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**Hexagonalization in AdS/CFT :
Classical Limit in AdS_5/CFT_4 and Mirror
Corrections in AdS_3/CFT_2**

Matheus Augusto Fabri

Orientador

Pedro Vieira

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“(...) A Constituição certamente não é perfeita. Ela própria o confessa ao admitir a reforma. Quanto a ela, discordar, sim. Divergir, sim. Descumprir, jamais. Afrontá-la, nunca.

Traidor da Constituição é traidor da Pátria. Conhecemos o caminho maldito. Rasgar a Constituição, trancar as portas do Parlamento, garrotear a liberdade, mandar os patriotas para a cadeia, o exílio e o cemitério.

Quando após tantos anos de lutas e sacrifícios promulgamos o Estatuto do Homem da Liberdade e da Democracia bradamos por imposição de sua honra. Temos ódio à ditadura. Ódio e nojo.

Amaldiçoamos a tirania aonde quer que ela desgrace homens e nações. Principalmente na América Latina.”

Dr. Ulysses Guimarães, presidente da Assembleia Nacional Constituinte.

Resumo

Um dos principais usos da dualidade AdS/CFT é o cálculo não perturbativo da dinâmica da teoria de campos conforme (CFT) na fronteira. Isto é possível nos casos integráveis, como por exemplo $AdS_3 \times S^3 \times T^4$ ou o caso mais explorado $AdS_5 \times S^5$. Neste contexto desenvolveu-se um formalismo não perturbativo denominado hexagonalização para o cálculo das constantes de estruturas da CFT dual no limite planar. Os objetos centrais nesta formulação são os chamados hexágonos, os quais podem ser derivados de primeiros princípios neste formalismo para então se calcular as constantes de estruturas à acoplamento finito. O presente trabalho aborda o formalismo dos hexágonos nas dualidades mencionadas anteriormente e este se divide em duas partes. Na Parte 1 analisamos as constantes de estruturas a acoplamento fraco em $\mathcal{N} = 4$ Yang-Mills supersimétrica. Aqui focaremos em dois setores da teoria gerados somente por operadores escalares ou com spin, respectivamente. Nós encontramos novas representações para os hexágonos tal que ambos os setores estão em pé de igualdade e por meio de um cuidadoso uso destas derivamos o limite clássico de funções de correlação nestes setores. Nossos resultados estão de acordo com cálculos da literatura realizados por meio de métodos indiretos. Já para a Parte 2 focaremos em hexágonos em $AdS_3 \times S^3 \times T^4$. Um importante problema em aberto nesta dualidade é o cálculo da dinâmica da CFT na teoria dual. Para este fim nós estendemos a proposta de hexagonalização dada em [B. Eden, D. I. Plat, and A. Sfondrini, J. High Energy Phys. 08 (2021) 049]. Neste trabalho completamos esta proposta definindo as correções *mirror* que permitem a descrição de constantes de estruturas para um número finito de contrações. Além disso também provamos que operadores protegidos nesta teoria não recebem estas correções. Por fim encerramos descrevendo como utilizar hexagonalização para calcular funções de correlação com quatro operadores nesta dualidade.

Palavras Chaves: AdS-CFT Correspondence; Integrability; Conformal Field Theory; Bethe Ansatz.

Áreas do conhecimento: Física matemática; Teoria quântica de campos; Supersimetria.

Abstract

One of the main uses of the AdS/CFT duality is the non-perturbative computation of the conformal field theory (CFT) data on the boundary. This is possible on integrable backgrounds like for example the $AdS_3 \times S^3 \times T^4$ or the most well known example of $AdS_5 \times S^5$. In these contexts it was developed a non-perturbative formalism called hexagonalization to compute structure constants of the dual CFT in the planar limit. The main objects in this framework are the so-called hexagons which can be bootstrapped within this formalism to then calculate the structure constants at finite coupling. This work concerns this hexagonalization framework in the aforementioned backgrounds and it is divided into two parts. In Part 1 we analyze structure constants at weak coupling in $\mathcal{N} = 4$ supersymmetric Yang-Mills. We focus on two sectors of the theory spanned only by scalar or spinning operators, respectively. We find new representations of the hexagons that put both sectors on equal footing and by judicious use of these new expressions we derive the classical limit of correlation functions in these sectors. Our results match previous computations done in the literature through more indirect means. Now Part 2 concerns hexagons in $AdS_3 \times S^3 \times T^4$. A big open problem in this holographic duality is to compute the CFT data of the dual theory. For this end we extend the hexagonalization proposal made in [B. Eden, D. I. Plat, and A. Sfondrini, J. High Energy Phys. 08 (2021) 049] for this background. In this work we complete this picture by computing the so-called mirror corrections that allow the description of structure constants for finite bridge lengths and as a byproduct we prove that the protected operators in the theory do not receive these corrections. We then close by describing how to use hexagonalization to compute four-point functions in this background.

Keywords: AdS-CFT Correspondence; Integrability; Conformal Field Theory; Bethe Ansatz.

Fields of knowledge: Mathematical physics; Quantum field theory; Supersymmetry.

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Chapter 1

Introduction

One of most fruitful ideas of the last decades in theoretical physics was the large- N expansion proposed by Gerardus t'Hooft in [1]. In this work it was proposed the organization of perturbation theory in matrix models, like $SU(N)$ gauge theories for instance, by the topology of the Feynman diagrams instead of the Yang-Mills coupling g_{YM} dependence as usual. By properly defining the t'Hooft coupling $\lambda = Ng_{YM}^2$ one sees that each graph is then accompanied by a N^{2-2g} factor with g being the genus of the Feynman graph. Then in the large- N limit the usual weak coupling expansion is done in λ and the graphs organized in a $1/N$ series with planar contributions ($g = 0$), torus contributions ($g = 1$), and so on. Nonetheless this latter series looks like a genus expansion of string theory in terms of the string coupling g_s . One could heuristically say that the Feynman diagrams are the triangulation of the string worldsheet and that somehow the $1/N$ expansion captures the genus expansion in g_s in string theory with $g_s \sim 1/N$. The concrete realization of this idea was given in the seminal work [2] by Juan Maldacena and in the follow up by E. Witten in [3], where it was developed the *holographic principle*.

The holographic principle, also called gauge-gravity or AdS/CFT duality, basically tell us that a specific quantum gravity theory on a d -dimensional anti-de Sitter space (AdS_d) is dual to a conformal quantum field theory on $(d - 1)$ -dimensional spacetime (CFT_{d-1}). By these theories being dual we mean that there is a dictionary mapping dynamical quantities in the AdS_d bulk to the conformal boundary, like for example energies of states in AdS_d to scaling dimensions of the dual CFT_{d-1} . We represent the holographic duality in Figure 1.1. We can hardly overestimate the impact of this principle in theoretical high-energy physics. For instance it yields to us a non-perturbative definition of quantum gravity on anti-de Sitter spacetimes in terms of (relatively) mundane CFTs on flat space. However this principle remains at the moment a conjecture, meaning that at the moment its underlying mechanics are not well understood even though there are many tests of its validity.

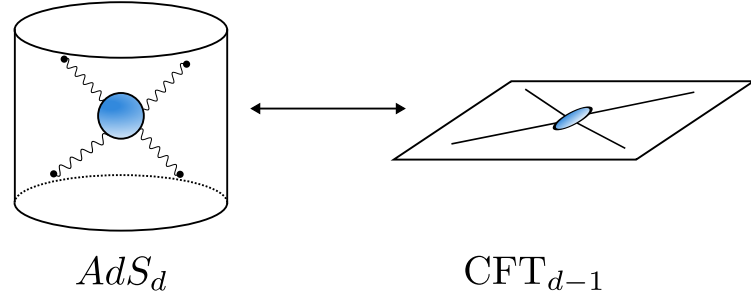


Figure 1.1: Holographic duality. Here we show an example of how some calculation on AdS/CFT would work. We represent the AdS_d space on the left by a cylinder and some calculation on the bulk gets translated to a calculation on a CFT_{d-1} calculation on the right. One can think of the CFT_{d-1} as living on the conformal boundary of AdS_d thus the name holographic, since all d -dimensional quantum gravity information would be coded into $d - 1$ dimensions.

In this work we will focus on the use of specific instances of this duality to compute quantities of the dual CFT and remarkably in the examples we will discuss this can be done at *finite coupling*. Let us detail these examples. The most well known instance of gauge-gravity duality is:

The Type IIB superstring theory in $AdS_5 \times S^5$, which can be seen as the near horizon limit of a system with N D3-branes, is dual to $\mathcal{N} = 4$ super Yang-Mills (SYM) in a flat four-dimensional spacetime with gauge group $SU(N)$. The string tension T , the t'Hooft coupling λ and the AdS_5 radius R are related by

$$TR^2 = \sqrt{\lambda}.$$

The quantum gravity theory in this case is a type of string theory on the product space $AdS_5 \times S^5$ and as G. t'Hooft told us $g_s \sim 1/N$, with N being the number of colors in the gauge theory¹. Thus the large- N limit picks only the sphere contributions in string perturbation theory which are related to the planar diagrams ($g = 0$) in the dual CFT_4 , then large- N limit is called planar limit.

The dual CFT_4 in this case is a non-abelian Yang-Mills theory with $\mathcal{N} = 4$ supersymmetry with all matter fields in the adjoint representation of the gauge group, that is, they are in the same supermultiplet of the gauge fields. Also the R-charge group is simply $SU(4)$ with the states organizing in representations of this group. The R-charge is related to the isometry group of the compact S^5 . This CFT at first looks as complicated as any usual quantum field theory (QFT)

¹For a review on the string and CFT sides of the duality and the duality itself see [4, 5, 6, 7].

since even at the planar limit one has to employ the usual perturbation theory to compute observables. This is where we can resort to the AdS/CFT duality. Note that it is a strong-weak coupling duality, which means that weak coupling in the CFT_4 side ($\lambda \rightarrow 0$) is related to strong coupling computations in the string side ($T \rightarrow 0$ in units of AdS_5 radius) and vice versa. Therefore to analyze complicated computations at strong coupling in one side we look at the easier weak coupling analysis in the other side of the duality.

Another duality that has been of great interest in the last couple of years and it will also be central for us is the following [2]:

The Type IIB superstring theory in $AdS_3 \times S^3 \times T^4$ with Neveu–Schwarz (NSNS) flux κ and Ramond–Ramond (RR) flux h , which can be seen as the near horizon limit of a F1-NS5-D1-D5 brane system, is dual to a two-dimensional $\mathcal{N} = (4,4)$ superconformal CFT.

The reader probably noted that in this case we are not as precise we were in the AdS_5/CFT_4 duality. The reason is that for most of the moduli space of the string theory, the dual CFT_2 is not known. Note that the moduli space is actually higher-dimensional for this duality, however only the fluxes h and κ are relevant for the spectrum [8]. The recent interest for this duality is due to the fact that for $\kappa = 1$ and $h = 0$ one can actually prove the duality [9, 10, 11], and with this get hints of how it works and maybe analyze the higher-dimensional cases which remain unproven.

Even though we do not now exactly the dual CFT_2 we have some information of it. First it has $\mathcal{N} = (4,4)$ supersymmetry with $SU(2)_L \otimes SU(2)_R$ R-charge, which as before it is the isometry of the compact S^3 . For $\kappa = 0$ we are in the pure RR limit and here the duality behaves much like the $AdS_5 \times S^5$ case with h being the analog of the t'Hooft coupling. However the CFT_2 dual here is completely unknown with some proposals given in [12, 13]. It turns out that for the pure NSNS limit with $h = 0$ and $\kappa = 1$ the dual CFT_2 is known and it is simply the $Sym_N(T^4)$ [10]. This theory is defined by taking N copies of free bosons and fermions and then identifying these by the permutation group S_N . This identification work as the gauge symmetry in $\mathcal{N} = 4$ SYM and N is the analog of the number of colors in the gauge group, thus as before the planar limit is the large- N limit². Some

²However the notion of planarity works a little different here and we will describe in the Conclusion (Chapter 6) how a computation of some observable works and the planarity count for $Sym_N(T^4)$.

proposal for the dual CFT₂ for $\kappa \geq 2$ and $h = 0$ are given in [14, 15], however for mixed flux (generic h and κ) this landscape remains uncharted.

In this work we will focus on the two dualities discussed above, however these are not the only examples known³. Another well known example is the AdS_4/CFT_3 duality. Here we have type IIA string on $AdS_4 \times CP^3$ dual to three-dimensional $\mathcal{N} = 6$ superconformal Chern-Simons with gauge group $U(N) \times U(N)$ with levels k and $-k$, respectively [17]. Now that we have these dualities what we want to do with them? It can be used in two ways. First as a non-perturbative definition of quantum gravity in AdS spacetimes. Then we compute dynamical quantities in the CFT _{d} and map them to gravity properties in the bulk. Another way to use it is to do computations on the string side of the duality and extract information of the CFT _{d} . It is the latter procedure we use here, that is, using AdS/CFT duality we try to “solve” the dual conformal field theory.

Let us define what we mean by solving a CFT. From the conformal bootstrap program it is well known that the entire dynamical content of a CFT is coded into the so-called CFT data [18]. For instance, to build representations of the conformal algebra we consider primary operators $\mathcal{O}_j$ which are just the highest weight states in this algebra. Thus the CFT data is given by $\{\Delta_j, C_{ijk}\}$ where Δ_j is the scaling dimension of a primary and C_{ijk} are the structure constants of three primaries, *id est*, constants that say how three operators interact. Then to solve the CFT we must compute these data, the idea then is to use the holographic duality to extract them. However usual computations in the string side are challenging since string theory on AdS spacetimes are not well understood. Although some advances were recently made in [19], computations of desired quantities remain for the moment out of reach. But there is one technique that comes to our rescue: *quantum integrability*.

Integrable models or exactly solved models are a class of nontrivial systems that have an exact solution, which means that they can be solved without resorting to perturbation theory. This concept seems kind of vague, however in classical mechanics it can be precisely defined. Indeed, if we have a system defined in a $2n$ -dimensional phase space and it possess n linearly independent conserved charges we can construct the action-angle variables and then integrate the equations of motion, thus solving it [20, 21]. This is the so-called Liouville integrability. Some examples of classical integrable models are the Euler, Lagrange and Kowalevski tops, the Kepler problem and the Neumann model.

³For a list with known examples AdS/CFT dualities see [16].

This concept of integrability can be extended to the realm of quantum mechanics. Here we do not have a precise meaning of it as in the classical case⁴, however if a system possess sufficient conserved charges it can be exactly solved. Therefore solvability is tied to the existence of conserved charges beyond the usual ones like energy and momentum. For quantum mechanical systems, being integrable means that we can extract the eigenvalues and eigenvectors of the hamiltonian and other quantities of interest, like correlation functions for instance. Some examples of integrable models are XXX, XXZ and XYZ one-dimensional Heisenberg spin chains, Hubbard model, one-dimensional Bose Gas, and so on. These are solved using a technique called quantum inverse scattering method which leads to the famous algebraic Bethe ansatz (ABA) and through it we can extract the physical quantities of interest. For some reviews in quantum integrable models see [23, 24].

Another realm we can apply these techniques is statistical mechanics. Here integrability works remarkably similar to the quantum mechanics case and one can use it to compute for example the partition function and from it extract all the thermodynamic content of the desired model. This was essential to the understanding of critical phenomena (see the reviews [25, 26]). Some examples of integrable systems in statistical mechanics are two-dimensional Ising model and six- and eight-vertex models. Here we see a common feature of all these integrable models in quantum and statistical mechanics, they are restricted to $1 + 1$ dimensions.

We can see the origin of their restriction to two-dimensional models through the example of integrable quantum field theories (IQFTs). This class of QFTs have the special property of possessing factorized S-matrices and an infinite number of conserved charges, they first appeared in the seminal work [27] by Zamolodchikov and Zamolodchikov. Consider for example a model with some global symmetry, like $O(N)$ for instance. Then we can use it to restrict scattering to $O(N)$ -invariant channels and that is it, we can not go beyond that. We thus use the usual QFT methods to compute the scattering matrix. Turns out that some models, like the $O(N)$ -invariant QFT in two dimensions, have an infinite tower of conserved charges that do not commute with the Lorentz group, that is, they are momentum dependent. We can use these to constrain the S-matrix and they imply no particle production [28]. Also the S-matrix is factorized, which means that $3 \rightarrow 3$ scattering decomposes into a sequence of $2 \rightarrow 2$ events [29]. This is the famous Yang-Baxter equation, which is a cubic constrain for the $2 \rightarrow 2$ S-matrix and by using it we

⁴See for example the discussion in the first two chapters of [22].

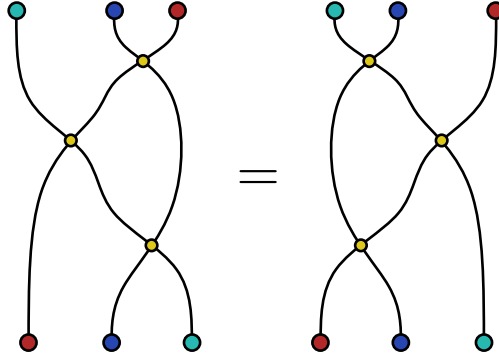


Figure 1.2: Yang-Baxter equation. Here we have represent graphically the Yang-Baxter equation. Each small yellow dot represent a $2 \rightarrow 2$ scattering event. Then this equation tell us that the two possible ways of decomposing a $3 \rightarrow 3$ event as a sequence of $2 \rightarrow 2$ scatterings are equal. In a similar way we can decompose any $n \rightarrow n$ event.

can completely fix the $O(N)$ S-matrix for example (after imposing some physical conditions like the spectrum of bound states and unitarity). The Yang-Baxter equation is represented in Figure 1.2. Then integrable quantum field theories have factorized scattering and due to it the S-matrix is fixed and the spectrum can be determined at *finite volume* using the thermodynamic Bethe ansatz [30]. Some examples of IQFTs are the $O(N)$ model, sine-Gordon model, massive Thirring model and so on. The reason for the restriction of integrable models to two-dimensional QFTs is due to the Coleman-Mandula theorem⁵ [31]. With this brief detour into IQFTs let us head back to holographic dualities.

This story of integrable models is very interesting and all, however how it links with AdS_{d+1}/CFT_d dualities if it is restricted to two-dimensional models? Note that the underlying theory describing quantum gravity in the bulk is a string theory which can be seen as a two-dimensional QFT in the worldsheet. Turns out that for some backgrounds this QFT is integrable and when this happens we have an integrable background. The previous discussed backgrounds AdS_5/CFT_4 , AdS_4/CFT_3 and AdS_3/CFT_2 are integrable [32, 33]. This means that we can use all the tools from quantum integrability to find dynamical quantities in the string theory and then use the AdS/CFT dictionary to find their dual in the CFT side. All of this is done at *finite coupling*, hence the power integrability in AdS/CFT .

As said before, in this work we will focus on the $AdS_5 \times S^5$ and $AdS_3 \times S^3 \times T^4$ dualities. The first use of integrability in AdS/CFT was in the computation of

⁵Basically if higher-dimensional QFTs had this infinite tower of conserved charges the theorem says that the scattering would be trivial and thus the QFT would be simply a free field theory.

the spectrum of the dual CFT. Let us start with the AdS_5/CFT_4 case. One of the first signs of integrability was that the string model at large tension T , that is classical string theory, is classically integrable for this background. Now on the CFT_4 side, one of the first hints of integrable structures was the remarkable observation that we can recast the one-loop dilatation operator of $\mathcal{N} = 4$ SYM as the hamiltonian of an integrable spin chain, for the $SU(2)$ sector of the theory this is just the one-dimensional XXX Heisenberg spin chain [34]. In this language single trace operators are dual to eigenstates of the chain, also called Bethe states, and the eigenvalues of this hamiltonian are the one-loop anomalous dimensions. The spectrum and thus the anomalous dimensions can be found using the algebraic Bethe ansatz. Then in a series of works this result was extended to higher and higher loops where the hamiltonian one obtains becomes of long-range. Over the last two decades, many techniques have been taken from exactly solved models and applied to the analysis of a plethora of observables in $\mathcal{N} = 4$ SYM at weak, strong, and even at finite coupling. An example of outstanding success is the computation of the spectrum of single trace operators to all loops in perturbation theory that revealed rich integrable structures such as the quantum spectral curve [35]. For reviews on these topics see [32, 36, 37].

For $AdS_3 \times S^3 \times T^4$ the integrable techniques for the spectrum are similar to the AdS_5/CFT_4 case. As before, this string background is integrable and the spectral problem has been analyzed in depth [33, 38, 39, 40, 41, 42, 43]. Even though we do not have the dual CFT_2 for mixed flux, the spectral problem can still be analyzed in terms of a spin chain. However there are some glaring differences with respect to $AdS_5 \times S^5$ integrability. One of them is that the IQFT in the worldsheet has massless excitations in contrast with its higher dimensional cousin where all excitations are massive. These massless modes make the comparison with the string theory a little tricky and introduces novel features to the integrable structure⁶. Nonetheless quantum integrability can still be applied to analyze the spectral problem in this background, and more recently even a quantum spectral curve formalism, as in AdS_5/CFT_4 , was developed⁷ [45, 46]. In this work we focus on the *pure RR* case whose integrable structure is very similar to the $AdS_5 \times S^5$

⁶For example, usually the ABA yields the spectrum of asymptotically large operators. Finite length corrections are given by the so-called wrapping corrections that are suppressed for large operators. However here this is not exactly true because massless modes' wrapping corrections to the spectrum enters at the same order of the ABA ones [44].

⁷However here the story still not complete here since the massless modes still elusive in this formalism.

one. Nonetheless it is worth noticing that the pure NSNS case is also integrable and in this exceptional situation the string theory is *exactly* given by a quantum integrable spin chain without any corrections [42]. The model can also be solved by worldsheet CFT techniques [47]. Note that for mixed flux the spectral problem remains open due to some technical problems [48, 49, 38].

One can see the spectral problem in AdS/CFT as the problem of defining an IQFT in the cylinder. This is due to the fact that a closed string moving in space defines a worldsheet that is topologically equivalent to a cylinder. However one can try to analyze an IQFT on more complicated topologies, like a pair of pants or a sphere with four punctures. These more complicated geometries are related to the computation of structure constants (pair of pants) and of four-point functions (sphere with four punctures) through a procedure called *hexagonalization*. This formalism is the central theme in this work and we will describe it now.

After the integrability for the spectral problem was worked out in details, the natural next step was the computation of the other half of the CFT data: the structure constants. With this in mind the tailoring procedure for $AdS_5 \times S^5$ emerged in a series of works [50, 51, 52, 53, 54, 55, 56]. It allowed to find three-point functions of single trace operators at weak coupling, the main idea was to map each operator to a Bethe state and cut these states into subchains. Then we join these subchains through inner products, computed using quantum integrability techniques, in a way that reproduces Wick contractions and then sum over all possible ways of cutting. The next link in the chain of developments was the hexagonalization formalism, which was introduced in [57] and it enabled the calculation of structure constants of $\mathcal{N} = 4$ SYM. Unlike tailoring, hexagonalization works at finite coupling for asymptotically large operators⁸.

Let us describe hexagonalization in more detail. Basically it consists in taking the pair of pants topology that describes the structure constant of the CFT_4 in the string side and cut it in half. Each side is the so-called hexagon form factor. When we cut the pair of pants, the excitations of each operator can be in the front or back hexagons and we sum over all the possible distributions of the physical excitations in each form factor. Note here the resemblance with the tailoring formalism. Also in moving an excitation from the front to the back hexagon we gain a propagation factor that we need to take into account. Each hexagon has three physical edges, where the operators are cut, and three mirror edges corresponding to where we cut the pair of pants. To obtain the structure constant we then glue back the hexagons

⁸A recent proposal for finite length operators was made in [58].

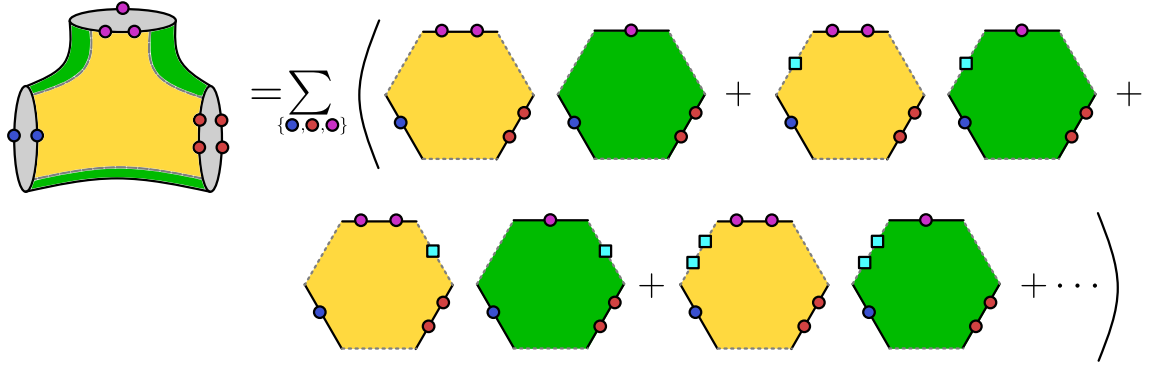


Figure 1.3: Hexagonalization of a structure constant. On the left we have a pair of pants representing a structure constant and on the right its hexagonalization. We cut the pair of pants into two hexagons. Each hexagon has physical and mirror edges denoted by the solid and dashed lines, respectively. Note that we sum over all possible ways of distributing the excitations in the back and front hexagons. The first contribution correspond to the asymptotic hexagon. The remaining ones are the mirror corrections where we add a single mirror particle (square) in a mirror edge, two mirror particles, and so on.

by inserting a complete basis of states in the mirror edges. The first contribution is the insertion of the vacuum which yields the so-called asymptotic hexagon. In this limit the structure constant is just the product of the two hexagons with the physical excitations distributed in both of them. The contributions corresponding to adding excitations in the mirror edges are the mirror corrections. These are exponentially suppressed in the large bridge length limit, therefore if all the bridge lengths are large the structure constant is given only by the asymptotic piece. The hexagonalization procedure is shown in Figure 1.3.

The only piece remaining is the hexagon itself. To fix it we bootstrap them by using four axioms that are inspired by the form factor program in IQFTs for local operators (see the reviews [29, 59]) and for the non-local operators introduced in [60]. We can understand these axioms as a way of defining QFT in nonstandard spacetimes. This hexagonalization formalism was then improved and tested in [61, 62, 63, 64, 65, 66, 67] and it passed in all the tests. Later the same idea was applied to more complicated topologies like the sphere with four or five punctures that correspond to four- and five-point functions in $\mathcal{N} = 4$ SYM, respectively [68, 69, 70, 71, 72, 73]. Hexagonalization works in a similar way for these, we make more cuts and sum over all the possible ways of cutting the worldsheet with some extra ingredients that we shall explain in this work. Although we discussed hexagonalization only for $AdS_5 \times S^5$ duality, it was recently introduced in the

AdS_3/CFT_2 context in [74]. It works remarkably similar as the higher dimensional case, however only the asymptotic hexagon was described in that work and no attempt for the hexagonalization of higher point functions was made.

With this introduction into the subject matter made, let us describe the content of this work. This thesis is divided into two parts. The first part covers hexagonalization in $AdS_5 \times S^5$ and the classical limit. It is an expansion of the work done in collaboration with Gabriel Lefundes in [75]. This part is described in Chapter 2. The second part covers hexagonalization in $AdS_3 \times S^3 \times T^4$, mirror corrections and four-point functions. This is an extension of the work done in [76] and it is in detailed in Chapters 3, 4 and 5. Below we expand on the content of each part.

Part 1: Hexagons and the classical limit

In the first part we are interested in structure constants of $\mathcal{N} = 4$ SYM at weak coupling. The tailoring and hexagonalization formalisms should compute the same structure constants at tree level in the $SU(2)$ sector of $\mathcal{N} = 4$ SYM, however their equivalence was not proven in full generality. In Chapter 2 we amend this by proving that they yield the same result at tree level for three excited operators, thus closing this hole in the hexagon literature. In this process we work out some analytic properties of the hexagons and find new representations for them at tree level in both the compact $SU(2)$ and non-compact $SL(2)$ sectors. These could hint to new representations of the hexagon form factor at finite coupling. An advantage of these new expressions is that they are computationally less costly, since we do not need to do multiple tensor contractions to compute them as the original definition of the hexagon in [57].

Another aspect we also consider in AdS_5/CFT_4 hexagonalization is the *classical limit* of structure constants. This is a central concept at strong coupling in $\mathcal{N} = 4$ SYM. As discussed before, in this regime the string side of the duality is described by classical strings. Indeed we can map three-point functions in this limit to the scattering of closed strings in $AdS_5 \times S^5$. The non-linear sigma model on the string dual is classically integrable and for strings with large charges the correlation functions can be evaluated by the saddle-point method. These “heavy” string states correspond in $\mathcal{N} = 4$ SYM to operators with large charges [77].

Nonetheless we are interested in the weak coupling regime and here the classical limit also plays a role. An example of this is the famous Frolov-Tseytlin limit, where at strong coupling and asymptotically large charges we find the same results at weak coupling under the expansion in a appropriate parameter [78].

Although it does not work at all loops, indeed there are some observables where this duality breaks at some loop order with one example described in [79]. The classical and quantum integrability descriptions appearing at weak and strong coupling also show some similarities when taking the classical limit of the spin chain associated with heavy single trace operators. In this limit, we have a large number of Bethe roots that condense into macroscopic branch cuts in the complex plane. These resemble the finite-gap solutions of the classical string sigma model described in [80, 81]. For operators in the $SU(2)$ sector of $\mathcal{N} = 4$ SYM the classical limit of spin chains is described by the Landau-Lifschitz model, which makes the relationship with the strong coupling framework explicit as shown in [82].

Here we compute the classical limit of a special class of structure constants in the $SU(2)$ sector. From the tailoring formalism we know that three-point functions belong to two classes: the type I-I-I and type I-I-II. Each correspond to specific choices of the external operators. The type I-I-II are simpler and they are in the same complexity class as structure constants with only two excited operators. Their classical limit was found [55, 82]. Now the type I-I-I correlators are more involved and the derivation of their classical limit from the tailoring formulas was an open problem, we then compute this classical limit in Chapter 2. Our derivation is done directly from the microscopic formulas using recursion relations we derived for the hexagons analogous to the ones in [67]. We use the same methods to also compute the classical limit of $SL(2)$ three-point functions for some fixed polarizations of the external operators.

Part 2: Hexagonalization $AdS_3 \times S^3 \times T^4$ and mirror corrections

As described before the dual CFT_2 for the $AdS_3 \times S^3 \times T^4$ is unknown beyond the pure NSNS case with $\kappa = 1$, therefore any method of extracting information about this CFT_2 is relevant. Due to this, the hexagonalization of structure constants for this background, introduced in [74], yields valuable information. In Chapter 3 we introduce the necessary integrability background and the hexagonalization formalism for this duality. We will work here in the *pure RR* limit. Also we compute some examples of structure constants.

Note however that in [74] it was introduced only the asymptotic part of the hexagon which computes structure constants for large bridge lengths. Thus it remained open the problem of defining the mirror corrections for finite length bridges. We attack this problem in Chapter 4, there we introduce the mirror corrections and demonstrate how they behave. These corrections are in general

coupling dependent. Note however that there is a set of operators in $AdS_3 \times S^3 \times T^4$ whose CFT data are protected. These operators form the so-called chiral ring. We then prove that all the chiral ring structure constants do not receive mirror corrections, which is a nontrivial fact in this model due to the intricate structure of the chiral ring.

The role of mirror corrections go beyond yielding the finite length corrections for structure constants, they are indeed paramount to the hexagonalization of n -point functions. The main reason is that independent of the choices made for the external operators we will always have small bridge lengths for correlators with more than three insertions. Therefore mirror corrections are weakly suppressed in the hexagonalization of n -point functions. In Chapter 5 we provide a first step on computing n -point correlation functions for the pure RR dual CFT_2 and we discuss the problems which appear when computing four-point functions by hexagonalization in this background.

Outline

This thesis is organized in the following way. As said before, Part 1 is covered in Chapter 2. In Sections 2.1, 2.2 and 2.3 we review the tailoring formalism, hexagonalization in $AdS_5 \times S^5$ and the classical limit, respectively. In Section 2.4 we then describe new representations for the hexagons and the classical limit of structure constants in the $SU(2)$ and $SL(2)$ sectors. Now Part 2 is covered in Chapters 3, 4 and 5. Section 3.1 is a review of integrability in $AdS_3 \times S^3 \times T^4$. In Section 3.2 we present the chiral ring and we end the review part of the chapter in Section 3.3 by presenting the hexagonalization formalism in this background. Then we show some simple structure constants that can be found using hexagonalization in Section 3.4. We move to mirror corrections in Section 4.1 and describe the transfer matrix formalism and mirror measure in Section 4.2. The cancellation of mirror corrections in the chiral ring is described in Section 4.3. We close by discussing hexagonalization of four-point functions in Chapter 5. In Section 5.1 we review the properties of the dual CFT_2 and the hexagonalization itself is described in Section 5.2. Then in Section 5.3 we end up by discussing an example of a simple correlator and some problems that appeared in the analysis. All chapters end with a summary of what was discussed in it.

Chapter 2

Hexagons and the classical limit

As described in the Introduction (Chapter 1), the spectrum of single trace operators in $\mathcal{N} = 4$ SYM at the planar level was described non-perturbatively using integrability techniques. This was achieved by writing the operator as an eigenstate (also called Bethe state) of a quantum mechanical integrable spin chain and then one diagonalizes this operator. Using this spin chain description, one then attempts to compute the structure constants.

In this chapter we will describe structure constants of $\mathcal{N} = 4$ SYM at weak coupling in the so-called classical limit in both $SU(2)$ and $SL(2)$ sectors. For this we introduce the double spin chain formalism for the $SU(2)$ sector in Section 2.1. Then we describe the spinning hexagons for the $SL(2)$ sector in Section 2.2. Following this we recall the properties of the classical limit in Section 2.3. Finally in Section 2.4 we close the chapter by computing the classical limit of a class of structure constants and as a byproduct we also give new representations of the hexagons in both sectors.

2.1 Tailoring and double spin chain formalism

An integrability framework to compute structure constants at tree level was introduced in [50, 51, 52, 53, 54] and was dubbed *tailoring*. The main idea is to take the Bethe states describing the three operators in a structure constant, cut and glue them in a way that reproduces the Wick contractions. This formalism worked well at tree level and was extended to one-loop using a technique called θ -morphism [53, 83]. The main drawbacks of this formalism are that it describes only the $SU(2)$ sector of $\mathcal{N} = 4$ SYM and it is limited to one-loop only.

The first problem was partially solved in [55] by another procedure called double spin chain formalism, which we will briefly describe below. However both of them, in some limits, were fully solved by hexagonalization. This latter approach will be quickly discussed in Section 2.2 for $AdS_5 \times S^5$ and fully detailed

in Chapter 3 for $AdS_3 \times S^3 \times T^4$. In the double spin chain formalism the operators are mapped to Bethe states and then cut and contracted as in tailoring. However the main conceptual change here is that these are mapped to double spin chain states. This allows the description of the $SO(4)$ sector of the theory and contains the previous description as a special case. Let us describe this formalism in more details now.

In this approach, we define a double spin chain using the decomposition of the fundamental $SO(4)$ representation as a tensor product of two doublet $SU(2)$ representations. This allows us to describe correlators with operators built on top of distinct vacua. More explicitly we can map each scalar in the $SO(4)$ sector to a double spin chain state:

$$\begin{aligned} Z &\rightarrow |\uparrow\rangle \otimes |\uparrow\rangle, & \bar{Z} &\rightarrow |\downarrow\rangle \otimes |\downarrow\rangle, \\ Y &\rightarrow |\uparrow\rangle \otimes |\downarrow\rangle, & -\bar{Y} &\rightarrow |\downarrow\rangle \otimes |\uparrow\rangle. \end{aligned} \quad (2.1)$$

Then to build the excited states (or operators) we put excitations on one sector or the other of the double spin chain. We consider $|\uparrow \cdots \uparrow\rangle$ as the vacuum state and excitations (also called magnons) are down spins $|\downarrow\rangle$ moving in the chain as a plane wave with amplitudes given by the scattering matrices [23].

Note that we can not excite both sectors at once since this would give an operator outside the $SO(4)$ sector. This yields two classes of operators: if we excite the left spin chain we have a *type I* operator, otherwise we have a *type II* operator. In the field language the vacuum of length ℓ is simply $\text{tr}(Z^\ell)$ (which is the vacuum considered in [50]). However by applying a $SU(2)_L \times SU(2)_R$ R-charge rotation we can go to general vacua, *id est*, polarized operators. This allows the description of more general correlators.

After defining the correspondence of spin chains and operators one then has to cut and then contract these states to build the structure constant. Then we first cut the spin chain state into left and right part as in [50]. An excited (non-BPS) operator in $\mathcal{N} = 4$ SYM with M excitations is given by a Bethe state described by the roots $\mathbf{u} = \{u_1, \dots, u_M\}$. Consider for example the left spin chain of some operator with this set of Bethe roots in the usual vacuum of ℓ up states which we denote by $|\mathbf{u}, \uparrow^\ell\rangle_L$. Then its cutting is given by

$$|\mathbf{u}, \uparrow^\ell\rangle_L = \sum_{\alpha_l \cup \alpha_r = \mathbf{u}} H_\ell(\alpha_l, \alpha_r) |\alpha_l, \uparrow^{\ell_l}\rangle \otimes |\alpha_r, \uparrow^{\ell_r}\rangle, \quad (2.2)$$

where ℓ_l and ℓ_r are the lengths of the left and right subchains, respectively. Here

we divided the set of Bethe roots into two disjoint subsets α_l and α_r . Clearly the number of terms in the sum is 2^M , which is just the number of partitions of $\mathbf{u}$. The explicit form of the cutting factor $H_\ell(\alpha_l, \alpha_r)$ is

$$H_\ell(\alpha_l, \alpha_r) = \prod_{u \in \alpha_l} \prod_{v \in \alpha_r} \frac{u - v + i}{u - v} \left(u - \frac{i}{2}\right)^{\ell_r} \left(v + \frac{i}{2}\right)^{\ell_l}. \quad (2.3)$$

This cutting factor can be understood physically as a propagating factor of roots from one subchain to another¹. Therefore in the final form of the structure constant one has three sums over partitions of the operators with $H_\ell(\alpha_l, \alpha_r)$ weighting the terms.

So far we constructed Bethe states on top of the usual vacuum of up spins $|\uparrow \cdots \uparrow\rangle$. Now we rotate this state to build a generalized vacuum. We have that half-BPS states in $\mathcal{N} = 4$ SYM in the $SO(4)$ sector can be defined in terms of a polarization vector $P \in \mathbb{C}^4$:

$$P \cdot \Phi = P_{a\dot{a}} \Phi^{a\dot{a}} \quad \text{with} \quad \Phi = \begin{pmatrix} Z & Y \\ -\bar{Y} & \bar{Z} \end{pmatrix}, \quad (2.4)$$

with $a, \dot{a} \in \{1, 2\}$. We note that this vector must be null to describe a half-BPS state, then by representation theory of $SO(4)$ we can write it as a product of two spinors

$$P_{a\dot{a}} = n_a \tilde{n}_{\dot{a}}. \quad (2.5)$$

For example, $n_a = \tilde{n}_{\dot{a}} = (1, 0)$ gives us the Z field. The $SO(4)$ rotation of the scalars act on the spinors as $SU(2)_L \otimes SU(2)_R$, respectively. Clearly the spinors transform in the spin $1/2$ representation of $SU(2)$ and there are many parametrizations we can consider for its action. Here we pick the coherent state representation, *id est*, let $g \in SU(2)_L$ then

$$g = e^{zS_-} e^{-\log(1+|z|^2)S_3} e^{\bar{z}S_+}, \quad (2.6)$$

where S_- , S_+ and S_3 are the usual lowering, raising and spin operators in $SU(2)$, respectively. Then $z, \bar{z} \in \mathbb{C}$ parametrize the rotation and for $SU(2)_R$ we use the

¹Due to this there are two ways of define this quantity depending on the normalization. Basically one can define states in the algebraic Bethe ansatz normalization or in the coordinate Bethe ansatz one. The only change on one or another is an overall function of the roots. We work here on the algebraic normalization.

analogous variables $\tilde{z}, \tilde{\bar{z}} \in \mathbb{C}$. Therefore n and $\tilde{n}$ are

$$n = g \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{1+|z|^2}} \begin{pmatrix} 1 \\ z \end{pmatrix} \quad \text{and} \quad \tilde{n} = \frac{1}{\sqrt{1+|\tilde{z}|^2}} \begin{pmatrix} 1 \\ \tilde{z} \end{pmatrix}. \quad (2.7)$$

Thus to build an excited state on top of a generalized vacuum $|\mathbf{u}, n^\ell\rangle_{L,R}$ we simply rotate $|\mathbf{u}, \uparrow^\ell\rangle_{L,R}$:

$$|\mathbf{u}, n^\ell\rangle_L = \frac{e^{zS_-}}{(1+|z|^2)^{\ell-M/2}} |\mathbf{u}, \uparrow^\ell\rangle_L \quad \text{and} \quad |\mathbf{u}, \tilde{n}^\ell\rangle_R = \frac{e^{\tilde{z}S_-}}{(1+|\tilde{z}|^2)^{\ell-M/2}} |\mathbf{u}, \uparrow^\ell\rangle_R. \quad (2.8)$$

To derive this we used the fact that the Bethe state $|\mathbf{u}, \uparrow^\ell\rangle$ is a highest weight with spin $\ell - M/2$. We can then apply the cutting (2.2) in the same way to the polarized Bethe state as

$$|\mathbf{u}, n^\ell\rangle_L = \frac{1}{(1+|z|^2)^{\ell-M/2}} \sum_{\alpha_l \cup \alpha_r = \mathbf{u}} H_\ell(\alpha_l, \alpha_r) \left(e^{zS_-^l} |\alpha_l, \uparrow^{\ell_l}\rangle \right) \otimes \left(e^{zS_-^r} |\alpha_r, \uparrow^{\ell_r}\rangle \right), \quad (2.9)$$

where we just divided the lowering operator for the right and left subchains using $e^{zS_-^r}$ and $e^{zS_-^l}$, respectively. In the tailoring procedure of [50] one constructed the Bethe states only on top of the $\text{tr}(Z^\ell)$ vacuum, which restricted the analysis to the $SU(2)$ sector of the theory. However, as advertised earlier, the double spin chain formalism allows the description of the larger $SO(4)$ sector.

We described the first step of the double spin chain formalism, the second step is the gluing or in field language the Wick contractions. The Wick contractions act on conjugated fields and we can see this action in terms of the double spin chain as an antisymmetric pairing. Indeed, consider two fields that are described by polarizations P_1 and P_2 , then their contraction $P_1 \cdot P_2$ is simply given by the spinor product

$$P_1 \cdot P_2 = (\epsilon^{ab} n_{1,a} n_{2,b}) (\epsilon^{\dot{a}\dot{b}} \tilde{n}_{1,\dot{a}} \tilde{n}_{2,\dot{b}}) = \langle n_1, n_2 \rangle \langle \tilde{n}_1, \tilde{n}_2 \rangle. \quad (2.10)$$

Where we see the antisymmetric pairing due to the ϵ -tensor. In the spin chain language we define a antisymmetric singlet state $\langle \mathbf{1}_{12} |$ such that its action on two spin chain states mimics the Wick contraction of the corresponding operators. Indeed, let us define an antisymmetric pairing in the left spin chain as

$$\langle |n_1\rangle, |n_2\rangle \rangle = \langle \mathbf{1} | (|n_1\rangle \otimes |n_2\rangle) = \langle n_1, n_2 \rangle, \quad \text{where} \quad \langle \mathbf{1} | = \epsilon^{ab} \langle a | \otimes \langle b |. \quad (2.11)$$

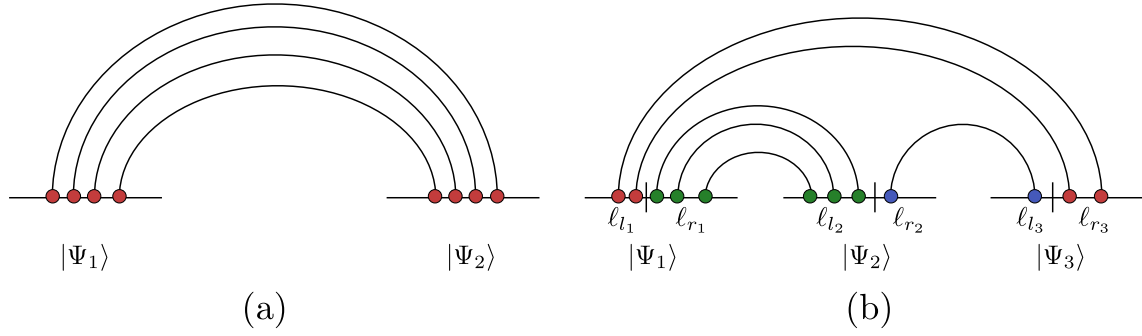


Figure 2.1: Representation of the singlet state. In (a) we have a schematic representation of how the singlet state acts on a two-point function and in (b) how it acts on three-point function. Here connecting two sites of chains denotes a Wick contraction. Note that that way the antisymmetric pairing is defined we have planarity of the three-point function.

We can do this for general spin chain states by defining $\langle \mathbf{1}_{12} |$:

$$\langle |\Psi_1\rangle, |\Psi_2\rangle \rangle = \langle \mathbf{1}_{12} | (|\Psi_1\rangle \otimes |\Psi_2\rangle) = \prod_{m=1}^l \langle 1_{k, \ell-k+1} | (|\Psi_1\rangle \otimes |\Psi_2\rangle), \quad (2.12)$$

where $\langle 1_{k, \ell-k+1} |$ is the antisymmetric pairing between the site k of $|\Psi_1\rangle$ and $\ell - k + 1$ of $|\Psi_2\rangle$. Which is done in this way such that we get only planar diagrams, which is fine since we are working in the large- N limit. This is represented schematically in Figure 2.1.

Analogously to the two states case we can define the three states contractions. Thus for the L spin chain we have

$$\langle |\mathcal{O}_1\rangle, |\mathcal{O}_2\rangle, |\mathcal{O}_3\rangle \rangle_L = \sum \langle |\mathcal{O}_1\rangle^l, |\mathcal{O}_3\rangle^r \rangle_L \langle |\mathcal{O}_3\rangle^l, |\mathcal{O}_2\rangle^r \rangle_L \langle |\mathcal{O}_2\rangle^l, |\mathcal{O}_1\rangle^r \rangle_L, \quad (2.13)$$

where the upper scripts l and r represent the left or right subchain of the $|\mathcal{O}_j\rangle_L$ state and we sum here over all partitions of the roots for each operator as stated before. We have similar contributions for the R spin chain. The ordering of contractions is such that the corresponding fatgraph contribution is planar as shown in Figure 2.1. Note that the length of the subchains are fixed by the bridge lengths, *id est*, the

number of contractions between operators. These are:

$$\ell_{l_1} = \ell_{r_3} = \ell_{13} = \frac{\ell_1 + \ell_3 - \ell_2}{2}, \quad (2.14)$$

$$\ell_{l_3} = \ell_{r_2} = \ell_{23} = \frac{\ell_2 + \ell_3 - \ell_1}{2}, \quad (2.15)$$

$$\ell_{l_2} = \ell_{r_1} = \ell_{12} = \frac{\ell_1 + \ell_2 - \ell_3}{2}. \quad (2.16)$$

In the tailoring formalism, the same result was achieved for the $SU(2)$ sector, however the authors did not introduce the singlet state. They defined the so-called flipping and gluing operations which in the end just yield the good old Wick contractions [50]. Another approach to contracting the fields is the so-called spin vertex introduced in [84]. It is similar in nature to the singlet state and it extends the analysis to all sectors of $\mathcal{N} = 4$ SYM by using an oscillator representation of the $\mathfrak{psu}(2,2|4)$ algebra. However it has some problems that were amended in [85], where the authors introduced a singlet state for the full $\mathfrak{psu}(2,2|4)$, but no application to the computation of structure constants was done and they restricted to the analysis of two-point functions.

We will not detail how to construct $\langle |\Psi_1\rangle, |\Psi_2\rangle \rangle$ here since it involves the tools of algebraic Bethe ansatz. We simply state the final result:

$$\langle |\Psi_1\rangle, |\Psi_2\rangle \rangle = (-1)^{M_1} \langle \downarrow^\ell | e^{(z_1 - z_2)S_-} | \mathbf{u}_1 \cup \mathbf{u}_2, \uparrow^\ell \rangle, \quad (2.17)$$

where M_j is the number of roots and z_j the polarization parametrization for $|\Psi_j\rangle$. This scalar product on the right can be written as

$$\langle \downarrow^\ell | e^{zS_-} | \mathbf{u}, \uparrow^\ell \rangle = \frac{z^{\ell-M}}{(\ell-M)!} \langle \downarrow^\ell | S_-^{\ell-M} | \mathbf{u}, \uparrow^\ell \rangle = z^{\ell-M} Z_p(\mathbf{u}|\ell). \quad (2.18)$$

Note that we must have S_3 spin conservation, then the only non-zero term is the one with $\ell - M$ descendant operators. The $Z_p(\mathbf{u}|\ell)$ is the so-called partial domain wall partition function (pDWPF) [55, 86, 87]. This is given by

$$Z_p(\mathbf{u}|\ell) = \frac{\prod_j (u_j - i/2)^\ell}{\prod_{i < j} (u_i - u_j)} \times \det \left(u_b^{a-1} - \left(\frac{u_b + i/2}{u_b - i/2} \right)^\ell (u_b - i)^{a-1} \right). \quad (2.19)$$

In the case at hand ℓ will be the bridge length of the contractions where the roots $\mathbf{u}$ lie. A physical interpretation for the pDWPF is that it is the usual domain wall

partition function when some rapidities are taken to infinity, *id est*, it is the scalar product of a Bethe state and a descendant of a Bethe state which is exactly what we have in (2.18).

With all these ingredients we can write the structure constant of three non-BPS operators in the $SO(4)$ of $\mathcal{N} = 4$ SYM at the planar level. We need simply to join the left and right contractions of the states and normalize accordingly. Then the structure constant is given by²

$$C_{123}^{\bullet\bullet\bullet} = \frac{\sqrt{\ell_1 \ell_2 \ell_3}}{N \mathcal{N}_1 \mathcal{N}_2 \mathcal{N}_3} \langle |\mathcal{O}_1\rangle, |\mathcal{O}_2\rangle, |\mathcal{O}_3\rangle \rangle_L \langle |\mathcal{O}_1\rangle, |\mathcal{O}_2\rangle, |\mathcal{O}_3\rangle \rangle_R, \quad (2.20)$$

where $\mathcal{N}_j$ is the normalization of each operator that is given by the Gaudin norm (see Appendix A). The left and right spin chain contributions, are given as a sum over partitions as aforementioned. For the L contribution

$$\langle |\mathcal{O}_1\rangle, |\mathcal{O}_2\rangle, |\mathcal{O}_3\rangle \rangle_L = \prod_{i=1}^3 \left(\frac{1}{1 + |z_i|^2} \right)^{\ell_i/2 - M_i} \sum_{\alpha_k \cup \bar{\alpha}_k = \mathbf{u}_k} z_{12}^{\ell_{12} - |\alpha_2| - |\bar{\alpha}_1|} z_{31}^{\ell_{13} - |\alpha_1| - |\bar{\alpha}_3|} z_{23}^{\ell_{23} - |\alpha_3| - |\bar{\alpha}_2|} \mathcal{D}(\alpha_j, \bar{\alpha}_j), \quad (2.21)$$

where

$$\begin{aligned} \mathcal{D}(\alpha_j, \bar{\alpha}_j) = & (-1)^{|\alpha_1| + |\alpha_2| + |\alpha_3|} \prod_{i=1}^3 H_{\ell_i}(\alpha_i, \bar{\alpha}_i) Z_p(\alpha_1 \cup \bar{\alpha}_3 | \ell_{13}) Z_p(\alpha_2 \cup \bar{\alpha}_1 | \ell_{12}) \times \\ & \times Z_p(\alpha_3 \cup \bar{\alpha}_2 | \ell_{23}). \end{aligned} \quad (2.22)$$

Also $\langle |\mathcal{O}_1\rangle, |\mathcal{O}_2\rangle, |\mathcal{O}_3\rangle \rangle_R$ is basically the same with the appropriate right sector variables. Here $z_{ij} = z_i - z_j$ and $|\alpha|$ denotes the length of the set α . This is the final form of the structure constant in the $SO(4)$ sector and we can separate them into two classes, if all operators are of the same type we have a *type I-I-I* or *pure* correlator. Otherwise if one of the operators is of type II we have a *mixed* or *type I-I-II* correlation function. Examples of representatives of these classes of correlators are given in Figure 2.2.

There are two technical details that will serve for our purposes later: the z_i dependence factorizes and the order of the roots $\mathbf{u}_k$ when you partition them does not matter. The first is expected since it is known that in the computation of the

²Here we use the common notation of $C_{123}^{\bullet\bullet\bullet}$ indicating the structure constant of three non-BPS operators, $C_{123}^{\bullet\bullet\circ}$ the one with only two non-BPS, and so on.

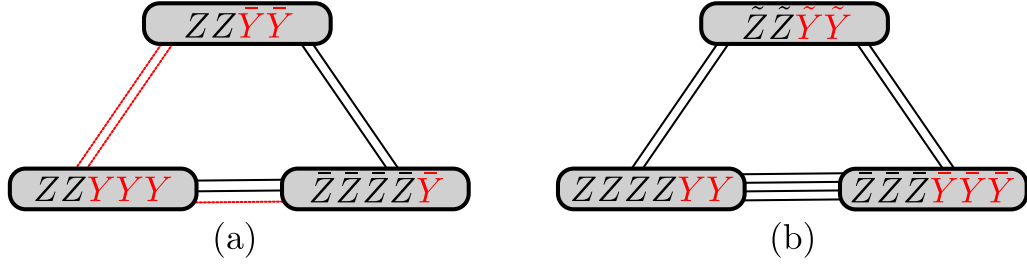


Figure 2.2: Examples of type I-I-II and I-I-I correlators. In (a) we have an example of a type I-I-II correlator analyzed in [52]. Now in (b) we have a type I-I-I correlator where $\tilde{Z} = Z + \bar{Z} + Y - \bar{Y}$ and $\tilde{Y} = Y + \bar{Z}$. This structure constant was analyzed in [57]. Note that to describe (b) we have to rotate the vacuum to $\tilde{Z}$ and $\tilde{Y}$, which was not possible in the tailoring formalism.

three-point function of scalar operators the polarizations factorizes [55]. This is not true for spinning operators (see Section 2.4.2). The independence of the root's ordering comes from the fact that any ordering define the same spin chain state in the algebraic Bethe ansatz. Both of these states are related to the fact that for physical operators the roots satisfy the Bethe equations, *id est*, they are on-shell. As proven in [55] this implies that then the z_{ij} dependence factorizes as

$$\langle |\mathcal{O}_1\rangle, |\mathcal{O}_2\rangle, |\mathcal{O}_3\rangle \rangle_L = \prod_{i=1}^3 \left(\frac{1}{1 + |z_i|^2} \right)^{\ell_i/2 - M_i} z_{12}^{\ell_{12} - M_{12}} z_{23}^{\ell_{23} - M_{23}} z_{31}^{\ell_{31} - M_{13}} \mathcal{G}_{123}^{(L)}, \quad (2.23)$$

where $M_{ij} = M_i + M_j - M_k$ and $\mathcal{G}_{123}^{(L)}$ is a function of the rapidities only. We have a similar function $\mathcal{G}_{123}^{(R)}$ for the R spin chain. These functions were known, prior to this work, only for type I-I-II correlators [55, 87]. Indeed, by simply comparing the powers of z_{ij} in (2.21) and (2.23) one can derive for type I-I-II correlators

$$\mathcal{G}_{123}^{(L)} = (-1)^{M_2} \prod_{j=1}^{M_1} \left(u_j + \frac{i}{2} \right)^{\ell_{13}} \prod_{j=1}^{M_2} \left(v_j - \frac{i}{2} \right)^{\ell_{23}} Z_p(\mathbf{u} \cup \mathbf{v} | \ell_{12}), \quad (2.24)$$

$$\mathcal{G}_{123}^{(R)} = \prod_{j=1}^{M_3} \left(w_j + \frac{i}{2} \right)^{\ell_{23}} Z_p(\mathbf{w} | \ell_{13}). \quad (2.25)$$

Where $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are the roots of the first, second and third operator, respectively. However for the more complicated case of type I-I-I it was unknown how to derive these functions from the double spin chain prior to this work. The main complication is that all three operators interact upon contracting. As will be detailed in Section 2.2, finding this function is equivalent to derive the hexagon

formalism at tree level from the double spin chain formalism, which was an open problem [57].

2.1.1 New expression for the structure constant

In Section 2.4 we will derive the classical limit of pure correlators. However first we need to put it on an appropriate form, that is, we need to eliminate the dependence on the polarization as in (2.23). Since we will work only with type I-I-I correlators, we have only the L part of these $\langle |\mathcal{O}_1\rangle, |\mathcal{O}_2\rangle, |\mathcal{O}_3\rangle \rangle_L$. Then we drop the labels on $\mathcal{G}_{123}^{(L)}$ to make the notation lighter:

$$\langle |\mathcal{O}_1\rangle, |\mathcal{O}_2\rangle, |\mathcal{O}_3\rangle \rangle_L = \prod_{i=1}^3 \left(\frac{1}{1 + |z_i|^2} \right)^{\ell_i/2 - M_i} z_{12}^{\ell_{12} - M_{12}} z_{23}^{\ell_{23} - M_{23}} z_{31}^{\ell_{13} - M_{13}} \mathcal{G}. \quad (2.26)$$

As explained before, if the roots satisfy the Bethe equations then $\mathcal{G}$ is polarization independent. However here we will use the Bethe equations to rewrite the structure constant, then we find $\mathcal{G}$ and we take the roots in the final expression to be off-shell. Thus we write

$$\mathcal{G} = \sum_{\alpha_k \cup \bar{\alpha}_k = \mathbf{u}_k} z_{12}^{M_{12} - |\alpha_2| - |\bar{\alpha}_1|} z_{23}^{M_{23} - |\alpha_3| - |\bar{\alpha}_2|} z_{31}^{M_{31} - |\alpha_1| - |\bar{\alpha}_3|} \tilde{\mathcal{D}}(\alpha_j, \bar{\alpha}_j), \quad (2.27)$$

where the new $\tilde{\mathcal{D}}(\alpha_j, \bar{\alpha}_j)$ assumes the following convoluted form

$$\begin{aligned} \tilde{\mathcal{D}}(\alpha_j, \bar{\alpha}_j) &= (-1)^{|\bar{\alpha}_1| + |\bar{\alpha}_2| + |\bar{\alpha}_3|} \sum_{\substack{\beta_1 \cup \bar{\beta}_1 = \bar{\alpha}_1 \cup \alpha_2 \\ \beta_2 \cup \bar{\beta}_2 = \bar{\alpha}_2 \cup \alpha_3 \\ \beta_3 \cup \bar{\beta}_3 = \bar{\alpha}_3 \cup \alpha_1}} (-1)^{|\beta_1| + |\beta_2| + |\beta_3|} \\ &\times a_{\ell_{12}}(\bar{\beta}_1 \cap \alpha_2) a_{\ell_{23}}(\bar{\beta}_2 \cap \alpha_3) a_{\ell_{31}}(\bar{\beta}_3 \cap \alpha_1) a_{\ell_{31}}(\beta_1 \cap \bar{\alpha}_1) a_{\ell_{12}}(\beta_2 \cap \bar{\alpha}_2) a_{\ell_{23}}(\beta_3 \cap \bar{\alpha}_3) \\ &\times S(\mathbf{u}_1, \beta_1 \cap \bar{\alpha}_1) S(\mathbf{u}_2, \beta_2 \cap \bar{\alpha}_2) S(\mathbf{u}_3, \beta_3 \cap \bar{\alpha}_3) \\ &\times f^+(\bar{\alpha}_1, \alpha_1) f^+(\bar{\alpha}_2, \alpha_2) f^+(\bar{\alpha}_3, \alpha_3) f^+(\beta_1, \bar{\beta}_1) f^+(\beta_2, \bar{\beta}_2) f^+(\beta_3, \bar{\beta}_3), \quad (2.28) \end{aligned}$$

and

$$f^+(\mathbf{x}, \mathbf{y}) = \prod_{u \in \mathbf{x}} \prod_{v \in \mathbf{y}} \left(\frac{u - v + i}{u - v} \right), \quad (2.29)$$

$$S(\mathbf{x}, \mathbf{y}) = \frac{f(\mathbf{x}, \mathbf{y})}{f(\mathbf{y}, \mathbf{x})} = \prod_{u \in \mathbf{x}} \prod_{v \in \mathbf{y}} \left(\frac{u - v + i}{u - v - i} \right), \quad (2.30)$$

$$a_\ell(\mathbf{x}) = \prod_{u \in \mathbf{x}} \left(\frac{u + i/2}{u - i/2} \right)^\ell. \quad (2.31)$$

Physically, $S(u, v)$ is the $SU(2)$ spin chain S-matrix, a_ℓ is $e^{iP\ell}$, where P is the total momentum of the excitations, and $f^+(u, v)$ can be seen as the hexagon scalar factor at tree level. The derivation of these formulas is done in details in Appendix B.

The reader can complain that we claimed $\mathcal{G}$ independence from the polarizations, however they remain present in (2.27). But if one takes this expression evaluate it for a few examples, it can be seen that indeed all the polarizations cancel. We will see explicitly this cancelling in Section 2.4 for the classical limit. With the description of the $SU(2)$ laid out, let us move it to the description of correlators in the $SL(2)$ sector.

2.2 Non-compact spin chains and hexagons

Now we will briefly review how the computation of structure constants of spinning operators was done in [67]. An operator in the so-called $SL(2)$ sector in $\mathcal{N} = 4$ SYM is given simply by

$$\mathcal{O}_J(x, L, R) = L_{\alpha_1} R_{\dot{\alpha}_1} \cdots L_{\alpha_J} R_{\dot{\alpha}_J} \mathcal{O}_J^{\alpha_1 \dot{\alpha}_1 \cdots \alpha_J \dot{\alpha}_J}(x), \quad (2.32)$$

where $\mathcal{O}_J^{\alpha_1 \dot{\alpha}_1 \cdots \alpha_J \dot{\alpha}_J}(x)$ is a BPS vacuum with J covariant derivatives $\mathcal{D}^{\alpha \dot{\alpha}}$ applied on it with $\alpha, \dot{\alpha} \in \{1, 2\}$ being spinor indices for each derivative. The polarization spinors L_α and $R_{\dot{\alpha}}$ play the same role as the polarization vectors in the $SU(2)$ sector by characterizing each operator. Due to $SL(2)$ invariance, the final structure constant must contain only the following spinor contractions

$$H_{ij} = \langle L_i, R_j \rangle \langle L_j, R_i \rangle \quad \text{and} \quad V_i = \langle L_i, R_i \rangle \quad \text{with} \quad \langle A, B \rangle = \epsilon^{\alpha \dot{\alpha}} A_\alpha B_{\dot{\alpha}},$$

where $\epsilon^{\alpha \dot{\alpha}} = -\epsilon^{\dot{\alpha} \alpha}$ is an invariant $SL(2)$ tensor. Up to now the story is similar to the $SU(2)$ sector, however, unlike the case of scalars operators, the spinning structure

for the three-point function *is not* fixed by $SL(2)$ invariance [88]. Consider for example a correlator with $J_1 = 2$, $J_2 = 4$ and $J_3 = 6$, then the normalized structure constant at tree level is given in [89] and it can be written as

$$\frac{C_{123}^{\bullet\bullet\bullet}}{C_{123}^{\circ\circ\circ}} = \frac{V_1^2 V_2^4 V_3^6}{84\sqrt{55}} \left(t_{13} t_{23} 960 + t_{12}^2 15 - t_{23}^4 t_{13} 2007 - t_{13}^2 t_{23}^2 7680 + \dots \right), \quad (2.33)$$

where we defined the quantity

$$t_{ij} = \frac{H_{ij}}{V_i V_j}.$$

With this redefinition we see that the structure constant in the $SL(2)$ sector is a polynomial in the variables t_{ij} with all possible tensor structures contributing and not just a single term like in the $SU(2)$ sector.

To compute these structure constants, the authors in [67] used the hexagonalization approach. To avoid repeating ourselves later we will restrain here to just briefly describe the expressions for these correlators and later in Chapter 3 we will show the origin of hexagonalization in the context of AdS_3/CFT_2 duality. As described in Chapter 1, in the hexagon formalism the three-point function in $\mathcal{N} = 4$ SYM can be described as

$$\begin{aligned} \frac{C_{123}^{\bullet\bullet\bullet}}{C_{123}^{\circ\circ\circ}} = & \sum_{\substack{\delta_1 \cup \bar{\delta}_1 = \mathbf{u}_1 \\ \delta_2 \cup \bar{\delta}_2 = \mathbf{u}_2 \\ \delta_3 \cup \bar{\delta}_3 = \mathbf{u}_3}} (-1)^{|\delta_1| + |\delta_2| + |\delta_3|} a_{\ell_{31}}(\delta_1) \mathcal{S}_{<}(\bar{\delta}_1, \delta_1) \times a_{\ell_{12}}(\delta_2) \mathcal{S}_{<}(\bar{\delta}_2, \delta_2) \\ & \times a_{\ell_{23}}(\delta_3) \mathcal{S}_{<}(\bar{\delta}_3, \delta_3) \mathcal{Z}(\delta_1, \delta_3, \delta_2) \mathcal{Z}(\bar{\delta}_1, \bar{\delta}_2, \bar{\delta}_3). \end{aligned} \quad (2.34)$$

Where $a_{\ell_\ell}(z)$ is a momentum propagation factor $e^{ip(z)\ell}$ from an hexagon to another and the form factors $\mathcal{Z}(\delta_1, \delta_3, \delta_2)$ are the so-called hexagons. The remaining factor $\mathcal{S}_{<}(\bar{\delta}_j, \delta_j)$ is an S-matrix factor gained from scattering excitations in one hexagon to bring them to the other hexagon. These are given by

$$\mathcal{S}_{<}(\bar{\delta}_j, \delta_j) = \prod_{\substack{u_m \in \bar{\delta}_j, u_n \in \delta_j \\ m < n}} \mathcal{S}(u_m, u_n), \quad (2.35)$$

with $\mathcal{S}(u, v)$ being the $SL(2)$ scattering matrix. In (2.34) we are considering only the asymptotic hexagon, thus it works at finite coupling and for asymptotically large operators and bridge lengths.

The momentum factor and the $SL(2)$ S-matrix are well known from the spectral problem. We have only to define the hexagon form factor now. As explained in [57], it is defined as tensor contractions of the Beisert S-matrix, that is, the $\mathfrak{su}(2|2)$ S-matrix of $\mathcal{N} = 4$ SYM. However the main insight of [67] was to note that this tensor contraction can be reinterpreted as a partition function with the Boltzmann weights being the $\mathfrak{su}(2|2)$ S-matrix. In this language the polarizations spinors defining the operators enter as boundary conditions of this partition function such that in the end we obtain a structure constant that is polynomial in the t_{ij} variables. The full computation of the hexagon at finite coupling for a generic configuration, even in this new reinterpretation, remains costly nonetheless. However we can restrict ourselves to tree level where this partition function has a much simpler analytical structure and can be efficiently computed.

At tree level the hexagon form factor is a rational function of the rapidities and it has poles when the roots of two operators coincide. This is due to the decoupling property of the hexagon and it implies recursion relations for them. Consider for example coinciding roots on the first two operators

$$\begin{aligned} \text{Res}_{v_j \rightarrow u_i} \mathcal{Z}(\mathbf{u}, \mathbf{v}, \mathbf{w}) &= i \langle L_1, R_2 \rangle \langle L_2, R_1 \rangle \prod_{i'=1}^{i-1} \frac{f^-(u_i, u_{i'})}{f^-(u_{i'}, u_i)} \prod_{j'=j+1}^{J_2} \frac{f^-(v_{j'}, u_i)}{f^-(u_i, v_{j'})} \times \\ &\quad \times \mathcal{Z}(\mathbf{u}/\{u_i\}, \mathbf{v}/\{v_j\}, \mathbf{w}), \end{aligned} \quad (2.36)$$

where J_1 , J_2 and J_3 are the number of excitations on each operator and $\mathbf{u}/\{x\}$ denote set difference. Here $f^-(u, v)$ is

$$f^-(u, v) = \frac{u - v - i}{u - v}. \quad (2.37)$$

Note here the sign change in $f^-(u, v)$ with respect to the previously defined $f^+(u, v)$ for $SU(2)$. This is due to the fact that the $SL(2)$ S-matrix is the inverse of the $SU(2)$ one. Also we have that when a root is taken to infinity that

$$\lim_{u_i \rightarrow \infty} \mathcal{Z}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (-1)^{J_1+J_2+J_3+1} \langle L_1, R_1 \rangle \times \mathcal{Z}(\mathbf{u}/\{u_i\}, \mathbf{v}, \mathbf{w}). \quad (2.38)$$

Since the hexagons are rational functions of the rapidities and we known the poles and asymptotic conditions at infinity we can totally fix them. This will be done in Section 2.4.2 where we find a new representation for this partition function at tree level.

For our convenience, we will rewrite the hexagon to match with our notation for the $SU(2)$ sector and as a byproduct the recursion relations will simplify considerably. We then work with $\mathcal{H}(\mathbf{u}, \mathbf{v}, \mathbf{w})$, which is defined by

$$\mathcal{H}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (-1)^{\frac{(J_1+J_2+J_3)(J_1+J_2+J_3-1)}{2}} \mathcal{Z}(\mathbf{u}, \mathbf{v}, \mathbf{w}) \frac{f_{<}^-(\mathbf{u}_1) f_{<}^-(\mathbf{u}_2) f_{<}^-(\mathbf{u}_3)}{V_1^{J_1} V_2^{J_2} V_3^{J_3}}, \quad (2.39)$$

where

$$f_{<}^-(\mathbf{u}) = \prod_{i < j} f^-(u_i, u_j). \quad (2.40)$$

Using this redefinition in the recursion relation (2.36) we obtain

$$\begin{aligned} & \frac{\text{Res}_{v_j \rightarrow u_i} \mathcal{H}(\mathbf{u}, \mathbf{v}, \mathbf{w})}{f_{<}^-(\mathbf{u}_1) f_{<}^-(\mathbf{u}_2 / \{v_j\}) f_{<}^-(\mathbf{u}_3)} \prod_{j'=1}^{j-1} \frac{1}{f^-(v_{j'}, u_i)} \prod_{j'=j+1}^{J_2} \frac{1}{f^-(u_i, v_{j'})} = \\ & -it_{12} \prod_{i'=1}^{i-1} \frac{f^-(u_i, u_{i'})}{f^-(u_{i'}, u_i)} \prod_{j'=j+1}^{J_2} \frac{f^-(v_{j'}, u_i)}{f^-(u_i, v_{j'})} \frac{\mathcal{H}(\mathbf{u} / \{u_i\}, \mathbf{v} / \{v_j\}, \mathbf{w})}{f_{<}^-(\mathbf{u}_1 / \{u_i\}) f_{<}^-(\mathbf{u}_2 / \{v_j\}) f_{<}^-(\mathbf{u}_3)}, \end{aligned} \quad (2.41)$$

which, after simplifications, become simply

$$\text{Res}_{y \rightarrow x} \mathcal{H}(\mathbf{u}_1 \cup \{x\}, \mathbf{u}_2 \cup \{y\}, \mathbf{u}_3) = -it_{12} f^-(x, \mathbf{u}_1) f^-(\mathbf{u}_2, x) \mathcal{H}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3), \quad (2.42)$$

$$\text{Res}_{y \rightarrow x} \mathcal{H}(\mathbf{u}_1, \mathbf{u}_2 \cup \{x\}, \mathbf{u}_3 \cup \{y\}) = -it_{23} f^-(x, \mathbf{u}_2) f^-(\mathbf{u}_3, x) \mathcal{H}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3), \quad (2.43)$$

$$\text{Res}_{y \rightarrow x} \mathcal{H}(\mathbf{u}_1 \cup \{y\}, \mathbf{u}_2, \mathbf{u}_3 \cup \{x\}) = -it_{31} f^-(x, \mathbf{u}_3) f^-(\mathbf{u}_1, x) \mathcal{H}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3). \quad (2.44)$$

Here we wrote all the possible recursion relations from decoupling. Also we can simplify the second recursion relation (2.38) and obtain

$$\lim_{x \rightarrow \infty} \mathcal{H}(\mathbf{u}_1 \cup \{x\}, \mathbf{u}_2, \mathbf{u}_3) = \mathcal{H}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3), \quad (2.45)$$

$$\lim_{x \rightarrow \infty} \mathcal{H}(\mathbf{u}_1, \mathbf{u}_2 \cup \{x\}, \mathbf{u}_3) = \mathcal{H}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3), \quad (2.46)$$

$$\lim_{x \rightarrow \infty} \mathcal{H}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \cup \{x\}) = \mathcal{H}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3). \quad (2.47)$$

As previously said, in Section 2.4.2 we found a solution for these and thus a new representation for the hexagon at tree level.

We can also rewrite the full structure constant and for it we find the following

simplification of (2.34):

$$\begin{aligned} \mathcal{G}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = & \sum_{\delta_i \cup \bar{\delta}_i = \mathbf{u}_i} (-1)^{|\delta_1| + |\delta_2| + |\delta_3|} f^-(\bar{\delta}_1, \delta_1) f^-(\bar{\delta}_2, \delta_2) f^-(\bar{\delta}_3, \delta_3) \\ & \times a_{\ell_{31}}(\delta_1) a_{\ell_{12}}(\delta_2) a_{\ell_{23}}(\delta_3) \mathcal{H}(\delta_1, \delta_3, \delta_2) \mathcal{H}(\bar{\delta}_1, \bar{\delta}_2, \bar{\delta}_3). \end{aligned} \quad (2.48)$$

With all these tools from the structure constant in the $SU(2)$ and $SL(2)$ sectors discussed, let us end the review part by discussing the classical limit.

2.3 What is the classical limit?

There are many limits that one can take on the observables of $\mathcal{N} = 4$ SYM that may be of interest depending on the context: weak coupling, strong coupling, lightcone limit, and so on. One of them is the so-called *classical limit*. In the simplest terms, in this regime we consider heavy operators with many excitations on top of it. This limit is natural when one considers the strong coupling limit of $\mathcal{N} = 4$ SYM, where it can be analyzed in terms of classical string integrability in $AdS_5 \times S^5$ [90]. In this semiclassical regime the operators one observe correspond to string states with asymptotically large quantum numbers, *id est*, semiclassical string states [77]. Nonetheless this regime is also interesting at weak coupling, here similar integrability structures between the weak and strong coupling regimes start to emerge. One instance of this is the classical limit of structure constants in the $SU(2)$ sector at weak and strong coupling analyzed in [82], where the same classical integrability tools were used in both of regimes. Another instance is the famous Frolov-Tseytlin limit, which is defined by $\lambda, J_i \rightarrow \infty$ such that $\lambda/J_i^2 \lll 1$ with λ and J_i being the 't Hooft coupling and some charge, respectively [78]. Remarkably for some observables we have that an expansion in λ/J_i^2 is equal to the weak coupling expansion [51, 91], although some mismatches may happen at higher loops as explored in [79]. Therefore the classical limit is an important tool to understand the AdS/CFT duality.

Let us then detail this limit. We consider here the classical limit at weak coupling described in [80]. The classical limit is defined as limit of asymptotically large charges for the operators or Bethe states. Consider an operator in the $SU(2)$ sector with M Bethe roots and length ℓ , then this corresponds to

$$\ell \sim M \rightarrow \infty.$$

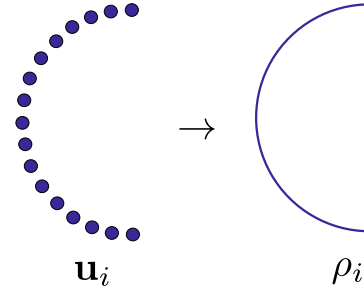


Figure 2.3: Condensation of roots. In the classical limit, the roots condense into cuts and we describe the set of roots $\mathbf{u}_i$ by a root density ρ_i in the complex plane.

In this limit we have that the Bethe roots are asymptotically large, *i.e.*, $u_i \sim \ell$. However their separation remain of order $\mathcal{O}(1)$. This means that if one rescales the roots as $u_i \rightarrow u_i/\ell$, then their separation is of order $\mathcal{O}(1/\ell)$ and the roots condense into a continuum, which is represented schematically in Figure 2.3.

This root condensation generates a cut in the density of roots. Indeed we can see this clearly by taking the classical limit of the Bethe equations. These in the $SU(2)$ sector are given by

$$\left(\frac{u_j + i/2}{u_j - i/2} \right)^\ell = \prod_{k \neq j} \frac{u_j - u_k + i}{u_j - u_k - i}. \quad (2.49)$$

We can take the logarithm of these equations, substitute $u_j = \ell x_j$ and then consider the large ℓ limit:

$$\frac{1}{x_j} = \frac{2}{\ell} \sum_{k \neq j} \frac{1}{x_j - x_k} - 2\pi n_j. \quad (2.50)$$

Here $n_j \in \mathbb{Z}$ is an integer that signals which branch of the logarithm one is considering. Note that we are taking a sum over all roots in the right hand side. In the classical limit, not only ℓ goes to infinity but also M . Since the root separation goes to zero we have a continuum as said before and the sum becomes

$$\frac{1}{\ell} \sum_k \dots = \int_{\mathcal{C}} dz \rho(z) \dots. \quad (2.51)$$

With $\rho(z)$ being the density of roots defined in the contour $\mathcal{C} = \cup_j \mathcal{C}_j$ where the roots condense. Note that each integer n_j that specifies the solution leads to a cut and the union of all of them describe the full operator. This means that the Bethe

equations in the classical limit become

$$\frac{G(z + i0) + G(z - i0)}{2} = \mathcal{G}(z) = \frac{1}{2z} + \pi n_p, \quad \text{where } z \in \mathcal{C}_p. \quad (2.52)$$

With $\mathcal{G}(z)$ being the principal value of $G(z)$. Here we used the resolvent function:

$$G(z) = \int_{\mathcal{C}} dx \frac{\rho(x)}{z - x}. \quad (2.53)$$

Note that since we are integrating on the rapidity cut $\mathcal{C}$ then if $z \in \mathcal{C}_p$ the function diverges. That is why in (2.52) one introduces the $\pm i0$ prescription which yields the principal value. This has its microscopic origin in the $k \neq j$ prescription in (2.50). Therefore the roots condense into cuts of the resolvent function, this fact will play a role later in the derivation of the classical limit of the structure constants in Section 2.4.

Up to now we discussed the classical limit for the $SU(2)$ sector. Nonetheless for the non-compact $SL(2)$ spin chain, in specific cases, it works very similarly to the compact case [81]. If we consider the large spin limit $S \rightarrow \infty$ and large length $\ell \rightarrow \infty$, with $\ell \sim S$, the discussion follows exactly as in the compact case. However this is not the only large spin limit one can consider for $SL(2)$ chains. Since here the spin S is not bounded by the length we can keep finite ℓ and take only $S \rightarrow \infty$. In this case we can not apply the same arguments as before for the condensation of roots. Here we will work strictly in the classical limit defined with both spin and length being large. There are also some notable differences between them. The $SL(2)$ case can be seen as the analytic continuation in spin of the compact $SU(2)$ one. This simple modification make the Bethe roots all real, *id est*, the cuts are real. At strong coupling these large spin states corresponds to folded string solutions (or GKP strings) and in this limit their structure function is known for specific polarizations [92, 93].

The idea now is to apply this classical limit to the structure constant. One sees that microscopically the characterization of the operator depends heavily on their Bethe roots and similarly for the structure constants. However as we have seen for the classical limit, the microscopic details about the Bethe roots become unimportant and the entire operator is described by a single density satisfying the functional equation (2.52). Similarly, we expect that such microscopic details at the classical limit for the expression (2.27) gets averaged out and we obtain then a clean expression for the structure constants as a function of the densities only. So

we need the classical limit of quantities appearing in (2.28). By directly taking the classical limit we have

$$f^\pm(z, \mathbf{u}) = e^{\pm iG(z)} \quad \text{and} \quad a_\ell(z) = e^{i\ell/z}. \quad (2.54)$$

Another useful function to introduce is the quasi-momentum:

$$p(z) = G(z) - \frac{\ell}{2z}. \quad (2.55)$$

Turns out that the classical limit of the structure constants depends on the densities only through the quasi-momenta. Note that the quasi-momenta are meromorphic functions with singularities at the origin and branch cuts at $\mathcal{C}$.

Up to now we considered the classical limit of the summand only, however we need to see how the sum over partitions is transformed in this limit. Since in the classical limit each set of Bethe roots is associated with a density, to each partition of the roots we also associate a density. Then given $\mathbf{u} = \alpha \cup \bar{\alpha}$ associated with $\rho(z)$ we have

$$\alpha \rightarrow \rho_\alpha(z) \quad \text{and} \quad \bar{\alpha} \rightarrow \rho_{\bar{\alpha}}(z), \quad \text{with the constraint} \quad \rho(z) = \rho_\alpha(z) + \rho_{\bar{\alpha}}(z).$$

Naturally the sum over partitions then turns into a functional integral over the densities $\rho_\alpha(z)$ and $\rho_{\bar{\alpha}}(z)$. Indeed:

$$\sum_{\alpha_k \cup \bar{\alpha}_k = \mathbf{u}_k} \cdots \rightarrow \int [d\rho_{\alpha_k}] [d\rho_{\bar{\alpha}_k}(z)] \cdots, \quad (2.56)$$

and in the summand we write everything as a function of the densities. In the end we can then compute this functional integral by saddle point method with the care of taking the constraint for the densities into account. This approach was developed in [52] and used to compute the classical limit of type I-I-II correlators for a single non-BPS operator. We will further detail the subtleties of this method for the type I-I-I correlators in Section 2.4. Note that for the mixed correlators this is not necessary since as shown in (2.24) and (2.25) the correlator can be written without the sum over partitions and the classical limit can be taken directly as it was did in [55].

Later we will compute the classical limit for type I-I-I correlators directly from the microscopic expression (2.27). However some results were already known in the literature. The classical limit of type I-I-II correlators was computed by many

methods and it is given by

$$\log \left(\frac{C_{123}^{\bullet\bullet\bullet}}{C_{123}^{\circ\circ\circ}} \right) = \text{pols} + \oint_{\mathcal{C}_1 \cup \mathcal{C}_2} \frac{dz}{2\pi} \text{Li}_2 \left(e^{ip_1 + ip_2 + i\ell_3/2z} \right) + \oint_{\mathcal{C}_3} \frac{dz}{2\pi} \text{Li}_2 \left(e^{ip_3 + i(\ell_2 - \ell_1)/2z} \right) - \frac{1}{2} \sum_{j=1}^3 \oint_{\mathcal{C}_j} \frac{dz}{2\pi} \text{Li}_2 \left(e^{2ip_j} \right), \quad (2.57)$$

where pols is just all the terms containing the polarizations, which are fixed by $SU(2)$ invariance

$$\begin{aligned} \text{pols} = & M_1 \log(1 + |z_1|^2) + M_2 \log(1 + |z_2|^2) + M_3 \log(1 + |\tilde{z}_3|^2) + \\ & - (M_1 + M_2) \log(z_{12}) + (M_1 - M_2) \log(z_{23}) + (M_2 - M_1) \log(z_{31}) + \\ & + M_3 \log(\tilde{z}_{12}) - M_3 \log(\tilde{z}_{23}) - M_3 \log(\tilde{z}_{31}), \end{aligned} \quad (2.58)$$

and $C_{123}^{\circ\circ\circ}$ is the structure constant of three BPS operators with the same lengths as the excited operators. Note that such configuration is considerably simpler than I-I-I, since each sector has at most two interacting operators. Indeed this class of correlators is equivalent in complexity to the two non-BPS correlators since in each sector at most two operators interact. To truly uncover the three-point function integrability one must also tackle the pure correlators, which we do here.

The classical limit for mixed correlators was first found using the aforementioned functional integral method for a single non-BPS in [52]. Then later in [94] this was extended for the full three non-BPS correlator by considering an expansion on $a_\ell(u)$ and then taking the classical limit on a specific ansatz. Both of these works were situated in the weak coupling regime, however in [95] using hexagonalization the authors were able to derive this result, but now at *finite coupling*. It still retains the same structure as in (2.57) with the quasi-momentum $p(u)$ being the finite coupling one, but they could explore only the type I-I-II correlators. This situation was partially remedied in [82], here for the first time the classical limit of the pure and mixed correlators were found at weak and strong coupling. However the method used was quite indirect and no limit of (2.27) was ever taken. The

classical limit of the type I-I-I correlator found in [82] is

$$\log \left(\frac{C_{123}^{\bullet\bullet\bullet}}{C_{123}^{\circ\circ\circ}} \right) = \text{pols} + \frac{1}{2} \sum_{\{i,j,k\}=\text{cperm}\{1,2,3\}} \oint_{\mathcal{C}_i \cup \mathcal{C}_j} \frac{dz}{2\pi} \text{Li}_2 \left(e^{ip_i + ip_j - ip_k} \right) + \\ - \frac{1}{2} \sum_{j=1}^3 \oint_{\mathcal{C}_j} \frac{dz}{2\pi} \text{Li}_2 \left(e^{2ip_j} \right). \quad (2.59)$$

Where the polarization term is given by:

$$\text{pols} = \sum_{j=1}^3 M_j \log(1 + |z_j|^2) - \sum_{\{i,j,k\}=\text{cperm}\{1,2,3\}} M_{ij} \log(z_{ij}). \quad (2.60)$$

Here $\text{cperm}\{1,2,3\}$ means all cyclic permutations of the set $\{1,2,3\}$. Note that unlike (2.57) we have that all the operators interact at once and not in a two-by-two fashion. In this work we attest the validity of this expression by finding the classical limit of pure correlators directly from the double spin chain formula. In the next section, we present such derivation.

Some results are also known in the $SL(2)$ sector. As said before we have two parameters to take the asymptotic limit which are ℓ and S . By taking both to infinity, in [91] it was computed the classical limit of $SL(2)$ structure constants with two non-BPS operators consisting of lightcone derivatives and a BPS one. This means that the derivatives lie in the plane and we are restricted to a specific polarization setting. Also it was observed that at leading order, it satisfied the weak/strong coupling relation in the Frolov-Tseytlin limit. With the review finally done, let us then discuss the classical limits of $SU(2)$ and $SL(2)$ correlators and the new hexagon representations in the next section.

2.4 New representations of the hexagons and classical limit

In this section we will compute the classical limit for type I-I-I correlators and of the hexagons for some choices of polarizations in the non-compact sector. A consequence of this is that we derive new representation of the hexagons.

2.4.1 $SU(2)$ sector

We will first derive our first new result, a new definition of the hexagon for the $SU(2)$ sector direct from the sum (2.27). Then we consider this formula and take the classical limit of pure correlators to derive (2.59) directly from the microscopic sum. We also derive recursion relations analogous to (2.42) - (2.47), but now for the $SU(2)$ case.

Deriving the hexagons from double spin chains

In the end we want to explore the classical limit of (2.27). Before doing this we will derive the tree level hexagon directly from the microscopic sum. In [94] it was introduced a method for computing the classical limit of the structure constants and we will use it here to prove this hexagon relation. The main idea of [94] consists in expanding $\mathcal{G}$ as a series on a_ℓ . Clearly, the first term of this series does not contain any a_ℓ and it is given by the following constraints:

$$\bar{\beta}_1 \cap \alpha_2 = \bar{\beta}_2 \cap \alpha_3 = \bar{\beta}_3 \cap \alpha_1 = \beta_1 \cap \bar{\alpha}_1 = \beta_2 \cap \bar{\alpha}_2 = \beta_3 \cap \bar{\alpha}_3 = \emptyset. \quad (2.61)$$

These conditions fix the subsets β_j and $\bar{\beta}_j$ of the subpartitions in (2.28) to be

$$\beta_1 = \alpha_2, \beta_2 = \alpha_3, \beta_3 = \alpha_1, \bar{\beta}_1 = \bar{\alpha}_1, \bar{\beta}_2 = \bar{\alpha}_2, \bar{\beta}_3 = \bar{\alpha}_3. \quad (2.62)$$

Therefore, plugging these back in (2.27), we get the zeroth order contribution that we denote by

$$\begin{aligned} \mathcal{H}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = & (-1)^{M_1+M_2+M_3} \sum_{\substack{\alpha_1 \cup \bar{\alpha}_1 = \mathbf{u}_1 \\ \alpha_2 \cup \bar{\alpha}_2 = \mathbf{u}_2 \\ \alpha_3 \cup \bar{\alpha}_3 = \mathbf{u}_3}} z_{12}^{M_2-M_3-|\alpha_2|+|\alpha_1|} z_{23}^{M_3-M_1-|\alpha_3|+|\alpha_2|} \times \\ & \times z_{31}^{M_1-M_2-|\alpha_1|+|\alpha_3|} f^+(\bar{\alpha}_1, \alpha_1) f^+(\bar{\alpha}_2, \alpha_2) f^+(\bar{\alpha}_3, \alpha_3) \\ & f^+(\alpha_2, \bar{\alpha}_1) f^+(\alpha_3, \bar{\alpha}_2) f^+(\alpha_1, \bar{\alpha}_3). \end{aligned} \quad (2.63)$$

Here by naming it as $\mathcal{H}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ we anticipate that in the end it will be equivalent to the hexagon form factor. This is independent of the polarizations too, indeed we can consider for example the hexagon with two excitations on a operator and

only one on the other two. We then find:

$$\begin{aligned} \mathcal{H}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = & \frac{i}{w_1 - v_1} - \frac{(u_1 - v_1 - i)(u_2 - v_1 - i)}{(u_1 - v_1)(v_1 - u_2)} + \\ & + i \frac{(u_1 - u_2 - i)(u_2 - v_1 - i)}{(u_1 - u_2)(u_2 - v_1)(u_1 - w_1)} + i \frac{(u_1 - u_2 + i)(u_1 - v_1 - i)}{(u_1 - u_2)(u_1 - v_1)(u_2 - w_1)}. \end{aligned} \quad (2.64)$$

For more complicated cases we can still check this independence on the polarizations, however we will not show this here because the expressions are too large and not illuminating. Note that $\mathcal{H}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ has a certain structure where we have weights $f^+(\bar{\alpha}_j, \alpha_j)$ associated with each operator and other weights $f^+(\alpha_i, \bar{\alpha}_j)$ related to interactions between “neighbor” partitions. Remarkably the same kind of structure remains for the for $SL(2)$ correlators as it will be shown in Section 2.4.2.

Now we need to analyze the rest of the terms in the $\mathcal{G}$ expansion and rewrite it in terms of $\mathcal{H}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$. Let us partition each root set as $\mathbf{u}_i = \delta_i \cup \bar{\delta}_i$ and consider the term in the expansion proportional to

$$a_{\ell_{31}}(\delta_1) a_{\ell_{12}}(\delta_2) a_{\ell_{23}}(\delta_3). \quad (2.65)$$

The partitions δ_i and $\bar{\delta}_i$ are defined such that given β_i we define $\gamma_i \cup \bar{\gamma}_i = \delta_i$ with $\bar{\beta}_i \cap \alpha_j = \bar{\gamma}_j$ and $\beta_i \cap \bar{\alpha}_i = \gamma_i$. These conditions fix the β_i and $\bar{\beta}_i$ partitions to be

$$\beta_i = \alpha_j / \bar{\gamma}_j \cup \gamma_i \text{ and } \bar{\beta}_i = \bar{\alpha}_i / \gamma_i \cup \bar{\gamma}_j. \quad (2.66)$$

We then insert these back into (2.27) and do some convoluted algebraic manipulations and groupings (detailed in Appendix C if the reader is interested) to obtain in the end the following expression for the structure constant

$$\begin{aligned} \mathcal{G}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = & \sum_{\substack{\delta_1 \cup \bar{\delta}_1 = \mathbf{u}_1 \\ \delta_2 \cup \bar{\delta}_2 = \mathbf{u}_2 \\ \delta_3 \cup \bar{\delta}_3 = \mathbf{u}_3}} (-1)^{|\delta_1| + |\delta_2| + |\delta_3|} f^+(\bar{\delta}_1, \delta_1) f^+(\bar{\delta}_2, \delta_2) f^+(\bar{\delta}_3, \delta_3) \\ & \times a_{\ell_{31}}(\delta_1) a_{\ell_{12}}(\delta_2) a_{\ell_{23}}(\delta_3) \mathcal{H}(\delta_1, \delta_3, \delta_2) \mathcal{H}(\bar{\delta}_1, \bar{\delta}_2, \bar{\delta}_3). \end{aligned} \quad (2.67)$$

This expression has exactly the same form as hexagonalization expression, as shown in Section 2.2, and we indeed identify $\mathcal{H}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ with the hexagon form factor at *tree level*³. This completes the proof in Appendix L of [57], where there it

³Note that in the hexagon formula we have the S-matrix $S(\bar{\delta}_1, \delta_1)$ and not $f^+(\bar{\delta}_1, \delta_1)$ for each

was derived the hexagon form factors from the double spin chain for at most one excitation in each operator. Here we worked it out for general excited operators thus proving the equality of these two formalisms at tree level. Turns out that the expression (2.63) lies at the heart of our derivation for the classical limit of type I-I-I correlator.

Decoupling and the classical limit

We saw in Section 2.2 that the hexagon form factors for the $SL(2)$ sector satisfy recursion relations related to the decoupling property. We will derive exactly the same relations but for the hexagon expression (2.63) in the $SU(2)$ sector. To derive these we look at (2.63) and note that it is a rational function with poles at coincident roots. There are two potential sources for these poles: $f^+(\bar{\alpha}_i, \alpha_i)$ and $f^+(\alpha_i, \bar{\alpha}_j)$. Note that if we take coincident roots x and y on the same operator there are two contributions for the residue $x \in \alpha_i$ and $y \in \bar{\alpha}_i$ or $x \in \bar{\alpha}_i$ and $y \in \alpha_i$. Since the function $f(u, v)$ is antisymmetric on the pole these two contributions cancel each other. Then we can have poles only for coincident roots on distinct operators. If we consider $x \in \mathbf{u}_i$ and $y \in \mathbf{u}_j$ we have the following equations

$$\text{Res}_{y \rightarrow x} \mathcal{H}(\mathbf{u}_1 \cup \{x\}, \mathbf{u}_2 \cup \{y\}, \mathbf{u}_3) = i f^+(x, \mathbf{u}_1) f^+(\mathbf{u}_2, x) \mathcal{H}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3), \quad (2.68)$$

$$\text{Res}_{y \rightarrow x} \mathcal{H}(\mathbf{u}_1 \cup \{x\}, \mathbf{u}_2, \mathbf{u}_3 \cup \{y\}) = i f^+(x, \mathbf{u}_3) f^+(\mathbf{u}_1, x) \mathcal{H}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3), \quad (2.69)$$

$$\text{Res}_{y \rightarrow x} \mathcal{H}(\mathbf{u}_1, \mathbf{u}_2 \cup \{x\}, \mathbf{u}_3 \cup \{y\}) = i f^+(x, \mathbf{u}_2) f^+(\mathbf{u}_3, x) \mathcal{H}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3). \quad (2.70)$$

Another decoupling condition we could derive is when a root goes to infinity, *id est*, the analogous of (2.45) - (2.47). It can be seen that these are

$$\lim_{x \rightarrow \infty} \mathcal{H}(\mathbf{u}_1 \cup \{x\}, \mathbf{u}_2, \mathbf{u}_3) = \mathcal{H}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3), \quad (2.71)$$

$$\lim_{x \rightarrow \infty} \mathcal{H}(\mathbf{u}_1, \mathbf{u}_2 \cup \{x\}, \mathbf{u}_3) = \mathcal{H}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3), \quad (2.72)$$

$$\lim_{x \rightarrow \infty} \mathcal{H}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \cup \{x\}) = \mathcal{H}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3). \quad (2.73)$$

Since at tree level the hexagon is fully rational, it can be fixed totally by these relations. In general it is hard to reduce the sum over partitions (2.63) to a single function, however in some cases it can be done. For example for one and two

operator. This is explained by the fact that $\mathcal{G}$ is equal to the hexagonalization expression up to some overall factor of $f^+(\mathbf{u}_i, \mathbf{u}_j)$. This doesn't affect the result since it is an overall factor and it was also observed in [96].

non-BPS operators it is simple to check that

$$\mathcal{H}(\mathbf{u}_1, \emptyset, \emptyset) = 1, \quad (2.74)$$

$$\mathcal{H}(\mathbf{u}_1, \mathbf{u}_2, \emptyset) = f^+(\mathbf{u}_2, \mathbf{u}_1), \quad (2.75)$$

solve all these decoupling relations. These cases when inserted on $\mathcal{G}$ yields the mixed correlators (2.24) and (2.25).

Just for completeness, we can check that (2.63) indeed satisfies the recursion relations (2.68) - (2.70). For example look at the first one, as said before the only contribution for the residues comes when $x \in \bar{\alpha}_1$ and $y \in \alpha_2$. Let $M_i = |\mathbf{u}_i|$ and redefine the partitions in (2.63) to be

$$\begin{aligned} \alpha_1 &= \gamma_1 & \alpha_2 &= \gamma_2 \cup \{x\} & \text{and} & \alpha_3 &= \gamma_3 \\ \bar{\alpha}_1 &= \bar{\gamma}_1 \cup \{x\} & \bar{\alpha}_2 &= \bar{\gamma}_2 & & \bar{\alpha}_3 &= \bar{\gamma}_3 \end{aligned},$$

then we obtain

$$\begin{aligned} \text{Res}_{y \rightarrow x} \mathcal{H}(\mathbf{u}_1 \cup \{x\}, \mathbf{u}_2 \cup \{y\}, \mathbf{u}_3) &= \sum_{\substack{\gamma_1 \cup \bar{\gamma}_1 = \mathbf{u}_1 \\ \gamma_2 \cup \bar{\gamma}_2 = \mathbf{u}_2 \\ \gamma_3 \cup \bar{\gamma}_3 = \mathbf{u}_3}} z_{12}^{M_2+1-M_3-|\gamma_2|-1+|\gamma_1|} \\ &\quad z_{23}^{M_3-M_1-1-|\gamma_3|+|\gamma_2|+1} z_{31}^{M_1+1-M_2-1-|\gamma_1|+|\gamma_3|} f^+(\bar{\gamma}_1 \cup \{x\}, \gamma_1) f^+(\bar{\gamma}_2, \gamma_2 \cup \{x\}) \\ &\quad f^+(\bar{\gamma}_3, \gamma_3) f^+(\gamma_3, \bar{\gamma}_2) f^+(\gamma_1, \bar{\gamma}_3) f^+(\gamma_2, \bar{\gamma}_1) f^+(\gamma_2, \{x\}) \\ &\quad f^+(\{x\}, \bar{\gamma}_1) \text{Res}_{y \rightarrow x} f^+(y, x), \quad (2.76) \end{aligned}$$

which after simplifications yield us exactly the recursion relation (2.68), with the factor of i coming from the residue of $f^+(y, x)$. Therefore all the recurrence relations (2.68) - (2.70) are satisfied and for the (2.71) - (2.73) the proof is direct. Note the cancelling of the extra roots' contribution in the polarizations, a step that does not happen in the $SL(2)$ case as we will show later.

We note here the power of the expression (2.63). As described before, in [67] the hexagon was described as a partition function for the $SL(2)$ sector. The same rewriting of the hexagon could be done in the $SU(2)$ sector by exchanging spinors to polarization vectors. Therefore, even at tree level the hexagon is a complicated partition function on a Kagome lattice. Here we shown that this complicated partition function can be written at tree level as a simple sum over partitions.

Up to now we did not relate these decoupling relations with the classical limit. Turns out that these are essential to our derivation of the hexagon's classical limit.

Based on the pole structure and the two non-BPS result, a natural guess for the hexagon for the full three non-BPS operators is

$$\mathcal{H}_{\text{ansatz}}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = f^+(\mathbf{u}_2, \mathbf{u}_1) f^+(\mathbf{u}_3, \mathbf{u}_2) f^+(\mathbf{u}_1, \mathbf{u}_3). \quad (2.77)$$

Despite its simplicity, the decoupling relations are not satisfied. Indeed for (2.68) we find

$$\text{Res}_{y \rightarrow x} \mathcal{H}_{\text{ansatz}}(\mathbf{u}_1 \cup \{x\}, \mathbf{u}_2 \cup \{y\}, \mathbf{u}_3) = i f^+(x, \mathbf{u}_1) f^+(\mathbf{u}_2, x) f^+(\mathbf{u}_3, x) f^+(x, \mathbf{u}_3) \mathcal{H}_{\text{ansatz}}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3). \quad (2.78)$$

It is now that the classical limit enters. Note that in this regime the Bethe roots are large and the roots of distinct operators are macroscopically distinct. This means that in the classical limit

$$f^+(\mathbf{u}_3, x) f^+(x, \mathbf{u}_3) \approx 1. \quad (2.79)$$

Note that this is only valid because $x \notin \mathbf{u}_3$, otherwise we would have roots with separation of $\mathcal{O}(1/\ell)$ and this argument would fail. Using this fact we derive that the classical limit of the hexagon satisfy the decoupling relations. Therefore using the decoupling conditions we can establish that our ansatz (2.77) is indeed the hexagon, however *only* in the classical limit.

Now we have all the tools to compute the classical limit of the pure correlators. Inserting our ansatz for the hexagon at the classical limit in (2.67) we derive the following expression for the structure constant:

$$\frac{\mathcal{G}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)}{\mathcal{H}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)} = \prod_{\{i,j,k\}=\text{cperm}\{1,2,3\}} \left(\sum_{\alpha_i \cup \tilde{\alpha}_i = \mathbf{u}_i} (-1)^{|\alpha_i|} a_{\ell_{ki}}(\alpha_i) \frac{f^+(\tilde{\alpha}_i, \alpha_i)}{f^+(\alpha_i, \mathbf{u}_k) f^+(\mathbf{u}_j, \alpha_i)} \right), \quad (2.80)$$

Remarkably the sum over partitions is decoupled now and the classical limit becomes much easier to take, indeed we will simply reuse the techniques of [52]. We will do it step-by-step below.

First we need to turn the sum over partitions in (2.80) into a functional integral

over the root's densities as aforementioned in Section 2.3. We then find

$$\sum_{\alpha_i \cup \bar{\alpha}_i = \mathbf{u}_i} (-1)^{|\alpha_i|} \frac{a_{\ell_{ki}}(\alpha_i) f^+(\bar{\alpha}_i, \alpha_i)}{f^+(\alpha_i, \mathbf{u}_k) f^+(\mathbf{u}_j, \alpha_i)} \rightarrow \int [D\rho_{\alpha_i}] \exp \left(\int_{\mathcal{C}_i} du \mathcal{F}[\rho_{\alpha_i}] + i\rho_{\alpha_i} q_i \right). \quad (2.81)$$

Here we already solved the constraint and integrate over ρ_{α_i} only. Also we recombined the classical limits of $a_{\ell_{ki}}(\alpha_i)$ and $f^+(\alpha_i, \mathbf{u}_k) f^+(\mathbf{u}_j, \alpha_i)$ to obtain $q_i(z)$ which contains the quasi-momenta and it is given by

$$q_i(z) = \pi + p_j(z) - p_k(z) - p_i(z). \quad (2.82)$$

and $q_i(z)$ denotes the principal value prescription for the quasi-momenta evaluated at the cut $\mathcal{C}_i$. The missing ingredient here is $\mathcal{F}[\rho_{\alpha_i}]$, which is called the stochastic anomaly.

When passing from the sum over partitions to the functional integral, there are many contributions one has to take into account. The classical limit of $f^+(u, v)$ for roots belonging to distinct operators was given in (2.54). However if the roots belong to the same operator as in $f^+(\bar{\alpha}_i, \alpha_i)$ we have a problem. Basically there are roots with separation of order $\mathcal{O}(1/\ell)$ such that the final contribution in the product is of order $\mathcal{O}(1)$. These contributions are harder to compute the classical limit, nonetheless this was achieved in [52] and encompassed in the so-called stochastic anomaly:

$$\mathcal{F}[\rho_{\alpha_i}] = F(\rho_i) - F(\rho_{\alpha_i}) - F(\rho_{\bar{\alpha}_i}) \quad \text{with} \quad F(\rho) = \int_0^\rho dx \log \sinh(\pi x). \quad (2.83)$$

The specialist reader may have missed the entropy contribution. This extra term arises from the fact that multiple *microscopic* contributions describe the same *macroscopic* configurations. The same term also appears in the formulation of Thermodynamic Bethe Ansatz [30]. However this contribution is already included in the stochastic anomaly, therefore we have the full classical limit of the structure constant and it remains just to compute these functional integrals.

To calculate the functional integrals (2.81) we use the saddle point method. The saddle point equations for $\rho_{\alpha_i}(z)$ are easily derivable and given by

$$\log \left(\frac{\sinh(\pi \rho_{\alpha_j})}{\sinh(\pi(\rho_j - \rho_{\alpha_j}))} \right) = i q_j. \quad (2.84)$$

These are very similar to the saddle point equations for a single non-BPS operator derived in [52]. We can invert these equations for $\rho_{\alpha_i}(z)$ and we find the following simple result after we substitute in the saddle point

$$\log \frac{\mathcal{G}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)}{\mathcal{H}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)} = - \sum_{\{i,j,k\}=\text{cperm}\{1,2,3\}} \oint_{\mathcal{C}_i} \frac{du}{2\pi} \text{Li}_2 \left(e^{-ip_k - ip_i + ip_j} \right). \quad (2.85)$$

These are different from the expected result (2.59) due to the presence of single contour terms and the hexagon prefactor. Now we convert this expression into the previously known result.

To find the same as in [82] we have to join the contours $\mathcal{C}_i$ in (2.85). For this end we will use the Bethe equations and some judicious contour manipulations. We can rewrite the classical limit of the Bethe equations (2.52) in terms of the quasi-momenta as

$$p_j(u + i0) + p_j(u - i0) = 2\pi n_j^{(k)} \quad \text{for } u \in \mathcal{C}_j^{(k)}. \quad (2.86)$$

Where as before $\cup_k \mathcal{C}_j^{(k)} = \mathcal{C}_j$ and $n_j^{(k)} \in \mathbb{Z}$. The nice thing about this form of Bethe equations is that we can use it to manipulate $p(u)$. It tell us that whenever one crosses a rapidity cut the following changes occur: $p(u) \rightarrow -p(u)$ modulo 2π and $\mathcal{C}_j \rightarrow -\mathcal{C}_j$. So any crossing of a cut changes the orientation of the integration contour and flips the exponential of the quasi-momenta.

Let us do these manipulations then. Consider for example the first term in (2.85), we start by splitting it in the following way

$$\begin{aligned} \oint_{\mathcal{C}_1} \frac{du}{2\pi} \text{Li}_2 \left(e^{-ip_3 - ip_1 + ip_2} \right) &= -\frac{1}{2} \oint_{\mathcal{C}_1} \frac{du}{2\pi} \text{Li}_2 \left(e^{ip_1 + ip_2 - ip_3} \right) + \\ &+ \frac{1}{2} \oint_{\mathcal{C}_1} \frac{du}{2\pi} \text{Li}_2 \left(e^{-ip_3 - ip_1 + ip_2} \right). \end{aligned} \quad (2.87)$$

In the first term we used the Bethe equations to go to the second sheet, then the sign of the integral and p_1 changed. To deal with the second term we consider the following dilogarithm identity

$$\text{Li}_2(x) = -\text{Li}_2(-1/x) - \frac{\pi^2}{6} - \frac{\log^2(1/x)}{2}. \quad (2.88)$$

With this we invert the p_j in the second term of (2.87) and find:

$$\oint_{\mathcal{C}_1} \frac{du}{2\pi} \text{Li}_2 \left(e^{-ip_3 - ip_1 + ip_2} \right) = -\frac{1}{2} \oint_{\mathcal{C}_1} \frac{du}{2\pi} \text{Li}_2 \left(e^{ip_1 + ip_2 - ip_3} \right) + \\ -\frac{1}{2} \oint_{\mathcal{C}_1} \frac{du}{2\pi} \text{Li}_2 \left(e^{ip_1 + ip_3 - ip_2} \right) - \frac{1}{4} \oint_{\mathcal{C}_1} \frac{du}{2\pi} \log^2 \left(-e^{ip_3 + ip_1 - ip_2} \right). \quad (2.89)$$

We can do similar manipulations in all the other terms of (2.85) and we find exactly the contributions that recombine to almost yield us the desired result

$$\log \frac{\mathcal{G}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)}{\mathcal{H}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)} = \text{rest} + \frac{1}{2} \sum_{\{i,j,k\}=\text{cperm}\{1,2,3\}} \oint_{\mathcal{C}_i \cup \mathcal{C}_j} \frac{dz}{2\pi} \text{Li}_2 \left(e^{ip_i + ip_j - ip_k} \right). \quad (2.90)$$

Here rest are all the terms obtained when flipping the dilogarithms and are simply

$$\text{rest} = \sum_{\{i,j,k\}=\text{cperm}\{1,2,3\}} \frac{1}{4} \oint_{\mathcal{C}_i} \frac{du}{2\pi} \log^2 \left(-e^{ip_i - ip_j + ip_k} \right). \quad (2.91)$$

After somewhat extensive, but simple, algebra one can derive that the hexagon prefactor $\log \mathcal{H}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ exactly cancels the term rest up to phases that can be reabsorbed by operator redefinitions.

Therefore we find our final result for the classical limit of type I-I-I correlators

$$\log \left(\frac{C_{123}^{\bullet\bullet\bullet}}{C_{123}^{\circ\circ\circ}} \right) = \text{pols} + \frac{1}{2} \sum_{\{i,j,k\}=\text{cperm}\{1,2,3\}} \oint_{\mathcal{C}_i \cup \mathcal{C}_j} \frac{dz}{2\pi} \text{Li}_2 \left(e^{ip_i + ip_j - ip_k} \right) + \\ - \frac{1}{2} \sum_{j=1}^3 \oint_{\mathcal{C}_j} \frac{dz}{2\pi} \text{Li}_2 \left(e^{2ip_j} \right). \quad (2.92)$$

Which is the same as (2.59). Note here that we included the polarizations' contributions and the classical limit of the determinant part of the operator's norm which are the single contour terms (see Appendix A). The main point of this derivation is that we used the decoupling relations and with them the classical limit's computation of this structure constant directly from the microscopic formula becomes straightforward. This then validates the result of [82] which were obtained in an indirect way.

2.4.2 $SL(2)$ sector

Here we derive the classical limit of the hexagon in the non-compact $SL(2)$ sector. As in the previous section we first find a new representation for the hexagon based on the recursion relations of Section 2.2 and with them we compute the hexagon's classical limit in some specific cases.

A new representation for the $SL(2)$ hexagon

For the compact sector, we found a new expression for the hexagon at tree level from tailoring in (2.63). The question now is that if we can do the same for the more complicated case of the $SL(2)$ sector. We remember that the recursion relations (2.42) - (2.47) uniquely define for us the hexagon partition function since it's a rational function, therefore if we find a solution for these we have solved the problem. Based on the $SU(2)$ case we propose for the non-compact scenario

$$\begin{aligned} \mathcal{H}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = & \sum_{\substack{\alpha_1 \cup \bar{\alpha}_1 = \mathbf{u}_1 \\ \alpha_2 \cup \bar{\alpha}_2 = \mathbf{u}_2 \\ \alpha_3 \cup \bar{\alpha}_3 = \mathbf{u}_3}} h_1^{|\alpha_1|} (1 - h_1)^{|\bar{\alpha}_1|} h_2^{|\alpha_2|} (1 - h_2)^{|\bar{\alpha}_2|} h_3^{|\alpha_3|} (1 - h_3)^{|\bar{\alpha}_3|} \\ & \times f^-(\bar{\alpha}_1, \alpha_1) f^-(\bar{\alpha}_2, \alpha_2) f^-(\bar{\alpha}_3, \alpha_3) f^-(\alpha_2, \bar{\alpha}_1) f^-(\alpha_3, \bar{\alpha}_2) f^-(\alpha_1, \bar{\alpha}_3), \end{aligned} \quad (2.93)$$

with the new variables h_1, h_2 and h_3 being defined as

$$h_2(1 - h_1) = t_{12}, \quad h_3(1 - h_2) = t_{23}, \quad \text{and} \quad h_1(1 - h_3) = t_{31}. \quad (2.94)$$

At first sight it is not clear that this new hexagon representation is a polynomial in the t_{ij} variables, however we can do some simple tests to check this. Consider for example an hexagon with 2, 2 and 1 excitations on each side, then we have

$$\mathcal{H}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = 1 + \frac{it_{23}(v_1 + v_2 - 2w_1 + i)}{(v_1 - w_1)(v_2 - w_1)} + \frac{it_{31}(u_1 + u_2 - 2w_1 - i)}{(u_1 - w_1)(w_1 - u_2)} + \dots \quad (2.95)$$

Note that if indeed (2.93) satisfies the recursion relations then it must be a polynomial of t_{ij} . The novelty of this expression is that it reduces the complicated sum over configurations in the partition function of [67] to a single sum over partitions. This simplifies the computation at tree level and maybe it can give a hint to an all loop expansion of it. Below we check that this expression indeed satisfy the recursion relations.

Let us first compute the residue in expression (2.93), we note that the only

nonzero contribution for the residue happens when $x \in \bar{\alpha}_1$ and $y \in \alpha_2$. The proof works exactly as in the $SU(2)$ case. The $y \rightarrow x$ residue is

$$\begin{aligned} \text{Res}_{y \rightarrow x} \mathcal{H}(\mathbf{u}_1 \cup \{x\}, \mathbf{u}_2 \cup \{y\}, \mathbf{u}_3) &= \sum_{\substack{\alpha_1 \cup \bar{\alpha}_1 = \mathbf{u}_1 \cup \{x\} \\ \alpha_2 \cup \bar{\alpha}_2 = \mathbf{u}_2 \cup \{x\} \\ \alpha_3 \cup \bar{\alpha}_3 = \mathbf{u}_3}} h_1^{|\alpha_1|} (1 - h_1)^{|\bar{\alpha}_1|} h_2^{|\alpha_2|} (1 - h_2)^{|\bar{\alpha}_2|} \\ &\quad h_3^{|\alpha_3|} (1 - h_3)^{|\bar{\alpha}_3|} f^-(\bar{\alpha}_1, \alpha_1) f^-(\bar{\alpha}_2, \alpha_2) f^-(\bar{\alpha}_3, \alpha_3) f^-(\alpha_3, \bar{\alpha}_2) f^-(\alpha_1, \bar{\alpha}_3) \\ &\quad f^-(\alpha_2 / \{x\}, \bar{\alpha}_1 / \{x\}) f^-(\alpha_2 / \{x\}, \{x\}) f^-(\{x\}, \bar{\alpha}_1 / \{x\}) \text{Res}_{y \rightarrow x} f^-(y, x), \quad (2.96) \end{aligned}$$

redefining the partitions to be

$$\begin{aligned} \alpha_1 &= \gamma_1 & \alpha_2 &= \gamma_2 \cup \{x\} & \alpha_3 &= \gamma_3 \\ \bar{\alpha}_1 &= \bar{\gamma}_1 \cup \{x\} & \bar{\alpha}_2 &= \bar{\gamma}_2 & \bar{\alpha}_3 &= \bar{\gamma}_3 \end{aligned}, \quad \text{and}$$

we find that

$$\begin{aligned} \text{Res}_{y \rightarrow x} \mathcal{H}(\mathbf{u}_1 \cup \{x\}, \mathbf{u}_2 \cup \{y\}, \mathbf{u}_3) &= -ih_2(1 - h_1) f^-(\mathbf{u}_2, x) f^-(x, \mathbf{u}_1) \\ &\quad \sum_{\substack{\gamma_1 \cup \bar{\gamma}_1 = \mathbf{u}_1 \\ \gamma_2 \cup \bar{\gamma}_2 = \mathbf{u}_2 \\ \gamma_3 \cup \bar{\gamma}_3 = \mathbf{u}_3}} h_1^{|\gamma_1|} (1 - h_1)^{|\bar{\gamma}_1|} h_2^{|\gamma_2|} (1 - h_2)^{|\bar{\gamma}_2|} h_3^{|\gamma_3|} (1 - h_3)^{|\bar{\gamma}_3|} f^-(\bar{\gamma}_1, \gamma_1) f^-(\bar{\gamma}_2, \gamma_2) \\ &\quad f^-(\bar{\gamma}_3, \gamma_3) f^-(\gamma_2, \bar{\gamma}_1) f^-(\gamma_3, \bar{\gamma}_2) f^-(\gamma_1, \bar{\gamma}_3). \quad (2.97) \end{aligned}$$

Which yields the recursion relation (2.42) if we use the definition of h_i variables and the new hexagon representation:

$$\text{Res}_{y \rightarrow x} \mathcal{H}(\mathbf{u}_1 \cup \{x\}, \mathbf{u}_2 \cup \{y\}, \mathbf{u}_3) = -it_{12} f^-(x, \mathbf{u}_1) f^-(\mathbf{u}_2, x) \mathcal{H}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3). \quad (2.98)$$

With similar manipulations we can prove the remaining recursion relations. Note that the proof of (2.45) - (2.47) is trivial due to the form of $f^-(u, v)$. Therefore the expression (2.93) is indeed the hexagon partition function since it solves all the recursion relations.

Another simple test we can do is to check the case of abelian hexagon. As explained in [67], one of the manifestations of this case is given by $\mathbf{u}_2 = \mathbf{u}_3 = \emptyset$ which yields

$$\mathcal{H}(\mathbf{u}, \emptyset, \emptyset) = 1. \quad (2.99)$$

We can prove this directly from (2.93). Here we have

$$\mathcal{H}(\mathbf{u}, \emptyset, \emptyset) = \sum_{\alpha \cup \bar{\alpha} = \mathbf{u}} h_1^{|\alpha|} (1 - h_1)^{|\bar{\alpha}|} f^-(\bar{\alpha}, \alpha). \quad (2.100)$$

This is a rational function with poles for coincident roots, however it is easy to see that the residue vanishes. Indeed, let $x, y \in \mathbf{u}$ and consider the residue $y \rightarrow x$. There are two contributions to it: $x \in \alpha$ and $y \in \bar{\alpha}$ or $x \in \bar{\alpha}$ and $y \in \alpha$. Since $f^-(u, v)$ is antisymmetric at the pole these two cancel each other. Then $\mathcal{H}(\mathbf{u}, \emptyset, \emptyset)$ is a polynomial function and due to the asymptotic conditions (2.45) - (2.47) it must be unity. Therefore we can properly match our new representation with this abelian case. Note that this abelian case can also be obtained if $(t_{12}, t_{23}, t_{31}) = (0, 0, 0)$ which is equivalent to $(h_1, h_2, h_3) = (0, 0, 0)$. Another less trivial abelian case we can match corresponds to $(t_{12}, t_{23}, t_{31}) = (1, 0, 0)$, which is equivalent to $(h_1, h_2, h_3) = (0, 1, 0)$ or $(h_1, h_2, h_3) = (0, 1, 1)$. These choices truncate the sum over partitions (2.93) to

$$\mathcal{H}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = f^-(\mathbf{u}_2, \mathbf{u}_1), \quad (2.101)$$

which matches the abelian case (at tree level) in [67]. These tests further validate our expression (2.93) for the hexagon partition function in the $SL(2)$ sector for general polarized operators.

For these abelian cases we can even write simple expressions for the full structure constant $\mathcal{G}$. For this we introduce the so-called $\mathcal{A}$ -functional by

$$\mathcal{A}_{\mathbf{u}}^{\pm}[g] = \sum_{\alpha \cup \bar{\alpha} = \mathbf{u}} (-1)^{|\alpha|} g(\alpha) f^{\pm}(\alpha, \bar{\alpha}), \quad (2.102)$$

which has a determinant expression as proved in [94]

$$\mathcal{A}_{\mathbf{u}}^{\pm}[g] = \frac{\det_{ab} (u_a^{b-1} - g(u_a)(u_a \pm i)^{b-1})}{\det_{ab} (u_a^{b-1})}. \quad (2.103)$$

Note that if choose $g(z) = a_{\ell}(z)$ we have the pDWPF of (2.19) minus the polynomial part⁴. If we pick $(t_{12}, t_{23}, t_{31}) = (0, 0, 0)$ for the polarizations, it can be seen that

$$\mathcal{G} = \mathcal{A}_{\mathbf{u}_1}^+[a_{\ell_{31}}] \mathcal{A}_{\mathbf{u}_2}^+[a_{\ell_{12}}] \mathcal{A}_{\mathbf{u}_3}^+[a_{\ell_{23}}]. \quad (2.104)$$

⁴Here we are committing a slight abuse of notation, since we introduce $\mathcal{A}$ also in Appendix B, however for a specific choice of $g(z)$.

Now for the other abelian choice $(t_{12}, t_{23}, t_{31}) = (1, 0, 0)$ we find

$$\mathcal{G} = \mathcal{A}_{\mathbf{u}_3}^+[a_{\ell_{23}}] \sum_{\substack{\alpha_1 \cup \bar{\alpha}_1 = \mathbf{u}_1 \\ \alpha_2 \cup \bar{\alpha}_2 = \mathbf{u}_2}} (-1)^{|\alpha_1| + |\alpha_2|} a_{\ell_{31}}(\alpha_1) a_{\ell_{12}}(\alpha_2) f^-(\bar{\alpha}_1, \alpha_1) f^-(\bar{\alpha}_2, \alpha_2) f^-(\alpha_2, \alpha_1) f^-(\bar{\alpha}_2, \bar{\alpha}_1). \quad (2.105)$$

Which is like the type I-I-II correlator in the $SU(2)$ sector. In [67] there are Pfaffian representations of this correlator.

A case that has no analog in the $SU(2)$ sector and we can analyze is the one with $(t_{12}, t_{23}, t_{31}) = (t, 0, 0)$ and $0 < t < 1$. This is equivalent to choosing $(h_1, h_2, h_3) = (0, t, 0)$ which then yields the following hexagon:

$$\mathcal{H}(\delta_1, \delta_2, \delta_3) = \sum_{\alpha_2 \cup \bar{\alpha}_2 = \delta_2} t^{|\alpha_2|} (1-t)^{|\bar{\alpha}_2|} f(\bar{\alpha}_2, \alpha_2) f(\alpha_2, \delta_1). \quad (2.106)$$

Clearly the limits $t \rightarrow 0$ and $t \rightarrow 1$ gives the same abelian cases discussed before. Using the $\mathcal{A}$ -functional we can write this in closed form

$$\mathcal{H}(\delta_1, \delta_2, \delta_3) = (1-t)^{|\delta_2|} \mathcal{A}_{\delta_2}^+ \left[\frac{t}{t-1} f^-(z, \delta_1) \right]. \quad (2.107)$$

Therefore the correlator becomes simply

$$\mathcal{G} = (1-t)^{M_2} \mathcal{A}_{\mathbf{u}_3}^+[a_{\ell_{23}}] \sum_{\substack{\alpha_1 \cup \bar{\alpha}_1 = \mathbf{u}_1 \\ \alpha_2 \cup \bar{\alpha}_2 = \mathbf{u}_2}} (-1)^{|\alpha_1| + |\alpha_2|} a_{\ell_{31}}(\alpha_1) a_{\ell_{12}}(\alpha_2) f^-(\bar{\alpha}_1, \alpha_1) f^-(\bar{\alpha}_2, \alpha_2) \mathcal{A}_{\delta_2}^+ \left[\frac{t}{t-1} f^-(z, \delta_1) \right] \mathcal{A}_{\bar{\delta}_2}^+ \left[\frac{t}{t-1} f^-(z, \bar{\delta}_1) \right]. \quad (2.108)$$

Which is a generalization of the previous formula for the abelian cases. The reader can worry that this expression is not polynomial in t , however the prefactor $(1-t)^{M_2}$ guarantees this property.

Another nice thing that we can do with (2.93) is to extract from the structure constant $\mathcal{G}$ the contribution of a specific tensor structure. Let us derive this now. The hexagon (2.93) can be rewritten as

$$\mathcal{H}(\delta_1, \delta_2, \delta_3) = \sum_{m_1=0}^{|\delta_1|} \sum_{m_2=0}^{|\delta_2|} \sum_{m_3=0}^{|\delta_3|} h_1^{m_1} (1-h_1)^{|\delta_1|-m_1} h_2^{m_2} (1-h_2)^{|\delta_2|-m_2} h_3^{m_3} \times (1-h_3)^{|\delta_3|-m_3} \mathcal{H}_{m_1 m_2 m_3}(\delta_1, \delta_2, \delta_3), \quad (2.109)$$

where

$$\mathcal{H}_{m_1 m_2 m_3}(\delta_1, \delta_2, \delta_3) = \sum_{\substack{\alpha_j \cup \bar{\alpha}_j = \delta_j \\ |\alpha_j| = m_j}} f^-(\bar{\alpha}_1, \alpha_1) f^-(\bar{\alpha}_2, \alpha_2) f^-(\bar{\alpha}_3, \alpha_3) f^-(\alpha_2, \bar{\alpha}_1) \times \\ f^-(\alpha_3, \bar{\alpha}_2) f^-(\alpha_1, \bar{\alpha}_3). \quad (2.110)$$

We then insert (2.109) into the structure constant (2.48) and we find the

$$\mathcal{G} = \sum_{p=0}^{M_1} \sum_{q=0}^{M_2} \sum_{r=0}^{M_3} t_{12}^p t_{23}^q t_{31}^r \mathcal{G}_{pqr}, \quad (2.111)$$

with $\mathcal{G}_{pqr}$ being a specific tensor contribution given by⁵

$$\mathcal{G}_{pqr} = \sum_{\delta_i \cup \bar{\delta}_i = \mathbf{u}_i} (-1)^{|\delta_1| + |\delta_2| + |\delta_3|} f^-(\bar{\delta}_1, \delta_1) f^-(\bar{\delta}_2, \delta_2) f^-(\bar{\delta}_3, \delta_3) a_{\ell_{31}}(\delta_1) a_{\ell_{12}}(\delta_2) \times \\ a_{\ell_{23}}(\delta_3) \sum_{m_1=0}^{|\delta_1|} \sum_{m_2=0}^{|\delta_2|} \sum_{m_3=0}^{|\delta_3|} \sum_{\bar{m}_1=0}^{|\bar{\delta}_1|} \sum_{\bar{m}_2=0}^{|\bar{\delta}_2|} \sum_{\bar{m}_3=0}^{|\bar{\delta}_3|} \mathcal{H}_{m_1 m_2 m_3}(\delta_1, \delta_2, \delta_3) \mathcal{H}_{\bar{m}_1 \bar{m}_2 \bar{m}_3}(\bar{\delta}_1, \bar{\delta}_2, \bar{\delta}_3) \\ \delta_{r, m_1 + \bar{m}_1} \delta_{p, m_2 + \bar{m}_2} \delta_{q, m_3 + \bar{m}_3} \delta_{m_1 + \bar{m}_1, M_3 - m_3 - \bar{m}_3} \delta_{m_2 + \bar{m}_2, M_1 - m_1 - \bar{m}_1} \delta_{m_3 + \bar{m}_3, M_2 - m_2 - \bar{m}_2}. \quad (2.112)$$

To derive this expression we just use the previous definitions and the fact that $\mathcal{G}$ is polynomial in t_{ij} . This implies that the only contributions to a specific tensor structure are the ones in (2.112) and that is why the Kronecker deltas appear. With this discussion about the new hexagon representation done, we now move to the classical limit in the $SL(2)$ sector.

Some classical limits

We can also consider the classical limit in the $SL(2)$ sector, however due to the dependence on the polarizations t_{12} , t_{23} and t_{31} we could not work in full generality as in the $SU(2)$ case. As we specified in Section 2.3 we consider the large spin $J_i \rightarrow \infty$ and large operators $\ell \sim J_i$. The simplest cases to workout the classical limit are the ones dual to pure and mixed correlators of the compact case.

⁵Here there is an abuse of notation, in the last line of this equation the $\delta_{a,b}$ factors are Kronecker deltas and not partitions.

Indeed the abelian cases considered in the previous section

$$\begin{aligned} (t_{12}, t_{23}, t_{31}) = (0, 0, 0) &\longleftrightarrow \mathcal{H}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = 1, \\ (t_{12}, t_{23}, t_{31}) = (1, 0, 0) &\longleftrightarrow \mathcal{H}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = f(\mathbf{u}_2, \mathbf{u}_1). \end{aligned}$$

The first case is the trivial hexagon, however the latter one is in fact dual to the type I-I-II correlators where in fact only two operators interact. Indeed we can see that the decoupling identities match. The more complicated case $(t_{12}, t_{23}, t_{31}) = (1, 1, 1)$ is dual to the type I-I-I correlator due to the same reasoning. The classical limit of these cases discussed here can be found using exactly the same methods as in the previous section since for these the polarization structure is fixed. Therefore they are not very interesting.

The more complicated case where we can analyze the classical limit is the one with polarizations $(t_{12}, t_{23}, t_{31}) = (t, 0, 0)$ and $0 < t < 1$, which was discussed before. We can take the classical limit of the hexagon (2.106) using the same tools as in Section 2.4.1. Thus we find

$$\log \mathcal{H}(\delta_1, \delta_2, \delta_3) = \oint_{\mathcal{C}_2} \frac{du}{2\pi} \text{Li}_2 \left(\frac{t}{t-1} e^{iG_{\delta_1}(u) - iG_{\delta_2}(u)} \right). \quad (2.113)$$

Where $G_{\delta_j}(u)$ is the following resolvent defined by:

$$G_{\delta_j}(u) = \int_{\mathcal{C}_j} \frac{dz}{2\pi} \frac{\rho_{\delta_j}(z)}{z - u}. \quad (2.114)$$

Using the Bethe equations we can rewrite the classical limit (2.113) as an integral on the cut and not around it

$$\begin{aligned} \log \mathcal{H}(\delta_1, \delta_2, \delta_3) = \int_{\mathcal{C}_2} \frac{du}{2\pi} \text{Li}_2 \left(\frac{t}{t-1} e^{iG_{\delta_1}(u) - iG_{\delta_2}(u)} \right) - \\ \text{Li}_2 \left(\frac{t}{t-1} e^{iG_{\delta_1}(u) + iG_{\delta_2}(u) - i\ell_2/u} \right). \end{aligned} \quad (2.115)$$

As seen in (2.108) the sum over partitions of $\mathbf{u}_3$ entirely decouple and we end up with the following result in the classical limit

$$\mathcal{G} = \exp \left(\oint_{\mathcal{C}_3} \frac{dz}{2\pi} \text{Li}_2(e^{ip_3}) \right) \int [d\rho_{\delta_1}] [d\rho_{\delta_2}] e^{S_{\text{eff}}}, \quad (2.116)$$

For the part that decouples we used the classical limit results of [52]. Now the

effective action S_{eff} is given by

$$S_{\text{eff}} = \mathcal{F}[\rho_{\delta_1}] + \mathcal{F}[\rho_{\delta_2}] + \int_{\mathcal{C}_1} du i q_1 \rho_{\delta_1} + \int_{\mathcal{C}_2} du i q_2 \rho_{\delta_2} + \log \mathcal{H}(\delta_1, \delta_2, \delta_3) + \log \mathcal{H}(\bar{\delta}_1, \bar{\delta}_2, \bar{\delta}_3), \quad (2.117)$$

with $\mathcal{F}[\rho_{\delta_j}]$ being the stochastic anomaly (see Section 2.4.1) and q_j is

$$q_1(u) = \pi + \frac{\ell_{31}}{u} + \int_{\mathcal{C}_1} \frac{dz}{2\pi} \frac{\rho_1(z)}{z-u} \quad \text{and} \quad q_2(u) = \pi + \frac{\ell_{12}}{u} + \int_{\mathcal{C}_2} \frac{dz}{2\pi} \frac{\rho_2(z)}{z-u}. \quad (2.118)$$

We now use the saddle point method to compute this functional integral. However we were not able to continue the computation of the three-point function from here. For example, if we try to compute the saddle point equations, we end up with a complicated system of coupled integral equations for the densities:

$$\begin{aligned} & \log \left(\frac{\sinh(\pi(\rho_1(v) - \rho_{\alpha_1}(v)))}{\sinh(\pi\rho_{\alpha_1}(v))} \right) + i q_1(v) + \\ & \int_{\mathcal{C}_2} \frac{dx}{2\pi i} \frac{1}{v-x} \log \left(\frac{t-1 - t e^{iG_{\delta_1}(x) - iG_{\delta_2}(x)}}{t-1 - t e^{iG_1(x) - iG_{\delta_1}(x) - iG_2(x) + iG_{\delta_2}(x)}} \right) + \\ & \int_{\mathcal{C}_2} \frac{dx}{2\pi i} \frac{1}{v-x} \log \left(\frac{t-1 - t e^{iG_1(x) - iG_{\delta_1}(x) + iG_2(x) - iG_{\delta_2}(x) - i\ell_2/x}}{t-1 - t e^{iG_{\delta_1}(x) + iG_{\delta_2}(x) - i\ell_2/x}} \right) = 0, \quad (2.119) \end{aligned}$$

$$\begin{aligned} & \log \left(\frac{\sinh(\pi(\rho_2(v) - \rho_{\alpha_2}(v)))}{\sinh(\pi\rho_{\alpha_2}(v))} \right) + i q_2(v) + \\ & \int_{\mathcal{C}_2} \frac{dx}{2\pi i} \frac{1}{v-x} \log \left(\frac{t-1 - t e^{iG_1(x) - iG_{\delta_1}(x) - iG_2(x) + iG_{\delta_2}(x)}}{t-1 - t e^{iG_{\delta_1}(x) - iG_{\delta_2}(x)}} \right) + \\ & \int_{\mathcal{C}_2} \frac{dx}{2\pi i} \frac{1}{v-x} \log \left(\frac{t-1 - t e^{iG_1(x) - iG_{\delta_1}(x) + iG_2(x) - iG_{\delta_2}(x) - i\ell_2/x}}{t-1 - t e^{iG_{\delta_1}(x) + iG_{\delta_2}(x) - i\ell_2/x}} \right) = 0. \quad (2.120) \end{aligned}$$

Which we could not solve for now, but in the limits $t \rightarrow 0$ or $t \rightarrow 1$ it reduces to the classical limit of the abelian cases.

2.5 Chapter summary

With this we conclude our explorations on hexagons in $AdS_5 \times S^5$ and the classical limit. Let us remember briefly what we achieved here. In Section 2.1

we reviewed the double spin chain formulation, introduced the pure and mixed correlators and rewrote the type I-I-I structure constants in a way that makes $SU(2)$ invariance manifest. Then in Section 2.2 we reviewed the hexagon description for spinning operators and rewrote them in a notation that resembles the $SU(2)$ case. We close the review part in Section 2.3 by describing the classical limit and citing some results in the literature. In Section 2.4 we concentrated the results. By rewriting the tailoring results we found new representations of the hexagons at tree level in the $SU(2)$ and $SL(2)$ sectors. These forms satisfy recurrence relations that resembles the ones found in [67]. With them we were able to compute directly from the microscopic formulas the classical limit of a class of structure constants in the $SU(2)$ sector: the type I-I-I correlators. In the $SL(2)$ sector we were able to explore the classical limit in some regimes with fixed polarizations. With this done we move now to hexagonalization of structure constants in $AdS_3 \times S^3 \times T^4$ in the next chapter.

Chapter 3

Hexagonalization in AdS_3/CFT_2

Now we do a change on the focus, we move from the classical limit of structure constants in $\mathcal{N} = 4$ SYM to hexagonalization in the $AdS_3 \times S^3 \times T^4$ duality. We start in Section 3.1 by addressing how integrability works in this background and highlight differences to the more well known $AdS_5 \times S^5$ case. Then in Section 3.2 we describe the structure of half-BPS operators in this theory, which is quite novel in comparison with $\mathcal{N} = 4$ SYM. With this integrability setup ready we then describe how to define hexagons and write the structure constants of the dual CFT_2 using them in Section 3.3. We wrap up in Section 3.4 by describing some simple structure constants using hexagonalization.

3.1 Integrability in $AdS_3 \times S^3 \times T^4$

The integrability description of $AdS_3 \times S^3 \times T^4$ works remarkably similar to the $AdS_5 \times S^5$ case, however we have here some extra details that have deep impact on the physics of the model. This background has been found to be classically and quantum integrable, with the spectral problem being analyzed in depth [33, 38, 39, 40, 41, 42, 43]. Below we remark some specifics of this duality that have no counterpart in $AdS_5 \times S^5$ and make it interesting to analyze.

As said in the introduction we consider a type IIB superstring theory on $AdS_3 \times S^3 \times T^4$ spacetime. Unlike in its higher dimensional cousin, it can support both Ramond-Ramond (RR) and Neveu-Schwarz (NSNS) fluxes [38]. The RR and NSNS fluxes, in units of AdS radius, are denoted by $h \in \mathbb{R}^+$ and $\kappa \in \mathbb{N}$, respectively. Also this target space can be seen as the near horizon limit of the F1-NS5-D1-D5 brane system [8]. There are two interesting regimes one can consider:

- **Pure NSNS:** Here we have $h = 0$ and the worldsheet theory is equivalent a $\mathfrak{sl}(2, \mathbb{R})$ WZW model with level κ [47]. The brane system here is F1-NS5 [8]. The dual CFT_2 is conjectured to be a deformation of the symmetric product orbifold of N copies of T^4 (denoted by $Sym_N(T^4)$). Some proposals for this

deformation are given in [14, 15]. The *only* point in the full moduli where the dual CFT_2 is known is $\kappa = 1$ which is exactly $Sym_N(T^4)$ [10].

- **Pure RR:** Here we have $\kappa = 0$ and a D1-D5 brane system [8]. Also the string model is very similar to the $AdS_5 \times S^5$ case, therefore due to its complexity not many results are known in this front in comparison with the simpler pure NSNS case. The dual CFT_2 theory in this case is not known, however it is known that it is in the same moduli as the $Sym_N(T^4)$. A proposal in terms of a Higgs branch CFT_2 is given in [13] and another proposal as a deformation of $Sym_N(T^4)$ is given in [12].

Note that the moduli space is higher dimensional, however we limit ourselves here to the near horizon limit and to states with no winding and momentum on the T^4 directions, therefore the only relevant moduli parameters are h and κ [8]. Note that this is not the only possible instance of the AdS_3/CFT_2 duality, indeed we can also have the $AdS_3 \times S^3 \times K3$ and $AdS_3 \times S^3 \times S^3 \times S^1$ target space which are integrable [97, 98, 99], however we focus here only in the aforementioned target space. We now describe the integrable structure of the model.

3.1.1 From string theory to the symmetry algebra

We will start by showing how the isometries of the superstring model reduces to the symmetry group after the lightcone gauge-fixing procedure. Note here that we do not present this subject matter in its full details, rather we refer the reader to [33, 38]. For example, rather than discuss the full coset action on $\mathfrak{psu}(1,1|2)^{\oplus 2}$ we will consider a simpler setup in terms of the bosonic non-linear sigma model (NLSM), this will suffice for our purposes.

The isometries of $AdS_3 \times S^3 \times T^4$ are $\mathfrak{psu}(1,1|2)^{\oplus 2} \oplus \mathfrak{u}(1)^{\oplus 4}$ which contains the bosonic subalgebras $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$, $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$, and $\mathfrak{u}(1)^{\oplus 4}$ [33]. The first two algebras can be interpreted as the conformal algebra and R-charge of the dual CFT_2 , here we nicely see the split into the chiral and antichiral factors of the conformal algebra. The latter are the isometries of T^4 . Now the NLSM action that describes the bosonic part of closed strings in $AdS_3 \times S^3 \times T^4$ is simply

$$S = -\frac{h}{2} \int_{-l/2}^{l/2} \int_{-\infty}^{+\infty} d\sigma d\tau \sqrt{|\gamma|} \gamma^{\alpha\beta} G_{MN}(X) \partial_\alpha X^M \partial_\beta X^N, \quad (3.1)$$

where l is the string length, $\gamma^{\alpha\beta}$ the worldsheet metric, $G_{MN}(X)$ the $AdS_3 \times S^3 \times T^4$ metric, and X^M the target space coordinates. Also (τ, σ) are the worldsheet

coordinates with τ and σ being the temporal and spatial coordinates, respectively. Note that the NLSM action has Weyl and diffeomorphism invariance, therefore we have non-physical degrees of freedom before the gauge-fixing. The simplest way of doing the gauge-fixing to obtain the spectrum and reduced symmetry algebra is by lightcone gauge-fixing¹. See [101, 102] for the general explanation of the gauge-fixing procedure and [33, 38, 103] for the application to AdS_3/CFT_2 context, which we will follow for the rest of section.

Let us denote the target space's coordinates by

$$AdS_3 : (t, z_1, z_2), \quad S^3 : (\phi, y_1, y_2) \quad \text{and} \quad T^4 : (X^1, X^2, X^3, X^4).$$

Here t is a time coordinate and ϕ an orbital angle in S^3 . Therefore we can define the conserved charges related to translations on these variables by

$$H^{t.s.} = - \int_{-l/2}^{l/2} d\sigma \, p_t \quad \text{and} \quad J = \int_{-l/2}^{l/2} d\sigma \, p_\phi. \quad (3.2)$$

Where p_t and p_ϕ are the conjugated canonical momentum to the t and ϕ coordinates, respectively. We interpret $H^{t.s.}$ as the target space hamiltonian, whose eigenvalues would correspond to the scaling dimensions in the dual CFT_2 and J to the magnetic R-charge. The neat part of the lightcone gauge-fixing is that it relates the target space hamiltonian with the worldsheet one that is related to τ -translations. Then the scaling dimensions of the dual CFT_2 states are related to the energies of the worldsheet excitations.

We define the lightcone coordinates to be:

$$x_- = \phi - t \quad \text{and} \quad x_+ = \frac{\phi + t}{2}, \quad (3.3)$$

and the lightcone gauge-fixing condition is simply:

$$x_- = \tau \quad \text{and} \quad p_+ = 1. \quad (3.4)$$

This completely fixes the gauge symmetry of the action and the remaining Hilbert space is physical. These conditions have two consequences. The first one implies

¹Here we deliberately ignore some subtleties of the lightcone gauge-fixing procedure that may appear as highlighted in [100] since we want to only demonstrate the method.

that the target space and worldsheet hamiltonian are related by

$$H^{t.s} = H + J, \quad (3.5)$$

then diagonalizing the worldsheet hamiltonian H is the same as finding the spectrum in the dual CFT_2 . This is where integrability techniques enter, we diagonalize the two-dimensional worldsheet hamiltonian, which is quantum integrable, using Bethe ansatz and then we match it with the dual CFT_2 scaling dimensions. The second gauge-fixing condition implies that

$$\int_{-l/2}^{l/2} d\sigma p_+ = l \Rightarrow \frac{H^{t.s.} + J}{2} = l. \quad (3.6)$$

Which means that we lost scaling invariance on the worldsheet theory, due to the gauge-fixing, since the string length l is fixed by the charges of the state.

Note that this gauge-fixing eliminates the unphysical propagation modes in the string. Indeed, using the Virasoro conditions we can fix p_- as a function of the physical coordinates. Therefore it can be seen that

$$H = -p_-(x_i, y_i, X_i, p_j). \quad (3.7)$$

The Virasoro constraints are non-linear due to the structure of the target space metric G_{MN} , thus we can expand p_- perturbatively as a series in h around $h \rightarrow \infty$. For the first term that is quadratic in the fields we find the following hamiltonian:

$$H = \sum_{j=1}^2 \left(\frac{p_{z_j}^2}{2} + \frac{(\partial_\sigma z_j)^2}{2} + \frac{z_j^2}{2} \right) + \sum_{j=1}^2 \left(\frac{p_{y_j}^2}{2} + \frac{(\partial_\sigma y_j)^2}{2} + \frac{y_j^2}{2} \right) + \sum_{j=1}^4 \left(\frac{p_{X_j}^2}{2} + \frac{(\partial_\sigma X_j)^2}{2} \right). \quad (3.8)$$

Then we see that the worldsheet theory has four massive excitations (of mass $m = 1$) corresponding to AdS_3 and S^3 modes and also four massless excitations ($m = 0$) corresponding to the T^4 modes. We see here the elimination of the two longitudinal string modes. These eight string modes are the fundamental excitations of the model as we will see in the next section. The massless T^4 modes are the main source of distinction to $AdS_5 \times S^5$ case where all excitations are massive, we will see that they bring some new features in the dual CFT_2 (see Section 3.2 for example) that are absent in its higher dimensional cousins.

Due to the lightcone gauge-fixing, not all the charges in the isometries will be conserved. Indeed, all the conserved charges must not have explicit dependence on x_- , which reduces the symmetry algebra to $\mathfrak{psu}(1|1)_{c.e.}^{\oplus 4}$. These will be the symmetries that our excitations will transform and in the next section we write representations of them. We can write this algebra as two copies of $\mathfrak{psu}(1|1)^{\oplus 2}$ centrally extended, indeed let $\{q, s, \tilde{q}, \tilde{s}, H\}$ span this algebra. Then it is given by

$$\begin{aligned} \{q, s\} &= H, & \{\tilde{q}, \tilde{s}\} &= \tilde{H}, \\ \{q, \tilde{q}\} &= C, & \{s, \tilde{s}\} &= C^+, \end{aligned} \quad (3.9)$$

where C is the so-called central extension that links the two factors of the $\mathfrak{psu}(1|1)^{\oplus 2}$ and H and $\tilde{H}$ are the left and right hamiltonian related to each $\mathfrak{sl}(2, \mathbb{R})$ factor in the CFT_2 conformal group. To describe the full $\mathfrak{psu}(1|1)_{c.e.}^{\oplus 4}$ algebra we define its generators by

$$\begin{aligned} Q^1 &= q \otimes \mathbb{I} & S_1 &= s \otimes \mathbb{I} & \tilde{Q}_1 &= \tilde{q} \otimes \mathbb{I} & \tilde{S}^1 &= \tilde{s} \otimes \mathbb{I} \\ Q^2 &= \Sigma \otimes q & S_2 &= \Sigma \otimes s & \tilde{Q}_2 &= \Sigma \otimes \tilde{q} & \tilde{S}^2 &= \Sigma \otimes \tilde{s} \end{aligned} \quad (3.10)$$

where $\mathbb{I}$ is the identity operator and Σ the graded identity operator. Then it just follows

$$\begin{aligned} \{Q^A, S_B\} &= \delta_B^A H, & \{\tilde{Q}_A, \tilde{S}^B\} &= \delta_A^B \tilde{H}, \\ \{Q^A, \tilde{Q}_B\} &= \delta_B^A C, & \{S_A, \tilde{S}^B\} &= \delta_A^B C^+. \end{aligned} \quad (3.11)$$

This is the full $\mathfrak{psu}(1|1)_{c.e.}^{\oplus 4}$ symmetry that is left after the lightcone gauge-fixing procedure. Also we have the conditions $(Q^A)^\dagger = S_A$ and $(\tilde{Q}_A)^\dagger = \tilde{S}^A$.

The central charge is the same in both the $\mathfrak{psu}(1|1)^{\oplus 2}$ factors and it is given by

$$C = \frac{i\hbar}{2}(e^{iP} - 1)e^{2i\zeta}. \quad (3.12)$$

Where P is the worldsheet momentum and $\zeta \in \mathbb{C}$ is a constant. Note that on physical states we have the level matching condition, *id est*, $P = 0$ for these states. Therefore the central extension vanishes for physical states, that means we have an off-shell central extension of $\mathfrak{psu}(1|1)^{\oplus 4}$. It is remarkable that the worldsheet momentum appears in the off-shell symmetry algebra, since it is not a charge in the original isometry group $\mathfrak{psu}(1, 1|2)^{\oplus 2}$. Here $\zeta \in \mathbb{C}$ is a number that labels the representation and it is related in the worldsheet theory to boundary conditions of the string. Note also that the algebra has a $\mathfrak{su}(2)_\bullet$ automorphism that acts on the supercharges. This automorphism is related to the $\mathfrak{so}(4) = \mathfrak{su}(2)_\bullet \oplus \mathfrak{su}(2)_\circ$ local

rotational symmetry of T^4 . This plays a role here since we are considering in the decompactified worldsheet, thus the T^4 is not probed and its $u(1)^{\oplus 4}$ isometry is enhanced to $\mathfrak{so}(4)$ [103, 104]. The supercharges organize in doublets of $\mathfrak{su}(2)_\bullet$ and the massless excitations are arranged in $\mathfrak{su}(2)_\circ$ multiplets.

The presence of the central charge has a deep impact in the theory. Let $\mathcal{V}_p$ be the one-particle space with worldsheet momentum p . Then we have that the central charge is not additive in the two-particle state $\mathcal{V}_{p_1} \otimes \mathcal{V}_{p_2}$, which modifies nontrivially the algebraic structure. Indeed, consider the usual coproduct map that defines a charge $\mathcal{Q}$ to act on $\mathcal{V}_{p_1} \otimes \mathcal{V}_{p_2}$ defined by

$$\mathcal{Q}_{(12)}(p_1, p_2) = \mathcal{Q}(p_1, \xi_1) \otimes \mathbb{I} + \Sigma \otimes \mathcal{Q}(p_2, \xi_2), \quad (3.13)$$

where $\mathcal{Q}(p, \xi)$ is a charge on a representation of $\mathfrak{psu}(1|1)_{c.e.}^{\oplus 4}$ labeled by worldsheet momentum p and ξ . For this to be consistent with the $\mathfrak{psu}(1|1)_{c.e.}^{\oplus 4}$ algebra we have two options: fix $\xi_j = 0$ and modify the coproduct structure or keep this trivial coproduct and pick ξ_j appropriate for each factor. Here we will pick the first choice and modify the coproduct, this makes $\mathfrak{psu}(1|1)_{c.e.}^{\oplus 4}$ to have the structure of a Hopf algebra². Then we pick the following modification for the coproduct:

$$\begin{aligned} Q_{(12)}^A(p_1, p_2) &= Q^A(p_1) \otimes \mathbb{I} + e^{ip_1/2} \Sigma \otimes Q^A(p_2), \\ \tilde{Q}_{A,(12)}(p_1, p_2) &= \tilde{Q}_A(p_1) \otimes \mathbb{I} + e^{ip_1/2} \Sigma \otimes \tilde{Q}_A(p_2), \\ S_{A,(12)}(p_1, p_2) &= S_A(p_1) \otimes \mathbb{I} + e^{-ip_1/2} \Sigma \otimes S_A(p_2), \\ \tilde{S}_{(12)}^A(p_1, p_2) &= \tilde{S}^A(p_1) \otimes \mathbb{I} + e^{-ip_1/2} \Sigma \otimes \tilde{S}^A(p_2). \end{aligned} \quad (3.14)$$

Remember that here $\xi_j = 0$ and then it can be checked that these choices realize the $\mathfrak{psu}(1|1)_{c.e.}^{\oplus 4}$ algebra in the two-particle space. This coproduct structure is fundamental in the derivation of the Borsato–Ohlsson–Sax–Sfondrini S-matrix, an object which will be central in the definition of the hexagons. Note also that the central charge is proportional to the RR flux h , therefore in the pure NSNS limit they vanish. So all these considerations we described here do not exist in this limit. We discuss more about the consequences of this in the Conclusion (Chapter 6).

A direct consequence of this nontrivial algebraic structure is that the energy dispersion relation can be found exactly in terms of the RR and NSNS fluxes. Let

²A Hopf algebra is an algebraic structure that generalizes Lie algebras and the most glaring distinction from the latter is the modification on the coproduct and non-linear Lie bracket. The appearance of Hopf algebras in this case is not an accident, in fact they are quite ubiquitous in the theory of integrable models. See for instance [105, 106].

the energy E and a $u(1)$ -charge M be

$$E = H + \tilde{H} \quad \text{and} \quad M = H - \tilde{H}. \quad (3.15)$$

Note that M in the dual CFT_2 is a combination of spin and R-charge, therefore it is quantized in integers. Then it possess the following eigenvalues

$$M = \frac{kp}{2\pi} + m, \quad (3.16)$$

where $m \in \mathbb{Z}$ and for a single particle $p = 0 \bmod 2\pi$. Thus if we consider representations of the $\mathfrak{psu}(1|1)_{c.e.}^{\oplus 4}$ algebra that satisfy the following shortening condition

$$H\tilde{H} = CC^\dagger, \quad (3.17)$$

we can totally fix the energy dispersion relation to be

$$E = \sqrt{\left(\frac{kp}{2\pi} + m\right)^2 + 4h^2 \sin(p/2)^2}, \quad (3.18)$$

where m has the interpretation of the mass of particles. So for $m = 0$ we have the T^4 massless modes. For $m = \pm 1$ we have the massive modes of AdS_3 and S^3 . These are the fundamental modes in the hamiltonian (3.8). Here we classify the massive particles into two classes: left (L) with $m \geq 1$ and right (R) ones with $m \leq -1$. The bound states are composite states of these and have mass $|m| \geq 2$. In the next section we will describe the spectrum in more detail.

From now on we will work out in the *pure RR* regime ($\kappa = 0$). A reason to do this is that the integrable structure is better controlled in this case. Indeed the dressing phases of the S-matrix are unknown in the mixed flux regime, making the definitions of the hexagon incomplete there. Also in this pure RR limit the analysis simplify and it is similar to the $AdS_5 \times S^5$ case, so we can make comparisons with it which is very instructive.

Instead of using the momentum parametrization, it is more convenient to use Zhukovsky variables defined by

$$x(u) + \frac{1}{x(u)} = \frac{u}{h}, \quad (3.19)$$

where u is the so-called spectral parameter or rapidity that was used extensively

in Chapter 2. The solution of this equation is

$$x(u) = \frac{u + i\sqrt{4h^2 - u^2}}{2h}. \quad (3.20)$$

The advantage of these over the momentum parametrization is that all quantities like the representation coefficients of $\mathfrak{psu}(1|1)_{c.e.}^{\oplus 4}$ and the S-matrix are rational functions of $x^\pm(u) = x(u \pm i/2)$ as we will show later. Another reason for this choice is that at weak coupling it has the expansion

$$x^\pm(u) = \frac{u \pm i/2}{h} - \frac{h}{u \pm i/2} - \frac{h^3}{(u \pm i/2)^3} + \mathcal{O}(h^5). \quad (3.21)$$

Which is the same weak coupling expansion as in [39] for the spin chain analysis of the model. This is exactly the same as in the $AdS_5 \times S^5$ case. We can also define

$$x^{[\pm a]}(u) = x\left(u \pm \frac{ia}{2}\right). \quad (3.22)$$

To describe massive fundamental particles we use only $a = 1$, bound states we use $a \geq 2$ and for massless particles we use $a \rightarrow 0$.

Now we can describe the physical quantities like the energy and momentum as a function of the rapidity. The energy and momentum for massive modes are given by

$$E(u) = \frac{h}{2i} \left(x^+(u) - \frac{1}{x^+(u)} - x^-(u) + \frac{1}{x^-(u)} \right), \quad (3.23)$$

$$p(u) = \frac{1}{i} \log \left(\frac{x^+(u)}{x^-(u)} \right). \quad (3.24)$$

To obtain the energy and momentum for bound states we just do $x^\pm \rightarrow x^{[\pm Q]}$. We choose the branches of the Zhukovsky variables to have $|x^\pm(u)| \geq 1$ for physical particles. This restricts $u \in \mathbb{R}$ for massive states, this is the so-called physical region. In this region is easy to see that energy is positive and momentum is real and periodic $-\pi < p < \pi$. The momentum and energy of a massive excitation are shown in Figure 3.1. Similar expressions can be found for massless particles. We denote their Zhukovsky variables by

$$x_o^+(u) = x(u) \quad \text{and} \quad x_o^-(u) = \frac{1}{x(u)}. \quad (3.25)$$

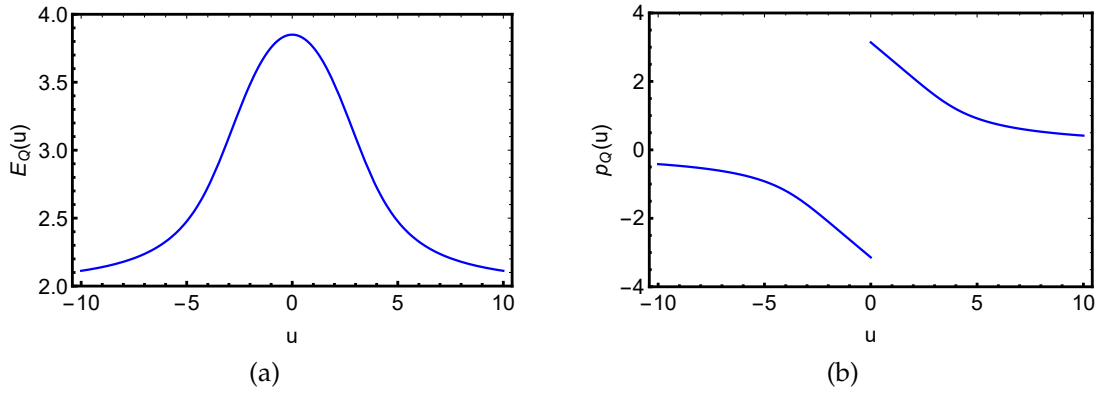


Figure 3.1: Energy and momentum of a massive excitation. These are the energy (a) and momentum (b) of a excitation with mass $m = 2$ at coupling $h = \pi^2/6$. Note that momentum is between $-\pi$ and π and that energy is positive in the physical region.

So they satisfy the constraint $x_o^+(u)x_o^-(u) = 1$ and then all functions of the massless Zhukovsky variables can be expressed only in terms of $x(u)$. Note that here we will choose to rescale u for massless modes to be hu such that then:

$$x(u) = \frac{u + i\sqrt{4 - u^2}}{2}. \quad (3.26)$$

So whenever we consider massless modes this is the parametrization we are taking into account. Thus the energy and momentum are simply the $a \rightarrow 0$ limit of the massive ones:

$$E(u) = \frac{h}{i} \left(x(u) - \frac{1}{x(u)} \right) \quad \text{and} \quad p(u) = \frac{1}{i} \log \left(x(u)^2 \right). \quad (3.27)$$

Here the physical region corresponds to $|x(u)| = 1$ and this restricts $u \in [-2, 2]$. Then in this region, energy is non-negative and momentum is periodic as expected and shown in Figure 3.2. Note that if we had considered mixed flux theories it would be necessary to define three sets of Zhukovsky variables, this further complicates the analysis, specially with respect to crossing and mirror transformations [38]. In the next section we describe the spectrum, *id est*, representations of $\mathfrak{psu}(1|1)_{c.e.}^{\oplus 4}$.

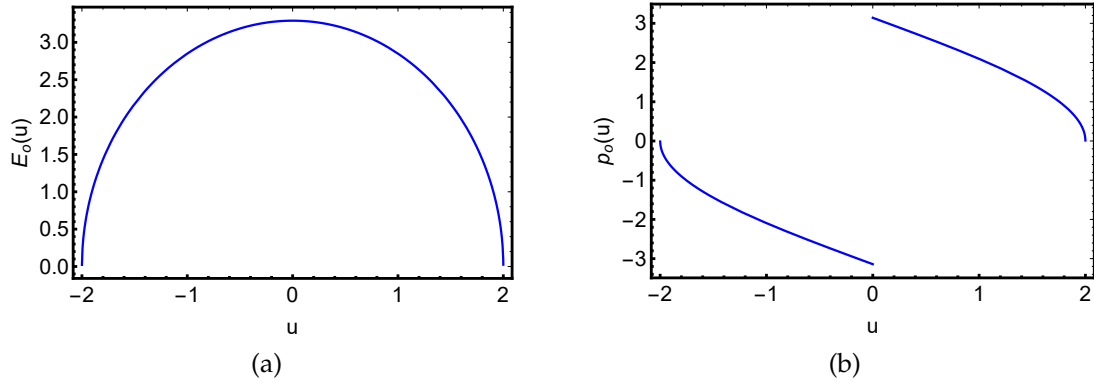


Figure 3.2: Energy and momentum of a massless excitation. On the left (a) we have the energy and on the right the momentum (b) of a massless excitation. Note that momentum is again between $-\pi$ and π and that energy is positive in the physical region $u \in [-2, 2]$.

3.1.2 Particle content

To describe the spectrum we can consider tensor products of representations of $\mathfrak{psu}(1|1)^{\oplus 2}$ that satisfy the shortening condition. The representations of this algebra are two-dimensional and are labeled by $(\mathbf{1}|\mathbf{1})$. We use the following notation for the states in a $(\mathbf{1}|\mathbf{1})$ representation: $|\phi\rangle$ and $|\varphi\rangle$ denote a highest and lowest weight state, respectively. Clearly the lowest weight state has the opposite statistics of the highest weight one and this can be either bosonic or fermionic denoted by superscripts B and F , respectively. There are four types of representations of interest for us³

$$\rho_L = \begin{pmatrix} \phi^B \\ \varphi^F \end{pmatrix}, \quad \rho_R = \begin{pmatrix} \phi^F \\ \varphi^B \end{pmatrix}, \quad \rho_\circ = \begin{pmatrix} \phi_\circ^B \\ \varphi_\circ^F \end{pmatrix}, \quad \text{and} \quad \rho'_\circ = \begin{pmatrix} \phi_\circ^F \\ \varphi_\circ^B \end{pmatrix}. \quad (3.28)$$

The ρ_L and ρ_R will be used to describe the left and right particles, respectively. Also $\rho_\circ$ and $\rho'_\circ$ will describe the massless particles. Contrast this with the case of $AdS_5 \times S^5$ where the excitations transform into a single $(\mathbf{2}|\mathbf{2})$ multiplet of $\mathfrak{psu}(2|2)$ for fundamental particles [107], so the $AdS_3 \times S^3 \times T^4$ is simpler in this regard.

Let us see how the $\mathfrak{psu}(1|1)^{\oplus 2}$ acts on these states, here we follow [74]. For ρ_L

³Since there are some distinct notations for the spectrum scattered throughout the literature we stick with the one in [74].

we have that

$$\begin{aligned} q|\phi^B(u)\rangle &= a(u)|\phi^F(u)\rangle, & s|\phi^F(u)\rangle &= a^*(u)|\phi^B(u)\rangle, \\ \tilde{q}|\phi^F(u)\rangle &= b(u)|\phi^B(u)\rangle, & \tilde{s}|\phi^B(u)\rangle &= b^*(u)|\phi^F(u)\rangle. \end{aligned} \quad (3.29)$$

Now for ρ_R we have that

$$\begin{aligned} q|\phi^F(u)\rangle &= b(u)|\phi^B(u)\rangle, & s|\phi^B(u)\rangle &= b^*(u)|\phi^F(u)\rangle, \\ \tilde{q}|\phi^B(u)\rangle &= a(u)|\phi^F(u)\rangle, & \tilde{s}|\phi^F(u)\rangle &= a^*(u)|\phi^B(u)\rangle. \end{aligned} \quad (3.30)$$

And for the massless $\rho_\circ$ and $\rho'_\circ$ representations:

$$\begin{aligned} q|\phi_\circ^B(u)\rangle &= a_\circ(u)|\phi_\circ^F(u)\rangle, & s|\phi_\circ^F(u)\rangle &= a_\circ^*(u)|\phi_\circ^B(u)\rangle, \\ \tilde{q}|\phi_\circ^F(u)\rangle &= b_\circ(u)|\phi_\circ^B(u)\rangle, & \tilde{s}|\phi_\circ^B(u)\rangle &= b_\circ^*(u)|\phi_\circ^F(u)\rangle, \end{aligned} \quad (3.31)$$

$$\begin{aligned} q|\phi_\circ^F(u)\rangle &= a_\circ(u)|\phi_\circ^B(u)\rangle, & s|\phi_\circ^B(u)\rangle &= a_\circ^*(u)|\phi_\circ^F(u)\rangle, \\ \tilde{q}|\phi_\circ^B(u)\rangle &= b_\circ(u)|\phi_\circ^F(u)\rangle, & \tilde{s}|\phi_\circ^F(u)\rangle &= b_\circ^*(u)|\phi_\circ^B(u)\rangle. \end{aligned} \quad (3.32)$$

Where the coefficients are

$$\begin{aligned} a(u) &= e^{i\tilde{\zeta}}\eta(u), & a^*(u) &= e^{-i\tilde{\zeta}}\sqrt{\frac{x^-(u)}{x^+(u)}}\eta(u), \\ b(u) &= -e^{i\tilde{\zeta}}\frac{\eta(u)}{\sqrt{x^+(u)x^-(u)}}, & b^*(u) &= -e^{-i\tilde{\zeta}}\frac{\eta(u)}{x^+(u)}, \\ a_\circ(u) &= e^{i\tilde{\zeta}}\eta_\circ(u), & a_\circ^*(u) &= e^{-i\tilde{\zeta}}\frac{\eta_\circ(u)}{x(u)}, \\ b_\circ(u) &= -e^{i\tilde{\zeta}}\eta_\circ(u), & b_\circ^*(u) &= -e^{-i\tilde{\zeta}}\frac{\eta_\circ(u)}{x(u)}, \end{aligned} \quad (3.33)$$

with

$$\eta(u) = \left(\frac{x_u^+}{x_u^-}\right)^{1/4} \sqrt{\frac{ih(x_u^- - x_u^+)}{2}} \quad \text{and} \quad \eta_\circ(u) = \sqrt{\frac{ih(1 - x_u^2)}{2}}. \quad (3.34)$$

Note that we used $x(u) = x_u$ to simplify notation. With this we can build representations of the full $\mathfrak{psu}(1|1)_{c.e.}^{\oplus 4}$ algebra which we do below.

The first representation we can construct is the one with L particles made out of $\rho_L \otimes \rho_L$. Then the particle content is:

$$\begin{aligned} |Y(u)\rangle &= |\phi^B(u) \otimes \phi^B(u)\rangle, & |Z(u)\rangle &= |\phi^F(u) \otimes \phi^F(u)\rangle, \\ |\Psi^1(u)\rangle &= |\phi^F(u) \otimes \phi^B(u)\rangle, & |\Psi^2(u)\rangle &= |\phi^B(u) \otimes \phi^F(u)\rangle. \end{aligned} \quad (3.35)$$

These have the dispersion relation (3.18) with mass $m = 1$ and $\kappa = 0$ since we are working in the pure RR limit. Also Y and Z correspond to the S^3 and AdS_3

massive modes of the string in the hamiltonian (3.8), respectively. The fermions Ψ^A are their superpartners and they form a doublet of $\mathfrak{su}(2)_\bullet$. Using the relations (3.29) we have that the L modes transform as

$$\begin{aligned} Q^A|Y(u)\rangle &= a(u)|\Psi^A(u)\rangle, & \tilde{S}^A|Y(u)\rangle &= b^*(u)|\Psi^A(u)\rangle, \\ Q^A|\Psi^B(u)\rangle &= \epsilon^{AB}a(u)|Z(u)\rangle, & S_A|\Psi^B(u)\rangle &= \delta_A^B a^*(u)|Y(u)\rangle, \\ \tilde{Q}_A|\Psi^B(u)\rangle &= \delta_A^B b(u)|Y(u)\rangle, & \tilde{S}^A|\Psi^B(u)\rangle &= \epsilon^{AB}b^*(u)|Z(u)\rangle, \\ \tilde{Q}_A|Z(u)\rangle &= -\epsilon_{AB}b(u)|\Psi^B(u)\rangle, & S_A|Z(u)\rangle &= -\epsilon_{AB}a^*(u)|\Psi^B(u)\rangle. \end{aligned} \quad (3.36)$$

Where $\epsilon_{AB} = -\epsilon_{BA}$ and $\epsilon^{12} = 1$ with ϵ_{AB} being the inverse of ϵ^{AB} . Also we used the supercharges of $\mathfrak{psu}(1|1)_{c.e.}^{\oplus 4}$ given in (3.10).

The second representation we can construct is the one with R particles built of the product $\rho_R \otimes \rho_R$. The particle content is:

$$\begin{aligned} |\tilde{Z}(u)\rangle &= |\phi^F(u) \otimes \phi^F(u)\rangle, & |\tilde{Y}(u)\rangle &= |\varphi^B(u) \otimes \varphi^B(u)\rangle, \\ |\tilde{\Psi}^1(u)\rangle &= |\varphi^B(u) \otimes \phi^F(u)\rangle, & |\tilde{\Psi}^2(u)\rangle &= |\phi^F(u) \otimes \varphi^B(u)\rangle. \end{aligned} \quad (3.37)$$

These have mass $m = -1$ and $\tilde{Y}$ and $\tilde{Z}$ correspond to the remaining S^3 and AdS_3 massive modes of (3.8), respectively, with $\tilde{\Psi}^A$ being their superpartners. Here we have:

$$\begin{aligned} Q^A|\tilde{Z}(u)\rangle &= a(u)|\tilde{\Psi}^A(u)\rangle, & \tilde{Q}_A|\tilde{Z}(u)\rangle &= \epsilon_{AB}b(u)|\tilde{\Psi}^B(u)\rangle, \\ Q^A|\tilde{\Psi}^B(u)\rangle &= -\epsilon^{AB}a(u)|\tilde{Y}(u)\rangle, & S_A|\tilde{\Psi}^B(u)\rangle &= \delta_A^B a^*(u)|\tilde{Z}(u)\rangle, \\ \tilde{Q}_A|\tilde{\Psi}^B(u)\rangle &= \delta_A^B b(u)|\tilde{Y}(u)\rangle, & \tilde{S}^A|\tilde{\Psi}^B(u)\rangle &= -\epsilon^{AB}b^*(u)|\tilde{Z}(u)\rangle, \\ S_A|\tilde{Y}(u)\rangle &= \epsilon_{AB}a^*(u)|\tilde{\Psi}^B(u)\rangle, & \tilde{S}^A|\tilde{Y}(u)\rangle &= b^*(u)|\tilde{\Psi}^B(u)\rangle. \end{aligned} \quad (3.38)$$

Where we used the transformations of the ρ_R representation of $\mathfrak{psu}(1|1)^{\oplus 2}$ in (3.30). Note that the counting of massive modes match the one coming from string theory in (3.8). Also note that we have a L/R -symmetry that comes from exchanging these two representations.

For the massless modes ($m = 0$) we have two possible particle multiplets: $\rho_\circ \otimes \rho'_\circ$ and $\rho'_\circ \otimes \rho_\circ$. Then we can organize these as

$$\begin{aligned} |\chi^1(u)\rangle &= |\phi_\circ^B(u) \otimes \phi_\circ^F(u)\rangle, & |\tilde{\chi}^1(u)\rangle &= |\varphi_\circ^F(u) \otimes \varphi_\circ^B(u)\rangle, \\ |T^{11}(u)\rangle &= |\varphi_\circ^F(u) \otimes \phi_\circ^F(u)\rangle, & |T^{12}(u)\rangle &= |\phi_\circ^B(u) \otimes \varphi_\circ^B(u)\rangle, \end{aligned} \quad (3.39)$$

$$\begin{aligned} |\chi^2(u)\rangle &= i|\phi_\circ^F(u) \otimes \phi_\circ^B(u)\rangle, & |\tilde{\chi}^2(u)\rangle &= -i|\varphi_\circ^B(u) \otimes \varphi_\circ^F(u)\rangle, \\ |T^{21}(u)\rangle &= i|\varphi_\circ^B(u) \otimes \phi_\circ^B(u)\rangle, & |T^{22}(u)\rangle &= -i|\phi_\circ^F(u) \otimes \varphi_\circ^F(u)\rangle. \end{aligned} \quad (3.40)$$

These two multiplets are isomorphic⁴ and we can organize them in a doublet of $\mathfrak{su}(2)_\circ$ by considering the index $\dot{A} \in \{1, 2\}$, remember that this is a subalgebra of the local rotational symmetry group of T^4 , i.e., of $\mathfrak{so}(4)$ as mentioned in the previous section. Note that unlike the L and R multiplets we have that now the highest weight states are fermionic and that the $T^{\dot{A}\dot{A}}$ represent the massless T^4 modes in (3.8). Therefore all fundamental (bosonic) string modes are accounted for. The massless states transform as:

$$\begin{aligned}
 Q^A |\chi^{\dot{A}}(u)\rangle &= a_\circ(u) |T^{\dot{A}\dot{A}}(u)\rangle, & \tilde{S}^A |\chi^{\dot{A}}(u)\rangle &= b_\circ^*(u) |T^{\dot{A}\dot{A}}(u)\rangle, \\
 Q^A |T^{\dot{A}\dot{B}}(u)\rangle &= \epsilon^{AB} a_\circ(u) |\tilde{\chi}^{\dot{A}}(u)\rangle, & S_A |T^{\dot{A}\dot{B}}(u)\rangle &= \delta_A^B a_\circ^*(u) |\chi^{\dot{A}}(u)\rangle, \\
 \tilde{Q}_A |T^{\dot{A}\dot{B}}(u)\rangle &= \delta_A^B b_\circ(u) |\chi^{\dot{A}}(u)\rangle, & \tilde{S}^A |T^{\dot{A}\dot{B}}(u)\rangle &= \epsilon^{AB} b_\circ^*(u) |\tilde{\chi}^{\dot{A}}(u)\rangle, \\
 \tilde{Q}_A |\tilde{\chi}^{\dot{A}}(u)\rangle &= -\epsilon_{AB} b_\circ(u) |T^{\dot{A}\dot{B}}(u)\rangle, & S_A |\tilde{\chi}^{\dot{A}}(u)\rangle &= -\epsilon_{AB} a_\circ^*(u) |T^{\dot{A}\dot{B}}(u)\rangle.
 \end{aligned} \tag{3.41}$$

Here we can see that these transformations are just two copies of the L transformations' rules where we take the massless limit in the coefficients of the representation. A similar structure appears in the S-matrix in the next section. Note that we can not have $\rho_L \otimes \rho_R$, $\rho_{L,R} \otimes \rho_\circ$ or $\rho_{L,R} \otimes \rho'_\circ$ since to be able to construct representations of $\mathfrak{psu}(1|1)_{c.e.}^{\oplus 4}$ out of representations of $\mathfrak{psu}(1|1)^{\oplus 2}$ we need to consider the tensor product of representations with the same central charge.

These four supermultiplets are not the full spectrum of the $AdS_3 \times S^3 \times T^4$ integrable model, there is one remaining piece left out: the *bound states*. Usually bound states appear as complex momentum poles in the S-matrix of $2 \rightarrow 2$ scattering of the fundamental excitations⁵. Then these two particles that scatter with complex momenta are arranged in a way that their combined energy and momentum describes a bound state with physical momentum and energy.

In the context of $AdS_3 \times S^3 \times T^4$, a bound state of n particles is formed when a composite state has energy and momentum of each constituent arranged in a way that it transforms in a single particle representation of $\mathfrak{psu}(1|1)_{c.e.}^{\oplus 4}$. As detailed in [109, 110] the bound states can be composed only of L or R particles. two-particle representations build of L - R or massless-massless or L / R -massless excitations do not make an appropriate one-particle representation of the superalgebra for any set of momenta. A nice fact of AdS_3/CFT_2 is that the bound states in this model transform in the same representation of the fundamental particles,

⁴The factors of i in the second massless multiplet are here only for convenience and are not necessary.

⁵This is well detailed in the context of two-dimensional quantum field theories in the review [28] and in the context of XXX spin chain in [108].

unlike the AdS_5/CFT_4 case where the bound states transform in antisymmetric representations of $\mathfrak{psu}(2|2)_{c,e}^{\oplus 2}$ [108].

Consider then n fundamental particles that built a bound state, these possess the following set of rapidities

$$u_j^{(n)} = u + i \frac{(n+1-2j)}{2}, \text{ where } j = 1, \dots, n. \quad (3.42)$$

Then the tensor product of n fundamental particles where each one has rapidity $u_j^{(n)}$ behaves as a single particle with rapidity u . The bound state has mass $m = n$ or $m = -n$ for L and R bound states, respectively. The spectrum of L and R bound states is then given by

$$\begin{array}{cc} |Y_n(u)\rangle & |\tilde{Z}_n(u)\rangle \\ |\Psi_n^1(u)\rangle & |\Psi_n^2(u)\rangle \quad \text{and} \quad |\tilde{\Psi}_n^1(u)\rangle & |\tilde{\Psi}_n^2(u)\rangle \\ |Z_n(u)\rangle & |\tilde{Y}_n(u)\rangle \end{array}, \quad (3.43)$$

which are remarkably similar to L and R representations, except by the difference in mass. To end the discussion on the bound states we note that by summing the energies and momenta of the constituents we obtain the respective quantities, $E_n(u)$ and $p_n(u)$, for the bound states which are given by

$$E_n(u) = \frac{h}{2i} \left(x^{[+n]}(u) - \frac{1}{x^{[+n]}(u)} - x^{[-n]}(u) + \frac{1}{x^{[-n]}(u)} \right), \quad (3.44)$$

$$p_n(u) = \frac{1}{i} \log \left(\frac{x^{[+n]}(u)}{x^{[-n]}(u)} \right). \quad (3.45)$$

With this we have the complete spectrum of the $AdS_3 \times S^3 \times T^4$ integrable model.

3.1.3 Crossing and mirror transformations

In this section we discuss the kinematics of the integrable model, more specifically we focus on the mirror and crossing transformations in this model. Due to the inclusion of massless modes this slightly differs from the AdS_5/CFT_4 case. This discussion follows the TBA recently analyzed in [43]. Let us consider the analytical structure in the u -plane induced by the Zhukovsky parametrization⁶.

⁶In [48] the authors introduced a slightly different parametrization of the Zhukovsky variables which leads to different positions for the cuts in the physical and mirror regions, however this is just a matter of convention and one can choose any of the two options to work with.

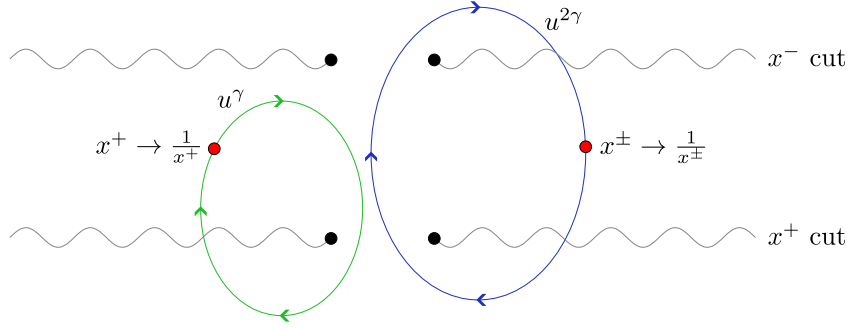


Figure 3.3: Analytical structure induced by the Zhukovsky variables in the u -plane. Here we have the $x^\pm$ branch cuts that go out to infinity. In the right (blue) we have a path that denotes the analytic continuation that corresponds to crossing. In the left (green) we have a path corresponding to a mirror transformation. Note that the branch cuts are separated by i and that in the massless limit of the Zhukovsky variables they collide and yield a single branch cut.

Note that for massive particles $x(u)$ has two long cuts starting at $u = \pm 2h$ and for massless particles they start at $u = \pm 2$. When one crosses these cuts a solution of (3.19) goes to the another one, so we have $x(u) \rightarrow 1/x(u)$.

Given this analytical structure for the Zhukovsky variables we note that both energy and momentum have two sets of long cuts for massive modes, a lower one induced by x^+ and an upper one by x^- . As usual when we cross both cuts we do a crossing transformation, this means that $E \rightarrow -E$ and $p \rightarrow -p$ which is directly visible from (3.23) - (3.24). So crossing for massive particles is implemented. For massless particles we note that there is a single cut in the energy and momentum, however we cross it only once and we have a crossing transformation just as the massive modes. Therefore the full rule for crossing of massive and massless modes is:

$$\textbf{Crossing: } x^+ \rightarrow \frac{1}{x^+}, \quad x^- \rightarrow \frac{1}{x^-}, \quad x \rightarrow \frac{1}{x}.$$

The analytical structure of the energy and momentum of massive modes in the u -plane is shown in Figure 3.3. We denote the analytic continuation of a crossing transformation by $u^{2\gamma}$ as the usual notation in $AdS_5 \times S^5$ integrability. Which means just that we crossed cuts twice, nonetheless we stick with the same notation for the massless modes. Note that the presence of a single cut for massless modes can be seen as the coalescence of the $x^\pm$ -cuts in the zero shift limit.

Also a crossing transformation takes a particle to its antiparticle. This means that the quantum numbers of a crossing pair are conjugated. So the crossing transformation inverts the $u(1)$ -charge M , that is, it takes L into R particles and

vice-versa. Thus the crossing of massive particles is simply:

$$\begin{aligned} |Y(u^{2\gamma})\rangle &= |\tilde{Y}(u)\rangle, & |Z(u^{2\gamma})\rangle &= |\tilde{Z}(u)\rangle, & |\Psi^A(u^{2\gamma})\rangle &= -|\tilde{\Psi}^A(u)\rangle, \\ |\tilde{Y}(u^{2\gamma})\rangle &= |Y(u)\rangle, & |\tilde{Z}(u^{2\gamma})\rangle &= |Z(u)\rangle, & |\Psi^A(u^{2\gamma})\rangle &= |\tilde{\Psi}^A(u)\rangle. \end{aligned} \quad (3.46)$$

For massless particles we have a slightly more complicated set of transformations given by:

$$\begin{aligned} |\chi^1(u^{2\gamma})\rangle &= i|\tilde{\chi}^2(u)\rangle, & |T^{11}(u^{2\gamma})\rangle &= i\sigma_p|T^{21}(u)\rangle, \\ |\chi^2(u^{2\gamma})\rangle &= i|\tilde{\chi}^1(u)\rangle, & |T^{21}u(u^{2\gamma})\rangle &= i\sigma_p|T^{11}(u)\rangle, \\ |\tilde{\chi}^1(u^{2\gamma})\rangle &= -i|\chi^2(u)\rangle, & |T^{12}u(u^{2\gamma})\rangle &= i\sigma_p|T^{22}(u)\rangle, \\ |\tilde{\chi}^2(u^{2\gamma})\rangle &= -i|\chi^1(u)\rangle, & |T^{22}(u^{2\gamma})\rangle &= i\sigma_p|T^{12}(u)\rangle. \end{aligned} \quad (3.47)$$

Note that crossing simply exchanged the two massless multiplets. Here σ_p is just:

$$\sigma_p = -\text{sgn}(\sin(p(u)/2)), \quad (3.48)$$

with sgn being the sign function. This set of transformations will be of particular use for us in Chapter 4 in deriving the mirror corrections for the hexagons.

Now the mirror kinematics. A mirror transformation is defined by $E \rightarrow i\tilde{p}$ and $p \rightarrow i\tilde{E}$ where $\tilde{E}$ and $\tilde{p}$ are the mirror energy and mirror momentum, respectively. The mirror energy must be positive, but the mirror momentum is real and non-periodic. For massive particles the mirror kinematics can be found by crossing only the lower x^+ -cut as shown in Figure 3.3. Therefore

$$\tilde{E}(u) = \log(x^+(u)x^-(u)), \quad (3.49)$$

$$\tilde{p}(u) = -\frac{h}{2} \left(\frac{1}{x^+(u)} - x^+(u) - x^-(u) + \frac{1}{x^-(u)} \right). \quad (3.50)$$

For $u \in \mathbb{R}$ these satisfy the physical requirements as shown in Figure 3.4. Then mirror kinematics here works just like in $AdS_5 \times S^5$ case. We denote a mirror analytic continuation in u by u^γ .

However for massless particles since both cuts merge it does not appear that there is a *half* crossing transformation, that is, a mirror transformation. Nonetheless one can analytic continue E and p to another region where the kinematics is the

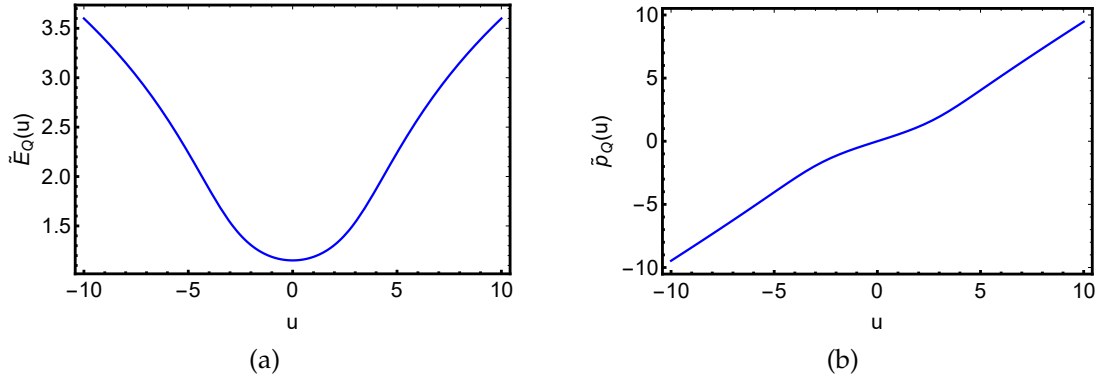


Figure 3.4: Energy and momentum of a mirror massive excitation. We show here the energy (a) and momentum (b) of a mass $m = 2$ mirror excitation at coupling $h = \pi^2/6$. The mirror momentum is non-periodic now and the mirror energy is positive.

one for the mirror theory. Indeed consider

$$\tilde{E}(u) = -\log \left(x(u)^2 \right), \quad (3.51)$$

$$\tilde{p}(u) = -h \left(x(u) - \frac{1}{x(u)} \right), \quad (3.52)$$

therefore the region where these satisfy the physical requirements is $|u| \geq 2$. But with the care that we are below the cuts, since this is the region where $\tilde{E}(u)$ is positive as shown in Figure 3.5. So the mirror transformation for massive and massless excitations is defined as:

$$\begin{aligned} \textbf{Mirror: } x^+ &\rightarrow \frac{1}{x^+} \text{ for } |m| \geq 1, \\ u \in [-2, 2] &\rightarrow u + i0^-, \quad |u| \geq 2 \text{ for } m = 0. \end{aligned}$$

This is the mirror kinematics that will be central to define the mirror corrections of the hexagon in Chapter 4. Note that for bound states the crossing and mirror transformations are similarly defined.

3.1.4 S-matrix

Now we will review the properties of the $\mathfrak{psu}(1|1)^{\oplus 2}$ S-matrix, also called Borsato–Ohlsson–Sax–Sfondrini S-matrix, how it is fixed by symmetry and also we will show some matrix elements explicitly, which will be useful for our purposes in later sections. As described before, the string excitations propagate in the two-

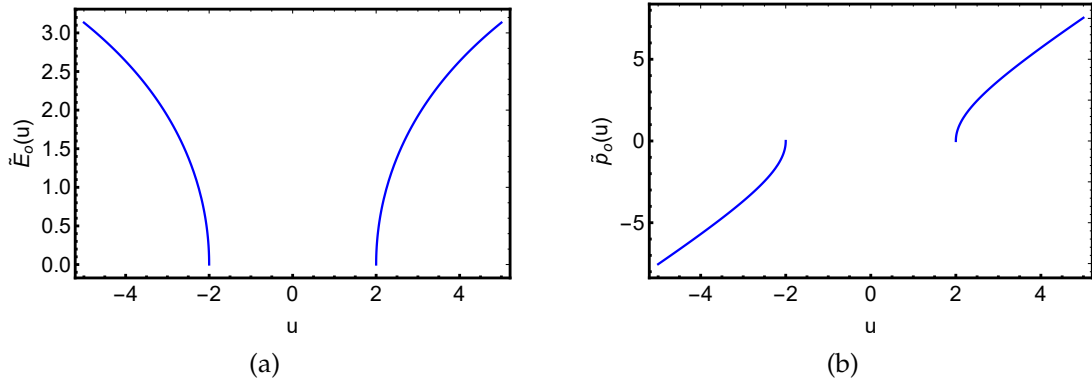


Figure 3.5: Energy and momentum of a massless mirror excitation. Here we have the energy (a) and momentum (b) of a massless mirror excitation. The mirror momentum is non-periodic now and the mirror energy is positive in the region where $|u| \geq 2$ and we are below the cuts.

dimensional worldsheet described by the closed string of length l . Due to the finite length, the scattering of string modes is not well defined since there are no asymptotically separated states. Therefore we consider the *decompactification limit* of $l \rightarrow \infty$. In this regime we can properly define the S-matrices and the momentum are quantized by the Bethe equations. Remember that l corresponds to the R-charge (that is length) of the pseudo-vacuum state in the spin chain language of the dual CFT_2 , therefore the spectrum is described by the algebraic Bethe ansatz only for asymptotically large chains. Finite length corrections enter through the thermodynamic Bethe ansatz procedure and, for massive states at least, are exponentially suppressed in l^7 [42, 43].

Then given the existence of the S-matrix in this decompactification limit let us describe it. We will focus on the description of the $\mathfrak{psu}(1|1)^{\oplus 2}$ S-matrix, since to find the full $\mathfrak{psu}(1|1)_{c.e.}^{\oplus 4}$ invariant S-matrix of the model we simply take the tensor product of the former. We have four two-dimensional supermultiplets $\{\rho_L, \rho_R, \rho_\circ, \rho'_\circ\}$ which means that our two-particle state has dimension 64. Therefore the $\mathfrak{psu}(1|1)^{\oplus 2}$ S-matrix has dimension 64×64 , that is, it has 4096 elements. The number of non-zero components drastically reduces since the S-matrix should satisfy some physical constraints, these are: $\mathfrak{psu}(1|1)^{\oplus 2}$ invariance, braiding unitarity, unitarity and crossing invariance [33, 38, 103]. Let $S_{(12)}(u, v)$ be the $2 \rightarrow 2$ S-matrix that acts on excitations 1 and 2 with rapidity u and v , respectively. Then

⁷For massless modes we have that the wrapping corrections to the spectrum enters at the same order of the ABA [44].

braiding unitarity and unitarity are given by

$$\mathbf{S}_{(12)}(u, v)\mathbf{S}_{(21)}(v, u) = \mathbb{I} \quad \text{and} \quad \mathbf{S}_{(12)}(u, v)\mathbf{S}_{(12)}(u, v)^\dagger = \mathbb{I}, \quad (3.53)$$

respectively. We will not describe here crossing invariance to not enter in some specific details that will not be of use for us, however the reader can consult [33, 38, 103]. Now the most important constraint is $\mathfrak{psu}(1|1)^{\oplus 2}$ invariance. For the $2 \rightarrow 2$ S-matrix it reads as

$$\mathbf{S}_{(12)}(u, v)Q_{(12)}(u, v) = Q_{(12)}(v, u)\mathbf{S}_{(12)}(u, v), \quad (3.54)$$

where $Q_{(12)}(u, v)$ is a charge of $\mathfrak{psu}(1|1)^{\oplus 2}$ that acts on a two-particle state composed with particles of rapidities u and v given in (3.14). Note that (3.54) is not exactly a commutation relation since we exchanged the momentum of particles when we act with the S-matrix on the state first. This is due to the fact that the S-matrix exchanges the momentum of the particles in the asymptotic outgoing state.

From the transformation rules (3.29) - (3.32) of $\mathfrak{psu}(1|1)^{\oplus 2}$ we see that the charges map a representation to itself, therefore the $2 \rightarrow 2$ S-matrix splits into blocks of defined chirality like $LL \rightarrow LL$, $LR \rightarrow LR$ and $RR \rightarrow RR$ for example. This makes many elements in the S-matrix to automatically vanish. Remarkably due to the nontrivial two-particle representations of the particles, modified co-product induced by the off-shell central extension, the entire matrix structure of the $2 \rightarrow 2$ S-matrix is fixed by symmetry alone. The only unknown factors are scalar functions that multiply each block of defined chirality, these are the so-called *dressing factors*. We will briefly comment on these later. We can compare this with the $AdS_5 \times S^5$ case for instance. There we have that the residual $\mathfrak{psu}(2|2)_{c.e.}$ also fixes the matrix part of the S-matrix, however we have a single dressing factor (the BES dressing phase) [107]. This is tied to the fact that we have only a single irreducible representation there, but here we have four irreducible representations of $\mathfrak{psu}(1|1)^{\oplus 2}$ which implies that we have four dressing factors, at least after identification under the L/R -symmetry.

The first block of the S-matrix that we show is the LL one. Here we follow the

conventions of [74]. We have the following non-zero S-matrix elements:

$$\begin{aligned}
\mathbf{S}_{(12)}|\phi_u^B\phi_v^B\rangle &= A_{uv}^{LL}|\phi_v^B\phi_u^B\rangle, \\
\mathbf{S}_{(12)}|\phi_u^B\phi_v^F\rangle &= B_{uv}^{LL}|\phi_v^F\phi_u^B\rangle + C_{uv}^{LL}|\phi_v^B\phi_u^F\rangle, \\
\mathbf{S}_{(12)}|\phi_u^F\phi_v^B\rangle &= D_{uv}^{LL}|\phi_v^B\phi_u^F\rangle + E_{uv}^{LL}|\phi_v^F\phi_u^B\rangle, \\
\mathbf{S}_{(12)}|\phi_u^F\phi_v^F\rangle &= F_{uv}^{LL}|\phi_v^F\phi_u^F\rangle.
\end{aligned} \tag{3.55}$$

Where each coefficient is given by:

$$\begin{aligned}
A_{uv}^{LL} &= 1, & D_{uv}^{LL} &= \sqrt{\frac{x_v^+}{x_v^-} \frac{x_u^-}{x_u^+} - \frac{x_v^-}{x_v^+}}, \\
B_{uv}^{LL} &= \sqrt{\frac{x_u^-}{x_u^+} \frac{x_u^+}{x_u^-} - \frac{x_v^+}{x_v^-}}, & E_{uv}^{LL} &= C_{uv}^{LL}, \\
C_{uv}^{LL} &= \sqrt{\frac{x_u^-}{x_u^+} \frac{x_v^+}{x_v^-} \frac{\eta_u}{\eta_v} \frac{x_v^-}{x_u^-} - \frac{x_v^+}{x_v^-}}, & F_{uv}^{LL} &= -\sqrt{\frac{x_u^-}{x_u^+} \frac{x_v^+}{x_v^-} \frac{x_u^+}{x_u^-} - \frac{x_v^-}{x_v^+}}.
\end{aligned} \tag{3.56}$$

Note here two things. First, we have an extra sign in F_{uv}^{LL} due to the permutation of fermions. Second, we have here only the matrix part of the LL block, each factor is then multiplied by the dressing factor Σ_{uv}^{LL} .

Another block of the S-matrix is the RR one. The following S-matrix elements are non-zero:

$$\begin{aligned}
\mathbf{S}_{(12)}|\phi_u^F\phi_v^F\rangle &= A_{uv}^{RR}|\phi_v^F\phi_u^F\rangle, \\
\mathbf{S}_{(12)}|\phi_u^F\phi_v^B\rangle &= D_{uv}^{RR}|\phi_v^B\phi_u^F\rangle + E_{uv}^{RR}|\phi_v^F\phi_u^B\rangle, \\
\mathbf{S}_{(12)}|\phi_u^B\phi_v^F\rangle &= B_{uv}^{RR}|\phi_v^F\phi_u^B\rangle + C_{uv}^{RR}|\phi_v^B\phi_u^F\rangle, \\
\mathbf{S}_{(12)}|\phi_u^B\phi_v^B\rangle &= F_{uv}^{RR}|\phi_v^B\phi_u^B\rangle.
\end{aligned} \tag{3.57}$$

Which each element being:

$$\begin{aligned}
A_{uv}^{RR} &= 1, & D_{uv}^{RR} &= \sqrt{\frac{x_v^+}{x_v^-} \frac{x_u^-}{x_u^+} - \frac{x_v^-}{x_v^+}}, \\
B_{uv}^{RR} &= \sqrt{\frac{x_u^-}{x_u^+} \frac{x_u^+}{x_u^-} - \frac{x_v^+}{x_v^-}}, & E_{uv}^{RR} &= C_{uv}^{RR}, \\
C_{uv}^{RR} &= \sqrt{\frac{x_u^-}{x_u^+} \frac{x_v^+}{x_v^-} \frac{\eta_u}{\eta_v} \frac{x_v^-}{x_u^-} - \frac{x_v^+}{x_v^-}}, & F_{uv}^{RR} &= \sqrt{\frac{x_u^-}{x_u^+} \frac{x_v^+}{x_v^-} \frac{x_u^+}{x_u^-} - \frac{x_v^-}{x_v^+}}.
\end{aligned} \tag{3.58}$$

Note here that in F_{uv}^{RR} the minus sign was removed because the statistics of the particles changed. Also note the L/R -symmetry in action since these coefficients are the same as in the LL block. As before each coefficient is multiplied by the

dressing factor Σ_{uv}^{RR} .

We also have the mixed LR and RL blocks. For the LR scattering we have the coefficients

$$\begin{aligned} \mathbf{S}_{(12)}|\phi_u^B\phi_v^F\rangle &= C_{uv}^{LR}|\phi_v^F\phi_u^B\rangle, \\ \mathbf{S}_{(12)}|\phi_u^B\phi_v^B\rangle &= A_{uv}^{LR}|\phi_v^B\phi_u^B\rangle + B_{uv}^{LR}|\phi_v^F\phi_u^F\rangle, \\ \mathbf{S}_{(12)}|\phi_u^F\phi_v^F\rangle &= E_{uv}^{LR}|\phi_v^F\phi_u^F\rangle + F_{uv}^{LR}|\phi_v^B\phi_u^B\rangle, \\ \mathbf{S}_{(12)}|\phi_u^F\phi_v^B\rangle &= D_{uv}^{LR}|\phi_v^B\phi_u^F\rangle. \end{aligned} \quad (3.59)$$

Now for the RL we have

$$\begin{aligned} \mathbf{S}_{(12)}|\phi_u^F\phi_v^B\rangle &= D_{uv}^{RL}|\phi_v^B\phi_u^F\rangle, \\ \mathbf{S}_{(12)}|\phi_u^F\phi_v^F\rangle &= E_{uv}^{RL}|\phi_v^F\phi_u^F\rangle + F_{uv}^{RL}|\phi_v^B\phi_u^B\rangle, \\ \mathbf{S}_{(12)}|\phi_u^B\phi_v^B\rangle &= A_{uv}^{RL}|\phi_v^B\phi_u^B\rangle + B_{uv}^{RL}|\phi_v^F\phi_u^F\rangle, \\ \mathbf{S}_{(12)}|\phi_u^B\phi_v^F\rangle &= C_{uv}^{RL}|\phi_v^F\phi_u^B\rangle. \end{aligned} \quad (3.60)$$

So for the LR block the S-matrix elements are simply

$$\begin{aligned} A_{uv}^{LR} &= \sqrt{\frac{x_u^-}{x_u^+}} \frac{1 - x_u^+ x_v^-}{1 - x_u^- x_v^-}, & D_{uv}^{LR} &= \sqrt{\frac{x_u^- x_v^-}{x_u^+ x_v^+}} \frac{1 - x_u^+ x_v^+}{1 - x_u^- x_v^-}, \\ B_{uv}^{LR} &= \sqrt{\frac{x_u^- x_v^-}{x_u^+ x_v^+}} \frac{2i}{h} \frac{\eta_u \eta_v}{1 - x_u^- x_v^-}, & E_{uv}^{LR} &= -\sqrt{\frac{x_v^-}{x_v^+}} \frac{1 - x_u^- x_v^+}{1 - x_u^- x_v^-}, \\ C_{uv}^{LR} &= 1, & F_{uv}^{LR} &= -B_{uv}^{LR}. \end{aligned} \quad (3.61)$$

And the coefficients in the RL block are related to the LR ones by braiding unitarity, therefore

$$\begin{aligned} A_{uv}^{RL} &= \sqrt{\frac{x_u^+}{x_u^-}} \frac{1 - x_u^+ x_v^-}{1 - x_u^+ x_v^+}, & D_{uv}^{RL} &= 1, \\ B_{uv}^{RL} &= \frac{2i}{h} \frac{\eta_u \eta_v}{1 - x_u^+ x_v^+}, & E_{uv}^{RL} &= -\sqrt{\frac{x_v^+}{x_v^-}} \frac{1 - x_u^- x_v^+}{1 - x_u^+ x_v^+}, \\ C_{uv}^{RL} &= \sqrt{\frac{x_u^+ x_v^+}{x_u^- x_v^-}} \frac{1 - x_u^- x_v^-}{1 - x_u^+ x_v^+}, & F_{uv}^{RL} &= -B_{uv}^{RL}. \end{aligned} \quad (3.62)$$

Again each block is multiplied by its dressing factor, with these being Σ_{uv}^{LR} and Σ_{uv}^{RL} .

Now for the massless-massless scattering we consider it as the massless limit

of the LL or RR blocks. Remember that this limit is defined as:

$$x^+(u) \rightarrow x(u) \quad \text{and} \quad x^-(u) \rightarrow 1/x(u). \quad (3.63)$$

There is one small caveat in taking this limit. We have to consider the change in the excitations' statistics that may happen when going from ρ_L to $\rho_\circ$ or $\rho'_\circ$. Indeed the $\rho_\circ$ and $\rho_\circ$ scattering is exactly the same as the LL block in (3.56), however for the $\rho'_\circ$ and $\rho'_\circ$ we have

$$\begin{aligned} S_{(12)}|\phi_{\circ,u}^F \phi_{\circ,v}^F\rangle &= -A_{uv}^{LL}|\phi_{\circ,v}^F \phi_{\circ,u}^F\rangle, \\ S_{(12)}|\phi_{\circ,u}^F \phi_{\circ,v}^B\rangle &= B_{uv}^{LL}|\phi_{\circ,v}^B \phi_{\circ,u}^F\rangle + C_{uv}^{LL}|\phi_{\circ,v}^F \phi_{\circ,u}^B\rangle, \\ S_{(12)}|\phi_{\circ,u}^B \phi_{\circ,v}^F\rangle &= D_{uv}^{LL}|\phi_{\circ,v}^F \phi_{\circ,u}^B\rangle + E_{uv}^{LL}|\phi_{\circ,v}^B \phi_{\circ,u}^F\rangle, \\ S_{(12)}|\phi_{\circ,u}^B \phi_{\circ,v}^B\rangle &= -F_{uv}^{LL}|\phi_{\circ,v}^B \phi_{\circ,u}^B\rangle, \end{aligned} \quad (3.64)$$

note the highlighted minus signs in cyan due to the change in statistics. Also note that we need to take the massless limit in the coefficients. For example, this limit for the LL coefficients is just

$$\begin{aligned} A_{uv}^{LL} &= 1, & D_{uv}^{LL} &= \frac{x_v - x_u}{1 - x_u x_v}, \\ B_{uv}^{LL} &= \frac{x_u - x_v}{1 - x_u x_v}, & E_{uv}^{LL} &= C_{uv}^{LL}, \\ C_{uv}^{LL} &= \frac{\eta_{\circ,u}}{\eta_{\circ,v}} \frac{1 - x_v^2}{1 - x_u x_v}, & F_{uv}^{LL} &= 1. \end{aligned} \quad (3.65)$$

We can similarly define the scattering of $\rho_\circ$ and $\rho'_\circ$ modes. Now to compute the scattering of massless and massive modes we simply consider the massless limit in LL , LR or RL blocks, appropriately taking care of the signs coming from statistics.

The dressing factors are given by solutions of the constraints coming from crossing. Before we show the solutions explicitly we use L/R -symmetry to reduce the number of independent dressing factors. For the massive modes we can write all dressing factors as a function of only two denoted by $\Sigma_{uv}^{\bullet\bullet}$ and $\tilde{\Sigma}_{uv}^{\bullet\bullet}$. Indeed we have:

$$\begin{aligned} \Sigma_{uv}^{LL} &= \Sigma_{uv}^{RR} = \Sigma_{uv}^{\bullet\bullet}, \\ \Sigma_{uv}^{LR} &= \tilde{\Sigma}_{uv}^{\bullet\bullet}, \\ \Sigma_{uv}^{RL} &= \sqrt{\frac{x_u^- x_v^-}{x_u^+ x_v^+} \frac{1 - x_u^+ x_v^+}{1 - x_u^- x_v^-}} \tilde{\Sigma}_{uv}^{\bullet\bullet}. \end{aligned} \quad (3.66)$$

The solutions for these and the massless dressing $\Sigma_{uv}^{\circ\circ}$ are⁸ [74]

$$\begin{aligned} (\Sigma_{uv}^{\bullet\bullet})^2 &= \frac{1}{(\sigma_{uv}^{\bullet\bullet})^2} \left(\frac{x_u^+ x_v^-}{x_u^- x_v^+} \right)^2 \frac{x_u^- - x_v^+}{x_u^+ - x_v^-} \frac{1 - x_u^- x_v^+}{1 - x_u^+ x_v^-}, \\ (\tilde{\Sigma}_{uv}^{\bullet\bullet})^2 &= \frac{1}{(\tilde{\sigma}_{uv}^{\bullet\bullet})^2} \left(\frac{x_u^+}{x_u^-} \right)^2 \frac{x_v^-}{x_v^+} \frac{1 - x_u^- x_v^-}{1 - x_u^+ x_v^+} \frac{1 - x_u^- x_v^+}{1 - x_u^+ x_v^-}, \\ (\Sigma_{uv}^{\circ\circ})^2 &= \frac{x_v^2}{(\sigma_{uv}^{\circ\circ})^2}, \end{aligned} \quad (3.67)$$

where $\sigma_{uv}^{\bullet\bullet}$, $\tilde{\sigma}_{uv}^{\bullet\bullet}$ and $\sigma_{uv}^{\circ\circ}$ are phases analogous to the BES dressing phase of $AdS_5 \times S^5$. To find these, one solves a functional equation coming from crossing symmetry. Note however that these are not rational functions of the Zhukovsky variables, in fact they possess a more complicated analytic structure consisting of an infinitely sheeted Riemann surface instead of simple square root branch cuts induced by $x^\pm$, just as the $AdS_5 \times S^5$ case [111]. However there are some unusual features of these with respect to $AdS_5 \times S^5$, one of them is that the one-loop result in the semiclassical string expansion (the so-called HL term) is not universal but the tree level term remains universal (AFS term) [112]. This is unusual since for AdS_4/CFT_3 and AdS_5/CFT_4 both terms are equal, here we need to consider the product $\sigma_{uv}^{\bullet\bullet} \tilde{\sigma}_{uv}^{\bullet\bullet}$ to obtain the same HL term as in these higher dimensional dualities. Another unusual feature is that the weak coupling expansion in the integrability coupling h starts at $\mathcal{O}(h)$ (at least for the proposal in [109]), contrast this with the first nontrivial term in the BES phase starting at $\mathcal{O}(h^6)$ [111]. This means that nontrivial contributions to the spectrum enters much sooner in the h -expansion for $AdS_3 \times S^3 \times T^4$.

One tentative to find the phases $\sigma^{**}(u, v)$ is in [109] for massive dressing phases and [40] for the massless ones, however there are some problems with this proposed solution. One of them is that the dispersion relation for massless modes differs from the semiclassical string result at two-loops. Another one is that some phases, like $\sigma^{\bullet\bullet}(u, v)$ for example, do not match the semiclassical string result at higher orders. Another proposal was recently made in [48]. It solved some issues of the previous proposal, however the problems of incompatibility with the semiclassical string result still there. Although we can not specifically pinpoint the origin of the discrepancies, a possible hint of their origin may be the massless modes. A problem for the future could be to clarify what exactly do we mean by the massless modes' S-matrix and how to properly renormalize the string

⁸For the dressing phases of the remaining S-matrices the interested reader can consult [74].

perturbation theory results to then make further progress in these matters. Turns out that for our purposes we do not need the full form of these dressing phases, we will use only their analytical properties which any proposed solution must possess.

The reader can worry that we fixed only the $2 \rightarrow 2$ S-matrix, what about the full $n \rightarrow m$ one? Remember from the Introduction (Chapter 1) that in a IQFT the S-matrix satisfies the Yang-Baxter equation and there is no particle production. This means that any $n \rightarrow n$ process can be written as a sequence of $2 \rightarrow 2$ scatterings (see Figure 1.2). Therefore by fixing the $2 \rightarrow 2$ S-matrix we fixed the entire S-matrix of the model. Note here the distinction to the $O(N)$ case discussed in the Introduction (Chapter 1), there we had that symmetry did not fix the S-matrix entirely. But here our symmetry algebra is so constraining that entirely fixes the S-matrix up to phases and we can check that it satisfies the Yang-Baxter equation.

3.2 Chiral ring

A major distinction between AdS_5/CFT_4 and AdS_3/CFT_2 dualities is the structure of half-BPS states. In the language of the dual CFT these are operators annihilated by half of the supersymmetry generators in the model. Consider for instance the case of AdS_5/CFT_4 , then we have that the highest weight state in a half-BPS supermultiplet is simply the pseudo-vacuum state $|0\rangle$ without any excitations. In the dual CFT_4 this corresponds to the single trace operator $\text{tr}(Z^J)$, where J is the length of the string, that is, the R-charge of the state. However in AdS_3/CFT_2 we have half of the amount of supersymmetries with respect to the $AdS_5 \times S^5$ case. This means that our half-BPS states are much less restricted and we have a richer set of possible states. Indeed we have 16 possible families of half-BPS operators and as proved in [113, 114] these have protected scaling dimensions and structure constants between themselves. So they yield a good laboratory to test hexagonalization's predictions or the duality itself. These families of half-BPS operators form the so-called *chiral ring*.

Why we have this special structure for the half-BPS modes in $AdS_3 \times S^3 \times T^4$? This is possible due to the existence of the massless modes. Indeed consider for example massive modes at zero momentum, they have an energy cost $|m|$ to be added to the state which can be seen directly from the dispersion relation (3.18) for $\kappa = 0$. However, for massless modes at zero momentum, we have no energy cost whatsoever. Therefore, from the point of view of integrability, we construct

this chiral ring by adding massless fermionic excitations with null momenta in the original half-BPS vacuum $|0\rangle$. There are four massless fermions $|\chi^A\rangle$ and $|\tilde{\chi}^A\rangle$ that we can add and these furnish the previously mentioned 16 families of the chiral ring. Note that the addition of zero momentum massive modes in the vacuum also yields a half-BPS state, however these are simply descendants of $|0\rangle$ by acting with the supercharges on the vacuum state. This is due to the energy cost and therefore the addition of zero momentum fermionic massless modes yields genuinely new half-BPS that are not descendants of any other state [39].

Now we describe the chiral ring in details. The dual CFT_2 has $\mathcal{N} = (4, 4)$ supersymmetry, this means that we have $\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$ as the R-charge algebra which is just the isometry of S^3 . Then our states carry $(m, \bar{m})$ magnetic quantum numbers that classify them in representations of the R-charge algebra [39]. We denote a half-BPS operator in the dual CFT_2 by $\mathbb{V}_{j,m,\bar{m}}^{\alpha\bar{\alpha}}$ where $\alpha, \bar{\alpha} \in \{+, -, 1, 2\}$ and j is related to the scaling dimension and R-charge of the operator. For half-BPS states the left and right scaling dimensions are equal to the highest weight R-charges since these operators are protected [113, 114]. As said before these states are obtained by insertions of massless fermionic zero momentum states, therefore these can be decomposed into singlets, doublets and triplets of $\mathfrak{su}(2)_\circ$. We have to take some care in the way we take the zero momentum limit, because the massless particles have chirality and we must keep the positivity condition for the energy. Indeed we must take the following zero momentum limit [39]

$$\begin{aligned} \lim_{u \rightarrow 2^-} |\chi^1(u)\rangle, \quad \lim_{u \rightarrow -2^+} |\chi^2(u)\rangle, \\ \lim_{u \rightarrow 2^-} |\tilde{\chi}^1(u)\rangle, \quad \lim_{u \rightarrow -2^+} |\tilde{\chi}^2(u)\rangle. \end{aligned} \tag{3.68}$$

The correspondence between the families of half-BPS operators and zero momentum fermion insertions is described in Figure 3.6.

The three-point functions of the chiral ring can be found purely from symmetry considerations as done in [115]. First there are the selection rules for the correlators. These must be singlets of the R-charge, therefore the magnetic quantum numbers must satisfy the following sum rules

$$m_1 + m_2 + m_3 = 0 \quad \text{and} \quad \bar{m}_1 + \bar{m}_2 + \bar{m}_3 = 0. \tag{3.69}$$

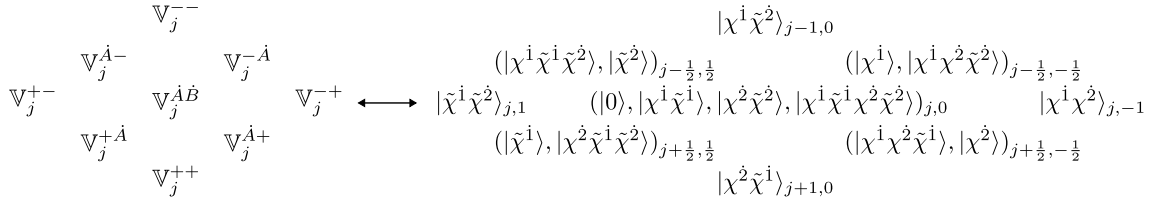


Figure 3.6: Structure of the chiral ring. Here we show on the left the entire chiral ring in operator notation and in the right the corresponding massless fermionic insertions at zero momentum. Here we suppressed the $(m, \bar{m})$ labels in the left for convenience. The subscripts on the states in the right denote the scaling dimension Δ and spin s respectively. Then the R-charge of each state is simply $(\frac{\Delta+s}{2}, \frac{\Delta-s}{2})$, so j is a label for it. Note that the operators transform in representations of $\mathfrak{su}(2)_\circ$, indeed $V_j^{\pm\pm}$ are singlets, $V_j^{\pm A}$ and $V_j^{A\pm}$ are doublets and V_j^{AB} splits into a singlet and triplet.

Basically this allows four possible inequivalent classes:

$$\begin{aligned} & \{(\dot{A}, \bar{\epsilon}_1), (\dot{B}, \bar{\epsilon}_2), (\epsilon_3, \bar{\epsilon}_3)\}, \quad \{(\dot{A}_1, \dot{A}_2), (\dot{B}_1, \dot{B}_2), (\epsilon_3, \bar{\epsilon}_3)\}, \\ & \{(\dot{A}, \bar{\epsilon}_1), (\dot{B}_1, \dot{B}_2), (\epsilon_3, \dot{C})\}, \quad \{(\epsilon_1, \bar{\epsilon}_1), (\epsilon_2, \bar{\epsilon}_2), (\epsilon_3, \bar{\epsilon}_3)\}, \end{aligned}$$

where $\dot{A} \in \{1, 2\}$ is a $\mathfrak{su}(2)_\circ$ index and $\epsilon, \bar{\epsilon} = \pm$. To describe the correlators we first define

$$f(j_i, m_i) = \begin{pmatrix} \frac{j_1+\epsilon_1}{2} & \frac{j_2+\epsilon_2}{2} & \frac{j_3+\epsilon_3}{2} \\ m_1 & m_2 & m_3 \end{pmatrix} \times \frac{\sqrt{(j_{12} + \epsilon_{12})!(j_{23} + \epsilon_{23})!(j_{13} + \epsilon_{13})! \left(\frac{j_1+j_2+j_3+\epsilon_1+\epsilon_2+\epsilon_3}{2} + 1\right)!}}{\sqrt{(j_1 + \epsilon_1)!(j_2 + \epsilon_2)!(j_3 + \epsilon_3)!}}. \quad (3.70)$$

Where the first term is the Wigner $3j$ -symbol and

$$j_{mn} = \frac{j_m + j_n - j_k}{2} \quad \text{and} \quad \epsilon_{mn} = \frac{\epsilon_m + \epsilon_n - \epsilon_k}{2}. \quad (3.71)$$

Therefore the nontrivial structure constants are simply

$$\begin{aligned}
\langle \mathbb{V}_{j_1, m_1, \bar{m}_1}^{A\bar{\epsilon}_1} \mathbb{V}_{j_2, m_2, \bar{m}_2}^{\dot{B}\bar{\epsilon}_2} \mathbb{V}_{j_3, m_3, \bar{m}_3}^{\epsilon_3 \bar{\epsilon}_3} \rangle &= \frac{f(j_i, m_i) f(\bar{j}_i, \bar{m}_i)}{\sqrt{N}} (1 - \delta^{A\dot{B}}) \frac{(\bar{\epsilon}_1 j_1 + \bar{\epsilon}_2 j_2 + \bar{\epsilon}_3 j_3 + 1)}{2\sqrt{j_3}}, \\
\langle \mathbb{V}_{j_1, m_1, \bar{m}_1}^{A_1 A_2} \mathbb{V}_{j_2, m_2, \bar{m}_2}^{\dot{B}_1 \dot{B}_2} \mathbb{V}_{j_3, m_3, \bar{m}_3}^{\epsilon_3 \bar{\epsilon}_3} \rangle &= \frac{f(j_i, m_i) f(\bar{j}_i, \bar{m}_i)}{\sqrt{N}} (1 - \delta^{A_1 \dot{B}_1}) (1 - \delta^{A_2 \dot{B}_2}) \sqrt{\frac{j_1 j_2}{j_3}}, \\
\langle \mathbb{V}_{j_1, m_1, \bar{m}_1}^{A\bar{\epsilon}_1} \mathbb{V}_{j_2, m_2, \bar{m}_2}^{\dot{B}_1 \dot{B}_2} \mathbb{V}_{j_3, m_3, \bar{m}_3}^{\epsilon_3 \dot{C}} \rangle &= \frac{f(j_i, m_i) f(\bar{j}_i, \bar{m}_i)}{\sqrt{N}} (1 - \delta^{A\dot{B}_1}) (1 - \delta^{\dot{B}_2 \dot{C}}) \sqrt{j_2}, \\
\langle \mathbb{V}_{j_1, m_1, \bar{m}_1}^{\epsilon_1 \bar{\epsilon}_1} \mathbb{V}_{j_2, m_2, \bar{m}_2}^{\epsilon_2 \bar{\epsilon}_2} \mathbb{V}_{j_3, m_3, \bar{m}_3}^{\epsilon_3 \bar{\epsilon}_3} \rangle &= \frac{f(j_i, m_i) f(\bar{j}_i, \bar{m}_i)}{\sqrt{N}} \times \\
&\quad \frac{(\epsilon_1 j_1 + \epsilon_2 j_2 + \epsilon_3 j_3 + 1)(\bar{\epsilon}_1 j_1 + \bar{\epsilon}_2 j_2 + \bar{\epsilon}_3 j_3 + 1)}{4\sqrt{j_1 j_2 j_3}}.
\end{aligned} \tag{3.72}$$

After an appropriate normalization given in [115]. Note here that we use conformal symmetry to fix the operators at positions 0, 1 and ∞ , that is why the correlators are position independent. Also note that the correlators also organize themselves into singlets of $\mathfrak{su}(2)_\circ$.

As mentioned before, these structure constants are protected and they can be tested against hexagon predictions. However there is a problem here, the half-BPS multiplets mix. Indeed, let us separate them into three classes:

$$\left\{ \begin{array}{c} \mathbb{V}_{j+1}^{--} \\ \mathbb{V}_j^{A\dot{B}} \\ \mathbb{V}_{j-1}^{++} \end{array} \right\}, \quad \left\{ \begin{array}{c} \mathbb{V}_{j+1}^{A-} \\ \mathbb{V}_j^{+A} \end{array} \right\}, \quad \text{and} \quad \left\{ \begin{array}{c} \mathbb{V}_{j+1}^{-\dot{A}} \\ \mathbb{V}_j^{\dot{A}+} \end{array} \right\}. \tag{3.73}$$

Using the equivalence described in Figure 3.6, it can be seen that these have the same scaling dimensions and spin. Nonetheless the remaining $\mathbb{V}_j^{+-}$ and $\mathbb{V}_j^{-+}$ do not mix with anyone. Due to this mixing, in [74] it was computed the structure constant of unmixed operators only, these are the following

$$\langle \mathbb{V}_{j_1, \frac{j_1-1}{2}, \frac{j_1+1}{2}}^{-+} \mathbb{V}_{j_2, -\frac{j_2-1}{2}, -\frac{j_2+1}{2}}^{-+} \mathbb{V}_{j_3, 0, 0}^{+-} \rangle \quad \text{and} \quad \langle \mathbb{V}_{j_1, \frac{j_1-1}{2}, \frac{j_1+1}{2}}^{-+} \mathbb{V}_{j_2, 0, 0}^{-+} \mathbb{V}_{j_3, -\frac{j_3-1}{2}, -\frac{j_3+1}{2}}^{-+} \rangle,$$

for these cases the hexagon prediction was tested, more specifically in [74] it was computed only the ratio of these structure constants. With the $AdS_3 \times S^3 \times T^4$ integrability foundations briefly described in the last sections we now finally turn our attention to the hexagons in the next section.

3.3 Hexagons and structure constants

We already gave a heuristic explanation about hexagonalization in the Introduction (Chapter 1) and we even use some properties of it in Chapter 2, now we explain in detail how this formalism works in the context of $AdS_3 \times S^3 \times T^4$. The main idea of hexagonalization is to compute the three-point, and later higher-point, correlation functions of operators described by integrable spin chains in the planar limit of AdS/CFT through a factorization in terms of the so-called hexagon form factors. As showed before in Section 2.2, in this framework the structure constant is given by

$$C_{123}^{\bullet\bullet\bullet} = \sum_{\alpha_j \cup \bar{\alpha}_j = \mathbf{u}_j} \prod_{j=1}^3 w(\alpha_j, \bar{\alpha}_j) \mathcal{H}(\alpha_1, \alpha_3, \alpha_2) \mathcal{H}(\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3), \quad (3.74)$$

where $\mathcal{H}(\alpha_1, \alpha_3, \alpha_2)$ is the hexagon form factor and

$$w(\alpha_j, \bar{\alpha}_j) = (-1)^{|\alpha_j|} a_{l_{ji}}(\alpha_j) \mathcal{S}_{<}(\bar{\alpha}_j, \alpha_j), \quad (3.75)$$

is a propagation factor that is related to moving an excitation of the j -th operator from one hexagon to another. Note that here we use the same notation as in Chapter 2. Let us go term by term, we have that $\mathcal{S}(u, v)$ is the finite coupling S-matrix of the excitations, which here is simply the $\mathfrak{psu}(1|1)_{c.e.}^{\oplus 4}$ S-matrix⁹. Also $a_l(u)$ is the momentum factor, which at finite coupling is given by

$$a_l(u) = e^{ip(u)l} = \left(\frac{x_u^+}{x_u^-} \right)^l \quad \text{or} \quad a_l(u) = e^{ip_{\circ}(u)l} = (x_u)^{2l}, \quad (3.76)$$

for massive and massless modes, respectively. Then we sum over all partitions of the operators excitations. In Figure 3.7 we show an example of what an hexagon computation looks like and we explain the role of each contribution.

The hexagonalization formalism is very powerful, however it has limitations. Indeed it works at *finite coupling*, however (3.74) is restricted to asymptotic large bridge lengths. We can understand this last fact physically as follows. Consider an excitation on the first operator, to move it to the the other hexagon we must propagate it through a bridge that is connecting two operators. As we will show

⁹Note that in Section 3.1 we presented only the $\mathfrak{psu}(1|1)^{\oplus 2}$ S-matrix. To obtain the $\mathfrak{psu}(1|1)_{c.e.}^{\oplus 4}$ invariant S-matrix we just take the tensor products of the former. These S-matrices are compiled on Appendix A of [74].

$$\begin{aligned}
C_{123}^{\bullet\circ\circ} = & \text{Hexagon}(u_1, u_2) - e^{ip_2 l_{31}} \text{Hexagon}(u_1) \text{Hexagon}(u_2) \\
& - e^{ip_1 l_{31}} \mathcal{S}(u_1, u_2) \text{Hexagon}(u_2) \text{Hexagon}(u_1) + e^{i(p_1+p_2)l_{31}} \text{Hexagon}(u_1, u_2)
\end{aligned}$$

Figure 3.7: Hexagonalization of a structure constant. Here we show how the hexagonalization formula (3.74) works for a structure constant with a single non-BPS operator. First we assume an ordering for the roots, given this we have that the first term has no propagation factors since no root went to the right hexagon. Now in the second and third contributions we moved one excitation to the right hexagon, this yields us propagation factors. For the second term we get only the minus and momentum factor, however for the third one we need to scatter first u_1 with u_2 , thus paying a S-matrix, and then we propagate u_1 to the right hexagon with the momentum and minus sign factors. The last term possess only the propagation factors. Note that the order of roots does not really matter since a physical operator satisfy the Bethe equations and we can consider any desired order [57].

in Chapter 4, the mirror particles have a propagation factor which decays exponentially in the bridge length. Therefore for small bridge lengths we must include these corrections and the hexagonalization prescription, called asymptotic hexagon, (3.74) works only for large bridge lengths. Also as said in Chapter 2 the hexagonalization formalism, even in the asymptotic limit, has its computational limitations when one consider many excitations on a operator, we will show in the following subsections why this happens.

Hexagons can be seen as a natural extension of the tailoring formalism, although both formalisms have similar expressions for the structure constants it is not direct to see the existence of the hexagons from tailoring. This demands some algebraic manipulations from the tailoring expressions which we detailed back in Section 2.4. Hexagonalization was first introduced in [57] for AdS_5/CFT_4 and in [74] for AdS_3/CFT_2 , it is the conventions and derivations of the latter that we follow in this section. Now we show in the next subsections how to constrain the hexagon itself using four properties: symmetry, Watson axiom, decoupling and monodromy. Also we will present some example of structure constants in the dual CFT_2 that we can compute rather directly.

3.3.1 Fixing the hexagon

In this section we will constrain the hexagon form factor. For this we impose four properties on them. The first one we consider is symmetry invariance, this will totally fix the tensor structure of one- and two-particle hexagons. It works remarkably similar to the S-matrix bootstrap from Section 3.1. When we fix the two-particle hexagon's matrix structure we note that each block has an undetermined scalar factor. So using the Watson, decoupling and monodromy axioms, we can fix these factors in terms of the phases σ_{uv}^{**} . Thus with all these properties in mind we finally show a solution for the hexagon form factor for n particles.

Symmetry axiom

The most important property we used to fix the $\mathfrak{psu}(1|1)^{\oplus 2}$ invariant S-matrix was symmetry. Remember that this is the residual symmetry left after the lightcone gauge-fixing. Another way we can also see this fact is from the point of view of the dual CFT_2 . Let us start by considering the two-point function that preserves the maximal amount of supersymmetry. This is composed by two half-BPS operators at positions 0 and ∞ [96]. One of the operators is the highest weight and the other one is the lowest weight in the same R-charge multiplet. Then the residual symmetry are all generators of isometries that keep this configuration invariant. Using an analogy to the Wigner's symmetry procedure in QFT (see [116]) we consider this as the canonical two-point function configuration and the excitations on top of this will transform under the residual symmetry group. This procedure yield us exactly the $\mathfrak{psu}(1|1)_{c.e.}^{\oplus 4}$ symmetry for the full spectral problem [33].

We can do the same procedure for three-point functions and in doing so we discover the symmetry group of the hexagon. So we first consider a canonical configuration that preserves as much symmetry as possible for the three-point functions. This is given by the three half-BPS operators in the same R-charge multiplet where now one is the highest weight, other the lowest weight and the remaining one is neither the lowest or highest weight state, it is in the middle of the multiplet [57, 96, 74]. Now the operators are localized in positions 0, 1 and ∞ . To put a generic configuration of three operators in this canonical frame we consider a supertranslation transformation that acts on a operator at position 0 by

$$\mathcal{O}_{t,k} = e^{tT_k} \mathcal{O}(0) e^{-tT_k}, \quad (3.77)$$

where t is the position of $\mathcal{O}_{t,k}$, T_k is the supertranslation generator and $k \in \mathbb{C}$ a free parameter. This generator is made of a specific combination of R-charge and conformal transformations of the dual CFT_2 and the residual symmetry group is the subalgebra of $\mathfrak{psu}(1|1)_{c.e.}^{\oplus 4}$ that is invariant with respect to this generator [74].

Let us workout this subalgebra. First define the following supercharges

$$\begin{aligned} Q_A &= S_A - \frac{i}{k} \epsilon_{AB} Q^B, \\ \tilde{Q}_A &= \tilde{Q}_A - i k \epsilon_{AB} \tilde{S}^B, \end{aligned} \quad (3.78)$$

then the invariant subalgebra with respect to T_k can be seen as the diagonal subalgebra of $\mathfrak{psu}(1|1)_{c.e.}^{\oplus 4}$. This is simply

$$\begin{aligned} \{Q_A, Q_B\} &= \{\tilde{Q}_A, \tilde{Q}_B\} = 0, \\ \{Q_A, \tilde{Q}_B\} &= -\epsilon_{AB} \frac{i}{k} (C - k^2 C^\dagger) \end{aligned} \quad (3.79)$$

with C being the same central charge as in (3.12). An analogous result was also found for the $AdS_5 \times S^5$ case, there the residual symmetry is the diagonal subalgebra of $\mathfrak{psu}(2|2)^{\oplus 2}$ [57]. Note that the algebra (3.79) is very similar to the $\mathfrak{psu}(1|1)^{\oplus 2}$ algebra and we assume that the hexagon form factor is invariant under it, this is the symmetry axiom.

Knowing the symmetry algebra we need to apply it to the hexagon. For simplicity, we will for now restrict ourselves to hexagons with a single excited operator, *id est*, only a single physical edge has excitations. Configurations of the hexagon with more than one excited operator can be related to these simpler cases by analytic continuation on the rapidities, which we will detail later. We now change the notation of the hexagon from (3.74) and denote the n -particle hexagon form factor as

$$\langle \mathfrak{h} | X_{a_1, u_1} \cdots X_{a_n, u_n} \rangle,$$

where X_{a_j, u_j} denotes a particle X_{a_j} with rapidity u_j . As X_{a_j} we can consider excitations on the L , R or massless multiplets given in (3.35), (3.37), (3.39) and (3.40). The assumption of invariance under (3.79) is then just

$$\langle \mathfrak{h} | Q | X_{a_1, u_1} \cdots X_{a_n, u_n} \rangle = 0, \quad (3.80)$$

where Q is any generator of (3.79). Until now the parameter k is free, however due to (3.80) the central charge annihilates all states being them physical or not, even

with its action being diagonal. This constrains the hexagon too much, therefore we obtain a nontrivial result only if we fix $k = 1$ [74].

Using the definition (3.78) and the actions of the supercharges on the states, we find for one-particle hexagons the following result

$$\begin{aligned} \langle \mathfrak{h} | Y_u \rangle &= 1, & \langle \mathfrak{h} | Z_u \rangle &= -i, & \langle \mathfrak{h} | \tilde{Y}_u \rangle &= 1, & \langle \mathfrak{h} | \tilde{Z}_u \rangle &= -i, \\ \langle \mathfrak{h} | \chi_u^1 \rangle &= 1, & \langle \mathfrak{h} | \tilde{\chi}_u^1 \rangle &= -i, & \langle \mathfrak{h} | \chi_u^2 \rangle &= 1, & \langle \mathfrak{h} | \tilde{\chi}_u^2 \rangle &= -i, \end{aligned} \quad (3.81)$$

with the remaining one-particle form factors vanishing. Note that in this case, the symmetry axiom reduces the number of independent form factors such that we can choose normalizations of the one-particle states that yield (3.81). The normalization factor is the so-called measure and it will play a central role to define finite bridge length corrections to the asymptotic hexagon in Chapter 4.

For the two-particle hexagons we note that the action of the supercharges split them into blocks of defined chirality like the S-matrix in Section 3.1. This is not surprising since the symmetry algebra is remarkably similar. More than that we find for the LL block the following structure

$$\begin{aligned} \langle \mathfrak{h} | Y_u Y_v \rangle &= h_{uv}^{LL} A_{uv}^{LL}, & \langle \mathfrak{h} | Z_u Y_v \rangle &= -i h_{uv}^{LL} D_{uv}^{LL}, \\ \langle \mathfrak{h} | Y_u Z_v \rangle &= -i h_{uv}^{LL} B_{uv}^{LL}, & \langle \mathfrak{h} | Z_u Z_v \rangle &= h_{uv}^{LL} F_{uv}^{LL}, \\ \langle \mathfrak{h} | \Psi_u^1 \Psi_v^2 \rangle &= -i h_{uv}^{LL} C_{uv}^{LL}, & \langle \mathfrak{h} | \Psi_u^2 \Psi_v^1 \rangle &= i h_{uv}^{LL} C_{uv}^{LL}, \end{aligned} \quad (3.82)$$

where h_{uv}^{LL} is an undetermined scalar factor. Note that this is basically the LL block of the $\mathfrak{psu}(1|1)^{\oplus 2}$ S-matrix up to phases. We find similar structures for the other blocks with their respective scalar factors h_{uv}^{**} . For example in the RR block we have:

$$\begin{aligned} \langle \mathfrak{h} | \tilde{Y}_u \tilde{Y}_v \rangle &= h_{uv}^{RR} A_{uv}^{RR}, & \langle \mathfrak{h} | \tilde{Z}_u \tilde{Y}_v \rangle &= -i h_{uv}^{RR} D_{uv}^{RR}, \\ \langle \mathfrak{h} | \tilde{Y}_u \tilde{Z}_v \rangle &= -i h_{uv}^{RR} B_{uv}^{RR}, & \langle \mathfrak{h} | \tilde{Z}_u \tilde{Z}_v \rangle &= h_{uv}^{RR} F_{uv}^{RR}, \\ \langle \mathfrak{h} | \tilde{\Psi}_u^1 \tilde{\Psi}_v^2 \rangle &= -i h_{uv}^{RR} C_{uv}^{RR}, & \langle \mathfrak{h} | \tilde{\Psi}_u^2 \tilde{\Psi}_v^1 \rangle &= i h_{uv}^{RR} C_{uv}^{RR}. \end{aligned} \quad (3.83)$$

Here we see the L/R -symmetry explicitly. Now for the LR and RL blocks we have:

$$\begin{aligned} \langle \mathfrak{h} | Y_u \tilde{Y}_v \rangle &= h_{uv}^{LR} A_{uv}^{LR}, & \langle \mathfrak{h} | Z_u \tilde{Y}_v \rangle &= -i h_{uv}^{LR} D_{uv}^{LR}, \\ \langle \mathfrak{h} | Y_u \tilde{Z}_v \rangle &= -i h_{uv}^{LR} C_{uv}^{LR}, & \langle \mathfrak{h} | Z_u \tilde{Z}_v \rangle &= h_{uv}^{LR} E_{uv}^{LR}, \\ \langle \mathfrak{h} | \Psi_u^1 \tilde{\Psi}_v^2 \rangle &= -h_{uv}^{LR} F_{uv}^{LR}, & \langle \mathfrak{h} | \Psi_u^2 \tilde{\Psi}_v^1 \rangle &= -h_{uv}^{LR} B_{uv}^{LR}, \end{aligned} \quad (3.84)$$

$$\begin{aligned}
\langle \mathfrak{h} | \tilde{Y}_u Y_v \rangle &= h_{uv}^{RL} A_{uv}^{RL}, & \langle \mathfrak{h} | \tilde{Z}_u Y_v \rangle &= -i h_{uv}^{RL} D_{uv}^{RL}, \\
\langle \mathfrak{h} | \tilde{Y}_u Z_v \rangle &= -i h_{uv}^{RL} C_{uv}^{RL}, & \langle \mathfrak{h} | \tilde{Z}_u Z_v \rangle &= h_{uv}^{RL} E_{uv}^{RL}, \\
\langle \mathfrak{h} | \tilde{\Psi}_u^1 \Psi_v^2 \rangle &= -h_{uv}^{RL} B_{uv}^{RL}, & \langle \mathfrak{h} | \tilde{\Psi}_u^2 \Psi_v^1 \rangle &= -h_{uv}^{RL} F_{uv}^{RL}.
\end{aligned} \tag{3.85}$$

And for the massless block we have:

$$\begin{aligned}
\langle \mathfrak{h} | \chi_u^{\dot{A}} \chi_v^{\dot{B}} \rangle &= h_{uv}^{\circ\circ} A_{uv}^{\circ\circ}, & \langle \mathfrak{h} | \tilde{\chi}_u^{\dot{A}} \tilde{\chi}_v^{\dot{B}} \rangle &= -i h_{uv}^{\circ\circ} D_{uv}^{\circ\circ}, \\
\langle \mathfrak{h} | \chi_u^{\dot{A}} \tilde{\chi}_v^{\dot{B}} \rangle &= -i h_{uv}^{\circ\circ} B_{uv}^{\circ\circ}, & \langle \mathfrak{h} | \tilde{\chi}_u^{\dot{A}} \chi_v^{\dot{B}} \rangle &= h_{uv}^{\circ\circ} F_{uv}^{\circ\circ}, \\
\langle \mathfrak{h} | T_u^{\dot{A}1} T_v^{\dot{B}2} \rangle &= i h_{uv}^{\circ\circ} C_{uv}^{\circ\circ}, & \langle \mathfrak{h} | T_u^{\dot{A}2} T_v^{\dot{B}1} \rangle &= -i h_{uv}^{\circ\circ} C_{uv}^{\circ\circ}.
\end{aligned} \tag{3.86}$$

Note here that these are blind to $\mathfrak{su}(2)_o$ index just like the S-matrix. This similarity of the tensor structure of the hexagon to the $\mathfrak{psu}(1|1)^{\oplus 2}$ S-matrix is not accidental since this is the only rapidity dependent two-particle tensor invariant under this algebra. Therefore this is the only quantity that could appear here.

Unfortunately the symmetry axiom does not fix n -particle hexagons, however this $\mathfrak{psu}(1|1)^{\oplus 2}$ S-matrix structure hints at an underlying structure for the two-particle hexagon that will allow generalization later. The main argument here is that we can understand the hexagon as a scattering of the $\mathfrak{psu}(1|1)^{\oplus 2}$ particles composing the excitations. Indeed, we can write a generic $\mathfrak{psu}(1|1)^{\oplus 4}_{c.e.}$ excitation as $|\Theta_u^{a\tilde{a}}\rangle = |\zeta_u^a \otimes \tilde{\zeta}_u^{\tilde{a}}\rangle$ where $|\zeta_u^a\rangle$ are simply states transforming in the ρ_L, ρ_R, ρ_o and ρ'_o representations of $\mathfrak{psu}(1|1)^{\oplus 2}$. Then the two-particle hexagon can compactly be written as

$$\langle \mathfrak{h} | \Theta_u^{a\tilde{a}} \Theta_v^{b\tilde{b}} \rangle = (-1)^{(F_a + F_{\tilde{a}})F_b} \mathbf{K}_u \mathbf{K}_v |\zeta_u^a \tilde{\zeta}_v^{\tilde{b}}\rangle \otimes \mathbf{S}_{(12)} |\tilde{\zeta}_u^{\tilde{a}} \zeta_v^{\tilde{b}}\rangle, \tag{3.87}$$

where $\mathbf{S}_{(12)}$ is the $\mathfrak{psu}(1|1)^{\oplus 2}$ S-matrix that acts on the right factor composing the excitations and $\mathbf{K}_u$ is a contraction operator that acts on $|\zeta_u^a\rangle$ and $|\tilde{\zeta}_u^{\tilde{a}}\rangle$ and yields $\langle \mathfrak{h} | \Theta_u^{a\tilde{a}} \rangle$, *id est*, it contracts the states and yields a scalar¹⁰. Also F_a is the fermion number, that is, it is 0 or 1 if $|\zeta_u^a\rangle$ is a boson or a fermion, respectively. Note that in the $\mathfrak{psu}(1|1)^{\oplus 2}$ S-matrix we should substitute $\Sigma_{uv}^{**} \rightarrow h_{uv}^{**}$ to obtain the two-particle hexagon. We can graphically understand expression (3.87) as exemplified in Figure 3.8. One can check then that (3.87) correctly reproduces all the two-particle hexagon form factors and that we can similarly write the mixed mass form factors which are just the massless limits of (3.82) - (3.86). Now we

¹⁰This contraction operator was first introduced in the AdS_3/CFT_2 context. This was not necessary in the AdS_5/CFT_4 case, however here we have fermionic one-particle form factors that do not vanish. Thus we need it to properly account for the statistics of the excitations which is nicely handled by these contraction operators.

$$\begin{aligned}
\langle \mathfrak{h} | Y_u Z_v \rangle &= \text{Diagram 1} = \text{Diagram 2} \\
&= h_{uv}^{LL} B_{uv}^{LL} \text{Diagram 3} + h_{uv}^{LL} C_{uv}^{LL} \text{Diagram 4} \\
&= h_{uv}^{LL} B_{uv}^{LL} \times (-i) = -i h_{uv}^{LL} B_{uv}^{LL}
\end{aligned}$$

Figure 3.8: two-particle hexagon computation. We show here how to compute the form factor $\langle \mathfrak{h} | Y_u Z_v \rangle$. We first denote the excitations on the hexagon as $|\xi_u^a \otimes \xi_v^{\tilde{a}}\rangle$. Then we move the right factors of the tensor product to the right to scatter them. Here we may have minus factors that we gain when we permute two fermions in the process of moving the particles. After the scattering we contract the states. In this example we note that the contractions in the first term in the second line yields $\langle \mathfrak{h} | Y_u \rangle \langle \mathfrak{h} | Z_v \rangle$ and the second one we obtain $\langle \mathfrak{h} | \Psi_u^1 \rangle \langle \mathfrak{h} | \Psi_v^2 \rangle$, thus only the former survive and we obtain the correct two-particle hexagon form factor. All this graphic procedure is encapsulated in (3.87).

move to the next hexagon property.

Watson axiom

Another property that the hexagon should satisfy is the Watson axiom, before we describe it we should first set some notation about the hexagon's scalar factors that multiply each block just as we did for the dressing factors Σ_{uv}^{**} . At first we have 9 independent functions that are

$$h_{uv}^{LL}, h_{uv}^{RR}, h_{uv}^{LR}, h_{uv}^{RL}, h_{uv}^{*\circ}, h_{uv}^{\circ*}, \text{ and } h_{uv}^{\circ\circ},$$

with $*$ = L, R . Using L/R -symmetry we can reduce the massive part of the scalar factors to only two independent functions $h_{uv}^{\bullet\bullet}$ and $\tilde{h}_{uv}^{\bullet\bullet}$, where these are related to the previous factors by

$$\begin{aligned}
h_{uv}^{LL} &= h_{uv}^{RR} = h_{uv}^{\bullet\bullet}, \\
h_{uv}^{LR} &= \tilde{h}_{uv}^{\bullet\bullet}, \\
h_{uv}^{RL} &= \sqrt{\frac{x_u^- x_v^-}{x_u^+ x_v^+} \frac{1 - x_u^+ x_v^+}{1 - x_u^- x_v^-}} \tilde{h}_{uv}^{\bullet\bullet}.
\end{aligned} \tag{3.88}$$

From now on we use the notation $h_{uv}^{*\circ} = h_{*,uv}^{\bullet\circ}$ and $h_{uv}^{\circ*} = h_{*,uv}^{\circ\bullet}$ for the mixed mass terms.

Given these definitions we can properly write the Watson axiom¹¹. This property tells us that when we exchange two particles in a hexagon form factor we pay a S-matrix. This can be stated as

$$\langle \mathfrak{h} | X_{a_1, u_1} \cdots X_{a_{j+1}, u_{j+1}} X_{a_j, u_j} \cdots X_{a_n, u_n} \rangle = \langle \mathfrak{h} | \mathcal{S}_{(j, j+1)}(u_j, u_{j+1}) | X_{a_1, u_1} \cdots X_{a_n, u_n} \rangle, \quad (3.89)$$

with $\mathcal{S}_{(j, j+1)}(u_j, u_{j+1})$ being the $\mathfrak{psu}(1|1)_{c.e.}^{\oplus 4}$ S-matrix that acts on particles j and $j+1$ only. One can see that the matrix part of the two-particle hexagon automatically satisfies this axiom, however it imposes conditions on the scalar factors. Indeed we have:

$$\begin{aligned} \frac{h_{uv}^{\bullet\bullet}}{h_{vu}^{\bullet\bullet}} &= (\Sigma_{uv}^{\bullet\bullet})^2, & \frac{h_{*,uv}^{\bullet\circ}}{h_{*,vu}^{\circ\bullet}} &= (\Sigma_{*,uv}^{\bullet\circ})^2, \\ \frac{\tilde{h}_{uv}^{\bullet\bullet}}{\tilde{h}_{vu}^{\bullet\bullet}} &= (\tilde{\Sigma}_{uv}^{\bullet\bullet})^2, & \frac{h_{*,uv}^{\circ\bullet}}{h_{*,vu}^{\bullet\circ}} &= (\Sigma_{*,uv}^{\circ\bullet})^2. \end{aligned} \quad (3.90)$$

Where the mixed mass $\Sigma_{*,uv}^{\bullet\circ}$ and $\Sigma_{*,uv}^{\circ\bullet}$ are defined in a similar way of the scalar factors. Note that for the massless scalar factor we have an extra minus sign. This is due to the fact that the highest weight particles in the massless supermultiplets are fermionic, so when we exchange them we gain a minus sign. These conditions together with the next axioms allow us to fix these scalar factors in terms of the dressing phases of the S-matrix. The Watson axiom is represented graphically in Figure 3.9.

Decoupling axiom

Another condition we impose on the hexagons is the decoupling property. This is heavily inspired and based on decoupling condition of the S-matrix, which we describe now. Consider a scattering event where we have two particles that are the crossing pair of each other, we denote them by $|X_u\rangle$ and $|\bar{X}_{v^{2\gamma}}\rangle$. If we take the limit $v \rightarrow u$, the composite state of these particles forms $|1_u\rangle$ which is called *singlet*. This state scatters trivially with all particles since, as the name suggests,

¹¹The name Watson axiom stems from a similar axiom in the form factor program in two-dimensional integrable quantum field theories, for more on these see [29, 59].

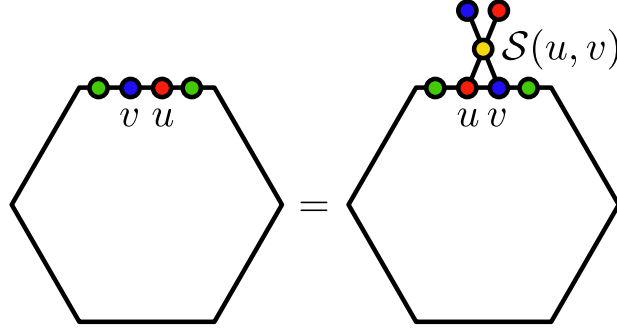


Figure 3.9: Watson axiom. Here we represent the Watson axiom, to permute the excitations with rapidities u and v we concatenate a $\mathfrak{psu}(1|1)_{c.e.}^{\oplus 4}$ S-matrix $\mathcal{S}(u, v)$ with the hexagon.

it is a singlet¹² of $\mathfrak{psu}(1|1)_{c.e.}^{\oplus 4}$ [107, 117]. This state has null energy and momenta and carries no R-charge. Since the singlet has trivial scattering with other states it decouples in any scattering event. We apply the same idea now to the hexagon and we have the decoupling property:

$$\text{Res}_{v \rightarrow u} \langle \mathfrak{h} | X_{a,u} \bar{X}_{a,v^2\gamma} X_{a_1,u_1} \cdots X_{a_n,u_n} \rangle = \frac{1}{\mu_{X_a}(u)} \langle \mathfrak{h} | X_{a_1,u_1} \cdots X_{a_n,u_n} \rangle, \quad (3.91)$$

where $\mu_{X_a}(u)$ is the *measure* factor associated with particle $|X_a\rangle$. Since we have $\langle \mathfrak{h} | \emptyset \rangle = 1$, the measure is defined as

$$\text{Res}_{v \rightarrow u} \langle \mathfrak{h} | X_{a,u} \bar{X}_{a,v^2\gamma} \rangle = \frac{1}{\mu_{X_a}(u)}. \quad (3.92)$$

This object will be central in our discussions of finite length corrections in Chapter 4, so we will abstain on comment it for now. Note that this decoupling property is the one that gave the simple poles to the tree level hexagons discussed previously in Chapter 2. We represent the decoupling property in Figure 3.10.

From the decoupling property we obtain functional equations relating distinct hexagon scalar factors. Indeed we obtain the following equations for the massive

¹²If we want to be more specific, the singlet is a linear combination of crossing pairs of the spectrum with some extra insertions called $\mathcal{Z}$ -markers. Note that crossing symmetry is a QFT concept, there is no analog in the spin chain picture of the integrable model. The role of the singlet and the decoupling condition is to implement crossing symmetry in the S-matrix of the spin chain picture.

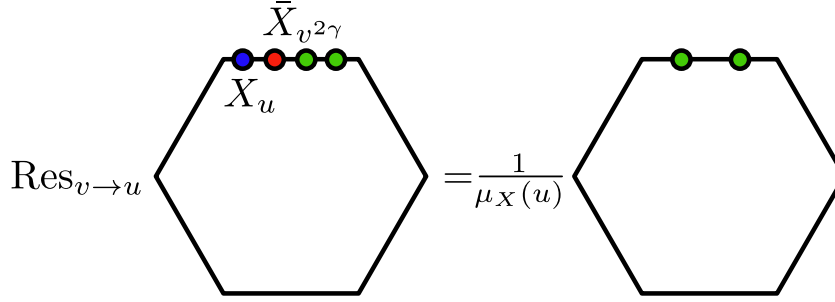


Figure 3.10: Decoupling axiom. We show here the decoupling axiom, when the rapidity of a crossing pair is equal we have a pole whose residue is the inverse of measure factor times the hexagon without the crossing pair.

and massless factors

$$\begin{aligned} h_{uv}^{\bullet\bullet} \tilde{h}_{u^{2\gamma}v}^{\bullet\bullet} &= \sqrt{\frac{x_u^+}{x_u^-} \frac{x_u^- - x_v^-}{x_u^+ - x_v^-}}, \\ h_{uv}^{\circ\circ} h_{u^{2\gamma}v}^{\circ\circ} &= \frac{x_v - x_u}{1 - x_u x_v}. \end{aligned} \quad (3.93)$$

Note that these equations are very similar to the crossing equations for the dressing phases [40, 109]. Therefore the solutions for the scalar factors will be written in terms of these phases. Another hint that this is the case is that these scalar factors have nontrivial monodromy just like Σ_{uv}^{**} . Now we move to the last hexagon axiom we will impose.

Monodromy axiom

We defined the hexagon form factor for a single excited operator, that is we put excitations only in a single physical edge. However we can construct a general hexagon configuration from analytic continuations of the one we defined previously. We define a jump of an excitation to a neighbor edge in the clockwise direction as a mirror transformation on the excitation [57, 74]. Therefore to jump one edge we do a mirror transformation ($u \rightarrow u^\gamma$), two edges a crossing transformation ($u \rightarrow u^{2\gamma}$), and so on. Similarly we can do counterclockwise moves by the inverse transformations. Therefore we can move excitations from the other operators by analytical continuations in the rapidity and put all excitations on a single edge. An example of this procedure is shown in Figure 3.11.

With this we impose the monodromy axiom. This is given by

$$\langle \mathfrak{h} | X_{a_1, u_1} \cdots X_{a_n, u_n} X_{a, u} \rangle = \langle \mathfrak{h} | \bar{X}_{a, u^{6\gamma}} X_{a_1, u_1} \cdots X_{a_n, u_n} \rangle, \quad (3.94)$$

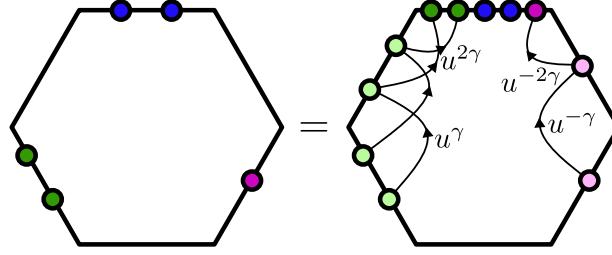


Figure 3.11: Analytic continuations in the hexagon. This is an example of how can we move excitations of the general excited hexagon to obtain a simpler hexagon with excitations only in a single edge using the rules described here. Note that there is not an unique way of doing these moves, we could do it in another way which would yield the same result.

where $\bar{X}_{a,v}$ denotes the crossing pair of particle $X_{a,v}$. We can understand this as the invariance of the hexagon if we take an excitation and move it clockwise back to the same edge we started it. This is represented in Figure 3.12. Just as the decoupling and Watson axioms this imposes functional constraints in the scalar factors. Consider for example the massive scalar factors, the monodromy axiom yields

$$h_{uv}^{\bullet\bullet} = \sqrt{\frac{x_v^- x_u^- - x_v^+}{x_v^+ x_u^- - x_v^-}} \tilde{h}_{v^{6\gamma}u}^{\bullet\bullet}. \quad (3.95)$$

We can obtain similar constraints for the remaining scalar factors. This is not the only constraint coming from analytical continuations. We have that the labels of the operators are not special, therefore we can cyclically rotate the hexagon and obtain the same form factor. This gives us the following constraint for the massive scalar factors

$$h_{uv}^{\bullet\bullet} = h_{u^{2\gamma}v^{2\gamma}}^{\bullet\bullet}. \quad (3.96)$$

With similar relations for the other scalar factors. This cyclic invariance is shown in Figure 3.12. Combining the Watson, decoupling and monodromy axioms we can completely fix the scalar factors in terms of the dressing phases as we will show below.

3.3.2 Full hexagon ansatz and partition function

Now we can present the final form for the n -particle hexagon form factor. As told before, we can fix the matrix structure of the two-particle hexagon by symmetry up to undetermined scalar factors. One can then solve the constraints given by Watson, decoupling and monodromy axioms to obtain these unknown

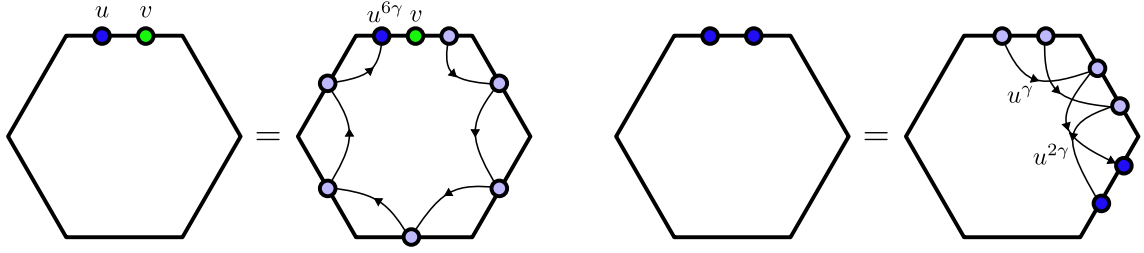


Figure 3.12: Monodromy axiom and cyclic invariance. On the left we show the monodromy axiom and in the right the cyclic invariance of the hexagon.

factors [74]. Indeed, for the massive and massless scalar factors one finds:

$$\begin{aligned}
 h_{uv}^{\bullet\bullet} &= \frac{1}{\sigma_{uv}^{\bullet\bullet}} \frac{x_u^+}{x_u^-} \sqrt{\frac{(x_u^+ - x_v^+)(x_u^- - x_v^-)}{(x_u^+ - x_v^-)^2} \frac{1 - x_u^- x_v^+}{1 - x_u^+ x_v^-}}, \\
 \tilde{h}_{uv}^{\bullet\bullet} &= \frac{1}{\tilde{\sigma}_{uv}^{\bullet\bullet}} \frac{x_u^+}{x_u^-} \sqrt{\frac{x_v^-}{x_v^+} \frac{1 - x_u^- x_v^+}{1 - x_u^+ x_v^-}}, \\
 h_{uv}^{\circ\circ} &= \frac{i}{\sigma_{uv}^{\circ\circ}} \frac{x_u - x_v}{1 - x_u x_v}.
 \end{aligned} \tag{3.97}$$

Similar formulas can be derived for the mixed mass scalar factors [74]. We can compare this with the $AdS_5 \times S^5$ case. There we have a single scalar factor whose expression is written in terms of the BES phase [57]. Therefore here we obtain a similar but more convoluted result due to the extra multiplets we have in the theory.

With these solutions for the hexagon scalar factors, the two-particle hexagon is then completely fixed. Now we can try to bootstrap the n -particle form factor. As said before symmetry is not enough to fix the matrix part, nonetheless based on the two-particle solution (3.87) one can propose the following expression for n particles

$$\langle h | \Theta_{u_1}^{a_1 \tilde{a}_1} \cdots \Theta_{u_n}^{a_n \tilde{a}_n} \rangle = (-1)^{\sum_{i < j} (F_{a_i} + F_{\tilde{a}_i}) F_{a_j}} \times \\
 \mathbf{K}_{u_1} \cdots \mathbf{K}_{u_n} | \tilde{\zeta}_{u_1}^{a_1} \cdots \tilde{\zeta}_{u_n}^{a_n} \rangle \otimes \mathbf{S}_{(1 \cdots n)} | \tilde{\zeta}_{u_1}^{\tilde{a}_1} \cdots \tilde{\zeta}_{u_n}^{\tilde{a}_n} \rangle. \tag{3.98}$$

This is just a generalization of (3.87) and we can check that it satisfy all axioms, so we take this as the n -particle ansatz. Here we have that the action of $\mathbf{S}_{(1 \cdots n)}$ on the n particles can be decomposed in a sequence of $2 \rightarrow 2$ scatterings using the Yang-Baxter equation as discussed in Section 3.1.

In Chapter 2 we said that the structure constants of spinning operators in

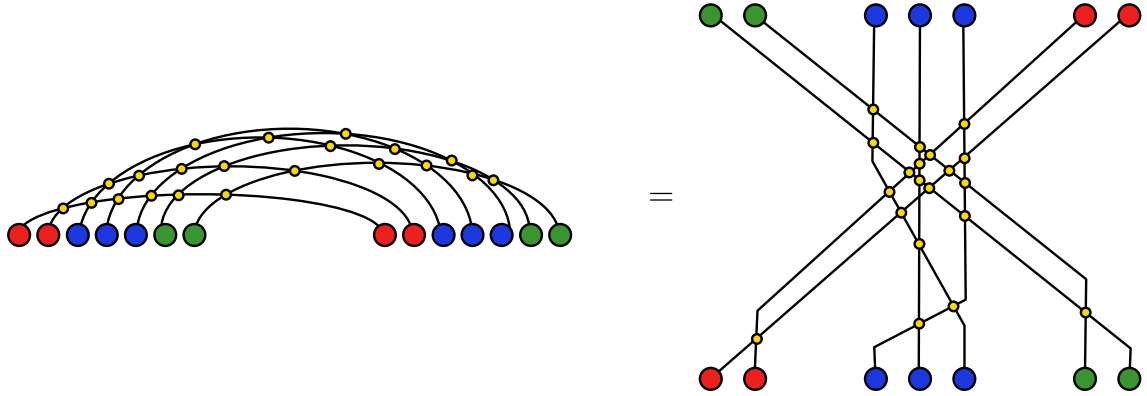


Figure 3.13: Hexagon partition function. In the left we have a standard hexagon computation for seven particles as it is described in the multiparticle ansatz (3.98). This can be rewritten as the partition function on the right with a judicious use of the Yang-Baxter equation. Here the Boltzmann weights are $\mathfrak{psu}(1|1)^{\oplus 2}$ S-matrices, and in AdS_5 these are $\mathfrak{psu}(2|2)$ ones, and the partition functions are defined on unusual Kagome lattices. Using this structure the authors in [67] were able to derive recursion relations for the $AdS_5 \times S^5$ hexagons as described in Section 2.2.

$AdS_5 \times S^5$ can be written in terms of the hexagon partition function. This concept was introduced in [67] for AdS_5/CFT_4 , however it could also be defined in $AdS_3 \times S^3 \times T^4$ duality. We will not do this here, however a partition function structure can be seen directly from the form of the multiparticle ansatz. We show how it appears in Figure 3.13. Therefore we derived the n -particle hexagon form factor and with it we can now try to compute some structure constants since this was the goal desired when introducing the hexagons.

3.4 Some examples of structure constants

The hexagonalization formalism is an integrability based bootstrap approach for the computation of structure constants. Then it must be tested against data in the dual CFT_2 . It is here that $AdS_3 \times S^3 \times T^4$ is weaker than its higher dimensional analog. Since we do not know the dual CFT_2 for generic values of the NSNS and RR fluxes, there is no data to compare with hexagonalization. However there exists a set of observables that we can compare: the chiral ring. As detailed in Section 3.2 the chiral ring has protected structure constants, therefore their value should be constant throughout the moduli. In [74] the authors set out to compute the structure constants in (3.72) from hexagonalization. There were some caveats however. Due to the mixing of the half-BPS supermultiplets they could compare

three-point functions involving only $\mathbb{V}_j^{+-}$ and $\mathbb{V}_j^{-+}$. Also they computed only the ratio

$$\frac{\langle \mathbb{V}_{j_1, \frac{j_1-1}{2}, \frac{j_1+1}{2}}^{-+} \mathbb{V}_{j_2, -\frac{j_2-1}{2}, -\frac{j_2+1}{2}}^{-+} \mathbb{V}_{j_3, 0, 0}^{+-} \rangle}{\langle \mathbb{V}_{j_1, \frac{j_1-1}{2}, \frac{j_1+1}{2}}^{-+} \mathbb{V}_{j_2, 0, 0}^{-+} \mathbb{V}_{j_3, -\frac{j_3-1}{2}, -\frac{j_3+1}{2}}^{-+} \rangle}. \quad (3.99)$$

The reason for this is that there is an ambiguity of how to take the zero momentum limit to define the half-BPS operators. One way of doing it is to take one excitation at a time to zero momentum. Another way, that was the one used in [74], is to consider coincident momenta and then take the null limit. This introduces some divergences in the correlator and due to this only the ratio (3.99) was compared to the known results and it matched with them up to normalization factors.

Even if we can not compare with known results, there are simple structure constants that we are able to compute from hexagonalization. We will show them here and we leave to future efforts the problem of matching them with results coming from other methods, therefore they stand now as predictions. From the multiparticle hexagon ansatz (3.98) we note that the hexagon form factor complexity is directly tied with the $\mathfrak{psu}(1|1)^{\oplus 2}$ S-matrix. It is simple to see from this that if we consider only highest ($|\phi_u^*\rangle$) or lowest ($|\phi_u^*\rangle$) excitations in the right factor of the tensor product $|\Theta_u^{a\tilde{a}}\rangle = |\zeta_u^a \otimes \tilde{\zeta}_u^{\tilde{a}}\rangle$ then the hexagon is diagonal, *id est*, there are no matrix contractions to be done or partition functions to be computed. Therefore the final result for the hexagon is just a product of functions and we obtain simple formulas for the structure constant in this case. In the language of [67] this case is the analog of the abelian hexagons.

We now compute these simple structure constants. The first example we consider is the three-point function with a single non-BPS operator which we denote by $C_Y^{\bullet\circ\circ}$. The other two BPS operators are simply the state $|0\rangle$. Using the n -particle hexagon expression (3.98) and the fact that the hexagon in this example involves only $|Y_u\rangle$ excitations, one sees that the hexagon acquires the following simple form

$$\langle h|Y_{u_1} \cdots Y_{u_n}\rangle = \langle h|Y\rangle^n \prod_{i<j} h_{u_i u_j}^{\bullet\bullet} A_{u_i u_j}^{LL} = \prod_{i<j} h_{u_i u_j}^{\bullet\bullet}. \quad (3.100)$$

Let $\mathbf{u}$ the set of roots describing the non-BPS operator involved in $C_Y^{\bullet\circ\circ}$, therefore using expression (3.74) we can write this structure constant as

$$C_Y^{\bullet\circ\circ} = \sum_{\alpha \cup \bar{\alpha} = \mathbf{u}} (-1)^{|\alpha|} a_{l_{12}}(\alpha) \mathcal{S}_{<}(\bar{\alpha}, \alpha) h_{<}^{\bullet\bullet}(\alpha, \alpha) h_{<}^{\bullet\bullet}(\bar{\alpha}, \bar{\alpha}). \quad (3.101)$$

Note that we used the same notation as in Section 2.2. Also here $\mathcal{S}(u, v)$ is the S-matrix coefficient of $|Y_u Y_v\rangle$ which is simply [74]

$$\mathcal{S}(u, v) = (\Sigma_{uv}^{\bullet\bullet})^2. \quad (3.102)$$

Using constraints coming from Watson axiom (3.90) and some algebraic manipulations akin to the ones in Appendix C, we can rewrite (3.101) as

$$C_Y^{\bullet\bullet\circ} = h_{<}^{\bullet\bullet}(\mathbf{u}, \mathbf{u}) \sum_{\alpha \cup \bar{\alpha} = \mathbf{u}} (-1)^{|\alpha|} \frac{e^{ip(\bar{\alpha})l_{12}}}{h^{\bullet\bullet}(\alpha, \bar{\alpha})}. \quad (3.103)$$

This formula is remarkably similar to the tailoring expressions described in Chapter 2, however unlike the latter it is valid at finite coupling and asymptotic large bridge lengths. This is a formula that one could compare with a proposal for the dual CFT_2 to the pure RR regime of the bulk theory.

We could look at a similar structure constant, but now we replace the S^3 excitations by the AdS_3 ones, that is, change $|Y_u\rangle$ to $|Z_u\rangle$. Now the three-point function is denoted by $C_Z^{\bullet\bullet\circ}$ and the hexagon can be computed directly from (3.98) which yields

$$\begin{aligned} \langle \mathfrak{h} | Z_{u_1} \cdots Z_{u_n} \rangle &= (-1)^{\binom{n}{2}} \langle \mathfrak{h} | Z \rangle^n \prod_{i < j} F_{u_i u_j}^{LL}, \\ &= (-i)^n \prod_{i < j} \left(h_{ij}^{\bullet\bullet} e^{-i(p_i - p_j)/2} \frac{x_i^+ - x_j^-}{x_i^- - x_j^+} \right), \\ &= (-i)^{|\mathbf{u}|} h_{<}^{\bullet\bullet}(\mathbf{u}, \mathbf{u}) g_{<}(\mathbf{u}, \mathbf{u}), \end{aligned} \quad (3.104)$$

where $g(u, v)$ is given by

$$g(u_i, u_j) = e^{-i(p_i - p_j)/2} \frac{x_i^+ - x_j^-}{x_i^- - x_j^+}. \quad (3.105)$$

Using the same manipulations as in the previous case we find the following structure constant

$$C_Z^{\bullet\bullet\circ} = h_{<}^{\bullet\bullet}(\mathbf{u}, \mathbf{u}) g_{<}(\mathbf{u}, \mathbf{u}) (-i)^{|\mathbf{u}|} \sum_{\alpha \cup \bar{\alpha} = \mathbf{u}} (-1)^{|\alpha|} \frac{e^{ip(\bar{\alpha})l_{12}}}{h^{\bullet\bullet}(\alpha, \bar{\alpha}) g(\alpha, \bar{\alpha})}. \quad (3.106)$$

Again we note the similarity with the tailoring expressions, however with an extra function $g(u, v)$ now. Note that we could apply the same procedures in the

analogues $C_{\tilde{Y}}^{\bullet\bullet\circ}$ and $C_{\tilde{Z}}^{\bullet\bullet\circ}$, but due to L/R -symmetry we have that $C_Y^{\bullet\bullet\circ} = C_{\tilde{Y}}^{\bullet\bullet\circ}$ and $C_Z^{\bullet\bullet\circ} = C_{\tilde{Z}}^{\bullet\bullet\circ}$.

We can also do a slightly more complicated case. Consider the three-point function consisting of three non-BPS operators. The first operator $\mathcal{O}_1$ has only excitations $|Y_u\rangle$, $\mathcal{O}_2$ has only $|\tilde{Y}_u\rangle$ and $\mathcal{O}_3$ is made of $|Y_u\rangle$ only. We denote this structure constant by $C_{Y,\tilde{Y},Y}^{\bullet\bullet\bullet}$. The hexagons in this case possess excitations on the three physical edges, however as we show in Figure 3.11 we can move the excitations using analytic continuations and put all of them in a single edge. For this we simply move the excitations on $\mathcal{O}_2$ two edges and the ones in $\mathcal{O}_3$ four edges. Then from the crossing rules (3.46) we have that the hexagon involved on this computation is simply given by

$$\begin{aligned} \langle h | Y_{w_1^{4\gamma}} \cdots Y_{w_m^{4\gamma}} Y_{v_1^{2\gamma}} \cdots Y_{v_n^{2\gamma}} Y_{u_1} \cdots Y_{u_q} \rangle &= h_{<}^{\bullet\bullet}(\mathbf{w}^{4\gamma}, \mathbf{w}^{4\gamma}) h_{<}^{\bullet\bullet}(\mathbf{v}^{2\gamma}, \mathbf{v}^{2\gamma}) h_{<}^{\bullet\bullet}(\mathbf{u}, \mathbf{u}) \\ &h^{\bullet\bullet}(\mathbf{u}, \mathbf{v}^{2\gamma}) h^{\bullet\bullet}(\mathbf{u}, \mathbf{w}^{4\gamma}) h^{\bullet\bullet}(\mathbf{v}^{2\gamma}, \mathbf{w}^{4\gamma}), \end{aligned} \quad (3.107)$$

where u_j , v_j and w_j are the excitations on $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$, respectively. With this we can do the same manipulations as before and write the following expression for $C_{Y,\tilde{Y},Y}^{\bullet\bullet\bullet}$

$$\begin{aligned} C_{Y,\tilde{Y},Y}^{\bullet\bullet\bullet} &= h_{<}^{\bullet\bullet}(\mathbf{u}, \mathbf{u}) h_{<}^{\bullet\bullet}(\mathbf{v}^{2\gamma}, \mathbf{v}^{2\gamma}) h_{<}^{\bullet\bullet}(\mathbf{w}^{4\gamma}, \mathbf{w}^{4\gamma}) \sum_{\alpha_j \cup \bar{\alpha}_j = \mathbf{u}_j} (-1)^{|\alpha_1| + |\alpha_2| + |\alpha_3|} \\ &\frac{e^{ip(\bar{\alpha}_1)l_{12}} e^{ip(\bar{\alpha}_2)l_{23}} e^{ip(\bar{\alpha}_3)l_{31}}}{h^{\bullet\bullet}(\alpha_1, \bar{\alpha}_1) h^{\bullet\bullet}(\alpha_2^{2\gamma}, \bar{\alpha}_2^{2\gamma}) h^{\bullet\bullet}(\alpha_3^{4\gamma}, \bar{\alpha}_3^{4\gamma})} h^{\bullet\bullet}(\alpha_1, \alpha_2^{2\gamma}) h^{\bullet\bullet}(\alpha_1, \alpha_3^{4\gamma}) h^{\bullet\bullet}(\alpha_2^{2\gamma}, \alpha_3^{4\gamma}) \\ &h^{\bullet\bullet}(\bar{\alpha}_1, \bar{\alpha}_2^{2\gamma}) h^{\bullet\bullet}(\bar{\alpha}_1, \bar{\alpha}_3^{4\gamma}) h^{\bullet\bullet}(\bar{\alpha}_2^{2\gamma}, \bar{\alpha}_3^{4\gamma}). \end{aligned} \quad (3.108)$$

This is remarkably similar to the abelian structure constants in [67]. We could also obtain a similar result for $C_{Z,\tilde{Z},Z}^{\bullet\bullet\bullet}$ using the same procedure.

Also using the weak coupling expansions of Appendix D we can write the tree

level expansion of the previous structure constants¹³. For $C_Y^{\bullet\circ\circ}$ we have

$$C_Y^{\bullet\circ\circ} = \prod_{i < j} \left(\sqrt{\frac{u_i + u_j + i/2}{u_i + u_j - i/2}} \frac{u_i - u_j}{u_i - u_j + i} \right) \sum_{\alpha \cup \bar{\alpha} = \mathbf{u}} (-1)^{|\alpha|} \prod_{u \in \bar{\alpha}} \left(\frac{u + i/2}{u - i/2} \right)^{l_{12}} \\ \prod_{u \in \alpha} \prod_{v \in \bar{\alpha}} \left(\sqrt{\frac{u + v - i/2}{u + v + i/2}} \frac{u - v + i}{u - v} \right). \quad (3.109)$$

Now for $C_Z^{\bullet\circ\circ}$ we have

$$C_Z^{\bullet\circ\circ} = (-i)^M \prod_{i < j} \left(\frac{u_j + i/2}{u_j - i/2} \frac{u_i - u_j}{u_i - u_j - i} \right) \sum_{\alpha \cup \bar{\alpha} = \mathbf{u}} (-1)^{|\alpha|} \prod_{u \in \bar{\alpha}} \left(\frac{u + i/2}{u - i/2} \right)^{l_{12}} \\ \prod_{u \in \alpha} \prod_{v \in \bar{\alpha}} \left(\frac{v - i/2}{v + i/2} \frac{u - v - i}{u - v} \right). \quad (3.110)$$

Similar expressions can be derived for $C_{Y,\tilde{Y},Y}^{\bullet\bullet\bullet}$ and $C_{Z,\tilde{Z},Z}^{\bullet\bullet\bullet}$.

3.5 Chapter summary

In this chapter we derived the hexagonalization formalism for $AdS_3 \times S^3 \times T^4$. First in Section 3.1 we considered how integrability worked in this spacetime, that is, we showed how to derive the symmetry algebra, described the particle content, detailed crossing and mirror transformations and finished with the derivation of the S-matrix. Then we described what is the chiral ring and how to define it from the point of view of integrability in Section 3.2. With this preliminary notions set up we finally detailed the hexagonalization formalism. We introduced the four axioms that fix the hexagon and solved them for the two-particle form factor in Section 3.3. Then a solution was proposed for the n -particle hexagon and we finished in Section 3.4 with some examples of computations of structure constants. We detailed here the asymptotic hexagon, which is valid for asymptotic large bridge lengths. To include finite length effects we have to consider the *mirror corrections*, this is what we do in the next chapter.

¹³Here weak coupling means $h \rightarrow 0$ and we call tree level is the order $\mathcal{O}(h^0)$ term. This is different to $AdS_5 \times S^5$, where weak coupling means small t'Hooft coupling λ which is the same as $h \rightarrow 0$ since $h \propto \sqrt{\lambda}$. Here we do not know the slope function at finite coupling, unlike in AdS_5/CFT_4 or AdS_4/CFT_3 [118]. Then there is no known independent notion of weak coupling in the unknown dual CFT_2 coupling.

Chapter 4

Hexagons and finite length corrections

After introducing the *asymptotic hexagons* of $AdS_3 \times S^3 \times T^4$ in Chapter 3, we are ready now to describe finite length corrections. These come into two types and we must add them to have the complete picture of the structure constant. One is due to finite size bridges and the other corresponds to finite size operators [57]. Here we deal with the ones coming from finite length bridges and we describe how to define them for hexagons in the AdS_3/CFT_2 duality. The corrections due to finite length operators are necessary to regularize poles corresponding to mirror magnons circling an excited operator, in the context of AdS_5/CFT_4 this was described in [119] with these poles starting at four loops. However these singularities exponentially decay in the length of the operators, therefore if we work with large operators we do not need to worry with these poles.

These finite length operators' corrections were not explored in the AdS_3/CFT_2 context and we will refrain to do so, therefore we will work in the regime of asymptotically large operators. It may be more useful to pursue this direction after we have more control over the Thermodynamic Bethe Ansatz of the model for this background (to describe finite length operators), more knowledge about the prescription to compute the dressing phases and control over the dual CFT_2 to have data to compare.

Nonetheless we will describe below the finite bridge length corrections, also named mirror corrections. In Section 4.1 we introduce the mirror corrections and how to define them. Then in Section 4.2 we compute the measure of the mirror corrections and write the transfer matrix formalism to define the integrand of these. We end up in Section 4.3 by showing up the cancelling of the mirror corrections for structure constants in the chiral ring.

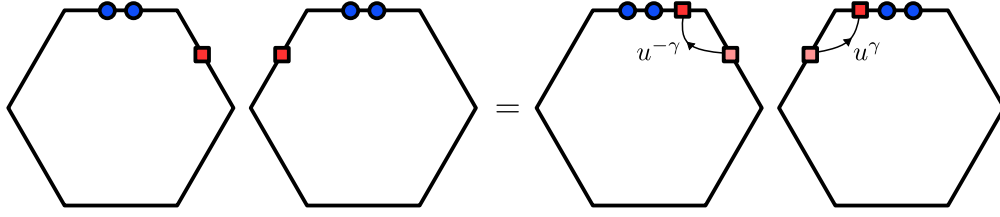


Figure 4.1: Insertion of a mirror particle in the hexagons. On the left we see the insertion of a mirror particle, denoted by the square, in a mirror edge. When moving the excitation to a nearby physical edge we do a mirror transformation as shown in the right.

4.1 Mirror corrections

The mirror corrections are added in the hexagon picture by introducing a complete pair of states of mirror particles in each cut of the hexagon, that is, we add these states in the mirror edges of the hexagon. Then, using the analytic continuations on the rapidity to move the excitations, we see that a mirror transformation suffices to put them in a physical edge, that is why these corrections are dubbed mirror corrections. In Figure 4.1 we show an example of this mirror move. Then we modify the hexagon prescription (3.74) to be simply:

$$\begin{aligned}
 C_{123}^{\bullet\bullet\bullet} = & \sum_{\alpha_j \cup \bar{\alpha}_j = \mathbf{u}_j} \prod_{j=1}^3 w(\alpha_j, \bar{\alpha}_j) \mathcal{H}(\alpha_1, \alpha_3, \alpha_2) \mathcal{H}(\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3) + \\
 & \sum_{\alpha_j \cup \bar{\alpha}_j = \mathbf{u}_j} \sum_{a=1}^{\infty} \sum_{X_a} \int \frac{du}{2\pi} \mu_a(u^\gamma) e^{-\tilde{E}_a(u)l} \prod_{j=1}^3 w(\alpha_j, \bar{\alpha}_j) \times \\
 & \mathcal{H}(\alpha_1, \alpha_2, \alpha_3 | X_a(u)) \mathcal{H}(\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3 | X_a(u)) + \dots, \quad (4.1)
 \end{aligned}$$

where the first term is the asymptotic hexagon given in (3.74), X_a is the mirror particle we added with quantum number a , $\mu_a(u^\gamma)$ is the integration measure associated with the mirror insertion, $e^{-\tilde{E}_a(u)l}$ is the propagation factor of the mirror particle with $\tilde{E}_a$ being the mirror energy and l the bridge length associated with the mirror edge where the excitation is inserted, and $\mathcal{H}(\alpha_1, \alpha_2, \alpha_3 | X_a(u))$ is the hexagon with a mirror insertion.

Let us explain the physics behind each term. Note that we inserted a complete basis of mirror particles, that is why we sum over all the states X_a that we can insert and if these carry a quantum number a we sum over it too. Also we integrated over all the values of the rapidity with the integration measure being $\mu_a(u^\gamma)$. This

$$\begin{aligned}
C_{123}^{\bullet\circ\circ} = & \text{asymptotic hexagon} - \sum_{a=1}^{\infty} \sum_{X_a} \int \frac{du}{2\pi} \mu_a(u^\gamma) e^{-\tilde{E}_a l_{13}} e^{ip_2 l_{31}} \text{Diagram 1} \\
& - \sum_{a=1}^{\infty} \sum_{X_a} \int \frac{du}{2\pi} \mu_a(u^\gamma) e^{-\tilde{E}_a l_{12}} e^{ip_2 l_{31}} \text{Diagram 2} \\
& - \sum_{a=1}^{\infty} \sum_{X_a} \int \frac{du}{2\pi} \mu_a(u^\gamma) e^{-\tilde{E}_a l_{23}} e^{ip_2 l_{31}} \text{Diagram 3} + \dots
\end{aligned}$$

Figure 4.2: Mirror corrections in the hexagonalization. Here we show the addition of mirror corrections to the structure constant in Figure 3.7. The first term is simply the previous asymptotic hexagon contribution. The first mirror correction comes from adding a single mirror particle to each bridge for each asymptotic hexagon contribution in Figure 3.7. Note here that we took only the second diagram contribution to exemplify and in ellipsis we have all the remaining infinite mirror corrections.

measure comes from the fact that we normalized the one-particle hexagons to be the ones in (3.81). The propagation factor $e^{-\tilde{E}_a(u)l}$ is the analytic continuation $e^{i\tilde{p}_a(u^\gamma)l}$ of the factor appearing in $w(\alpha_j, \bar{\alpha}_j)$. Since the mirror energy is positive in the mirror region we have an exponential decaying of the mirror corrections in l , that is why these are suppressed in the asymptotic hexagon. The ellipsis in (4.1) are just the insertions of n mirror particles with $n \rightarrow \infty$. Therefore the mirror corrections are an infinite series of terms that we add on top of the asymptotic hexagon. In Figure 4.2 we show how these corrections appear graphically.

Up to now we described mirror corrections in general, then let us specialize now to AdS_3/CFT_2 . Unlike the AdS_5/CFT_4 case, here we have three sets of particles X_a that we must insert: L , R and massless. Note that only the massive particles form bound states that needed to be summed over, therefore we identify Q and $\bar{Q}$ with the bound state number of the L and R particles, respectively. The general form of the mirror corrections for a single particle insertion in this case are

then

$$\begin{aligned} & \sum_{Q=1}^{\infty} \int_{\mathbb{R}} \frac{du}{2\pi} \mu_Q(u^\gamma) e^{-\tilde{E}_Q(u)l} \text{int}_Q^L(u^\gamma|\{u_i\}) + \\ & \sum_{\bar{Q}=1}^{\infty} \int_{\mathbb{R}} \frac{du}{2\pi} \mu_{\bar{Q}}(u^\gamma) e^{-\tilde{E}_{\bar{Q}}(u)l} \text{int}_{\bar{Q}}^R(u^\gamma|\{u_i\}) + \int_{|u|\geq 2} \frac{du}{2\pi} \mu_\circ(u) e^{-\tilde{E}_\circ(u)l} \text{int}^\circ(u|\{u_i\}), \end{aligned} \quad (4.2)$$

where $\text{int}_a^{L,R}(u^\gamma|\{u_i\})$ and $\text{int}^\circ(u|\{u_i\})$ are the integrand for massive and massless corrections, respectively. There are two details that we should consider here. First we have that the mirror integration is done over the region where the mirror energy is positive, thus for massive particles this is just $u \in \mathbb{R}$ and for massless ones is $|u| \geq 2$ with u below the cut as described in Section 3.1. Second the integrands are just the sum over mirror particle insertions X_a in the hexagon factors. For a single excited operator these are given then by

$$\begin{aligned} \text{int}_Q^L(u^\gamma|\{u_i\}) &= \sum_{\alpha \cup \bar{\alpha} = \mathbf{u}} w(\alpha, \bar{\alpha}) \sum_{X=Y, Z, \Psi^A} \mathcal{H}(\alpha|X_Q(u)) \mathcal{H}(\bar{\alpha}|X_Q(u)), \\ \text{int}_{\bar{Q}}^R(u^\gamma|\{u_i\}) &= \sum_{\alpha \cup \bar{\alpha} = \mathbf{u}} w(\alpha, \bar{\alpha}) \sum_{X=\bar{Y}, \bar{Z}, \bar{\Psi}^A} \mathcal{H}(\alpha|X_{\bar{Q}}(u)) \mathcal{H}(\bar{\alpha}|X_{\bar{Q}}(u)), \\ \text{int}^\circ(u^\gamma|\{u_i\}) &= \sum_{\alpha \cup \bar{\alpha} = \mathbf{u}} w(\alpha, \bar{\alpha}) \sum_{X=\chi^{\dot{A}}, \bar{\chi}^{\dot{A}}, T^{A\dot{A}}} \mathcal{H}(\alpha|X(u)) \mathcal{H}(\bar{\alpha}|X(u)), \end{aligned} \quad (4.3)$$

with similar expressions for excitations on the other operators. Then there are two ingredients for us to find: the mirror measure and the integrand. We will cover both in the next section.

4.2 Measure and transfer matrix formalism

We will begin by computing the measure. As described in Section 3.3, the measure enters in the decoupling axiom as a proportionality factor. It was defined in (3.92), which we reproduce below

$$\text{Res}_{v \rightarrow u} \langle \mathfrak{h} | X_{a,u} \bar{X}_{a,v^{2\gamma}} \rangle = \frac{1}{\mu_{X_a}(u)}. \quad (4.4)$$

Where $|X_a\rangle$ is any particle flavour and $|\bar{X}_a\rangle$ the corresponding crossing pair. A major difference to the $AdS_5 \times S^5$ case is that we now have in principle four

independent measures due to the model possessing four supermultiplets. Note that due to the L/R -symmetry we have that:

$$\langle \mathfrak{h} | \tilde{Y}_u Y_v \rangle = \langle \mathfrak{h} | Y_u \tilde{Y}_v \rangle \quad \text{and} \quad \langle \mathfrak{h} | \tilde{Z}_u Z_v \rangle = \langle \mathfrak{h} | Z_u \tilde{Z}_v \rangle. \quad (4.5)$$

Which can be verified from (3.84) and (3.85). Therefore the measures μ_Y and $\mu_{\tilde{Y}}$ are equal and the same is true for μ_Z and $\mu_{\tilde{Z}}$. Also we have that, due to supersymmetry, particles in the same supermultiplet have the same measure. Thus $\mu_Y = \mu_{\tilde{Z}}$ and this implies that the L and R sectors have the same measure which we denote by

$$\mu(u) \equiv \mu_Y(u). \quad (4.6)$$

Also the hexagon elements are blind to $su(2)_\circ$ indices, this makes the measure of massless supermultiplets equal. We denote it by

$$\mu_\circ(u) \equiv \mu_{\chi^A}(u). \quad (4.7)$$

In the end we have two independent measures to compute. Note here that we defined the measure only for fundamental particles, nonetheless since the bound state multiplets are in the same representation as the fundamental ones, but with distinct mass, we can define the bound state measure by simply doing $x_u^\pm \rightarrow x_u^{[\pm Q]}$. Now let us find these measures.

For the massive part of the spectrum of fundamental particles we just use (4.6). However there is a caveat here, when we do the crossing of a $|Y_u\rangle$ particle it is necessary to add an extra momentum factor $e^{ip(u)}$. This is needed since we are in the spin chain picture of the model. Basically to go from the string picture to the spin chain one we need to do a momentum dependent redefinition of the states. This slightly changes the states and the crossing rule gains this extra momentum factor¹. The measure is then computed by

$$\frac{1}{\mu(v)} = \text{Res}_{u \rightarrow v} \frac{x_u^+}{x_u^-} \langle \mathfrak{h} | \tilde{Y}_{u^{2\gamma}} Y_v \rangle = \frac{x_v^+}{x_v^-} \text{Res}_{u \rightarrow v} A^{RL}(u^{2\gamma}, v). \quad (4.8)$$

Note that to compute this measure we need $\sigma^{\bullet\bullet}(u, u)$ and this can be found from the crossing equations and braiding unitarity given in [74]. Therefore the computed measures do not rely on the explicit form of the dressing phases and thus are independent of the issues raised in [48], which we explained in Section

¹This is detailed in the Appendix F of [57] for $AdS_5 \times S^5$ and is the same for us here.

3.1, since we used only these properties and not the explicitly form of the dressing factors. Then we obtain in the end the following measure for a bound state with quantum number a :

$$\mu_a(u) = \frac{ix_u^{[+a]}x_u^{[-a]}}{h \left(x_u^{[+a]} - x_u^{[-a]} \right) \sqrt{\left(1 - \left(x_u^{[-a]} \right)^2 \right) \left(1 - \left(x_u^{[+a]} \right)^2 \right)}}. \quad (4.9)$$

However this is not the measure that appears in the mirror corrections (4.1), but the mirror transformed one. Using the mirror transformation of the Zhukovsky variables (see Section 3.1) we find

$$\mu_a(u^\gamma) = \frac{x_u^{[+a]}x_u^{[-a]}}{h \left(1 - x_u^{[+a]}x_v^{[-a]} \right) \sqrt{\left(1 - \left(x_u^{[-a]} \right)^2 \right) \left(1 - \left(x_u^{[+a]} \right)^2 \right)}}. \quad (4.10)$$

There are two significant facts that we can highlight from these expressions. First of all $\mu_a(u^\gamma)$ can be seen as the square root of the $AdS_5 \times S^5$ measure given in [57]. Second it can be checked that (4.9) is crossing invariant as it should be, since the measure can be defined on any physical edge of the hexagon and we can go from one to another by doing crossing transformations.

The massless particles' measure is computed similarly. For it we have:

$$\frac{1}{\mu_\circ(v)} = \text{Res}_{u \rightarrow v} i \langle \mathfrak{h} | \tilde{\chi}_{u^{2\gamma}}^{\dot{A}} \chi_v^{\dot{B}} \rangle = \text{Res}_{u \rightarrow v} D^{\circ\circ}(u^{2\gamma}, v). \quad (4.11)$$

Note that there are no momentum factor $e^{ip(u)}$ for crossed particles since these are fermionic, which can also be seen in the $AdS_5 \times S^5$ case. The factor of i comes from the crossing rules (3.47). Thus the massless measure is simply:

$$\mu_\circ(u) = -\frac{(x_u)^2}{(1 - (x_u)^2)^2}. \quad (4.12)$$

Which is significantly simpler than the massive case. It can be checked that this is crossing invariant even though we do not have the crossing factor as before. Also to go to the mirror region we just change the range of u as detailed in Section 3.1, then the form of $\mu_\circ(u)$ remains the same for the mirror measure. Due to this we have that $\mu_\circ(u)$ is the massless limit of $\mu_a(u)$ and not of its mirror transformed version (up to the $1/h$ factor). This makes sense since there is no

mirror transformation in the Zhukovsky variables for massless particles.

With these measures we can analyze their behavior at weak coupling. Consider the $\mathcal{N} = 4$ SYM case, there we have that mirror corrections enter at $2 + 2l$ -loops for the insertion of a mirror particle in a bridge of length l [57]. Again we see that these are delayed to higher and higher orders at large l and then the asymptotic hexagon is valid up to the aforementioned loop order. For AdS_3/CFT_2 we use the weak coupling expansions in Appendix D and find for the (mirror transformed) measure and mirror energy:

$$\begin{aligned}\mu_Q(u^\gamma) &= -\frac{h}{2(u^2 + Q^2/4)} + \frac{(Q^2 - 8u^2)}{4(u^2 + Q^2/4)^3}h^3 + \mathcal{O}(h^5), \\ e^{-\tilde{E}_Q(u)} &= \frac{h^2}{u^2 + Q^2/4} - \frac{(Q^2 - 4u^2)}{2(u^2 + Q^2/4)^3}h^4 + \mathcal{O}(h^6).\end{aligned}\tag{4.13}$$

Now for the massless case it can be seen that $\tilde{E}_o$ and μ_o are coupling independent. Then we have that the mirror corrections for a bridge of length l , if we disregard the integrand, at weak coupling start as

$$\textbf{Massive particles: } \mathcal{O}(h^{1+2l}) \quad \text{and} \quad \textbf{Massless particles: } \mathcal{O}(h^0). \tag{4.14}$$

Which differs significantly from the $AdS_5 \times S^5$ case since there all loop corrections come in even powers of the integrability coupling. Also note that since the massless mirror corrections enter at order $\mathcal{O}(h^0)$, therefore we could expect them to contribute to the structure constants of the chiral ring, since these are protected, and to correct the asymptotic hexagon even at tree level. However it turns out that all mirror corrections to the chiral ring vanish which will be proved in Section 4.3. In Chapter 5 we explain how these weak coupling expansions are compatible with the OPE of four-point functions.

Since the measure has been dealt with we now turn to the integrand. Following what was done for the $AdS^5 \times S^5$ case in [57] we can write it as the eigenvalues of the transfer matrix of $\mathfrak{psu}(1|1)^{\oplus 2}$. The transfer matrix is an operator that acts on a spin chain state and returns the state and the corresponding eigenvalue. These eigenvalues are central in the integrable structure of the model². Before we rewrite

²For instance if we demand that they are regular functions on the rapidities we find the Bethe equations as null residue conditions, thus we can solve the spectral problem. Also these eigenvalues are the generating functions of all the conserved charges of the integrable model. For the discussion of the transfer matrix in the simpler XXX model one can see [23] and in the AdS_3/CFT_2 context see [120, 121].

the integrands in terms of these transfer matrices, we define them below. We denote the eigenvalues by $\hat{\Delta}^*$ with $*$ = $L, R, \circ$ and these are [121]

$$\begin{aligned}\hat{\Delta}^L(u|\{u_j\}) &= \prod_{j=1}^{K_L} A_{uu_j}^{LL} \prod_{j=1}^{K_R} C_{uu_j}^{LR} \prod_{j=1}^{K_\circ} A_{uu_j}^{L\circ} - (-1)^{K_R} \prod_{j=1}^{K_L} D_{uu_j}^{LL} \prod_{j=1}^{K_R} E_{uu_j}^{LR} \prod_{j=1}^{K_\circ} D_{uu_j}^{L\circ}, \\ \hat{\Delta}^R(u|\{u_j\}) &= (-1)^{K_R+1} \prod_{j=1}^{K_L} D_{uu_j}^{RL} \prod_{j=1}^{K_R} F_{uu_j}^{RR} \prod_{j=1}^{K_\circ} D_{uu_j}^{R\circ} + \prod_{j=1}^{K_L} A_{uu_j}^{RL} \prod_{j=1}^{K_R} B_{uu_j}^{RR} \prod_{j=1}^{K_\circ} A_{uu_j}^{R\circ}, \\ \hat{\Delta}^\circ(u|\{u_j\}) &= 0.\end{aligned}\tag{4.15}$$

Here K_* is the number of particles of type $*$ = $L, R, \circ$ described by the roots $\mathbf{u} = \{u_j\}$ and the coefficients here are the matrix part of the $\mathfrak{psu}(1|1)^{\oplus 2}$ S-matrix defined in Section 3.1. Note that the transfer matrix is a function of a set of rapidities $\mathbf{u}$ that will be the Bethe roots of the operators in the hexagon and on a spectral parameter u that will correspond to the rapidity of the mirror particle. For massless modes the transfer matrix vanishes since the two massless multiplets $\rho_\circ$ and $\rho'_\circ$ in $\mathfrak{psu}(1|1)^{\oplus 2}$ are equal but come with opposite statistics for the excitations. Also to find the transfer matrices for bound states $\hat{\Delta}_a^*$ we just change $x^\pm(u) \rightarrow x^{[\pm a]}(u)$ in the S-matrix elements.

With this setup let us see the integrand, consider for example a single mirror particle insertion. We have to sum over particle flavour of the inserted mirror particle as seen in (4.1). This flavour sum can be recast as a transfer matrix of $\mathfrak{psu}(1|1)^{\oplus 2}$, in Figure 4.3 we show this procedure graphically. After this rewriting of the flavour sum we obtain for the insertion of a mirror particle of type $*$ and bound state number Q :

$$\text{int}_Q^*(u^\gamma|\{u_i\}) = \mathcal{A}_{\text{asymptotic}} \prod_j h_{Q,1}^{**j}(u^\gamma, u_j) \hat{\Delta}_Q^*(u^\gamma|\{u_i\}). \tag{4.16}$$

Where $\mathcal{A}_{\text{asymptotic}}$ is the asymptotic part of the hexagon and $h_{Q,1}^{**j}$ is the hexagon scalar factor between a fundamental particle and a particle with bound state number Q and with $*_j$ being the representation of the j -th excitation (these scalar factors are given in Appendix E). From this rewriting in terms of transfer matrices, we can check that the wrapping corrections for massless modes *vanish* since the corresponding massless transfer matrix also vanishes as shown in (4.15).

We can also compute the integrand when we insert multiple mirror particle corrections as it was done for AdS_5/CFT_4 in [61]. As before each sum over flavour

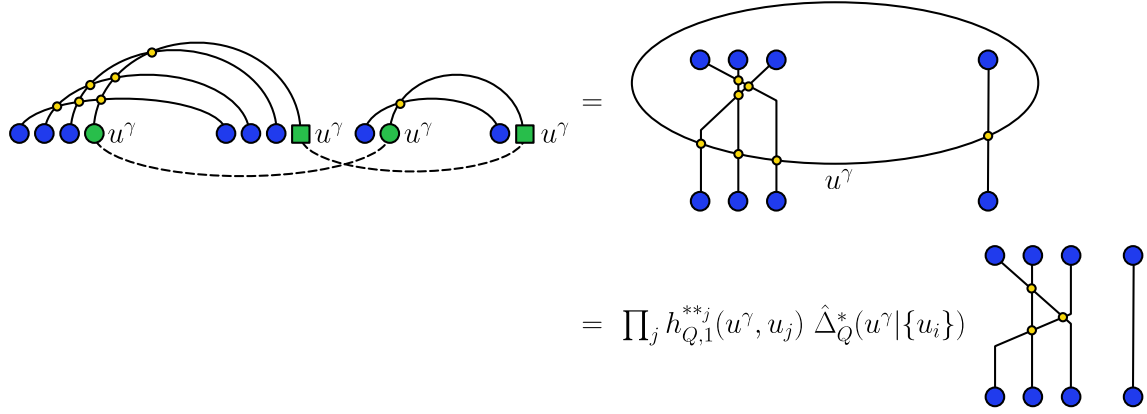


Figure 4.3: Flavour sum and transfer matrix. Here we show how the flavour sum in (4.1) is recast as an eigenvalue of the transfer matrix. On the left we have a typical hexagon computation of left and right hexagons with a single mirror insertion with rapidity u^γ . The flavour sum is denoted here as the dashed lines which have the same structure of a transfer matrix, *id est*, a sequence of scatterings of the mirror and physical particles with the mirror insertion traced out (see [121]). This is shown on the top right. Since the transfer matrix acts on Bethe states and return themselves, we have that the flavour sum is simply the asymptotic hexagon result times the eigenvalue of the Bethe state as shown in the bottom right. Also note that in the definition of the hexagons we have the S-matrix elements with the hexagon scalar factors as the dressing phases, this is why $h_{Q,1}^{**j}(u^\gamma, u_j)$ appears in (4.16).

turns into a transfer matrix, however now we must combine braiding unitarity for the S-matrix and Yang-Baxter equation to obtain the transfer matrices. Also for each pair of mirror insertions we gain the symmetric factor $h(u^\gamma, v^\gamma)h(v^\gamma, u^\gamma)$. The recasting of the flavour sum for the insertion of three mirror particles in terms of the transfer matrix is shown in Figure 4.4. An example of a mirror integrand with multiple L -particles is³

$$\text{int}_{Q_n}^L(\{u_n^\gamma\}|\{u_i\}) = \prod_k \prod_j h_{Q_k,1}^{L*j}(u_k^\gamma, u_j) \prod_{k \neq j} h_{Q_k, Q_j}^{LL}(u_k^\gamma, u_j^\gamma) h_{Q_j, Q_k}^{LL}(u_j^\gamma, u_k^\gamma) \prod_k \hat{\Delta}_{Q_k}^L(u_k^\gamma | \{u_i\}) \mathcal{A}_{\text{asymptotic}}. \quad (4.17)$$

The integrand with distinct multiplets and particle content can easily be generalized. A final point that we want to make is that even though the massless corrections vanish for the structure constant they could play a role in the four-point function hexagonalization as we will explain in Chapter 5. With this we fully

³For h_{Q_j, Q_k}^{LL} and similar scalar factors see Appendix E.

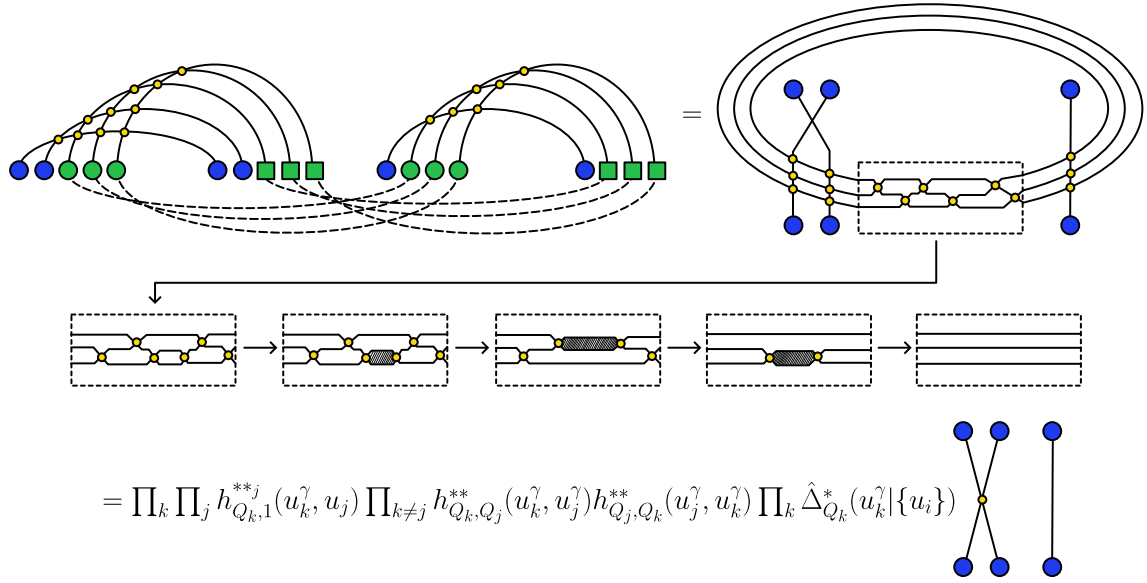


Figure 4.4: Multiple mirror corrections and transfer matrix. Here we show how the insertion of three mirror corrections is turned into (4.17). As in Figure 4.3 we denote the flavour sums as the dashed lines (top left) and as before they turn into traces (top right). However we have now three mirror particles that scatter between themselves as shown in the rectangle. We first use the Yang-Baxter equations to rewrite the scattering, then we form the highlighted bubble. These bubbles are just the left side of braiding unitarity (3.53) and then we get the identity times the symmetric factors. We keep using braiding unitarity until we arrive on just the untangled lines which is just a simple composition of the traces. After this we use the definition of the transfer matrices and we end up with (4.17).

described the mirror corrections for $AdS_3 \times S^3 \times T^4$ in full generality, we now turn to their effect on the chiral ring.

4.3 Mirror corrections and the chiral ring

Since the structure constants of the chiral ring are protected one can reasonably expect finite length corrections to vanish, at least for massive mirror particles. Here we will prove that in fact they *vanish for all mirror insertions*. The proof is similar to the cancellation of wrapping corrections in the TBA of half-BPS states [39]. First we need to see the effect of adding these massless fermions at zero momentum. By analyzing the action of $\mathfrak{psu}(1|1)^{\oplus 2}$ symmetry on the massless representations $\rho_\circ$ and $\rho'_\circ$ we can see that these multiplets are annihilated by the algebra for zero momentum particles. Indeed, all the coefficients of the representation are

proportional to $\eta_\circ(u)$ and it vanishes for null momentum (see Section 3.1):

$$\eta_\circ(u) = \sqrt{\frac{ih(1-x_u^2)}{2}} \Rightarrow \lim_{u \rightarrow \pm 2^\mp} \eta_\circ(u) = 0. \quad (4.18)$$

This vanishing implies that massless particles with zero momentum transform as singlets of $\mathfrak{psu}(1|1)^{\oplus 2}$ and thus scatter purely by transmission with another states. Also, since they are singlets, the scattering is independent of the flavour of the particle inside each representation that we scatter with.

Let us see these S-matrices explicitly and check that they are pure transmission. As explained in Section 3.2 we had to take care in the way we take the null momentum limit of the massless $\mathfrak{psu}(1|1)^{\oplus 2}$ excitations that compose the $|\chi_u^{\dot{A}}\rangle$ and $|\tilde{\chi}_u^{\dot{A}}\rangle$ fermions. Note that $p \rightarrow 0^\pm$ corresponds to $u \rightarrow \pm 2^\mp$, then the correct null momentum limit for $\rho_\circ$ and $\rho'_\circ$ excitations as a function of the rapidity is:

$$\phi_\circ^F(2^-), \phi_\circ^B(2^-), \phi_\circ^B(-2^+), \text{ and } \phi_\circ^F(-2^+). \quad (4.19)$$

Now we derive the S-matrix of these modes with all the $\mathfrak{psu}(1|1)^{\oplus 2}$ excitations. Let $|X_u^{(L,R)}\rangle$ be a particle in a massive L - or R -multiplet. Their scattering with the zero momentum insertions can be computed by just doing the appropriate zero momentum limit on each S-matrix element in Section 3.1. Then we find:

$$\mathbf{S}_{(12)}|\phi_\circ^B X_u^{(L,R)}\rangle = |X_u^{(L,R)}\phi_\circ^B\rangle, \quad (4.20)$$

$$\mathbf{S}_{(12)}|\phi_\circ^F X_u^{(L,R)}\rangle = (-1)^{|X^{(L,R)}|} |X_u^{(L,R)}\phi_\circ^F\rangle, \quad (4.21)$$

$$\mathbf{S}_{(12)}|\phi_\circ^F X_u\rangle = \begin{cases} (-1)^{|X^{(L)}|} \sqrt{\frac{x_u^+}{x_u^-} \frac{1+x_u^-}{1+x_u^+}} |X_u^{(L)}\phi_\circ^F\rangle \\ (-1)^{|X^{(R)}|} \sqrt{\frac{x_u^-}{x_u^+} \frac{1+x_u^+}{1+x_u^-}} |X_u^{(R)}\phi_\circ^F\rangle \end{cases}, \quad (4.22)$$

$$\mathbf{S}_{(12)}|\phi_\circ^B X_u\rangle = \begin{cases} \sqrt{\frac{x_u^+}{x_u^-} \frac{1-x_u^-}{1-x_u^+}} |X_u^{(L)}\phi_\circ^B\rangle \\ \sqrt{\frac{x_u^-}{x_u^+} \frac{1-x_u^+}{1-x_u^-}} |X_u^{(R)}\phi_\circ^B\rangle \end{cases}. \quad (4.23)$$

With $|X| = 0$ or $|X| = 1$ if $|X_u\rangle$ is a boson or a fermion, respectively. For massless

particles $|X_u^\circ\rangle$ at non-zero momentum we find:

$$\mathbf{S}_{(12)}|\phi_\circ^B X_u^\circ\rangle = |X_u^\circ \phi_\circ^B\rangle, \quad (4.24)$$

$$\mathbf{S}_{(12)}|\phi_\circ^F X_u^\circ\rangle = (-1)^{|X^\circ|} |X_u^\circ \phi_\circ^F\rangle, \quad (4.25)$$

$$\mathbf{S}_{(12)}|\varphi_\circ^F X_u^\circ\rangle = (-1)^{|X^\circ|} |X_u^\circ \varphi_\circ^F\rangle, \quad (4.26)$$

$$\mathbf{S}_{(12)}|\varphi_\circ^B X_u^\circ\rangle = -|X_u^\circ \varphi_\circ^B\rangle. \quad (4.27)$$

Clearly these S-matrices are pure transmission and flavour independent as expected.

Due to the form of the multiparticle hexagon ansatz (3.98) in terms of the $\mathfrak{psu}(1|1)^{\oplus 2}$ S-matrix, the hexagon form factor with these zero momentum insertions have their contribution factorized. Indeed let $|X_\circ\rangle$ be some fermion at zero momentum. Then

$$\langle \mathfrak{h} | X_\circ X_{a_1, u_1} \cdots X_{a_n, u_n} \rangle = \prod_j S_{\circ X_{a_j}}(u_j) \langle \mathfrak{h} | X_{a_1, u_1} \cdots X_{a_n, u_n} \rangle. \quad (4.28)$$

Where $S_{\circ X_{a_j}}(u_j)$ are one of the S-matrices in (4.20) - (4.27) that depend on the multiplet and momentum of the scattered particle $|X_{a_j, u_j}\rangle$ and which massless fermion was added. If there are more than one massless fermion at zero momentum, the hexagon form factor still factorizes. However now this it splits into a product of the phases $S_{\circ X_{a_j}}(u_j)$ times the hexagon of only the massless zero momentum fermions insertions:

$$\begin{aligned} \langle \mathfrak{h} | X_\circ^{(1)} \cdots X_\circ^{(j)} X_{a_1, u_1} \cdots X_{a_n, u_n} \rangle &= \prod_j S_{\circ X_{a_j}}(u_j) \langle \mathfrak{h} | X_\circ^{(1)} \cdots X_\circ^{(j)} \rangle \times \\ &\quad \langle \mathfrak{h} | X_{a_1, u_1} \cdots X_{a_n, u_n} \rangle. \end{aligned} \quad (4.29)$$

This is shown in Figure 4.5. As described in Section 3.4, the hexagon with only massless fermions at zero momenta demands a more careful treatment, but the full hexagon form factor factorizes nonetheless⁴.

Now that massless fermions can be added in the hexagon in a simple way we can look at the cancellation of wrapping corrections. We have that the phase $S_{\circ X_{a_j}}(u_j)$ is independent of the flavour of the scattered particles and it depends only on the rapidity u_j and the multiplet where the particle lies, thus when a

⁴This factorization resembles the charged hexagon formulation that appears in the hexagonalization of the fishnet theory in [122].

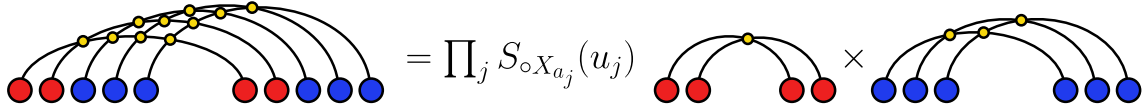


Figure 4.5: Factorization of the hexagon form factor with the insertion of fermionic massless modes at zero momentum. On the left we have an example of a hexagon form factor with massless fermions at null momenta (in red) and some other insertions (in blue). Since these fermions transform as singlets their scattering with the other modes is pure transmission. Then the hexagon factorizes as a product of phases and the corresponding hexagon form factors.

massive mirror magnon is inserted one has that the flavour's sum shown in Figure 4.3 decouples from the Bethe roots due to the pure transmission scattering. Therefore we have

$$\hat{\Delta}_a^*(u^\gamma | \{u_i\}) = \prod_j S_{\circ}(u_j) \hat{\Delta}_a^*(u^\gamma | \emptyset) = 0 \quad \text{if } * = L, R. \quad (4.30)$$

Where $\hat{\Delta}_a^*(u^\gamma | \emptyset)$ vanishes due to supersymmetry [121]. Then all mirror corrections when L and R mirror particles are on top of the half-BPS states vanish. As said before, for massless mirror modes the wrapping corrections always vanish so we disregard them. With this we conclude that all half-BPS three-point functions do not receive mirror corrections and are given only by the asymptotic hexagon prescription.

4.4 Chapter summary

In this chapter we completed the $AdS_3 \times S^3 \times T^4$ hexagonalization formalism detailed in Chapter 3 by computing the mirror corrections. We first introduced them in Section 4.1 and we had two undefined terms: the measure and the integrand. We then fixed the measure in Section 4.2 for massive and massless modes and showed its weak coupling expansions. Also we introduced the transfer matrix method and then rewrote the integrand in terms of the transfer matrix's eigenvalues. With these we showed then how all mirror corrections of the chiral ring's structure constants vanish in Section 4.3.

In the next chapter we move to the hexagonalization of n -point functions and in this context the mirror corrections become essential. For three-point functions we can always delay the mirror corrections to high loop orders for large bridge lengths. Also in this case l_{ij} are completely fixed by the lengths of each operator, so

we may pick these such that we have large bridges. However for n -point functions (with $n \geq 4$) we have that not all the bridge lengths are fixed and then we need to sum over all choices which includes small l_{ij} , thus mirror corrections always enter at low orders in the h -expansion. Since we have the mirror corrections sorted out let us move to the hexagonalization of n -point functions.

Chapter 5

Four-point functions and hexagons

Having described how to compute structure constants in $AdS_3 \times S^3 \times T^4$ using hexagonalization in Chapter 3 and the mirror corrections in Chapter 4, we now move to the next logical step: computation of four point functions. The hexagonalization of four-point functions was introduced in [69, 68] for $AdS_5 \times S^5$ and later it was improved in a series of works [62, 63, 70, 71, 73], which ultimately led to the formulation of the octagon [72, 123, 124, 125]. The logic of hexagonalization for higher point functions is the same as in the structure constant case of Chapter 3, we first divide the n -point functions into $2(n - 2)$ hexagons and glue them using the procedures detailed in the previous chapters.

However there is two problems in doing so. The first is that in the hexagon prescription we described did not account for the spacetime dependence on the positions of the insertions. Due to superconformal symmetry this is fixed for the three point function, however for $n > 3$ we must account for this nontrivial dependence. Second, we computed the hexagons in the canonical frame, which is always attainable for the three-point function by superconformal transformations. However this is not the case for $n \geq 4$, since by using these transformations we can put only three operators, that is, only one hexagon in the canonical frame. These two problems are related to each other and are solved by introducing a weight $\mathcal{W}_{ij}$ to the hexagonalization prescription. Below we will compute this weight and develop an hexagonalization prescription for n -point functions in the $AdS_3 \times S^3 \times T^4$ case, however we were not able to complete this program due to some technical details as we will describe later.

The outline of this chapter is the following. In Section 5.1 we describe general properties of the dual CFT_2 that will be necessary for us. With these properties, in Section 5.2 we then detail the hexagonalization of n -point functions and introduce the weight $\mathcal{W}_{ij}$. We end up in Section 5.3 by showing an example of a four-point function and then we describe some open problems in this direction.

5.1 Four-point functions in the dual CFT₂

Before proceeding to the hexagonalization of n -point functions we should establish some facts of the dual CFT₂ first. This theory has $\mathcal{N} = (4, 4)$ superconformal symmetry, therefore we can classify its operators by weights of the Virasoro algebra and of the R-charge symmetry which is given by $\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$ [115, 126]. Let $\mathcal{O}(x, \bar{x}|y, \bar{y})$ be some operator with $\{x, \bar{x}\}$ being its position in holomorphic variables and $\{y, \bar{y}\}$ its polarizations in S^3 . These polarizations are similar to the spinors n_a and $\tilde{n}_{\dot{a}}$ describing the operators in the $SO(4)$ sector of $\mathcal{N} = 4$ SYM in Section 2.1.

The operators transform in multiplets of the R-charge, therefore they fit into representations of $\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$. These have total spin $J \geq 0$ and $\bar{J} \geq 0$ for each factor of R-charge and the corresponding $(m, \bar{m})$ magnetic charges. Then we can decompose the operators in modes with definite m and $\bar{m}$:

$$\mathcal{O}(x, \bar{x}|y, \bar{y}) = \sum_{m=-J}^J \sum_{\bar{m}=-\bar{J}}^{\bar{J}} \sqrt{c_{J,m} c_{\bar{J},\bar{m}}} y^{J-m} \bar{y}^{\bar{J}-\bar{m}} \mathcal{O}_{m,\bar{m}}(x, \bar{x}) \quad \text{with } c_{a,b} = \binom{2a}{a+b}, \quad (5.1)$$

where $\mathcal{O}_{m,\bar{m}}$ denotes an operator in the $(2\mathbf{J} + 1) \oplus (2\bar{\mathbf{J}} + 1)$ representation with $(m, \bar{m})$ magnetic charges. Note that $y = 0$ picks the highest weight state in the $\mathfrak{su}(2)_L$, i.e., $m = J$ and the same happens for $\mathfrak{su}(2)_R$ if we take $\bar{y} = 0$. Also $y = \infty$ takes the lowest weight state which is just $m = -J$. Now if we want to pick descendant states we can apply derivatives in y and $\bar{y}$. Indeed it is simple to see that

$$\partial_y^{J-m} \partial_{\bar{y}}^{\bar{J}-\bar{m}} \mathcal{O}(x, \bar{x}|y, \bar{y})|_{y=\bar{y}=0} = \sqrt{c_{J,m} c_{\bar{J},\bar{m}}} (J-m)! (\bar{J}-\bar{m})! \mathcal{O}_{m,\bar{m}}(x, \bar{x}). \quad (5.2)$$

A similar trick is done in $\mathcal{N} = 4$ SYM to take the descendants, there we compute derivatives with respect to the polarizations vectors and then obtain R-charge descendants [71].

We will be interested in n -point functions of these operators. The two- and three-point functions have fixed conformal structure [115, 18]. Given the normalization in [115], the two-point correlator is simply:

$$\langle \mathcal{O}_1(x_1, \bar{x}_1|y_1, \bar{y}_1) \mathcal{O}_2(x_2, \bar{x}_2|y_2, \bar{y}_2) \rangle = \frac{y_{12}^{2J} \bar{y}_{12}^{2\bar{J}}}{x_{12}^{2h} \bar{x}_{12}^{2\bar{h}}}, \quad (5.3)$$

where $x_{ij} = x_i - x_j$ and $y_{ij} = y_i - y_j$. Also both operators have opposite R-charge, otherwise the correlator vanishes. Now for the three-point functions we have a slightly more complicated structure, however it is also fixed by symmetry as we showed for the chiral ring in Section 3.2. There we should have total null magnetic charges.

For four-point functions however the conformal structure is not fixed. We will consider the case of four half-BPS operators with identical dimensions. In the chiral ring these operators are dual to the $|0\rangle$ state. Since there are no excitations, the correlator will be given purely by the mirror corrections. We have four operator insertions in the correlator, then it is possible to form conformal and R-charge cross-ratios which are invariant under superconformal transformations [18]. These are

$$\eta = \frac{x_{12}x_{34}}{x_{13}x_{24}}, \quad \bar{\eta} = \frac{\bar{x}_{12}\bar{x}_{34}}{\bar{x}_{13}\bar{x}_{24}}, \quad \alpha = \frac{y_{12}y_{34}}{y_{13}y_{24}}, \quad \text{and} \quad \bar{\alpha} = \frac{\bar{y}_{12}\bar{y}_{34}}{\bar{y}_{13}\bar{y}_{24}}. \quad (5.4)$$

Then the four-point function of scalars with equal total dimension Δ can be written as

$$G(x_i, y_i) = \left(\frac{y_{12}\bar{y}_{12}}{x_{12}\bar{x}_{12}} \frac{y_{34}\bar{y}_{34}}{x_{34}\bar{x}_{34}} \right)^\Delta \mathcal{G}(\eta, \bar{\eta}, \alpha, \bar{\alpha}). \quad (5.5)$$

Here the **highlighted** prefactor carries all the conformal weight of the correlator and the function $\mathcal{G}(\eta, \bar{\eta}, \alpha, \bar{\alpha})$ is invariant. It is the computation of this invariant that we are interested in.

The conformal invariant part of the correlator must satisfy some consistency properties. The first of them is that it must be invariant under $\eta \leftrightarrow \bar{\eta}$ and $\alpha \leftrightarrow \bar{\alpha}$ permutations. The fact that we have a CFT₂ demands that both exchanges must be made together to have invariance, unlike in higher dimensions. Another requirement is that it needs to satisfy the superconformal Ward identities given in [126, 127]. These reduce to the following differential constraints:

$$(\partial_\eta + \partial_\alpha)\mathcal{G}(\eta, \bar{\eta}, \alpha, \bar{\alpha})|_{\eta=\alpha} = 0, \quad \text{and} \quad (\partial_{\bar{\eta}} + \partial_{\bar{\alpha}})\mathcal{G}(\eta, \bar{\eta}, \alpha, \bar{\alpha})|_{\bar{\eta}=\bar{\alpha}} = 0. \quad (5.6)$$

A four-point function that satisfy these can be split in the general form

$$\mathcal{G}(\eta, \bar{\eta}, \alpha, \bar{\alpha}) = \mathcal{C} + (\eta - \alpha)S(\eta, \alpha) + (\bar{\eta} - \bar{\alpha})S(\bar{\eta}, \bar{\alpha}) + (\eta - \alpha)(\bar{\eta} - \bar{\alpha}) \mathcal{H}(\eta, \bar{\eta}, \alpha, \bar{\alpha}). \quad (5.7)$$

Where $\mathcal{C}$ is a constant, $S(\eta, \alpha)$ is a function related to short multiplets of the superalgebra, and $\mathcal{H}(\eta, \bar{\eta}, \alpha, \bar{\alpha})$ is related to long multiplets. These conditions are *necessary*, but not *sufficient*. Therefore whatever the correlator one finds through

hexagonalization it must satisfy (5.6), but it does not tell us that it corresponds to the desired dual CFT₂.

The Ward identities are not sufficient to say that a computed correlator is correct, however to check this we can use the operator product expansion (OPE) in the four-point function and extract the CFT data. Indeed, consider the previous four-point function (5.7). If we fix the polarizations y_j , we can do the OPE on $\mathcal{O}_1 - \mathcal{O}_2$ and $\mathcal{O}_3 - \mathcal{O}_4$ channels and write the four point correlator in terms of the conformal blocks, which are fixed by symmetry [18]. Thus the dynamical inputs are the OPE coefficients (structure constants) and scaling dimensions of the model. Then the correlator (5.7) is simply given by

$$\mathcal{G}(\eta, \bar{\eta}, \alpha, \bar{\alpha}) = \sum_{\mathcal{O}} (C_{h_{\mathcal{O}}, \bar{h}_{\mathcal{O}}})^2 g_{h_{\mathcal{O}}, \bar{h}_{\mathcal{O}}}^{\Delta_{12}, \Delta_{34}}(\eta, \bar{\eta}), \quad (5.8)$$

where h and $\bar{h}$ are the chiral and antichiral scaling dimensions of the conformal algebra (thus $\Delta = h + \bar{h}$), $C_{h_{\mathcal{O}}, \bar{h}_{\mathcal{O}}}$ is the OPE coefficient of $\mathcal{O}_1, \mathcal{O}_2$ and $\mathcal{O}$, and $\Delta_{ij} = \Delta_i - \Delta_j$. We do the sum over all primary operators $\mathcal{O}$ in the theory. Note that we are considering equal operators, thus $\Delta_i = \Delta$ and the OPE coefficients in the $\mathcal{O}_1 - \mathcal{O}_2$ and $\mathcal{O}_3 - \mathcal{O}_4$ channels are equal and thus the square on (5.8). The conformal blocks are given by

$$g_{h, \bar{h}}^{\Delta, \Delta'}(\eta, \bar{\eta}) = \frac{\eta^h \bar{\eta}^{\bar{h}} k_h(\Delta, \Delta' | \eta) k_{\bar{h}}(\Delta, \Delta' | \bar{\eta}) + \bar{\eta}^h \eta^{\bar{h}} k_h(\Delta, \Delta' | \bar{\eta}) k_{\bar{h}}(\Delta, \Delta' | \eta)}{(-2)^{h-\bar{h}} (1 + \delta_{h, \bar{h}})}, \quad (5.9)$$

with $k_h(\Delta, \Delta' | \eta)$ being the following hypergeometric function

$$k_h(\Delta, \Delta' | \eta) = {}_2F_1 \left(h - \frac{\Delta}{2}, h + \frac{\Delta'}{2}, 2h \middle| \eta \right). \quad (5.10)$$

What we could do to test a correlator is to compute (5.7) through hexagonalization and then extract the OPE data on the right of (5.8) by just comparing both sides.

This can be done at tree level in $\mathcal{N} = 4$ SYM to extract the OPE coefficients of twist-two operators and at one-loop to extract their anomalous dimensions if we consider the lightcone limit, after this the result can be compared with hexagonalization [57]. Another case where this procedure was taken was in [71], here the author considered octagon like four-point functions and extracted the structure constants. Then with this information one simply computes the same OPE coefficients from the hexagons and compare them, which in this case was

verified that they matched. One could also use (5.8) in reverse, *id est*, compute the CFT data using hexagonalization and the asymptotic Bethe ansatz and then reconstruct the correlator. This can not be done in full generality since we have non-planar contributions in the OPE sum in (5.8) and these are not calculable by hexagonalization¹. However one can look for OPE channels where these contributions are suppressed and then we can reconstruct the correlator purely using the lowest genus CFT data. Unfortunately we were not able to complete the hexagonalization of four-point functions for $AdS_3 \times S^3 \times T^4$ due to some ambiguities that we will explain later, however we wanted to explain here what could one do once four-point functions are available in the AdS_3/CFT_2 duality. Then let us construct the hexagonalization for them in the next section.

5.2 Hexagonalization of four point functions

The hexagonalization of n -point functions works in a similar way to the structure constant one. Note that now correlators have two types of contributions: connected and disconnected. Then a n -point function is written as

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = G_{\text{disc}}(x_1, \cdots, x_n) + \frac{\sqrt{l_1 \cdots l_n}}{N^{n-2}} G_{\text{con}}(x_1, \cdots, x_n), \quad (5.11)$$

where $G_{\text{disc}}(x_1, \cdots, x_n)$ and $G_{\text{con}}(x_1, \cdots, x_n)$ are the disconnected and connected components of the correlator², respectively. The connected contributions are the ones computed by hexagonalization and in the discussion below we are always referring to the connected piece of the correlator. Note that by convenience we factored the genus contributions N^{n-2} and the factor $\sqrt{l_1 \cdots l_n}$ that comes from Wick contractions.

As in the structure constant case, hexagonalization consists in taking the world-sheet corresponding to the correlator and cut into hexagons. Consider n point functions, then we can split it into $2(n-2)$ hexagonal patches. Unlike the case of structure constants, there are now many ways to cut these higher point correlators into hexagons. For now let us focus on four point functions. The prescription

¹At least in its simplest form the hexagon can not compute higher non-planar contributions, some attempts to do so are [64, 65].

²This factorization comes from the clustering property of correlators in QFT [116]. Basically when we take two operators and consider the limit of their separation to infinity, the correlator splits into products of lower point functions with these two operators belonging to distinct clusters. Then the connected piece is defined as the piece of the correlator that vanishes in this limit.

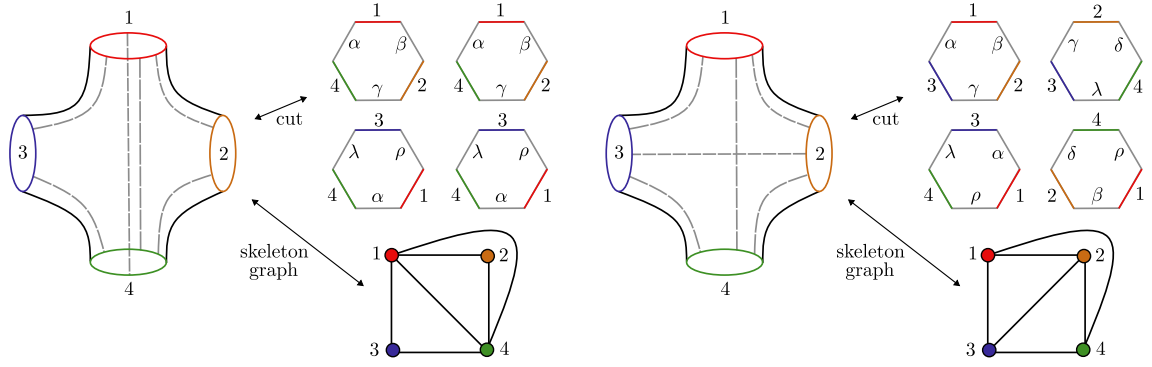


Figure 5.1: Hexagonalization of a four-point function and the related skeleton graph and hexagon decomposition. Here we consider two examples of hexagonalization cuttings for a four point function. On the left of each example we have the correlator, where it is the dashed lines that we cut it. In both examples we have all the four hexagons that decompose the correlator such that we must identify and then glue all edges that have the same Greek label. Also related to each cutting we have the specific skeleton graph.

proposed in [69, 68] for $AdS_5 \times S^5$ is to associate a specific cutting to a skeleton graph and then sum over all possible graphs. These skeleton graphs come from contracting the correlators at weak coupling. Note that each skeleton graph corresponds to a class of Wick contractions, we still have to sum over all possible bridge lengths which are not fixed for $n \geq 3$ (we will detail this later in Section 5.3). As described in the end of Chapter 4, this sum over all possible bridge lengths makes the mirror corrections absolutely essential here since they would contribute at low orders in small h -expansion. In Figure 5.1 we show two examples of specific cuts of a four-point function and the related skeleton graphs.

However now we have an extra element that was not present in the previous hexagonalization procedure. It turns out that now there are hexagons with distinct operators in their physical edges, then we can not put all of them in the canonical frame at the same time. This implies the existence of a weight $\mathcal{W}_{ij}$ in all bridges $i - j$ related to the factor one gains from gluing an hexagon in the canonical frame and another one that is not in this frame. Based on [68], we propose the following prescription for the hexagonalization of correlators in the dual CFT_2 of $AdS_3 \times S^3 \times T^4$:

$$G_{\text{con}}(x_i, y_i) = \sum_{\Gamma} \left(\prod_{(ij)} d_{ij}^{l_{ij}} \right) \left[\sum_{\psi_{(ij)}} \int d\mu_{\psi_{ij}}(u_{ij}) \mathcal{W}_{\psi_{ij}} \prod_{(ijk)} \mathcal{H}_{(ijk)} \right], \quad (5.12)$$

where Γ is the set of all skeleton graphs, $\mathcal{H}_{(ijk)}$ is the hexagon corresponding to

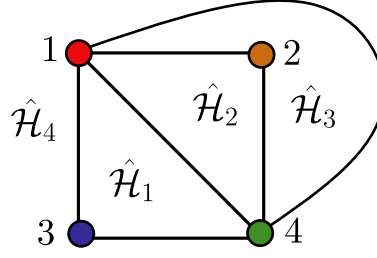


Figure 5.2: Example of an hexagon partition. Here we show an example of skeleton graph corresponding to a cutting of the four-point function (which is the left one in Figure 5.1). We labeled the four faces of the graph by its corresponding hexagons.

face (ijk) with mirror particles inserted, and $d\mu_{\psi_{ij}}$ the associated measure to the mirror particle. Note that each term corresponding to a skeleton graph comes with a weight (highlighted in (5.12)) that is just the contribution of the skeleton graph at tree level. For an usual CFT_2 a conservative proposal for d_{ij} would be

$$d_{ij} = \frac{y_{ij} \bar{y}_{ij}}{x_{ij} \bar{x}_{ij}}. \quad (5.13)$$

Which is just the contraction of scalar fields of R-charges $J_3 = \tilde{J}_3 = 1/2$ and dimension $\Delta = 1^3$. For the moment we assume that the prescription (5.12) of assigning skeleton graphs with the Wick contraction result as a prefactor is also valid here in $AdS_3 \times S^3 \times T^4$. The validity of this assumption will be further discussed in the Conclusion (Chapter 6).

Now let us see the origin of $\mathcal{W}_{ij}$. Consider the hexagonalization cutting in Figure 5.2. We can put the hexagon containing operators $\mathcal{O}_1, \mathcal{O}_3$ and $\mathcal{O}_4$ (denoted by $\hat{\mathcal{H}}_1$) in the canonical configuration, however when doing this we find that the operator $\mathcal{O}_2$ can not be fixed⁴. For it we have:

$$\eta = x_2, \quad \bar{\eta} = \bar{x}_2, \quad \text{and} \quad \alpha = y_2, \quad \bar{\alpha} = \bar{y}_2. \quad (5.14)$$

Therefore we can put either $\hat{\mathcal{H}}_1$ or $\hat{\mathcal{H}}_2$ in Figure 5.2 in the canonical frame, but not both at once. Let g_j be the superconformal transformation that takes the hexagon

³Usually in two-dimensional CFTs scalar fields have dimension zero. However they can gain a geometric dimension, this was showed for the dual CFT_2 proposed in [13] for the pure RR model.

⁴This is analogous to the case of conformal transformations. We can use them to put four generic points in the plane with three of them at fixed positions. The coordinates of the free point are the invariant conformal cross ratios and thus the fourth insertion is free.

form factor $\hat{\mathcal{H}}_j$ and put it in the canonical frame $\hat{\mathcal{H}}$, then

$$\hat{\mathcal{H}}_j = g_j^{-1} \hat{\mathcal{H}} g_j. \quad (5.15)$$

Using this we can see how $\mathcal{W}_{ij}$ appears in the hexagon expansion. Consider a mirror insertion in the bridge 1 – 4, the gluing at this bridge is given by

$$\begin{aligned} e^{-l_{14}\tilde{E}_\Psi} \hat{\mathcal{H}}_2|\Psi\rangle\langle\Psi|\hat{\mathcal{H}}_1 &= e^{-l_{14}\tilde{E}_\Psi} \hat{\mathcal{H}}|\Psi\rangle\langle\Psi|g_2g_1^{-1}|\Psi\rangle\langle\Psi|\hat{\mathcal{H}}, \\ &= e^{-l_{14}\tilde{E}_\Psi} \mathcal{W}_{14} \hat{\mathcal{H}}|\Psi\rangle\langle\Psi|\hat{\mathcal{H}}. \end{aligned} \quad (5.16)$$

Note that we have the propagation factor $e^{-l_{14}\tilde{E}_\Psi}$ of the mirror excitation. Here $\mathcal{W}_{14}$ is the **highlighted** term, which will simply be the eigenvalue of the rotation matrix between a canonical and a non-canonical hexagon. We have similar definitions when putting excitations on other bridges.

Now let us compute this weight. To find it we just consider the transformation that takes $\hat{\mathcal{H}}_2$ and put it in the canonical frame. It is clear from (5.14) that in order to find it we have to do a superconformal transformation to take $\mathcal{O}_2$ to position 1 and to lie at the middle of the R-charge multiplet. Let the dilation operators on the CFT₂ Virasoro algebra be $\{L_0, \tilde{L}_0\}$ and $\{J_3, \tilde{J}_3\}$ be the R-charge generators that yield the quantum numbers $(m, \tilde{m})$, thus the former transform $\{\eta, \bar{\eta}\}$ and the latter acts $\{\alpha, \bar{\alpha}\}$. The weight is thus given by:

$$\mathcal{W}_{14} = \langle\Psi|e^{-L_0 \log \eta} e^{-\tilde{L}_0 \log \bar{\eta}} e^{J_3 \log \alpha} e^{\tilde{J}_3 \log \bar{\alpha}}|\Psi\rangle. \quad (5.17)$$

To find the contribution in any other bridge, we simply exchange the operator indices in (5.4) which implies redefinitions of the cross ratios $\{\eta, \bar{\eta}, \alpha, \bar{\alpha}\}$. We can rewrite (5.17) in terms of more adequate variables that are the energy E and mass M of the magnons. We use the fact that [74]

$$E = L_0 + \tilde{L}_0 - J_3 - \tilde{J}_3 \quad \text{and} \quad M = L_0 - \tilde{L}_0 - J_3 + \tilde{J}_3. \quad (5.18)$$

So the weight acquires a convenient form for us:

$$\mathcal{W}_\psi = e^{-\frac{E_\Psi}{2} \log(\eta\bar{\eta})} e^{-\frac{M_\Psi}{2} \log(\eta/\bar{\eta})} e^{(J_3)_\Psi \log(\alpha/\eta)} e^{(\tilde{J}_3)_\Psi \log(\bar{\alpha}/\bar{\eta})}. \quad (5.19)$$

This is the final form of the weight $\mathcal{W}_{ij}$ for the hexagonalization of four point functions. Note that (5.19) is valid if the magnon is physical, if it is a mirror



Figure 5.3: One edge irreducible graph. We show on the left a one edge irreducible graph and on the right a non irreducible one. If we cut the 2 – 4 bridge of the right graph we get a disconnected graph.

magnon we have $E = i\tilde{p}$ and thus

$$\mathcal{W}_\psi = e^{-\frac{i\tilde{p}_\Psi}{2} \log(\eta\bar{\eta})} e^{-\frac{M_\Psi}{2} \log(\eta/\bar{\eta})} e^{(J_3)_\Psi \log(\alpha/\eta)} e^{(\tilde{J}_3)_\Psi \log(\tilde{\alpha}/\bar{\eta})}. \quad (5.20)$$

Note that this derivation is based on symmetry alone and does not depend on the correctness or not of the skeleton prescription. Also it assumes a similar form as in $\mathcal{N} = 4$ SYM where the R-charge term mixes the sphere and spatial cross-ratios just like (5.19) and (5.20) [57].

Also there one extra detail in the hexagon prescription (5.12). For $\mathcal{N} = 4$ SYM it was found that we do not sum over all skeleton graphs when adding mirror corrections [68]. The prescription gave there was that for the asymptotic contributions we sum over all possible connected graphs, however when adding the mirror corrections, in fact we consider only one edge irreducible graphs. These are the graphs where if we delete an edge we still have a connected graph. In Figure 5.3 we see the distinction between a graph that is one edge irreducible and one that it is not. In [70] it was proved that the mirror corrections in non one edge irreducible graphs vanish. Here we will consider the same prescription. With the hexagonalization of four-point functions detailed we now try to compute an example in the next section.

5.3 Four-point function of half-BPS operators

Now we focus on describing a simple example of a four-point function composed of length two half-BPS operators computed through hexagonalization. We will start with tree level and then move to $\mathcal{O}(h)$. However here we could not describe the correlator in its entirety due to three problems we encountered, in the course of the computations we will detail these difficulties.

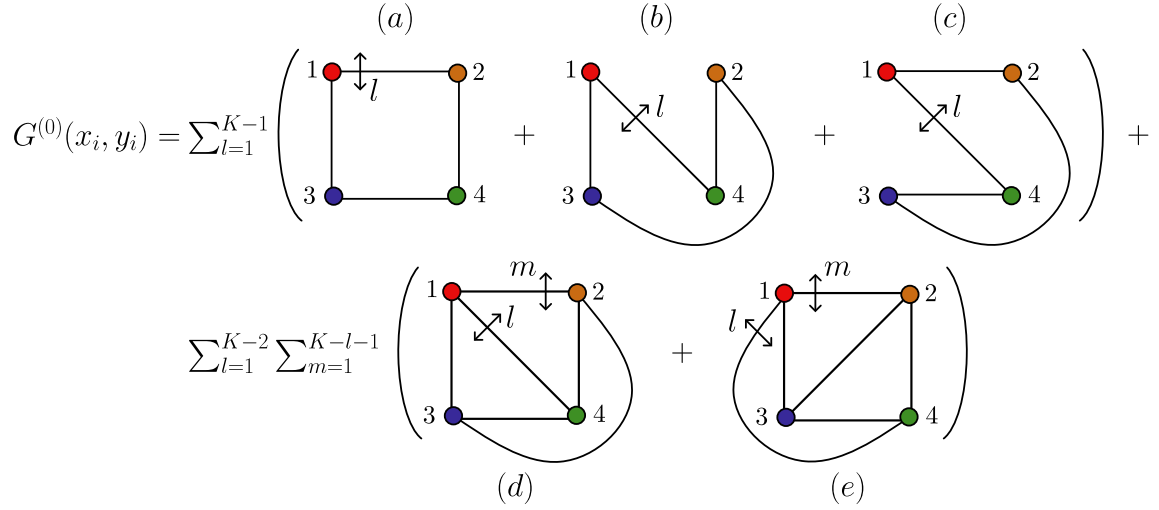


Figure 5.4: Skeleton graphs for polarized four point function. These are all the Wick contractions for the polarized four-point function of half-BPS operators of length K .

5.3.1 Order $\mathcal{O}(h^0)$ correlator

We showed in Section 4.2 that the massless mirror corrections vanish and the massive mirror ones start at order $\mathcal{O}(h^{1+2l})$ for a bridge of length l . Then given the prescription (5.12), to find the order $\mathcal{O}(h^0)$ contribution of the four-point function made only of $|0\rangle$ half-BPS operators we just disregard the mirror corrections. Thus the correlator at order $\mathcal{O}(h^0)$ is given by Wick contractions

$$G^{(0)}(x_i, y_i) = \sum_{\mathcal{G}} \prod_{(ij)} d_{ij}^{l_{ij}}, \quad (5.21)$$

where the bridge lengths l_{ij} satisfy the following constraints

$$\sum_{(ij) \cap k} l_{ij} = l_k. \quad (5.22)$$

Which tell us that all the contractions of the k -th operator add up to its length. Now we will consider the general case of length K half-BPS operators and then specialize to $K = 2$ later. We first write all the skeleton graphs for the connected part of the correlator as seen in Figure 5.4 [71]. We then label the five contributions as (a) – (e) and by doing the Wick contractions we find for each graph:

$$\begin{aligned}
(a) &= d_{12}^K d_{34}^K \sum_{l=1}^{K-1} \left(\frac{u}{\sigma}\right)^{K-l}, & (c) &= d_{12}^K d_{34}^K \sum_{l=1}^{K-1} \left(\frac{u}{\sigma} \frac{\tau}{v}\right)^l, \\
(b) &= d_{12}^K d_{34}^K \left(\frac{u}{\sigma}\right)^K \sum_{l=1}^{K-1} \left(\frac{\tau}{v}\right)^l, & (d) = (e) &= d_{12}^K d_{34}^K \sum_{l=1}^{K-2} \sum_{m=1}^{K-l-1} \left(\frac{u}{\sigma}\right)^{K-m} \left(\frac{\tau}{v}\right)^l.
\end{aligned} \tag{5.23}$$

Here we wrote them in terms of the simpler invariant cross ratios

$$\begin{aligned}
u &= \eta \bar{\eta}, & \sigma &= \alpha \bar{\alpha}, \\
v &= (1 - \eta)(1 - \bar{\eta}), & \tau &= (1 - \alpha)(1 - \bar{\alpha}).
\end{aligned} \tag{5.24}$$

To obtain the full correlator (5.21) we just sum over all these contributions:

$$\begin{aligned}
G^{(0)}(x_i, y_i) &= d_{12}^K d_{34}^K \left(\sum_{l=1}^{K-1} \left(\left(\frac{u}{\sigma}\right)^{K-l} + \left(\frac{u}{\sigma}\right)^K \left(\frac{\tau}{v}\right)^l + \left(\frac{u}{\sigma} \frac{\tau}{v}\right)^l \right) + \right. \\
&\quad \left. 2 \sum_{l=1}^{K-2} \sum_{m=1}^{K-l-1} \left(\frac{u}{\sigma}\right)^{K-m} \left(\frac{\tau}{v}\right)^l \right). \tag{5.25}
\end{aligned}$$

Which can be written in closed form by computing the geometric sums. Note that $d_{12}^K d_{34}^K$ is the conformal weight prefactor in (5.5). Now we specialize to length $K = 2$ operators. Therefore the tree level correlator (5.25) is the following rational function:

$$G^{(0)}(x_i, y_i) = d_{12}^K d_{34}^K \left(\frac{(\alpha \bar{\alpha} + \eta \bar{\eta})(\bar{\alpha}((2\alpha - 1)\eta \bar{\eta} - \alpha \eta - \alpha \bar{\eta} + \alpha) - (\alpha - 1)\eta \bar{\eta})}{\alpha^2 \bar{\alpha}^2 (\eta - 1)(\bar{\eta} - 1)} \right). \tag{5.26}$$

This is the sum of graphs (a) – (c) with the remaining ones vanish. It can be checked that this satisfy the $\mathcal{N} = (4, 4)$ superconformal Ward identities (5.6) and it is symmetric in $\eta \leftrightarrow \bar{\eta}$ and $\alpha \leftrightarrow \bar{\alpha}$ exchanges⁵ and thus it can be written as (5.7).

We could try to use the OPE expansion (5.8) to extract CFT data. Remember that to use it we need to fix the polarizations, which implies fixed α and $\bar{\alpha}$. If we want to keep these free we could use the superconformal blocks of [127]. However here we did not extract CFT data from this correlator. The main reason is that the hexagonalization formula for the structure constants is not yet in a form that allows us to do this. Indeed, in [71] the CFT data of $\mathcal{N} = 4$ SYM was extracted from the hexagonalization of four-point functions using the OPE formula. However it

⁵This is clear since the correlator (5.25) is written in terms of $\{u, v, \sigma, \tau\}$ which are symmetric themselves.

was used a slightly modified hexagon formula (the so-called BKV hexagons) to compute the structure constants for comparison⁶. Another reason is that although (5.12) is conservative it may be wrong, indeed there is an example of a CFT_2 where the correlator is not given as a sum of Wick contractions: the $\text{Sym}_N(T^4)$. This is significant since it is in the same moduli of the pure RR dual CFT_2 . Thus before extracting the CFT data would be better to have at least an independent test for the correlator (at tree level at least) to be able to assess its correctness or not. Now we move to $\mathcal{O}(h)$ corrections in the next subsection.

5.3.2 Order $\mathcal{O}(h)$ correlator and some issues

The first order beyond tree level for our four-point function of length two operators comes from putting a mirror excitation on a bridge of length zero. Consider the skeleton graph (a), we can put a mirror excitation either on the outside or inner 1 – 4 bridge, both have length zero. Note that the graphs (b) and (c) are topologically equivalent to (a) by relabelling the indices, therefore we focus on (a) for now. Following (4.1) we have in principle three distinct contributions for the insertion of a single mirror magnon

$$\mathcal{M}_L(\eta, \bar{\eta}, \alpha, \bar{\alpha}) = \sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} \frac{du}{2\pi} \mu_Q(u^\gamma) \text{hex}_Q^L(\eta, \bar{\eta}, \alpha, \bar{\alpha}), \quad (5.27)$$

$$\mathcal{M}_R(\eta, \bar{\eta}, \alpha, \bar{\alpha}) = \sum_{\bar{Q}=1}^{\infty} \int_{-\infty}^{\infty} \frac{du}{2\pi} \mu_{\bar{Q}}(u^\gamma) \text{hex}_{\bar{Q}}^R(\eta, \bar{\eta}, \alpha, \bar{\alpha}), \quad (5.28)$$

$$\mathcal{M}_o(\eta, \bar{\eta}, \alpha, \bar{\alpha}) = \int_{|u| \geq 2} \frac{du}{2\pi} \mu_o(u) \text{hex}_o(\eta, \bar{\eta}, \alpha, \bar{\alpha}), \quad (5.29)$$

with these denoting the L , R and massless contributions, respectively. As in (4.1) we inserted a completed basis of mirror states, thus we sum over bound states and the integrand is the contribution after summing over particle flavour, however with the insertion of the weight (5.20). This sum over flavour yields a transfer matrix like structure as described in Figure 4.3. Note also that we do not have the propagation factor $e^{-\tilde{E}_* l}$ since we are considering bridges of length zero.

Let us look at the integrand more carefully. By the same procedure described in Chapter 4 the sum over flavours is converted into a trace. Since there are no excitations on the operators one could expect that this sum vanishes, however we

⁶The main reason is that one has to consider structure constants not only with the level 0 excitations (in the Nested Bethe Ansatz language) but also with the level 1 or 2 excitations which demands a small modification of the hexagon prescription [62].

have flavour operators in the weight (5.20) that acts on the particles being traced over. Therefore we obtain the following integrands now

$$\text{hex}_Q^L(\eta, \bar{\eta}, \alpha, \bar{\alpha}) = e^{-\frac{i\vec{p}_Q}{2} \log(\eta\bar{\eta})} e^{-\frac{Q}{2} \log(\eta/\bar{\eta})} \text{Tr}_Q \left((-1)^F e^{2J_3 \log(\alpha/\eta)} e^{2\tilde{J}_3 \log(\bar{\alpha}/\bar{\eta})} \right), \quad (5.30)$$

$$\text{hex}_Q^R(\eta, \bar{\eta}, \alpha, \bar{\alpha}) = e^{-\frac{i\vec{p}_Q}{2} \log(\eta\bar{\eta})} e^{\frac{Q}{2} \log(\eta/\bar{\eta})} \text{Tr}_Q \left((-1)^F e^{2J_3 \log(\alpha/\eta)} e^{2\tilde{J}_3 \log(\bar{\alpha}/\bar{\eta})} \right), \quad (5.31)$$

$$\text{hex}_o(\eta, \bar{\eta}, \alpha, \bar{\alpha}) = e^{-\frac{i\vec{p}_o}{2} \log(\eta\bar{\eta})} \text{Tr}_o \left((-1)^F e^{2J_3 \log(\alpha/\eta)} e^{2\tilde{J}_3 \log(\bar{\alpha}/\bar{\eta})} \right), \quad (5.32)$$

where $(-1)^F$ is the fermion number operator and the trace is computed over the corresponding $\mathfrak{psu}(1|1)^{\oplus 2}$ multiplet. These traces can be seen as twisted transfer matrices, like the ones recently introduced in [121]. Also it is clear that these traces are independent of the rapidity. Thus we rewrite (5.30) - (5.32) as

$$\text{hex}_Q^L(\eta, \bar{\eta}, \alpha, \bar{\alpha}) = \mathcal{Z}_L(\eta, \bar{\eta}, \alpha, \bar{\alpha}) \left(\frac{\bar{\eta}}{\eta} \right)^{Q/2} e^{-\frac{i\vec{p}_Q}{2} \log(\eta\bar{\eta})}, \quad (5.33)$$

$$\text{hex}_Q^R(\eta, \bar{\eta}, \alpha, \bar{\alpha}) = \mathcal{Z}_R(\eta, \bar{\eta}, \alpha, \bar{\alpha}) \left(\frac{\eta}{\bar{\eta}} \right)^{Q/2} e^{-\frac{i\vec{p}_Q}{2} \log(\eta\bar{\eta})}, \quad (5.34)$$

$$\text{hex}_o(\eta, \bar{\eta}, \alpha, \bar{\alpha}) = \mathcal{Z}_o(\eta, \bar{\eta}, \alpha, \bar{\alpha}) e^{-\frac{i\vec{p}_o}{2} \log(\eta\bar{\eta})}, \quad (5.35)$$

where $\mathcal{Z}_*(\eta, \bar{\eta}, \alpha, \bar{\alpha})$ are simply the aforementioned traces. Note that the eigenvalue of M in (5.20) for L particles is the reverse of the R states, this is why the ratio $\eta/\bar{\eta}$ is reversed in (5.33) with respect to (5.34).

An ingredient that we left undefined is the trace, turns out that its computation is more an art than science. If one consider the excitations in the string frame we have that they do not carry R-charge, because due to the lightcone gauge-fixing the string length (or R-charge) is fixed (see Section 3.1). Then this trace vanishes due to supersymmetry, which is undesired. However working in the spin chain frame one has that the excitations carry R-charge and in fact we have a dynamical spin chain that changes length when acting with the supercharges. To deal with these length changing effects in the states in [107] it was introduced the so-called $\mathcal{Z}$ -markers that delete or add units of R-charge. Now for the computation of n point correlation functions in $AdS_5 \times S^5$ from hexagonalization it appears that the same $\mathcal{Z}$ -markers are necessary. In [68] the authors showed that without assigning $\mathcal{Z}$ -markers to mirror states one could not reproduce the universal R-charge prefactor

that appeared from perturbation theory and is a consequence of supersymmetry. Turns out that this assignment is *ad hoc* and there is no reason, up to now, for why it works [70, 73]. Since we do not have perturbative data such a procedure is not possible for us. We can also understand this $\mathcal{Z}$ -marker prescription as a R-charge assignment on the states being traced over. However it goes even beyond that since the procedure in [68] is to take the average over two charge assignments. In summary the hexagon formalism is some sort of hybrid between the string and spin chain frames and understanding this $\mathcal{Z}$ -marker better in $\mathcal{N} = 4$ SYM would allow us to analyze its effect on our setup. We creatively name this ambiguity the $\mathcal{Z}$ -marker problem.

Let us for the moment ignore this $\mathcal{Z}$ -marker *conundrum*. We can compute the mirror integral on the rapidity at order $\mathcal{O}(h)$ using the weak coupling expansions of Appendix D. Consider for example the contribution of L particles (5.27) and assume that $\mathcal{Z}_L$ is Q -independent. We have:

$$\mathcal{M}_L(\eta, \bar{\eta}, \alpha, \bar{\alpha}) = \mathcal{Z}_L(\eta, \bar{\eta}, \alpha, \bar{\alpha}) \sum_{Q=1}^{\infty} \left(\frac{\bar{\eta}}{\eta} \right)^{Q/2} \int_{-\infty}^{\infty} \frac{du}{2\pi} \mu_Q(u^\gamma) e^{-\frac{i\bar{p}_Q}{2} \log(\eta\bar{\eta})}. \quad (5.36)$$

Then we can expand the integrand at weak coupling and in doing so we find:

$$\begin{aligned} I_Q(\eta, \bar{\eta}) &= \int_{-\infty}^{\infty} \frac{du}{2\pi} \mu_Q(u^\gamma) e^{-\frac{i\bar{p}_Q}{2} \log(\eta\bar{\eta})}, \\ &= - \int_{-\infty}^{\infty} \frac{du}{2\pi} \frac{h}{2(u^2 + Q^2/4)} e^{-iu \log(\eta\bar{\eta})}, \\ &= \frac{e^{-Q|\log(\eta\bar{\eta})|}}{Q}. \end{aligned} \quad (5.37)$$

Where we computed the integral by residue and the result depends if one has $|\eta| > 1$ or $|\eta| < 1$, that is why we have the absolute value. Therefore the L particle contribution at order $\mathcal{O}(h)$ for $\eta\bar{\eta} < 1$ is

$$\begin{aligned} \mathcal{M}_L(\eta, \bar{\eta}, \alpha, \bar{\alpha}) &= -\frac{h}{2} \mathcal{Z}_L(\eta, \bar{\eta}, \alpha, \bar{\alpha}) \sum_{Q=1}^{\infty} \left(\frac{\bar{\eta}}{\eta} \right)^{Q/2} \frac{(\eta\bar{\eta})^{-Q/2}}{Q}, \\ &= \frac{h}{2} \mathcal{Z}_L(\eta, \bar{\eta}, \alpha, \bar{\alpha}) \log \left(1 - \frac{1}{\eta} \right). \end{aligned} \quad (5.38)$$

This can be found assuming that whatever the $\mathcal{Z}$ -marker prescription gives, it is Q -independent. For the R particle contribution (5.28) we have a similar result, but with conjugated variables.

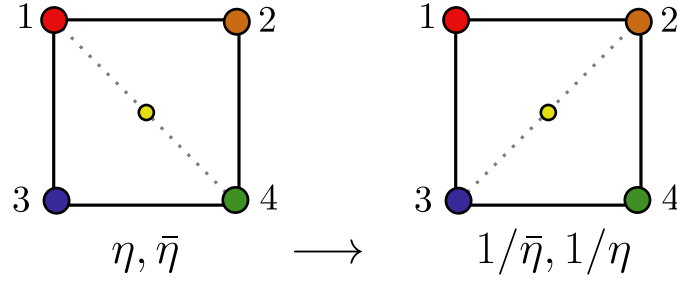


Figure 5.5: Flip invariance. Consider the graph (a), then the contribution of a mirror excitation on the bridge 1 – 4 of length zero is the same as if we had inserted the excitation on the 2 – 3 bridge of length also zero. For the equality to be true we need to redefine the cross ratios as indicated in the figure.

It is here that we face our second problem. Since we are in a two-dimensional theory, it is expected that the correlators possess singularities like $\log(\eta\bar{\eta})$ in perturbation theory due to the $(\eta\bar{\eta})^\Delta$ term in the conformal block expansion (5.8). This could be true here only after one joins the L and R contributions because each part contains only $\log(\eta)$ or $\log(\bar{\eta})$, as it can be seen in (5.38). However there is the problem of the undetermined $\mathcal{Z}$ -marker prescription when doing this, then in principle we can not join these correlators. Nonetheless it is consistent that the first mirror correction is of order $\mathcal{O}(h)$ since this would correspond to the massless modes' anomalous dimensions that are of the same order, which can be seen from the energy dispersion (3.18) for massless modes in the pure RR limit.

The third problem are the massless modes themselves. One can see that unless the $\mathcal{Z}$ -marker prescription for the two massless multiplets differs, all the massless mirror corrections would vanish. Just like for structure constants in Chapter 4. This would be desirable since otherwise we would have an infinite tower of mirror corrections that enters at order $\mathcal{O}(h^0)$. But at the moment there is no way to settle which option would be correct.

So we could not derive the full four-point function, however not everything is lost since we can constrain it. The correlators must be invariant under superconformal Ward identities and flip invariance. The former was described in Section 5.1. The latter is simply a condition imposed on specific mirror corrections. Consider for example the mirror correction of a single particle on the 1 – 4 bridge of length zero in graph (a). Flip invariance imposes that this is the same as inserting a mirror corrections on the 2 – 3 bridge of the same graph, also of length zero. But for this to be valid we need to redefine the cross ratios to be $\eta \rightarrow \bar{\eta}^{-1}$ and $\bar{\eta} \rightarrow \eta^{-1}$. Flip invariance is demonstrated in Figure 5.5. We could use flip invariance and

superconformal Ward identities to test if any proposal for the $\mathcal{Z}$ -markers is valid. If we disregard the $\mathcal{Z}_L$ term in (5.36) we can see that the L contribution is flip invariant. To prove this we just use crossing invariance of the measure and the exchange of L and R representations under crossing.

Even though we do not possess the explicit form of the $\mathcal{M}_*$ contributions we can write what would be the correlator at order $\mathcal{O}(h)$ if these were found. Assume for the moment that the massless mirror corrections vanish, then we need to add both the L and R contributions for each graph. Thus we define

$$\mathcal{M}(\eta, \bar{\eta}, \alpha, \bar{\alpha}) = \mathcal{M}_R(\eta, \bar{\eta}, \alpha, \bar{\alpha}) + \mathcal{M}_L(\eta, \bar{\eta}, \alpha, \bar{\alpha}). \quad (5.39)$$

The full $\mathcal{O}(h)$ contribution is obtained after summing the contributions of skeleton graphs (a), (b) and (c). Using the prescription (5.12) we multiply each graph by the tree level contribution, thus finding

$$\begin{aligned} G^{(1)}(x_i, y_i) = 2 & \left(d_{12}d_{24}d_{34}d_{13} \mathcal{M}(\eta, \bar{\eta}, \alpha, \bar{\alpha}) + \right. \\ & d_{13}d_{23}d_{24}d_{14} \mathcal{M}(1 - \eta, 1 - \bar{\eta}, 1 - \alpha, 1 - \bar{\alpha}) + \\ & \left. d_{12}d_{23}d_{34}d_{14} \mathcal{M}\left(\frac{\eta}{\eta - 1}, \frac{\bar{\eta}}{\bar{\eta} - 1}, \frac{\alpha}{\alpha - 1}, \frac{\bar{\alpha}}{\bar{\alpha} - 1}\right) \right). \quad (5.40) \end{aligned}$$

Note the overall factor of 2, it comes from adding the mirror contributions in the inner and outer channels for each skeleton graph. As said before the graphs (b) and (c) are topologically equivalent to (a) by relabeling the operators' indices, that is why the cross ratios are transformed in the second and third terms of (5.40). After factoring out $d_{12}^2 d_{34}^2$ we obtain

$$\begin{aligned} G^{(1)}(x_i, y_i) = 2d_{12}^2 d_{34}^2 & \left(\frac{\eta \bar{\eta}}{\alpha \bar{\alpha}} \mathcal{M}(\eta, \bar{\eta}, \alpha, \bar{\alpha}) + \right. \\ & \frac{(1 - \eta)(1 - \bar{\eta})}{(1 - \alpha)(1 - \bar{\alpha})} \mathcal{M}(1 - \eta, 1 - \bar{\eta}, 1 - \alpha, 1 - \bar{\alpha}) + \\ & \left. \frac{\eta \bar{\eta}(1 - \alpha)(1 - \bar{\alpha})}{(1 - \eta)(1 - \bar{\eta})\alpha \bar{\alpha}} \mathcal{M}\left(\frac{\eta}{\eta - 1}, \frac{\bar{\eta}}{\bar{\eta} - 1}, \frac{\alpha}{\alpha - 1}, \frac{\bar{\alpha}}{\bar{\alpha} - 1}\right) \right). \quad (5.41) \end{aligned}$$

Which is in the form (5.5). Then one just needs to check superconformal Ward identities in this final correlator for any proposal of the $\mathcal{Z}$ -marker and thus of $\mathcal{M}_*$.

5.4 Chapter summary

In this chapter we described the hexagonalization of higher point correlation functions in $AdS_3 \times S^3 \times T^4$. We first reviewed in Section 5.1 some properties of correlators in the dual CFT_2 that any correlator must obey, like the classification of operators in R-charge multiplets and the Ward identities. We also described a procedure to extract the CFT data from the correlator using the OPE, such that we could then compare any hexagonalization proposal with independent computations for these data.

Then we moved to the hexagonalization procedure *per se* in Section 5.2. We described how to cut and glue four-point functions and in doing so we introduced an extra ingredient in the hexagonalization: the weight $\mathcal{W}_{ij}$. We detailed the origin of this factor and how to fix it by symmetry. With all this done we exemplified the procedure by computing a four-point function of length two half-BPS operators in Section 5.3. We started by analyzing the tree level correlator and checked that it satisfy the Ward identities. Then we attempted to compute the $\mathcal{O}(h)$ of this correlator, which is given by a single mirror insertion on a bridge of length zero. We then detailed how to calculate these corrections, however in this process we discovered three problems: the $\mathcal{Z}$ -marker prescription, analytic structure and massless modes mirror corrections. To fix these it is necessary to have input from perturbative data, which is absent for AdS_3/CFT_2 for most of the moduli space.

In the next chapter we will conclude this work by describing some future prospects concerning the classical limit of structure constants and $AdS_5 \times S^5$ hexagons, hexagonalization of structure constants in $AdS_3 \times S^3 \times T^4$ and computation of four-point functions in this duality. We will conclude by discussing what was done and what could be done to further advance in these matters.

Chapter 6

Conclusion

We will finish this thesis by discussing what was achieved and some possible future directions for Part 1 which concerns hexagonalization and the classical limit and for Part 2 about AdS_3/CFT_2 hexagons and mirror corrections.

Part 1: Hexagons and the classical limit

Since its conception in [57], the hexagonalization formalism was thoroughly checked against data coming from many sources like perturbation theory for instance. These tests were important to establish the validity of the formalism and also to improve it. One of these tests was the match to the tailoring method at tree level. However their equivalence was not fully established. In this work we resolved this issue by proving, for the $SU(2)$ sector, that hexagonalization and tailoring yield the same results. This was achieved in Section 2.4 and the final result is shown in (2.67). Through this analysis we even found a new representation for the hexagon at tree level as a sum over partitions given in (2.63) that bypasses all the tensor contractions involved in the hexagon computation.

We also did a similar analysis on the non-compact $SL(2)$ sector. We note that the new hexagon representation (2.63) for $SU(2)$ is simply a rational function of the rapidities. Then we derived recursion relations for it akin to the $SL(2)$ sector ones discussed in Section 2.2. Thus having the expression (2.63) in mind for the $SU(2)$ hexagons we also proposed a new representation for the spinning hexagons in (2.93). Both of these new representations share a remarkable similar structure, which hints at an unified description for all sectors. As seen in [67] the hexagon can be seen as a complicated sum over configurations of a partition function defined in a Kagome lattice. Nonetheless, at weak coupling, our expressions remarkably reduce this sum to simpler sums over partitions (2.63) and (2.93).

We also tackled the classical limit of correlation functions in both $SU(2)$ and $SL(2)$ sectors. Making a careful use of the aforementioned recursion relations and more standard classical limit techniques from the literature, we were able

to compute the classical limit in the compact sector for type I-I-I three-point functions at weak coupling in Section 2.4. After some algebraic manipulations these match exactly the results in [82]. Unfortunately we were not able to analyze the classical limit for the non-compact $SL(2)$ sector for generic polarizations. We could compute it only for some fixed polarizations that resemble correlators in the $SU(2)$ sector. Nonetheless we were able to go slightly beyond these special cases and we started to see some complicated structures that may appear. Clearly not everything is settled in these matters and below we show some research prospects connected with this work which we think are relevant and could be fruitful for the future.

One direction to look is the Wilson loop (WL) and OPE duality. Before we discuss the connection with this work, let us briefly remark what it is this relation. The main idea of the duality is that the large spin limit of structure constants involving three spinning operators are given by null hexagonal WLs. This was first introduced in [128] and a precise dictionary for this duality was worked out in [129]. This is a remarkable connection purely from the point of view of QFT since it relates the CFT data of local insertions to a class of nonlocal operators. Now from the integrability side it is also significant since null hexagonal Wilson loops are described by the pentagon form factor program, as seen in [130, 131, 132, 133], and the structure constants are given by hexagonalization. To understand this pentagon-hexagon relation is a very exciting challenge in quantum integrability.

The classical limit of the $SL(2)$ sector done in this work partially captures this large spin analysis. A null hexagonal WL is described by three invariant cross ratios $\{U_1, U_2, U_3\}$. Using the map in [129], the polarizations of the cases analyzed in Section 2.4 are related to $(U_1, U_2, U_3) = (1, 1, 1)$, for $(t_{12}, t_{23}, t_{31}) = (1, 1, 1)$; and to $(U_1, U_2, U_3) = (0, 0, 0)$, for all other cases. Note that the instance with all null cross ratios is singular, however the $(t_{12}, t_{23}, t_{31}) = (1, 1, 1)$ example corresponds to the null hexagonal Wilson loop which is regular and can be embedded in AdS_3 [134]. There is the problem however that our large spin limit do not overlap with the one from [129]. Indeed, there the map was established for the OPE of the lightest single trace operators, therefore we are not in the limit of large R-charge that we explored in Section 2.4. To extend our results and cover the ones described WL/OPE duality we either work out this duality in the limit of large R-charge or modify the classical limit methods to account for large spin only. Note also that for these null WLs at tree level are purely given by kinematics and thus trivial, so we should also extend our methods to at least one-loop to obtain non-trivial results.

Another direction to explore is to find a way of considering the classical limit directly in the macroscopic regions, doing this we bypass all the microscopic sum over partitions and the process of taking their limits. This was done for the $SU(2)$ sector in [82]. The main idea is to map the classical limit problem to a Landau-Lifschitz model, then this can be solved by using sophisticated classical integrability tools at weak and strong coupling. One could then try to extend these methods to the full $SL(2)$ sector and not only for a restricted set of polarizations as it was done in [92, 93], there the polarizations are constrained to the same plane. With this we hope to bypass all the trouble of finding the saddle point equations and solving them in this non-compact sector in the classical limit as described in Section 2.4. We suspect that the extension of [82] to the $SL(2)$ sector to be very reminiscent of the computation of Wilson loops living in AdS_5 due to the WL/OPE duality. For WL this was done in [134] where we can write the answer in terms of the free energy of a thermodynamics Bethe ansatz and it was later generalized to a Y -system in [135].

We could also look to our new representations from another angle as coming from a partition function on Kagome lattices. One could try to derive for instance the expression for the $SL(2)$ sector (2.93) from the partition function itself, which is an interesting statistical mechanics problem. On the other hand we have no counterpart in the partition function picture to the hexagon expression (2.63) for the $SU(2)$ sector, so it would be good to also generalize this picture to this compact sector and further to all sectors of $\mathcal{N} = 4$ SYM.

As mentioned before it is important to extend our results beyond tree level for the WL/OPE duality. For instance, one could look at the hexagon partition function at finite coupling. However as detailed in [67] this partition function is not simply a rational function and it has non-trivial branch cuts. Besides that we even have extra poles at finite coupling in other sheets that are not taken into account in the recursion relations of Chapter 2. A possible direction to analyze our expressions at one-loop is to use the θ -morphism, which was a technique applied in the tailoring formalism to describe structure constants at higher loops [53, 83].

Part 2: Hexagonalization $AdS_3 \times S^3 \times T^4$ and mirror corrections

Now we move to hexagons in $AdS_3 \times S^3 \times T^4$. In this work we set out to complete the hexagonalization program, in the pure RR limit, introduced in [74]. In Chapter 3 we first introduced the necessary AdS_3/CFT_2 integrability framework for hexagonalization and then this formalism itself. We discussed also some

simple structure constants at tree level. Then we analyzed the mirror corrections in Chapter 4. We first introduced them in terms of two unknowns: the mirror measure and the integrand. We computed this measure and analyzed their coupling dependence. For the integrand we discussed how to write it in terms of the $\mathfrak{psu}(1|1)^{\oplus 2}$ transfer matrices. With the mirror corrections fully described we then proved that these cancel for the chiral ring structure constants that are protected. For this we simply used the fact that massless fermionic particles at zero momentum are singlets, therefore these decouple in the transfer matrices and their trace vanishes.

A direction that we could extend this work is to consider pure NSNS or mixed fluxes cases. Note that the extension of our computations for mixed flux is mostly direct, it amounts to redefine the Zhukovsky variables and slightly change crossing rules. But hexagonalization would not be well defined in this background since the hexagon scalar factors are written in terms of the dressing phases and these are unknown for mixed flux. Now the pure NSNS regime is more interesting to analyze. The integrability description is simpler in this case and as said before the thermodynamic Bethe ansatz can be worked out in exact form. Also the S-matrix in this case is similar to the flat space superstrings scattering [42]. Thus describing hexagonalization in this case would probably give hints about how to defining hexagons for flat space superstrings. But in spite of this the hexagons can not be defined as we did for the pure NSNS case. The main reason is that the central charge of the symmetry algebra of the hexagon vanishes and then we can not constrain the two-particle hexagon as we did in Section 3.3. Nonetheless this is an interesting and promising problem to pursue.

We also attempted to describe the hexagonalization of n -point functions in Chapter 5. We started by defining the hexagonalization prescription with the assignment of the skeleton graphs and then we derive the weights $\mathcal{W}_\psi$ as a function of the charges in the $\mathfrak{psu}(1|1)^{\oplus 2}$. To exemplify the method we then computed the tree level and $\mathcal{O}(\hbar)$ terms of the for the simplest four-point function of length $K = 2$ half-BPS states. In the course of the computation we faced three problems: ambiguity of the $\mathcal{Z}$ -marker prescription, the analytic structure, and the role of the massless modes. Due to the lack of perturbative data we could not solve these issues.

We speculate a common origin for these problems. Note that for $\mathcal{N} = 4$ SYM the prescription of assigning skeleton graphs as prefactors in the hexagonalization is well justified if we remember that the correlators at tree level are simply given

by Wick contractions. This may not be true here, indeed this is not the fact for the $\text{Sym}_N(T^4)$ which is the dual CFT_2 for $h = 0$ and $\kappa = 1$. In this theory we have basically N copies of a free lagrangian with the S_N identification. We can interpret the action of the permutation group as the existence of a twist field that generates a branch cut in spacetime. When we cross this cut we go to another copy of the theory. Therefore the correlators are not defined in the Riemann sphere, but in branched covering of it¹. One can see in [115, 12] that each correlator is then given in terms of a *finite* number of diagrams with each one of these corresponding to a certain covering of the sphere. We then see that these are similar to the skeleton graph procedure, but they do not correspond to a certain Wick contraction. We can see this distinction clearly in the large- N counting. Indeed each Feynman diagram in $\mathcal{N} = 4$ SYM comes with a N factor that counts the genus, however here each of the diagrams in [115, 12] come with a non-trivial function of N which only at large- N yields the genus counting plus subleading terms in $1/N$. Now how to change the prescription (5.12) to accommodate these facts is not clear. One could first understand how the pure RR CFT_2 is a deformation of $\text{Sym}_N(T^4)$ and then check if the correlators are correct by extracting the OPE data as it was done for $\mathcal{N} = 4$ SYM in [71]. Then one compares the result with the data from ABA and hexagonalization. If everything matches we then can move to the hexagonalization of n point functions.

Summary

The hexagonalization formalism was an important achievement in AdS/CFT since it allowed to access the structure constants and even higher point functions of the dual CFT at finite coupling. Nonetheless it is full of technical details that limit its applicability. Even if we consider the limit of large operators and bridges lengths, where the hexagon acquires its simplest form, we are still in practice limited to the study of operators with few excitations. This is due to the sheer complexity of the tensor contractions necessary to compute the hexagon form factors. This is where our new hexagon representations may be of use. Despite all of these difficulties the hexagons provide valuable data that one could use to explore important aspects of holographic dualities, like the WL/OPE duality for example.

¹Indeed in [11] these covering surfaces were identified with the localization points of the integral over the moduli space in the string side of the duality. This is roughly how AdS_3/CFT_2 was proved for $\kappa = 1$.

Also note that the hexagonalization formalism is an *ad hoc* solution to the problem of computing structure constants. It was not derived from first principles and it must be always tested against data and improved. This becomes a central problem in the $AdS_3 \times S^3 \times T^4$ case, since here we lack perturbative data for the dual CFT_2 at generic fluxes. One should better understand it in order to amend the problems we detailed and make progress in these matters.

Integrability is very powerful, it allowed us to compute quantities in QFT at limits we can not usually probe using other methods. In the context of AdS/CFT it became relevant to analyze very non-trivial aspects of CFTs at dimensions higher than two. However it is not *all-powerful*, its applicability remains limited to special QFTs and specific sectors within these theories. Nonetheless we hope to further explore the landscape of AdS/CFT using these tools to study quantum gravity and also QFTs, and as a byproduct discover new integrability tools and mathematical structures to apply in other areas of physics.

Appendix A

Gaudin norm of Bethe states

The norm of a Bethe state can be computed directly from the algebraic Bethe ansatz [136]. It is also called the Gaudin norm. Given a set of roots $\mathbf{u}$ that describes a certain Bethe state, the Gaudin norm is given simply by

$$\mathcal{N}(\mathbf{u}) = \sqrt{\prod_{x \in \mathbf{u}} (x - i/2)^\ell \prod_{y \in \mathbf{u}} (y + i/2)^\ell} \mathcal{B}(\mathbf{u}), \quad (\text{A.1})$$

where ℓ is the length of the spin chain and

$$\mathcal{B}(\mathbf{u}) = \sqrt{\prod_{n \neq m} \left(\frac{u_n - u_m + i}{u_n - u_m} \right) \det \mathcal{M}}, \quad (\text{A.2})$$

$$\mathcal{M}_{nm} = \frac{2}{(u_n - u_m)^2 + 1} + \delta_{nm} \left(\frac{\ell}{u_n^2 + 1/4} - \sum_{i=1}^M \frac{2}{(u_n - u_i)^2 + 1} \right). \quad (\text{A.3})$$

The polynomial factors in the norm can be absorbed in the definition of $\mathcal{D}(\alpha_j, \bar{\alpha}_j)$ as we do in Appendix B. It remains to compute the classical limit of the determinant factor $\mathcal{B}(\mathbf{u})$. It was already found in [52, 94] and we state it here for reference

$$\log \mathcal{B}(\mathbf{u}) = \frac{1}{2} \oint_{\mathcal{C}_{\mathbf{u}}} \frac{dz}{2\pi} \text{Li}_2 \left(e^{2ip} \right). \quad (\text{A.4})$$

This explain the presence of the single contour terms in equations (2.57) and (2.59).

Appendix B

Bethe equations and rewriting $\mathcal{D}(\alpha_j, \bar{\alpha}_j)$

In this appendix we rewrite $\mathcal{D}(\alpha_j, \bar{\alpha}_j)$ using Bethe equations such that the final structure constant will then be independent of the polarizations z_{ij} for off-shell roots. First we define the Baxter polynomials as usual and a different normalization for the pDWPF:

$$Q_\ell^\pm(\mathbf{x}) = \prod_{u \in \mathbf{x}} (u \pm i/2)^\ell, \quad (\text{B.1})$$

$$Z_p(\alpha_i \cup \bar{\alpha}_j | \ell_{ij}) = Q_{\ell_{ij}}^+(\bar{\alpha}_j) Q_{\ell_{ij}}^-(\alpha_i) \zeta(\alpha_i \cup \bar{\alpha}_j | \ell_{ij}). \quad (\text{B.2})$$

This rewriting of the scalar products suits our purposes since it is $\zeta(\alpha_i \cup \bar{\alpha}_j | \ell_{ij})$ that has a good classical limit and not the former definition [55]. Therefore we can write $\mathcal{D}(\alpha_j, \bar{\alpha}_j)$ as

$$\begin{aligned} \mathcal{D}(\alpha_j, \bar{\alpha}_j) = (-1)^{|\alpha_1|+|\alpha_2|+|\alpha_3|} \prod_{i=1}^3 f(\alpha_i, \bar{\alpha}_i) \prod_{i=1}^3 Q_{\ell_i}^-(\alpha_i) Q_{\ell_i}^+(\bar{\alpha}_i) \zeta(\alpha_1 \cup \bar{\alpha}_3 | \ell_{13}) \times \\ \zeta(\alpha_2 \cup \bar{\alpha}_1 | \ell_{12}) \zeta(\alpha_3 \cup \bar{\alpha}_2 | \ell_{23}). \end{aligned} \quad (\text{B.3})$$

Now one defines the useful quantity

$$\mathcal{A}(\mathbf{u} | \ell) = \frac{\det \left(u_b^{a-1} - \left(\frac{u_b+i/2}{u_b-i/2} \right)^\ell (u_b-i)^{a-1} \right)_{ab}}{\prod_{i < j} (u_i - u_j)}. \quad (\text{B.4})$$

This is exactly the pDWPF minus the the polynomial factor multiplying it in (2.19). Turns out that it can be also defined in terms of a sum over partitions, which will be useful later [94, 50]. This is also called the $\mathcal{A}$ -functional (see Section 2.4.2). Then

we can write the summand as

$$\mathcal{D}(\alpha_j, \bar{\alpha}_j) = (-1)^{|\alpha_1|+|\alpha_2|+|\alpha_3|} \prod_{i=1}^3 f(\alpha_i, \bar{\alpha}_i) \frac{\prod_{i=1}^3 Q_{\ell_i}^-(\alpha_i) Q_{\ell_i}^+(\bar{\alpha}_i)}{a_{\ell_{13}}(\bar{\alpha}_3) a_{\ell_{12}}(\bar{\alpha}_1) a_{\ell_{23}}(\bar{\alpha}_2)} \mathcal{A}(\alpha_1 \cup \bar{\alpha}_3 | \ell_{13}) \times \\ \mathcal{A}(\alpha_2 \cup \bar{\alpha}_1 | \ell_{12}) \mathcal{A}(\alpha_3 \cup \bar{\alpha}_2 | \ell_{23}). \quad (\text{B.5})$$

This rewriting is useful since it allows us to cancel the polynomial factors in the norm and write an expression with a well defined classical limit.

Now we will redefine $\mathcal{D}(\alpha_j, \bar{\alpha}_j)$ to contain factors from the norm. As said in Appendix A the determinant part of the norm has a well defined classical limit, then we define $\tilde{\mathcal{D}}(\alpha_j, \bar{\alpha}_j)$ as the previous summand but divided by the polynomial part of the norms. Therefore

$$\tilde{\mathcal{D}}(\alpha_j, \bar{\alpha}_j) = \frac{(-1)^{|\alpha_1|+|\alpha_2|+|\alpha_3|}}{a_{\ell_{13}}(\bar{\alpha}_3) a_{\ell_{12}}(\bar{\alpha}_1) a_{\ell_{23}}(\bar{\alpha}_2)} \prod_{i=1}^3 f(\alpha_i, \bar{\alpha}_i) \prod_{i=1}^3 \sqrt{\frac{a_{\ell_i}(\bar{\alpha}_i)}{a_{\ell_i}(\alpha_i)}} \mathcal{A}(\alpha_1 \cup \bar{\alpha}_3 | \ell_{13}) \times \\ \mathcal{A}(\alpha_2 \cup \bar{\alpha}_1 | \ell_{12}) \mathcal{A}(\alpha_3 \cup \bar{\alpha}_2 | \ell_{23}). \quad (\text{B.6})$$

Since we are dealing with physical operators, the total momentum of each one vanishes. This is related to the fact that the physical operators we are dealing are on a trace or from the point of view of the string theory side, the worldsheet momentum vanishes for physical states. This imposes the following constraint on $a_{\ell_j}(x)$:

$$a_{\ell_j}(\mathbf{u}_j) = a_{\ell_j}(\alpha_j) a_{\ell_j}(\bar{\alpha}_j) = 1, \quad \text{for } \alpha_j \cup \bar{\alpha}_j = \mathbf{u}_j. \quad (\text{B.7})$$

We can take this identity and the relation between the bridge lengths to redefine the variables in the sum and obtain the new summand

$$\tilde{\mathcal{D}}(\alpha_j, \bar{\alpha}_j) = (-1)^{|\alpha_1|+|\alpha_2|+|\alpha_3|} \prod_{i=1}^3 f(\alpha_i, \bar{\alpha}_i) a_{\ell_{23}}(\bar{\alpha}_3) a_{\ell_{13}}(\bar{\alpha}_1) a_{\ell_{12}}(\bar{\alpha}_2) \mathcal{A}(\alpha_1 \cup \bar{\alpha}_3 | \ell_{13}) \times \\ \mathcal{A}(\alpha_2 \cup \bar{\alpha}_1 | \ell_{12}) \mathcal{A}(\alpha_3 \cup \bar{\alpha}_2 | \ell_{23}). \quad (\text{B.8})$$

This is now ready to be manipulated and obtain equation (2.28).

The full structure constant $\mathcal{G}$ is z_{ij} independent if the roots satisfy the Bethe equations. Before we apply them to $\tilde{\mathcal{D}}(\alpha_j, \bar{\alpha}_j)$ we note that $\mathcal{A}(\mathbf{u}|\ell)$ can be written

as a sum over partitions [94, 50]

$$\mathcal{A}(\mathbf{u}|\ell) = \sum_{\alpha \cup \bar{\alpha} = \mathbf{u}} (-1)^{|\alpha|} a_\ell(\alpha) f(\bar{\alpha}, \alpha). \quad (\text{B.9})$$

So the full summand is just

$$\begin{aligned} \mathcal{D} = & (-1)^{|\bar{\alpha}_1|+|\bar{\alpha}_2|+|\bar{\alpha}_3|} \sum_{\substack{\beta_1 \cup \bar{\beta}_1 = \bar{\alpha}_1 \cup \alpha_2 \\ \beta_2 \cup \bar{\beta}_2 = \bar{\alpha}_2 \cup \alpha_3 \\ \beta_3 \cup \bar{\beta}_3 = \bar{\alpha}_3 \cup \alpha_1}} (-1)^{|\beta_1|+|\beta_2|+|\beta_3|} \frac{a_{\ell_{12}}(\alpha_2) a_{\ell_{23}}(\alpha_3) a_{\ell_{31}}(\alpha_1)}{a_{\ell_{12}}(\beta_1) a_{\ell_{23}}(\beta_2) a_{\ell_{31}}(\beta_3)} \\ & \times f(\bar{\alpha}_1, \alpha_1) f(\bar{\alpha}_2, \alpha_2) f(\bar{\alpha}_3, \alpha_3) f(\beta_1, \bar{\beta}_1) f(\beta_2, \bar{\beta}_2) f(\beta_3, \bar{\beta}_3). \quad (\text{B.10}) \end{aligned}$$

Note that:

$$\frac{a_{\ell_{12}}(\alpha_2) a_{\ell_{23}}(\alpha_3) a_{\ell_{31}}(\alpha_1)}{a_{\ell_{12}}(\beta_1) a_{\ell_{23}}(\beta_2) a_{\ell_{31}}(\beta_3)} = \frac{a_{\ell_{12}}(\bar{\beta}_1 \cap \alpha_2) a_{\ell_{23}}(\bar{\beta}_2 \cap \alpha_3) a_{\ell_{31}}(\bar{\beta}_3 \cap \alpha_1)}{a_{\ell_{12}}(\beta_1 \cap \bar{\alpha}_1) a_{\ell_{23}}(\beta_2 \cap \bar{\alpha}_2) a_{\ell_{31}}(\beta_3 \cap \bar{\alpha}_3)}. \quad (\text{B.11})$$

Since $\beta_j \cap \bar{\alpha}_j \subset \mathbf{u}_j$ we can use the Bethe equations to invert the $a_\ell(x)$ factors in the denominator. In our notation the Bethe equations are simply

$$a_{\ell_j}(x) = S(\{x\}, \mathbf{u}_j). \quad (\text{B.12})$$

In the end we obtain the final form of the summand in equation (2.28)

$$\begin{aligned} \tilde{\mathcal{D}}(\alpha_j, \bar{\alpha}_j) = & (-1)^{|\bar{\alpha}_1|+|\bar{\alpha}_2|+|\bar{\alpha}_3|} \sum_{\substack{\beta_1 \cup \bar{\beta}_1 = \bar{\alpha}_1 \cup \alpha_2 \\ \beta_2 \cup \bar{\beta}_2 = \bar{\alpha}_2 \cup \alpha_3 \\ \beta_3 \cup \bar{\beta}_3 = \bar{\alpha}_3 \cup \alpha_1}} (-1)^{|\beta_1|+|\beta_2|+|\beta_3|} a_{\ell_{12}}(\bar{\beta}_1 \cap \alpha_2) \\ & a_{\ell_{23}}(\bar{\beta}_2 \cap \alpha_3) a_{\ell_{31}}(\bar{\beta}_3 \cap \alpha_1) a_{\ell_{31}}(\beta_1 \cap \bar{\alpha}_1) a_{\ell_{12}}(\beta_2 \cap \bar{\alpha}_2) a_{\ell_{23}}(\beta_3 \cap \bar{\alpha}_3) \\ & S(\mathbf{u}_1, \beta_1 \cap \bar{\alpha}_1) S(\mathbf{u}_2, \beta_2 \cap \bar{\alpha}_2) S(\mathbf{u}_3, \beta_3 \cap \bar{\alpha}_3) \\ & f(\bar{\alpha}_1, \alpha_1) f(\bar{\alpha}_2, \alpha_2) f(\bar{\alpha}_3, \alpha_3) f(\beta_1, \bar{\beta}_1) f(\beta_2, \bar{\beta}_2) f(\beta_3, \bar{\beta}_3). \quad (\text{B.13}) \end{aligned}$$

Similar manipulations were made in [57] to check that $\mathcal{G}$ is independent of the polarizations and relate the hexagon formalism to the double spin chain formulation. However there the authors only verified it for one excitation on each operator. Here we work for a general number of them and establish the above summand.

Appendix C

Hexagons from double spin chain

Here we will rewrite the sum over partitions (2.27) to recover the hexagonalization formula. Remember that in Section 2.4.1 we defined the partitions $\{\gamma_j, \bar{\gamma}_j\}$ such that

$$\bar{\beta}_i \cap \alpha_j = \bar{\gamma}_j, \quad \beta_i \cap \bar{\alpha}_i = \gamma_i, \quad \text{and} \quad \gamma_i \cup \bar{\gamma}_i = \delta_i.$$

Where $\delta_i \cup \bar{\delta}_i = \mathbf{u}_i$. This is defined as such that all the the containing a_ℓ are joined to obtain

$$a_{\ell_{31}}(\delta_1)a_{\ell_{12}}(\delta_2)a_{\ell_{23}}(\delta_3).$$

Given these constraints we can solve them for $\{\beta_j, \bar{\beta}_j\}$ and find

$$\beta_i = \alpha_j / \bar{\gamma}_j \cup \gamma_i \text{ and } \bar{\beta}_i = \bar{\alpha}_i / \gamma_i \cup \bar{\gamma}_j. \quad (\text{C.1})$$

We then substitute this into (2.28), which results in the following complicated expression

$$\begin{aligned} \tilde{\mathcal{D}}(\alpha_i, \bar{\alpha}_i) &= (-1)^{|\bar{\delta}_1|+|\bar{\delta}_2|+|\bar{\delta}_3|} \times a_{\ell_{31}}(\delta_1)a_{\ell_{12}}(\delta_2)a_{\ell_{23}}(\delta_3) \\ &\times \frac{f(\bar{\gamma}_1, \gamma_1)}{f(\gamma_1, \bar{\gamma}_1)} \frac{f(\bar{\gamma}_2, \gamma_2)}{f(\gamma_2, \bar{\gamma}_2)} \frac{f(\bar{\gamma}_3, \gamma_3)}{f(\gamma_3, \bar{\gamma}_3)} \frac{f(\bar{\delta}_1, \gamma_1)}{f(\gamma_1, \bar{\delta}_1)} \frac{f(\bar{\delta}_2, \gamma_2)}{f(\gamma_2, \bar{\delta}_2)} \frac{f(\bar{\delta}_3, \gamma_3)}{f(\gamma_3, \bar{\delta}_3)} \\ &\times f(\bar{\alpha}_1 / \gamma_1, \alpha_1) f(\gamma_1, \alpha_1) f(\bar{\alpha}_2 / \gamma_2, \alpha_2) f(\gamma_2, \alpha_2) f(\bar{\alpha}_3 / \gamma_3, \alpha_3) f(\gamma_3, \alpha_3) \\ &\times f(\alpha_2 / \bar{\gamma}_2, \bar{\alpha}_1 / \gamma_1) f(\gamma_1, \bar{\alpha}_1 / \gamma_1) f(\alpha_2 / \bar{\gamma}_2, \bar{\gamma}_2) f(\gamma_1, \bar{\gamma}_2) \\ &\times f(\alpha_3 / \bar{\gamma}_3, \bar{\alpha}_2 / \gamma_2) f(\gamma_2, \bar{\alpha}_2 / \gamma_2) f(\alpha_3 / \bar{\gamma}_3, \bar{\gamma}_3) f(\gamma_2, \bar{\gamma}_3) \\ &\times f(\alpha_1 / \bar{\gamma}_1, \bar{\alpha}_3 / \gamma_3) f(\gamma_3, \bar{\alpha}_3 / \gamma_3) f(\alpha_1 / \bar{\gamma}_1, \bar{\gamma}_1) f(\gamma_3, \bar{\gamma}_1). \quad (\text{C.2}) \end{aligned}$$

If we define the partitions $\{\omega_j, \bar{\omega}_j\}$ by $\bar{\delta}_i = \omega_i \cup \bar{\omega}_i$ such that $\alpha_i = \bar{\omega}_i \cup \bar{\gamma}_i$ and $\bar{\alpha}_i = \omega_i \cup \gamma_i$, we can see that many ratios of $f(u, v)$ simplify and we find

$$\begin{aligned} \tilde{\mathcal{D}}(\alpha_i, \bar{\alpha}_i) &= (-1)^{|\bar{\delta}_1|+|\bar{\delta}_2|+|\bar{\delta}_3|} \times a_{\ell_{31}}(\delta_1) a_{\ell_{12}}(\delta_2) a_{\ell_{23}}(\delta_3) \times f(\bar{\delta}_1, \delta_1) f(\bar{\delta}_2, \delta_2) f(\bar{\delta}_3, \delta_3) \\ &\quad \times f(\omega_1, \bar{\omega}_1) f(\bar{\omega}_1, \omega_3) f(\omega_3, \bar{\omega}_3) f(\bar{\omega}_3, \omega_2) f(\omega_2, \bar{\omega}_2) f(\bar{\omega}_2, \omega_1) \\ &\quad \times f(\bar{\gamma}_1, \gamma_1) f(\gamma_1, \bar{\gamma}_2) f(\bar{\gamma}_2, \gamma_2) f(\gamma_2, \bar{\gamma}_3) f(\bar{\gamma}_3, \gamma_3) f(\gamma_3, \bar{\gamma}_1). \end{aligned} \quad (\text{C.3})$$

Here we are starting to see the hexagons, one in **red** and another in **blue** and also the self interacting terms $f(\bar{\delta}_j, \delta_j)$. We need now only to split the polarization factors and absorb it on each factor. This can be done if we note that

$$z_{12}^{M_{12}-|\alpha_2|-|\bar{\alpha}_1|} = z_{12}^{|\bar{\delta}_1|+|\bar{\delta}_2|-|\bar{\delta}_3|-|\bar{\omega}_2|-|\omega_1|} z_{12}^{|\delta_1|+|\delta_2|-|\delta_3|-|\bar{\gamma}_2|-|\gamma_1|}, \quad (\text{C.4})$$

$$z_{23}^{M_{23}-|\alpha_3|-|\bar{\alpha}_2|} = z_{23}^{|\bar{\delta}_2|+|\bar{\delta}_3|-|\bar{\delta}_1|-|\bar{\omega}_3|-|\omega_2|} z_{12}^{|\delta_2|+|\delta_3|-|\delta_1|-|\bar{\gamma}_3|-|\gamma_2|}, \quad (\text{C.5})$$

$$z_{31}^{M_{31}-|\alpha_1|-|\bar{\alpha}_3|} = z_{31}^{|\bar{\delta}_3|+|\bar{\delta}_1|-|\bar{\delta}_2|-|\bar{\omega}_1|-|\omega_3|} z_{31}^{|\delta_3|+|\delta_1|-|\delta_2|-|\bar{\gamma}_1|-|\gamma_3|}. \quad (\text{C.6})$$

Noting that $\gamma_i \cup \bar{\gamma}_i = \delta_i$ and $\omega_i \cup \bar{\omega}_i = \bar{\delta}_i$ we then find directly

$$\begin{aligned} \mathcal{G}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) &= \sum_{\substack{\delta_1 \cup \bar{\delta}_1 = \mathbf{u}_1 \\ \delta_2 \cup \bar{\delta}_2 = \mathbf{u}_2 \\ \delta_3 \cup \bar{\delta}_3 = \mathbf{u}_3}} (-1)^{|\delta_1|+|\delta_2|+|\delta_3|} f(\bar{\delta}_1, \delta_1) f(\bar{\delta}_2, \delta_2) f(\bar{\delta}_3, \delta_3) \\ &\quad \times a_{\ell_{31}}(\delta_1) a_{\ell_{12}}(\delta_2) a_{\ell_{23}}(\delta_3) \mathcal{H}(\delta_1, \delta_3, \delta_2) \mathcal{H}(\bar{\delta}_1, \bar{\delta}_2, \bar{\delta}_3). \end{aligned} \quad (\text{C.7})$$

Where $\mathcal{H}$ is the tree level hexagon given in (2.63). Therefore we established the equivalence between double spin chain formalism and hexagonalization

Appendix D

Weak coupling expansions in AdS_3/CFT_2 hexagonalization

In this appendix we compile the weak coupling expressions of some useful quantities. For the measure described in Chapter 4 we get:

$$\mu_Q(u) = \frac{1}{2Q} - \frac{Q}{4(u^2 + Q^2/4)^2} h^2 + \frac{Q(3Q^2 - 32u^2)}{16(u^2 + Q^2/4)^4} h^4 + \mathcal{O}(h^6), \quad (D.1)$$

$$\mu_Q(u^\gamma) = -\frac{h}{2(u^2 + Q^2/4)} + \frac{(Q^2 - 8u^2)}{4(u^2 + Q^2/4)^3} h^3 + \mathcal{O}(h^5), \quad (D.2)$$

$$\mu_o(u) = \frac{1}{4 - u^2}. \quad (D.3)$$

Now for the energy and momentum in the physical region we have:

$$E_Q(u) = Q + \frac{2Q}{u^2 + Q^2/4} h^2 - \frac{Q(Q^2 - 12u^2)}{2(u^2 + Q^2/4)^3} h^4 + \mathcal{O}(h^6), \quad (D.4)$$

$$e^{ip_Q(u)} = \frac{u + iQ/2}{u - iQ/2} + \frac{2iQu}{(u^2 + Q^2/4)(u - iQ/2)^2} h^2 + \mathcal{O}(h^4), \quad (D.5)$$

$$E_o(u) = h\sqrt{4 - u^2}, \quad (D.6)$$

$$e^{ip_o(u)} = \frac{u^2 - 2 + iu\sqrt{4 - u^2}}{2}. \quad (D.7)$$

Now in the mirror kinematics we have:

$$e^{-\tilde{E}_Q(u)} = \frac{h^2}{u^2 + Q^2/4} - \frac{(Q^2 - 4u^2)}{2(u^2 + Q^2/4)^3} h^4 + \mathcal{O}(h^6), \quad (D.8)$$

$$\tilde{p}_Q(u) = 2u - \frac{4u}{u^2 + Q^2/4} h^2 + \frac{(3Q^2u - 4u^3)}{(u^2 + Q^2/4)^3} h^4 + \mathcal{O}(h^6). \quad (D.9)$$

For massless mirror particles we just do the analytic continuation in u (see Section 3.1) and obtain similar expressions to (D.6) and (D.7) but in the mirror region.

The final pieces we need are the weak coupling expansions of the function $g(u, v)$ and scalar factor $h^{\bullet\bullet}(u, v)$ for Section 3.4. These are

$$g(u, v) = \frac{v + i/2}{v - i/2} \sqrt{\frac{u + v - i/2}{u + v + i/2}} + \mathcal{O}(h), \quad (D.10)$$

$$h^{\bullet\bullet}(u, v) = \sqrt{\frac{u + v + i/2}{u + v - i/2}} \frac{u - v}{u - v + i} + \mathcal{O}(h). \quad (D.11)$$

We assumed here that the phases σ_{uv}^{**} are trivial at tree level, which is reasonable and compatible with [48].

Appendix E

Bound states and hexagon scalar factors

The hexagon scalar factor for states with non-trivial bound state number can be found by *fusion*. This is the process where we join fundamental particles with complex momenta and then build a bound state as it was explained in Section 3.1. Then the fusion formula for these scalar factors are simply given by:

$$h_{Q_1 Q_2}^{\bullet\bullet}(u, v) = \prod_{m=1}^{Q_1} \prod_{n=1}^{Q_2} h^{\bullet\bullet}(u_m^{(Q_1)}, v_n^{(Q_2)}). \quad (\text{E.1})$$

$$\tilde{h}_{Q\bar{Q}}^{\bullet\bullet}(u, v) = \prod_{m=1}^Q \prod_{n=1}^{\bar{Q}} \tilde{h}^{\bullet\bullet}(u_m^{(Q)}, v_n^{(\bar{Q})}). \quad (\text{E.2})$$

$$h_{Q\bar{Q}}^{RL}(u, v) = \prod_{m=1}^Q \prod_{n=1}^{\bar{Q}} \left(\sqrt{\frac{x^-(u_m^{(Q)})x^-(v_n^{(\bar{Q})})}{x^+(u_m^{(Q)})x^+(v_n^{(\bar{Q})})} \frac{1 - x^+(u_m^{(Q)})x^+(v_n^{(\bar{Q})})}{1 - x^-(u_m^{(Q)})x^-(v_n^{(\bar{Q})})}} \right) \tilde{h}_{Q\bar{Q}}^{\bullet\bullet}(u, v). \quad (\text{E.3})$$

$$h_{*,Q}^{\bullet\circ}(u, v) = \prod_{m=1}^Q h_{*}^{\bullet\circ}(u_m^{(Q)}, v). \quad (\text{E.4})$$

$$h_{*,Q}^{\circ\bullet}(u, v) = \prod_{n=1}^Q h_{*}^{\circ\bullet}(u, v_n^{(Q)}). \quad (\text{E.5})$$

Note that the $u_n^{(Q)}$ are defined in Section 3.1. Similar fusion formulas could be found for the dressing factors of the S-matrix. These are useful to compute higher orders in the h -expansion for structure constants or correlation functions. For $AdS_5 \times S^5$ we have a structure that resembles the one described here [61].

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