New thought experiment to test the generalized second law of thermodynamics

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We propose an extension of the original thought experiment proposed by Geroch, which sparked much of the actual debate and interest on black hole thermodynamics, and show that the generalized second law of thermodynamics is in compliance with it.

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In 1970 Geroch [1] raised the possibility of violating the ordinary second law of thermodynamics with help of classical black holes. The idea was to bring *slowly* from infinity a box with proper energy E_b over the event horizon and throw it eventually inside the hole. The cycle would be closed by lifting back the ideal rope. Because static asymptotic observers would ascribe zero energy to the box at the event horizon, the hole would remain the same after engulfing the box. This would challenge the ordinary second law of thermodynamics, since eventually all entropy associated with the box would be vanished from the Universe with no entropy increase counterpart.

As an objection to Geroch's process, Bekenstein argued that quantum mechanics would constraint the size and energy of the box accordingly. This would prevent the box from reaching the event horizon as a whole and, thus, the black hole would necessarily gain mass after engulfing the box. Then, Bekenstein [2] conjectured that black holes would have a nonzero entropy $S_\mathrm{bh} = kc^3 A/(4\hbar G)$ proportional to the event horizon area A and formulated the *Generalized Second Law* (GSL), namely, that the total entropy of a closed system (including that one associated with black holes) would never decrease. Now, because the GSL would be violated when the box entropy satisfied $S > 2\pi k E_\mathrm{b} R/(\hbar c)$, where R is the proper radius of the smallest sphere which circumscribes the box (see Ref. [3] for a comprehensive discussion), Bekenstein conjectured in addition the existence of a new thermodynamical law, namely, that every system should have an entropy-to-energy ratio satisfying $S/E_\mathrm{b} \le 2\pi k R/(\hbar c)$.

Notwithstanding, in 1982 Unruh and Wald showed [4] that by taking into account the buoyancy force induced by the Hawking radiation [5], as a comprehensive semiclassical gravity analysis would demand (notice that S_bh depends on G, c and $\hbar$), the GSL would *not* be violated irrespective of the imposition of the constraint $S/E_\mathrm{b} \le 2\pi k R/(\hbar c)$. The thermal ambiance outside the hole would prevent the box from descending beyond the point after which the energy delivered to the black hole would be too small to guarantee $\delta S_\mathrm{bh} \ge S$ as demanded by the GSL.

Unruh and Wald's resolution depends crucially on the precise point where the box finds its hydrostatic equilibrium: were it *lower*, the GSL would be violated. This circumstance led us to analyze an extension of the Geroch process in which the box is given some angular momentum before it enters the hole. In this case, one can decompose the force on the box into four distinct components. The first two ones correspond to the gravitational and buoyancy forces, which are already present when the box is static outside the hole. The remaining ones correspond to the centrifugal force and to an extra one, denominated here *kinetic gravitational force* (see Ref. [6]), which effectively increases the gravitational force on the box. Close enough to the black hole, i.e., $r < 3GM/c^2$, the kinetic gravitational force surpasses the centrifugal one [7], and the equilibrium point is lower than when the box is at rest. Thus, to rescue the GSL we must rely on the box's kinetic energy, which is the single new ingredient added to the original Geroch process. Indeed, we show here that, the kinetic energy given to the box increases enough its total energy to compensate the reduction of the potential energy caused by the lowering of the equilibrium point. In this way, the energy given to the hole is enough to guarantee $\delta S_\mathrm{bh} \ge S$. The precise increase of the total entropy in this process is displayed. We use natural units $c = \hbar = G = k = 1$ throughout the rest of the paper.

Let us describe our static black hole by the line element $ds^2 = -\chi^2 dt^2 + \chi^{-2} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$, where $\chi = \sqrt{1 - 2M/r}$ is the gravitational redshift factor. The hole which is assumed to be in thermal equilibrium with Hawking radiation can be thought as being enclosed in a large container made of adiathermal walls [8].

Our thermodynamical analysis will be carried out by Killing observers at rest with the thermal radiation which is treated as a perfect fluid (see Refs. [9–11] and references therein for a recent discussion). This is characterized by the stress-energy tensor $T^{\mu\nu} = (e + p)u^\mu u^\nu + p g^{\mu\nu}$, where $e = e(r)$ and $p = p(r)$ are the proper energy density and pressure, respectively, and $u^\mu = \chi^\mu/\chi$ is the corresponding 4-velocity with $\chi^\mu = (\partial_t)^\mu$. The associated proper acceleration $a_\mathrm{s} = \sqrt{a_\mathrm{s}^\alpha a_\alpha^\mathrm{s}}$ (with $a_\mathrm{s}^\alpha = u^\nu \nabla_\nu u^\alpha$) can be written as $a_\mathrm{s} = M/\chi r^2$. From the condition $\nabla_\mu T^{\mu\nu} = 0$, we obtain $\nabla^\mu p + (e + p) a_\mathrm{s}^\mu = 0$, which leads to

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$$ed\chi/dl + d(\chi p)/dl = 0, \tag{1}$$

where $l(r) \equiv \int_{2M}^{r} dr'/\sqrt{1 - 2M/r'}$.

For the sake of simplicity, we assume that our box is rectangular, has proper volume V and is *thin*, i.e. $\delta l d\chi/dl \ll \chi$ everywhere in the box, where δl is the box's proper height. This condition will be not only physically desirable as a way to minimize *turbulence* and *shear* effects but also technically convenient as will be seen further.

The process which we consider here is as follows. Firstly the box is lowered slowly from infinity by some agent towards the black hole up to the point where $r \equiv r_{\rm uw}$, in which place it finds its hydrostatic equilibrium. As shown by Unruh and Wald [4], $r_{\rm uw}$ is the solution of the equation $E_{\rm b} = Ve$. In this step some work $W_{\rm uw} > 0$ is gained by the asymptotic static agent. Now, he/she spends some energy to put the box in uniform circular motion with angular velocity $\omega_0 = d\phi/dt = \text{const}$ (at $\theta = \pi/2$). We argue further that this can be done without significantly disturbing the background radiation. The energy spent by the agent in this part of the process is denoted by K_1, where $K_1 < 0$ in our convention. Clearly, the hydrostatic equilibrium point changes as the consequence of the motion (see Fig. 1). In the process of bringing the box to its new equilibrium point the asymptotic agent gains some extra work $W > 0$, where we assume here that the angular momentum J is kept constant. Next, we suppose that the box is released and allowed to fall into the black hole, which is supposed to remain in equilibrium with its thermal atmosphere [10]. Any entropy increase in the dropping process will be disregarded here because we are interested in analyzing the most challenging situation for the GSL in the context of rotating boxes, i.e. the one where the final total entropy is the least. This is also the reason why we release the box at the equilibrium point, since this is where the minimum amount of energy is delivered to the black hole. At the end, the angular momentum and energy delivered to the hole are $\delta L = J$ and

$$\delta M = E_{\rm b} - W_{\rm net}, \tag{2}$$

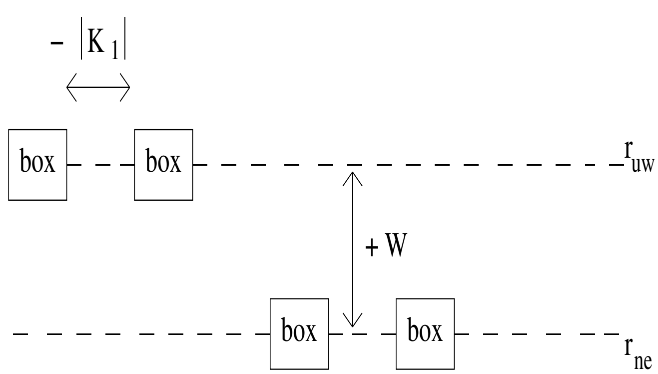

FIG. 1. A sketch of our thought experiment is depicted above. Note that for $r_{\rm uw} < 3M$, we have $r_{\rm ne} < r_{\rm uw}$.

respectively, where $W_{\rm net} = W_{\rm uw} + K_1 + W$ is the net work gained by the asymptotic agent.

Because we assume $E_{\rm b} \ll M$, we only consider first-order terms in the expression for the black hole entropy increase

$$\delta S_{\rm bh} = \frac{\delta M}{T_{\rm bh}} = \frac{E_{\rm b} - W_{\rm uw}}{T_{\rm bh}} + \frac{|K_1| - W}{T_{\rm bh}}, \tag{3}$$

where $T_{\rm bh} = 1/(8\pi M)$ (see Refs. [12,13]). This was obtained by differentiating $S_{\rm bh} = S_{\rm bh}(E, L) = 2\pi E^2(1 + \sqrt{1 - L^2/E^4})$ around $E = M$ and $L = 0$ and using Eq. (2). Now assuming the most challenging case, where the box is filled with thermal radiation, it can be shown that $(E_{\rm b} - W_{\rm uw})/T_{\rm bh} = S_{\rm b}$ [4] (see also Refs. [9,14]). In our process with the moving box, the change in the generalized total entropy is, thus,

$$\delta S_{\rm g} = \delta S_{\rm bh} - S_{\rm b} = (|K_1| - W)/T_{\rm bh}. \tag{4}$$

As a result, the GSL will be satisfied depending whether or not $|K_1| - W \geq 0$. In order to decide on it, we have to analyze more carefully the subprocess, where the box gains the kinetic energy K_1 and moves towards its new equilibrium point ($r = r_{\rm ne}$) along which the asymptotic agent gains the work W. (Naturally, if the box is not put in motion, $|K_1| = W = 0$ and we recover Unruh and Wald's result.) At this point, we would like to make two remarks about the process of setting the box in motion. First, we do not want that the moving box disturbs much the thermal atmosphere because the associated entropy increase would be difficult to compute. This should be partly achieved by using *thin* boxes or by considering a set of boxes rather than a single one. They would be lowered from infinity to $r_{\rm uw}$ and fitted one with the other forming a closed ring around the black hole. This would eliminate front and rear particle shocks with the box walls, which would disrupt the energy distribution (and entropy) of the thermal bath. Particle shocks with the up and down walls (which would still exist) are not source of concernings, since they do not lead to energy or momentum transfer. In this paper, the assumption of a single thin box will suffice. After all, the existence of other sources of entropy increase would only help to render the GSL valid. Now, the use of thin boxes is also useful to solve our second concerning. In order to keep the box uncorrupted during the initial acceleration interval, we impose that the 4-velocity v^μ of the box's points satisfy the *no expansion condition*: $\Theta \equiv \nabla_\mu v^\mu = 0$. This can be realized by choosing $v^\mu(x^\alpha) = [\chi^\mu + \omega(x^\alpha)\phi^\mu]/|\chi^\mu + \omega(x^\alpha)\phi^\mu)|$ with $\phi^\mu = (\partial_\phi)^\mu$ and $\omega(x^\alpha) = \chi^2 t/r^2\phi \leq \omega_0$ for $0 \leq t/\phi \leq \omega_0 r^2/\chi^2$, where $0 < \omega_0 < \chi/r$. This is necessary but not sufficient to guarantee the validity of the rigid body condition $\sigma_{\mu\nu} + (\Theta/3)h_{\mu\nu} = 0$, i.e., that the proper distance among the box's points are kept the same, where $h_{\mu\nu} \equiv g_{\mu\nu} + v_\mu v_\nu$ and $\sigma_{\mu\nu} \equiv h_\mu^\alpha h_\nu^\beta \nabla_{(\alpha} v_{\beta)} - (\Theta/3)h_{\mu\nu}$ is the shear tensor. Happily, however, the use of our thin box assumption leads to an approximate verification of the rigid body condition (see Ref. [6] for a comprehensive discussion). (The thinner the box, the more the rigid body equation is satisfied.) Finally, we also stress that as the box reaches its uniform circular motion the rigid body condition is fully verified and no distortion appears at all.

In order to compute Eq. (4), we begin recalling that in the uniform motion regime, $t/\phi > \omega_0 r^2/\chi^2$, the box's points have 4-velocity $v^\mu = \eta^\mu/\eta$, where $\eta^\mu = \chi^\mu + \omega_0 \phi^\mu$ with $\eta = \sqrt{\chi^2 - r^2\omega_0^2}$ and proper acceleration $a_{\rm m} = \sqrt{a_{\rm m}^\alpha a_\alpha^{\rm m}}$ (with $a_{\rm m}^\alpha = v^\nu \nabla_\nu v^\alpha$), which can be rewritten as $a_{\rm m} = \eta^{-1} d\eta/dl$. The box's angular momentum and kinetic energy at $r = r_{\rm uw}$ as defined asymptotically are

$$J \equiv E_{\rm b} v^\mu \phi_\mu|_{r=r_{\rm uw}} = E_{\rm b}\omega_0 r^2/\eta|_{r=r_{\rm uw}} \quad (5)$$

and

$$K_1 \equiv E_{\rm b}[\chi^\mu(u_\mu - v_\mu)]_{r=r_{\rm uw}} = -E_{\rm b}|\chi(1-\chi/\eta)|_{r=r_{\rm uw}}. \quad (6)$$

The local force on the box is

$$F_{\rm loc} \equiv E_{\rm b} a_{\rm m} \quad (7)$$

$$= \frac{ME_{\rm b}}{\chi r^2} - \frac{\chi J^2}{E_{\rm b} r^3} + \frac{MJ^2}{\chi E_{\rm b} r^4}. \quad (8)$$

The first two terms in the right-hand side should be identified with the gravitational force on the box when it lies at rest and with the centrifugal force, respectively. The last term (which involves M as well as J) is what we have called kinetic gravitational force. By using Eq. (5), one can verify that for $r < 3M$ the kinetic gravitational force is larger than (the absolute value of) the centrifugal force. As a result, for $r < 3M$ the *new* equilibrium point for the *moving* box will be closer to the black hole, i.e., $r_{\rm ne} < r_{\rm uw}$ (see Fig. 1).

In order to obtain $r_{\rm ne}$, we must calculate the buoyancy force on the moving box. The proper hydrostatic pressures on the top ($r = r_\top$) and at the bottom ($r = r_\perp$) of the box are $P_{\top/\perp} \equiv T_{\mu\nu} n^\mu_{\top/\perp} n^\nu_{\top/\perp} = p(r_{\top/\perp})$, where $n^\mu_{\top/\perp} = \chi(r_{\top/\perp})(\partial_r)^\mu$ are unit vectors orthogonal to the box's 4-velocity. Consequently, the hydrostatic scalar forces on the top and at the bottom of the box are $F_\top = -Ap_\top$ and $F_\perp = Ap_\perp$, respectively, where A is the corresponding proper area. In order to obtain the buoyancy force, we transmit both $F_\top$ and $F_\perp$ to the point $\mathcal{O}$, where the local force (8) is calculated. Let us assume that the forces are transmitted through *ideal* cables and rods characterized by the stress-energy tensor $\mathcal{T}^{\mu\nu} = P_{\rm c/r} h^{\mu\nu}$ satisfying $\nabla_\mu \mathcal{T}^{\mu\nu} = 0$, where $P_{\rm c/r}$ stands for pressure. Thus, from $F_{\top/\perp}$ we obtain the transmitted forces $F^{\mathcal{O}}_{\top/\perp}$ at $\mathcal{O}$ as $F^{\mathcal{O}}_{\top/\perp} = [\eta(r_{\top/\perp})/\eta(r_{\mathcal{O}})]F_{\top/\perp}$. The buoyancy force is, then, written as

$$F^{\mathcal{O}}_{\rm buo} = F^{\mathcal{O}}_\top + F^{\mathcal{O}}_\perp = \frac{V}{\eta}\frac{d(\eta p)}{dl}|_{r=r_{\mathcal{O}}}, \quad (9)$$

where we have used our thin box assumption, namely, that $\delta l d(\eta p)/dl \ll \eta p$ everywhere in the box so that we can neglect higher derivative terms in Eq. (9).

Now, by adding up Eqs. (7) and (9) we obtain the total local force on the box as

$$F^{\mathcal{O}}_{\rm tot} = V\left[\frac{\rho_{\rm b}}{\eta}\frac{d\eta}{dl} + \frac{1}{\eta}\frac{d(\eta p)}{dl}\right]_{r=r_{\mathcal{O}}}, \quad (10)$$

where $\rho_{\rm b} = E_{\rm b}/V$. Now, we must note that the corresponding total local 4-force points along $(\partial_r)^\mu$, and so it also lies in the spacelike section of the static observers. As a result, the static observers ascribe the same force $F^{\mathcal{O}}_{\rm tot}$ acting on the box. Hence, the force which the asymptotic agent must apply to sustain the box is $F^\infty_{\rm tot} = \chi(r_{\mathcal{O}})F^{\mathcal{O}}_{\rm tot}$, which can be recast in the form [see Eq. (1)]

$$F^\infty_{\rm tot} = V\left[\frac{M(\rho_{\rm b} - e)}{r^2} + \frac{J^2(\rho_{\rm b} + p)}{E_{\rm b}^2 r^3}\left(\frac{3M}{r} - 1\right)\right]_{r=r_{\mathcal{O}}}. \quad (11)$$

Clearly in the limit where $J \to 0$, this expression is equal to Unruh and Wald's result

$$F^\infty_{\rm uw} = V(\rho_{\rm b} - e)\chi a_{\rm s} \quad (12)$$

(see Refs. [4,15]). Note that if $r_{\rm uw} = r_{\mathcal{O}} < 3M$, then $F^\infty_{\rm tot} > 0$ (where we recall that $F^\infty_{\rm uw} = 0$) and the box is pulled downwards. The new equilibrium point at $r = r_{\rm ne}$ is obtained as the solution of $F^\infty_{\rm tot}(r_{\rm ne}) = 0$, i.e.

$$[M(\rho_{\rm b} - e)\eta^2(r_{\rm uw})r^2 + \omega_0^2(\rho_{\rm b} + p)r^4_{\rm uw}(3M - r)]_{r=r_{\rm ne}} = 0, \quad (13)$$

where the radial dependence of $p = p(e) = p[e(r)]$ is required.

The work W gained by the asymptotic agent as the box is lowered from $r = r_{\rm uw}$ to $r = r_{\rm ne}$ is

$$W = -\int_{r_{\rm uw}}^{r_{\rm ne}} F^\infty_{\rm tot} dr/\chi, \quad (14)$$

where $F^\infty_{\rm tot}$ is given in Eq. (11).

Before using Eqs. (6) and (14) in Eq. (4) to calculate explicitly $\delta S_{\rm g}$, let us use first a shortcut to show that $\delta S_{\rm g} > 0$. For this purpose, let us add two extra steps in our original cycle as follows. Rather than throwing the box to the black hole at $r = r_{\rm ne}$, we (i) stop the box and (ii) bring it back to $r = r_{\rm uw}$ (see Fig. 2). In the process of stopping it, the asymptotic agent gains an energy

$$K_{\rm ne} = E_{\rm b}|\chi(1 - \chi/\eta)|_{r=r_{\rm ne}} \quad (15)$$

and in the process of pulling it back from $r_{\rm ne}$ to $r_{\rm uw}$, he/she also gains an extra energy

$$W_{\rm ne} = -\int_{r_{\rm ne}}^{r_{\rm uw}} F^\infty_{\rm uw} dr/\chi, \quad (16)$$

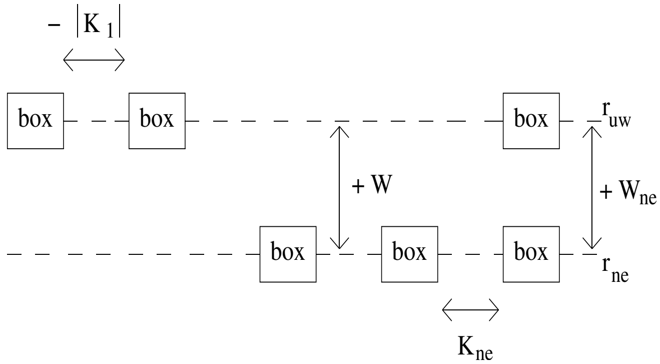

FIG. 2. A sketch of our auxiliary closed cycle is depicted above.

where F_{uw}^{∞} is given in Eq. (12). By assuming that the closed cycle which brings the box from r_{uw} to r_{ne} and back to r_{uw} is conservative, we must have $K_1 + W + K_{ne} + W_{ne} = 0$, i.e. $|K_1| - W = K_{ne} + W_{ne}$. Then from Eq. (4), we obtain

$$\delta S_g = (K_{ne} + W_{ne})/T_{bh} > 0. \tag{17}$$

This guarantees that the box's energy increase of kinetic origin $|K_1|$ is enough to compensate the energy decrease of gravitational origin W [see Eq. (4)], saving the GSL.

Now, we proceed to calculate explicitly δS_g. For this purpose, we assume (for simplicity) that the Hawking radiation and the box only contain a single free massless bosonic field, say, photons, in which case $p = e(r)/3$ with $e = (\pi^2/15)T^4$ and $T = T_{bh}/\chi$ is the Tolman's relation [16]. In this case

$$r_{uw} = \frac{2M}{1 - \sqrt{(\pi^2 T_{bh}^4)/(15\rho_b)}}, \tag{18}$$

where we impose $\rho_b > 9\pi^2 T_{bh}^4/15$ to guarantee that $2M < r_{uw} < 3M$ [and we recall that r_{ne} is given in Eq. (13)]. Finally, we are in position to evaluate numerically δS_g. As a check, we use independently Eqs. (4) and (18). The results are plotted in Fig. 3.

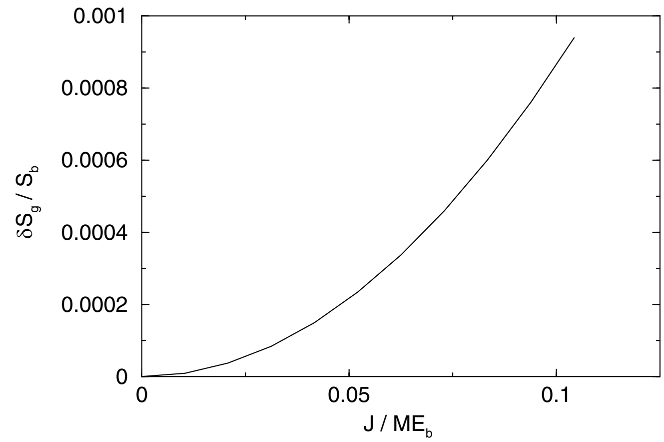

FIG. 3. Here we plot the generalized total entropy increase δS_g as a function of the box angular momentum J.

The existence of Hawking radiation has allowed us to ascribe temperature to black holes. This in addition with the laws of black hole mechanics led us to associate entropy to these objects. However, in order to treat black holes as legitimate thermodynamical systems it is necessary to conjecture the GSL. Since it is not possible to develop direct tests for the GSL, the best we can do is to verify its validity through thought experiments devised in contexts, where our well known theories can be safely used. In these vein, we have offered here a new thought experiment and shown that the GSL complies with it.

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[1] R. Geroch, Colloquium at Princeton University, 1971 (unpublished).
[2] J. D. Bekenstein, Phys. Rev. D **7**, 2333 (1973).
[3] R. Bousso, Rev. Mod. Phys. **74**, 825 (2002).
[4] W. G. Unruh and R. M. Wald, Phys. Rev. D **25**, 942 (1982).
[5] S. W. Hawking, Nature (London) **248**, 30 (1974); S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).
[6] G. E. A. Matsas, Phys. Rev. D **68**, 027701 (2003).
[7] M. A. Abramowicz and J. P. Lasota, Acta Phys. Pol. B **5**, 327 (1974).
[8] S. W. Hawking, Phys. Rev. D **13**, 191 (1976); P. C. W. Davies, Rep. Prog. Phys. **41**, 1313 (1978).
[9] R. M. Wald, Living Rev. Relativity **4**, 2001-6 (2001).
[10] S. Gao and R. M. Wald, Phys. Rev. D **64**, 084020 (2001).
[11] É. É. Flanagan, D. Marolf, and R. M. Wald, Phys. Rev. D **62**, 084035 (2000).
[12] R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).
[13] R. M. Wald, *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics* (University of Chicago Press, Chicago, 1994).
[14] W. G. Unruh and R. M. Wald, Phys. Rev. D **27**, 2271 (1983).
[15] M. A. Pelath and R. M. Wald, Phys. Rev. D **60**, 104009 (1999).
[16] R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Oxford University, Oxford, England, 1934).