



Nonsmooth Continuous-Time Optimization Problems: Necessary Conditions

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Abstract—We consider Lipschitz continuous-time nonlinear optimization problems and provide first-order necessary optimality conditions of both Fritz John and Karush-Kuhn-Tucker types. © 2001 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Consider the continuous-time nonlinear programming problem

$$\begin{aligned}
&\text{minimize} && \phi(x) = \int_0^T f(t, x(t)) dt, \\
&\text{subject to} && g_i(t, x(t)) \leq 0, \quad \text{a.e. } t \in [0, T], \\
&&& i \in I = \{1, \dots, m\}, \quad x \in X.
\end{aligned}
\tag{CNP}$$

Here X is a nonempty open convex subset of the Banach space $L_\infty^n[0, T]$ of all n -dimensional vector-valued Lebesgue measurable essentially bounded functions defined on the compact interval $[0, T] \subset \mathbb{R}$, with the norm $\|\cdot\|_\infty$ defined by

$$\|x\|_\infty = \max_{1 \leq j \leq n} \text{ess sup}\{|x_j(t)|, 0 \leq t \leq T\},$$

where for each $t \in [0, T]$, $x_j(t)$ is the j^{th} component of $x(t) \in \mathbb{R}^n$, ϕ is a real-valued function defined on X , $g(t, x(t)) = \gamma(x)(t)$, and $f(t, x(t)) = \Gamma(x)(t)$, where γ is a map from X into the

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normed space $\Lambda_1^m[0, T]$ of all Lebesgue measurable essentially bounded m -dimensional vector functions defined on $[0, T]$, with the norm $\|\cdot\|_1$ defined by

$$\|y\|_1 = \max_{1 \leq j \leq m} \int_0^T |y_j(t)| dt,$$

and Γ is a map from X into the normed space $\Lambda_1^1[0, T]$.

This class of problems was introduced in 1953 by Bellman [1] in connection with production-inventory “bottleneck processes”. He considered a type of optimization problem, which is now known as continuous-time linear programming, formulated its dual, and provided duality relations. He also suggested some computational procedures.

Since then, a lot of authors have extended his theory to wider classes of continuous-time linear problems (see, e.g., [2–10]).

On the other hand, optimality conditions of Karush-Kuhn-Tucker type for continuous nonlinear problems were first investigated by Hanson and Mond [11]. They considered a class of linearly constrained nonlinear programming problems. Assuming a nonlinear integrand in the cost function was twice differentiable, they linearized the cost function and applied Levinson’s duality theory [3] to obtain the Karush-Kuhn-Tucker optimality conditions. Also applying linearization, Farr and Hanson [12] obtained necessary and sufficient optimality conditions for a more general class of continuous-time nonlinear problems (both cost function and constraints were nonlinear).

Assuming some kind of constraint qualification and using direct methods, further generalizations of the theory of optimality conditions for continuous-time nonlinear problems were discussed by Scott and Jefferson [13], Abraham and Buie [14], Reiland and Hanson [15], and Zalmai [16–20]. However, the development of nonsmooth necessary optimality conditions for problem (CNP) is not yet satisfactory.

Our aim in this paper is to provide first-order necessary optimality conditions in the form of Fritz John and Karush-Kuhn-Tucker theorems for a general class of nonsmooth continuous-time Lipschitz programming problems. This is accomplished through generalizations of the differentiable versions of Fritz John and Karush-Kuhn-Tucker theorems in [16] to the Lipschitz case. Similar results on necessary conditions of optimality can be obtained as a consequence of the maximum principle of Pontryagin for optimal control problems. However, we avoid this route and provide a direct proof exploiting the connections between optimization theory and theorems of the alternative. Sufficient conditions have been pursued in another paper [21].

Related results can be found in [22]. However, his arguments are via smooth approximation of nondifferentiable functions rather than alternative theorems.

This work is organized as follows. In Section 2, we recall some basic properties of Lipschitz nonsmooth analysis, support functions, integration of multifunctions and state the generalized Gordan Theorem. In Section 3, we establish the nonsmooth geometric optimality conditions for (CNP). The nonsmooth versions of Fritz John and Karush-Kuhn-Tucker continuous-time optimality conditions are obtained in Sections 4 and 5, respectively.

2. PRELIMINARIES

In this section, we summarize basic concepts and tools from nonsmooth analysis, including support functions and integration of multifunctions. Most of the material included here can be found in [23]. We also state the generalized Gordan Theorem, which has been a very useful tool in this work.

In what follows, $\mathcal{B}$ denotes a real linear space with norm $\|\cdot\|$ and $\mathcal{B}^*$ its topological dual with norm given by

$$\|\xi\|_* = \sup \{ \langle \xi, v \rangle : v \in \mathcal{B}, \|v\| \leq 1 \},$$

where $\langle \cdot, \cdot \rangle$ is the canonical duality pairing between $\mathcal{B}^*$ and $\mathcal{B}$.

Let $f : \mathcal{B} \rightarrow \mathbb{R}$ be locally Lipschitz, i.e., for all $x \in \mathcal{B}$ there is $\epsilon > 0$ and constant K depending on ϵ such that

$$|f(x_1) - f(x_2)| \leq K\|x_1 - x_2\|, \quad \forall x_1, x_2 \in x + \epsilon B.$$

Here B denotes the open unit ball of $\mathcal{B}$. We also say that f is Lipschitz of rank K near x .

Let $v \in \mathcal{B}$. The *generalized directional derivative* of f at x in the direction v , denoted by $f^0(x; v)$, is defined as follows:

$$f^0(x; v) = \limsup_{\substack{y \rightarrow x \\ s \rightarrow 0^+}} \frac{f(y + sv) - f(y)}{s}.$$

Here $y \in \mathcal{B}$ and $s \in (0, +\infty)$.

The *generalized gradient* of f at x , denoted by $\partial f(x)$, is the subset of $\mathcal{B}^*$ given by

$$\{\xi \in \mathcal{B}^* : f^0(x; v) \geq \langle \xi, v \rangle, \forall v \in \mathcal{B}\}.$$

For every $v \in \mathcal{B}$, one has

$$f^0(x; v) = \max \{\langle \xi, v \rangle : \xi \in \partial f(x)\}.$$

We say that f is *regular* at $x \in \mathcal{B}$ if

- (i) for all $v \in \mathcal{B}$, the directional derivative $f'(x; v)$ exists;
- (ii) for all $v \in \mathcal{B}$, $f'(x; v) = f^0(x; v)$.

2.1. Support Functions

We recall that the *support function* of a nonempty subset D of $\mathcal{B}$ is the function $\sigma_D : \mathcal{B}^* \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$\sigma_D(\xi) = \sup \{\langle \xi, x \rangle : x \in D\}.$$

We now state some basic known results for support functions which are needed in the sequel.

PROPOSITION 2.1. (See [24].) *Let C, D be nonempty closed convex subsets of $\mathcal{B}$, and let Σ, Δ be nonempty weak*-closed convex subsets of $\mathcal{B}^*$. Then*

$$\begin{aligned} C \subseteq D & \text{ iff } \sigma_C(\xi) \leq \sigma_D(\xi), & \forall \xi \in \mathcal{B}^*, \\ \Delta \subseteq \Sigma & \text{ iff } \sigma_\Delta(x) \leq \sigma_\Sigma(x), & \forall x \in \mathcal{B}. \end{aligned}$$

PROPOSITION 2.2. (See [24].) *Let C, D be nonempty closed convex subsets of $\mathcal{B}$, and Σ, Δ be nonempty weak*-closed convex subsets of $\mathcal{B}^*$. Let also $\mu, \lambda \geq 0$ be given scalars. Then*

$$\begin{aligned} \mu\sigma_C(\xi) + \lambda\sigma_D(\xi) &= \sigma_{\{\mu C + \lambda D\}}(\xi), & \forall \xi \in \mathcal{B}^*, \\ \mu\sigma_\Delta(x) + \lambda\sigma_\Sigma(x) &= \sigma_{\{\mu\Delta + \lambda\Sigma\}}(x), & \forall x \in \mathcal{B}. \end{aligned}$$

2.2. Integration of Multifunctions

Given a multifunction $G : [0, T] \rightarrow \mathbb{R}^n$, denote by $S^1(G)$ the following set:

$$S^1(G) = \{f \in L_1^n[0, T], f(t) \in G(t) \text{ a.e. } t \in [0, T]\}.$$

We define the integral of G , denoted by $\int_0^T G(t) dt$, as the following subset of $\mathbb{R}^n$:

$$\int_0^T G(t) dt := \left\{ \int_0^T f(t) dt : f \in S^1(G) \right\}.$$

A multifunction G is said to be integrably bounded if G is measurable and there exists an integrable function $z : [0, T] \rightarrow \mathbb{R}_+$ such that

$$|G(t)| = \sup_{x \in G(t)} \|x\| \leq z(t), \quad \text{a.e. } t \in [0, T].$$

THEOREM 2.3. *If G is an integrably bounded multifunction with compact values, then*

$$\sigma_{\int_0^T G(t) dt}(v) = \int_0^T \sigma_{G(t)}(v) dt, \quad \forall v \in \mathbb{R}^n.$$

The proof of this theorem can be found, for example, in [25].

2.3. The Generalized Gordan Theorem

In this section, we state a transposition theorem, known as the Generalized Gordan Theorem [18]. It is the key to moving from the geometric optimality condition obtained in Section 3 to the main results on first-order necessary optimality conditions in this work.

For the next result, the domain of definition of the elements of the spaces $L_\infty^n[0, T]$, $L_\infty^m[0, T]$, $\Lambda_1^m[0, T]$ are replaced with a nonzero Lebesgue measurable set $A \subset [0, T]$.

THEOREM 2.4. *Let $A \subset [0, T]$ be a set of positive Lebesgue measure, X be a nonempty convex subset of $L_\infty^n(A)$, and $p_i : A \times V \rightarrow \mathbb{R}$, $i \in I$, be defined by $p_i(t, x(t)) = \pi_i(x)(t)$, where V is an open subset of $\mathbb{R}^n$, $\pi = (\pi_1, \dots, \pi_m)$ is a map from X to $\Lambda_1^m(A)$, and suppose that p_i is convex with respect to its second argument throughout A . Then exactly one of the following systems is consistent:*

- (i) *there is $x \in X$ such that $p_i(t, x(t)) < 0$ a.e. $t \in A$, $i \in I$,*
- (ii) *there is a nonzero m -vector function $u \in L_\infty^m(A)$, $u_i(t) \geq 0$ a.e. $t \in A$, $i \in I$, such that*

$$\int_0^T \sum_{i \in I} u_i(t) p_i(t, x(t)) dt \geq 0,$$

for all $x \in X$.

PROOF. The proof of Theorem 2.4 follows in similar fashion as that of Theorem 3.2 in [18], replacing $[0, T]$ by A . ■

3. GEOMETRIC CHARACTERIZATION OF A MINIMUM

Let $\mathbb{F}$ be the set of all feasible solutions to problem (CNP) (we suppose it is nonempty), i.e.,

$$\mathbb{F} = \{x \in X : g_i(t, x(t)) \leq 0 \text{ a.e. } t \in [0, T], i \in I\}.$$

Let V be an open subset of $\mathbb{R}^n$ containing the set

$$\{x(t) \in \mathbb{R}^n : x \in \mathbb{F}, t \in [0, T]\}.$$

f and g_i , $i \in I$, are real functions defined on $[0, T] \times V$. The function $t \rightarrow f(t, x(t))$ is assumed to be Lebesgue measurable and integrable for $x \in X$.

For all $\bar{x} \in \mathbb{F}$ and $i \in I$, let $A_i(\bar{x})$ denote the set

$$\{t \in [0, T] : g_i(t, \bar{x}(t)) = 0\}.$$

Hereafter, we assume that, given $a \in V$, there exist an $\epsilon > 0$ and a positive number k such that $\forall t \in [0, T]$, and $\forall x_1, x_2 \in a + \epsilon B$ (B denotes the unit ball of $\mathbb{R}^n$), we have

$$|f(t, x_1) - f(t, x_2)| \leq k\|x_1 - x_2\|.$$

Similar hypotheses are assumed for g_i , $i \in I$. Hence, $f(t, \cdot)$ and $g_i(t, \cdot)$, $i \in I$, are Lipschitz near every $\bar{x} \in V$, throughout $[0, T]$.

We suppose the Lipschitz constant is the same for all functions involved.

Now, assume $\bar{x} \in X$ and $h \in L_\infty^n[0, T]$ are given. So, the Clarke generalized directional derivatives

$$g_i^0(t, \bar{x}(t); h(t)) := \gamma_i^0(\bar{x}; h)(t) := \limsup_{\substack{y \rightarrow \bar{x} \\ \lambda \rightarrow 0^+}} \frac{\gamma_i(y + \lambda h)(t) - \gamma_i(y)(t)}{\lambda}$$

and

$$f^0(t, \bar{x}(t); h(t)) := \Gamma^0(\bar{x}; h)(t) := \limsup_{\substack{y \rightarrow \bar{x} \\ \lambda \rightarrow 0^+}} \frac{\Gamma(y + \lambda h)(t) - \Gamma(y)(t)}{\lambda}$$

are finite a.e. $t \in [0, T]$ and $\phi^0(\bar{x}; h)$ is finite.

For ease of reading and presentation, in the rest of this work, we use the notations $f^0(t, \bar{x}(t); h(t))$ and $g_i^0(t, \bar{x}(t); h(t))$ rather than $\Gamma^0(\bar{x}; h)(t)$ and $\gamma_i^0(\bar{x}; h)(t)$. It follows easily from the assumptions that

$$\begin{aligned} t &\rightarrow f^0(t, \bar{x}(t); h(t)), \\ t &\rightarrow g_i^0(t, \bar{x}(t); h(t)), \quad i \in I, \end{aligned}$$

are Lebesgue measurable and integrable for all $\bar{x} \in X$, and $h \in L_\infty^n[0, T]$.

Consider the following cones in $L_\infty^n[0, T]$ with zero vertices:

$$\begin{aligned} \mathcal{K}(\phi; \bar{x}) &= \{h \in L_\infty^n[0, T] : \phi^0(\bar{x}; h) < 0\}, \\ \mathcal{K}(g_i; \bar{x}) &= \{h \in L_\infty^n[0, T] : g_i^0(t, \bar{x}(t); h(t)) < 0 \text{ a.e. } t \in A_i(\bar{x})\}, \quad i \in I. \end{aligned}$$

We are now in a position to provide a geometric characterization of a local minimum for problem (CNP).

THEOREM 3.1. *Let $\bar{x}$ be an optimal solution of problem (CNP). Then*

$$\bigcap_{i \in I} \mathcal{K}(g_i; \bar{x}) \cap \mathcal{K}(\phi; \bar{x}) = \emptyset. \quad (1)$$

PROOF. Suppose the intersection of cones in (1) is nonempty and take $h \in L_\infty^n[0, T]$ in this intersection. It follows from the limsup properties and continuity of the functions involved that there is a real number $\delta > 0$ such that, $\forall \lambda \in (0, \delta)$, $\bar{x} + \lambda h \in X$,

$$\begin{aligned} g_i(t, \bar{x}(t) + \lambda h(t)) &\leq 0, \quad \text{a.e. } t \in [0, T], \quad i \in I, \\ \phi(\bar{x} + \lambda h) &< \phi(\bar{x}). \end{aligned}$$

But that means $\bar{x} + \lambda h$, $\lambda \in (0, \delta)$, is a feasible solution for (CNP) with a better objective function value. This contradicts the optimality of $\bar{x}$ for problem (CNP). Therefore, the intersection in (1) is empty. ■

4. FRITZ JOHN TYPE OPTIMALITY CONDITIONS

In this section, we derive a new continuous-time analogue of the Fritz John necessary optimality conditions by translating the geometric optimality conditions into algebraic statements. This is made possible through the use of the Generalized Gordan Theorem. We also point out that the new Fritz John necessary conditions generalize the smooth case treated by Zalmai (see [16, Theorem 3.3]).

THEOREM 4.1. Let $\bar{x} \in \mathbb{F}$. Let $f(t, \cdot)$ and $g(t, \cdot)$ be Lipschitz near $\bar{x}(t)$. If $\bar{x}$ is a local optimal solution of (CNP), then there exist $\bar{u}_0 \in \mathbb{R}$, $\bar{u}_i \in L^\infty_\infty[0, T]$, $i \in I$, such that

$$0 \leq \int_0^T \left[\bar{u}_0 \partial_x f(t, \bar{x}(t)) + \sum_{i \in I} \bar{u}_i(t) \partial_x g_i(t, \bar{x}(t)) \right] dt, \quad (2)$$

$$\bar{u}_0 \geq 0, \quad \bar{u}(t) \geq 0, \quad \text{a.e. } t \in [0, T], \quad (3)$$

$$(\bar{u}_0, \bar{u}(t)) = (\bar{u}_0, \bar{u}_1(t), \dots, \bar{u}_m(t)) \neq 0, \quad \text{a.e. } t \in [0, T], \quad (4)$$

$$\bar{u}_i(t) g_i(t, \bar{x}(t)) = 0, \quad \text{a.e. } t \in [0, T], \quad i \in I. \quad (5)$$

PROOF. We shall proceed under the *Interim Hypothesis*: (CNP) has only one constraint

$$g(t, x(t)) \leq 0, \quad \text{a.e. } t \in [0, T].$$

The removal of this interim hypothesis will be done at the end of the proof.

We denote

$$A(\bar{x}) = \{t \in [0, T] : g(t, \bar{x}(t)) = 0\},$$

$$K(g, \bar{x}) = \{h \in L^\infty_\infty[0, T] : g^0(t, \bar{x}(t); h(t)) < 0, t \in A(\bar{x})\}.$$

LEMMA 4.2. Let $\bar{x} \in \mathbb{F}$. Let $f(t, \cdot)$, $g(t, \cdot)$ be Lipschitz near $\bar{x}(t)$ throughout $[0, T]$. If $\bar{x}$ is a local optimal solution of (CNP), then there exist $\bar{u}_0 \in \mathbb{R}$, $\bar{u} \in L^\infty[0, T]$, such that

$$0 \leq \int_0^T [\bar{u}_0 f^0(t, \bar{x}(t); h(t)) + \bar{u}(t) g^0(t, \bar{x}(t); h(t))] dt, \quad \forall h \in L^\infty_\infty[0, T], \quad (6)$$

$$\bar{u}_0 \geq 0, \quad \bar{u}(t) \geq 0, \quad \text{a.e. } t \in [0, T], \quad (7)$$

$$(\bar{u}_0, \bar{u}(t)) \neq 0, \quad \text{a.e. } t \in [0, 1], \quad (8)$$

$$\bar{u}(t) g(t, \bar{x}(t)) = 0, \quad \text{a.e. } t \in [0, T], \quad i \in I. \quad (9)$$

PROOF. If $\bar{x}$ is a local optimal solution to problem (CNP), then by Theorem 3.1,

$$\mathcal{K}(g; \bar{x}) \cap \mathcal{K}(\phi; \bar{x}) = \emptyset.$$

Hence, there is no $h \in L^\infty_\infty[0, T]$ such that

$$\phi^0(\bar{x}; h) < 0,$$

$$g^0(t, \bar{x}(t); h(t)) < 0, \quad \text{a.e. } t \in A(\bar{x}).$$

We can conclude, by making use of Theorem 2.4, that there are $\bar{u}_0 \in \mathbb{R}$, $u \in L^\infty[0, T]$, with $\bar{u}_0 \geq 0$, $u(t) \geq 0$ a.e. $t \in [0, T]$, not all identically zero, such that

$$0 \leq \bar{u}_0 \phi^0(\bar{x}; h) + \int_{A(\bar{x})} u(t) g^0(t, \bar{x}(t); h(t)) dt, \quad \forall h \in L^\infty_\infty[0, T].$$

Setting $\bar{u}(t) = u(t)$, if $t \in A(\bar{x})$ and $\bar{u}(t) = 0$, otherwise, we obtain

$$0 \leq \bar{u}_0 \phi^0(\bar{x}; h) + \int_0^T \bar{u}(t) g^0(t, \bar{x}(t); h(t)) dt$$

$$\leq \int_0^T \{ \bar{u}_0 f^0(t, \bar{x}(t); h(t)) + \bar{u}(t) g^0(t, \bar{x}(t); h(t)) \} dt,$$

for all $h \in L_\infty^n[0, T]$. (Fatou's Lemma is used in the last inequality.) Thus, (6) is proved. The remaining assertions of Lemma 4.2 follow immediately. ■

Let $\bar{x}$ be an optimal solution to (CNP). It follows from Lemma 4.2 that there exist $\bar{u}_0 \in \mathbb{R}$ and $\bar{u} \in L^\infty[0, T]$, satisfying (6)–(9).

It remains to prove assertion (2) to conclude the proof of the theorem. Statement (6) can be rewritten as follows:

$$\begin{aligned} 0 &\leq \int_0^T \{ \bar{u}_0 \sigma_{\partial_x f(t, \bar{x}(t))}(h(t)) + \bar{u}(t) \sigma_{\partial_x g(t, \bar{x}(t))}(h(t)) \} dt, \\ &= \int_0^T [\sigma_{\{ \bar{u}_0 \partial_x f(t, \bar{x}(t)) + \bar{u}(t) \partial_x g(t, \bar{x}(t)) \}}(h(t))] dt, \end{aligned}$$

for all $h \in L_\infty^n[0, T]$ (the equality above follows from Proposition 2.2). Since the above inequality holds for all $h \in L_\infty^n[0, T]$, it holds, in particular, for constant functions $h(t) = v \in \mathbb{R}^n, \forall t \in [0, T]$.

It can be easily verified that the multifunction

$$t \rightarrow \bar{u}_0 \partial_x f(t, \bar{x}(t)) + \bar{u}(t) \partial_x g(t, \bar{x}(t))$$

is integrably bounded and compact-valued. By Theorem 2.3, we have

$$\begin{aligned} 0 &\leq \int_0^T [\sigma_{\{ \bar{u}_0 \partial_x f(t, \bar{x}(t)) + \bar{u}(t) \partial_x g(t, \bar{x}(t)) \}}(v)] dt \\ &= \sigma_{\int_0^T [\bar{u}_0 \partial_x f(t, \bar{x}(t)) + \bar{u}(t) \partial_x g(t, \bar{x}(t))] dt}(v). \end{aligned}$$

But, by Proposition 2.1, this is equivalent to

$$0 \in \int_0^T [\bar{u}_0 \partial_x f(t, \bar{x}(t)) + \bar{u}(t) \partial_x g(t, \bar{x}(t))] dt,$$

which finishes the proof of the theorem under the interim hypothesis.

REMOVAL OF THE INTERIM HYPOTHESIS. Suppose (CNP) has m constraints $g_i(t, x(t)) \leq 0$ a.e. $t \in [0, T]$, and $\bar{x}$ is a local optimal solution. Reduce the m constraints of (CNP) to just one by defining

$$g(t, x(t)) = \max_{1 \leq i \leq m} g_i(t, x(t)), \quad \text{a.e. } t \in [0, T].$$

The point $\bar{x}$ is also an optimal solution of the modified problem. Let $I(t, x) := \{i \in I : g_i(t, x(t)) = g(t, x(t))\}$. From what has been proved under the interim hypothesis, there exist $\bar{u}_0 \in \mathbb{R}$, $u \in L^\infty[0, T]$, satisfying

$$0 \in \int_0^T [\bar{u}_0 \partial_x f(t, \bar{x}(t)) + u(t) \partial_x g(t, \bar{x}(t))] dt \quad (10)$$

and (7)–(9). It can be deduced from (10) and the definition of integration of multifunctions that there exists a measurable function $e(t) \in \partial_x g(t, \bar{x}(t))$ a.e. $t \in [0, T]$ such that

$$0 \in \int_0^T [\bar{u}_0 \partial_x f(t, \bar{x}(t)) + u(t)e(t)] dt. \quad (11)$$

We have the following lemma.

LEMMA 4.3. *There exists $v \in L_\infty^m[0, T]$, $v(t) \geq 0$ a.e. $t \in [0, T]$, satisfying*

1. $v_i(t) = 0$ whenever $g_i(t, \bar{x}(t)) \neq g(t, \bar{x}(t))$, $i = 1, \dots, m$;
2. $\sum_{i=1}^m v_i(t) = 1$ a.e. $t \in [0, T]$;
3. $e(t) \in \sum_{i=1}^m v_i(t) \partial_x g_i(t, \bar{x}(t))$ a.e. $t \in [0, T]$.

PROOF. For each t where $\partial_x g(t, \bar{x}(t))$ is well defined, it follows from [23] that

$$\partial_x g(t, \bar{x}(t)) \subset \text{co}\{\partial_x g_i(t, \bar{x}(t)) : i \in I(t, \bar{x})\}.$$

Since $e(t) \in \partial_x g(t, \bar{x}(t))$ a.e. $t \in [0, T]$, we obtain

$$e(t) \in \text{co}\{\partial_x g_i(t, \bar{x}(t)) : i \in I(t, \bar{x})\}.$$

Define

$$V(t) := \left\{ (v_1, \dots, v_m) \in \mathbb{R}^m : \sum_{i=1}^m v_i = 1, v_i \geq 0, \right. \\ \left. v = 0 \text{ if } g_i(t, \bar{x}(t)) < g(t, \bar{x}(t)), e(t) \in \sum_{i=1}^m v_i \partial_x g_i(t, \bar{x}(t)) \right\}.$$

The set $V(t)$ is obviously nonempty and closed a.e., and V is a measurable set-valued function defined a.e. on $[0, T]$. It follows from standard measurable selection theorems (see, e.g., [23]) that we can choose measurable functions $v_1(t), \dots, v_m(t)$ defined on $[0, T]$ such that $(v_1(t), \dots, v_m(t)) \in V(t)$ a.e. $t \in [0, T]$. The proof of the lemma follows immediately. ■

Now defining $\bar{u}_i(t) := u(t)v_i(t)$, it follows easily from Lemma 4.3 and (11) that assertions (2)–(5) of Theorem 4.1 are valid. ■

In Theorem 4.1, if $f(t, \cdot)$ and $g_i(t, \cdot)$ are Clarke regular, then condition (2) can be replaced by

$$0 \in \partial_x L(\bar{x}, \bar{u}_0, \bar{u}),$$

where

$$L(x, u_0, u) := \int_0^T \left[u_0 f(t, x(t)) + \sum_{j=1}^m u_j(t) g_j(t, x(t)) \right] dt.$$

5. KARUSH-KUHN-TUCKER TYPE OPTIMALITY CONDITIONS

In the necessary conditions, proved in the previous section, there is no guarantee that the Lagrange multiplier associated with the objective function will be nonzero. It is usual to assume some kind of regularity condition on the restrictions of the problem to make sure that the multiplier is in fact nonzero. These regularity conditions are usually referred to as constraint qualifications. We assume the following natural constraint qualification:

$$\bigcap_{i \in I} \mathcal{K}(g_i, \bar{x}) \neq \emptyset. \quad (12)$$

We now state and prove the following Karush-Kuhn-Tucker type theorem.

THEOREM 5.1. *Let $\bar{x} \in \mathbb{F}$ and suppose the constraint qualification (12) is satisfied. If $\bar{x}$ is a local minimum of problem (CNP), then there exist $\bar{u}_i \in L^\infty[0, T]$, $i \in I$, such that*

$$0 \in \int_0^T \left[\partial_x f(t, \bar{x}(t)) + \sum_{i \in I} \bar{u}_i(t) \partial_x g_i(t, \bar{x}(t)) \right] dt, \quad (13)$$

$$\bar{u}_i(t) \geq 0, \quad \text{a.e. } t \in [0, T], \quad i \in I, \quad (14)$$

$$\bar{u}_i(t) g_i(t, \bar{x}(t)) = 0, \quad \text{a.e. } t \in [0, T], \quad i \in I. \quad (15)$$

PROOF. We first prove Theorem 5.1 under the *Interim Hypothesis*: (CNP) has only one constraint $g(t, x(t)) \leq 0$ a.e. $t \in [0, T]$.

If $\bar{x}$ is a local optimal solution to problem (CNP), then by Lemma 4.2, there exist $\bar{u}_0 \in \mathbb{R}$, $\bar{u}(t) \in L^\infty[0, T]$, such that (6)–(9) hold true. If $\bar{u}_0 = 0$, then (6) would reduce to

$$0 \leq \int_0^T u(t)g^0(t, \bar{x}(t); h(t)) dt, \quad \forall h \in L_\infty^n[0, T].$$

Hence, by the Generalized Gordan Theorem, there is no $h \in X$ such that

$$g^0(t, \bar{x}(t); h(t)) < 0, \quad \text{a.e. } t \in [0, T],$$

contradicting the constraint qualification (12). So, $\bar{u}_0 \neq 0$. Set

$$\bar{u}_0 = \frac{\bar{u}_0}{\bar{u}_0}, \quad \bar{u}(t) = \frac{\bar{u}(t)}{\bar{u}_0},$$

and the theorem follows from the inequality

$$0 \leq \int_0^T [f^0(t, \bar{x}(t); h(t)) + \bar{u}(t)g^0(t, \bar{x}(t); h(t))] dt, \quad \forall h \in L_\infty^n[0, T],$$

by using arguments similar to those in the proof of condition (2) of Theorem 4.1.

REMOVAL OF THE INTERIM HYPOTHESIS. Let $g(t, x(t)) := \max\{g_i(t, x(t)) : i \in I\}$. We need the following technical result.

LEMMA 5.2. *The constraint qualification (12) for m constraints implies $K(g, \bar{x}) \neq \emptyset$.*

PROOF. It follows from the measurable selection theorem [23], by using standard arguments, that there exists $\xi \in L_\infty^n[0, T]$ such that $\xi(t) \in \partial_x g(t, \bar{x}(t))$ and

$$g^0(t, \bar{x}(t); h(t)) = \sum_{i=1}^n \xi_i(t)h_i(t), \quad \text{a.e. } t \in [0, T], \quad \forall h \in L_\infty^n[0, T].$$

Let $h \in L_\infty^n[0, T]$ be given but arbitrary. An application of Lemma 4.3 and Proposition 2.1 implies that there exists an essentially bounded function $u(t) \in V(t)$ a.e. (V as defined in the proof of Lemma 4.3) such that

$$\begin{aligned} \xi(t) &\in \sum_{i \in I} u_i(t) \partial_x g_i(t, \bar{x}(t)), \\ g^0(t, \bar{x}(t); h(t)) &\leq \sigma_{\sum_{i \in I} u_i(t) \partial_x g_i(t, \bar{x}(t))}(h(t)), \end{aligned}$$

a.e. $t \in [0, T]$. It follows from the above inequality and Proposition 2.2 that

$$\begin{aligned} g^0(t, \bar{x}(t); h(t)) &\leq \sum_{i \in I} u_i(t) \sigma_{\partial_x g_i(t, \bar{x}(t))}(h(t)), \quad \text{a.e. } t \in [0, T], \\ &= \sum_{i \in I} u_i(t) g_i^0(t, \bar{x}(t); h(t)), \quad \text{a.e. } t \in [0, T]. \end{aligned}$$

Therefore,

$$\bigcap_{i \in I} K(g_i, \bar{x}) \neq \emptyset \Rightarrow K(g, \bar{x}) \neq \emptyset.$$

Now, if $\bar{x}$ is optimal for (CNP) with m constraints, then it is also optimal for (CNP) with the constraint $g(t, x(t)) \leq 0$ a.e. $t \in [0, T]$. It follows from Lemma 5.2 and the theorem in question as proved so far, that there exists $u \in L^\infty[0, T]$, $u \geq 0$ such that

- (i) $u(t)g(t, \bar{x}(t)) = 0$ a.e. $t \in [0, T]$;
- (ii) $0 \in \int_0^T \partial_x f(t, \bar{x}(t)) dt + \int_0^T u(t) \partial_x g(t, \bar{x}(t)) dt$.

Arguments similar to those in the proof of Theorem 4.2 yield the desired result. ■

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