+1} z &= (z + \beta_{m+1}) \frac{Q_m(z)}{Q_{m-1}(z)} - \frac{Q_{m+1}(z)}{Q_{m-1}(z)}, \quad m \geqslant 2, \end{aligned} \tag{2.1}$$

one obtains

Lemma 2.1. *For any $m \geqslant 1$, the two consecutive polynomials Q_m and Q_{m+1} do not have common zeros.*

Proof. If $\omega \in \mathbb{C}$ is such that $Q_1(\omega) = Q_2(\omega) = 0$ then $\alpha_2\omega = 0$. Since $\omega = 0$ is not a zero of any Q_m, this is a contradiction. Hence $Q_1(z)$ and $Q_2(z)$ do not have any common zeros.

Now let $m \geqslant 2$ and suppose that Q_{m-1} and Q_m do not have any common zeros. If $Q_m(\omega) = 0$ then $Q_{m-1}(\omega) \neq 0$ and from (2.1),

$$Q_{m+1}(\omega) = -\alpha_{m+1}\omega Q_{m-1}(\omega) \neq 0.$$

This means Q_m and Q_{m+1} do not have any common zeros. Hence by induction the lemma follows. $\square$

From (1.1) it is also easily verified that for any $n \geqslant 1$,

$$Q_n(z) = \begin{vmatrix} z+\beta_1 & -\alpha_2 & 0 & \cdots & 0 & 0 \\ -z & z+\beta_2 & -\alpha_3 & \cdots & 0 & 0 \\ 0 & -z & z+\beta_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & z+\beta_{n-1} & -\alpha_n \\ 0 & 0 & 0 & \cdots & -z & z+\beta_n \end{vmatrix}. \tag{2.2}$$

Hence we can state the following theorem.

Theorem 2.1. *The zeros of Q_n are the eigenvalues of the lower Hessenberg matrix*

$$\mathbf{H}_n = \begin{bmatrix} \eta_1 & \alpha_2 & 0 & \cdots & 0 & 0 \\ \eta_1 & \eta_2 & \alpha_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \eta_1 & \eta_2 & \eta_3 & \cdots & \alpha_{n-1} & 0 \\ \eta_1 & \eta_2 & \eta_3 & \cdots & \eta_{n-1} & \alpha_n \\ \eta_1 & \eta_2 & \eta_3 & \cdots & \eta_{n-1} & \eta_n \end{bmatrix},$$

where $\eta_m = \alpha_m - \beta_m$, $m = 1, 2, \ldots, n$, with $\alpha_1 = 0$.

Proof. From (2.2),

$$Q_n(z) = \det(z\mathbf{A}_n - \mathbf{B}_n),$$

where $\mathbf{A}_n$ and $\mathbf{B}_n$ are, respectively, the $n \times n$ matrices

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \quad \begin{bmatrix} -\beta_1 & \alpha_2 & 0 & \cdots & 0 \\ 0 & -\beta_2 & \alpha_3 & \cdots & 0 \\ 0 & 0 & -\beta_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & -\beta_n \end{bmatrix}.$$

Note that, since $\mathbf{A}_n$ is non-singular,

$$Q_n(z) = \det(\mathbf{A}_n)\det(z\mathbf{I} - \mathbf{A}_n^{-1}\mathbf{B}_n).$$

Since $\det(\mathbf{A}_n) = 1$ and

$$\mathbf{A}_n^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix},$$

we see that Q_n is the (monic) characteristic polynomial of the lower Hessenberg matrix $\mathbf{H}_n = \mathbf{A}_n^{-1}\mathbf{B}_n$. This completes the proof of the theorem. □

We can also write, $Q_n(z) = \det(z\mathbf{I} - \mathbf{B}_n\mathbf{A}_n^{-1})\det(\mathbf{A}_n)$. Hence Q_n is the characteristic polynomial of the matrix $\mathbf{B}_n\mathbf{A}_n^{-1}$ and also of the matrix $(\mathbf{B}_n\mathbf{A}_n^{-1})^{\mathrm{T}} = (\mathbf{A}_n^{-1})^{\mathrm{T}}\mathbf{B}_n^{\mathrm{T}}$. This means the zeros of Q_n are also the eigenvalues of the upper Hessenberg matrix

$$(\mathbf{A}_n^{-1})^{\mathrm{T}}\mathbf{B}_n^{\mathrm{T}} = \begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \cdots & \gamma_{n-1} & -\beta_n \\ \alpha_2 & \gamma_2 & \gamma_3 & \cdots & \gamma_{n-1} & -\beta_n \\ 0 & \alpha_3 & \gamma_3 & \cdots & \gamma_{n-1} & -\beta_n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \gamma_{n-1} & -\beta_n \\ 0 & 0 & 0 & \cdots & \alpha_n & -\beta_n \end{bmatrix},$$

where $\gamma_m = \alpha_{m+1} - \beta_m$, $m = 1, 2, \ldots$ See [15] for a use of the above eigenvalue representation to obtained information about the orthogonal L-polynomials associated with measures on the positive real line.

In the remaining part of this section we look at in detail the eigenvalue representation associated with the matrix $\mathbf{H}_n$. Again the analysis is very much similar to that used in [15] but with the assumptions made in (1.1) and one further assumption that the eigenvalues $z_{n,j}$, $j = 1, 2, \ldots, n$, of $\mathbf{H}_n$ (i.e., the zeros of Q_n) are distinct.

With $\tilde{Q}_0(z) = \hat{Q}_0(z) = Q_0(z)$ and for $1 \leqslant m \leqslant n$, $\tilde{Q}_m(z) = z^{-m}Q_m(z)$ and $\hat{Q}_m(z) = (\alpha_2\alpha_3\cdots\alpha_{m+1})^{-1}Q_m(z)$, the recurrence relation (1.1) for $m = 1, 2, \ldots, n-1$ can be given in the following two different forms:

$$z[-\hat{Q}_{m-1}(z) + \hat{Q}_m(z)] = -\beta_{m+1}\hat{Q}_m(z) + \alpha_{m+2}\hat{Q}_{m+1}(z), \quad 1 \leqslant m \leqslant n-1$$

and

$$z[\tilde{Q}_m(z) - \tilde{Q}_{m+1}(z)] = \alpha_{m+1}\tilde{Q}_{m-1}(z) - \beta_{m+1}\tilde{Q}_m(z), \quad 1 \leqslant m \leqslant n-1,$$

with $z[\hat{Q}_0(z)] = -\beta_1\hat{Q}_0(z) + \alpha_2\hat{Q}_1(z)$ and $z[\tilde{Q}_0(z) - \tilde{Q}_1(z)] = -\beta_1\tilde{Q}_0(z)$. In matrix representation these are

$$z\mathbf{A}_n\hat{\mathbf{b}}(z) = \mathbf{B}_n\hat{\mathbf{b}}(z) + \alpha_{n+1}\hat{Q}_n(z)\mathbf{e}_n \tag{2.3}$$

and

$$z\tilde{\mathbf{b}}(z)^{\mathrm{T}}\mathbf{A}_n = \tilde{\mathbf{b}}(z)^{\mathrm{T}}\mathbf{B}_n + z\tilde{Q}_n(z)\mathbf{e}_n^{\mathrm{T}}, \tag{2.4}$$

respectively, where $\mathbf{e}_n$ is the nth column of the $n \times n$ identity matrix,

$$\hat{\mathbf{b}}(z) = [\hat{Q}_0(z), \hat{Q}_1(z), \ldots, \hat{Q}_{n-1}(z)]^{\mathrm{T}} \quad \text{and}$$
$$\tilde{\mathbf{b}}(z) = [\tilde{Q}_0(z), \tilde{Q}_1(z), \ldots, \tilde{Q}_{n-1}(z)]^{\mathrm{T}}.$$

Hence from the matrix representation (2.3), $\hat{\mathbf{b}}(z_{n,j})$ is a right eigenvector of $\mathbf{H}_n = \mathbf{A}_n^{-1}\mathbf{B}_n$ associated with the eigenvalue $z_{n,j}$. Moreover, since $z_{n,j}$, $j = 1, 2, \ldots, n$ are assumed to be distinct, the corresponding right eigenvectors $\hat{\mathbf{b}}(z_{n,j})$, $j = 1, 2, \ldots, n$, are all linearly independent.

Likewise, from the matrix representation (2.4), the vector $\mathbf{A}_n^{\mathrm{T}}\tilde{\mathbf{b}}(z_{n,j})$ is a left eigenvector of $\mathbf{H}_n$ associated with the eigenvalue $z_{n,j}$. Even though it is not relevant here, it may be more appropriate to say that $\tilde{\mathbf{c}}(z_{n,j})$, where $\tilde{\mathbf{c}}(z) = \overline{\mathbf{A}_n^{\mathrm{T}}\tilde{\mathbf{b}}(z)}$, is a left eigenvector of $\mathbf{H}_n$ as we have in this case $\tilde{\mathbf{c}}(z_{n,j})^*\mathbf{H}_n = \tilde{\mathbf{c}}(z_{n,j})^* z_{n,j}$. Here, $*$ represents the conjugate transpose. Again, there are n linearly independent left eigenvectors $\tilde{\mathbf{c}}(z_{n,j})$, $j = 1, 2, \ldots, n$, associated with the n distinct $z_{n,j}$.

Multiplication of (2.3) evaluated at w on the right by $\tilde{\mathbf{b}}(z)^{\mathrm{T}}$ and multiplication of (2.4) on the left by $\hat{\mathbf{b}}(w)$ we obtain the systems

$$w\mathbf{A}_n\hat{\mathbf{b}}(w)\tilde{\mathbf{b}}(z)^{\mathrm{T}} = \mathbf{B}_n\hat{\mathbf{b}}(w)\tilde{\mathbf{b}}(z)^{\mathrm{T}} + \alpha_{n+1}\hat{Q}_n(w)\mathbf{e}_n\tilde{\mathbf{b}}(z)^{\mathrm{T}} \tag{2.5}$$

and

$$z\hat{\mathbf{b}}(w)\tilde{\mathbf{b}}(z)^{\mathrm{T}}\mathbf{A}_n = \hat{\mathbf{b}}(w)\tilde{\mathbf{b}}(z)^{\mathrm{T}}\mathbf{B}_n + z\tilde{Q}_n(z)\hat{\mathbf{b}}(w)\mathbf{e}_n^{\mathrm{T}}. \tag{2.6}$$

Since $\operatorname{tr}\mathbf{AB} = \operatorname{tr}\mathbf{BA}$, subtracting the trace of (2.6) from the trace of (2.5) gives

$$\sum_{m=0}^{n-2}\{\tilde{Q}_m(z) - \tilde{Q}_{m+1}(z)\}\hat{Q}_m(w) + \tilde{Q}_{n-1}(z)\hat{Q}_{n-1}(w)$$
$$= \frac{z\tilde{Q}_n(z)\hat{Q}_{n-1}(w) - \alpha_{n+1}\hat{Q}_n(w)\tilde{Q}_{n-1}(z)}{z - w}. \tag{2.7}$$

This is a kind of Cristoffel–Darboux formula. The term on the left of this formula can be identified as $\tilde{\mathbf{b}}(z)^{\mathrm{T}}\mathbf{A}_n\hat{\mathbf{b}}(w) = \tilde{\mathbf{c}}(z)^*\hat{\mathbf{b}}(w)$, and this confirms the orthogonality of the left eigenvector $\tilde{\mathbf{c}}(z_{n,j})$ and the right eigenvector $\hat{\mathbf{b}}(z_{n,k})$, when $j \neq k$. In (2.7) letting $w \to z$,

$$\sum_{m=0}^{n-2}\{\tilde{Q}_m(z) - \tilde{Q}_{m+1}(z)\}\hat{Q}_m(z) + \tilde{Q}_{n-1}(z)\hat{Q}_{n-1}(z)$$
$$= \alpha_{n+1}\hat{Q}_n'(z)\tilde{Q}_{n-1}(z) - z\tilde{Q}_n(z)\hat{Q}_{n-1}'(z).$$

Thus for $1 \leqslant j \leqslant n$ and $1 \leqslant k \leqslant n$, we have

$$[\mathbf{A}_n^{\mathrm{T}}\tilde{\mathbf{b}}(z_{n,j})]^{\mathrm{T}}\hat{\mathbf{b}}(z_{n,k}) = \hat{\mathbf{b}}(z_{n,k})^{\mathrm{T}}[\mathbf{A}_n^{\mathrm{T}}\tilde{\mathbf{b}}(z_{n,j})] = \lambda_{n,j}^{-1}\delta_{j,k}, \tag{2.8}$$

where $\lambda_{n,j}^{-1} = \alpha_{n+1}\hat{Q}_n'(z_{n,j})\tilde{Q}_{n-1}(z_{n,j})$. Since the zeros of Q_n are assumed to be simple (distinct) and, from Lemma 2.1, that Q_n and Q_{n-1} do not have any common zeros, we also find that $\lambda_{n,j}^{-1} \neq 0$, for $j = 1, 2, \ldots, n$. Thus we can say that the two finite sequences $\{\hat{\mathbf{b}}(z_{n,j})\}_{j=1}^{n}$ and $\{\mathbf{A}_n^{\mathrm{T}}\tilde{\mathbf{b}}(z_{n,j})\}_{j=1}^{n}$ are biorthogonal to each other.

If we define the $n \times n$ non-singular matrix $\mathbf{P}_n$ by

$$\mathbf{P}_n = \left[\frac{\hat{\mathbf{b}}(z_{n,1})}{\alpha_{n+1}\hat{Q}_n'(z_{n,1})}, \frac{\hat{\mathbf{b}}(z_{n,2})}{\alpha_{n+1}\hat{Q}_n'(z_{n,2})}, \ldots, \frac{\hat{\mathbf{b}}(z_{n,n})}{\alpha_{n+1}\hat{Q}_n'(z_{n,n})} \right], \tag{2.9}$$

then clearly its inverse (the matrix that satisfy $\mathbf{P}_n^{-1}\mathbf{P}_n = \mathbf{I}_n$) is

$$\mathbf{P}_n^{-1} = \left[\frac{\mathbf{A}_n^{\mathrm{T}}\tilde{\mathbf{b}}(z_{n,1})}{\tilde{Q}_{n-1}(z_{n,1})}, \frac{\mathbf{A}_n^{\mathrm{T}}\tilde{\mathbf{b}}(z_{n,2})}{\tilde{Q}_{n-1}(z_{n,2})}, \ldots, \frac{\mathbf{A}_n^{\mathrm{T}}\tilde{\mathbf{b}}(z_{n,n})}{\tilde{Q}_{n-1}(z_{n,n})} \right]^{\mathrm{T}}. \tag{2.10}$$

If $\hat{p}_{k,j}$ and $\tilde{p}_{k,j}$ are the elements in the kth row and jth column of the matrices $\mathbf{P}_n$ and $\mathbf{P}_n^{-1}$, respectively, then

$$\hat{p}_{k,j} = \frac{\hat{Q}_{k-1}(z_{n,j})}{\alpha_{n+1}\hat{Q}_n'(z_{n,j})} \quad \text{and} \quad \tilde{p}_{k,j} = \frac{\tilde{Q}_{j-1}(z_{n,k}) - \tilde{Q}_j(z_{n,k})}{\tilde{Q}_{n-1}(z_{n,k})}.$$

With the matrices $\mathbf{P}_n$ and $\mathbf{P}_n^{-1}$, we can write the eigenvalue problem associated with $\mathbf{H}_n$ as

$$\mathbf{P}_n^{-1}\mathbf{H}_n\mathbf{P}_n = \mathbf{\Xi}_n, \tag{2.11}$$

where $\mathbf{\Xi}_n$ is the diagonal matrix containing the eigenvalues $z_{n,1}, z_{n,2}, \ldots, z_{n,n}$.

From $\mathbf{P}_n\mathbf{P}_n^{-1} = \mathbf{I}_n$, we also obtain

$$\sum_{j=1}^{n} \hat{Q}_l(z_{n,j})[\tilde{Q}_k(z_{n,j}) - \tilde{Q}_{k+1}(z_{n,j})]\lambda_{n,j} = \delta_{l,k}$$

for $0 \leqslant l \leqslant n-1$ and $0 \leqslant k \leqslant n-1$. Here the $\lambda_{n,j}$ are as in (2.8). This is the biorthogonality of the two finite sequences $\{\hat{Q}_m\}_{m=0}^{n-1}$ and $\{\tilde{Q}_m - \tilde{Q}_{m+1}\}_{m=0}^{n-1}$ on the point set $z_{n,1}, z_{n,2}, \ldots, z_{n,n}$.

If we define the linear functional $\mathscr{L}_n[\cdot]$, on the space of Laurent polynomials, by

$$\mathscr{L}_n[f] = \sum_{j=1}^{n} \lambda_{n,j} f(z_{n,j}),$$

then $\mathscr{L}_n[\hat{Q}_l(\tilde{Q}_k - \tilde{Q}_{k+1})] = \delta_{l,k}$ for $0 \leqslant l \leqslant n-1$ and $0 \leqslant k \leqslant n-1$. Moreover, since the Laurent polynomials $\{\tilde{Q}_k - \tilde{Q}_{k+1}\}_{k=0}^{l-1}$ form a basis for the linear space spanned by $\{z^{-1}, z^{-2}, \ldots, z^{-l}\}$, we can also write for the polynomials Q_l, $l = 1, 2, \ldots, n$,

$$\mathscr{L}_n[z^{-l+k}Q_l(z)] = 0, \quad k = 0, 1, \ldots, l-1, \quad l \leqslant n,$$

an L-orthogonality property as in (1.3). Also note that $\mathscr{L}_n[z^{-l-1}Q_l(z)] \neq 0$ for $0 \leqslant l < n$.

We mention that Hendriksen and Njåstad [8] have established the biorthogonality of the two sequences $\{\hat{Q}_m\}$ and $\{\tilde{Q}_m - \tilde{Q}_{m+1}\}$ in relation to a moment functional $\mathscr{L}$, when $\{Q_m\}$ satisfy the L-orthogonality $\mathscr{L}\big(z^{-m+k}Q_m(z)\big) = 0$, $0 \leqslant k \leqslant$

$m-1$ and $\mathscr{L}(z^{-m-1}Q_m(z)) \neq 0$. For other biorthogonal relations associated with the recurrence relation (1.1) we cite [3] and [20].

3. Bounds for the zeros

First we define the region $W(\lambda_1, \lambda_2, \tau)$, where $\lambda_2 \geqslant \lambda_1$ and $\tau > 0$, by

$$\begin{aligned} &W(\lambda_1, \lambda_2, \tau) \\ &\quad \equiv \left\{ z = x + \mathrm{i}y \in \mathbb{C} : |y| \leqslant \tau, \lambda_1 - \sqrt{\tau^2 - y^2} \leqslant x \leqslant \lambda_2 + \sqrt{\tau^2 - y^2} \right\}. \end{aligned}$$

Now for $\delta \neq 0$, let $\mathbf{D}_\delta$ be the $n \times n$ diagonal matrix with diagonal entries $(\delta, \delta^2, \ldots, \delta^n)$. Then the eigenvalues of $\mathbf{H}_n$ are the same as that of the matrix $\mathbf{D}_\delta \mathbf{H}_n \mathbf{D}_\delta^{-1}$, given by

$$\begin{bmatrix} \eta_1 & \alpha_2\delta^{-1} & \cdots & 0 & 0 \\ \eta_1\delta & \eta_2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \eta_1\delta^{n-3} & \eta_2\delta^{n-4} & \cdots & \alpha_{n-1}\delta^{-1} & 0 \\ \eta_1\delta^{n-2} & \eta_2\delta^{n-3} & \cdots & \eta_{n-1} & \alpha_n\delta^{-1} \\ \eta_1\delta^{n-1} & \eta_2\delta^{n-2} & \cdots & \eta_{n-1}\delta & \eta_n \end{bmatrix}.$$

We can state the following theorem.

Theorem 3.1. *For any $n \geqslant 2$, set $\alpha_n^M = \max\{|\alpha_k|, k = 2, 3, \ldots, n\}$ and $\eta_n^M = \max\{|\eta_k|, k = 2, 3, \ldots, n\}$. Let*

$$d_1 = \frac{\sqrt{\alpha_n^M}}{\sqrt{\alpha_n^M} + \sqrt{\eta_n^M}}, \quad d_2 = \frac{\alpha_n^M}{\alpha_n^M + |\eta_1| - \eta_n^M} \quad \textit{and}$$
$$d_3 = \frac{2\alpha_n^M}{\sqrt{(\alpha_n^M)^2 + 4\alpha_n^M(|\eta_1| - \eta_n^M)} + \alpha_n^M}.$$

Then the zeros of Q_k, $1 \leqslant k \leqslant n$, are inside the disk $|z| \leqslant \hat{\rho}_n$, where

$$\hat{\rho}_n = \begin{cases} \frac{\alpha_n^M + |\eta_1| - \eta_n^M}{|\eta_1| - \eta_n^M} |\eta_1|, & \textit{if } 0 < d_2 < d_1, \\ (\sqrt{\alpha_n^M} + \sqrt{\eta_n^M})^2, & \textit{otherwise}. \end{cases} \tag{3.1}$$

If η_k, for $k = 1, 2, \ldots, n$, are all real then the zeros of Q_k, $1 \leqslant k \leqslant n$, are inside the region $W(\eta_{n,1}, \eta_{n,2}, \tilde{\rho}_n)$, where

$$\eta_{n,1} = \min\{\eta_k, k = 1, 2, \ldots, n\}, \quad \eta_{n,2} = \max\{\eta_k, k = 1, 2, \ldots, n\}$$

and

$$\tilde{\rho}_n = \begin{cases} \frac{\sqrt{(\alpha_n^M)^2+4\alpha_n^M(|\eta_1|-\eta_n^M)}\ +\alpha_n^M}{2(|\eta_1|-\eta_n^M)}|\eta_1|, & \textit{if } 0 < d_3 < d_1, \\ (\sqrt{\alpha_n^M}+\sqrt{\eta_n^M})^2 - \eta_n^M, & \textit{otherwise}. \end{cases}$$

Proof. We assume that $0 < \delta < 1$. Also recall that $\alpha_1 = 0$. Application of the Gerschgorin theorem gives that the eigenvalues of the matrix $\mathbf{D}_\delta \mathbf{H}_n \mathbf{D}_\delta^{-1}$, that is the matrix $\mathbf{H}_n$, are inside $\bigcup_{k=1}^n \Delta_k^{(n)}(\delta)$, where $\Delta_k^{(n)}(\delta)$ is the closed disk with center η_k and radius $\rho_k(\delta) = |\eta_k| \sum_{r=1}^{n-k} \delta^r + |\alpha_k|\delta^{-1}$. Clearly, $\rho_1(\delta) < \tilde{\rho}_1(\delta) = \frac{\delta}{1-\delta}|\eta_1|$ and for $2 \leqslant k \leqslant n$,

$$\begin{aligned}\rho_k(\delta) \leqslant \frac{\delta}{1-\delta}|\eta_k| + \frac{1}{\delta}|\alpha_k| \leqslant \tilde{\rho}_n(\delta) &= \frac{\delta}{1-\delta}\eta_n^M + \frac{1}{\delta}\alpha_n^M \\ &= -\eta_n^M + \frac{1}{1-\delta}\eta_n^M + \frac{1}{\delta}\alpha_n^M.\end{aligned}$$

Thus all $\Delta_k^{(n)}(\delta)$, $2 \leqslant k \leqslant n$, are inside the disk

$$|z| \leqslant \hat{\rho}_n(\delta) = \frac{1}{1-\delta}\eta_n^M + \frac{1}{\delta}\alpha_n^M$$

and, likewise, $\Delta_1^{(n)}(\delta)$ is inside the disk $|z| < \hat{\rho}_1(\delta) = \frac{1}{1-\delta}|\eta_1|$. We choose the value of δ that minimizes $\max\{\hat{\rho}_1(\delta), \hat{\rho}_n(\delta)\}$, which turns out to be d_2 or d_1 depending on $\eta_n^M \ll |\eta_1|$ or not. This gives the first part of the theorem.

To obtain the latter part of the theorem, we note that the disk $\Delta_1^{(n)}(\delta)$ is inside the disk $|z - \eta_1| < \tilde{\rho}_1(\delta)$ and for any k, $2 \leqslant k \leqslant n$, the disk $\Delta_k^{(n)}(\delta)$ is inside the disk $|z - \eta_k| \leqslant \tilde{\rho}_n(\delta)$. We choose the δ that minimizes $\max\{\tilde{\rho}_1(\delta), \tilde{\rho}_n(\delta)\}$, which turns out to be d_3 or d_1 depending on $\eta_n^M << |\eta_1|$ or not. Since the centers η_k of all the disks $\Delta_k^{(n)}(\delta)$ are on a straight line (in this case the real line), we obtain the last part of the theorem for the polynomial Q_n.

Finally, since the zero of $Q_1(z)$ is η_1, it is certainly inside $\bigcup_{k=1}^n \Delta_k^{(n)}(\delta)$. If p is such that $1 < p < n$ then the eigenvalues of $\mathbf{H}_p$ (i.e., the zeros of Q_p) are inside $\bigcup_{k=1}^p \Delta_k^{(p)}(\delta)$. Since $\Delta_k^{(p)}(\delta) \subset \Delta_k^{(n)}(\delta)$ for $1 \leqslant k \leqslant p$, we note that the zeros of Q_p are also inside $\bigcup_{k=1}^n \Delta_k^{(n)}(\delta)$. Hence the results of the theorem are valid also for all the polynomials Q_k, $1 \leqslant k < n$. □

We now state a theorem shown in Saff and Varga [14].

Theorem 3.2 (Saff and Varga). *Let* α_k, $k = 2, 3, \ldots, n$ *and* β_k, $k = 1, 2, 3, \ldots, n$ *be positive real numbers. Set*

$$\tau = \min\{-\eta_k = \beta_k - \alpha_k : k = 1, 2, \ldots, n\},$$

with $\alpha_1 = 0$. *Then if* $\tau > 0$, *the parabolic region*

$$\mathscr{P}^+(\tau) \equiv \{z = x + \mathrm{i}y \in \mathbb{C} : y^2 \leqslant 4\tau(\tau + x),\ x > -\tau\},$$

contains no zeros of the polynomials $Q_1, Q_2, \ldots, Q_n$.

The following theorem gives some information on the zeros of the polynomials Q_n, when all α_{n+1} are positive and all β_n are equal to some positive constant.

Theorem 3.3. *Let $\beta_k = \beta > 0$ for $k = 1, 2, \ldots, n$ and $\alpha_k > 0$ for $k = 2, \ldots, n$. Then the zeros of any Q_k, $1 \leqslant k \leqslant n$, are distinct (except for a possible double zero at $z = \beta$) and lie on $\mathscr{C}(\beta) \cup (0, \infty)$, where*

$$\mathscr{C}(\beta) \equiv \{z : z = \beta e^{i\theta}, 0 < \theta < 2\pi\}.$$

In particular, if $\{\frac{\alpha_{k+1}}{4\beta}\}_{k=1}^{n-1}$ is a positive chain sequence then all the zeros are distinct and lie on the open circle $\mathscr{C}(\beta)$.

Proof. A proof of this theorem when $\beta = 1$ is given in [4]. Let

$$x = x(\beta; z) = \frac{1}{2}(\beta^{-1/2} z^{1/2} + \beta^{1/2} z^{-1/2}).$$

Then the polynomials $P_k(x) = (4\beta z)^{-k/2} Q_k(z)$, $0 \leqslant k \leqslant n$, satisfy $P_0(x) = 1$, $P_1(x) = x$ and

$$P_{k+1}(x) = x P_k(x) - \frac{\alpha_{k+1}}{4\beta} P_{k-1}(x), \quad 1 \leqslant k \leqslant n-1. \tag{3.2}$$

From this it is well known that the zeros of P_k are real, distinct and lie symmetrically about the origin. If we write $P_{2k}(x) = \prod_{j=1}^{k}(x^2 - x_{2k,j}^2)$ and $P_{2k+1}(x) = x \prod_{j=1}^{k}(x^2 - x_{2k+1,j}^2)$ then from $Q_k(z) = (4\beta z)^{k/2} P_k(x(\beta; z))$ we see that

$$Q_{2k}(z) = \prod_{j=1}^{k}[(z - z_{2k,j})(z - \beta^2/z_{2k,j})],$$

$$Q_{2k+1}(z) = (z + \beta) \prod_{j=1}^{k}[(z - z_{2k+1,j})(z - \beta^2/z_{2k+1,j})],$$

where $z_{k,j}/\beta = (2x_{k,j}^2 - 1) + 2\sqrt{x_{k,j}^2(x_{k,j}^2 - 1)}$. Hence, if $x_{k,j}^2 < 1$ then $z_{k,j}$ and $\beta^2/z_{k,j}$ are a conjugate pair of zeros of Q_k on the circle $\mathscr{C}(\beta)$ and if $x_{k,j}^2 > 1$ then they are two positive zeros of Q_k inverse to each other about the point β. Hence, we conclude that the zeros of Q_k are either on the circle $\mathscr{C}(\beta)$ or on the positive real line. If $z = \beta$ is a zero of Q_k then it is a zero of multiplicity 2.

Now we assume $\{\frac{\alpha_{k+1}}{4\beta}\}_{k=1}^{n-1}$ to be a positive chain sequence. That is there exists a second sequence $\{g_k\}_{k=0}^{n-1}$, where $0 \leqslant g_0 < 1$ and $0 < g_k < 1$ for $1 \leqslant k \leqslant n-1$, such that $\frac{\alpha_{k+1}}{4\beta} = (1 - g_{k-1})g_k$, $1 \leqslant k \leqslant n-1$.

From (3.2), since $\frac{\alpha_{k+1}}{4\beta x^2} = \frac{P_k(x)}{x P_{k-1}(x)}\left(1 - \frac{P_{k+1}(x)}{x P_k(x)}\right)$, it is well known (see for example [6]) that all the zeros of P_k, $1 \leqslant k \leqslant n$, are inside $(-1, 1)$. Hence, in this case, all the zeros of Q_k, $1 \leqslant k \leqslant n$, are on the open circle $\mathscr{C}(\beta)$. $\square$

Considering special cases of this theorem we have

Corollary 3.3.1. *Let $\beta_k = \beta > 0$, $1 \leqslant k \leqslant n$. Then the following hold*:

1. *If $0 < \alpha_{k+1} \leqslant \beta$, $1 \leqslant k \leqslant n-1$, then all the zeros of Q_k, $1 \leqslant k \leqslant n$, are on $\mathscr{C}(\beta)$.*
2. *In particular, for $\epsilon > 0$, if $0 < \alpha_{k+1} \leqslant \beta - \epsilon$, $1 \leqslant k \leqslant n-1$, then the zeros of any Q_k, $1 \leqslant k \leqslant n$, are on the ark of $\mathscr{C}(\beta)$ that stays outside the parabolic region*
$$\mathscr{P}^+(\epsilon) \equiv \{z = x + \mathrm{i}y \in \mathbb{C} : y^2 \leqslant 4\epsilon(\epsilon + x), x > -\epsilon\}.$$
3. *Again if $0 < \alpha_{k+1} \leqslant \beta$, $1 \leqslant k \leqslant n-1$, let $\kappa_n = \max_{1\leqslant k \leqslant n-1} \sqrt{\alpha_{k+1}/\beta}$. Then all the zeros of Q_k, $1 \leqslant k \leqslant n$, are on the open ark*
$$\mathscr{C}(\beta, \theta_n) \equiv \{z : z = \beta \mathrm{e}^{\mathrm{i}\theta}, \theta_n < \theta < 2\pi - \theta_n\},$$
where $\theta_n = 2\arccos(\kappa_n \cos(\frac{\pi}{n+1}))$.
4. *For $\epsilon > 0$ suppose that $0 < \alpha_{k+1} \leqslant \beta + \epsilon$, $1 \leqslant k \leqslant n-1$. If Q_n has any zeros outside $\mathscr{C}(\beta)$ then these zeros lie inside the interval $(\beta^2/b, b)$, where $b = (\sqrt{\beta + \epsilon} + \sqrt{\epsilon})^2$.*

Proof. The first part of this corollary is an immediate consequence of Theorem 3.3. The second part of this corollary follows from Theorems 3.3 and 3.2.

To obtain the third part of this corollary, we consider the polynomials $P_k(x) = (4\beta z)^{-k/2} Q_k(z)$, where $x = x(\beta; z) = \frac{1}{2}(\beta^{-1/2} z^{1/2} + \beta^{1/2} z^{-1/2})$. As we have already pointed out these satisfy
$$P_{k+1}(x) = x P_k(x) - \frac{\alpha_{k+1}}{4\beta} P_{k-1}(x), \quad 1 \leqslant k \leqslant n-1,$$
with $P_0(x) = 1$ and $P_1(x) = x$. Hence from Ismail and Li [9–Theorems 2 and 3] the zeros of $P_k(x)$, $1 \leqslant k \leqslant n$ are inside the interval $(-\hat{x}, \hat{x})$, where $\hat{x} = \cos(\frac{\pi}{n+1}) \max\limits_{1\leqslant k \leqslant n-1} \sqrt{\alpha_{k+1}/\beta}$. Hence the application $Q_k(z) = (4\beta z)^{k/2} P_k(x(\beta; z))$ gives the required result of the corollary.

To obtain the last part, we note that all the zeros of $P_k(x) = (4\beta z)^{-k/2} Q_k(z)$ are inside the interval $\left(-\sqrt{\frac{\beta+\epsilon}{\beta}}, \sqrt{\frac{\beta+\epsilon}{\beta}}\right)$. Hence the application $Q_k(z) = (4\beta z)^{k/2} P_k(x(\beta; z))$ gives the required result of the corollary. $\square$

With a typical application of the eigenvalue representation (2.11) we can state the following theorem.

Theorem 3.4. *Given the three term recurrence relation (1.1) suppose that all the zeros $z_{n,j}$ of Q_n (eigenvalues of $\mathbf{H}_n$) are distinct and that the matrices $\mathbf{P}_n$, $\mathbf{P}_n^{-1}$ and $\mathbf{\Xi}_n$ are those given respectively by Eqs. (2.9), (2.10) and (2.11).*

Consider the perturbed three term recurrence relation
$$Q^\epsilon_{m+1}(z) = (z + \beta^\epsilon_{m+1}) Q^\epsilon_m(z) - \alpha^\epsilon_{m+1} z Q^\epsilon_{m-1}(z), \quad m = 1, 2, \ldots, n-1,$$

with $Q^{\epsilon}_0 = 1$ and $Q^{\epsilon}_1(z) = z + \beta^{\epsilon}_1$, where the complex numbers α^{ϵ}_m and β^{ϵ}_m are such that $|\alpha^{\epsilon}_m - \alpha_m| \leqslant \epsilon$, $m = 2, 3, \ldots, n$ and $|\beta^{\epsilon}_m - \beta_m| \leqslant \epsilon$, $m = 1, 2, 3, \ldots, n$. Here ϵ is a positive number.

Let $\mathbf{H}^{\epsilon}_n$ be the Hessenberg matrix associated with this three term recurrence relation, defined according to Theorem 2.1, whose eigenvalues are the zeros of the polynomial Q^{ϵ}_n. Then for any zero ω of Q^{ϵ}_n

$$\min_{1\leqslant j\leqslant n} |\omega - z_{n,j}| \leqslant \|\mathbf{H}^{\epsilon}_n - \mathbf{H}_n\| \|\mathbf{P}_n\| \|\mathbf{P}^{-1}_n\|,$$

where $\|.\|$ is any subordinate norm in $\mathbb{C}^n$.

Proof. If ω is equal to any of the zeros of Q_n then the result is trivial. Hence we assume that ω is not equal to any of the zeros of Q_n. Let $\mathbf{u}$ be the eigenvector of $\mathbf{H}^{\epsilon}_n$ associated with the eigenvalue ω. Then

$$\mathbf{H}_n\mathbf{u} - \omega\mathbf{u} = -(\mathbf{H}^{\epsilon}_n - \mathbf{H}_n)\mathbf{u}.$$

Using (2.11) this can be written as $\mathbf{P}_n(\mathbf{\Xi}_n - \omega\mathbf{I}_n)\mathbf{P}^{-1}_n\mathbf{u} = -(\mathbf{H}^{\epsilon}_n - \mathbf{H}_n)\mathbf{u}$, and hence

$$\mathbf{u} = -\mathbf{P}_n(\mathbf{\Xi}_n - \omega\mathbf{I}_n)^{-1}\mathbf{P}^{-1}_n(\mathbf{H}^{\epsilon}_n - \mathbf{H}_n)\mathbf{u}.$$

Taking any subordinate norm gives

$$\|(\mathbf{\Xi}_n - \omega\mathbf{I}_n)^{-1}\|^{-1} \leqslant \|\mathbf{H}^{\epsilon}_n - \mathbf{H}_n\| \|\mathbf{P}_n\| \|\mathbf{P}^{-1}_n\|.$$

Since the matrix $(\mathbf{\Xi}_n - \omega\mathbf{I}_n)^{-1}$ is diagonal the results of the theorem follows. □

It is easily verified that

$$\|\mathbf{H}^{\epsilon}_n - \mathbf{H}_n\|_\infty \leqslant (2n-1)\epsilon \quad \text{and} \quad \|\mathbf{H}^{\epsilon}_n - \mathbf{H}_n\|_1 \leqslant (2n-1)\epsilon. \tag{3.3}$$

Corollary 3.4.1. *Let $\beta > 0$ and consider the three term recurrence relation*

$$Q^{\epsilon}_{m+1}(z) = (z + \beta^{\epsilon}_{m+1})Q^{\epsilon}_m(z) - \alpha^{\epsilon}_{m+1} z Q^{\epsilon}_{m-1}(z), \quad m = 1, 2, \ldots, n-1,$$

with $Q^{\epsilon}_0 = 1$ and $Q^{\epsilon}_1(z) = z + \beta^{\epsilon}_1$, where the complex numbers α^{ϵ}_m and β^{ϵ}_m are such that $|\mathrm{Im}(\alpha^{\epsilon}_m)| \leqslant \epsilon$, $0 < \mathrm{Re}(\alpha^{\epsilon}_m) \leqslant \beta$ for $m = 2, 3, \ldots, n$ and $|\beta^{\epsilon}_m - \beta| \leqslant \epsilon$ for $m = 1, 2, 3, \ldots, n$. Here ϵ is a positive number.

Let $\mathbf{H}^{\epsilon}_n$ be the Hessenberg matrix associated with this three term recurrence relation, defined according to Theorem 2.1, whose eigenvalues are the zeros of the polynomial Q^{ϵ}_n.

Consider the three term recurrence relation

$$Q_{m+1}(z) = (z + \beta)Q_m(z) - \mathrm{Re}(\alpha^{\epsilon}_{m+1}) z Q_{m-1}(z), \quad m = 1, 2, \ldots, n-1,$$

with $Q_0 = 1$, $Q_1(z) = z + \beta$. For this three term recurrence relation let $\mathbf{H}_n$, $\mathbf{P}_n$ and $\mathbf{P}^{-1}_n$ be the associated matrices defined according to Theorem 2.1 and Eqs. (2.9) and (2.10).

Then all the zeros of Q_n^ϵ *are inside the ring*

$$\zeta(\beta - \|\mathbf{H}_n^\epsilon - \mathbf{H}_n\|\|\mathbf{P}_n\|\|\mathbf{P}_n^{-1}\|) \leqslant |z| \leqslant \beta + \|\mathbf{H}_n^\epsilon - \mathbf{H}_n\|\|\mathbf{P}_n\|\|\mathbf{P}_n^{-1}\|,$$

where the real function $\zeta(x)$ *equals to* x *if* $x > 0$ *and equals to* 0 *otherwise.*

Proof. The proof follows from the fact that the zeros of Q_n are distinct and lie on the circle $|z| = \beta$, which is a consequence of $0 < \mathrm{Re}(\alpha_m^\epsilon) \leqslant \beta$ for $m = 2, 3, \ldots, n$ (see Corollary 3.3.1). □

Remark 3.1. In addition to $|\beta_m^\epsilon - \beta| \leqslant \epsilon$, $m = 1, 2, 3, \ldots, n$, the condition $|\mathrm{Im}(\alpha_m^\epsilon)| \leqslant \epsilon$ for $m = 2, 3, \ldots, n$, also enables us to make use of the upper bound in (3.3) if the appropriate norms are chosen.

Now we consider the following special case of the three term recurrence relation for which the matrices $\mathbf{\Xi}_n$, $\mathbf{P}_n$ and $\mathbf{P}_n^{-1}$ can be explicitly found. With $\beta > 0$, let the polynomials $\{Q_m(\beta; z)\}_{m=0}^n$ be given by

$$Q_{m+1}(\beta; z) = (z + \beta)Q_m(\beta; z) - \beta z Q_{m-1}(\beta; z), \quad m = 1, 2, \ldots, n-1,$$

with $Q_0(\beta; z) = 1$, $Q_1(\beta; z) = z + \beta$. We denote the Hessenberg matrix associated with this recurrence relation (given according to Theorem 2.1) by $\mathbf{H}_n^{(\beta)}$.

We have $Q_k(\beta; z) = (z^{k+1} - \beta^{k+1})/(z - \beta)$, $k = 0, 1, \ldots, n$. The zeros of $Q_n(\beta; z)$ (i.e. the eigenvalues of $\mathbf{H}_n^{(\beta)}$) are $z_{n,j} = z_{n,j}^{(\beta)} = \beta e^{i\frac{2j\pi}{n+1}}$, $j = 1, 2, \ldots, n$. Consequently,

$$\hat{Q}_k(\beta; z_{n,j}) = \beta^{-k} Q_k(\beta; z_{n,j}) = \frac{\sin\left(\frac{(k+1)j\pi}{n+1}\right)}{\sin\left(\frac{j\pi}{n+1}\right)} e^{i\frac{kj\pi}{n+1}},$$

$$\tilde{Q}_k(\beta; z_{n,j}) = z_{n,j}^{-k} Q_k(\beta; z_{n,j}) = \frac{\sin\left(\frac{(k+1)j\pi}{n+1}\right)}{\sin\left(\frac{j\pi}{n+1}\right)} e^{-i\frac{kj\pi}{n+1}}.$$

Thus the elements $\hat{p}_{k,j}^{(\beta)}$ and $\tilde{p}_{k,j}^{(\beta)}$ in the kth row and jth column of the corresponding matrices $\mathbf{P}_n = \mathbf{P}_n^{(\beta)}$ and $(\mathbf{P}_n)^{-1} = (\mathbf{P}_n^{(\beta)})^{-1}$, respectively, are

$$\hat{p}_{k,j}^{(\beta)} = \frac{2i}{n+1} \sin\left(\frac{kj\pi}{n+1}\right) e^{-i\frac{(2n-k)j\pi}{n+1}} \quad \text{and} \quad \tilde{p}_{k,j}^{(\beta)} = e^{-i\frac{2(j+1)k\pi}{n+1}}.$$

Hence we can state

$$\frac{2i}{n+1} \sum_{j=1}^{n} \sin\left(\frac{kj\pi}{n+1}\right) e^{i\frac{(k-2l)j\pi}{n+1}} = \delta_{kl}, \quad 1 \leqslant k, l \leqslant n$$

and

Lemma 3.1

$$\|(\mathbf{P}_n^{(\beta)})^{-1}\|_1 = \|(\mathbf{P}_n^{(\beta)})^{-1}\|_\infty = n,$$
$$\|\mathbf{P}_n^{(\beta)}\|_1 = \|\mathbf{P}_n^{(\beta)}\|_\infty = \frac{2}{n+1}\sigma_n,$$

where $\sigma_n = \max_{1\leqslant k\leqslant n}\{\sum_{j=1}^{n} |\sin(\frac{kj\pi}{n+1})|\} \leqslant \sqrt{n(n+1)/2}$.

Proof. Clearly $\|(\mathbf{P}_n^{(\beta)})^{-1}\|_1 = \|(\mathbf{P}_n^{(\beta)})^{-1}\|_\infty = n$ and $\sigma_n = \max_{1\leqslant k\leqslant n}\sum_{j=1}^{n} |\sin(\frac{kj\pi}{n+1})|$. The bound for σ_n is obtained by application of the Cauchy–Schwartz inequality. $\square$

Remark 3.2. We believe that $\sigma_n = \sum_{j=1}^{n}\sin(\frac{j\pi}{n+1}) = \cot(\frac{\pi}{2n+2})$.

Corollary 3.4.2. *Let $\beta > 0$ and consider the three term recurrence relation*

$$Q_{m+1}^{\epsilon}(z) = (z+\beta_{m+1}^{\epsilon})Q_m^{\epsilon}(z) - \alpha_{m+1}^{\epsilon} z Q_{m-1}^{\epsilon}(z), \quad m = 1,2,\ldots,n-1,$$

with $Q_0^{\epsilon} = 1$ and $Q_1^{\epsilon}(z) = z + \beta_1^{\epsilon}$, where the complex numbers α_m^{ϵ} and β_m^{ϵ} are such that $|\alpha_m^{\epsilon} - \beta| \leqslant \epsilon$ for $m = 2,3,\ldots,n$ and $|\beta_m^{\epsilon} - \beta| \leqslant \epsilon$ for $m = 1,2,3,\ldots,n$. Here ϵ is a positive number.

Let $\mathbf{H}_n^{\epsilon}$ be the Hessenberg matrix associated with this three term recurrence relation, defined according to Theorem 2.1, whose eigenvalues are the zeros of the polynomial Q_n^{ϵ}.

Then all the zeros of Q_n^{ϵ} are inside the ring

$$\zeta\left(\beta - \tfrac{2n\sigma_n}{n+1}\|\mathbf{H}_n^{\epsilon} - \mathbf{H}_n^{(\beta)}\|\right) \leqslant |z| \leqslant \beta + \tfrac{2n\sigma_n}{n+1}\|\mathbf{H}_n^{\epsilon} - \mathbf{H}_n^{(\beta)}\|,$$

where the real function $\zeta(x)$ is equal to x if $x > 0$ and equal to 0 otherwise. The number σ_n is as in Lemma 3.1.

4. Szegö and para orthogonal polynomials

Let $\mathrm{d}\nu(z)$ be a positive measure on the unit circle. This means $\nu(\mathrm{e}^{\mathrm{i}\theta})$, defined on $0 \leqslant \theta \leqslant 2\pi$, is a real, bounded and non-decreasing function with infinitely many points of increase such that the moments

$$\mu_m = \int z^m \mathrm{d}\nu(z) = \int_0^{2\pi} \mathrm{e}^{\mathrm{i}m\theta}\mathrm{d}\nu(\mathrm{e}^{\mathrm{i}\theta}), \quad m = 0,1,2,\ldots,$$

all exit. We consider the Szegö polynomials $\{S_n\}$ associated with the measure $\mathrm{d}\nu(z)$ defined by

$$\int S_n(z)\overline{S_m(z)}\,\mathrm{d}\nu(z) = 0, \quad n \neq m.$$

These polynomials were introduced by Szegö (see for example [16]). See also [17] for interesting basic information on these polynomials.

The Szegö polynomials are known to satisfy the system of recurrence relations (given here in terms of monic polynomials)

$$\begin{aligned} S_{n+1}(z) &= zS_n(z) + a_{n+1}S_n^*(z), \\ \left(1 - |a_{n+1}|^2\right) zS_n(z) &= S_{n+1}(z) - a_{n+1}S_{n+1}^*(z), \end{aligned} \tag{4.1}$$

for $n \geqslant 0$. Here $S_n^*(z) = z^n \overline{S}_n(1/z)$ are the reciprocal polynomials. The numbers $a_n = S_n(0), n \geqslant 1$, are known as the reflection coefficients of the Szegö polynomials.

The reflection coefficients have the property $|a_n| < 1$ for $n \geqslant 1$. Furthermore, the zeros of the Szegö polynomials are all inside the open unit disk.

If the reflection coefficients satisfy $0 < |a_n| < 1$, then the Szegö polynomials also satisfy

$$S_{n+1}(z) = \left(z + \frac{a_{n+1}}{a_n}\right) S_n(z) - \frac{a_{n+1}}{a_n}(1 - |a_n|^2)zS_{n-1}(z), \quad n \geqslant 1, \tag{4.2}$$

where $S_1(z) = (z + \frac{a_1}{a_0})$, with $a_0 = 1$. Hence these Szegö polynomials satisfy a three term recurrence relation of the form (1.1) with $\beta_n = \frac{a_n}{a_{n-1}}$, $n \geqslant 1$ and $\alpha_{n+1} = \frac{a_{n+1}}{a_n}(1 - |a_n|^2)$, $n \geqslant 1$.

In [10], Jones et al. studied the polynomials $S_n(z) + \omega_n S_n^*(z)$, where $|\omega_n| = 1$, which they called para-orthogonal polynomials. We consider the two special cases of para-orthogonal polynomials

$$R_n^{(1)}(z) = \frac{S_n(z) + S_n^*(z)}{1 + S_n(0)} \quad \text{and} \quad (z-1)R_n^{(2)}(z) = \frac{S_{n+1}(z) - S_{n+1}^*(z)}{1 - S_{n+1}(0)}, \tag{4.3}$$

for $n \geqslant 0$. The denominators are chosen in order to make the polynomials monic.

Clearly, $2S_n(z) = (1 + a_n)R_n^{(1)}(z) + (1 - a_n)(z - 1)R_{n-1}^{(2)}(z)$, $n \geqslant 1$, which from (4.1) leads to (if $-1 < a_n < 1$)

$$2zS_{n-1}(z) = R_n^{(1)}(z) + (z - 1)R_{n-1}^{(2)}(z), \quad n \geqslant 1. \tag{4.4}$$

If $-1 < a_n < 1$ then the monic polynomials $R_n^{(i)}$, $(i = 1, 2)$, satisfy $R_0^{(i)} = 1$, $R_1^{(i)}(z) = z + 1$ and

$$R_{n+1}^{(i)}(z) = (z + 1)R_n^{(i)}(z) - \alpha_{n+1}^{(i)} zR_{n-1}^{(i)}(z), \quad n \geqslant 1, \tag{4.5}$$

with $\alpha_{n+1}^{(1)} = (1 + a_{n-1})(1 - a_n) > 0$ and $\alpha_{n+1}^{(2)} = (1 - a_n)(1 + a_{n+1}) > 0$, $n \geqslant 1$. This last result should be attributed to Delsarte and Genin [7]. Note that the reflection coefficients a_n are real if and only if the measure $\mathrm{d}\nu(z)$ satisfies the symmetry $\mathrm{d}\nu(1/z) = -\mathrm{d}\nu(z)$.

Hence, if $-1 < a_n < 1$, then the para orthogonal polynomials $R_n^{(i)}$ satisfy a recurrence relation of the form (1.1) with $\beta_n = 1$, $n \geqslant 1$ and $\alpha_{n+1} = \alpha_{n+1}^{(i)}$, $n \geqslant 1$.

The following theorem gives the relation between the Szegö polynomials $\{S_n\}$ and orthogonal polynomials $\{P_n^{(i)}\}$ obtained through $P_n^{(i)}(x(z)) = (4z)^{-n/2} R_n^{(i)}(z)$, where

$$x(z) = \frac{1}{2}(z^{1/2} + z^{-1/2}).$$

Theorem 4.1. (1) *Let* $\mathrm{d}\nu(z)$ *be a positive measure on the unit circle such that the associated Szegö polynomials* $\{S_n\}$ *are all real* (*i.e.* $-1 < S_n(0) = a_n < 1$ *for* $n \geqslant 1$). *Set*

$$\alpha_{n+1}^{(1)} = (1 + a_{n-1})(1 - a_n) > 0 \quad \textit{and}$$
$$\alpha_{n+1}^{(2)} = (1 - a_n)(1 + a_{n+1}) > 0, \quad n \geqslant 1.$$

Define the positive measures $\mathrm{d}\phi^{(1)}$ *and* $\mathrm{d}\phi^{(2)}$ *by*

$$\mathrm{d}\phi^{(1)}(x) = -\mathrm{d}\nu(z) \quad \textit{and} \quad \mathrm{d}\phi^{(2)}(x) = -(1 - x^2)\mathrm{d}\phi^{(1)}(x),$$

where $x = x(z)$. *The supports of* $\mathrm{d}\phi^{(1)}$ *and* $\mathrm{d}\phi^{(2)}$ *are inside* $[-1, 1]$. *Then* (*for* $i = 1, 2$) *the sequence of polynomials* $\{P_n^{(i)}\}$, *given by* $P_0^{(i)} = 1$, $P_1^{(i)}(z) = x$ *and*

$$P_{n+1}^{(i)}(z) = x P_n^{(i)}(z) - \frac{1}{4}\alpha_{n+1}^{(i)} P_{n-1}^{(i)}(z), \quad n \geqslant 1,$$

are the monic orthogonal polynomials with respect to the measure $\mathrm{d}\phi^{(i)}$.

(2) *Conversely, let* $\mathrm{d}\phi^{(1)}$ *and* $\mathrm{d}\phi^{(2)}$ *be two positive measures defined inside* $[-1, 1]$ *such that* $\mathrm{d}\phi^{(2)}(x) = (1 - x^2)\mathrm{d}\phi^{(1)}(x)$. *Let the respective monic orthogonal polynomials* $P_n^{(1)}$ *and* $P_n^{(2)}$ *associated with these measures satisfy* $P_{n+1}^{(i)}(x) = x P_n^{(i)}(x) - \frac{1}{4}\alpha_{n+1}^{(i)} P_{n-1}^{(i)}(x)$, $n \geqslant 1$. *Then the reflection coefficients* $a_n = S_n(0)$ *of the Szegö polynomials* S_n *associated with the positive measure* $\mathrm{d}\nu(z) = -\mathrm{d}\phi^{(1)}(x(z))$ *satisfy*

$$a_n = 1 - \alpha_{n+1}^{(1)}/(1 + a_{n-1}) \quad \textit{and} \quad a_{n+1} = -1 + \alpha_{n+1}^{(2)}/(1 - a_n), \quad n \geqslant 1,$$

with $a_0 = 1$. *Given explicitly* (*with* $\mu_0^{(i)}$ *moment of order zero associated with* $\mathrm{d}\phi^{(i)}$),

$$a_{2n-1} = 2\frac{\alpha_{2n-1}^{(2)}\alpha_{2n-3}^{(2)}\cdots\alpha_3^{(2)}\mu_0^{(2)}}{\alpha_{2n-1}^{(1)}\alpha_{2n-3}^{(1)}\cdots\alpha_3^{(1)}\mu_0^{(1)}} - 1 \quad \textit{and}$$
$$a_{2n} = 2\frac{\alpha_{2n}^{(2)}\alpha_{2n-2}^{(2)}\cdots\alpha_2^{(2)}}{\alpha_{2n}^{(1)}\alpha_{2n-2}^{(1)}\cdots\alpha_2^{(1)}} - 1, \quad n \geqslant 1.$$

Moreover,

$$2zS_{n-1}(z) = R_n^{(1)}(z) + (z - 1)R_{n-1}^{(2)}(z), \quad n \geqslant 1,$$

where $R_n^{(i)}(z) = (4z)^{n/2} P_n^{(i)}(x(z))$.

A proof of this theorem follows from results given in [4,19].

We now consider the three term recurrence relation (4.2). From Theorem 2.1 the zeros of S_n are the eigenvalues of the lower Hessenberg matrix

$$\mathbf{H}_n = \begin{bmatrix} \eta_1 & \alpha_2 & 0 & \cdots & 0 & 0 \\ \eta_1 & \eta_2 & \alpha_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \eta_1 & \eta_2 & \eta_3 & \cdots & \alpha_{n-1} & 0 \\ \eta_1 & \eta_2 & \eta_3 & \cdots & \eta_{n-1} & \alpha_n \\ \eta_1 & \eta_2 & \eta_3 & \cdots & \eta_{n-1} & \eta_n \end{bmatrix},$$

where $\eta_r = -a_r\bar{a}_{r-1}$, $r = 1, 2, \ldots, n$ and $\alpha_r = \frac{a_r}{a_{r-1}} - a_r\bar{a}_{r-1} = \frac{a_r}{a_{r-1}}[1 - |a_{r-1}|^2]$, $r = 2, \ldots, n$.

Thus, with $\eta_r = -a_r\bar{a}_{r-1}$, $r = 1, 2, \ldots, n$ and $\alpha_r = \frac{a_r}{a_{r-1}} - a_r\bar{a}_{r-1}$, $r = 2, \ldots, n$, we can use Theorem 3.1 to obtain bounds for the zeros of Szegö polynomials. However, the first part of Theorem 3.1 does not seem to improve on the fact that the zeros of the Szegö polynomials are known to be within the unit disk.

Now applying Theorem 3.2 (theorem of Saff and Varga) to the recurrence relation (4.2) we obtain

Theorem 4.2

(1) *Suppose that* $0 < a_k < 1$ *for* $k = 1, 2, \ldots, n$. *Let*

$$\tau = \min\{-\eta_k = a_k a_{k-1}, k = 1, 2, \ldots, n\}.$$

Then the parabolic region

$$\mathscr{P}^+(\tau) \equiv \{z = x + iy \in \mathbb{C} : y^2 \leqslant 4\tau(\tau + x),\ x > -\tau\},$$

contains no zeros of $S_1, S_2, \ldots, S_n$.

(2) *Suppose that* $0 < (-1)^k a_k < 1$ *for* $k = 1, 2, \ldots, n$. *Let*

$$\tau = \min\{-a_k a_{k-1}, k = 1, 2, \ldots, n\}.$$

Then the parabolic region

$$\mathscr{P}^-(\tau) \equiv \{z = x + iy \in \mathbb{C} : y^2 \leqslant 4\tau(\tau - x), x < \tau\},$$

contains no zeros of $S_1, S_2, \ldots, S_n$.

Proof. The first part of this theorem is a restatement of Theorem 3.2 for the recurrence relation (4.2). To obtain the second part, we only need to write down the recurrence relation (4.2) in term of the polynomials $\{(-1)^n S_n(-z)\}$ and then apply Theorem 3.2. □

It is known (see [10]) that the zeros of para orthogonal polynomials stay on the unit circle. We verify this for the two sequences of para orthogonal polynomial $R_n^{(1)}$ and $R_n^{(2)}$ defined by (4.3). Since we must assume $-1 < a_n < 1$, in the recurrence relations given by (4.5) for these para orthogonal polynomials the coefficients

$\{\frac{1}{4}\alpha^{(1)}_{n+1}\}$ and $\{\frac{1}{4}\alpha^{(2)}_{n+1}\}$ are positive chain sequences with the respective parameter sequences $\{g^{(1)}_n = \frac{1-a_n}{2}\}$ and $\{g^{(2)}_n = \frac{1+a_{n+1}}{2}\}$. That is,

$$(1 - g^{(1)}_{n-1})g^{(1)}_n = \tfrac{1}{4}\alpha^{(1)}_{n+1}, \quad (1 - g^{(2)}_{n-1})g^{(2)}_n = \tfrac{1}{4}\alpha^{(2)}_{n+1}, \tag{4.6}$$

for $n \geqslant 1$, with $g^{(1)}_0 = 0$ and $0 < g^{(2)}_0 = 1 - \alpha^{(1)}_2/4 = (1 + a_1)/2 < 1$. Hence from Theorem 3.3 the zeros of $R^{(1)}_n$ and $R^{(2)}_n$ are all distinct and lie on the open unit circle $\mathscr{C}(1) \equiv \{z : z = \mathrm{e}^{\mathrm{i}\theta}, 0 < \theta < 2\pi\}$.

5. Polynomials with non-zero coefficients

Let

$$P_n(z) = \sum_{k=0}^{n} b_k z^k = b_n \sum_{k=0}^{n} \tilde{b}_k z^k = b_n \tilde{P}_n(z),$$

where b_k, $k = 0, 1, \ldots, n$ are complex, different from zero and $\tilde{b_k} = b_k/b_n$ for $k = 0, 1, \ldots, n$. Let

$$t_k = \frac{b_{k-1}}{b_k} = \frac{\tilde{b}_{k-1}}{\tilde{b}_k}, \quad k = 1, 2, \ldots, n.$$

The following results are easily verified.

$$\frac{1}{\tilde{P}_n(z)}\left[z^{n-1} + \frac{b_{n-1}}{b_n}z^{n-2} + \cdots + \frac{b_2}{b_n}z + \frac{b_1}{b_n}\right] = \frac{1}{z + t_1} \;-\; \frac{t_1 z}{z + t_2} \;-\; \frac{t_2 z}{z + t_3} \;-\; \cdots \;-\; \frac{t_{n-1} z}{z + t_n} \tag{5.1}$$

and

$$\frac{1}{\tilde{P}_n(z)}\left[z^{n-1} + \frac{b_{n-2}}{b_{n-1}}z^{n-2} + \cdots + \frac{b_1}{b_{n-1}}z + \frac{b_0}{b_{n-1}}\right] = \frac{1}{z + t_n} \;-\; \frac{t_{n-1} z}{z + t_{n-1}} \;-\; \frac{t_{n-2} z}{z + t_{n-2}} \;-\; \cdots \;-\; \frac{t_1 z}{z + t_1}. \tag{5.2}$$

Hence based on the continued fraction (5.1), if one considers the sequence of polynomials $\{Q^{(1)}_k\}^n_{k=0}$ generated by $Q^{(1)}_0 = 1$ and $Q^{(1)}_1(z) = z + t_1$ and

$$Q^{(1)}_{k+1}(z) = (z + t_{k+1})Q^{(1)}_k(z) - t_k z Q^{(1)}_{k-1}(z), \quad k = 1, 2, \ldots, n-1,$$

then $Q^{(1)}_k$ is the denominator of the k^{th} convergent of this finite continued fraction and, in particular, $Q^{(1)}_n = \tilde{P}_n$.

When the coefficients b_k, $k = 0, 1, \ldots, n$ of the polynomial P_n are all positive, Saff and Varga [14] has shown how to explore the consequence of the above three term recurrence relation using the parabolic region Theorem 3.2. In [14], the authors go even further by exploring the denominators of the convergents of Padé approximants associated with powers series expansions.

Now on the basis of the continued fraction (5.2), we consider the polynomials $\{Q_k^{(2)}\}_{k=0}^n$ generated by $Q_0^{(2)} = 1$ and $Q_1^{(2)}(z) = z + t_n$ and

$$Q_{k+1}^{(2)}(z) = (z + t_{n-k})Q_k^{(2)}(z) - t_{n-k}zQ_{k-1}^{(2)}(z), \quad k = 1, 2, \ldots, n-1.$$

The polynomial $Q_k^{(2)}$ is the denominator of the k^{th} convergent of the continued fraction (5.2) and, to our interest, $Q_n^{(2)} = \tilde{P}_n$. Hence using the Theorem 2.1 to the above three term recurrence relation we obtain the following result.

Theorem 5.1. *The zeros of P_n are the eigenvalues of the lower Hessenberg matrix*

$$\mathbf{H}_n = \begin{bmatrix} -t_n & t_{n-1} & 0 & \cdots & 0 & 0 \\ -t_n & 0 & t_{n-2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -t_n & 0 & 0 & \cdots & t_2 & 0 \\ -t_n & 0 & 0 & \cdots & 0 & t_1 \\ -t_n & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

This result is of no surprise as the above matrix can be obtained by a similarity transformation from the companion matrix of the polynomial P_n.

Now an analysis in the same spirit of the proof of Theorem 3.1 with the above matrix gives the following

Theorem 5.2. *Let $t_n^M = \max\{|t_k|, k = 1, 2, \ldots, n-1\}$ and $\tilde{t}_n^M = \max\{|t_k^{-1}|, k = 2, 3, \ldots, n\}$. For $0 < \delta < 1$ consider the disks*

$$\Delta_{0,1}(\delta) \equiv \{z : |z + t_n| \leqslant \tilde{\rho}_1(\delta)\} \quad \textit{and} \quad \Delta_{0,2}(\delta) \equiv \{z : |z| \leqslant \tilde{\rho}_2(\delta)\},$$

where $\tilde{\rho}_1(\delta) = \delta(1-\delta)^{-1}|t_n|$ and $\tilde{\rho}_2(\delta) = \delta^{-1}t_n^M$. Then the zeros of the polynomial P_n are inside the region $\Delta_{0,1}(\delta) \cup \Delta_{0,2}(\delta)$.

In particular, the following hold:

(1) *With $\tau_1 = |t_n| + t_n^M$, the zeros of the polynomial P_n are inside the disk*

$$\Delta_1 \equiv \{z : |z| \leqslant \tau_1\}.$$

(2) *With $\tau_2 = \frac{|t_n| - t_n^M + \sqrt{\{|t_n| - t_n^M\}^2 + 8t_n^M|t_n|}}{3|t_n| + t_n^M - \sqrt{\{|t_n| - t_n^M\}^2 + 8t_n^M|t_n|}}|t_n|$, the zeros of the polynomial P_n are inside the disk*

$$\Delta_2 \equiv \{z : |z + t_n| \leqslant \tau_2\}$$

(3) *With $\tau_3 = \frac{1}{2}[\sqrt{(t_n^M)^2 + 4t_n^M|t_n|} + t_n^M]$, the zeros of the polynomial P_n are inside the union $\Delta_3 \equiv \Delta_{3,1} \cup \Delta_{3,2}$ of the disks*

$$\Delta_{3,1} \equiv \{z : |z + t_n| \leqslant \tau_3\} \quad \textit{and} \quad \Delta_{3,2} \equiv \{z : |z| \leqslant \tau_3\}.$$

(4) *With* $\tau_4 = [|t_1^{-1}| + \tilde{t}_n^M]^{-1}$, *the zeros of the polynomial* P_n *are outside the open disk*

$$\tilde{\Delta}_4 \equiv \{z : |z| < \tau_4\}.$$

Proof. From the matrix $\mathbf{H}_n$ in Theorem 5.1, the disks $\Delta_{0,1}(\delta)$ and $\Delta_{0,2}(\delta)$ are obtained by applying the Gerschgorin theorem to $\mathbf{D}_\delta \mathbf{H}_n \mathbf{D}_\delta^{-1}$. Now the result of part 1 is obtained if we choose δ such that $\tilde{\rho}_1(\delta) + |t_n| = \tilde{\rho}_2(\delta)$, of part 2 is obtained if δ is such that $\tilde{\rho}_1(\delta) = \tilde{\rho}_2(\delta) + |t_n|$ and of part 3 is obtained if δ is such that $\tilde{\rho}_1(\delta) = \tilde{\rho}_2(\delta)$.

Finally the result of part 4 is obtained just the same as part 1 by considering the polynomial $z^n P_n(1/z)$. □

Remark 5.1. The intersections of two or more of the regions given in each part of the above theorem provide smaller regions where all the zeros of the polynomial P_n are contained.

Remark 5.2. Condition 1 of the above theorem, which also follows from the maximum norm of $\mathbf{H}_n$, is somewhat better than the bound obtained by Kojima (see [12–Exercise 30.6]) if $|t_0| + |t_n| < 2|t_n^M|$.

A result of Ostrowski (see [13–Appendix A]) which gives information on how the zeros of a polynomial vary locally and quantitatively with respect to the coefficients can be stated as follows. Let $\tilde{P}_n(z) = \sum_{k=0}^n \tilde{b}_k z^k$ and $\tilde{Q}_n(z) = \sum_{k=0}^n \tilde{c}_k z^k$ be such that $\tilde{b}_n = \tilde{c}_n = 1$ and $|\tilde{b}_k - \tilde{c}_k| \leqslant \epsilon$ for $k = 0, 1, \ldots, n-1$. Then the zeros x_j of $\tilde{P}_n(z)$ and the zeros y_j of $\tilde{Q}_n(z)$ can be ordered in such a way that we have $|x_j - y_j| < Cn\epsilon^{1/n}$, where C is some positive number. See also the recent paper [5] which re-examines the result of Ostrowski in terms of the so called Bombieri norm.

Now the following theorem gives some information on the local variation of the zeros when the coefficients of a polynomial vary from a geometric progression.

Theorem 5.3. *Let* $|\beta| > 0$ *and given the polynomial* $P_n(z) = \sum_{k=0}^n b_k z^k$ *the ratios* $t_k = b_{k-1}/b_k$ *are such that*

$$|t_k - \beta| \leqslant \epsilon, \quad k = 1, 2, \ldots, n.$$

Here ϵ *is a positive number. Then all the zeros of* P_n *are inside the ring*

$$\zeta\left(|\beta| - \tfrac{4n\sigma_n}{n+1}\epsilon\right) \leqslant |z| \leqslant |\beta| + \tfrac{4n\sigma_n}{n+1}\epsilon, \tag{5.3}$$

where the real function $\zeta(x)$ *equals* x *if* $x > 0$ *and equals* 0 *otherwise. The number* σ_n *is as in Lemma* 3.1.

Proof. If $\beta > 0$ the proof of this theorem follows from Corollary 3.4.2, where in this case we must take

$$\mathbf{H}_n^{\epsilon} - \mathbf{H}_n^{(\beta)} = \begin{bmatrix} -t_n+\beta & t_{n-1}-\beta & 0 & \cdots & 0 & 0 \\ -t_n+\beta & 0 & t_{n-2}-\beta & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -t_n+\beta & 0 & 0 & \cdots & t_2-\beta & 0 \\ -t_n+\beta & 0 & 0 & \cdots & 0 & t_1-\beta \\ -t_n+\beta & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

and hence $\|\mathbf{H}_n^{\epsilon} - \mathbf{H}_n^{(\beta)}\|_{\infty} \leqslant 2\epsilon$.

To obtain the result when $\beta = |\beta|e^{i\theta}$, we work with the polynomial $\tilde{P}_n(z) = P_n(ze^{-i\theta})$. This completes the proof. □

Remark 5.3. If no perturbation is allowed on t_n then (5.3) can be replaced by

$$\zeta\left(|\beta| - \frac{2n\sigma_n}{n+1}\epsilon\right) \leqslant |z| \leqslant |\beta| + \frac{2n\sigma_n}{n+1}\epsilon.$$

Remark 5.4. From condition 1 of Theorem 5.2 the zeros of P_n are inside the circle $|z| \leqslant 2|\beta| + 2\epsilon$. Hence the bounds given by (5.3) are not good if n is large. However, it is a better result than what is obtained from conditions 1 and 4 of Theorem 5.2 in the sense that the upper and lower bounds approach the value $|\beta|$ as we shrink ϵ.

6. Some examples

Example 1. Here we simply look at the zeros of the polynomials Q_n by taking different values for α_n and β_n in the recurrence relation (1.1).

- Fig. 6.1a shows the zeros of Q_{20} when $\beta_n = 1$, $\alpha_{2n} = -0.05$ and $\alpha_{2n+1} = 0.05$ for $n \geqslant 1$. As indicated by Theorem 3.1 all the zeros are inside the region $W(\lambda_1, \lambda_2, \tau)$, where $\lambda_1 = -1.05$, $\lambda_2 = -0.95$ and $\tau \approx 0.50825757$.
- Fig. 6.1b shows the zeros of Q_{20} when $\beta_n = 1$, $\alpha_{2n} = 0.05$ and $\alpha_{2n+1} = -0.05$ for $n \geqslant 1$. The zeros are within the region $W(\lambda_1, \lambda_2, \tau)$, again with $\lambda_1 = -1.05$, $\lambda_2 = -0.95$ and $\tau \approx 0.50825757$.
- Fig. 6.1c shows the zeros of Q_{20} when $\beta_1 = \beta_2 = -0.1$, $\beta_{n+2} = 1$ and $\alpha_n = 0.05$ for $n \geqslant 1$. All the zeros are inside the region $W(\lambda_1, \lambda_2, \tau)$, where $\lambda_1 = -0.95$, $\lambda_2 = 0.15$ and $\tau \approx 0.4858899$.
- Fig. 6.1d gives the zeros of Q_{20} when $\beta_1 = \beta_2 = 0.1$, $\beta_{n+2} = -1$ and $\alpha_n = 0.05$ for $n \geqslant 1$. Once again, as given by Theorem 3.1, all the zeros are inside the region $W(\lambda_1, \lambda_2, \tau)$, where $\lambda_1 = -0.10$, $\lambda_2 = 1.05$ and $\tau \approx 0.50825757$.

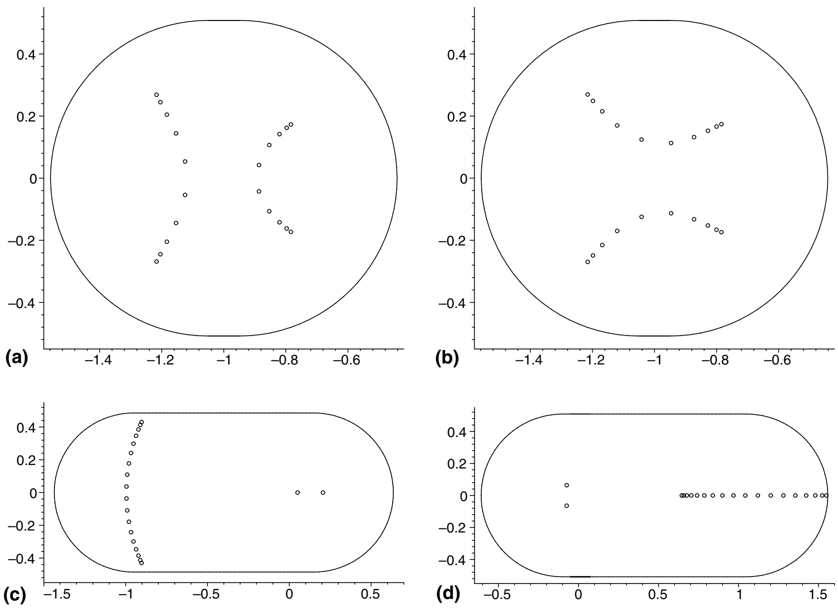

Fig. 6.1. Zeros of the polynomial Q_{20} generated from the recurrence relation (1.1) with four different combinations of the sequences $\{\alpha_n\}$ and $\{\beta_n\}$.

Example 2. Let $p > 0$ and $q > 0$ be such that $\sqrt{p} + \sqrt{q} \leqslant 1$. Consider the Szegö polynomials $S_n^{(p,q)}$ associated with the probability measure $\mathrm{d}\nu(z)$, where

$$\begin{aligned}
&\int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\nu(\mathrm{e}^{\mathrm{i}\theta}) \\
&\quad = \frac{B(p,q)}{A(p,q)} f(1) + \frac{1}{2\pi A(p,q)} \\
&\qquad \times \int_{\theta(b)}^{\theta(a)} f(\mathrm{e}^{\mathrm{i}\theta}) \frac{\sqrt{b^2 - \cos^2(\theta/2)}\sqrt{\cos^2(\theta/2) - a^2}}{|\sin(\theta)|} \mathrm{d}\theta \\
&\qquad + \frac{\Delta(q-p)}{A(p,q)} f(-1) + \frac{1}{2\pi A(p,q)} \\
&\qquad \times \int_{2\pi-\theta(a)}^{2\pi-\theta(b)} f(\mathrm{e}^{\mathrm{i}\theta}) \frac{\sqrt{b^2 - \cos^2(\theta/2)}\sqrt{\cos^2(\theta/2) - a^2}}{|\sin(\theta)|} \mathrm{d}\theta.
\end{aligned}$$

Here $\theta(x) = 2\arccos(x)$, $\Delta(x)$ is equal to 0 for $x \leqslant 0$ and equal to x for $x > 0$, $a = |\sqrt{p} - \sqrt{q}|$, $b = \sqrt{p} + \sqrt{q}$, $B(p,q) = \sqrt{1-b^2}\sqrt{1-a^2}$ and $A(p,q) = [1 - (p-q) + B(p,q)]/2$.

Note that if $q > p$ there is a jump in the measure at the point $z = -1$. Also note that if $\sqrt{p} + \sqrt{q} < 1$ then there is also a jump at $z = 1$. This jump vanishes when $\sqrt{p} + \sqrt{q} = 1$.

It was shown in [4], that the reflection coefficients satisfy

$$S_{2n-1}^{(p,q)}(0) = -B(p,q) - (p-q) \quad \text{and} \quad S_{2n}^{(p,q)}(0) = -B(p,q) + (p-q),$$

for $n \geqslant 1$.

Suppose that p and q are such that $(p-q) > B(p,q)$. Hence the reflection coefficients $a_n = S_n^{(p,q)}(0)$ satisfy $(-1)^n a_n > 0$, for $n \geqslant 1$. Thus we can apply Theorem 4.2 to study the zeros of $S_n(p,q)$.

The inequality $(p-q) > B(p,q)$ also means $p > q$ and thus the associated measure has no jump at the point $z = -1$.

In addition to satisfying $(p-q) > B(p,q)$, we also restrict p and q so that $\sqrt{p} - \sqrt{q} = a$ for a given a, where $0 < a < 1$. This means the measure is supported within the arc $\{z : z = \mathrm{e}^{\mathrm{i}\theta}, -2\arccos(a) \leqslant \theta \leqslant 2\arccos(a)\}$. It is easily verified that this choice for p and q limits their values in the range

$$\frac{1}{2}\left(a + \sqrt{1-a^2}\right) < \sqrt{p} \leqslant \frac{1}{2}(1+a) \quad \text{and} \quad 0 < \sqrt{p} - a = \sqrt{q}. \tag{6.1}$$

For example, if we take $a = 0.6$, then

$$0.7 < \sqrt{p} \leqslant 0.8 \quad \text{and} \quad \sqrt{q} = \sqrt{p} - 0.6.$$

For a particular choice of p and q in the range given by (6.1), we then obtain from Theorem 4.2 that the zeros of the Szegö polynomials $S_n^{(p,\,q)}$ for all $n \geqslant 1$ are outside the parabolic region

$$\mathscr{P}^-(\tau) \equiv \{z = x + \mathrm{i}y \in \mathbb{C} : y^2 \leqslant 4\tau(\tau - x), x < \tau\},$$

where $\tau = a^2b^2 - (1-a^2)(1-b^2)$, $a = \sqrt{p} - \sqrt{q}$ and $b = \sqrt{p} + \sqrt{q}$.

With $a = 0.6$, the choice $\sqrt{p} = 0.8$ and $\sqrt{q} = 0.2$ gives the Szegö polynomials $S_n^{(0.64,\,0.04)}$, where $S_{2n-1}^{(0.64,\,0.04)}(0) = -0.6$ and $S_{2n}^{(0.64,\,0.04)}(0) = 0.6$ for $n \geqslant 1$. Note that in this case $b = 1$ and therefore the measure does not have a jump at the point $z = 1$. The support of the measure is highlighted on the unit circle given in Fig. 6.2a.

The polynomials $S_n^{(0.64,\,0.04)}$, $n \geqslant 1$, must have all their zeros lie outside the parabolic region $\mathscr{P}^-(0.36)$. As shown in Fig. 6.2a, this is true for the polynomial $S_{10}^{(0.64,\,0.04)}$.

Again with $a = 0.6$, the choice $\sqrt{p} = 0.75$ and $\sqrt{q} = 0.15$ gives the Szegö polynomials $S_n^{(0.75^2,\,0.15^2)}$, where $S_{2n-1}^{(0.75^2,\,0.15^2)}(0) \approx -0.8887119$ and $S_{2n}^{(0.75^2,\,0.15^2)}(0) \approx 0.191288$ for $n \geqslant 1$. The support of the measure is highlighted on the unit circle given in Fig. 6.2b. The measure also has jump at the point $z = 1$.

Now the polynomials $S_n^{(0.75^2,\,0.15^2)}$, $n \geqslant 1$, must have all their zeros lie outside the parabolic region $\mathscr{P}^-(0.17)$. This is illustrated in Fig. 6.2b where we have plotted the zeros of $S_{10}^{(0.75^2,\,0.15^2)}$. We clarify that the zero of $S_{10}^{(0.75^2,\,0.15^2)}$ nearer to $z = 1$ is approximately equal to 0.999991502995.

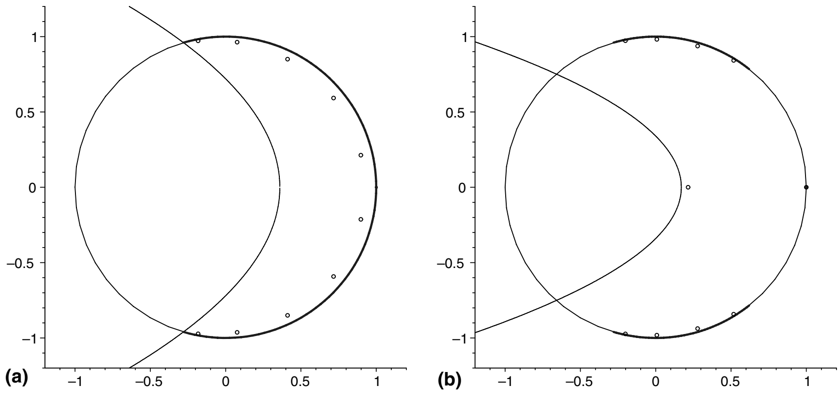

Fig. 6.2. (a) Zeros of $S_{10}^{(0.64,\,0.04)}$ are shown to be lying outside the parabolic region $\mathscr{P}^{-}(0.36)$. (b) Zeros of $S_{10}^{(0.75^2,\,0.15^2)}$ are shown to be lying outside the parabolic region $\mathscr{P}^{-}(0.17)$.

Example 3. Suppose that $a_n = q^n$, $n \geqslant 1$, where $0 \leqslant q < 1$, and we consider the associated Szegö polynomials. From (4.2), when $q > 0$ these Szegö polynomials (we denote by $S_n^{(q)}$) satisfy the three term recurrence relation

$$S_{n+1}^{(q)}(z) = (z+q)S_n^{(q)}(z) - q(1-q^{2n})zS_{n-1}^{(q)}(z), \quad n \geqslant 1,$$

with $S_1^{(q)}(z) = z + q$. In fact this recurrence is also valid for $q = 0$, when we obtain $S_{n+1}^{(0)}(z) = zS_n^{(0)}(z) = z^{n+1}$. The polynomials $S_n^{(0)}$ are the Chebyshev-Szegö polynomials, orthogonal in relation to the Lebesgue measure.

Consider the para orthogonal polynomials $R_n^{(1,q)} = R_n^{(1)}$ and $R_n^{(2,q)} = R_n^{(2)}$ defined by (4.3). These polynomials satisfy

$$\left.\begin{aligned} R_{n+1}^{(1,q)}(z) &= (z+1)R_n^{(1,q)}(z) - (1+q^{n-1})(1-q^n)zR_{n-1}^{(1,q)}(z),\\ R_{n+1}^{(2,q)}(z) &= (z+1)R_n^{(2,q)}(z) - (1-q^n)(1+q^{n+1})zR_{n-1}^{(2,q)}(z), \end{aligned}\right. \quad n \geqslant 1,$$

with $R_1^{(1,q)}(z) = R_1^{(2,q)}(z) = z + 1$. Hence from Theorem 4.1, the polynomials $P_n^{(1,q)}(x) = (4z)^{-n/2}R_n^{(1,q)}(z)$, $n \geqslant 1$, where $x = x(z) = \frac{1}{2}(z^{1/2} + z^{-1/2})$, satisfy

$$P_{n+1}^{(1,q)}(x) = xP_n^{(1,q)}(x) - \frac{1}{4}(1+q^{n-1})(1-q^n)P_{n-1}^{(1,q)}(x), \quad n \geqslant 1,$$

with $P_1^{(1,q)}(x) = x$. This recurrence relation shows that (see [1,2]) the polynomials $\{P_n^{(1,q)}\}$ are certain Al-Salam–Chihara polynomials. These polynomials are orthogonal in relation to the measure

$$\mathrm{d}\phi^{(1,q)}(x) = \frac{(q^2;q^2)_\infty}{\pi}\frac{1}{\sqrt{1-x^2}}h(x,q^{1/2})h(x,-q^{1/2})\mathrm{d}x,$$

defined on $[-1,1]$, where $h(x,c) = \prod_{k=0}^{\infty}[1 - 2cxq^k + c^2q^{2k}]$ and $(a;q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k)$. Hence, also from Theorem 4.1, we obtain that

$$\int_{\mathscr{C}} S_n^{(q)}(z)\overline{S_m^{(q)}(z)}\,\mathrm{d}\nu^{(q)}(z) = 0, \quad n \neq m,$$

where $\mathrm{d}\nu^{(q)}(\mathrm{e}^{\mathrm{i}\theta}) = \frac{(q^2;q^2)_\infty}{\pi}(q^{1/2}\mathrm{e}^{\mathrm{i}\theta/2}, q^{1/2}\mathrm{e}^{-\mathrm{i}\theta/2}; q)_\infty(-q^{1/2}\mathrm{e}^{\mathrm{i}\theta/2}, -q^{1/2}\mathrm{e}^{-\mathrm{i}\theta/2}; q)_\infty \mathrm{d}\theta$. We can also write

$$\mathrm{d}\nu^{(q)}(z) = \frac{(q^2;q^2)_\infty}{\pi}(qz;q^2)_\infty(qz^{-1};q^2)_\infty \frac{\mathrm{d}z}{2\mathrm{i}z}.$$

Note that $\{\frac{1}{4}(1+q^{n-1})(1-q^n)\}_{n=1}^{\infty}$ and $\{\frac{1}{4}(1-q^n)(1+q^{n+1})\}_{n=1}^{\infty}$ are positive chain sequences with respective parameter sequences $\{\frac{1}{2}(1-q^n)\}_{n=0}^{\infty}$ and $\{\frac{1}{2}(1+q^{n+1})\}_{n=0}^{\infty}$. Hence, for $j = 1, 2$, from the three term recurrence relation for $R_n^{(j,q)}$ and from Theorem 3.3, all the zeros of $R_n^{(j,q)}$ are on the open unit circle $\mathscr{C}(1) \equiv \{z : z = \mathrm{e}^{\mathrm{i}\theta}, 0 < \theta < 2\pi\}$.

Since $(1-q^{2n}) < (1+q^{n-1})(1-q^n)$ for $n \geqslant 1$, the sequence $\{\frac{1}{4}(1-q^{2n})\}_{n=1}^{\infty}$ is also a positive chain sequence. Hence, from the three term recurrence relation for $S_n^{(q)}$ and from Theorem 3.3, all the zeros of $S_n^{(q)}$ are on the open circle $\mathscr{C}(q) \equiv \{z : z = q\mathrm{e}^{\mathrm{i}\theta}, 0 < \theta < 2\pi\}$.

We can say more. In the three term recurrence relation for $S_n^{(q)}$ we have $q(1-q^{2n}) \leqslant q - q^{2m-1}$ for $1 \leqslant n \leqslant m-1$. Hence from the second part of Corollary 3.3.1, the zeros of any $S_n^{(q)}$, $1 \leqslant n \leqslant m$, must be on the arc of the circle $\mathscr{C}(q)$ which stays outside the parabolic region $\mathscr{P}^+(q^{2m-1})$.

Likewise, in the three term recurrence relation for $R_n^{(2,q)}$ we also have for example, $(1-q^n)(1+q^{n+1}) < (1-q^{2m-1})$ for $1 \leqslant n \leqslant m-1$. Hence from the second part of Corollary 3.3.1, the zeros of any $R_n^{(2,q)}$, $1 \leqslant n \leqslant m$, must be on the arc of the circle $\mathscr{C}(1)$ which stays outside the parabolic region $\mathscr{P}^+(q^{2m-1})$.

These results are confirmed in Fig. 6.3a and b where the zeros of $S_{10}^{(q)}$ and the zeros of $R_{10}^{(2,q)}$ are plotted for $q = 0.9$ and $q = 0.8$, respectively. The zeros of $R_{10}^{(2,q)}$, marked by small circles, are on the unit circle and the zeros of $S_{10}^{(q)}$, indicated by small boxes, are on a circle with radius q. These zeros are found to be outside the parabolic region $\mathscr{P}^+(q^{19})$.

We mention that the polynomials $\{S_n^{(\sqrt{q})}(z)\}$ are also a special case of the q-Appel Laurent biorthogonal polynomials considered in [18].

Example 4. We consider the polynomial $P_n(z) = \sum_{k=0}^{n} b_k z^k$, where $|b_k| = 2^k$, $k = 0, 1, \ldots, n$. For example, according to part 1 of Theorem 5.2 all the zeros of P_n must be within the disk $\Delta_1 \equiv \{|z| \leqslant 1\}$. Clearly this is true if $b_k = 2^k$, $k = 0, 1, \ldots, n$, as the zeros of P_n are all on the circle $|z| = 1/2$.

In Fig. 6.4a and b we plot the zeros of P_n for two other choices of $\{b_k\}$ satisfying the condition $|b_k| = 2^k$ and compare with the results given by Theorem 5.2. In these figures the Larger circle is the boundary of the region Δ_2, the intermediary circle is

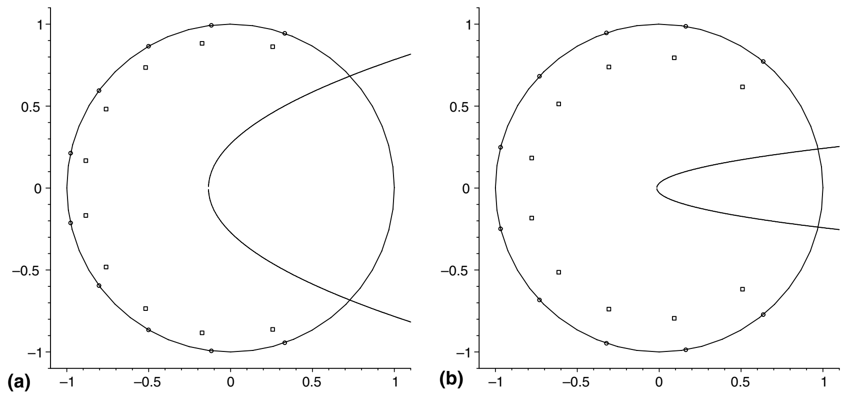

Fig. 6.3. The zeros of $S_{10}^{(q)}$ (indicated by the boxes) and the zeros of $R_{10}^{(2,q)}$ (indicated by the circles), when (a) $q = 0.9$ and (b) $q = 0.8$.

the boundary of the region Δ_1 and the smaller circle is the boundary of the region $\tilde{\Delta}_4 \equiv \{|z| < 0.25\}$.

In Fig. 6.4a the zeros of P_{20} are given when $b_k = 2^k$, $k = 2, \ldots, 18$ and $b_0 = -i$, $b_1 = i * 2$, $b_{19} = i * 2^{19}$ and $b_{20} = -i * 2^{20}$. The zeros are found to be within the intersection of all 4 of the regions given by Theorem 5.2.

In Fig. 6.4b the zeros of P_{20} are given when $b_k = 2^k$, $k = 1, \ldots, 19$, $b_0 = -1$ and $b_{20} = -2^{20}$. Again the zeros of P_{20} are within the intersection of all the regions given by Theorem 5.2. We clarify that the zero closer to the boundary of the disk Δ_1 has the approximate value 0.99999857 and the zero closer to the boundary of the disk $\tilde{\Delta}_4$ has the approximate value 0.25000036. These indicate that the results (part 1 and part 4) of Theorem 5.2 are fairly sharp.

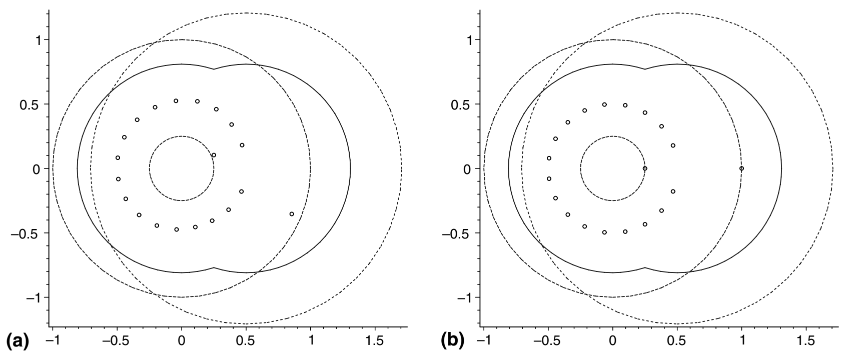

Fig. 6.4. The zeros of P_{20} when $|b_k| = 2^k$, $k = 0, 1, \ldots, 20$.

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