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Application of effective field theory and Grassmann variables to model compact binaries with spin

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Resumo

Começamos por revisar um modo de calcular a energia potencial entre dois corpos sem spin até segunda ordem nas velocidades, através da aplicação de uma Teoria de Campos Efetiva que modela a interação entre eles por meio do campo gravitacional. Estendendo este método para que leve em conta também os graus de liberdade de spin, e descrevendo tais graus de liberdade através de variáveis de Grassmann, calculamos termos adicionais no potencial advindos das interações spin-spin e spin-orbital. Finalmente, calculamos a energia como função da frequência de rotação do sistema binário, incluindo o efeito destes termos relacionados a spin. Tais resultados devem tornar-se cada vez mais relevantes para possíveis medições em observatórios de ondas gravitacionais tais como LIGO [1] e VIRGO [2].

Palavras-chave: Teoria de Campos Efetiva, Supersimetria, Variáveis de Grassmann, Sistemas Binários

Abstract

We review how an approximation of the potential between two spin-less bodies is calculated, up to second order in the velocities, by employing effective field theory methods to model their interaction though the gravitational field. By extending this methodology to take into account their spin degrees of freedom and by describing such spin degrees of freedom in terms of Grassmannian variables, we calculate the additional terms in the potential due to spin-spin and spin-orbital interactions. Finally, we calculate the energy as a function of rotational frequency of a binary system, including the effect of these spin related terms. Such results are becoming increasingly relevant in possible measurements by gravitational observatories like LIGO [1] or VIRGO [2].

Keywords: Effective Field Theory, Supersymmetry, Grassmann Variables, Binary Systems

Contents

Chapter 1

Introduction

Our goal in this work is to calculate classical quantities associated with a compact binary system, such as potential and total energy. We'll apply two methods that have been separately developed and put them together to tackle this problem: effective field theories and the application of Grassmann variables to model spin degrees of freedom.

We begin by briefly reviewing the effective field theory (EFT) method in chapter 2 and applying it to a toy model of two point particles interacting through a scalar field which will serve to illustrate the basic concept of generating an effective action by "integrating out" the field and revealing potential terms that depend only on the particles' trajectories. This will also serve as a prototype to the problem involving the real gravitational field.

Then, in chapter 3 we model a system of two point particles, still spinless, interacting through a real gravitational field. We employ EFT to integrate out the mediating gravitational field and give corrections to the Newtonian potential energy, which will be calculated here up to order 1PN. The obtained result is, effectively, the Einstein Infeld Hoffmann post-Newtonian potential as a function of particles' positions and velocities.

In chapter 4, section 4.1, we review a technique by which spin degrees of freedom of point particles can be modeled by anticommuting Grassmann variables in the Lagrangian. We show that they seem to describe spin by making rotation transformations and seeing them appear in a sum with orbital angular momentum that is preserved along rotations. From this result, in section 4.2, we construct a reparametrization invariant action and then employ EFT to a system of two spinning particles whose interaction is mediated by the gravitational field. With that in hand we are able to generate new pairings which will give rise to new potential energy terms in the effective action related to spin-spin and spin-orbit couplings between the two bodies.

Finally, in chapter 5, we use the previously computed results to calculate the energy associated with system of two spinning bodies with all corrections associated both with 1PN corrections and terms coming from the spin-spin and spin-orbit coupling. Then, in the particular case of circular orbits, we show that we can express this energy in the more useful format of a function of the rotational frequency without dependence on coordinates such as the distance between the two bodies.

Chapter 2

Summary of Effective Field Theory

2.1 Conceptual Basis

Consider the partition function:

$$Z[J] = \int \mathcal{D}\phi e^{(\frac{i}{\hbar})\int d^4x\mathcal{L}(\phi,\partial_\mu\phi,J)}$$

Suppose that the action $S = \int d^4x\mathcal{L}(\phi, \partial_\mu\phi, J)$ varies by large amounts of $\hbar$ over small displacements in ϕ. Then, the partition function will be dominated by the trajectories near the classical one, which minimizes the action. Further from these trajectories, the exponent oscillates very rapidly and contributions cancel out. So, in the classical limit:

$$\lim_{\hbar\to 0} Z[J] = \mathcal{N}e^{(\frac{i}{\hbar})S(\phi_{cl},J)}$$

Now, making $\hbar = 1$ and defining

$$iS_{eff}(J) \equiv \ln(Z[J]) \tag{2.1}$$

We can see that

$$iS_{eff}(J) = iS(\phi_{cl}, J) + \text{quantum corrections} + \text{const.}$$

In this way, we have achieved a result which, apart from quantum corrections, gives us the classical action in terms of the sources as the logarithm of the partition function. If the $\hbar$ dependent terms are very small, this effectively enables us to use all the techniques developed for path integrals, such as perturbation theory and Feynman diagrams, to extract terms from the action such as the kinetic energy, potential energy etc.

2.2 A toy model

In this section we'll apply the concepts just described above to the case of sources whose interaction is mediated by a scalar field with no mass and no self-interaction. That is,

$$S = S_{scalar} + S_{int} + S_{source}$$

Where

$$S_{scalar} = \int d^4x(-\frac{1}{2}\partial^\mu\phi\partial_\mu\phi)$$

And

$$S_{int} = \int d^4x\,\phi(x)J(x)$$

In this case, the partition function can be calculated exactly by Gaussian integration. We have:

$$Z[J] = e^{iSsource}\int \mathcal{D}\phi\, e^{i\int d^4x\{+\frac{1}{2}\phi\underbrace{(\partial_\mu\partial^\mu)}_{\equiv\Delta}\phi+\frac{i\epsilon}{2}\phi^2+\phi J\}}$$
$$= \mathcal{N}e^{iSsource}e^{-\frac{i}{2}J\cdot\Delta^{-1}\cdot J}$$
$$\text{where}\quad J\cdot\Delta^{-1}\cdot J \equiv \iint d^4x_1 d^4x_2 J(x_1)\Delta(x_1,x_2)J(x_2) \tag{2.2}$$

And $\mathcal{N}$ is a normalization constant. The term $+\frac{i\epsilon}{2}\phi^2$ has been added to make the integral converge and will result in the $i\epsilon$ term in the denominator below, which implies the time symmetric solution for Δ^{-1}:

$$\Delta^{-1}(x_1,x_2) = \int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot(x_1-x_2)}}{-k^2+i\epsilon} \tag{2.3}$$

$$\Rightarrow \Delta\Delta^{-1}(x_1,x_2) = \int \frac{d^4k}{2\pi}e^{-ik\cdot(x_1-x_2)} = \delta^4(x_1-x_2)$$

Where $k^2 \equiv k^\mu k_\mu$, therefore $-k^2 = (k^0)^2 - \mathbf{k}^2$. Now we want to model the presence of two point particles. Therefore we want:

$$J(\mathbf{x},t) \propto m_1\delta^3(\mathbf{x}-\mathbf{x_1}(t)) + m_2\delta^3(\mathbf{x}-\mathbf{x_2}(t))$$

From the scalar term we see that $[\phi] = L^{-1}$ is required to make the action term dimensionless. So we need that $[J] = L^{-3}$ to make the interaction term dimensionless as well. Since $\delta^3(\mathbf{x})$ is already of dimension L^{-3} we need to offset the dimension of the mass factor. Let's do that by dividing m by an arbitrary constant Λ of dimension L^{-1}. We choose:

$$\Lambda \equiv (32\pi G_N)^{-1/2} \tag{2.4}$$

Where G_N is Newton's gravitational constant. Note that Λ is arbitrary and that's OK as long as we aren't trying to create an experimentally proven theory; we're just playing with a toy model to demonstrate the methodology of effective field theory. So now we define:

$$J(\mathbf{x}, t) = \frac{m_1}{\Lambda}\delta^3(\mathbf{x} - \mathbf{x_1}(t)) + \frac{m_2}{\Lambda}\delta^3(\mathbf{x} - \mathbf{x_2}(t))$$

And the source term is:

$$S_{source} = -m_1 \int d\tau_1 - m_2 \int d\tau_2$$

From (2.1), (2.2) and (2.3) we conclude that:

$$\begin{aligned}
S_{eff} &= -i\, ln(Z[J]) \\
&= S_{source} - \\
&- i(-\frac{i}{2})\frac{m_1 m_2}{\Lambda^2} \iint d^4x\, d^4x' \delta^3(\mathbf{x} - \mathbf{x_1}(t))\delta^3(\mathbf{x'} - \mathbf{x_2}(t')) \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot(x-x')}}{-k^2 + i\epsilon} - \\
&- i(-\frac{i}{2})\frac{m_1^2}{\Lambda^2} \iint d^4x\, d^4x' \delta^3(\mathbf{x} - \mathbf{x_1}(t))\delta^3(\mathbf{x'} - \mathbf{x_1}(t')) \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot(x-x')}}{-k^2 + i\epsilon} + \\
&+ 1 \leftrightarrow 2
\end{aligned}$$

It can be seen that the terms proportional to m_1^2 (m_2^2) are constants with no dependence on the particles' trajectories, so they can be discarded because they won't influence the equations of motion. So, apart from these innocuous terms we have:

$$S_{eff} = S_{source} - \frac{m_1 m_2}{\Lambda^2} \int dt\, dt' \int \frac{dk^0}{2\pi} e^{+i\, k^0(t-t')} \int_{\mathbf{k}} \frac{e^{-i\,\mathbf{k}\cdot(\mathbf{x_1}(t) - \mathbf{x_2}(t'))}}{(k^0)^2 - \mathbf{k}^2} \tag{2.5}$$

Now we make use of the expansion:

$$\frac{1}{\mathbf{k}^2 - (k^0)^2} = \frac{1}{\mathbf{k}^2}\left(1 + \frac{(k^0)^2}{\mathbf{k}^2} + \frac{(k^0)^4}{\mathbf{k}^4} + \dots\right)$$

We shall see below that this expansion can be done as each successive term will generate higher orders in v^2. So, in zeroth order in k^0:

$$
\begin{aligned}
S_{eff} \supset & \int dt \frac{m_1 m_2}{\Lambda^2} \int_{\mathbf{k}} \frac{e^{-i\,\mathbf{k}\cdot(\mathbf{x_1}(t)-\mathbf{x_2}(t))}}{\mathbf{k}^2} \\
= & \int dt \frac{m_1 m_2}{\Lambda^2} \frac{1}{4\pi r} \\
= & \int dt \frac{8\, G_N m_1 m_2}{r}
\end{aligned}
$$

Where

$$
\int_{\mathbf{k}} \equiv \int \frac{d^3\mathbf{k}}{(2\pi)^3}
$$

And

$$
r \equiv |\mathbf{r}| \equiv |\mathbf{x_1} - \mathbf{x_2}|
$$

This gives the following potential term:

$$
V \supset -\frac{8\, G_N m_1 m_2}{r}
$$

Which, apart from a factor of 8, resembles the Newtonian gravitational potential. We don't need to worry about this extra factor because as said before we're *not* trying to deduce that gravity is a scalar field, which we know is not. We are performing this calculation simply to exercise the method of calculating potential terms starting from partition functions in a simpler scenario before we go to the more complicated (and real) gravitational case.

We can also calculate the higher order terms in $(k^0)^2$. For example, we can calculate the term of first order in $(k^0)^2$. To do that, first notice that

$$
\frac{1}{\mathbf{k}^2 - (k^0)^2} = \frac{1}{\mathbf{k}^2}\left(1 + \frac{(k^0)^2}{\mathbf{k}^2} + \frac{(k^0)^4}{\mathbf{k}^4} + \ldots\right)
$$

can be written inside the integral in (2.5) as

$$
\frac{1}{\mathbf{k}^2}\left(1 + \frac{\partial_t \partial_{t'}}{\mathbf{k}^2} + \frac{(\partial_t \partial_{t'})^2}{\mathbf{k}^4} + \ldots\right)
$$

Where the time derivatives are applied to $e^{+i\,k^0(t-t')}$. Then, to first order in $(k^0)^2/\mathbf{k}^2$ we'll have the term:

$$
S_{eff} \supset \frac{m_1 m_2}{\Lambda^2} \int dt\, dt' \int \frac{dk^0}{2\pi} (\partial_t \partial_{t'} e^{+i\,k^0(t-t')}) \int_{\mathbf{k}} \frac{e^{-i\,\mathbf{k}\cdot(\mathbf{x_1}(t)-\mathbf{x_2}(t'))}}{\mathbf{k}^4}
$$

After integrating by parts both with respect to t and t':

$$S_{eff} \supset \frac{m_1 m_2}{\Lambda^2} \int dt\, dt' \delta(t-t') \int_{\mathbf{k}} k_i v_{1i}(t) k_j v_{2j}(t') \frac{e^{-i\,\mathbf{k}\cdot(\mathbf{x_1}(t)-\mathbf{x_2}(t'))}}{\mathbf{k}^4}$$
$$= \frac{m_1 m_2}{\Lambda^2} \int dt\, v_{1i}(t) v_{2j}(t) (-\partial_{r_i}\partial_{r_j}) \int_{\mathbf{k}} \frac{e^{-i\,\mathbf{k}\cdot\mathbf{r}(t)}}{\mathbf{k}^4} \tag{2.6}$$

Here we can see, as stated previously, that each consecutive term in the expansion in $(k^0)^2/\mathbf{k}^2$ generates a higher order term in velocities.

We use that

$$\int_{\mathbf{k}} \frac{e^{-i\,\mathbf{k}\cdot\mathbf{r}}}{\mathbf{k}^{2\alpha}} = r^{2\alpha-3}\pi^{-3/2}2^{-2\alpha}\frac{\Gamma(3/2-\alpha)}{\Gamma(\alpha)}$$

Which, for $\alpha = 2$, gives:

$$\int_{\mathbf{k}} \frac{e^{-i\,\mathbf{k}\cdot\mathbf{r}}}{\mathbf{k}^4} = -\frac{r}{8\pi}$$

Also, since

$$\partial_{r_i}\partial_{r_j} r = \frac{\delta_{ij}}{r} - \frac{r_i r_j}{r^3}$$

Then (2.6) becomes:

$$S_{eff} \supset 32\pi G_N m_1 m_2 \int dt\, v_{1i} v_{2j} \frac{1}{8\pi}\left(\frac{\delta_{ij}}{r} - \frac{r_i r_j}{r^3}\right)$$

From which we get the following additional potential terms:

$$V \supset 4 G_N m_1 m_2 \left(-\frac{\mathbf{v_1}\cdot\mathbf{v_2}}{r} + \frac{r_i r_j}{r^3} v_i v_j \right)$$

Although seemingly this potential appears to imply action at a distance (since velocities and positions of the particles are measured at the same time), this actually comes from trading knowledge about the particles' positions at all times for knowing all higher derivatives in the Taylor expansion and, then, since we're assuming velocities much smaller than 1, we're allowing ourselves to treat the problem perturbatively and consider only the first leading orders.

Chapter 3

EFT for two gravitationally interacting spin-less particles

In the previous section we illustrated the usage of EFT to calculate the potential in a simple case involving a scalar field that mediates the interaction between two spin-less particles. Despite some resemblance to the Newtonian potential obtained in the end, we know that gravity is a rank 2 tensor field, not a scalar. In this chapter we will take this into account and calculate the potential terms given by all degrees of freedom of the gravitational field.

In order to tackle this more complicated problem we will break it into three parts:

First obtain the quadratic field terms of the action in terms of components of gravity taken from a useful ansatz (section 3.1).

Second, we will calculate the source terms in the action, which, contrary to the last chapter, now play a role in the potential because of the non-flat metric (section 3.2).

Third, calculate the potential terms up to 1PN order by contracting the fields in the composite (field + source) action (section 3.3)

3.1 Quadratic field terms in the action

Here we'll use the following variable redefinition for the metric as prescribed by [4] and is inspired by Kaluza-Klein, although it is used here for a *different* purpose than adding an extra dimension to unify gravity and electromagnetism, but to create a hierarchy *among* the gravity component fields in 4 dimensions:

$$g_{\mu\nu} = e^{2\phi/\Lambda} \begin{pmatrix} -1 & A_j/\Lambda \\ A_i/\Lambda & e^{-c_d\phi/\Lambda}\gamma_{ij} - A_iA_j/\Lambda^2 \end{pmatrix} \tag{3.1}$$

with $\gamma_{ij} \equiv \delta_{ij} + \sigma_{ij}/\Lambda$ and $c_d \equiv 2\frac{d-1}{d-2}$. So far we are simply rewriting $g_{\mu\nu}$ in terms of new components ϕ, A_j and σ_{ij}. The total action is given by:

$$S = S_{EH} + S_{GF} + S_{pp} = S_{quad} + S_{pp}$$

Where S_{EH} refers to the standard Einstein Hilbert action, expanded to quadratic order in the fields, S_{GF} is the harmonic gauge fixing term which we can find, for example, in [6] and S_{pp} refers to the two point particles' world-lines which will be detailed in the next section. In this section we'll calculate the sum $S_{quad} = S_{EH} + S_{GF}$ up to quadratic order where

$$S_{EH} = \frac{1}{16\pi G_N}\int dt\, d^d x\sqrt{-g}R \tag{3.2}$$

$$S_{GF} = \frac{-1}{32\pi G_N}\int dt\, d^d x g_{\mu\nu}\Gamma^\mu\Gamma^\nu \tag{3.3}$$

With $\Gamma_\mu \equiv \Gamma^\mu_{\nu\rho} g^{\nu\rho}$. The calculations were performed with the help of the Mathematica notebook as explained in Appendix A. We got:

$$\begin{aligned} S_{quad} = \int dt\, d^d x \Bigg\{ 4\Big[-(\vec{\nabla}\phi)^2 + \dot{\phi}^2\Big] + \Big[\frac{F_{ij}^2}{2} + (\vec{\nabla}\cdot\vec{A})^2 - \dot{\vec{A}}^2\Big] \\ + \Big[-\frac{1}{4}\dot{\sigma}^2 + \frac{1}{4}(\vec{\nabla}\sigma)^2 + \frac{1}{2}\dot{\sigma}_{ij}^2 - \frac{1}{2}(\vec{\nabla}\sigma_{ij})^2\Big]\Bigg\} \end{aligned} \tag{3.4}$$

Now we calculate the Green's functions associated with each field in the above action. These Green's functions arise when fields in two different space-time coordinates are "paired up" such as, for example, in the following expression:

$$\begin{aligned} G_F\big(\mathbf{x_1}(t_1); t_1, \mathbf{x_2}(t_2); t_2\big) \equiv \frac{1}{Z[\phi, J=0]}\int \mathcal{D}\phi\, e^{iS[\phi]}\phi\big(\mathbf{x_1}(t_1), t_1\big)\phi\big(\mathbf{x_2}(t_2), t_2\big) \\ = \frac{1}{Z[\phi, J=0]}\left\{\frac{\delta}{i\delta J\big(\mathbf{x_1}(t_1), t_1\big)}\frac{\delta}{i\delta J\big(\mathbf{x_2}(t_2), t_2\big)}\bigg|_{J=0} Z[\phi, J]\right\} \end{aligned}$$

Where

$$Z[\phi, J] \equiv \int \mathcal{D}\phi\, e^{i\left(S[\phi] + \int dt\, d^d x\, \phi(\mathbf{x},t)J(\mathbf{x},t)\right)}$$

In the last chapter, we have calculated the partition function for a scalar field. Given:

$$S_{scalar} = \int d^4 x(-\frac{1}{2}\partial^\mu\phi\partial_\mu\phi)$$

the partition function can be calculated exactly giving:

$$Z[J] = \int \mathcal{D}\phi \, e^{i\int d^4x\{+\frac{1}{2}\phi \underbrace{\left(\partial_\mu \partial^\mu\right)}_{\equiv \Delta} \phi + \phi J\}}$$

$$= \mathcal{N} e^{-\frac{i}{2} J\cdot\Delta^{-1}\cdot J}$$

$$\text{where} \quad J\cdot\Delta^{-1}\cdot J \equiv \iint d^4x_1 d^4x_2 J(x_1)\Delta(x_1,x_2)J(x_2) \tag{3.5}$$

And

$$\Delta^{-1}(x_1,x_2) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot(x_1-x_2)}}{-k^2+i\epsilon}$$

In this case the Green's function will be:

$$\begin{aligned}
&G_F\big(\mathbf{x_1}(t_1);t_1,\mathbf{x_2}(t_2);t_2\big) \\
=&\frac{1}{Z[\phi,J=0]}\left\{\frac{\delta}{i\delta J\big(\mathbf{x_1}(t_1),t_1\big)}\frac{\delta}{i\delta J\big(\mathbf{x_2}(t_2),t_2\big)}\bigg|_{J=0} Z[\phi,J]\right\} \\
=&\frac{1}{\mathcal{N}}\frac{1}{i^2}\mathcal{N}\Big(-i\Delta^{-1}\big(\mathbf{x_1}(t_1);t_1,\mathbf{x_2}(t_2);t_2\big)\Big) \\
=&-i\int\frac{dk^0}{2\pi}\int_{\mathbf{k}}\frac{e^{-ik^0(t_2-t_1)+i\mathbf{k}\cdot\big(\mathbf{x_2}(t_2)-\mathbf{x_1}(t_1)\big)}}{\mathbf{k}^2-(k^0)^2-i\epsilon}
\end{aligned} \tag{3.6}$$

Observe that G_F as defined obeys $G_F = i\Delta^{-1}$ so that it obeys $G_F\Delta(x_1,x_2) = i\delta^4(x_1-x_2)$ as expected for Green's functions in Field Theory.

Note, also, that we're simply reusing the results obtained in the last chapter in (2.2) and (2.3) but this time S_{source} is not factored out. It will be expanded in powers of the fields and, when summed to the action, will cause the appearance of 2-point and 4-point functions etc. modifying the potential as will become clear in section 3.3.

So we follow ahead with the plan and calculate the Green's functions associated with each field appearing in the quadratic action (3.4).

3.1.1 ϕ Green's function

The only difference between the quadratic term in ϕ and the scalar field example of last section is the factor of 4 instead of 1/2. Since the Green's function is the inverse of the operator, it should be one eighth of the one obtained in (3.6). Then

$$G_F^{(\phi)}(\mathbf{x},t) = -\frac{i}{8}\int\frac{dk^0}{2\pi}\int_{\mathbf{k}}\frac{e^{-ik^0t+i\mathbf{k}\cdot\mathbf{x}}}{\mathbf{k}^2-(k^0)^2-i\epsilon} \tag{3.7}$$

3.1.2 $\vec{A}$ Green's function

Take a look at the $\vec{A}$ term in the integrand in the quadratic field action (3.4). It is:

$$\left[\frac{F_{ij}^2}{2} + (\vec{\nabla}\cdot\vec{A})^2 - \dot{\vec{A}}^2\right]$$

First note that:

$$\begin{aligned}\frac{F_{ij}^2}{2} =&\frac{1}{2}(\partial_i A_j - \partial_j A_i)(\partial_i A_j - \partial_j A_i)\\ =&\partial_i A_j \partial_i A_j - \partial_j A_i \partial_j A_i\\ =& - A_i(\delta_{ij}\nabla^2 - \partial_i\partial_j)A_j\end{aligned}$$

Where we integrated by parts in the last step. We also have:

$$(\vec{\nabla}\cdot\vec{A})^2 = \partial_i A_i \partial_j A_j = -A_i\partial_i\partial_j A_j$$

And finally:

$$-\dot{\vec{A}}^2 = +\vec{A}\cdot\frac{\partial^2}{\partial t^2}\vec{A} = A_i\frac{\partial^2}{\partial t^2}\delta_{ij}A_j$$

Putting it all together we find that:

$$\left[\frac{F_{ij}^2}{2} + (\vec{\nabla}\cdot\vec{A})^2 - \dot{\vec{A}}^2\right] = \frac{1}{2}(-2)A_i\partial^\mu\partial_\mu\delta_{ij}A_j$$

Here the operator $\frac{1}{2}\partial^\mu\partial_\mu$ is multiplied by a factor of -2 whose inverse is $-1/2$ and there's also an outer product with δ_{ij} whose inverse is δ_{ij} itself. So we conclude that:

$$G_F^{(A)}(\mathbf{x},t)_{ij} = \frac{i}{2}\delta ij\int\frac{dk^0}{2\pi}\int_{\mathbf{k}}\frac{e^{-ik^0t+i\mathbf{k}\cdot\mathbf{x}}}{\mathbf{k}^2-(k^0)^2-i\epsilon} \tag{3.8}$$

3.1.3 σ Green's function

After integrating by parts, the quadratic terms in σ present in the action (3.4) can be written in the following form:

$$\begin{aligned}&\left[\sigma_{ii}\left(-\frac{1}{4}\partial^\mu\partial_\mu\right)\sigma_{kk} + \sigma_{ij}\left(\frac{1}{2}\partial^\mu\partial_\mu\right)\sigma_{ij}\right]\\ =&\sigma_{ij}\left(-\frac{1}{2}\delta_{ij}\delta_{kl}\right)\left(\frac{1}{2}\partial^\mu\partial_\mu\right)\sigma_{kl} + \sigma_{ij}\left(\frac{1}{2}\delta_{ik}\delta jl + \frac{1}{2}\delta_{il}\delta_{jk}\right)\left(\frac{1}{2}\partial^\mu\partial_\mu\right)\sigma_{kl}\\ =&\sigma_{ij}M_{ijkl}\left(\frac{1}{2}\partial^\mu\partial_\mu\right)\sigma_{jk}\end{aligned}$$

With

$$M_{ijkl} \equiv \frac{1}{2}\delta_{ik}\delta jl + \frac{1}{2}\delta_{il}\delta_{jk} - \frac{1}{2}\delta_{ij}\delta_{kl}$$

To find the inverse operator, we have to multiply the already found inverse of $(1/2\,\partial^\mu\partial_\mu)$ by the inverse of M.

Such inverse should by a matrix M^{-1}_{ijkl} such that

$$M^{-1}_{ijkl}M_{klmn} = \frac{1}{2}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jn})$$

The right hand side is the identity matrix with double and symmetric indices ij and mn. It can be checked that the solution is given by:

$$M^{-1}_{ijkl} = \frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{d-2}\delta_{ij}\delta_{kl}\right)$$

Where d is the number of spatial dimensions. Therefore the Green's function for $d = 3$ is:

$$G_F^{(\sigma)}(\mathbf{x},t)_{ijkl} = \frac{-i}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - 2\delta_{ij}\delta_{kl})\int\frac{dk^0}{2\pi}\int_{\mathbf{k}}\frac{e^{-ik^0t+i\mathbf{k}\cdot\mathbf{x}}}{\mathbf{k}^2-(k^0)^2-i\epsilon} \quad (3.9)$$

3.2 Source term

We will now write the source term for each particle which will be summed with the quadratic action term calculated in last section.

The expression for a particle's action is

$$S_{pp} = -m\int d\tau = -m\int\sqrt{-g_{\mu\nu}dx^\mu dx^\nu} = -m\int dt\sqrt{-g_{00} - 2g_{0i}v^i - g_{ij}v^iv^j}$$

Using the definition of the metric components given in (3.1) this action becomes:

$$S_{pp} = -m\int dt\, e^{\phi/\Lambda}\sqrt{1 - 2\frac{A_i}{\Lambda}v^i - e^{-c_d\phi/\Lambda}(v^2 + \frac{\sigma_{ij}}{\Lambda}v^iv^j - \frac{A_iA_j}{\Lambda^2}v^iv^j)}$$

What we're going to do next is expand the integrand in powers of the fields ϕ, A and σ. In this way, as will become apparent in the next section, expressions of type $\int\mathcal{D}\phi\, e^{iS_{quad}}e^{iS_{pp}}$ can be expanded into ones of type $\int\mathcal{D}\phi\, e^{iS_{quad}}\phi(x_1)\phi(x_2)$ which in turn are equal to the Green's function calculated in the section above.

Up to second order we have:

$$e^x \approx 1 + x + \frac{x^2}{2}$$
$$\sqrt{1+x} \approx 1 + \frac{x}{2} - \frac{x^2}{8}$$

We then get, up to order v^2:

$$\begin{aligned}
S_{pp} \approx & -m \int dt \left(1 + \frac{\phi}{\Lambda} + \frac{\phi^2}{2\Lambda^2}\right) \times \\
& \times \left(1 - \frac{A_i}{\Lambda} v^i + \frac{3A_iA_j}{4\Lambda^2} v^i v^j - \frac{v^2}{2} - \frac{\sigma_{ij}}{2\Lambda} v^i v^j + \frac{c_d \phi}{2\Lambda} v^2 + \frac{c_d \phi\, \sigma_{ij}}{2\Lambda^2} v^i v^j - \frac{c_d^2 \phi^2}{4\Lambda^2} v^2\right) + \dots \\
& = -m \int dt \left[\sqrt{1-v^2} + \frac{\phi}{\Lambda}\left(1 + \frac{3}{2}v^2\right) + \frac{\phi^2}{2\Lambda^2}\left(1 + \mathcal{O}(v^2)\right) - \frac{A_i}{\Lambda} v^i \left(1 - \frac{3A_j}{4\Lambda} v^j\right) \right. \\
& \left. - \frac{\sigma_{ij}}{2\Lambda} v^i v^j \left(1 - \frac{c_d \phi}{\Lambda}\right)\right] \qquad (3.10)
\end{aligned}$$

Where we used $c_d = 4$. Only terms up to order v^2 are displayed. We also omitted terms with products of 3 or more fields, such as, for example, $\phi A_i A_j$ or $\phi^2 \sigma_{ij}$. This omission is fine because, as will be detailed in the following section, these terms give rise to vanishing or quantum contributions, which can be neglected.

3.3 Calculation of potential terms

Now we will find the effective action terms which will be present in the potential. We start from (2.1) substituting the results from (3.10):

$$\begin{aligned}
iS_{eff} = & \, ln \int \mathcal{D}\phi \mathcal{D}A \mathcal{D}\sigma \, e^{iS_{quad}} \left\{ 1 + \dots + \frac{i^2}{2} \right. \\
& \times \left[-\frac{m_1}{\Lambda} \int dt_1 \left(\phi\left(1 + \frac{3}{2}v_1^2\right) + \frac{\phi^2}{2\Lambda} - A_i v_1^i\right) \right. \\
& \left. - \frac{m_2}{\Lambda} \int dt_2 \left(\phi\left(1 + \frac{3}{2}v_2^2\right) + \frac{\phi^2}{2\Lambda} - A_i v_2^i\right) + \dots \right]^2 \\
& \left. + \frac{i^3}{6}\left[-\frac{m_1}{\Lambda} \int dt_1 (\phi(1 + \dots)) - \frac{m_2}{\Lambda} \int dt_2 \left(\dots + \frac{\phi^2}{2\Lambda} + \dots\right)\right]^3 + \dots \right\}
\end{aligned} \qquad (3.11)$$

We see that Green's functions will start to appear when we combine ("pair up") fields of the same type at different space-time points. For example, taking the term linear in ϕ in the second line and multiplying by the analog term in the third line will give rise to the following term:

$$
\begin{aligned}
e^{iS_{eff}} \supset & \int \mathcal{D}\phi\mathcal{D}A\mathcal{D}\sigma \, e^{iS_{quad}}\Big[1 - \frac{m_1 m_2}{\Lambda^2}\iint dt_1 dt_2 \phi\big(\mathbf{x_1}(t_1), t_1\big)\phi\big(\mathbf{x_2}(t_2), t_2\big) \\
& \times \Big(1 + \frac{3}{2}v_1^2\Big)\Big(1 + \frac{3}{2}v_2^2\Big)\Big] \\
& = \mathcal{N} + \mathcal{N}\Big[- \frac{m_1 m_2}{\Lambda^2}\iint dt_1 dt_2 G_F^{(\phi)}\big(\mathbf{x_1}(t_1) - \mathbf{x_2}(t_2), t_1 - t_2\big) \\
& \times \Big(1 + \frac{3}{2}v_1^2\Big)\Big(1 + \frac{3}{2}v_2^2\Big)\Big]
\end{aligned}
$$

Using what we found for ϕ Green's function in (3.7) and approximating $ln(1+x) \approx 1 + x$, we get, apart from an additive constant:

$$
S_{eff} \supset \frac{m_1 m_2}{8\Lambda^2}\iint dt_1 dt_2 \int \frac{dk^0}{2\pi}\int_{\mathbf{k}} \frac{e^{-ik^0(t_1 - t_2) + i\mathbf{k}\cdot\big(\mathbf{x_1}(t_1) - \mathbf{x_2}(t_2)\big)}}{\mathbf{k}^2 - (k^0)^2}\Big(1 + \frac{3}{2}v_1^2\Big)\Big(1 + \frac{3}{2}v_2^2\Big)
$$

We now use the same prescription of section 2.2 with

$$
\begin{aligned}
\frac{1}{\mathbf{k}^2 - (k^0)^2} &= \frac{1}{\mathbf{k}^2}\left(1 + \frac{(k^0)^2}{\mathbf{k}^2} + \frac{(k^0)^4}{\mathbf{k}^4} + \dots\right) \\
&= \frac{1}{\mathbf{k}^2}\left(1 + \frac{\partial_{t_1}\partial_{t_2}}{\mathbf{k}^2} + \frac{(\partial_{t_1}\partial_{t_2})^2}{\mathbf{k}^4} + \dots\right)
\end{aligned}
$$

Up to order v^2 will we get:

$$
\begin{aligned}
S_{eff} &\supset \int dt \, \frac{m_1 m_2}{8\Lambda^2}\left[\frac{1}{4\pi r}\Big(1 + \frac{3}{2}v_1^2 + \frac{3}{2}v_2^2\Big) - v_1^i v_2^i \partial_{r_i}\partial_{r_j}\Big(- \frac{r}{8\pi}\Big)\right] \\
&= \int dt \, \frac{G_N m_1 m_2}{r}\left[1 + \frac{3}{2}v_1^2 + \frac{3}{2}v_2^2 + \frac{1}{2}\big(\mathbf{v_1}\cdot\mathbf{v_2} - (\mathbf{v_1}\cdot\hat{\mathbf{r}})(\mathbf{v_2}\cdot\hat{\mathbf{r}})\big)\right]
\end{aligned} \tag{3.12}
$$

Where we have used that

$$
v_1^i v_2^i \partial_{r_i}\partial_{r_j} r = v_1^i v_2^j \left(\frac{\delta_{ij}}{r} - \frac{r_i r_j}{r^3}\right)
$$

And the definition of Λ given in equation (2.4).

Notice that in (3.11) we're hiding a lot of terms. That's because their pairing won't generate relevant contributions. These are the criteria according to which we include pairings:

1. We only want terms which contain pairs of the same field. Terms with an odd power of any field will give zero because $\int d[\phi(x)] e^{iS_{quad}}(\phi(x))^n$ vanishes for n odd.

2. We only want contributions up to order v^2. Because of this, for example, terms proportional to σ_{ij} from (3.10) aren't considered because their pairing is of order v^4.

3. We aren't including quantum contributions. The power of $\hbar$ is proportional to the number of vertices minus the number of internal lines. Then, for example, pairing ϕ^2 from two different terms will give a power of $\hbar$ superior to the other terms. Notice, however, that the cubic term which includes one ϕ^2 and two different ϕ's will give a non-quantum contribution because the extra vertex will be compensated by an additional internal line, as we will see below.

Leaving aside the contributions excluded by the above considerations, we have the term calculated in (3.12) plus others. The next one will come from the pairing of the $\mathbf{A} \cdot \mathbf{v}$ terms:

$$
\begin{aligned}
e^{iS_{eff}} \supset & \int \mathcal{D}\phi \mathcal{D}A \mathcal{D}\sigma \, e^{iS_{quad}} \Bigg[1 + \frac{i^2}{2} \\
& \times 2 \frac{m_1 m_2}{\Lambda^2} \iint dt_1 dt_2 A_i A_j v_1^i(t_1) v_2^j(t_2) \Bigg] \\
& = \mathcal{N} + \mathcal{N} \Bigg\{ - \frac{m_1 m_2}{\Lambda^2} \iint dt_1 dt_2 \, G_F^{(A)} \big(\mathbf{x_1}(t_1) - \mathbf{x_2}(t_2), t_1 - t_2 \big)_{ij} \, v_1^i(t_1) v_2^j(t_2) \Bigg\}
\end{aligned}
$$

From (3.8) we substitute the Green's function for the A field. We see that only the order zero expansion in $(k^0)^2$ is needed here since the expression is already quadratic in v. Then:

$$
\begin{aligned}
iS_{eff} \supset & - \int dt \frac{m_1 m_2}{\Lambda^2} \int dt_1 dt_2 \, \frac{i}{2} \delta_{ij} \, \delta(t_1 - t_2) \frac{1}{4\pi r} v_1^i(t_1) v_2^j(t_2) \\
& = - \int \frac{i}{2} \frac{m_1 m_2}{\Lambda^2} \frac{1}{4\pi r} \mathbf{v_1} \cdot \mathbf{v_2}
\end{aligned}
$$

From which we obtain:

$$
S_{eff} \supset - \int dt \, \frac{4 \, G_N m_1 m_2}{r} \mathbf{v_1} \cdot \mathbf{v_2} \tag{3.13}
$$

There is yet another way to combine terms in (3.11) which is to contract two ϕ's from the fourth line with the ϕ^2 from the same line:

$$iS_{eff} \supset ln \int \mathcal{D}\phi\mathcal{D}A\mathcal{D}\sigma e^{iS_{quad}}\Big[1+\frac{i^3}{6} \times \frac{-3m_1^2 m_2}{2\Lambda^4}\iiint dt_A\, dt_B\, dt_C\, \phi(\mathbf{x_1},t_A)\phi(\mathbf{x_1},t_B)\phi^2(\mathbf{x_2},t_C)\Big] \tag{3.14}$$

Where the factor of 3 in the second line is due to the three different ways to generate the product in the integrand from the cubic term in (3.11).

To save space let's use the following notations:

$$A \leftrightarrow (\mathbf{x_1},t_A) \qquad B \leftrightarrow (\mathbf{x_1},t_B) \qquad C \leftrightarrow (\mathbf{x_2},t_C)$$
$$\phi_A\phi_B\phi_C^2 \equiv \phi(\mathbf{x_1},t_A)\phi(\mathbf{x_1},t_B)\phi^2(\mathbf{x_2},t_C)$$
$$\delta_{J_A} \equiv \frac{\delta}{\delta J(\mathbf{x_1},t_A)} \qquad \delta_{J_B} \equiv \frac{\delta}{\delta J(\mathbf{x_1},t_B)} \qquad \delta_{J_C} \equiv \frac{\delta}{\delta J(\mathbf{x_2},t_C)}$$
$$J\cdot G_F \cdot J \equiv \iint dt_1\, dt_2\, d^3x_1\, d^3x_2\, J(\mathbf{x_1},t_1) G_F\big(\mathbf{x_1}(t_1)-\mathbf{x_2}(t_2),t_1-t_2\big)J(\mathbf{x_2},t_2)$$
$$G_F\cdot J(A) \equiv \int dt\, d^dx\, G_F\big(\mathbf{x_1}(t_A)-\mathbf{x}(t),t_A-t\big)J(\mathbf{x},t)$$
$$G_F(A,C) \equiv G_F\big(\mathbf{x_1}(t_A)-\mathbf{x_2}(t_C),t_A-t_C\big)$$

Then

$$Z[J] = \int \mathcal{D}\phi\mathcal{D}A\mathcal{D}\sigma e^{iS_{quad}+\int dt\, d^dx \phi J} = \mathcal{N}e^{-\frac{1}{2}J\cdot G_F\cdot J}$$

And

$$\int \mathcal{D}\phi\mathcal{D}A\mathcal{D}\sigma e^{iS_{quad}}\phi_A\phi_B\phi_C^2 = \frac{1}{i^4}\delta_{J_A}\delta_{J_B}\delta^2_{J_C}Z[J]\Big|_{J=0}$$
$$=\mathcal{N}\,\delta_{J_A}\delta_{J_B}\delta^2_{J_C}e^{-\frac{1}{2}J\cdot G_F\cdot J} = \mathcal{N}\Big[\delta_{J_A}\delta_{J_B}\delta_{J_C}\Big(-G_F\cdot J(C)\,e^{-\frac{1}{2}J\cdot G_F\cdot J}\Big)\Big]$$
$$=\mathcal{N}\,\delta_{J_A}\delta_{J_B}\Big\{\Big[\Big(G_F\cdot J(C)\Big)^2 - G_F(C,C)\Big]e^{-\frac{1}{2}J\cdot G_F\cdot J}\Big\}$$
$$=\mathcal{N}\,\delta_{J_A}\Big\{\Big[2\Big(G_F\cdot J(C)\Big)G_F(B,C)+G_F(C,C)\Big(G_F\cdot J(B)\Big)+\mathcal{O}(J^2)\Big]e^{-\frac{1}{2}J\cdot G_F\cdot J}\Big\}\Big|_{J=0}$$
$$=\mathcal{N}\Big[2\,G_F(A,C)\,G_F(B,C)+G_F(C,C)\,G_F(A,B)\Big]$$

Now substituting the result above, together with ϕ Green's function obtained in (3.7) into (3.14) we have, to order v^0:

$$iS_{eff} \supset ln\Bigg\{\mathcal{N} - \mathcal{N}\frac{i\, m_1^2 m_2}{128\Lambda^4}\iiint dt_A\, dt_B\, dt_C \int \frac{dk^0}{2\pi} e^{-ik^0(t_A - t_C)} \int \frac{dk^{0\prime}}{2\pi} e^{-ik^{0\prime}(t_B - t_C)}$$

$$\int_{\mathbf{k}} \frac{e^{i\mathbf{k}\cdot\big(\mathbf{x_1}(t_A) - \mathbf{x_2}(t_C)\big)}}{\mathbf{k}^2} \int_{\mathbf{k}'} \frac{e^{i\mathbf{k}'\cdot\big(\mathbf{x_1}(t_B) - \mathbf{x_2}(t_C)\big)}}{\mathbf{k}'^2}\Bigg\}$$

$$\begin{aligned} iS_{eff} &\supset -i\frac{m_1^2 m_2}{128\Lambda^4}\int dt\Bigg[\int_{\mathbf{k}} \frac{e^{i\mathbf{k}\cdot(\mathbf{x_1} - \mathbf{x_2})}}{\mathbf{k}^2}\Bigg]^2 \\ &= -\frac{i}{128} m_1^2 m_2 (32\pi G_N)^2 \times \frac{1}{(4\pi r)^2} \\ &= -i\frac{G_N^2 m_1^2 m_2}{2r^2} \end{aligned}$$

Including the $1 \leftrightarrow 2$ combination we get:

$$S_{eff} \supset -\int dt \frac{G_N^2 m_1 m_2 (m_1 + m_2)}{2r^2} \tag{3.15}$$

Putting together equations (3.12), (3.13) and (3.15) we find the Einstein Infeld Hoffman corrections to the Newtonian potential:

$$V_{EIH} = -\frac{G_N m_1 m_2}{2r}\Bigg[3(v_1^2 + v_2^2) - 7\mathbf{v_1}\cdot\mathbf{v_2} - (\mathbf{v_1}\cdot\hat{\mathbf{r}})(\mathbf{v_2}\cdot\hat{\mathbf{r}})\Bigg] + \frac{G_N^2 m_1 m_2 (m_1 + m_2)}{2r^2} \tag{3.16}$$

Essentially, what we've done so far is calculating the $\mathcal{O}(v^2)$ (1PN) corrections to the Newtonian potential using field theoretic methods. We will now extend this technique to the case of spinning bodies. Note, also, that the last term has scale proportional to $(GM/r)^2$ and is, therefore, of the same PN order since each additional factor of (GM/r) is of the same order of v^2.

Chapter 4

The Case of Spinning Particles

4.1 Usage of anti-commuting Grassmann variables for spin degrees of freedom

The problem of calculating additional terms in the above potential due to spinning degrees of freedom has already been solved in EFT as seen, for example, in [7]. However, here we will devise a different approach leading to the same results, which is based on the modeling of the spin degrees of freedom by anti-commuting Grassmann variables as depicted in [8]. In this section we will make a brief summary of this method, and in the following section we will apply it to our problem of two bodies interacting through the gravitational field.

The point particle action written in a reparametrization invariant way is given by:

$$S_{pp} = \int d\tau \sqrt{-\dot{x}^{\mu}\dot{x}_{\mu}}$$

As explained, for example, in [9], the action can be put in another useful form by introducing a one-dimensional world-line metric $\gamma_{\tau\tau}(\tau)$ and its square root $\eta(\tau) \equiv \sqrt{-\gamma_{\tau\tau}(\tau)}$. In 4 dimensions we would call η a *tetrad* or *vierbein*; since it depends only on the world-line parameter τ we call it the *einbein*. The action:

$$S'_{pp} = \frac{1}{2} \int d\tau \left(\eta^{-1}\dot{x}^{\mu}\dot{x}_{\mu} - \eta\, m^2\right)$$

also has reparametrization invariance. By varying η we find the equation of motion:

$$\eta^2 = -\dot{x}^{\mu}\dot{x}_{\mu}/m^2 \tag{4.1}$$

When substituting this back into S'_{pp} we get back the first form S_{pp}.

If τ is the proper time then from the equation of motion $\eta = 1/m$. In this case we can write

$$S'_{pp} = \frac{1}{2}\int d\tau(m\dot{x}^{\mu}\dot{x}_{\mu} - m)$$

Where we no longer have reparametrization invariance. In order to be consistent with the notation in [8] we will use another parameter $s = \tau/2m$ which is also proportional to the proper time. We can then rewrite the action as

$$S'_{pp} = \int ds\left(\frac{1}{4}\dot{x}^{\mu}\dot{x}_{\mu} - m^2\right) \tag{4.2}$$

So that

$$L = \frac{1}{4}\dot{x}^{\mu}\dot{x}_{\mu}$$

We don't need to include the m^2 term in the Lagrangian since, in the particular case when s is proportional do the proper time, it is constant and thus doesn't influence the equations of motion. But we should keep this in mind as we'll put it back later when we will need to express the action in another parameter different from the particle's proper time.

Ravndal proposes in [8] that we modify the Lagrangian in the following way to describe the additional spinning degrees of freedom, first in the case of a free particle ($g_{\mu\nu} = \eta_{\mu\nu}$):

$$L = \frac{1}{4}(\dot{x}^{\mu}\dot{x}_{\mu} - i\xi^{\mu}\dot{\xi}_{\mu})$$

Where he introduces the four Grassmann variables $\xi^{\mu} = \xi^{\mu}(s)$ which anticommute:

$$\{\xi_{\mu}, \xi_{\nu}\} = 0$$

The hint that ξ_{μ} expresses the spin degrees of freedom comes from observing that the Lagrangian is invariant under Lorentz transformations:

$$\delta x_{\mu} = \epsilon_{\mu\nu}x^{\nu} \qquad \delta\xi_{\mu} = \epsilon_{\mu\nu}\xi^{\nu}$$

Where $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$ are infinitesimal parameters.

By Noether's theorem the conserved quantity associated with this symmetry is given by

$$K = \delta x^{\mu}p_{\mu} + \delta\xi^{\mu}\zeta_{\mu}$$

Where p_{μ} and ζ_{μ} are the conjugate momenta:

$$p_{\mu} \equiv \frac{\partial L}{\partial x^{\mu}} \qquad \zeta_{\mu} \equiv \frac{\partial L}{\partial \xi^{\mu}}$$

From which we get then the conserved quantity K as:

$$K = -\frac{1}{2}\epsilon^{\mu\nu}\Big[(x_\mu p_\nu - x_\nu p_\mu) + \frac{i}{4}(\xi_\mu\xi_\nu - \xi_\nu\xi_\mu)\Big]$$

We see then that $J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}$ is conserved where

$$L_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu$$

is the orbital angular momentum and

$$S_{\mu\nu} = \frac{i}{4}[\xi_\mu, \xi_\nu] \tag{4.3}$$

is the spin angular momentum.

This gauge fixed action is also verified to be SUSY invariant by expressing it in the form of super-variables:

$$X^\mu(s,\theta) = x^\mu(s) + i\theta\xi^\mu(s)$$

Where the SUSY transformations are induced by:

$$\theta \to \theta + \epsilon \qquad s \to s - i\epsilon\theta$$

So that

$$\begin{aligned} x^\mu + i\theta\xi^\mu &\to x^\mu - i\epsilon\theta\dot{x}^\mu + i(\theta+\epsilon)(\xi^\mu - i\epsilon\theta\dot{\xi}^\mu) \\ &= x^\mu + \underbrace{i\epsilon\xi^\mu}_{\delta x_\mu} + i\theta(\xi^\mu + \underbrace{\epsilon\dot{x}^\mu}_{\delta\xi^\mu}) \end{aligned}$$

Which makes evident the component transformations:

$$\delta x_\mu = i\epsilon\xi^\mu \qquad \delta\xi^\mu = \epsilon\dot{x}^\mu \tag{4.4}$$

We define the fermionic operator Q such that

$$\delta_Q X^\mu = \delta x^\mu + i\theta\delta\xi^\mu = i\epsilon\xi^\mu - i\epsilon\theta\dot{x}^\mu \equiv -i\epsilon Q X^\mu$$

Implying

$$Q = \theta\frac{\partial}{\partial s} + i\frac{\partial}{\partial\theta}$$

We see that Q satisfies the supersymmetry algebra:

$$\{Q, Q\} = 2i\partial_s$$

And it is also convenient to define the operator:

$$D \equiv \theta\partial_s - i\partial_\theta$$

satisfying the anti-commutation relation:

$$\{Q, D\} = 0$$

Finally, we can show that our Lagrangian can be written as:

$$L = \frac{1}{4}(\dot{x}^\mu \dot{x}_\mu - i\xi^\mu \dot{\xi}_\mu) = \frac{1}{4}\int d\theta \dot{X}^\mu D X_\mu$$

which can be checked by substituting D, X^μ and using the standard rules for integration of Grassmann variables:

$$\int d\theta = 0 \qquad \int d\theta\, \theta = 1$$

Following the rationale in [8], once we have the Lagrangian for the free particle using super-variables, under the presence a gravitational field we must simply exchange $\eta_{\mu\nu} \to g_{\mu\nu}$. That is,

$$L = \frac{1}{4}\int d\theta\, g_{\mu\nu}(X)\dot{X}^\mu D X^\nu$$

obtaining the correct equations in motion.

When expanding the Lagrangian in component form, remember that $g_{\mu\nu}(X)$ is expressed with a super-variable in the argument, therefore it has to be expanded like:

$$g_{\mu\nu}(x + i\theta\xi) = g_{\mu\nu}(x) + i\theta\xi^\lambda \partial_\lambda g_{\mu\nu}(x)$$

Then the Lagrangian in component form is expressed as

$$L = \frac{1}{4}g_{\mu\nu}(\dot{x}^\mu \dot{x}^\nu - i\xi^\mu \dot{\xi}^\nu) - \frac{i}{4}g_{\mu\nu,\lambda}\dot{x}^\mu \xi^\nu \xi^\lambda \tag{4.5}$$

With the conjugate momenta being given by

$$\begin{aligned} p_\mu &= \frac{\partial L}{\partial x^\mu} = \frac{1}{2}\dot{x^\mu} - \frac{1}{2}S^{\lambda\nu}\Gamma_{\lambda\nu\mu} \\ \zeta_\mu &= \frac{\partial L}{\partial \xi^\mu} = \frac{i}{4}\xi_\mu \end{aligned}$$

One important fact which will be used later on is that there is a conserved quantity associated with symmetries (4.4). Under these transformations, the Lagrangian (4.5) changes by the total derivative:

$$\delta L = \frac{i}{4}\epsilon\frac{d}{ds}(g_{\mu\nu}\xi^\mu\dot{x}^\nu)$$

Implying the conserved quantity:

$$K = \delta x^\mu p_\mu + \delta\xi^\mu\zeta_\mu - \frac{i}{4}\epsilon\frac{d}{ds}(g_{\mu\nu}\xi^\mu\dot{x}^\nu) = \frac{1}{2}i\epsilon\xi^\mu\dot{x}_\mu$$

That is, $\xi^\mu\dot{x}_\mu$ is conserved.

4.2 Reparametrization invariant action and new pairings

Now we take a detour from [8]. Here we have a different objective. Instead of deriving the equations of motion we want to write the action from the reference frame of an observer far away from the two particles. Therefore we cannot use the proper time so we have to go back to the reparametrization invariant form of the action.

First remember we had a m^2 term in (4.2) coming from the reparametrization invariant expression, so we put it back in the Lagrangian (4.5):

$$S = \frac{1}{4}\int ds\Big[g_{\mu\nu}(\dot{x}^\mu\dot{x}^\nu - i\xi^\mu\dot{\xi}^\nu) - ig_{\mu\nu,\lambda}\dot{x}^\mu\xi^\nu\xi^\lambda - 4\,m^2\Big]$$

Then change the parameter to $\tau = 2\,m\,s$ obtaining:

$$S = \frac{1}{2}\int d\tau\Big[g_{\mu\nu}(m\,\dot{x}^\mu\dot{x}^\nu - \frac{i}{2}\xi^\mu\dot{\xi}^\nu) - \frac{i}{2}g_{\mu\nu,\lambda}\dot{x}^\mu\xi^\nu\xi^\lambda - m\Big]$$

Here we are still using the proper time as the parameter for the world-line of the particle and the m term wouldn't change the equations of motion. To go back to the reparametrization invariant way, first notice that $\eta(\tau) = 1/m$ and since, under any change of parameter, $\eta(\tau)d\tau = \eta'(t)dt$, we have:

$$d\tau = m\,\eta'(t)dt \Rightarrow \frac{d}{d\tau} = \frac{1}{m\,\eta'(t)}\frac{d}{dt}$$

Therefore, making the change of variable from τ to t (and dropping the prime from η):

$$S = \frac{1}{2}\int dt\Big[g_{\mu\nu}\Big(\eta^{-1}\dot{x}^\mu\dot{x}^\nu - \frac{i}{2}\xi^\mu\dot{\xi}^\nu\Big) - \frac{i}{2}g_{\mu\nu,\lambda}\dot{x}^\mu\xi^\nu\xi^\lambda - \eta\,m^2\Big]$$

Now we can see that the m^2 term influences the equations of motion since it couples with η. Also, comparing to (4.1) we see that the equation of motion of η is unchanged since the new terms containing ξ don't couple with η. So we get the same result which we repeat here:

$$\eta^2 = \frac{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}{m^2}$$

Substituting η back into the action we get:

$$S = \int dt \Big[- m\sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} - \frac{i}{4} g_{\mu\nu}\xi^\mu \dot{\xi}^\nu - \frac{i}{4} g_{\mu\nu,\lambda}\dot{x}^\mu \xi^\nu \xi^\lambda \Big]$$

The two new terms in the particle's action, compared to the spin-less case, will generate new pairings in the partition function and therefore modify the potential. In order to calculate this effect we have to expand them into the gravitational component fields as below:

$$-\frac{i}{4} g_{\mu\nu}\xi^\mu \dot{\xi}^\nu =$$
$$-\frac{i}{4} e^{2\phi/\Lambda} \Big[- \xi^0 \dot{\xi}^0 + \frac{A_i}{\Lambda}(\xi^0 \dot{\xi}^i + \xi^i \dot{\xi}^0) + e^{-c_d \phi/\Lambda} \Big(\xi^a \dot{\xi}^a + \frac{\sigma_{ij}}{\Lambda}\xi^i \dot{\xi}^j \Big) - \frac{A_i A_j}{\Lambda^2}\xi^i \dot{\xi}^j \Big] \tag{4.6a}$$

$$-\frac{i}{4} g_{\mu\nu,\lambda}\dot{x}^\mu \xi^\nu \xi^\lambda =$$
$$-\frac{i}{4} \Big[g_{00,i}\xi^0 \xi^i + g_{0i,0}\xi^i \xi^0 + g_{0i,j}\xi^i \xi^j + g_{i0,j} v^i \xi^0 \xi^j + g_{ij,0} v^i \xi^j \xi^0 + g_{ij,k} v^i \xi^j \xi^k \Big] \tag{4.6b}$$

Let us first focus on the $\mathcal{O}(v^0)$ terms. Let's consider that the conserved quantity $\xi^\mu \dot{x}_\mu$ is zero, which is allowed given that its square will always be zero. Then, we should have $\xi^0 = \xi^i v_i$. Since, for now, we're interested only in $\mathcal{O}(v^0)$ corrections, the terms containing ξ^0 don't matter.

Continuing further to the second part (4.6b) we see that after disconsidering terms containing ξ^0 or of $\mathcal{O}(v)$ we're left only with:

$$\begin{aligned} &-\frac{i}{4}\Big[g_{0i,j}\xi^i \xi^j \Big] \\ &= -\frac{i}{4}\Big[\frac{2}{\Lambda}\partial_j \phi \, e^{2\phi/\lambda}\frac{A_i}{\Lambda} + e^{2\phi/\Lambda}\frac{\partial_j A_i}{\Lambda} \Big] \xi^i \xi^j \end{aligned}$$

We see above that the only term whose pairing generates non quantum contributions will be the one in zeroth order in ϕ and first order in A, that is:

$$-\frac{i}{4}\frac{\partial_j A_i}{\Lambda}\xi^i \xi^j \tag{4.7}$$

This pairing can be calculated as follows:

$$\begin{aligned}
&\int \mathcal{D}\phi\mathcal{D}A\mathcal{D}\sigma e^{iS_{quad}}\frac{\partial_j A_i(x)}{\Lambda}\frac{\partial_{j'}A_{i'}(x')}{\Lambda}\\
&=\frac{1}{\Lambda^2}\partial_j\partial_{j'}\int \mathcal{D}\phi\mathcal{D}A\mathcal{D}\sigma e^{iS_{quad}A_i(x)}A_{i'}(x')\\
&=\frac{1}{\Lambda^2}\partial_j\partial_{j'}\Big(-\frac{1}{2}\delta_{ii'}\Big)\Big[-i\int_{\mathbf{k}}\frac{dk^0}{2\pi}\frac{e^{-ik^0(t-t')+i\mathbf{k}\cdot\big(\mathbf{x}(t)-\mathbf{x}'(t')\big)}}{\mathbf{k}^2-(k^0)^2}\Big]\\
&=\frac{i}{2\Lambda^2}\delta_{ii'}\int_{\mathbf{k}}\frac{dk^0}{2\pi}\frac{e^{-ik^0(t-t')+i\mathbf{k}\cdot\big(\mathbf{x}(t)-\mathbf{x}'(t')\big)}}{\mathbf{k}^2-(k^0)^2}k_jk_{j'}
\end{aligned} \tag{4.8}$$

Where we used that

$$\partial_j\partial_{j'} = ik_j(-ik_{j'}) = k_jk_{j'}$$

We'll also use (4.3) to define the spatial components of spin:

$$\begin{aligned}
S_k &= \frac{1}{2}\epsilon_{ijk}S_{ij}\\
&\Rightarrow \frac{1}{2}i\,\xi_i\xi_j = S_{ij} = \epsilon_{ijk}S_k\\
&\Rightarrow \xi_i\xi_j = -2i\,\epsilon_{ijk}S_k
\end{aligned}$$

Now we are in conditions to start evaluating pairings. The first one we can look at is the pairing between two terms containing $\partial_j A_i$. Substituting (4.7) and the above definition for the spatial components of spin:

$$\begin{aligned}
iS_{eff} = ln\int \mathcal{D}\phi\mathcal{D}A\mathcal{D}\sigma\, e^{iS_{quad}}\Bigg\{1+\dots+\frac{i^2}{2}\\
\times\Bigg[-\frac{i}{4}\int dt\frac{\partial_j A_i}{\Lambda}(-2i)\epsilon_{ijk}S_k\\
-\frac{i}{4}\int dt'\frac{\partial_{j'}A_{i'}}{\Lambda}(-2i)\epsilon_{i'j'k'}S_{k'}+\dots\Bigg]^2+\dots\Bigg\}
\end{aligned}$$

(Note that we're using unprimed and primed indices to denote the two particles to make notation cleaner). Using (4.8) we get:

$$\begin{aligned}
iS_{eff} \supset -\frac{i}{8\Lambda^2}\delta_{ii'}\int dt\,dt'\delta(t-t')\int_{\mathbf{k}}\Big(1+\frac{\partial_t\partial_{t'}}{\mathbf{k}^2}+\dots\Big)\frac{e^{i\mathbf{k}\cdot\big(\mathbf{x}(t)-\mathbf{x}'(t')\big)}}{\mathbf{k}^2}k_jk_{j'}\\
\times\,\epsilon_{ijk}S_k\big(\mathbf{x}(t),t\big)\epsilon_{i'j'k'}S'_{k'}\big(\mathbf{x}'(t'),t'\big)
\end{aligned}$$

Using that:

$$\delta_{ii'}\,\epsilon_{ijk}\,\epsilon_{i'j'k'} = \epsilon_{ijk}\,\epsilon_{ij'k'} = \delta_{jj'}\,\delta_{kk'} - \delta_{jk'}\,\delta_{j'k}$$

We then get the following terms with zeroth order in velocities:

$$\begin{aligned} iS_{eff} \supset & -\frac{i}{8\Lambda^2}\mathbf{S}\cdot\mathbf{S}'\int dt\int_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} + \frac{i}{8\Lambda^2}S_j S'_k\int dt\int_{\mathbf{k}}\frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{\mathbf{k}^2}k_k k_j \\ = & \frac{i}{8\Lambda^2}\mathbf{S}\cdot\mathbf{S}'\int dt\,\delta^3(\mathbf{x}-\mathbf{x}') - \frac{i}{8\Lambda^2}S_j S'_k\int dt\,\partial_{r_j}\partial_{r_k}\left(\frac{1}{4\pi r}\right) \end{aligned}$$

Where we have used that $\partial_{r_j}\partial_{r_k}e^{i\mathbf{k}\cdot\mathbf{r}} = -k_j k_k\,e^{i\mathbf{k}\cdot\mathbf{r}}$.

The first term can be ignored since we're dealing with bodies interacting over a long distance compared to their own radii. To calculate the second term we note that:

$$\partial_{r_j}\partial_{r_k}\left(\frac{1}{r}\right) = -\frac{\delta_{jk}}{r^3} + \frac{3\,r_j r_k}{r^5}$$

So we obtain:

$$iS_{eff} \supset \int dt\frac{i}{32\pi\Lambda^2}\left[\frac{\mathbf{S}\cdot\mathbf{S}'}{r^3} - \frac{3}{r^3}(\mathbf{S}\cdot\hat{\mathbf{r}})(\mathbf{S}'\cdot\hat{\mathbf{r}})\right]$$

Substituting $\Lambda = (32\pi G_N)^{-1/2}$ we finally obtain the spin-spin contribution to the potential (changing notation for particles to indices 1 and 2):

$$V_{SS} = \frac{G_N}{r^3}\big[-\mathbf{S_1}\cdot\mathbf{S_2} + 3(\mathbf{S_1}\cdot\hat{\mathbf{r}})(\mathbf{S_2}\cdot\hat{\mathbf{r}})\big] \tag{4.9}$$

Matching the result previously obtained for the spin-spin potential by [7].

Now we look at contributions of $\mathcal{O}(v)$. The first possibility arises from combining the $\mathbf{A}\cdot\mathbf{v}$ term from (3.11) with the $\mathcal{O}(v^0)$ term due to the spinning degrees of freedom (4.7):

$$\begin{aligned} iS_{eff} = ln\int \mathcal{D}\phi\mathcal{D}A\mathcal{D}\sigma\,e^{iS_{quad}}\Bigg\{ & 1 + \cdots + \frac{i^2}{2} \\ & \times\left[-\frac{i}{4}\int dt_1\frac{\partial_j A_i}{\Lambda}(-2i)\epsilon_{ijk}S_k^{(1)}\right. \\ & \left. + \frac{m_2}{\Lambda}\int dt_2\big(A_l v_l^{(2)}\big) + \ldots\right]^2 + \ldots\Bigg\} + 1\leftrightarrow 2 \end{aligned}$$

Using the Green's function for A calculated in (3.8) and that $\partial/\partial x_j^{(1)}\,e^{i\mathbf{k}\cdot(\mathbf{x^{(1)}}-\mathbf{x^{(2)}})} = -\partial/\partial x_j^{(2)}\,e^{i\mathbf{k}\cdot(\mathbf{x^{(1)}}-\mathbf{x^{(2)}})} = \partial/\partial r_j\,e^{i\mathbf{k}\cdot(\mathbf{x^{(1)}}-\mathbf{x^{(2)}})}$ with $\mathbf{r}\equiv\mathbf{x^{(1)}}-\mathbf{x^{(2)}}$:

$$iS_{eff} \supset -\frac{1}{2}\cdot 2\Big(-\frac{i}{4}\Big)\frac{m_2}{\Lambda^2}\iint dt_1\, dt_2\Big[(-2i)\,\epsilon_{ijk}S_k^{(1)}v_l^{(2)}\partial_{r_j}\frac{i}{2}\delta_{il}\Big(1+\frac{\partial_{t_1}\partial_{t_2}}{\mathbf{k}^2}+\dots\Big)$$

$$\times\,\delta\big(t_1-t_2\big)\int_{\mathbf{k}}\frac{e^{i\mathbf{k}\cdot\big(\mathbf{x_1}(t_1)-\mathbf{x_2}(t_2)\big)}}{\mathbf{k}^2}\Big] - 1\leftrightarrow 2$$

$$\Rightarrow iS_{eff} \supset \int dt\Big[\frac{i}{4\Lambda^2}m_2 v_i^{(2)}\epsilon_{ijk}S_k^{(1)}\partial_{r_j}\frac{1}{4\pi r}\Big] - 1\leftrightarrow 2$$

$$= i\int dt\, 2\,G_N\Big[\frac{-1}{r^3}\epsilon_{ijk}S_k^{(1)}m_2 r_j v_i^{(2)}\Big] - 1\leftrightarrow 2 \qquad (4.10)$$

There is yet another possible contribution of $\mathcal{O}(v)$. It comes from pairing $\mathcal{O}(v^0)$ terms in (3.11) with $\mathcal{O}(v)$ terms in (4.6b). We see that the non quantum contributions will come from the pairing of ϕ. Expanding the $\mathcal{O}(v)$ terms in (4.6b) to first order in ϕ we obtain:

$$-\frac{i}{4}\Big[g_{00,i}\xi^0\xi^i + g_{0i,0}\xi^i\xi^0 + g_{ij,k}v^i\xi^j\xi^k\Big]$$

$$= -\frac{i}{4}\Big\{\partial_i(-e^{2\phi/\Lambda})\xi^0\xi^i + \partial_t\Big(e^{2\phi/\Lambda}\frac{A_i}{\Lambda}\Big)\xi^i\xi^0$$

$$+\,\partial_k\Big[e^{(2-c_d)\phi/\Lambda}\Big(\delta_{ij}+\frac{\sigma_{ij}}{\Lambda}\Big) - e^{2\phi/\Lambda}\frac{A_iA_j}{\Lambda^2}\Big]v^i\xi^j\xi^k\Big\}$$

$$= -\frac{i}{4}\Big[-\frac{2}{\Lambda}(\partial_i\phi)\xi^0\xi^i + \frac{2-c_d}{\Lambda}(\partial_k\phi)\delta_{ij}v^i\xi^j\xi^k\Big]$$

$$= -\frac{i}{4}\Big[-\frac{2}{\Lambda}(\partial_i\phi)(-2i)S_{ji}v_j - \frac{2}{\Lambda}(\partial_k\phi)v_i(-2i)S_{ik}\Big]$$

$$= \frac{2}{\Lambda}(\partial_i\phi)\epsilon_{jik}S_kv_j$$

In the first line, we neglected the terms of orders different than $\mathcal{O}(v)$. In the fourth line, we expanded only to first order in ϕ and no orders in the other fields (non quantum contributions). In the fifth line, we used that $\xi^\mu\xi^\nu = (-2i)S^{\mu\nu}$ and $c_d = 4$. In the last line, we used that $S_{ji} = \epsilon_{jik}S_k$.

The pairing with $-\frac{m_2}{\Lambda}\int dt\,\phi$ will produce:

$$iS_{eff} \supset \int \mathcal{D}\phi\mathcal{D}A\mathcal{D}\sigma e^{iS_{quad}} \Bigg\{$$

$$\frac{i^2}{2} \cdot 2 \cdot \frac{2m_2}{\Lambda^2}\,\partial_i^{(1)} \iint dt_1\, dt_2\, \phi(x_1)\epsilon_{jik} S_k^{(1)} v_j^{(1)} \phi(x_2) \Bigg\} + 1 \leftrightarrow 2$$

Where x_1 and x_2 are the four-vector coordinates of particles 1 and 2.

Making use of the Green's function for ϕ calculated in (3.7), and noting that we don't want to go beyond $\mathcal{O}(v)$, so that we can approximate $1/(\mathbf{k}^2 - (k^0)^2) \approx 1/\mathbf{k}^2$ we obtain:

$$\begin{aligned} iS_{eff} &\supset -\frac{im_2}{4\Lambda^2}\partial_{r_i} \int dt \frac{1}{4\pi r}\epsilon_{jik} S_k^{(1)} v_j^{(1)} - 1 \leftrightarrow 2 \\ &= i\int dt\, 2\, G_N \left[\frac{1}{r^3}\epsilon_{jik} S_k^{(1)} m_2 r_i v_j^{(1)}\right] - 1 \leftrightarrow 2 \end{aligned} \tag{4.11}$$

Where we used that $\partial_i^{(1)} = -\partial_i^{(2)} = \partial_{r_i}$

Summing the two contributions (4.10) and (4.11) we obtain:

$$\begin{aligned} iS_{eff} &\supset i\int dt\, 2\, G_N \left[\frac{1}{r^3}\epsilon_{ijk} S_i^{(1)} m_2 r_j \big(v_k^{(2)} - v_k^{(1)}\big)\right] - 1 \leftrightarrow 2 \\ &= -i\int dt \frac{2\, G_N m_2}{r^3}\mathbf{S^{(1)}} \cdot \mathbf{L^{(2,1)}} + 1 \leftrightarrow 2 \end{aligned} \tag{4.12}$$

Where we define $\mathbf{L^{(2,1)}} \equiv m_2(\mathbf{r_2} - \mathbf{r_1}) \times (\mathbf{v_2} - \mathbf{v_1})$ to be the orbital angular momentum of particle 2 relative to particle 1 and vice-versa for $\mathbf{L^{(1,2)}}$. This is the spin-orbit contribution to the potential:

$$V_{SO} = \frac{2\, G_N}{r^3}(\mathbf{S^{(1)}} \cdot \mathbf{L^{(2,1)}} + \mathbf{S^{(2)}} \cdot \mathbf{L^{(1,2)}}) \tag{4.13}$$

We'll see later on that this formula is incomplete. Another contribution will come out from the zeroth order term proportional to $\xi^0\dot{\xi}^0$ in the action. We'll see how it changes the spin-orbit potential in (5.11).

Chapter 5

Energy as a function of frequency

Having achieved the results for the Einstein Infeld Hoffmann corrections to the Newtonian potential in Chapter 3 (3.16); and spin-spin and spin-orbit corrections in Chapter 4 (4.9) and (4.13) it is straightforward to compute the energy by simply adding these corrections to the Newtonian expression. However, this is not very useful since we would be expressing the energy in terms of the radius r, which is not an observable. If we restrict ourselves to circular orbits, we can express the energy as a function of the frequency of rotation. This can be done in the following way:

1. We compute the equations of motion from the Lagrangian with corrections
2. The radial component of the acceleration is, by definition, $-\omega^2(\mathbf{r}, \mathbf{v})$ where ω is the angular frequency
3. We use $\mathbf{v} = \omega \mathbf{r}$ to eliminate $\mathbf{v}$ and thus we can express r as a function of ω and therefore we obtain $E(\omega)$.

We will apply this method first to the EIH corrections and then to the spin corrections.

5.1 Calculation of EIH corrections

As outlined above we start by computing the equations of motion. Other than the potential terms. From the point particle action (3.10) we see that a term $-m \int dt\sqrt{1-v^2}$ goes unaltered to the effective action S_{eff}. We see from (3.16) that V_{EIH} is of order 1PN. To match that we expand the kinetic term to $\mathcal{O}(v^4)$ so that:

$$S = \int dt \Bigg\{ \frac{1}{2}(m_1 v_1^2 + m_2 v_2^2) + \frac{1}{8}(m_1 v_1^4 + m_2 v_2^4) + \frac{G_N m_1 m_2}{r}$$
$$+ \frac{G_N m_1 m_2}{2r}\Bigg[3(v_1^2 + v_2^2) - 7\mathbf{v_1}\cdot\mathbf{v_2} - (\mathbf{v_1}\cdot\hat{\mathbf{r}})(\mathbf{v_2}\cdot\hat{\mathbf{r}})\Bigg] + \frac{G_N^2 m_1 m_2(m_1+m_2)}{2r^2}\Bigg\}$$

The equation of motion for particle 1 is, applying Euler Lagrange:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v_1^i}\right) = \frac{\partial L}{\partial x_1^i} \Rightarrow$$

$$m_1 a_1^i + m_1 a_1^j v_1^j v_1^i + \frac{1}{2} m_1 v_1^2 a_1^i - 3G_N \frac{m_1 m_2}{r^3} r_j (v_1^j - v_2^j) v_1^i + 3G_N \frac{m_1 m_2}{r} a^i +$$
$$\frac{7}{2} G_N \frac{m_1 m_2}{r^3} r_j (v_1^j - v_2^j) v_2^i - \frac{7}{2} G_N \frac{m_1 m_2}{r} a_2^i + \frac{d}{dt}\left(-\frac{1}{2} G_N \frac{m_1 m_2}{r^3} r^i v_2^j r^j \right)$$
$$= \mp G_N \frac{m_1 m_2}{r^3} r^i \mp \frac{1}{2} G_N \frac{m_1 m_2}{r^3} r^i \big[3(v_1^2 + v_2^2) - 7\mathbf{v_1}\cdot\mathbf{v_2} - (\mathbf{v_1}\cdot\hat{\mathbf{r}})(\mathbf{v_2}\cdot\hat{\mathbf{r}})\big]$$
$$\mp \frac{1}{2} G_N \frac{m_1 m_2}{r}\Bigg[-(\mathbf{v_1}\cdot\hat{\mathbf{r}})(\mathbf{v_2}\cdot\hat{\mathbf{r}}) + \frac{v_1^i}{r}(\mathbf{v_2}\cdot\hat{\mathbf{r}})\Bigg] \pm G_N^2 \frac{m_1 m_2 (m_1+m_2)}{r^4} r_1^i$$

Where the upper sign applies to particle 1, and the lower sign applies to particle 2 (also exchanging indices 1 and 2 for particle 2).

These equations of motion seem complicated but by recurring to the hypothesis of pure circular motion, by which $r_i(v_1^i - v_2^i) = 0$, many terms cancel out.

Another simplification can be done by using center of mass (CM) coordinates. They are defined by:

$$\mathbf{r_{CM}} \equiv \frac{m_1 \mathbf{r_1} + m_2 \mathbf{r_2}}{m_1 + m_2} \qquad M \equiv m_1 + m_2 \qquad \mu \equiv \frac{m_1 m_2}{M}$$

Then

$$\mathbf{v_1} = \mathbf{v_{CM}} + \frac{m_2}{M}\mathbf{v} \qquad \mathbf{v_2} = \mathbf{v_{CM}} - \frac{m_1}{M}\mathbf{v}$$

By symmetry under boosts we can assume that $\mathbf{v_{CM}} = 0$.

After subtracting EOMs for particles 1 and 2, canceling multiples of $\mathbf{r}\cdot\mathbf{v}$ and converting to CM coordinates we reach the following equation:

$$2\mu a^i + \mu a^j (1-2\eta) v^j v^i + \frac{1}{2}\mu a^i (1-2\eta) v^2 + \frac{13}{2}\mu \frac{GM}{r} a^i$$
$$= -\mu r^i \left[2\frac{GM}{r^3} + 3\frac{GM}{r^3} v^2 + \frac{G\mu}{r^3} v^2 - 2\frac{(GM)^2}{r^4} \right]$$

Where $\eta \equiv \mu/M$.

Note that we're at level 1PN so, in the second and third terms, we can substitute the Newtonian solution for a^i and a^j. With that substitution, the second term cancels out because it is proportional to $r^j v^j$. Replacing, in the third term, the Newtonian acceleration $a^i = -GMr^i/r^3$ and canceling factors of μ we have:

$$\begin{aligned} &a^i\left[2+\frac{13}{2}\frac{GM}{r}\right] \\ =&-r^i\left[2\frac{GM}{r^3}+3\frac{GM}{r^3}v^2+\eta\frac{GM}{r^3}v^2-2\frac{(GM)^2}{r^4}+\left(\eta-\frac{1}{2}\right)\frac{GM}{r^3}v^2\right] \end{aligned} \tag{5.1}$$

Let us define two auxiliary dimensionless variables:

$$\begin{aligned} \alpha &\equiv \frac{GM}{r} \\ x &\equiv (GM\omega)^{2/3} \end{aligned}$$

Our task is, then, to find α as a function of x. Note that, at Newtonian level:

$$\begin{aligned} a^i = -\omega^2 r^i &= -\frac{GM}{r^3}r^i \\ \Rightarrow (GM\omega)^2 &= \left(\frac{GM}{r}\right)^3 \\ \Rightarrow x &= \alpha \end{aligned}$$

So we're looking for perturbative corrections for $\alpha(x)$, that is,

$$\alpha(x) = x + \mathcal{O}(x^2) + \dots$$

Since we want only the next to leading order correction we want to find $d^2\alpha/dx^2|_{x=0}$ in the expansion:

$$\alpha(x) \approx \left.\frac{d\alpha}{dx}\right|_{x=0} x + \frac{1}{2}\left.\frac{d^2\alpha}{dx^2}\right|_{x=0} x^2 \tag{5.2}$$

Where we already know that $d\alpha/dx|_{x=0} = 1$. Replacing α and x in (5.1) we get:

$$2\alpha^4 - 2\alpha^3 + (4-2\eta)x^3\alpha + 2x^3 = 0$$

If we derive this equation twice, apply (5.2) and set $x = 0$ we find:

$$\left.\frac{d^2\alpha}{dx^2}\right|_{x=0} = 2 - \frac{2}{3}\eta$$

So (5.2) now reads as:

$$\alpha = x\left[1 + \left(1 - \frac{1}{3}\eta\right)x\right] \tag{5.3}$$

Another useful formula is:

$$\begin{aligned} v^2 = \omega^2 r^2 = \frac{x^3}{\alpha^2} &= \frac{x}{\left[1 + \left(1 - \frac{1}{3}\eta\right)x\right]^2} \\ &= x\left[1 - 2\left(1 - \frac{\eta}{3}\right)x\right] \end{aligned} \tag{5.4}$$

Now we have everything at hand to compute the energy as a function of frequency with corrections up to 1PN using (3.16).

First, it's important to take into account the terms up to $\mathcal{O}(v^4)$ in the expansion of $-m\sqrt{1-v^2}$ because we're working at level 1PN. This will generate the following terms in the Lagrangian:

$$\begin{aligned} &\frac{1}{2}(m_1 v_1^2 + m_2 v_2^2) + \frac{1}{8}(m_1 v_1^4 + m_2 v_2^4) \\ =&\frac{1}{2}\mu\frac{m_2 + m_1}{M}v^2 + \frac{1}{8}\mu\frac{m_2^3 + m_1^3}{M^3}v^4 \\ =&-\frac{1}{2}\mu\left[-v^2 + \left(\frac{3}{4}\eta - \frac{1}{4}\right)v^4\right] \end{aligned}$$

Where we have used center of mass coordinates with $\mathbf{v_{CM}} = 0$.

Summing with the potential at (3.16) we get the Lagrangian:

$$L = -\frac{1}{2}\mu\left[-2\frac{GM}{r} - \frac{GM}{r}(3+\eta)v^2 + \left(\frac{GM}{r}\right)^2 - v^2 + \left(\frac{3}{4}\eta - \frac{1}{4}\right)v^4\right]$$

Where we're assuming that $\mathbf{v}\cdot\mathbf{r} = 0$ (circular orbit).

Now, since energy is given by

$$E = \frac{\partial L}{\partial v}v - L$$

And substituting (5.3) and (5.4) we obtain, up to $\mathcal{O}(x^2)$,

$$\begin{aligned} E &= -\frac{1}{2}\mu x\left[1 - \left(\frac{3}{4} + \frac{\eta}{12}\right)x\right] \\ &= -\frac{1}{2}\mu(GM\omega)^{2/3}\left[1 - \left(\frac{3}{4} + \frac{\mu}{12M}\right)(GM\omega)^{2/3}\right] \end{aligned} \tag{5.5}$$

Which is the Newtonian energy plus the 1PN correction.

5.2 Calculation of corrections related to Spin

Let's start with the spin-spin contribution (4.9). Stating the Euler Lagrange equation in terms of center of mass coordinates we obtain:

$$
\begin{aligned}
\frac{\partial L}{\partial v_i} =&\mu a^i = \frac{\partial}{\partial r_i}\left\{\frac{G}{r^3}\big[\mathbf{S_1}\cdot\mathbf{S_2} - 3(\mathbf{S_1}\cdot\hat{\mathbf{r}})(\mathbf{S_2}\cdot\hat{\mathbf{r}})\big]\right\} + \dots \\
=& -\frac{3Gr^i}{r^5}\big[\mathbf{S_1}\cdot\mathbf{S_2} - 3(\mathbf{S_1}\cdot\hat{\mathbf{r}})(\mathbf{S_2}\cdot\hat{\mathbf{r}})\big] \\
&- \frac{3G}{r^3}\left[-2(\mathbf{S_1}\cdot\hat{\mathbf{r}})(\mathbf{S_2}\cdot\hat{\mathbf{r}})\frac{r^i}{r^2} + \frac{S_1^i}{r}(\mathbf{S_2}\cdot\hat{\mathbf{r}}) + \frac{S_2^i}{r}(\mathbf{S_1}\cdot\hat{\mathbf{r}})\right] + \dots
\end{aligned}
$$

So we conclude that

$$
\mathbf{a_{SS}} = -\frac{3G}{\mu r^4}\big\{\hat{\mathbf{r}}\big[\mathbf{S_1}\cdot\mathbf{S_2} - 5(\mathbf{S_1}\cdot\hat{\mathbf{r}})(\mathbf{S_2}\cdot\hat{\mathbf{r}})\big] + \mathbf{S_1}(\mathbf{S_2}\cdot\hat{\mathbf{r}}) + \mathbf{S_2}(\mathbf{S_1}\cdot\hat{\mathbf{r}})\big\} \quad (5.6)
$$

One could ask why we hadn't included terms with higher orders of v in the left hand side, that is, why we considered only the Newtonian kinetic term $\frac{1}{2}\mu v^2$ in $\frac{\partial L}{\partial v_i}$. The answer is that the right hand side is entirely (at least) of order 1PN. If the objects are small, that is $R_{1,2}$ is of order $m_{1,2}$, and, let's say, their rotational speeds are closer to unity, then their spins are of order $m_{1,2}v_{1,2}R_{1,2} = m_{1,2}^2$. Then:

$$
\frac{G}{r^3}S_1S_2 \sim \mu\frac{G}{r^3}Mm_1m_2 \sim \mu\left(\frac{GM}{r}\right)^3
$$

Which is already of order 2PN. If the rotational speeds are small, then the correction is even less significant. If the problem involved objects with sizes comparable to r, which isn't the case here, one could show that the correction would still be at least of order 1PN.

Coming back to (5.6), the next step is to look at the radial component which is what interests us here, since we want to find ω though the identity $-\omega^2\mathbf{r} = \mathbf{a}\cdot\hat{\mathbf{r}}$. Projecting (5.6) along the radial direction see that:

$$
\mathbf{a_{SS}}\cdot\hat{\mathbf{r}} = -\frac{3G}{\mu r^4}\big[\mathbf{S_1}\cdot\mathbf{S_2} - 3(\mathbf{S_1}\cdot\hat{\mathbf{r}})(\mathbf{S_2}\cdot\hat{\mathbf{r}})\big] \quad (5.7)
$$

In order to express the spin projections along the axis perpendicular to the orbit (i.e., along the orbital angular momentum of the system), we first see that:

$$
\mathbf{S_1}\cdot\mathbf{S_2} = (\mathbf{S_1}\cdot\hat{\mathbf{r}})(\mathbf{S_2}\cdot\hat{\mathbf{r}}) + (\mathbf{S_1}\cdot\hat{\mathbf{v}})(\mathbf{S_2}\cdot\hat{\mathbf{v}}) + (\mathbf{S_1}\cdot\hat{\mathbf{l}})(\mathbf{S_2}\cdot\hat{\mathbf{l}})
$$

But, taking the angular average, the first two terms are equal. Then we conclude that

$$(\mathbf{S_1}\cdot\hat{\mathbf{r}})(\mathbf{S_2}\cdot\hat{\mathbf{r}}) = \frac{1}{2}\big[\mathbf{S_1}\cdot\mathbf{S_2} - (\mathbf{S_1}\cdot\hat{\mathbf{l}})(\mathbf{S_2}\cdot\hat{\mathbf{l}})\big]$$

In this way (5.7) becomes:

$$\mathbf{a_{SS}}\cdot\hat{\mathbf{r}} = +\frac{3G}{2\mu r^4}\big[\mathbf{S_1}\cdot\mathbf{S_2} - 3(\mathbf{S_1}\cdot\hat{\mathbf{l}})(\mathbf{S_2}\cdot\hat{\mathbf{l}})\big] \tag{5.8}$$

Just for abbreviation let's define

$$d \equiv \mathbf{S_1}\cdot\mathbf{S_2} - 3(\mathbf{S_1}\cdot\hat{\mathbf{l}})(\mathbf{S_2}\cdot\hat{\mathbf{l}})$$

Now we can calculate the change in radius due to the spin-spin potential, relative to the Newtonian case, by adding the new radial acceleration term:

$$\begin{aligned} &-\omega^2 r = (\mathbf{a_{NEWTON}} + \mathbf{a_{SS}})\cdot\hat{r} = -\frac{GM}{r^2} + \frac{3G}{2\mu r^4}d \\ &\Rightarrow \omega^2 - \frac{GM}{r^3} + \frac{3G}{2\mu r^5}d = 0 \end{aligned}$$

We can solve this equation perturbatively since we're looking for the first order correction. We can then state the equation above as:

$$\omega^2 - \frac{GM}{r^3} + \frac{3G}{2\mu r^3}\frac{1}{r_{NEWTON}^2}d = 0 \tag{5.9}$$

Where we use the Newtonian level solution

$$(GM\omega)^2 = \left(\frac{GM}{r_{NEWTON}}\right)^3$$

The solution to (5.9) is given by:

$$\begin{aligned} r &= \frac{(GM)^{1/3}}{\omega^{2/3}}\left(1 - \frac{3}{2}\frac{\omega^{4/3}}{\mu\, G^{2/3}M^{5/3}}d\right)^{1/3} \\ &= \frac{(GM)^{1/3}}{\omega^{2/3}}\left(1 - \frac{1}{2}\frac{\omega^{4/3}}{\mu\, G^{2/3}M^{5/3}}d\right) \end{aligned} \tag{5.10}$$

Where in the last line we expanded $(1+x)^{1/3} \approx 1 + x/3$ since we want only the first order term in d.

Now we turn to the spin-orbit contribution. We could just start from (4.13) but there is something else to consider. When looking at the terms added to the action in (4.6a), consider what happens to the term:

$$\frac{i}{4}\xi^0\dot{\xi}^0$$

Where it is taken at leading order. It gets unaltered to the effective action and it does have a contribution. From the constraint $\xi^0 = \xi^i v_i$ we find (e.g. for particle 1):

$$\xi_1^0\,\dot{\xi}_1^0 = \xi_1^i\,v_1^i\,\dot{\xi}_1^j\,v_1^j + \xi_1^i\,v_1^i\,\xi_1^j\,a_1^j$$

From the EOM of ξ one can see that the first term has contributes in higher order than $\mathcal{O}(v^2)$ but the second term does have a contribution to be considered when taking the Newtonian order for acceleration.

Taking the center of mass coordinate where $v_1^i = \frac{m_2}{M}v^i$ we get:

$$\frac{i}{4}\xi_1^0\,\dot{\xi}_1^0 = \frac{i}{4}\xi_1^i\,v_1^i\,\xi_1^j\,a_1^j = \frac{1}{2}\epsilon_{ijk}S_k^{(1)}\frac{m_2^2}{M^2}v^i\left(-\frac{GM}{r^3}r^j\right)$$

Now it becomes evident that the spin-orbit potential deducted from (4.12) is incomplete. We have to add this new term. After this correction the formula for V_{SO} becomes:

$$V_{SO} = \frac{2G}{r^3}\mu(\mathbf{r}\times\mathbf{v})\cdot\left[\left(1+\frac{3m_1}{4m_2}\right)\mathbf{S_2}+\left(1+\frac{3m_2}{4m_1}\right)\mathbf{S_1}\right] \equiv \frac{G}{r^3}\mu(\mathbf{r}\times\mathbf{v})\cdot\vec{\chi} \quad (5.11)$$

Where, following [7], we define:

$$\vec{\chi} \equiv \left(2+\frac{3m_1}{2m_2}\right)\mathbf{S_2}+\left(2+\frac{3m_2}{2m_1}\right)\mathbf{S_1}$$

Now we can follow the same path we did for the spin-spin contribution. We find the equation of motion:

$$\mathbf{a_{SO}} = \frac{G}{r^3}\left[3\hat{\mathbf{r}}(\hat{\mathbf{r}}\times\mathbf{v})\cdot\vec{\chi} - 2\mathbf{v}\times\vec{\chi} + 3\hat{\mathbf{r}}\cdot\mathbf{v}(\hat{\mathbf{r}}\times\vec{\chi})\right] \quad (5.12)$$

Taking the radial component of this acceleration plus the Newtonian one and comparing with $-\omega\mathbf{r}$ we find:

$$\omega^2 - \frac{GM}{r^3} + \frac{G\omega}{r^3}\hat{\mathbf{l}}\cdot\vec{\chi} = 0$$

Giving the first order correction to the radius:

$$r = \frac{(GM)^{1/3}}{\omega^{2/3}}\left(1-\frac{1}{3}\frac{\omega}{M}\hat{\mathbf{l}}\cdot\vec{\chi}\right) \quad (5.13)$$

Now we can start calculating $E(\omega)$. First let's do it taking into account the spin-spin and spin-orbit contributions which have higher orders. Noticing from (5.12) that $\mathbf{a_{SO}} \cdot \mathbf{v} = 0$ the conserved energy is:

$$E = \frac{1}{2}\mu v^2 - \mu\frac{GM}{r} + V_{SS}$$

For any power of r we just need to approximate $[r_{NEWTON}(1+\delta)]^n \approx {r_{NEWTON}}^n(1+n\delta)$ so it becomes straightforward to get all contributions from the spin-spin and spin-orbit to the energy:

$$E(\omega) = -\frac{1}{2}\mu(GM\omega)^{2/3}\left\{1+\frac{4}{3}\frac{\omega}{M}\mathbf{\hat{l}}\cdot\vec{\chi}+\frac{\omega^{4/3}}{\mu\, G^{2/3}M^{5/3}}\left[\mathbf{S_1}\cdot\mathbf{S_2}-3(\mathbf{S_1}\cdot\mathbf{\hat{l}})(\mathbf{S_2}\cdot\mathbf{\hat{l}})\right]\right\}$$

Now summing the contribution previously calculated in (5.5) we can state the final result:

$$E(\omega) = -\frac{1}{2}\mu(GM\omega)^{2/3}\left\{1+\frac{4}{3}\frac{\omega}{M}\mathbf{\hat{l}}\cdot\vec{\chi}+\frac{\omega^{4/3}}{\mu\, G^{2/3}M^{5/3}}\left[\mathbf{S_1}\cdot\mathbf{S_2}-3(\mathbf{S_1}\cdot\mathbf{\hat{l}})(\mathbf{S_2}\cdot\mathbf{\hat{l}})\right] - \left(\frac{3}{4}+\frac{\mu}{12M}\right)(GM\omega)^{2/3}\right\} \tag{5.14}$$

And that concludes our work.

Appendix A

Mathematica notebook calculations

In this section we detail how we got to (3.4) using a Mathematica notebook whose original file can be downloaded from `http://tiny.cc/MautnersMasterThesis`. The notebook makes use of the "EinS" package ([5]) which can be obtained at `http://rcswww.urz.tu-dresden.de/~klioner/eins.html`.

Observe that the object definitions of the fields A, ϕ and σ are made such that the Λ denominator is incorporated into them. Therefore, the obtained expressions in quadratic order have to be multiplied by $(1/\Lambda^2)$. For example,

$$\sqrt{-g}R \to \frac{1}{\Lambda^2}\sqrt{-g}R = \frac{1}{32\pi G_N}\sqrt{-g}R$$

Therefore, the expression obtained for $\sqrt{-g}R$ has to be multiplied by 2 to give the Einstein Hilbert action integrand since according to (3.2):

$$S_{EH} = \int dt\, d^d x \frac{1}{16\pi G_N}\sqrt{-g}R$$

Accordingly, the expression $-\sqrt{-g}(g_{\mu\nu}\Gamma^\mu\Gamma^\nu)$ in the Mathematica file gives, after adjusting by the factor $1/\Lambda^2$, the correct factor of $1/(32\pi G_N)$ in (3.3). So, in the Mathematica file, the gauge fixing term is divided by 2 so that it has the same overall factor of $(1/2)$ as the Einstein Hilbert term. To summarize: the results of the Mathematica notebook are half the correct ones so we have to multiply everything by 2.

We now explain how we got the results for each of the component fields.

A.1 ϕ field

A.1.1 Einstein Hilbert term of order ϕ^2

This is located in the "RicciPAS0" entry in the file, which shows the expansion of $\sqrt{-g}R$ up to second order in the A field and zeroth order in σ. The terms quadratic in the ϕ field can be extracted as the ones proportional to $epsP^2$. Note that $E^{4\,epsP\,\phi}$ when expanded gives additional factors of $epsP$. We find the terms:

$$-2(\vec{\nabla}\phi)^2 + 18\dot{\phi}^2 - 6\ddot{\phi}(-4\phi) = \\ -2(\vec{\nabla}\phi)^2 - 6\dot{\phi}^2$$

Where we integrated the third term by parts.

A.1.2 Gauge fixing term of order ϕ^2

This is located in the "GaugeFixPAS0" entry and the only term quadratic in $epsP^2$ is:

$$8\dot{\phi}^2$$

Therefore, after summing the EH and GF terms quadratic in ϕ and multiplying by 2 as explained at the beginning of this section we obtain:

$$S_{quad} \supset \int dt\, d^dx\, 4[-(\vec{\nabla}\phi)^2 + \dot{\phi}^2]$$

A.2 A field

A.2.1 Einstein Hilbert term of order A^2

We look again at the "RicciPAS0" entry in the file, which shows the expansion of $\sqrt{-g}R$ up to second order in the A field and zeroth order in σ. The terms quadratic in the A field can be extracted as the ones proportional to $epsA^2$ and they are:

$$-2\dot{\vec{A}}^2 + \frac{1}{2}A_{a,b}A_{a,b} - 2A_a\ddot{A}_a - \frac{1}{2}A_{a,b}A_{b,a}$$

The first and third terms cancel each other after integrating by parts. The second and fourth terms can be simplified in the following way:

$$\begin{aligned}
&\frac{1}{2}\left(A_{a,b}A_{a,b} - A_{a,b}A_{b,a}\right) \\
=&\frac{1}{4}\left(A_{a,b}A_{a,b} + A_{b,a}A_{b,a} - A_{a,b}A_{b,a} - A_{b,a}A_{a,b}\right) \\
=&\frac{1}{4}(A_{a,b} - A_{b,a})^2 \\
=&\frac{1}{4}F_{ab}^2
\end{aligned}$$

A.2.2 Gauge fixing term of order A^2

This is located in the "GaugeFixPAS0" entry and the terms quadratic in $epsA^2$ are:

$$-\frac{1}{2}\dot{\vec{A}}^2 + \frac{1}{2}(\vec{\nabla}\cdot\vec{A})^2$$

Summing the EH and GF terms and multiplying by 2 we obtain:

$$S_{quad} \supset \int dt\, d^dx \left[\frac{F_{ij}^2}{2} + (\vec{\nabla}\cdot\vec{A})^2 - \dot{\vec{A}}^2\right]$$

A.3 σ field

A.3.1 Einstein Hilbert term of order σ^2

The calculations involving σ are more complicated, so here we calculate separately $\sqrt{-g}$ and R and then multiply them manually.

The calculation for $\sqrt{-g}$ can be found at entry "SDetgS2" which expands it only up to second order in σ without any expansion in A and ϕ. It gives:

$$1 + \frac{\sigma}{2} - \frac{1}{4}\sigma_{ab}^2 + \frac{1}{8}\sigma^2$$

Where it is understood that σ_{ab}^2 is equal to the contraction $\sum_{a,b}\sigma_{ab}\sigma_{ab}$ and $\sigma \equiv \operatorname{Tr}\sigma = \sum_a \sigma_{aa}$.

The calculation of R up to second order in σ and no expansions in A and ϕ is located at entry "RiccisclS2". It has terms of order σ, which get multiplied by the order σ term in $\sqrt{-g}$, and terms of order σ^2, which get multiplied by the 1 from $\sqrt{-g}$. The desired terms quadratic in σ in $\sqrt{-g}R$, after simplifying, are:

$$
\begin{aligned}
&\sqrt{-g}R \text{ in order } \sigma^2\\
=&-\frac{1}{4}\dot{\sigma}^2+\frac{1}{4}(\vec{\nabla}\sigma)^2+\frac{1}{4}\dot{\sigma}_{ij}^2-\frac{1}{4}(\vec{\nabla}\sigma_{ij})^2+\frac{1}{2}\sigma\sigma_{bc,bc}+(\sigma_{ab,a})^2-\frac{1}{2}(\sigma_{ab,c}\sigma_{ac,b})
\end{aligned}
$$

A.3.2 Gauge fixing term of order σ^2

First we calculate $\Gamma^\alpha \equiv \Gamma^\alpha_{\mu\nu}g^{\mu\nu}$. This is performed at cell "GamGFS2". We can see from the result that only terms of order σ and beyond are obtained. Since we're interested in the contraction $g_{\mu\nu}\Gamma^\mu\Gamma^\nu$, only the $\mathcal{O}(\sigma)$ term from Γ^α matters. Also, only the zeroth order of $g_{\mu\nu}$ (that is, $\eta_{\mu\nu}$) can be used if we truncate the result to orders less than 2 in σ. The same applies to $\sqrt{-g}$ which is therefore considered to be 1 for this calculation.

Then, in order σ^2 we'll have:

$$
\begin{aligned}
&\frac{1}{2}(-1)\sqrt{-g}g_{\mu\nu}\Gamma^\mu\Gamma^\nu\\
=&-\frac{1}{2}\Gamma_\alpha\Gamma^\alpha\\
=&-\frac{1}{2}\left(\frac{-\partial_\alpha\sigma}{2}+\partial_a\sigma_{\alpha a}\right)\left(\frac{-\partial^\alpha\sigma}{2}+\partial^a\sigma^{\alpha a}\right)\\
=&\frac{\dot{\sigma}^2}{8}-\frac{(\vec{\nabla}\sigma)^2}{8}+\frac{1}{2}\partial_b\sigma\partial_a\sigma_{ab}-\frac{1}{2}(\partial_a\sigma_{ab})^2
\end{aligned}
$$

Where α is a 4-vector index and a is a 3D vector index. Also, indices are lowered and uppered with the Lortentz metric.

Remember, also, that the (1/2) factor accounts for the fact that the Einstein Hilbert term obtained in the notebook is half the correct one, so we'll divide the GF term by 2, sum it with the EH term and after multiply the final result by 2.

Now we can sum the EH and GF terms. After integrating by parts, the fifth term in the EH expression cancel with the third term in the GF expression. The last two terms of EH plus the last term of GF give

$$
\begin{aligned}
&(1/2)[(\sigma_{ab,a})^2-\sigma_{ab,c}\sigma_{ac,b}]\\
=&(1/2)[\sigma_{ba,a}\sigma_{bc,c}-\sigma_{ba,c}\sigma_{bc,a}]\\
=&(1/2)[-\sigma_{ba,ca}\sigma_{bc}+\sigma_{ba,ca}\sigma_{bc}]\\
=&0
\end{aligned}
$$

Where we integrated by parts in the third line.

Therefore, summing the remaining terms we're left with (after multiplying by 2 as explained before):

$$S_{quad} \supset \int dt\, d^d x \Big[-\frac{1}{4}\dot{\sigma}^2 + \frac{1}{4}(\vec{\nabla}\sigma)^2 + \frac{1}{2}\dot{\sigma}_{ij}^2 - \frac{1}{2}(\vec{\nabla}\sigma_{ij})^2 \Big]$$

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