UNIVERSIDADE ESTADUAL PAULISTA
“JÚLIO DE MESQUITA FILHO”
Campus de São José do Rio Preto

Gino Gustavo Maqui Huamán

Interval Analysis and Applications

PhD. Thesis

São José do Rio Preto - Brazil

2018

Gino Gustavo Maqui Huamán

Interval Analysis and Applications

Tese apresentada como parte dos requisitos para obtenção do título de Doutor em Matemática, junto ao Programa de Pós-Graduação em Matemática, do Instituto de Biociências, Letras e Ciências Exatas da Universidade Estadual Paulista "Júlio de Mesquita Filho", Câmpus de São José do Rio Preto.

Financiadora: CAPES

Orientador: Prof. Dr. Geraldo Nunes Silva

São José do Rio Preto
2018

Huamán, Gino Gustavo Maqui.
Interval Analysis and Applications / Gino Gustavo Maqui Huamán. -- São José do Rio Preto, 2018
82 f. : il.

Orientador: Geraldo Nunes Silva
Tese (doutorado) – Universidade Estadual Paulista “Júlio de Mesquita Filho”, Instituto de Biociências, Letras e Ciências Exatas

1. Matemática. 2. Otimização matemática. 3. Teoria do controle. 4. Análise de intervalos (Matemática) I. Universidade Estadual Paulista "Júlio de Mesquita Filho". Instituto de Biociências, Letras e Ciências Exatas. III. Título.

CDU – 518.734

Ficha catalográfica elaborada pela Biblioteca do IBILCE
UNESP - Câmpus de São José do Rio Preto

Gino Gustavo Maqui Huamán

Interval Analysis and Applications

Tese apresentada como parte dos requisitos para obtenção do título de Doutor em Matemática, junto ao Programa de Pós-Graduação em Matemática, do Instituto de Biociências, Letras e Ciências Exatas da Universidade Estadual Paulista “Júlio de Mesquita Filho”, Câmpus de São José do Rio Preto.

Financiadora: CAPES

Comissão Examinadora

Prof. Dr. Geraldo Nunes Silva
UNESP – Câmpus de São José do Rio Preto
Orientador

Prof. Dr. Weldon A. Lodwick
CU Denver– USA

Prof. Dr. Yurilev Chalco Cano
UTA – Arica - Chile

Profa. Dra. Ulcilea Alves Severino Leal
UFTM – Câmpus de Iturama

Prof. Dr. Valeriano Antunes de Oliveira
UNESP – Câmpus de São José do Rio Preto

São José do Rio Preto
02 de abril de 2018

To my beloved parents, Benedicta and Julián,
to my sisters, Luz Marina and Carmen Rosa,
to all my family and friends,
in special to my sister Marcia *(in memory)*
I dedicate.

Acknowledgments

To my parents Benedicta and Julián for all the love they gave me, without their teachings, nothing of what I got would be reality, my sisters Luz Marina and Carmen Rosa who supported me unconditionally in every decision of my life, my brother-in-law Hernan who in silence became a fundamental part of my formation, to my nephews Heisell, Hernan Rodrigo, Mariell, Pilar del Carmen, Franccesco and Jacob who were sources of unconditional motivation for me to be able to improve despite the adversities. My dear family, I thank you in a very special way for all the family support you gave me to reach my goals.

I would also like to thank in a special way all those who contributed to make this dream of becoming a doctor come true. To Yurilev Chalco for the recommendation to study in Brazil and for the teachings, to Geraldo Nunes for being my advisor during the master's and doctorate, to Weldon Lodwick for all his teachings. To UNESP for the opportunity, to all the teachers of the Postgraduate Program of Mathematics of IBILCE and CAPES for the scholarship.

And how not to thank all those people who directly or indirectly supported me during this long period, among which I would like to mention: my fellow students, Ulcilea, José Renato, Tiago, Jairo, Daniel, Junior, Willian, Matheus, Gustavo, Cheienne, Paula, Paola, Yen, Mariana, Luana, Eduardo, Gislaine, Ismael, Guto, Jéssica, Rodrigo and Diego. To my housemates, Rodiak and Tatiane. To my friends from Rio Preto, Stefani, Nathalia, Heloísa, Ana Livia, Eliton, Otávio, Bruno and Fabrício.

"Dream it.

Believe it.

Achieve it."

Anonymous

Resumo

Esta Tese trabalha com alguns conceitos fundamentais da analise intervalar e suas aplicações. Em primeiro lugar, a Tese aborda a álgebra de funções de valor intervalar gH-diferenciáveis. Especificamente, damos condições para a gH- diferenciabilidade da soma e gH-diferença de duas funções de valor intervalar gH-diferenciáveis; também para o produto e composição de uma função real diferenciável e uma função de valor intervalar gH-diferenciável. Em segundo lugar, a Tese e dedicada a obtenção de condições necessárias e suficientes para problemas de otimização com funções objetivas de valor intervalar. Essas funções objetivas são obtidas a partir de funções contínuas usando aritmética intervalar restrita. Damos um conceito de derivada para esta classe de funções de valor intervalar e, em seguida, introduzimos o conceito de ponto estacionário. Encontramos as condições necessárias com base na definição dos pontos estacionários e provamos que essas condições também são suficientes nas noções de convexidade generalizada. Obtemos também condições necessárias e suficientes para o problema de otimização intervalar com restrições. E, finalmente, lidamos com o espaço quociente de intervalos $\mathbb{I}$ em relação a família de intervalos simétricos e dado um conceito de diferenciabilidade para funções de classes de equivalência, fazemos uma comparação com outros conceitos de diferenciabilidade. Alguns exemplos e contraexemplos ilustram os resultados obtidos.

Palavras-chave: Aritmética intervalar standard. gH-derivada. Aritmética intervalar restrita. Otimização intervalar. Espaço quociente de intervalos.

Abstract

This Thesis works with some fundamentals concepts of interval analysis and it applications. First of all, the thesis deals with the algebra of gH-differentiable interval-valued functions. Specifically, we give conditions for the gH-differentiability of the sum and gH-difference of two gH-differentiable interval-valued functions; also for the product and composition of a differentiable real function and a gH-differentiable interval-valued function. Second, the thesis is devoted to obtaining necessary and sufficient conditions for optimization problems with interval-valued objective functions. These objective functions are obtained from continuous functions by using constrained interval arithmetic. We give a concept of derivative for this class of interval-valued functions and then we introduce the concept of stationary point. We find necessary conditions based on the stationary points definition and we prove that these conditions are also sufficient under generalized convexity notions. We obtain the necessary and sufficient conditions for constrained interval-valued optimization problem. And finally, we deal with the quotient space of intervals $\mathbb{I}$ with respect to the family of symmetric intervals and given a concept of differentiability for equivalence classes-valued functions, we make a comparison with other concepts of differentiability. Some examples and counterexamples illustrates the obtained results.

Keywords: Standard interval arithmetic. gH-derivative. Constraint interval arithmetic. Interval optimization. Quotient space of intervals.

Contents

List of Figures

List of Symbols

$\mathbb{R}$:	set of real numbers;
$\mathbb{I}$:	proper interval space;
$\bar{\mathbb{I}}$:	improper interval space;
SIA:	standard interval arithmetic;
CIA:	constrained interval arithmetic;
$\underline{a}$:	lower end point of the interval A;
$\overline{a}$:	upper end point of the interval A;
w_a:	width of the interval A;
$+_s$:	standard interval addition operator;
$-_s$:	standard interval subtraction operator;
$\times_s$:	standard interval multiplication operator;
$\div_s$:	standard interval division operator;
$\ominus_{gh}$:	generalized Hukuhara subtraction operator;
$H(A, B)$:	Pompeu-Hausdorff metric;
A^I:	real single-valued function;
$\underline{f}$:	lower end point function of F $(F(x) = [\underline{f}(x), \overline{f}(x)])$
$\overline{f}$:	upper end point function of F $(F(x) = [\underline{f}(x), \overline{f}(x)])$
$\preceq_{LU}$:	lower-upper partial order relation;
$\mathcal{C}^k$:	k-tuple of k intervals;
$SP(F_{\mathcal{C}^k})$:	set of all stationary points of $F_{\mathcal{C}^k}$;
$\sim$:	equivalence relation;
$\langle\cdot\rangle$:	equivalence class;
$\mathcal{S}$:	class of symmetric intervals;
$\mathbb{I}/\mathcal{S}$:	set of equivalence classes.

Chapter 1

Introduction

Parameters associated with optimization and control models are not always precise (see, e.g., [10, 11, 17]). In most situations they take on some kind of uncertainty in the parameters and/or in the variables, uncertainties that are inherent to the associated real problem [28]. In particular in this Thesis, we are interested in studying interval-type uncertainties and the associated optimization or control problems will be called interval optimization problems, as well as interval control problems. If we intend to obtain optimality conditions for an interval optimization problem, it is important to consider some type of interval structure to work on. In recent years standard interval arithmetic together with gH-difference ([30, 35, 14] and [15]), turned out to be of vital importance in new developments of interval structures, not only to the studies of interval analysis, but also to fuzzy analysis [7, 17]. In the literature, if we look for works related to interval or fuzzy optimization, we will find studies related to the mathematical structure derived from standard interval arithmetic (SIA) and gH-difference [21, 22, 26, 39, 40]. In addition, we can find a great number of papers where necessary and sufficient conditions of optimality have been obtained [13, 31] for the unconstrained problems. This is basically due to the fact that basic and fundamental tools, such as the algebra of gH-differentiable interval-valued functions, were little studied. In this sense, the following questions arise. 1) Is it possible that the standard interval arithmetic together with the gH-difference gives a structure with appropriate properties to study optimization and control models involving interval uncertainty? 2) Is it possible to obtain some kind of optimality conditions for interval optimization and interval control problems, involving some mathematical structure that works with an interval arithmetic which differs from the standard one? 3) Can we create some kind of interval structure with appropriate properties (vector space structure)?

These questions in light of the things that have previously been studied in the literature, we can see that:

Regarding the first question, we must see carefully the works done by: B. Bede and L. Stefanini - Generalized Hukuhara differentiability of interval-valued functions and interval differential equations [35], Allahviranloo - Note on Generalized Hukuhara differentiability of interval-valued functions and interval differential equations [2], Chalco -Cano et.al. - Generalized and π derivative for set valued functions [14], and other important works such as [29, 4, 5, 30], since they provide important tools for the development of theories in optimization and control in both the interval context and the fuzzy context (see [21, 39, 40, 13, 9, 31, 1] and [24]). Despite of these studies and given the complexity of the interval structures it is of utmost importance to provide a new algebra of gH-differentiable functions. For example, considering the work of Stefanini and Bede [35], Remark 19 was corrected by Chalco-Cano in [15] and this would imply the need to correct Proposition 24 in [35]. Although Allahviranloo [2] tried to correct it, we still have to work on the other cases where the functions are $(iii)gH$ and $(iv)gH$ differentiable. In fact, there is no study done regarding the other operations with gH-differentiable functions and the aforementioned was only done for the addition of two gH-differentiable functions and the product of a gH-differentiable function by a scalar. In this sense, we will present a complete study on the algebra of gH-differentiable functions, starting from the operation of addition, passing through the gH-difference, the product, the chain rule, among other operations, and considering the four cases of gH-differentiability.

Regarding the second question, we note that there are very few studies that deal with interval optimization problems with a different structure than that determined by standard interval arithmetic and gH-difference. Among these works we would like to mention, for example, Campos, et al [10], where the authors use the single level constrained interval arithmetic, proposed by Chalco-Cano et.al. [12] to address interval control problems, as well as the Thesis of Ulcilea Alves Severino Leal [23], where she works on interval uncertainty in optimization and control. We point out once again that in order to approach an interval optimization problem, for example, it is necessary to make use of an interval aritimetic structure which is well defined and has a rich set of properties. In this Thesis we will choose the constrained interval arithmetic proposed by Weldon A. Lodwick

[25] to answer this second question. We define some important concepts, such as order relation to establish some criterion for minimization or maximization and of derivative to characterize optimal points. That is, we provide necessary optimality conditions for the unconstrained interval optimization problem. We also establish sufficient conditions for this unconstrained problem under interval invexity assumption concept. For the constrained interval optimization problem, we will give Fritz-John type conditions and then, under constraint qualifications, the Karush-Kuhn-Tucker conditions. All these results are in an interval version involving the constrained interval arithmetic.

An important feature of this work is the practicality of how we can characterize and find the optimal points. This fact is of vital importance for the consolidation of interval optimization as an area because that will help future research resulting from it.

Lastly, considering the third question, it is no doubt interesting to pursue an algebraic interval structure that has a vector space structure [27]. It is, of course, not natural to find such a structure working directly with interval arithmetic operations, but, as it will be shown later, this can be achieved by introducing appropriate interval equivalence classes to approach to this problem. In order to make analysis on the quotient space of intervals we first define a metric on this space. Once the limit is defined, it is possible to define the concept of derivatives and integrals for functions defined over this space. After that, it is interesting to see how this new derivative is related to the gH-derivative, in the sense that the classes generated by the gH-derivatives have some kind of relation to the derivative functions defined over the interval quotient space.

We now set forth how this Thesis is developed. Indeed it is divided into three separate parts whose link is the interval analysis. The first part focuses on the algebraic study of generalized Hukuhara differentiable interval-valued functions, in order to understand the behavior of this type of functions. This is developed in Chapter 3. The second part, Chapter 4, focuses on the study of the necessary and sufficient conditions for interval optimization problems involving constrained interval arithmetic to find necessary and sufficient optimality conditions for interval optimization problems, both unconstrained and constrained. Finally, the third part, appearing in Chapter 5, will discuss without losing an isomorphic copy of the intervals the quotient space of intervals in which we are able to find a structure of vector space which makes it possible to define the concepts of derivative and integral, and to analyze its relationship with what has already been done.

Chapter 2

Preliminaries of interval analysis

This chapter we studies some preliminaries of the interval analysis, which includes the spaces in which we work, the arithmetic operations most used and their properties. Next, we define some types of functions where the domain is or not interval, and where the range is an interval or an interval $n-$tuple. In addition some properties and some geometric notions of these functions is presented. Finally, some topological properties of the interval space involving the arithmetic and the functions is developed.

2.1 Interval Space

Let A denote a closed and bounded interval. The endpoints of A are denoted by $\underline{a}$ and $\overline{a}$, where $\underline{a} \leq \overline{a}$ ($A = [\underline{a}, \overline{a}]$) and $\mathbb{I}$ denotes the family of all closed and bounded intervals, i.e.

$$\mathbb{I} = \left\{ [\underline{a}, \overline{a}] \ : \ \underline{a} \leq \overline{a}, \ \forall \underline{a}, \overline{a} \in \mathbb{R} \right\}.$$

In a natural way to represent intervals is by an ordered pairs, i.e., if $A = [\underline{a}, \overline{a}] \in \mathbb{I}$ then $(\underline{a}, \overline{a}) \in \mathbb{R}^2$. In this case, this respresentation of the space $\mathbb{I}$ will be a particular case of the interval space presented by Tiago Mendonça in [16], where $\mathbb{I}$ it is called proper interval space. In Tiago's work an extension of the proper interval space was studied, for expressions like $[\overline{a}, \underline{a}]$, with $\overline{a} \leq \underline{a}$. The family of these expressions was called improper interval space and it is denoted by $\overline{\mathbb{I}}$, i.e.,

$$\overline{\mathbb{I}} = \{ [\overline{a}, \underline{a}] : [\underline{a}, \overline{a}] \in \mathbb{I} \}$$

It is clear that with this extension there exist an isomorphism between $\mathbb{I} \cup \overline{\mathbb{I}}$ and $\mathbb{R}^2$, as shown in the following figure.

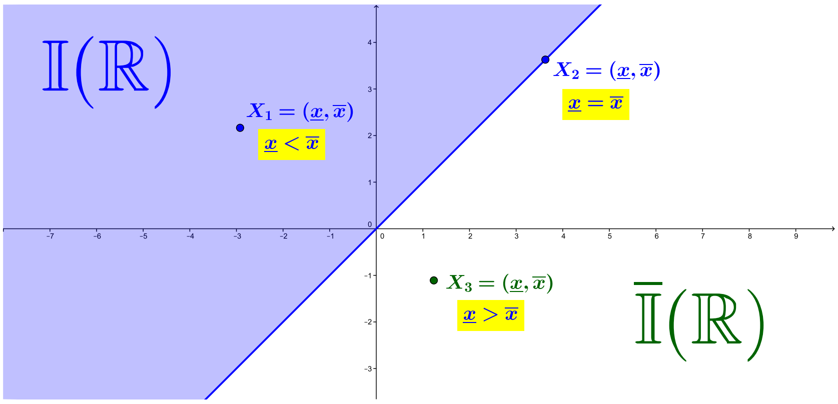

Figure 2.1: Isomorphism between $\mathbb{I} \cup \overline{\mathbb{I}}$ and $\mathbb{R}^2$

In this Thesis, we only study different results over the proper interval space, which will be called interval space.

Considering the interval space, we will define in the next sections two arithmetic structures. Then we will analyze its properties, which will be used in the main results of this research. We next will present two interval arithmetic that will be used in this Thesis.

2.2 Standard Interval Arithmetic

This arithmetic sometimes known as "Minkowski operations" [18], was strongly studied by Moore since 1959. Moore in [30] proposes the use the Minkowsky arithmetic operations but not using all the points of the interval, but only their ends to define the arithmetic operations. Next we define these operations using the Moore notation.

Given the intervals $A, B, C \in \mathbb{I}$ where $A = [\underline{a}, \overline{a}]$, $B = [\underline{b}, \overline{b}]$, $C = [\underline{c}, \overline{c}]$, and $\lambda \in \mathbb{R}$ we define the *standard* interval operations as the following, where the operation $*$ in the standard interval arithmetic is denoted $*_s$:

Addition:

$$A +_s B = [\underline{a}, \overline{a}] +_s [\underline{b}, \overline{b}] = [\underline{a} + \underline{b}, \overline{a} + \overline{b}]. \tag{2.1}$$

Multiplication by a scalar

$$\beta \cdot_s A = \beta \cdot_s [\underline{a}, \overline{a}] = \begin{cases} [\beta\underline{a}, \beta\overline{a}] & \text{if} \quad \beta \geq 0; \\ [\beta\overline{a}, \beta\underline{a}] & \text{if} \quad \beta < 0. \end{cases} \tag{2.2}$$

From this, if $\beta = -1$ we obtain, $(-1) \cdot_s A = [-\overline{a}, -\underline{a}]$, so that, we can define the subtraction by,

Subtraction:

$$\begin{aligned} A -_s B &= A +_s ((-1) \cdot_s B) \\ &= [\underline{a}, \overline{a}] +_s [-\overline{b}, -\underline{b}] \\ &= [\underline{a} - \overline{b}, \overline{a} - \underline{b}]. \end{aligned}$$

A seriously problem with the definition of subtraction is that we have no additive inverses when $A \in \mathbb{I}$ is nondegenerate, i.e. $A -_s A \neq 0$ if $A = [\underline{a}, \overline{a}]$, $\underline{a} < \overline{a}$.

Multiplication:

$$\begin{aligned} A \times_s B &= [\underline{a}, \overline{a}] \times_s [\underline{b}, \overline{b}] \\ &= [\min G, \max G], \end{aligned}$$

where $G = \{\underline{a} \cdot \underline{b}, \underline{a} \cdot \overline{b}, \overline{a} \cdot \underline{b}, \overline{a} \cdot \overline{b}\}$.

Division: If $0 \notin B$,

$$\begin{aligned} A/_s B &= [\underline{a}, \overline{a}]/_s[\underline{b}, \overline{b}] \\ &= [\min M, \max M], \end{aligned}$$

where $M = \{\underline{a}/\underline{b}, \underline{a}/\overline{b}, \overline{a}/\underline{b}, \overline{a}/\overline{b}\}$. If we consider $A \in \mathbb{I}$ nondegenerated, $0 \notin A$, then $A/_s A \neq [1, 1]$.

With these operations, we obtain the next proposition.

Proposition 2.2.1. *Given $A, B, C \in \mathbb{I}$ and considering the last four operations we obtain the next algebraic properties:*

1. *Commutativity for interval addition and multiplication:*

- $A +_s B = B +_s A$;
- $A \times_s B = B \times_s A$.

2. *Associativity for interval addition and multiplication:*

- $(A +_s B) +_s C = A +_s (B +_s C)$;
- $(A \times_s B) \times_s C = A \times_s (B \times_s C)$.

3. *Additive and multiplicative identity elements: these elements are* $0 = [0, 0]$ *and* $1 = [1, 1]$ *respectively, and both are degenerate elements,*

- $0 +_s A = A +_s 0 = A$;
- $1 \times_s A = A \times_s 1 = A$;
- $0 \times_s A = A \times_s 0 = 0$.

4. *Subdistributivity:*

- $A \times_s (B +_s C) \subseteq A \times_s B +_s A \times_s C$.

5. *Cancellation law:*

- $A +_s C = B +_s C \;\Rightarrow\; A = B$.

Stefanini and Bede proposed an interesting way to obtain a difference that makes sense in the concept of derivatives. The idea is to have a concept of subtraction which has more suitable properties. Stefanini and Bede in [35] have introduced the following difference between two intervals.

Definition 2.2.2 ([35]). *The generalized Hukuhara difference (gH-difference, for short) of two intervals A and B is defined as follows*

$$A \ominus_{gH} B = C \Leftrightarrow \begin{cases} (a) \;\; A = B +_s C, & or \\ (b) \;\; B = A +_s (-1) \cdot_s C. \end{cases}$$

This difference has many interesting new properties, for example $A \ominus_{gH} A = \{0\} = [0, 0]$. Also, the gH-difference of two intervals $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$ always exists and it is equal to (see Proposition 4 in [35])

$$A \ominus_{gH} B = \left[\min\left\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\right\}, \max\left\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\right\}\right].$$

The following properties were obtained in [35].

Proposition 2.2.3. *Let $A, B, C \in \mathbb{I}$, the gH-difference has the following properties.*

1. $A \ominus_{gH} A = 0$;
2. $(A +_s B) \ominus_{gH} B = A$;
3. $A \ominus_{gH} (A -_s B) = B$;
4. $A \ominus_{gH} (A +_s B) = -B$;
5. $-(A \ominus_{gH} B) = (-B) \ominus_{gH} (-A)$;
6. $(A -_s B) +_s B = C \Leftrightarrow A -_s B = C \ominus_{gH} B$;
7. *In general,* $B -_s A = A -_s B$ *does not imply* $A = B$; *but* $(A \ominus_{gH} B) = (B \ominus_{gH} A) = C$ *if and only if* $C = \{0\}$ *and* $A = B$;
8. $A + (B \ominus_{gH} A) = B$ *or* $B -_s (B \ominus_{gH} A) = A$ *and both equalities hold if and only if* $B \ominus_{gH} A$ *is a singleton set.*

Example 2.2.4. *Let* $A = [\frac{1}{2}, \frac{3}{2}],\ B = [-3, 11]$ *and* $C = [1, 5]$.

- $B \times_s B = [-33, 121]$;
- $C -_s C = [-4, 4]$;
- $C \ominus_{gH} C = [0, 0]$;
- $C /_s C = [\frac{1}{5}, 5]$;
- $A \times_s A -_s 2A = [-\frac{11}{4}, \frac{5}{4}]$;
- $A \times_s (A -_s 2) = [-\frac{9}{4}, -\frac{1}{4}]$.

2.3 Constrained interval arithmetic

This section presents constrained interval arithmetic (CIA). Lodwick [25, 26] was primarily concerned with an interval arithmetic that encodes the united extension of [36] in an explicit, direct way. To this end, an interval is redefined into an equivalent form as the real-valued function of one variable and two coefficients or parameters over the compact domain $[0, 1]$.

Definition 2.3.1. *An interval* $A = [\underline{a}, \overline{a}]$ *is the real single-valued function* $A^I(\lambda_a)$*,*

$$A^I(\lambda_a) = (1 - \lambda_a)\underline{a} + \lambda_a \bar{a}, \tag{2.3}$$
$$= w_a\lambda_a + \underline{a}, \quad 0 \leq \lambda_a \leq 1.$$

Here $w_a = \bar{a} - \underline{a} \geq 0$ *is the width of the interval.*

Strictly speaking, in (2.3), since the numbers $\underline{a}$ and $\bar{a}$ (consequently w_a) are known (inputs or data), they are *coefficients*, whereas λ_a is varying, although constrained between 0 and 1. Hence the name "constrained interval arithmetic". This means that $A^I(\lambda_a)$ is a single-valued real function with two coefficients. Moreover, we write λ_a to denote the parameter associated to the interval A. To simplify the notation we will write $\lambda, \lambda_1, \lambda_2, ...$ to denote the parameters associated to each interval. So the constrained parametric representation of an interval A will be (see [9])

$$A = [\underline{a}, \overline{a}] = \{a(\lambda) = w_a\lambda + \underline{a} \;:\; \lambda \in [0,1]\} = \{a(\lambda) = (\overline{a} - \underline{a})\lambda + \underline{a} \;:\; \lambda \in [0,1]\} \tag{2.4}$$

The algebraic operations for CIA are defined as follows. We consider two intervals $A = \{a(\lambda_1) \;:\; \lambda_1 \in [0,1]\}$ and $B = \{b(\lambda_2) \;:\; \lambda_2 \in [0,1]\}$, then

$$\begin{aligned} A * B &= C \\ &= [\underline{c}, \bar{c}] \\ &= \{a(\lambda_1) * b(\lambda_2) \;:\; \lambda_1, \lambda_2 \in [0,1]\} \\ &= \{c \;:\; c = a(\lambda_1) * b(\lambda_2),\ \lambda_1, \lambda_2 \in [0,1]\} \\ \text{where } \underline{c} &= \min\{c\},\ \ \bar{c} = \max\{c\},\ \ 0 \leq \lambda_1 \leq 1, 0 \leq \lambda_2 \leq 1 \\ \text{and } * &\in \{+, -, \times, \div\}. \end{aligned} \tag{2.5}$$

It is clear from (2.5) that constrained interval arithmetic is a constrained global optimization problem.

Remark 2.3.2. *From CIA [25] we know that, for dependent operations, we consider the same constrained parametric representation for the same intervals involved in the algebraic operations, i.e.* $A * A = \{a(\lambda) * a(\lambda) \;:\; \lambda \in [0,1]\}$*, where* $* \in \{+, -, \times, \div\}$.

CIA is the complete implementation of the united extension, and possesses an algebra which has addition desired properties compared to standard interval arithmetic. For instance $A - A = [0,0] = \{0\}$, $A \div A = [1,1] = \{1\}$ when $0 \notin A$ and possess a distributive law $A \times (B + C) = A \times B + A \times C$. The properties of the constrained parametric representation of an interval and CIA see [25, 26], are

- $+$ is associative, in fact, let $A, B, C \in \mathbb{I}$, using their constrained parametric representation,

$$\begin{aligned} (A+B)+C &= \{(A(\lambda_1)+B(\lambda_2))+C(\lambda_3) : \lambda_1, \lambda_2, \lambda_3 \in [0,1]\} \\ &= \{A(\lambda_1)+(B(\lambda_2)+C(\lambda_3)) : \lambda_1, \lambda_2, \lambda_3 \in [0,1]\} \\ &= A+(B+C) \end{aligned}$$

- $+$ has an identity element, this element is $\mathbf{0} = [0,0] = \{0\}, 0 \in \mathbb{R}$. Let $A \in \mathbb{I}$,

$$\begin{aligned} A+\mathbf{0} &= \{A(\lambda_1)+0 : \lambda_1 \in [0,1]\} \\ &= \{A(\lambda_1) : \lambda_1 \in [0,1]\} \\ &= A \end{aligned}$$

- Not all the elements $A \in \mathbb{I}$ have an inverse $B \in \mathbb{I}$, such that $A + B = B + A = \mathbf{0}$, in effect, suppose that all the elements $A \in \mathbb{I}$ has an inverse $B \in \mathbb{I}$ such that $A + B = B + A = \mathbf{0}$, if it will be true,

$$\begin{aligned} A+B &= \{0\} \\ &= \{A(\lambda_1)+B(\lambda_2)=0, \forall \lambda_1 \lambda_2 \in [0,1]\} \end{aligned}$$

  then $B(\lambda_2) = -A(\lambda_1)$, this equality is true only when $A \in \mathbb{I}$ is a degenerated interval, because their parametric representation do not depend of any λ parameter, on the other cases, we will have different type of monotonicity on their parametric representation (if $A(\lambda_1)$ is increasing then $B(\lambda_2)$ is not increasing), and it is absurd. That is, CIA like SIA has no additive inverses. However this true of any Minkowsky-based set arithmetic, where in our case the set are intervals.

- $+$ is commutative. Let $A, B \in \mathbb{I}$, consequently

$$\begin{aligned} A+B &= \{A(\lambda_1)+B(\lambda_2) : \ \lambda_1, \lambda_2 \in [0,1]\} \\ &= \{B(\lambda_2)+A(\lambda_1) : \ \lambda_1, \lambda_2 \in [0,1]\} \\ &= B+A \end{aligned}$$

Next, we will show some interesting examples involving the constrained interval arithmetic.

Example 2.3.3. *Let* $A = [\frac{1}{2}, \frac{3}{2}]$, $B = [-3, 11]$ *and* $C = [1, 5]$.

- $B \times B = [0, 121]$;
- $C - C = [0, 0]$;
- $C/C = [1, 1]$;
- $A \times A - 2A = [-1, -\frac{3}{4}]$;
- $A \times (A - 2) = [-1, -\frac{3}{4}]$

2.4 Interval differentiability

We next present one of the most useful concept, called gH-differentiability, for this concept is important to define an appropriate metric. The metric structure is given usually by the Pompeiu-Hausdorff metric H in $\mathbb{I}$ which is defined by

$$H(A, B) = \max \left\{ |\underline{a} - \underline{b}| \ , \ \left|\overline{a} - \overline{b}\right| \right\}.$$

Where $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$.

For the Pompeiu-Hausdorff metric, the following properties are well-known.

Proposition 2.4.1 ([35])**.** *For* $A, B \in \mathbb{I}$, *we have*

1. $H(kA, kB) = |k|H(A, b), \ \forall k \in \mathbb{R}$;
2. $H(A + C, B + C) = H(A, B)$;
3. $H(A + B, C + D) \leq H(A, C) + H(B, D)$;
4. $H(A, B) = H(A \ominus_{gH} B, \{0\})$.

Proposition 2.4.2 ([4, 5])**.** $(\mathbb{I}, H)$ *is a complete metric space.*

Based on the gH-difference Stefanini and Bede proposed the concept of gH-derivative.

Definition 2.4.3 ([35]). *Let $x_0 \in]a,b[$ and h such that $x_0 + h \in]a,b[$, then the gH-derivative of a function $F :]a,b[\to \mathbb{I}$ at x_0 is defined as*

$$F'(x_0) = \lim_{h \to 0} \frac{1}{h}[F(x_0 + h) \ominus_{gH} F(x_0)]. \tag{2.6}$$

If $F'(x_0) \in \mathbb{I}$ satisfying (2.6) exists, we say that F is generalized Hukuhara differentiable (gH-differentiable for short) at x_0.

The next chapter presents some properties and interesting examples about gH-differentiable functions

Chapter 3

Algebra of generalized Hukuhara differentiable interval-valued functions

It is well known that one of the most recently concepts of interval analysis is the concept of derivative given by Stefanini and Bede [35], and that many other authors used this concept to be worked in different areas, such as, interval calculus, interval optimization, interval differential equations, interval dynamical systems, among many others. But we can also see that there are still results from this definition that have not yet been consolidated. And one of these results is the algebra of gH-differentiable functions, a study that undoubtedly is important in solving problems related to the aforementioned areas and that we will develop next.

Henceforth T will denote an open subset of $\mathbb{R}$. Let $F : T \to \mathbb{I}$ be an interval-valued function with $F(x) = [\underline{f}(x), \overline{f}(x)]$, where $\underline{f}(x) \leq \overline{f}(x)$, $\forall x \in T$. The functions $\underline{f}$ and $\overline{f}$ are called the lower and the upper endpoint functions of F, respectively.

Example 3.0.1. *Let $F : \mathbb{R} \to \mathbb{I}$ be an interval valued function defined by*

$$F(x) = [-|x + 2|, |x - 2|].$$

Here, the endpoints functions are given by:

- $\underline{f}(x) = -|x + 2|$;
- $\overline{f}(x) = |x - 2|$;

and are graphically represented by Figure 3.1.

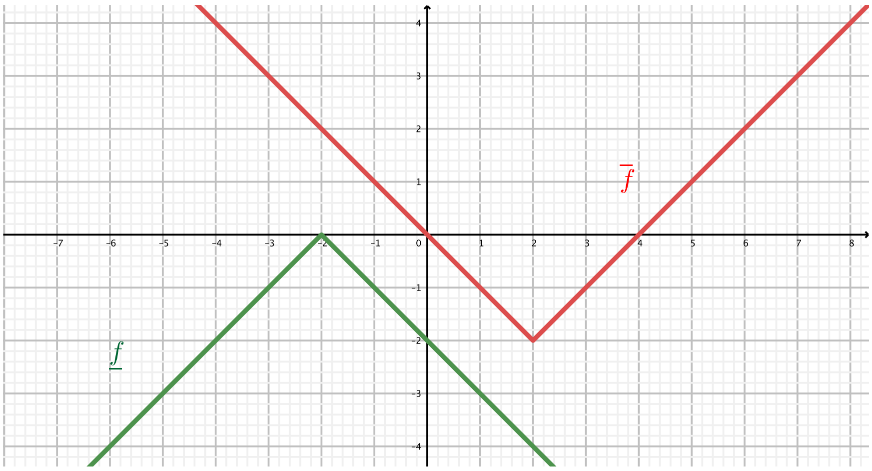

Figure 3.1: Endpoint functions of the interval valued function F.

Chalco-Cano et al. [14] showed that the gH-difference and the subtraction introduced by Markov [29] are equivalent concepts.

Obtaining the gH-derivative of an interval-valued function via (2.6) is a rather complex problem. However, the next result, given in [15], characterizes the gH-differentiability of F in terms of the differentiability of its endpoint functions $\underline{f}$ and $\overline{f}$.

Theorem 3.0.2 ([14]). *Let $F : T \to \mathbb{I}$ be an interval-valued function. Then F is gH-differentiable at $x_0 \in T$ if and only if one of the following cases holds:*

(a) $\underline{f}$ and $\overline{f}$ are differentiable at x_0 and

$$F'(x_0) = \left[\min\left\{(\underline{f})'(x_0), (\overline{f})'(x_0)\right\}, \max\left\{(\underline{f})'(x_0), (\overline{f})'(x_0)\right\}\right].$$

(b) The lateral derivatives $(\underline{f})'_-(x_0)$, $(\underline{f})'_+(x_0)$, $(\overline{f})'_-(x_0)$, $(\overline{f})'_+(x_0)$ exist and satisfy $(\underline{f})'_-(x_0) = (\overline{f})'_+(x_0)$, $(\underline{f})'_+(x_0) = (\overline{f})'_-(x_0)$, and

$$\begin{aligned} F'(x_0) &= \left[\min\left\{(\underline{f})'_-(x_0), (\underline{f})'_+(x_0)\right\}, \max\left\{(\underline{f})'_-(x_0), (\underline{f})'_+(x_0)\right\}\right] \\ &= \left[\min\left\{(\overline{f})'_-(x_0), (\overline{f})'_+(x_0)\right\}, \max\left\{(\overline{f})'_-(x_0), (\overline{f})'_+(x_0)\right\}\right]. \end{aligned}$$

Theorem 3.0.2 distinguishes two cases: one corresponding to (a), which implies the differentiability of the endpoint functions, and the other corresponding to (b), which implies the existence of the lateral derivatives of the endpoint functions. Such distinction will be useful in future results. Because of this reason, we next rewrite the previous theorem in the following way.

Theorem 3.0.3. *Let $F : T \to \mathbb{I}$ an interval-valued function. Then F is gH-differentiable at $x_0 \in T$ if and only if one of following cases hold:*

(i) $\underline{f}$ *and* $\overline{f}$ *are differentiable at* t_0 *and* $F'(x_0) = \left[\underline{f}'(x_0), \overline{f}'(x_0)\right]$.

(ii) $\underline{f}$ *and* $\overline{f}$ *are differentiable at* x_0 *and* $F'(x_0) = \left[\overline{f}'(x_0), \underline{f}'(x_0)\right]$.

(iii) $\underline{f}'_-(x_0), \underline{f}'_+(x_0), \overline{f}'_-(x_0)$, $\overline{f}'_+(x_0)$ *exist and satisfy* $\underline{f}'_-(x_0) = \overline{f}'_+(x_0)$, $\underline{f}'_+(x_0) = \overline{f}'_-(x_0)$, *and* $F'(x_0) = \left[\underline{f}'_-(x_0), \underline{f}'_+(x_0)\right]$.

(iv) $\underline{f}'_-(x_0), \underline{f}'_+(x_0), \overline{f}'_-(x_0)$, $\overline{f}'_+(x_0)$ *exist and satisfy* $\underline{f}'_-(x_0) = \overline{f}'_+(x_0)$, $\underline{f}'_+(x_0) = \overline{f}'_-(x_0)$, *and* $F'(x_0) = \left[\underline{f}'_+(x_0), \underline{f}'_-(x_0)\right]$.

Theorem 3.0.3 distinguishes four cases. We say that an interval-valued function $F : T \to \mathbb{I}$ is $(k)gH$-differentiable if case k in Theorem 3.0.3 holds for $k \in \{i, ii, iii, iv\}$.

Example 3.0.4. *In this example we show the four cases of* $(k)gH$*-diffrenetiability.*

(k=i) *Let $F : \mathbb{R} \to \mathbb{I}$ be an interval valued function, defined by*

$$F(x) = [-2, e^x].$$

It is clear that $F'(x) = [0, e^x]$, *consequently F is $(i)gH$-differentiable at $x \in \mathbb{R}$. Graphically this is show in Figure 3.2.*

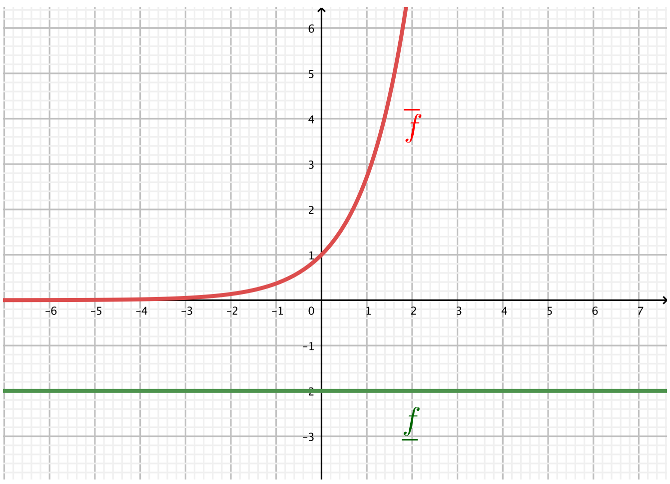

Figure 3.2: Interval valued function $(i)gH$-differentiable at $x \in \mathbb{R}$

(k=ii) *Let $F : \mathbb{R} \to \mathbb{I}$ be an interval valued function, defined by*

$$F(x) = [-2, e^{-x}].$$

It is clear that $F'(x) = [-e^{-x}, 0]$, *consequently* F *is* $(ii)gH$*-differentiable at* $x \in \mathbb{R}$. *Graphically, this is show in Figure 3.3.*

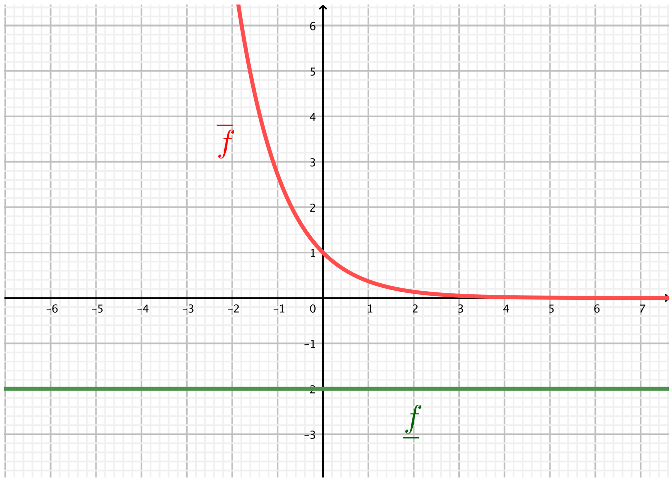

Figure 3.3: Interval valued function $(ii)gH$-differentiable at $x \in \mathbb{R}$

(k=iii) *Let* $F : (-3, 3) \to \mathbb{I}$ *be an interval valued function, defined by*

$$F(x) = [-|x| + 4, |x| - 4].$$

It is clear that $\underline{f}'_-(0) = \overline{f}'_+(0) = -1$, $\underline{f}'_+(0) = \overline{f}'_-(0) = 1$, *and* $F'(0) = [-1, 1]$. *Consequently* F *is* $(iii)gH$*-differentiable at* $x = 0$. *Graphically, this is show in Figure 3.4.*

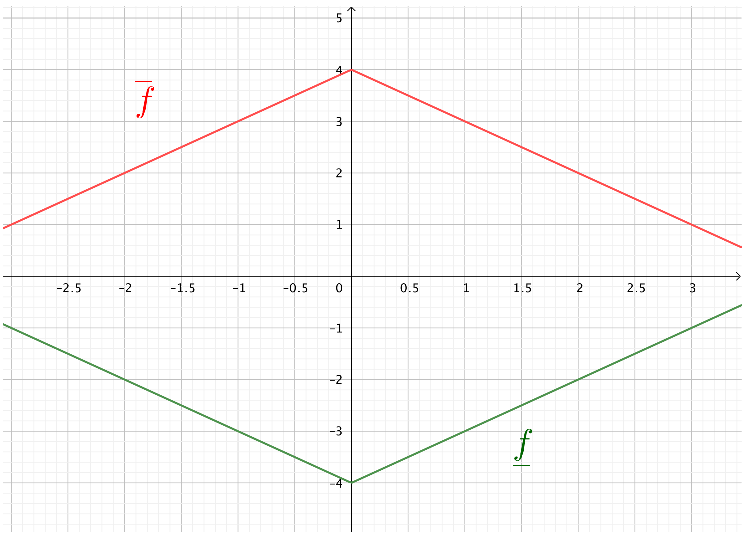

Figure 3.4: Interval valued function $(iii)gH$-differentiable at $x = 0$

(k=iv) *Let $F : \mathbb{R} \to \mathbb{I}$ be an interval valued function, defined by*

$$F(x) = [-|x|, |x|].$$

It is clear that $\underline{f}'_-(0) = \overline{f}'_+(0) = 1$, $\underline{f}'_+(0) = \overline{f}'_-(0) = -1$, and $F'(0) = [-1, 1]$. Consequently F is $(iv)gH$-differentiable at $x = 0$. Graphically, this is show in Figure 3.5.

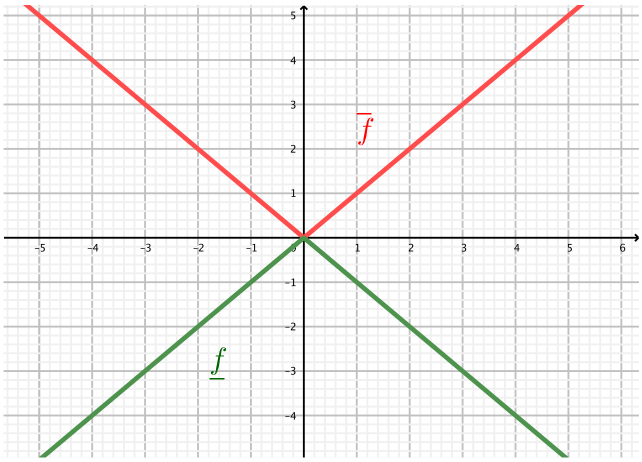

Figure 3.5: Interval valued function $(iv)gH$-differentiable at $x = 0$

Note that if $F : T \to \mathbb{I}$ is gH-differentiable at x_0 in more than one case then F is gH-differentiable at x_0 in all four cases and $F'(x_0)$ is a trivial interval or singleton, i.e. $F'(x_0) = \{a\} = [a, a]$, for some $a \in \mathbb{R}$. To show this, we will present the following example.

Example 3.0.5. *Let us consider two interval valued functions $G, F : \mathbb{R} \to \mathbb{I}$, defined by*

$$G(x) = [-x^2, x^2 + 1],$$

and

$$F(x) = [-(x-2)^2 + 2, x^2 + 1].$$

From gH-differentiable definition, we obtain that both functions are four cases gH-differentiable for some x_0. i.e. :

- *G is four cases gH-differentiable at $x_0 = 0$, and $G'(0) = [0, 0]$. Graphically, this is show in* Figure 3.6.

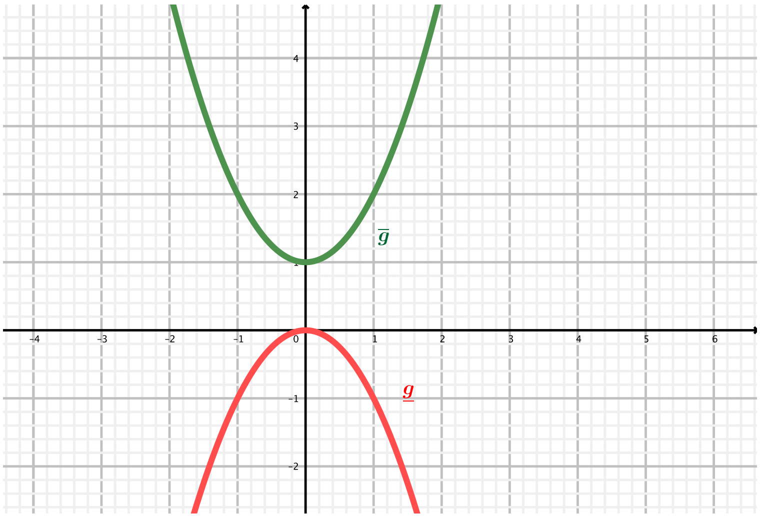

Figure 3.6: Four cases gH-differentiable Interval valued function G at $x = 0$

- F is four cases gH-differentiable at $x_0 = 1$, and $F'(1) = [2, 2]$. Graphically, this is seen in Figure 3.7.

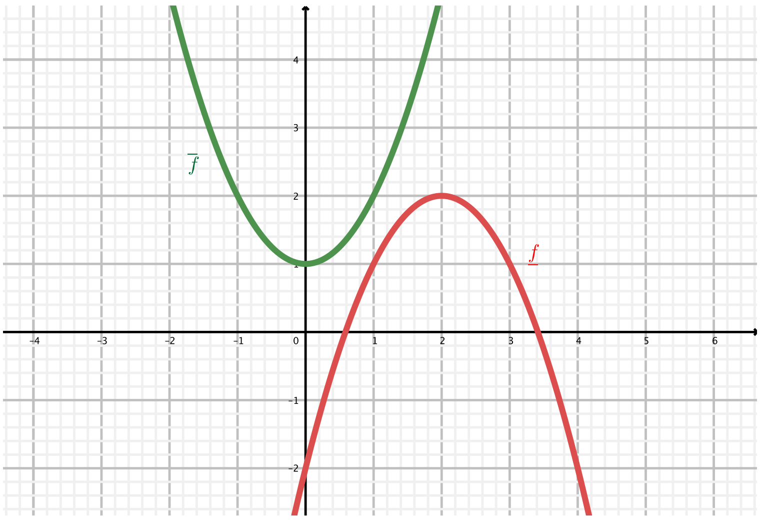

Figure 3.7: Four cases gH-differentiable Interval valued function F at $x = 1$

The cases of $(i)gH$-differentiability and $(ii)gH$-differentiability have been studied in various topics of interval-valued mathematical analysis. In contrast, cases of $(iii)gH$-differentiability and $(iv)gH$-differentiability have not been considered in the literature [2, 8, 35, 37].

3.1 Algebra of gH-differentiable interval-valued functions

Let $F, G : T \to \mathbb{I}$ be two interval-valued functions with $F(x) = \left[\underline{f}(x), \overline{f}(x)\right]$ and $G(x) = \left[\underline{g}(x), \overline{g}(x)\right]$. Let ξ, ζ be two real-valued functions so that $\xi : T \to \mathbb{R}$ and $\zeta : V \to T$, for some $V \subseteq \mathbb{R}$. The basic operations involving interval functions, addition $(+)$, gH-subtraction$(\ominus_{gH})$, multiplication by a scalar $(\cdot)$, and composition by a real function $(\circ)$ are defined by:

$$
\begin{aligned}
(F+G)(x) = F(x) + G(x) &= \left[\underline{f}(x) + \underline{g}(x), \overline{f}(x) + \overline{g}(x)\right], \\
(F \ominus_{gH} G)(x) = F(x) \ominus_{gH} G(x) &= \left[\min\left\{\underline{f}(x) - \underline{g}(x), \overline{f}(x) - \overline{g}(x)\right\},\right. \\
&\quad \left. \max\left\{\underline{f}(x) - \underline{g}(x), \overline{f}(x) - \overline{g}(x)\right\}\right], \\
(\xi \cdot F)(x) = \xi(x) \cdot F(x) &= \left[\min\left\{\xi(x)\underline{f}(x), \xi(x)\overline{f}(x)\right\},\right. \\
&\quad \left. \max\left\{\xi(x)\underline{f}(x), \xi(x)\overline{f}(x)\right\}\right], \\
(F \circ \zeta)(x) &= \left[\min\left\{\underline{f}(\zeta(x), \overline{f}(\zeta(x))\right\},\right. \\
&\quad \left. \max\left\{\underline{f}(\zeta(x)), \overline{f}(\zeta(x))\right\}\right].
\end{aligned}
$$

This section presents some properties of the algebra of gH-differentiable interval-valued functions. Specifically, we study conditions for the gH-differentiability of the sum and gH-difference of two gH-differentiable interval-valued functions, as well as the product and composition of a differentiable real-valued function and a gH-differentiable interval-valued function.

3.1.1 Sum of gH-differentiable interval-valued functions

The sum of two gH-differentiable interval-valued functions may not be a gH-differentiable interval-valued function. In fact, if we consider two $(k)gH$-differentiable interval-valued functions, both with different k, we do not necessarily obtain a gH-differentiable interval-valued function. Example 3.1.1 below illustrates this point.

Example 3.1.1. *Let $F, G : T \to \mathbb{I}$ be two interval-valued functions defined by $F(x) = [-|x|, |x|]$ and $G(x) = [0, e^{-x}]$. It is easy to verify that F is (iv)gH-differentiable at 0 and G is (ii)gH-differentiable at 0. The sum $(F+G)(x) = [-|x|, |x| + e^{-x}]$ is not gH-differentiable at 0. In fact, $\left(\underline{f+g}\right)'_{-}(0) = 1$, $\left(\underline{f+g}\right)'_{+}(0) = -1$, $\left(\overline{f+g}\right)'_{-}(0) = -2$ and $\left(\overline{f+g}\right)'_{+}(0) = 0$. Therefore, from Theorem 3.0.3, $F+G$ is not gH-differentiable at 0. Graphically the end points functions $\underline{f+g}$ and $\overline{f+g}$ of the sum $F+G$ are given by*

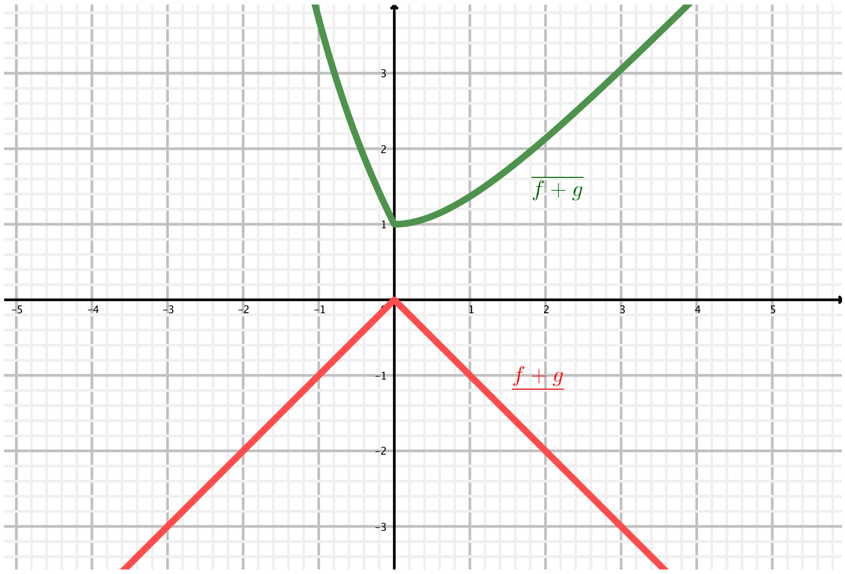

Figure 3.8: Not gH-differentiable Interval valued function $F+G$ at $x=0$

We next present results on the gH-differentiability of $F+G$ and give rules for calculating it as well. First we will show that if F and G are gH-differentiable at x_0 with equal type of gH-differentiability, then $F+G$ is also gH-differentiable at x_0 with the same type of gH-differentiability as F and G. This result was derived in [2] for cases (i) and (ii).

Theorem 3.1.2. *Let $F, G : T \to \mathbb{I}$ be two interval-valued functions. If F and G are $(k)gH$-differentiable at x_0 then $F+G$ is $(k)gH$-differentiable at x_0, for $k \in \{i, ii, iii, iv\}$. Moreover,*

$$(F+G)'(x_0) = F'(x_0) + G'(x_0). \tag{3.1}$$

Proof. Let F, G be interval-valued functions such that $F(x) = \left[\underline{f}(x), \overline{f}(x)\right]$ and $G(x) = \left[\underline{g}(x), \overline{g}(x)\right]$. To prove the result we will separately consider the four possible cases for k.

$(k=i)$ If F and G are $(i)gH$-differentiable interval-valued functions at x_0 then, from Theorem 3.0.3, the endpoint functions $\underline{f}+\underline{g}$ and $\overline{f}+\overline{g}$ of the sum $F+G$ are differentiable. Now, from Theorem 3.0.2(a) we have that

$$\begin{aligned}(F+G)'(x_0) &= \left[\min\left\{\underline{f}'(x_0)+\underline{g}'(x_0), \overline{f}'(x_0)+\overline{g}'(x_0)\right\},\right.\\ &\qquad \left.\max\left\{\underline{f}'(x_0)+\underline{g}'(x_0), \overline{f}'(x_0)+\overline{g}'(x_0)\right\}\right]\\ &= \left[\underline{f}'(x_0)+\underline{g}'(x_0), \overline{f}'(x_0)+\overline{g}'(x_0)\right]\end{aligned}$$

Therefore, $F+G$ is $(i)gH$-differentiable at x_0. In addition,

$$F'(x_0)+G'(x_0) = \left[\underline{f}'(x_0)+\underline{g}'(x_0), \overline{f}'(x_0)+\overline{g}'(x_0)\right] = (F+G)'(x_0).$$

$(k = ii)$ The proof for $k = ii$ is similar to that given for $k = i$.

$(k = iii)$ If F and G are $(iii)gH$-differentiable interval-valued functions at x_0 then, from Theorem 3.0.3 and properties of lateral derivatives, the lateral derivatives $(\underline{f} + \underline{g})'_-(x_0), (\underline{f} + \underline{g})'_+(x_0)$, $(\overline{f} + \overline{g})'_-(x_0)$ and $(\overline{f} + \overline{g})'_+(x_0)$ exist and satisfy

$$(\underline{f} + \underline{g})'_-(x_0) = (\overline{f} + \overline{g})'_+(x_0), \quad (\underline{f} + \underline{g})'_+(x_0) = (\overline{f} + \overline{g})'_-(x_0).$$

Now, from Theorem 3.0.2(b) we have that

$$\begin{aligned} (F + G)'(x_0) &= \Big[\min\left\{(\underline{f + g})'_-(x_0), (\underline{f + g})'_+(x_0)\right\}, \\ &\quad \max\left\{(\underline{f + g})'_-(x_0), (\underline{f + g})'_+(x_0)\right\}\Big]; \\ &= \left[(\underline{f + g})'_-(x_0), (\underline{f + g})'_+(x_0)\right]. \end{aligned}$$

Therefore, from Theorem 3.0.3(iii), $F+G$ is $(iii)gH$-differentiable at x_0. In addition, from properties of lateral derivatives,

$$\begin{aligned} F'(x_0) + G'(x_0) &= \left[\underline{f}'_-(x_0) + \underline{g}'_-(x_0), \underline{f}'_+(x_0) + \underline{g}'_+(x_0)\right] \\ &= (F + G)'(x_0). \end{aligned}$$

$(k = iv)$ The proof for $k = iv$ is similar to that given for $k = iii$.

$\square$

Example 3.1.1 showed that the sum of two $(k)gH$-differentiable interval-valued functions, having different values of k, may not be differentiable. Nevertheless, this is not always the case, as Example 3.1.3 below shows.

Example 3.1.3. *Let $F, G : T \to \mathbb{I}$ be two interval-valued functions, defined by $F(x) = [-|x| - 1, |x| + 1]$ and $G(x) = [|x| - 1, -|x| + 1]$. F is $(iv)gH$-differentiable at 0, and G is $(iii)gH$-differentiable at 0. Note that $(F + G)(x) = [-2, 2]$, and therefore $(F + G)'(0) = \{0\}$.*

Next we study the gH-differentiability of the sum of two interval-valued functions with different types of gH-differentiability.

Theorem 3.1.4. *Let $F, G : T \to \mathbb{I}$ be two interval-valued functions.*

(a) If F is $(i)gH$-differentiable at x_0 and G is $(ii)gH$-differentiable at x_0 then $F + G$ is either $(i)gH$-differentiable or $(ii)gH$-differentiable at x_0.

(b) *If F is $(iii)gH$-differentiable at x_0 and G is $(iv)gH$-differentiable at x_0 then $F+G$ is either $(iii)gH$-differentiable or $(iv)gH$-differentiable at x_0.*

Moreover, in all cases we have

$$(F+G)'(x_0) = F'(x_0) \ominus_{gH} (-1)G'(x_0), \tag{3.2}$$

and

$$(F+G)'(x_0) \subseteq F'(x_0) + G'(x_0), \tag{3.3}$$

with equality in (3.3) if and only if either $F'(x_0)$ or $G'(x_0)$ or both of them are a singleton.

Proof. (a) Since F is $(i)gH$-differentiable and G is $(ii)gH$-differentiable at x_0 we have that the endpoint functions $\underline{f}$, $\overline{f}$, $\underline{g}$ and $\overline{g}$ are differentiable functions at x_0. Thus $\underline{f}+\underline{g}$ and $\overline{f}+\overline{g}$ are differentiable functions at x_0 and so, from Theorem 3.0.2(a) and Theorem 3.0.3, $F+G$ is either $(i)gH$-differentiable or $(ii)gH$-differentiable at x_0 and

$$(F+G)'(x_0) = \left[\min\{\underline{f}'(x_0)+\underline{g}'(x_0), \overline{f}'(x_0)+\overline{g}'(x_0)\}\ ,\ \max\{\underline{f}'(x_0)+\underline{g}'(x_0), \overline{f}'(x_0)+\overline{g}'(x_0)\}\right]. \tag{3.4}$$

On the other hand,

$$\begin{aligned} & F'(x_0) \ominus_{gH} (-1)G'(x_0) \\ = & \left[\underline{f}'(x_0), \overline{f}'(x_0)\right] \ominus_{gH} (-1)\left[\overline{g}'(x_0), \underline{g}'(x_0)\right] \\ = & \left[\underline{f}'(x_0), \overline{f}'(x_0)\right] \ominus_{gH} \left[-\underline{g}'(x_0), -\overline{g}'(x_0)\right] \\ = & \left[\min\{\underline{f}'(x_0)+\underline{g}'(x_0), \overline{f}'(x_0)+\overline{g}'(x_0)\}\ ,\ \max\{\underline{f}'(x_0)+\underline{g}'(x_0), \overline{f}'(x_0)+\overline{g}'(x_0)\}\right]. \end{aligned} \tag{3.5}$$

Comparing (3.5) with (3.4) we obtain (3.2). In addition,

$$F'(x_0) + G'(x_0) = \left[\underline{f}'(x_0)+\overline{g}'(x_0), \underline{g}'(x_0)+\overline{f}'(x_0)\right]. \tag{3.6}$$

(a1) If $F+G$ is $(i)gH$-differentiable then $(F+G)'(x_0) = \left[\underline{f}'(x_0)+\underline{g}'(x_0), \overline{f}'(x_0)+\overline{g}'(x_0)\right]$. Comparing this result with (3.6) we have

$$(F+G)'(x_0) \subseteq F'(x_0) + G'(x_0).$$

(a2) If $F+G$ is $(ii)gH$-differentiable then $(F+G)'(x_0) = \left[\overline{f}'(x_0)+\overline{g}'(x_0), \underline{f}'(x_0)+\underline{g}'(x_0)\right]$. Comparing this result with (3.6) we also have that

$$(F+G)'(x_0) \subseteq F'(x_0) + G'(x_0).$$

Finally, from (3.2) it readily follows that the equality in (3.3) holds if and only if either $F'(x_0)$ or $G'(x_0)$ or both of them are a singleton.

(b) If the functions F and G are $(iii)gH$-differentiable and $(iv)gH$-differentiable at x_0, respectively, then the lateral derivatives $\underline{f}'_-(x_0), \underline{f}'_+(x_0), \overline{f}'_-(x_0), \overline{f}'_+(x_0), \underline{g}'_-(x_0), \underline{g}'_+(x_0), \overline{g}'_-(x_0)$ and $\overline{g}'_+(x_0)$ exist and satisfy $\underline{f}'_-(x_0) = \overline{f}'_+(x_0)$, $\underline{f}'_+(x_0) = \overline{f}'_-(x_0)$, $\underline{g}'_-(x_0) = \overline{g}'_+(x_0)$, $\underline{g}'_+(x_0) = \overline{g}'_-(x_0)$. Therefore, the derivatives $\left(\underline{f}+\underline{g}\right)'_-(x_0)$, $\left(\underline{f}+\underline{g}\right)'_+(x_0)$, $\left(\overline{f}+\overline{g}\right)'_-(x_0)$, $\left(\overline{f}+\overline{g}\right)'_+(x_0)$ exist and satisfy

$$\left(\underline{f}+\underline{g}\right)'_-(x_0) = \left(\overline{f}+\overline{g}\right)'_+(x_0), \quad \left(\underline{f}+\underline{g}\right)'_+(x_0) = \left(\overline{f}+\overline{g}\right)'_-(x_0).$$

From Theorem 3.0.2, $F+G$ is either $(iii)gH$-differentiable or $(iv)gH$-differentiable at x_0. From properties of lateral derivatives, similar steps to those given in the proof of (a), let us obtain (3.2) and (3.3). $\square$

Theorem 3.1.4 states that the sum $F+G$ of two gH-differentiable functions, F and G with different types of gH-differentiability ((i) and (ii) or (iii) and (iv)) is gH-differentiable with the same type of gH-differentiability as either F or G. In Example 3.1.3 we saw an instance of the sum of an $(iii)gH$-differentiable function and an $(iv)gH$-differentiable function. The next example illustrates the case of the sum of an $(i)gH$-differentiable function and an $(ii)gH$-differentiable function.

Example 3.1.5. *Let us consider the interval-valued functions $F, G : [0,1] \to \mathbb{I}$ defined by*

$$F(x) = [0, x], \quad G(x) = \left[0, 1-x^2\right].$$

Thus, F is $(i)gH$-differentiable and G is $(ii)gH$-differentiable at any $x \in (0,1)$. So

$$F'(x) = [0,1] \quad \text{and} \quad G'(x) = [-2x, 0],$$

and, from (3.2),

$$\begin{aligned} (F+G)'(x_0) &= F'(x_0) \ominus_{gH} (-1)G'(x_0) \\ &= [0,1] \ominus_{gH} (-1)[-2x, 0] \\ &= [0,1] \ominus_{gH} [0, 2x] \\ &= [\min\{0, 1-2x\}, \max\{0, 1-2x\}]. \end{aligned}$$

On the other hand,

$$(F+G)(x) = \left[0, x+1-x^2\right],$$

and graphically the end point functions are given by

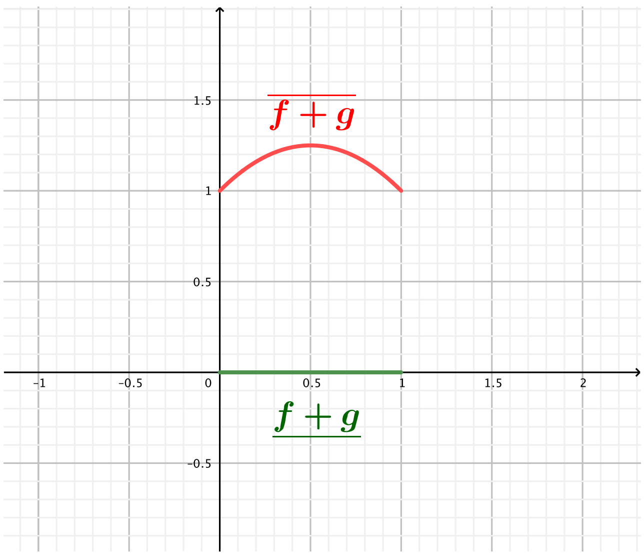

Figure 3.9: End point functions for the Example 3.1.5

Note that this function is $(i)gH$*-differentiable in* $(0, 1/2]$ *and* $(ii)gH$*-differentiable in* $[1/2, 1)$*. In any case, we always have*

$$(F+G)'(x) \subset F'(x) + G'(x),$$

for all $x \in (0,1)$*. Since neither* $F'(x)$ *nor* $G'(x)$ *are trivial intervals, for all* $x \in (0,1)$*, the equality* $(F+G)'(x) = F'(x) + G'(x)$ *is not possible.*

The following result shows that with other combinations of gH-differentiability for F and G, different from those studied in Theorems 3.1.2 and 3.1.4, the sum $F+G$ is not differentiable.

Theorem 3.1.6. *Let* $F, G : T \to \mathbb{I}$ *be two interval-valued functions, so that both of them are* gH*-differentiable at* x_0 *and neither* $F'(x_0)$ *nor* $G'(x_0)$ *are nontrivial intervals.*

(a) *If* F *is* $(i)gH$*-differentiable at* x_0 *and* G *is* $(iii)gH$*-differentiable (or* $(iv)gH$*-differentiable) at* x_0*, then* $F+G$ *is not* gH*-differentiable at* x_0*.*

(b) *If* F *is* $(ii)gH$*-differentiable at* x_0 *and* G *is* $(iii)gH$*-differentiable (or* $(iv)gH$*-differentiable) at* x_0*, then* $F+G$ *is not* gH*-differentiable at* x_0*.*

Proof. (a) If F is $(i)gH$-differentiable at x_0 and $F'(x_0)$ is a nontrivial interval then there exist $\underline{f}'(x_0)$ and $\overline{f}'(x_0)$, with $\underline{f}'(x_0) < \overline{f}'(x_0)$. Also, if G is $(iii)gH$-differentiable at x_0 and $G'(x_0)$ is a nontrivial interval then $\underline{g}'_-(x_0), \underline{g}'_+(x_0), \overline{g}'_-(x_0)$ and $\overline{g}'_+(x_0)$ exist and satisfy $\underline{g}'_-(x_0) = \overline{g}'_+(x_0)$ and $\underline{g}'_+(x_0) = \overline{g}'_-(x_0)$, with $\underline{g}'_-(x_0) < \underline{g}'_+(x_0)$. Thus, the endpoint functions $\underline{f} + \underline{g}$ and $\overline{f} + \overline{g}$ of $F + G$ are not differentiable at x_0 and

$$(\underline{f} + \underline{g})'_-(x_0) \neq (\overline{f} + \overline{g})'_+(x_0).$$

Therefore, from Theorem 3.0.2, $F+G$ is not gH-differentiable. If F is $(i)gH$-differentiable and G is $(iv)gH$-differentiable at x_0, the proof is similar.

(b) The proof is analogous to that of part (a). □

Example 3.1.1 illustrates the previous result.

The results in this subsection corrects, complements and generalizes those obtained in [2] and corrects Proposition 24 in [35].

3.1.2 gH-difference of gH-differentiable interval-valued functions

The gH-difference of two gH-differentiable interval-valued functions is not necessarily gH-differentiable, as in the case of the sum of two gH-differentiable interval-valued functions, the next example shows this.

Example 3.1.7. *Let us consider the functions of Example 3.1.1. We have that F is $(iv)gH$-differentiable at 0 and G is $(ii)gH$-differentiable at 0. The gH-difference is defined by*

$$\begin{aligned}(F \ominus_{gH} G)(x) &= \left[\min\left\{-|x|, |x| - e^{-x}\right\}, \max\left\{-|x|, |x| - e^{-x}\right\}\right] \\ &= \begin{cases} [-x - e^{-x}, x] & \text{if } x < 0, \\ [x - e^{-x}, -x] & \text{if } 0 \leq x < W(1/2), \\ [-x, x - e^{-x}] & \text{if } x \geq W(1/2), \end{cases}\end{aligned}$$

where $W(\cdot)$ is the W Lambert function. Note that $F \ominus_{gH} G$ is not gH-differentiable at 0. In fact the lateral derivatives at 0 are

$$\left(\underline{F \ominus_{gH} G}\right)'_-(0) = 0, \ \left(\underline{F \ominus_{gH} G}\right)'_+(0) = 2, \ \left(\overline{F \ominus_{gH} G}\right)'_-(0) = 1, \ \left(\overline{F \ominus_{gH} G}\right)'_+(0) = -1,$$

and so $F \ominus_{gH} G$ is not $(k)gH$-differentiable at 0 for any $k \in \{i, ii, iii, iv\}$. We can see this fact in Figure 3.10.

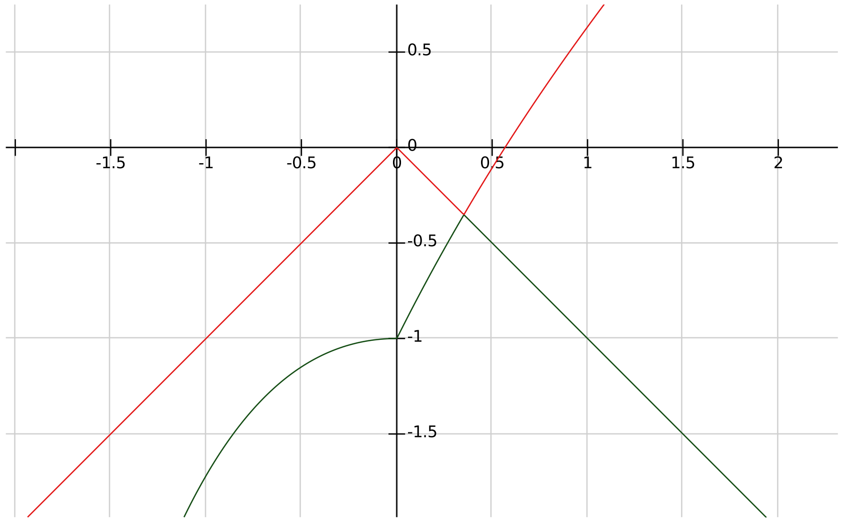

Figure 3.10: End point functions for the Example 3.1.7

Therefore, from Theorem 3.0.3, $F \ominus_{gH} G$ is not gH-differentiable at 0.

Before stating our results, it is necessary to consider the following equivalences and notation.

Remark 3.1.8. *Let $F, G : T \to \mathbb{I}$ be two interval-valued functions. For each $x \in T$, define $len(F)(x) = \overline{f}(x) - \underline{f}(x)$ and $len(G)(x) = \overline{g}(x) - \underline{g}(x)$. Let $F(x) \ominus_{gH} G(x) = H(x) = \left[\underline{h}(x), \overline{h}(x)\right]$, then*

$$\begin{aligned}
\underline{h}(x) &= \min\left\{\underline{f}(x) - \underline{g}(x), \overline{f}(x) - \overline{g}(x)\right\} \\
&= \frac{\underline{f}(x) - \underline{g}(x) + \overline{f}(x) - \overline{g}(x)}{2} - \frac{\left|\underline{f}(x) - \underline{g}(x) - \overline{f}(x) + \overline{g}(x)\right|}{2} \\
&= \frac{\underline{f}(x) + \overline{f}(x)}{2} - \frac{\underline{g}(x) + \overline{g}(x)}{2} - \frac{|len(G)(x) - len(F)(x)|}{2}
\end{aligned}$$

and

$$\begin{aligned}
\overline{h}(x) &= \max\left\{\underline{f}(x) - \underline{g}(x), \overline{f}(x) - \overline{g}(x)\right\} \\
&= \frac{\underline{f}(x) - \underline{g}(x) + \overline{f}(x) - \overline{g}(x)}{2} + \frac{\left|\underline{f}(x) - \underline{g}(x) - \overline{f}(x) + \overline{g}(x)\right|}{2} \\
&= \frac{\underline{f}(x) + \overline{f}(x)}{2} - \frac{\underline{g}(x) + \overline{g}(x)}{2} + \frac{|len(G)(x) - len(F)(x)|}{2}.
\end{aligned}$$

Thus,

$$\text{if } len(G)(x) > len(F)(x), \quad \text{then} \quad H(x) = [\overline{f}(x) - \overline{g}(x), \underline{f}(x) - \underline{g}(x)], \tag{3.7}$$

$$\text{if } len(G)(x) < len(F)(x), \quad \text{then} \quad H(x) = [\underline{f}(x) - \underline{g}(x), \overline{f}(x) - \overline{g}(x)], \tag{3.8}$$

$$\text{if } len(G)(x) = len(F)(x), \quad \text{then} \quad H(x) = \{\overline{f}(x) - \overline{g}(x)\} = \{\underline{f}(x) - \underline{g}(x)\}. \tag{3.9}$$

Theorem 3.1.9. *Let $F, G : T \to \mathbb{I}$ be two interval-valued functions. If F and G are both $(k)gH$-differentiable at x_0, for some $k \in \{i, ii, iii, iv\}$, then $F \ominus_{gH} G$ is gH-differentiable at x_0. Moreover,*

$$(F \ominus_{gH} G)'(x_0) = F'(x_0) \ominus_{gH} G'(x_0). \tag{3.10}$$

Proof. Assume that F and G are both $(i)gH$-differentiable at x_0. We will consider two cases: (a) $len(G)(x_0) - len(F)(x_0) \neq 0$ and (b) $len(G)(x_0) - len(F)(x_0) = 0$.

(*a*) If $len(G)(x_0) - len(F)(x_0) \neq 0$, then there exists an open neighborhood V_{x_0} of x_0 such that $len(G)(x) - len(F)(x) \neq 0$, $\forall x \in V_{x_0}$. So, from either (3.7) or (3.8), it follows that $\underline{h}$ and $\overline{h}$ are differentiable at x_0. From Theorem 3.0.2 $F \ominus_{gH} G$ is gH-differentiable at x_0 and

$$(F \ominus_{gH} G)'(x_0) = H'(x_0) = \left[\min\left\{\underline{h}'(x_0), \overline{h}'(x_0)\right\}, \max\left\{\underline{h}'(x_0), \overline{h}'(x_0)\right\}\right].$$

On the other hand, since F and G are $(i)gH$-differentiable at x_0, it follows that

$$\begin{aligned}
& F'(x_0) \ominus_{gH} G'(x_0) \\
= \ & \left[\underline{f}'(x_0), \overline{f}'(x_0)\right] \ominus_{gH} \left[\underline{g}'(x_0), \overline{g}'(x_0)\right] \\
= \ & \left[\min\left\{\underline{f}'(x_0) - \underline{g}'(x_0), \overline{f}'(x_0) - \overline{g}'(x_0)\right\}, \max\left\{\underline{f}'(x_0) - \underline{g}'(x_0), \overline{f}'(x_0) - \overline{g}'(x_0)\right\}\right] \\
= \ & \left[\min\left\{\underline{h}'(x_0), \overline{h}'(x_0)\right\}, \max\left\{\underline{h}'(x_0), \overline{h}'(x_0)\right\}\right] \\
= \ & (F \ominus_{gH} G)'(x_0).
\end{aligned}$$

(*b*) If $len(G)(x_0) - len(F)(x_0) = 0$, then there exists an open neighborhood V_{x_0} of x_0 where either $(b1)$ or $(b2)$ below holds. Next we will consider these two cases separately.

($b1$) $len(G)(x) - len(F)(x) = 0$, $\forall x \in V_{x_0}$. From (3.9), $\underline{h}(x) = \overline{h}(x)$ is differentiable at x_0 and from Theorem 3.0.2 $F \ominus_{gH} G$ is gH-differentiable at x_0. In addition

$$(F \ominus_{gH} G)'(x_0) = H'(x_0) = \{\underline{f}'(x_0) - \underline{g}'(x_0)\} = \{\overline{f}'(x_0) - \overline{g}'(x_0)\} = F'(x_0) \ominus_{gH} G'(x_0).$$

($b2$) x_0 is the unique root of $len(G)(x) - len(F)(x)$ in V_{x_0}. The following four subcases can be considered:

($b21$) There exists $\varepsilon > 0$ such that $len(G)(x) - len(F)(x) < 0$, $\forall x \in (x_0 - \varepsilon, x_0)$ and $len(G)(x) - len(F)(x) > 0$, $\forall x \in (x_0, x_0 + \varepsilon)$.

$(b22)$ There exists $\varepsilon > 0$ such that $len(G)(x) - len(F)(x) > 0,\ \forall x \in (x_0 - \varepsilon, x_0)$ and $len(G)(x) - len(F)(x) < 0,\ \forall x \in (x_0, x_0 + \varepsilon)$.

$(b23)$ $len(G)(x) - len(F)(x) > 0,\ \forall x \in V_{x_0} \setminus x_0$.

$(b24)$ $len(G)(x) - len(F)(x) < 0,\ \forall x \in V_{x_0} \setminus x_0$.

Now, if $(b21)$ holds then from properties of lateral derivatives

$$\begin{aligned} \underline{h}'_-(x_0) &= \underline{f}'_-(x_0) - \underline{g}'_-(x_0), \quad \underline{h}'_+(x_0) = \overline{f}'_+(x_0) - \overline{g}'_+(x_0), \\ \overline{h}'_-(x_0) &= \overline{f}'_-(x_0) - \overline{g}'_-(x_0), \quad \overline{h}'_+(x_0) = \underline{f}'_+(x_0) - \underline{g}'_+(x_0). \end{aligned}$$

From the differentiability of $\underline{f}, \overline{f}, \underline{g}$ and $\overline{g}$ at x_0, we obtain that $\underline{h}'_-(x_0) = \overline{h}'_+(x_0)$, $\underline{h}'_+(x_0) = \overline{h}'_-(x_0)$. Consequently, from Theorem 3.0.2, $F \ominus_{gH} G$ is gH-differentiable at x_0. Moreover

$$\begin{aligned} & (F \ominus_{gH} G)'(x_0) = H'(x_0) \\ = & \left[\min\left\{\underline{h}'_-(x_0), \underline{h}'_+(x_0)\right\}, \max\left\{\underline{h}'_-(x_0), \underline{h}'_+(x_0)\right\}\right] \\ = & \Big[\min\left\{\underline{f}'(x_0) - \underline{g}'(x_0), \overline{f}'(x_0) - \overline{g}'(x_0)\right\}, \\ & \max\left\{\underline{f}'(x_0) - \underline{g}'(x_0), \overline{f}'(x_0) - \overline{g}'(x_0)\right\}\Big] \\ = & F'(x_0) \ominus_{gH} G'(x_0). \end{aligned}$$

The proof of case $(b22)$ is analogous to that of case $(b21)$.

If $(b23)$ holds, from (3.7), $\underline{h}$ and $\overline{h}$ are differentiable at x_0 and thus, from Theorem 3.0.3, $F \ominus_{gH} G$ is gH-differentiable at x_0. In addition,

$$\begin{aligned} (F \ominus_{gH} G)'(x_0) &= H'(x_0) \\ &= \left[\min\left\{\underline{h}'(x_0), \overline{h}'(x_0)\right\}, \max\left\{\underline{h}'(x_0), \overline{h}'(x_0)\right\}\right] \\ &= F'(x_0) \ominus_{gH} G'(x_0). \end{aligned}$$

The proof of case $(b24)$ is analogous to that of case $(b23)$.

The proofs for the other types of gH-differentiability at x_0 are analogous to that given for $k = i$, so we omit them. □

Next, we study the differentiability of the gH-difference of two interval-valued functions with different types of gH-differentiability.

Theorem 3.1.10. *Let $F, G : T \to \mathbb{I}$ be two interval-valued functions.*

(a) *If F is (i)gH-differentiable at x_0 and G is (ii)gH-differentiable at x_0, then $F \ominus_{gH} G$ is gH-differentiable at x_0.*

(b) *If F is (ii)gH-differentiable at x_0 and G is (i)gH-differentiable at x_0, then $F \ominus_{gH} G$ is gH-differentiable at x_0.*

(c) *If F is (iii)gH-differentiable at x_0 and G is (iv)gH-differentiable at x_0, then $F \ominus_{gH} G$ is gH-differentiable at x_0.*

(d) *If F is (iv)gH-differentiable at x_0 and G is (iii)gH-differentiable at x_0, then $F \ominus_{gH} G$ is gH-differentiable at x_0.*

Moreover, in all cases we have

$$(F \ominus_{gH} G)'(x_0) = F'(x_0) - G'(x_0) \tag{3.11}$$

and

$$F'(x_0) \ominus_{gH} G'(x_0) \subseteq (F \ominus_{gH} G)'(x_0), \tag{3.12}$$

with equality in (3.12) if and only if either $F'(x_0)$ or $G'(x_0)$ or both of them are a singleton.

Proof. The proofs of (a)–(d) follow similar steps to those given in the proofs of Theorem 3.1.9, so we omit them. We will just prove (3.11) and (3.12) in case (a) since the proofs in the other cases are done in the same way.

If F is $(i)gH$-differentiable at x_0 and G is $(ii)gH$-differentiable at x_0, then $\underline{f}'(x_0) \leq \overline{f}'(x_0)$, $\overline{g}'(x_0) \leq \underline{g}'(x_0)$, which implies that

$$\underline{f}'(x_0) - \underline{g}'(x_0) \leq \overline{f}'(x_0) - \overline{g}'(x_0) \tag{3.13}$$

and therefore

$$(F \ominus_{gH} G)'(x_0) = \left[\underline{f}'(x_0) - \underline{g}'(x_0), \overline{f}'(x_0) - \overline{g}'(x_0)\right]. \tag{3.14}$$

Clearly, (3.11) holds. From (3.13) and (3.14), it readily follows that we obtain (3.12). By using (3.13), easy calculations show that the equality in (3.12) holds if and only if either $F'(x_0)$ or $G'(x_0)$ or both of them are a singleton. □

Theorems 3.1.9 and 3.1.10 give conditions for the gH-differentiability of the gH-difference of two interval-valued functions, but they do not indicate the type of gH-differentiability of the resulting interval-valued function. The next examples show that even when the two functions are gH-differentiable with the same type, the gH-difference of them may have many different types.

Example 3.1.11. *Let $F, G : \mathbb{R}^+ \to \mathbb{I}$ be two interval-valued functions defined by*

$$F(x) = \begin{cases} [e^x - 1, 1 - \sin(x)] & \text{if } -\frac{1}{2} \leq x \leq 0, \\ [1 - \sin(x), x + 1] & \text{if } 0 \leq x \leq \frac{1}{2}, \end{cases} \quad \text{and}$$

$$G(x) = \begin{cases} [-x, x + 1] & \text{if } -\frac{1}{2} \leq x \leq 0, \\ [x, 1 - x] & \text{if } 0 \leq x \leq \frac{1}{2}. \end{cases}$$

F is $(iii)gH$-differentiable at 0 and G is $(iv)gH$-differentiable at 0. We have that

$$F \ominus_{gH} G(x) = \begin{cases} [e^x, 1 - \sin(x)] & \text{if } -\frac{1}{2} \leq x \leq 0, \\ [1 - \sin(x), x] & \text{if } 0 \leq x \leq \frac{1}{2}, \end{cases}$$

which is $(iv)gH$-differentiable at 0.

Example 3.1.12. *Let $F, G : \mathbb{R}^+ \to \mathbb{I}$ be two interval-valued functions defined by $F(x) = [0, e^x + e^{-x}]$, $G(x) = [0, e^x]$. Note that $F \ominus_{gH} G(x) = [0, e^{-x}]$. The functions F and G are both $(i)gH$-differentiable functions, but $F \ominus_{gH} G$ is $(ii)gH$-differentiable at any $x \in \mathbb{R}^+$.*

Example 3.1.13. *Let $F, G : \mathbb{R}^+ \to \mathbb{I}$ be two interval-valued functions defined by $F(x) = \left[-2, \dfrac{x^2}{8} - \dfrac{1}{2}\right]$, $G(x) = [-2, e^{x-2} - 1]$. Note that*

$$F \ominus_{gH} G(x) = \left[\min\left\{0, \frac{x^2}{8} - \frac{1}{2} - e^{x-2} + 1\right\}, \max\left\{0, \frac{x^2}{8} - \frac{1}{2} - e^{x-2} + 1\right\}\right].$$

The functions F and G are both $(i)gH$-differentiable functions at $x = 2$, and $F \ominus_{gH} G$ is $(iv)gH$-differentiable at $x = 2$.

3.1.3 Product of a differentiable real-valued function and a gH-differentiable interval-valued function

This subsection studies the gH-differentiability of the product of a differentiable real-valued function, $g : T \to \mathbb{R}$, and a gH-differentiable interval-valued function, $F : T \to \mathbb{I}$. We start by considering the case where g is a constant.

Theorem 3.1.14. *Let $F : T \to \mathbb{I}$ be an interval-valued function. If F is $(k)gH$-differentiable at x_0, then αF is $(k)gH$-differentiable at x_0 with*

$$(\alpha \cdot F)'(x_0) = \alpha \cdot F'(x_0),$$

$\forall \alpha \in \mathbb{R}$, $\forall k \in \{i, ii, iii, iv\}$.

Proof. Assume that F is $(i)gH$-differentiable at x_0. Then, $F'(x_0) = \left[\underline{f}'(x_0), \overline{f}'(x_0)\right]$. Multiplying F by a scalar $\alpha \in \mathbb{R}$ we have,

$$\alpha \cdot F(x) = \begin{cases} \left[\alpha \underline{f}(x), \alpha \overline{f}(x)\right] & if\ \alpha \geq 0, \\ \left[\alpha \overline{f}(x), \alpha \underline{f}(x)\right] & if\ \alpha < 0. \end{cases}$$

We also have that

$$(\alpha \cdot F)'(x_0) = \begin{cases} \left[\alpha \underline{f}'(x_0), \alpha \overline{f}'(x_0)\right] & if\ \alpha \geq 0, \\ \left[\alpha \overline{f}'(x_0), \alpha \underline{f}'(x_0)\right] & if\ \alpha < 0. \end{cases} \tag{3.15}$$

On the other hand, if F' is multiplied by $\alpha \in \mathbb{R}$ we get

$$\alpha \cdot F'(x_0) = \begin{cases} \left[\alpha \underline{f}'(x_0), \alpha \overline{f}'(x_0)\right] & if\ \alpha \geq 0, \\ \left[\alpha \overline{f}'(x_0), \alpha \underline{f}'(x_0)\right] & if\ \alpha < 0. \end{cases} \tag{3.16}$$

From (3.15) and (3.16) we have that αF is $(i)gH$-differentiable and $(\alpha \cdot F)'(x_0) = \alpha \cdot F'(x_0)$. The proofs for $k \in \{ii, iii, iv\}$ are very similar, so we omit them. $\square$

If $g, f : T \to \mathbb{R}$ are two real-valued functions which are differentiable at x_0, then it is well-known that $(gf)'(x_0) = g(x_0)f'(x_0) + g'(x_0)f(x_0)$. The following result shows that, in some cases, a similar expression for the derivative of the product of a differentiable real-valued function and a gH-differentiable interval-valued function holds true.

Theorem 3.1.15. *Let $F : T \to \mathbb{I}$ be an interval-valued function and let $g : T \to \mathbb{R}$ be a real-valued function which is differentiable at x_0.*

(a) If F is $(i)gH$-differentiable at x_0, $g(x_0) \geq 0$ and $g'(x_0) \geq 0$, then $g \cdot F$ is $(i)gH$-differentiable at x_0.

(b) If F is $(ii)gH$-differentiable at x_0, $g(x_0) \geq 0$ and $g'(x_0) < 0$, then $g \cdot F$ is $(ii)gH$-differentiable at x_0.

(c) If F is $(i)gH$-differentiable at x_0, $g(x_0) < 0$ and $g'(x_0) < 0$, then $g \cdot F$ is $(i)gH$-differentiable at x_0.

(d) If F is $(ii)gH$-differentiable at x_0, $g(x_0) < 0$ and $g'(x_0) \geq 0$, then $g \cdot F$ is $(ii)gH$-differentiable at x_0.

Moreover, in all cases we have that

$$(g \cdot F)'(x_0) = g(x_0) \cdot F'(x_0) + g'(x_0) \cdot F(x_0). \tag{3.17}$$

Proof. Let $F(x) = \left[\underline{f}(x), \overline{f}(x)\right]$, the product of F and g is given by

$$(g \cdot F)(x) = g(x) \cdot F(x) = \begin{cases} \left[g(x)\underline{f}(x), g(x)\overline{f}(x)\right] & \text{if } g(x) \geq 0, \\ \left[g(x)\overline{f}(x), g(x)\underline{f}(x)\right] & \text{if } g(x) < 0. \end{cases}$$

If $g(x_0) \geq 0$, $g'(x_0) \geq 0$ and F is $(i)gH$-differentiable, then $g \cdot F$ is $(i)gH$-differentiable and

$$\begin{aligned} g'(x_0) \cdot F(x_0) + g(x_0) \cdot F'(x_0) &= g'(x_0)\left[\underline{f}(x_0), \overline{f}(x_0)\right] + g(x_0)\left[\underline{f}'(x_0), \overline{f}'(x_0)\right] \\ &= \left[g'(x_0)\underline{f}(x_0), g'(x_0)\overline{f}(x_0)\right] + \left[g(x_0)\underline{f}'(x_0), g(x_0)\overline{f}'(x_0)\right] \\ &= \left[g'(x_0)\underline{f}(x_0) + g(x_0)\underline{f}'(x_0), g'(x_0)\overline{f}(x_0) + g(x_0)\overline{f}'(x_0)\right] \\ &= (g \cdot F)'(x_0). \end{aligned}$$

The other cases can be dealt with similarly, so we omit their proofs. □

If $F : T \to \mathbb{I}$ is an interval-valued function $(i)gH$-differentiable at x_0, for $k \in \{i, ii\}$, and $g : T \to \mathbb{R}$ is a real-valued function differentiable at x_0, then trivially the product gF is either $(i)gH$-differentiable or $(ii)gH$-differentiable at x_0. Nevertheless, only in the cases considered in Theorem 3.1.15 the gH-derivative has the "nice" expression (3.17), which "imitates" the well-known one for the derivative of the product of two real-valued functions. Example 3.1.16 below gives an instance where expression (3.17) does not hold.

Example 3.1.16. *Let $F : (0.5, 1) \to \mathbb{I}$ be an interval-valued function defined by $F(x) = [0, x]$, and let $g : (0.5, 1) \to \mathbb{R}$ be an real-valued function defined by $g(x) = 1 - x^2$. Note that $F(x)$ is $(i)gH$-differentiable at every $x \in (0.5, 1)$, and that $(g \cdot F)(x) = [0, x - x^3]$ is also $(i)gH$-differentiable at every $x \in (0.5, 1)$, with*

$$(g.F)'(x) = \begin{cases} [0, 1 - 3x^2] & \text{if } x \in (0.5, 1/\sqrt{3}], \\ [1 - 3x^2, 0] & \text{if } x \in (1/\sqrt{3}, 1). \end{cases}$$

On the other hand, we have that

$$g(x) \cdot F'(x) + g'(x) \cdot F(x) = [1 - 2x, 1 - x^2],$$

and thus expression (3.17) does not hold.

Let $F : T \to \mathbb{I}$ be an interval-valued function and let $g : T \to \mathbb{R}$ be a real-valued function which is differentiable at x_0. If F is $(k)gH$-differentiable for $k \in \{iii, iv\}$, the product $g \cdot F$ is not necessarily gH-differentiable. Example 3.1.17 below illustrates this fact.

Example 3.1.17. *Let $F : T \to \mathbb{I}$ be an interval-valued function defined by $F(x) = [-|x|, |x| + 1]$ and $g : T \to \mathbb{R}$ a real-valued function defined by $g(x) = e^x$. F is $(iv)gH$-differentiable at 0, and g differentiable at 0, the product of F and g is given by $(g \cdot F)(x) = [-|x| e^x, (|x| + 1)e^x]$. We have that*

$$\left(\underline{g \cdot F}\right)'_{-}(0) = 0, \ \left(\overline{g \cdot F}\right)'_{-}(0) = 1, \ \left(\underline{g \cdot F}\right)'_{+}(0) = -1, \ \left(\overline{g \cdot F}\right)'_{+}(0) = 2.$$

Therefore $g \cdot F$ is not gH-differentiable at $x = 0$.

The following theorem give conditions for $g \cdot F$ to be gH-differentiable, when F is $(k)gH$-differentiable for $k \in \{iii, iv\}$.

Theorem 3.1.18. *Let $F : T \to \mathbb{I}$ be an interval-valued function so that $len(F)(x_0) = 0$ and let $g : T \to \mathbb{R}$ be a real-valued function which is differentiable at x_0.*

(a) If F is $(iii)gH$-differentiable at x_0 and $g(x_0) \geq 0$, then $g \cdot F$ is $(iii)gH$-differentiable at x_0.

(b) If F is $(iv)gH$-differentiable at x_0 and $g(x_0) \geq 0$, then $g \cdot F$ is $(iv)gH$-differentiable at x_0.

(c) If F is $(iii)gH$-differentiable at x_0 and $g(x_0) < 0$, then $g \cdot F$ is $(iv)gH$-differentiable at x_0.

(d) If F is $(iv)gH$-differentiable at x_0 and $g(x_0) < 0$, then $g \cdot F$ is $(iii)gH$-differentiable at x_0.

Moreover, in all cases we have

$$(g \cdot F)'(x_0) = g(x_0) \cdot F'(x_0) + g'(x_0) \cdot F(x_0). \tag{3.18}$$

Proof. If F is $(iii)gH$-differentiable or $(iv)gH$-differentiable and g is differentiable at x_0, it follows that $\underline{f}'_-(x_0), \underline{f}'_+(x_0), \overline{f}'_-(x_0)$ and $\overline{f}'_+(x_0)$ exist and satisfy $\underline{f}'_-(x_0) = \overline{f}'_+(x_0)$ and $\underline{f}'_+(x_0) = \overline{f}'_-(x_0)$. Multiplying these equalities by $g(x_0)$, we get that $g(x_0)\underline{f}'_-(x_0) = g(x_0)\overline{f}'_+(x_0)$ and $g(x_0)\underline{f}'_+(x_0) = g(x_0)\overline{f}'_-(x_0)$. If $len(F)(x_0) = 0$, then $\underline{f}_+(x_0) = \overline{f}_-(x_0)$, and therefore $\left(g\underline{f}\right)'_+(x_0) = \left(g\overline{f}\right)'_-(x_0)$ and $\left(g\underline{f}\right)'_-(x_0) = \left(g\overline{f}\right)'_+(x_0)$. This implies that $g \cdot F$ is gH-differentiable at x_0

(*a*) If F is $(iii)gH$-differentiable at x_0, $g(x_0) \geq 0$ and $len(F)(x_0) = 0$, then

$$\begin{aligned} g'(x_0) \cdot F(x_0) + g(x_0) \cdot F'(x_0) &= g'(x_0)\left[\underline{f}_-(x_0), \underline{f}_+(x_0)\right] + g(x_0)\left[\underline{f}'_-(x_0), \underline{f}'_+(x_0)\right] \\ &= \left[g'(x_0)\underline{f}_-(x_0), g'(x_0)\underline{f}_+(x_0)\right] \\ &\quad + \left[g(x_0)\underline{f}'_-(x_0), g(x_0)\underline{f}'_+(x_0)\right] \\ &= \left[\left(g\underline{f}\right)'_-(x_0), \left(g\underline{f}\right)'_+(x_0)\right] \\ &= (g \cdot F)'(x_0). \end{aligned}$$

Therefore $g \cdot F$ is $(iii)gH$-differentiable at x_0 and (3.18) holds.

The other cases can be dealt with similarly, so we omit their proofs. □

3.1.4 Composition of a differentiable real-valued function and a gH-differentiable interval-valued function

This section deals with the gH-differentiability of the composition of a gH-differentiable interval-valued function, F, and another function. We begin by considering the case where F is composed with a real-valued differentiable function and derive a *chain rule*.

Theorem 3.1.19. *Let $F : T \to \mathbb{I}$ be an interval-valued function gH-differentiable at y_0, and $U \subseteq \mathbb{R}$ be an open set, let $g : U \to \mathbb{R}$ be a real-valued function differentiable at x_0 so that $g(U) \subseteq T$ with $y_0 = g(x_0)$. Then the composite function $(F \circ g) = F(g(x))$ is gH-differentiable at x_0 and $(F \circ g)'(x_0) = F'(y_0).g'(x_0)$.*

Proof. Let us first assume that F is $(k)gH$-differentiable at y_0, for some $k \in \{i, ii\}$. Then $\underline{f} \circ g$ and $\overline{f} \circ g$ are differentiable at x_0. From Theorem 3.0.2, $F \circ g$ is gH-differentiable and

$$\begin{aligned}
(F\circ g)'(x_0) &= \left[\min\left\{(\underline{f}\circ g)'(x_0),(\overline{f}\circ g)'(x_0)\right\},\max\left\{(\underline{f}\circ g)'(x_0),(\overline{f}\circ g)'(x_0)\right\}\right]\\
&= \Big[\min\left\{\underline{f}'(g(x_0))\cdot g'(x_0),\overline{f}'(g(x_0))\cdot g'(x_0)\right\},\\
&\quad \max\left\{\underline{f}'(g(x_0))\cdot g'(x_0),\overline{f}'(g(x_0))\cdot g'(x_0)\right\}\Big]\\
&= g'(x_0)\cdot\left[\min\left\{\underline{f}'(y_0),\overline{f}'(y_0)\right\},\max\left\{\underline{f}'(y_0),\overline{f}'(y_0)\right\}\right]\\
&= F'(y_0)\cdot g(x_0).
\end{aligned}$$

Now, let us assume that F is $(k)gH$-differentiable at y_0, for some $k\in\{iii,iv\}$. Then the lateral derivatives $(\underline{f}\circ g)'_-(x_0),(\underline{f}\circ g)'_+(x_0),(\overline{f}\circ g)'_-(x_0)$ and $(\overline{f}\circ g)'_+(x_0)$ exist. We also have that

$$(\underline{f}\circ g)'_-(x_0)=\underline{f}'_-(g(x_0))\cdot g'(x_0)=\overline{f}'_+(g(x_0))\cdot g'(x_0)=(\overline{f}\circ g)'_+(x_0)$$

and

$$(\underline{f}\circ g)'_+(x_0)=\underline{f}'_+(g(x_0))\cdot g'(x_0)=\overline{f}'_-(g(x_0))\cdot g'(x_0)=(\overline{f}\circ g)'_-(x_0)$$

Therefore $F\circ g$ is gH-differentiable. In addition,

$$\begin{aligned}
(F\circ g)'(x_0) &= \left[\min\left\{(\underline{f}\circ g)'_-(x_0),(\overline{f}\circ g)'_-(x_0)\right\},\max\left\{(\underline{f}\circ g)'_-(x_0),(\overline{f}\circ g)'_-(x_0)\right\}\right]\\
&= \Big[\min\left\{\underline{f}'_-(g(x_0))\cdot g'(x_0),\overline{f}'_-(g(x_0))\cdot g'(x_0)\right\},\\
&\quad \max\left\{\underline{f}'_-(g(x_0))\cdot g'(x_0),\overline{f}'_-(g(x_0))\cdot g'(x_0)\right\}\Big]\\
&= g'(x_0)\cdot\left[\min\left\{\underline{f}'_-(y_0),\overline{f}'_-(y_0)\right\},\max\left\{\underline{f}'_-(y_0),\overline{f}'(y_0)_-\right\}\right]\\
&= F'(y_0)\cdot g(x_0).
\end{aligned}$$

□

Next we consider other types of composite functions and derive their associated chain rule.

Definition 3.1.20. *Let $\varphi:\mathbb{I}\to\mathbb{R}^2$ be a vector-valued function of an interval variable $Y=\left[\underline{y},\overline{y}\right]$ defined by $\varphi\left(\left[\underline{y},\overline{y}\right]\right)=(\varphi_1(\underline{y},\overline{y}),\varphi_2(\underline{y},\overline{y}))$, were $\varphi_1,\varphi_2:\mathbb{R}^2\to\mathbb{R}$ are real-valued functions. We say that φ is differentiable at $Y_0\in\mathbb{I}$, if φ_1 and φ_2 are differentiable at $(\underline{y_0},\overline{y_0})$ and*

$$\varphi'(Y_0)=\begin{pmatrix}\dfrac{\partial\varphi_1}{\partial\underline{y}}(\underline{y_0},\overline{y_0}) & \dfrac{\partial\varphi_1}{\partial\overline{y}}(\underline{y_0},\overline{y_0})\\ \dfrac{\partial\varphi_2}{\partial\underline{y}}(\underline{y_0},\overline{y_0}) & \dfrac{\partial\varphi_2}{\partial\overline{y}}(\underline{y_0},\overline{y_0})\end{pmatrix}.$$

We denote by $\mathcal{M}_{m\times n}(\mathbb{R})$ the set of $m\times n$ matrices with elements taking values in $\mathbb{R}$.

Definition 3.1.21. *Let $C \in \mathcal{M}_{2\times 2}(\mathbb{R})$ have elements*

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

and $A = [\underline{a}, \overline{a}] \in \mathbb{I}$. The $*$-product of C and A, denoted by $C * A$, is the element of $\mathcal{M}_{2\times 1}(\mathbb{R})$ defined by

$$C * A = \begin{pmatrix} c_{11}.\underline{a} + c_{12}.\overline{a} \\ c_{21}.\underline{a} + c_{22}.\overline{a} \end{pmatrix}. \tag{3.19}$$

Theorem 3.1.22. *Let $V \subseteq \mathbb{I}$ be an open set. Suppose that $F : T \to \mathbb{I}$ is $(k)gH$ differentiable at x_0 for some $k \in \{i, ii\}$, $\varphi = (\varphi_1, \varphi_2) : V \to \mathbb{R}^2$ is differentiable at $Y_0 = [\underline{y_0}, \overline{y_0}] := F(x_0)$ and $F(T) \subset V$. Then $\varphi \circ F : T \to \mathbb{R}^2$ is differentiable at x_0, with*

$$(\varphi \circ F)'(x_0) = \left[(-1)^{m+1}\varphi'(Y_0)\right] * \left[(-1)^{m+1} \cdot F'(x_0)\right],$$

where $m = 1$ if $k = i$ and $m = 2$ if $k = ii$.

Proof. By hypotheses, if $F := \left[\underline{f}, \overline{f}\right]$ is $(k)gH$-differentiable for $k \in \{i, ii\}$ at x_0, then $\underline{f}, \overline{f}$ are differentiable at x_0. Since φ_1, φ_2 are also differentiable at $(\underline{y_0}, \overline{y_0})$, from the classic chain rule we get that

$$(\varphi_1(\underline{f}, \overline{f}), \varphi_2(\underline{f}, \overline{f})) \tag{3.20}$$

is differentiable at x_0 and

$$\begin{aligned}
(\varphi_1(\underline{f}, \overline{f}), \varphi_2(\underline{f}, \overline{f}))'(x_0) &= (\varphi_1, \varphi_2)'(\underline{y_0}, \overline{y_0}).(\underline{f}, \overline{f})'(x_0) \\
&= \begin{pmatrix} \dfrac{\partial \varphi_1}{\partial \underline{y}}(\underline{y_0}, \overline{y_0}) & \dfrac{\partial \varphi_1}{\partial \overline{y}}(\underline{y_0}, \overline{y_0}) \\ \dfrac{\partial \varphi_2}{\partial \underline{y}}(\underline{y_0}, \overline{y_0}) & \dfrac{\partial \varphi_2}{\partial \overline{y}}(\underline{y_0}, \overline{y_0}) \end{pmatrix} \cdot \begin{pmatrix} \dfrac{\partial \underline{y}}{\partial \underline{x}}(x_0) \\ \dfrac{\partial \overline{y}}{\partial x}(x_0) \end{pmatrix} \\
&= \begin{pmatrix} \dfrac{\partial \varphi_1}{\partial \underline{y}}(\underline{y_0}, \overline{y_0}).\dfrac{\partial \underline{y}}{\partial x}(x_0) + \dfrac{\partial \varphi_1}{\partial \overline{y}}(\underline{y_0}, \overline{y_0}).\dfrac{\partial \overline{y}}{\partial x}(x_0) \\ \dfrac{\partial \varphi_2}{\partial \underline{y}}(\underline{y_0}, \overline{y_0}).\dfrac{\partial \underline{y}}{\partial x}(x_0) + \dfrac{\partial \varphi_2}{\partial \overline{y}}(\underline{y_0}, \overline{y_0}).\dfrac{\partial \overline{y}}{\partial x}(x_0) \end{pmatrix} \\
&= \begin{pmatrix} \dfrac{\partial \varphi_1}{\partial \underline{y}}(\underline{y_0}, \overline{y_0}).\underline{f}'(x_0) + \dfrac{\partial \varphi_1}{\partial \overline{y}}(\underline{y_0}, \overline{y_0}).\overline{f}'(x_0) \\ \dfrac{\partial \varphi_2}{\partial \underline{y}}(\underline{y_0}, \overline{y_0}).\underline{f}'(x_0) + \dfrac{\partial \varphi_2}{\partial \overline{y}}(\underline{y_0}, \overline{y_0}).\overline{f}'(x_0) \end{pmatrix}
\end{aligned} \tag{3.21}$$

From (3.20), $\varphi \circ F$ is differentiable, because

$$(\varphi \circ F)(x_0) = (\varphi_1(\underline{f}(x_0), \overline{f}(x_0)), \varphi_2(\underline{f}(x_0), \overline{f}(x_0))).$$

Next we consider separately the cases $k = i$ and $k = ii$.

$(k = i)$ The $(i)gH$-differentiability of F implies that $F'(x_0) = \left[\underline{f}'(x_0), \overline{f}'(x_0)\right]$. By the $*$-product definition we have that

$$\begin{aligned}
\varphi'(Y_0) * F'(x_0) &= \begin{pmatrix} \frac{\partial \varphi_1}{\partial \underline{y}}(\underline{y_0}, \overline{y_0}) & \frac{\partial \varphi_1}{\partial \overline{y}}(\underline{y_0}, \overline{y_0}) \\ \frac{\partial \varphi_2}{\partial \underline{y}}(\underline{y_0}, \overline{y_0}) & \frac{\partial \varphi_2}{\partial \overline{y}}(\underline{y_0}, \overline{y_0}) \end{pmatrix} * \left[\underline{f}'(x_0), \overline{f}'(x_0)\right] \\
&= \begin{pmatrix} \frac{\partial \varphi_1}{\partial \underline{y}}(\underline{y_0}, \overline{y_0}).\underline{f}'(x_0) + \frac{\partial \varphi_1}{\partial \overline{y}}(\underline{y_0}, \overline{y_0}).\overline{f}'(x_0) \\ \frac{\partial \varphi_2}{\partial \underline{y}}(\underline{y_0}, \overline{y_0}).\underline{f}'(x_0) + \frac{\partial \varphi_2}{\partial \overline{y}}(\underline{y_0}, \overline{y_0}).\overline{f}'(x_0) \end{pmatrix} \qquad (3.22)
\end{aligned}$$

From (3.21) and (3.22) we obtain $(\varphi \circ F)'(x_0) = [\varphi'(Y_0)] * [F'(x_0)]$.

$(k = ii)$ The $(ii)gH$-differentiability of F implies that $F'(x_0) = \left[\overline{f}'(x_0), \underline{f}'(x_0)\right]$. By the $*$-product definition we have that

$$\begin{aligned}
\varphi'(Y_0) * F'(x_0) &= \begin{pmatrix} -\frac{\partial \varphi_1}{\partial \underline{y}}(\underline{y_0}, \overline{y_0}) & -\frac{\partial \varphi_1}{\partial \overline{y}}(\underline{y_0}, \overline{y_0}) \\ -\frac{\partial \varphi_2}{\partial \underline{y}}(\underline{y_0}, \overline{y_0}) & -\frac{\partial \varphi_2}{\partial \overline{y}}(\underline{y_0}, \overline{y_0}) \end{pmatrix} * \left[-\underline{f}'(x_0), -\overline{f}'(x_0)\right] \\
&= \begin{pmatrix} \frac{\partial \varphi_1}{\partial \underline{y}}(\underline{y_0}, \overline{y_0}).\underline{f}'(x_0) + \frac{\partial \varphi_1}{\partial \overline{y}}(\underline{y_0}, \overline{y_0}).\overline{f}'(x_0) \\ \frac{\partial \varphi_2}{\partial \underline{y}}(\underline{y_0}, \overline{y_0}).\underline{f}'(x_0) + \frac{\partial \varphi_2}{\partial \overline{y}}(\underline{y_0}, \overline{y_0}).\overline{f}'(x_0) \end{pmatrix}. \qquad (3.23)
\end{aligned}$$

From, (3.21) and (3.23) we obtain $(\varphi \circ F)'(x_0) = [(-1)\varphi'(Y_0)] * (-1) \cdot [F'(x_0)]$.

Therefore $(\varphi \circ F)'(x_0) = [(-1)^{m+1}\varphi'(Y_0)] * [(-1)^{m+1} \cdot F'(x_0)]$. □

3.2 Conclusion

This chapter dealt with differentiability properties over the algebra of gH-differentiable interval-valued functions. Specifically, we gave conditions for the gH-differentiability of the sum and gH-difference of two gH-differentiable interval-valued functions; also for the product and composition of a differentiable real function and a gH-differentiable interval-valued function. Surprisingly, some expected facts, such us that the sum of two

gH-differentiable interval-valued functions is a gH-differentiable interval-valued function, among others, do not necessarily hold. Some examples and counterexamples illustrated the obtained results.

Chapter 4

Necessary and sufficient conditions for interval optimization problems involving constrained interval arithmetic

This chapter studies optimization conditions for interval optimization problems, for this, we use the constrained interval arithmetic and we began defining some interval partial order in the sense of minimization or maximization problem. Several partial order relations on $\mathbb{I}$ have been introduced in the literature. For instance, the usual order relation is $\preceq_{LU}$ defined by (see [21, 22, 30, 39, 40]).

$$A \preceq_{LU} B \text{ iff } \underline{a} \leq \underline{b} \text{ and } \overline{a} \leq \overline{b}.$$

Now considering the constrained parametric representation of an interval we consider the following order relations on $\mathbb{I}$.

Definition 4.0.1. *For $A,\ B\ \in \mathbb{I}$ recall that $a(\lambda), b(\lambda)$ denote the constrained interval representation of the interval A and B respectively. We write*

(i) $A \overset{\preceq}{=} B$ iff $a(\lambda) \leq b(\lambda),\ \forall \lambda \in [0,1]$;

(ii) $A \preceq B$ iff $A \overset{\preceq}{=} B$ and $A \neq B$; equivalently
$A \preceq B$ iff $a(\lambda) \leq b(\lambda)\ \forall \lambda \in [0,1]$, and there exists $\lambda_0 \in [0,1]$ such that $a(\lambda_0) < b(\lambda_0)$;

(iii) $A \prec B$ *iff* $a(\lambda) < b(\lambda),\ \forall \lambda \in [0,1]$.

The idea of the previous definition of order is to compare parameter value by parameter value (level by level). That is, if we consider two intervals A and B, we take the same level $\lambda \in [0,1]$ in each interval, i.e. we take $a(\lambda)$ and $b(\lambda)$ and compare on the level $a(\lambda)$ and $b(\lambda)$ in $\mathbb{R}$.

4.1 Interval valued function and differentiability

This section considers interval-valued functions $F : \mathbb{R} \to \mathbb{I}$, which are generated from a real-valued function considering the parameters as intervals. For this, we denote by $\mathbb{I}^k$ the product space, i.e.

$$\mathbb{I}^k = \underbrace{\mathbb{I} \times \mathbb{I} \times ... \times \mathbb{I}}_{k \text{ times}}.$$

We also denote by $\mathcal{C}^k$ a k-tuple of k intervals. That is $\mathcal{C}^k \in \mathbb{I}^k$, where

$$\mathcal{C}^k = (C_1, ..., C_k), \quad C_j = [\underline{c_j}, \overline{c_j}], \quad j = 1, ..., k.$$

Since each interval C_j has a constrained parametric representation $c_j(\lambda_j)$ we can write the constrained parametric representation of $\mathcal{C}^k$ by

$$\begin{aligned} \mathcal{C}^k \;\; = \;\; & \Big\{ c(\lambda) \;:\; c(\lambda) = (c_1(\lambda_1), ..., c_k(\lambda_k)), \;\; c_j(\lambda_j) = (\overline{c_j} - \underline{c_j})\lambda_j + \underline{c_j}, \\ & \lambda = (\lambda_1, ..., \lambda_k),\ 0 \le \lambda_j \le 1,\ j = 1, ..., k \Big\}. \end{aligned}$$

Let us consider the function $f : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$. For each $c = (c_1, ..., c_k) \in \mathbb{R}^k$, which are parameters involved with function f, we can write $f_c : \mathbb{R} \to \mathbb{R}$. For instance, f_c can represent the objective function of an optimization problem which has k-parameters (k-coefficient) $c_1, ..., c_k$, with $c_j \in \mathbb{R}$. How can we translate (extend) the function f_c to interval context if the parameters c are intervals $\mathcal{C}^k$? We use (SIA) to obtain an interval-valued function $F_{\mathcal{C}^k} : \mathbb{R} \to \mathbb{I}$ from f_c according Moore [30]. Here we are going to consider constrained interval arithmetic to obtain $F_{\mathcal{C}^k}$ from f_c.

Definition 4.1.1. *([9]) Let $f : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$ be a function and let $c = (c_1, ..., c_k) \in \mathbb{R}^k$ be parameters involved with f. For each vector of intervals $\mathcal{C}^k$, we define a constrained parametric representation of $F_{\mathcal{C}^k}(x)$ by*

$$F_{\mathcal{C}^k}(x) = \left\{ f_{c(\lambda)}(x) \;:\; f_{c(\lambda)} : \mathbb{R} \to \mathbb{R},\ c(\lambda) \in \mathcal{C}^k \right\}. \tag{4.1}$$

Proposition 4.1.2. *Let $f : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$ be a continuous function in the second argument $c \in \mathbb{R}^k$. Then the interval-valued functions $F_{\mathcal{C}^k} : \mathbb{R} \to \mathbb{I}$ given by expression (4.1) is well defined and*

$$F_{\mathcal{C}^k}(x) = \left[\min_{\lambda\in[0,1]^k} f_{c(\lambda)}(x), \max_{\lambda\in[0,1]^k} f_{c(\lambda)}(x) \right], \tag{4.2}$$

for all $x \in \mathbb{R}$.

Proof. Since f is a continuous function in the second argument and $c(\lambda) = (c_1(\lambda_1), ..., c_k(\lambda_k))$, with $\lambda = (\lambda_1, ..., \lambda_k) \in [0,1]^k$, then for each x fixed $f_{c(\lambda)}(x)$ is continuous in λ. So we have that $\min_{c(\lambda)\in\mathcal{C}^k} f_{c(\lambda)}(x)$ and $\max_{c(\lambda)\in C^k} f_{c(\lambda)}(x)$ exist and

$$\min_{\lambda\in[0,1]^k} f_{c(\lambda)}(x) = \min_{c(\lambda)\in\mathcal{C}^k} f_{c(\lambda)}(x) \quad \text{and} \quad \max_{\lambda\in[0,1]^k} f_{c(\lambda)}(x) = \max_{c(\lambda)\in C^k} f_{c(\lambda)}(x).$$

Thus we obtain (4.2). □

Note that if f is continuous in the second argument then the interval-valued function $F_{\mathcal{C}^k}$ is well defined and the interval $F_{\mathcal{C}^k}(x)$ is well defined (characterized) via its constrained parametric representation (4.1).

Example 4.1.3. *Consider the interval-valued function $F_{\mathcal{C}^1} : \mathbb{R} \to \mathbb{I}$ defined by*

$$F_{\mathcal{C}^1}(x) = [1,3]x^2 - 2x.$$

Clearly $F_{\mathcal{C}^1}$ is obtained from $f_c(x) = cx^2 - 2x$ by applying (4.1). In fact, in this case $\mathcal{C}^1 = [1,3]$, $c(\lambda) = 2\lambda + 1$ and the constrained parametric representation of $F_{\mathcal{C}^1}(x)$ is given by

$$F_{\mathcal{C}^1}(x) = \left\{ f_{c(\lambda)}(x) \ : \ \lambda \in [0,1] \right\} = \left\{ (2\lambda+1)x^2 - 2x \ : \ \lambda \in [0,1] \right\}.$$

Since $f_{c(\lambda)}(x)$ is linear in λ, from (4.2), we have

$$F_{\mathcal{C}^1}(x) = [x^2 - 2x, 3x^2 - 2x] = [1,3]x^2 - 2x.$$

Example 4.1.4. *Consider the interval-valued function $F_{\mathcal{C}^2} : \mathbb{R} \to \mathbb{I}$ defined by*

$$F_{\mathcal{C}^2}(x) = [1,2]x^2 - [3,5]x.$$

In this case, the constrained parametric representation of $F_{\mathcal{C}^2}(x)$ is given by

$$\begin{aligned} F_{\mathcal{C}^2}(x) &= \left\{ f_{c(\lambda)}(x) \ : \ \lambda = (\lambda_1, \lambda_2) \in [0,1]^2 \right\} \\ &= \left\{ (\lambda_1 + 1)x^2 - (2\lambda_2 + 3)x \ : \ \lambda_1, \lambda_2 \in [0,1] \right\}. \end{aligned}$$

Next we will give a concept of derivative for an interval-valued function. This concept is based on the differentiability of each element of the constrained parametric representation.

Definition 4.1.5. *Let $X \subset \mathbb{R}$ be an open set and let $F_{\mathcal{C}^k} : X \to \mathbb{I}$ be an interval-valued function. Suppose that $f_{c(\lambda)}$ is differentiable at x_0 for each $\lambda \in [0,1]^k$. Then we define the derivative of $F_{\mathcal{C}^k}$ at x_0, denoted by $F'_{\mathcal{C}^k}(x_0)$, by the constrained parametric representation*

$$F'_{\mathcal{C}^k}(x_0) = \left\{ f'_{c(\lambda)}(x_0) \ : \ c(\lambda) \in \mathcal{C}^k, \ \lambda \in [0,1]^k \right\}.$$

We say that $F_{\mathcal{C}^k}$ is differentiable at $x_0 \in X$ iff $F'_{\mathcal{C}^k}(x_0) \in \mathbb{I}$.

Proposition 4.1.6. *Let $X \subset \mathbb{R}$ be an open set and let $F_{\mathcal{C}^k} : X \to \mathbb{I}$ be an interval-valued function. Suppose that $f_{c(\lambda)}$ is differentiable at x_0 for each $\lambda \in [0,1]^k$ and $f'_{c(\lambda)}(x_0)$ is continuous at λ. Then $F_{\mathcal{C}^k}$ is differentiable and*

$$F'_{\mathcal{C}^k}(x_0) = \left[\min_{\lambda \in [0,1]^k} f'_{c(\lambda)}(x_0) \ , \ \max_{\lambda \in [0,1]^k} f'_{c(\lambda)}(x_0) \right]. \tag{4.3}$$

Proof. Since $f'_{c(\lambda)}(x_0)$ is continuous as a function of λ then we have that $\min_{\lambda \in [0,1]^k} f'_{c(\lambda)}(x_0)$ and $\max_{\lambda \in [0,1]^k} f'_{c(\lambda)}(x_0)$ exist and (4.3) holds. □

4.2 Interval optimization problems

This section considers the following (scalar) interval optimization problem

$$\begin{array}{lll} (\textbf{IO}) & \min & F_{\mathcal{C}^k}(x) \\ & \text{subject to} & x \in X \subseteq \mathbb{R}, \end{array}$$

where $F_{\mathcal{C}^k} : X \to \mathbb{I}$ is an interval-valued function with its constrained parametric representation given by (4.2) and X is an open subset of $\mathbb{R}$.

A way to interpret a solution for problem (IO) is to use the partial order relations given in Definition 4.0.1 and the constrained parametric representation of an interval-valued function (4.1) following a similar solution concept to the Pareto optimal solution. For this we denote by $N_\delta(x^*)$ the δ-neighborhood of x^*.

Definition 4.2.1. *Let $x^* \in X$.*

(i) x^* is said to be a (local) strict minimum for $F_{\mathcal{C}^k}$ iff there does not exist another $x \in X$, $x \neq x^*$, $(x \in X \cap N_\delta(x^*))$ such that $F_{\mathcal{C}^k}(x) \overset{\preceq}{=} F_{\mathcal{C}^k}(x^*)$.

(ii) x^* *is said to be a (local) minimum for* $F_{\mathcal{C}^k}$ *iff there does not exist another* $x \in X$, $x \neq x^*$, $(x \in X \cap N_\delta(x^*))$ *such that* $F_{\mathcal{C}^k}(x) \preceq F_{\mathcal{C}^k}(x^*)$.

(iii) x^* *is said to be a (local) weak minimum for* $F_{\mathcal{C}^k}$ *iff there does not exist another* $x \in X$, $x \neq x^*$, $(x \in X \cap N_\delta(x^*))$ *such that* $F_{\mathcal{C}^k}(x) \prec F_{\mathcal{C}^k}(x^*)$.

Lemma 4.2.2. *If* $x^* \in X$ *is a strict minimum, then* x^* *is a minimum, and consequently* x^* *is a weak minimum.*

Proof. The proof follows immediately from Definition 4.2.1 .

□

Note that the previous minimum definition for interval-valued function are a generalization of minimum concepts for a real function. In fact, if $F_{\mathcal{C}^0}(x) = \{f(x)\}$, where $f : X \to \mathbb{R}$ is a function, we have that:

- x^* is a (local) strict minimum for $F_{\mathcal{C}^0}$ iff x^* is a (local) strict minimum for f; and
- x^* is a (local) (weak) minimum for $F_{\mathcal{C}^0}$ iff x^* is a (local)(weak) minimum for f.

Example 4.2.3. *Let* $F_{\mathcal{C}^1} : \mathbb{R} \to \mathbb{I}$ *be defined, as in the Example 4.1.3, by*

$$F_{\mathcal{C}^1}(x) = [1,3]x^2 - 2x.$$

In this case, the constrained parametric representation of $F_{\mathcal{C}^1}$ *is*

$$F_{\mathcal{C}^1}(x) = \{f_{c(\lambda)}(x) = (2\lambda+1)x^2 - 2x \ : \ \lambda \in [0,1]\}.$$

Then $x^* = 1$ *is a strict minimum for* $F_{\mathcal{C}^1}$. *In fact, if there exists another* $x \in \mathbb{R}$, $x \neq 1$, *such that* $F_{\mathcal{C}^1}(x) \overset{\preceq}{=} F_{\mathcal{C}^1}(1)$ *then*

$$f_{c(\lambda)}(x) \leq f_{c(\lambda)}(1), \ \ \forall \lambda \in [0,1];$$

equivalently, for all $\lambda \in [0,1]$,

$$(2\lambda+1)x^2 - 2x \leq (2\lambda+1) - 2 \Leftrightarrow (2\lambda+1)(x-1)(x+1) \leq 2(x-1).$$

If $x > 1$ *we have, for all* $\lambda \in [0,1]$,

$$(2\lambda+1)(x+1) \leq 2 \Leftrightarrow x \leq \frac{2}{2\lambda+1} - 1 \leq 1,$$

which is absurd. In the same way, if $x < 1$ we have, for all $\lambda \in [0,1]$,

$$(2\lambda + 1)(x + 1) \geq 2 \Leftrightarrow x \geq \frac{2}{2\lambda + 1} - 1,$$

so $x \geq 1$ which is absurd.

Example 4.2.4. *Let $F_{\mathcal{C}^1} : \mathbb{R} \to \mathbb{I}$ be defined by*

$$F_{\mathcal{C}^1}(x) = [1,2] \cdot g(x),$$

where $g : \mathbb{R} \to \mathbb{R}$ is a function defined by

$$g(x) = \begin{cases} x^2 & \textit{if} \quad x \leq 0; \\ 0 & \textit{if} \quad 0 \leq x \leq 1; \\ (x-1)^2 & \textit{if} \quad x \geq 1. \end{cases}$$

Here we have that $x^* = 0$ is a weak minimum for $F_{\mathcal{C}^1}$ but it is not a strict minimum for $F_{\mathcal{C}^1}$. In fact, if $x^* = 0$ is not a weak minimum there exists another $x \in \mathbb{R}$, $x \neq x^*$, such that

$$f_{c(\lambda)}(x) < 0, \quad \forall \lambda \in [0,1],$$

i.e. $(\lambda + 1)g(x) < 0, \quad \forall \lambda \in [0,1]$ which is absurd since $g(x) \geq 0$ for all $x \in \mathbb{R}$. Thus $x^* = 0$ is a weak minimum for $F_{\mathcal{C}^1}$. On the other hand, there exists $x = 1$ such that $F_{\mathcal{C}^1}(1) = F_{\mathcal{C}^1}(0)$ and so $x^* = 0$ is not a strict minimum for $F_{\mathcal{C}^1}$.

4.3 Necessary conditions for interval optimization problems

The stationary point notion plays an important role in classical optimization from both, theoretical and practical point of view. In particular, stationary point is a concept which is used to obtain a necessary condition in optimization. So, in this section we propose a concept of stationary points to differentiable interval-valued functions and then give a necessary condition to (IO) based on this concept.

Definition 4.3.1. *Let $F_{\mathcal{C}^k} : X \to \mathbb{I}$ be a differentiable interval-valued function. Then $x^* \in X$ is a stationary point for $F_{\mathcal{C}^k}$ iff $0 \in F'_{\mathcal{C}^k}(x^*)$.*

We denote by $SP(F_{\mathcal{C}^k})$ the set of all stationary points of $F_{\mathcal{C}^k}$. Note that if $F_{\mathcal{C}^k}$ is differentiable then the constrained parametric representation

$$F'_{\mathcal{C}^k}(x) = \left\{ f'_{c(\lambda)}(x) \; : \; \lambda \in [0,1]^k, c(\lambda) \in \mathcal{C}^k \right\}. \tag{4.4}$$

is well defined (see Proposition 4.1.6). So we have the following characterization of stationary points.

Proposition 4.3.2. *Let $F_{\mathcal{C}^k} : X \to \mathbb{I}$ be an differentiable interval-valued function. Then $x^* \in X$ is a stationary point for $F_{\mathcal{C}^k}$ iff there exists $\lambda_0 \in [0,1]$ such that*

$$f'_{c(\lambda_0)}(x^*) = 0.$$

Proof. It is a consequence of the constrained parametric representation of $F'_{\mathcal{C}^k}(x)$ (4.4) and Definition 4.3.1. □

Proposition 4.3.2 is a characterization of stationary point and it is a very useful tool to obtain these points. The following examples show this fact.

Example 4.3.3. *Let $F_{\mathcal{C}^k}$ be an interval-valued function defined as in the Example 4.1.3. Then the constrained parametric representation is given by*

$$F_{\mathcal{C}^1}(x) = \{ f_{c(\lambda)}(x) = (2\lambda+1)x^2 - 2x \; : \; \lambda \in [0,1] \}.$$

Since $f_{c(\lambda)}(x) = (2\lambda+1)x^2 - 2x$ is differentiable at $x \in \mathbb{R}$ and $f'_{c(\lambda)}(x) = 2(2\lambda+1)x - 2$ is continuous at λ then $F_{\mathcal{C}^1}$ is differentiable. Taking into account Proposition 4.3.2 we find the stationary points of $F_{\mathcal{C}^1}$. In fact, the stationary points are such that

$$f'_{c(\lambda)}(x) = 0 \Leftrightarrow 2(2\lambda+1)x - 2 = 0 \Leftrightarrow x = \frac{1}{2\lambda+1},$$

with $\lambda \in [0,1]$. Therefore,

$$SP\left(F_{\mathcal{C}^1}\right) = \left[\frac{1}{3}, 1\right].$$

Next we present a necessary condition for the interval optimization problem (IO).

Theorem 4.3.4. *Let $F_{\mathcal{C}^k} : X \to \mathbb{I}$ be a differentiable interval-valued function. If $x^* \in X$ is a local weak minimum for $F_{\mathcal{C}^k}$ then x^* is a stationary point for $F_{\mathcal{C}^k}$.*

Proof. Suppose that x^* is not a stationary point for $F_{\mathcal{C}^k}$, i.e $0 \notin F'_{\mathcal{C}^k}(x^*)$. Then either $F'_{\mathcal{C}^k}(x^*) \subset \mathbb{R}^+$ or $F'_{\mathcal{C}^k}(x^*) \in \mathbb{R}^-$, where $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ and $\mathbb{R}^- = \{x \in \mathbb{R} : x < 0\}$. If $F'_{\mathcal{C}^k}(x^*) \subset \mathbb{R}^+$ we have

$$f'_{c(\lambda)}(x^*) > 0, \quad \forall \lambda \in [0,1]^k.$$

So, for each $\lambda \in [0,1]^k$, we have that $f_{c(\lambda)}$ is an increasing function in a neighborhood $N_{\delta_\lambda}(x^*)$. Considering ϵ small enough such that $\epsilon < \delta = \min\{\delta_\lambda \ : \ \lambda \in [0,1]^k\}$, we get $f_{c(\lambda)}(x^* - \epsilon) < f_{c(\lambda)}(x^*)$ for all $\lambda \in [0,1]^k$. Thus $F_{\mathcal{C}^k}(x^* - \epsilon) \prec F_{\mathcal{C}^k}(x^*)$, which is a contradiction to the hypotheses.

Similarly, if $F'_{\mathcal{C}^k}(x^*) \subset \mathbb{R}^-$ we have $F_{\mathcal{C}^k}(x^* + \epsilon) \prec F_{\mathcal{C}^k}(x^*)$, which is also a contradiction with the hypotheses. $\square$

Note that the converse of Theorem 4.3.4 is not true, that is, a stationary point for $F_{\mathcal{C}^k}$ is not necessarily a local weak minimum for $F_{\mathcal{C}^k}$. In fact, if we consider $F_{\mathcal{C}^1}(x) = [1,2]x^3$ we have that $x^* = 0$ is a stationary point for $F_{\mathcal{C}^1}$ but it is not a local weak minimum for $F_{\mathcal{C}^1}$.

4.4 Sufficient conditions to interval optimization problem

This Section presents sufficient conditions to problem (IO) using conditions of convexity. For that, we give the following definition.

Definition 4.4.1. *Let $F_{\mathcal{C}^k} : X \to \mathbb{I}$ be a differentiable interval-valued function. We say that $F_{\mathcal{C}^k}$ is invex iff there exists a function $\eta : X \times X \to X$ such that*

$$F_{\mathcal{C}^k}(x) - F_{\mathcal{C}^k}(y) \stackrel{\succeq}{=} F'_{\mathcal{C}^k}(y) \cdot \eta(x,y),$$

for all $x, y \in X$.

Note that, from previous definition and from constrained parametric representation of $F_{\mathcal{C}^k}(x)$ we have the following equivalence: $F_{\mathcal{C}^k}$ is invex iff there exists a function $\eta : X \times X \to X$ such that

$$f_{c(\lambda)}(x) - f_{c(\lambda)}(y) \geq f'_{c(\lambda)}(y) \cdot \eta(x,y),$$

for all $x, y \in X$ and for all $\lambda \in [0,1]^k$. So we have the following result.

Proposition 4.4.2. *Let $F_{\mathcal{C}^k} : X \to \mathbb{I}$ be a differentiable interval-valued function. Then $F_{\mathcal{C}^k}$ is invex with respect to η iff $f_{c(\lambda)}$ is invex with respect to same η for all $\lambda \in [0,1]^k$.*

Proof. The proof is immediate. □ □

Theorem 4.4.3. *Let $F_{\mathcal{C}^k} : X \to \mathbb{I}$ be a differentiable interval-valued function. If $F_{\mathcal{C}^k}$ is invex then each stationary point is a local weak minimum for $F_{\mathcal{C}^k}$.*

Proof. We suppose that x^* is a stationary point for $F_{\mathcal{C}^k}$ and it is not a local weak minimum for $F_{\mathcal{C}^k}$. Then from Proposition 4.3.2 there exists $\lambda_0 \in [0,1]^k$ such that

$$f'_{c(\lambda_0)}(x^*) = 0$$

and there is another $x \in X \cap N_\delta(x^*)$ such that $F_{\mathcal{C}^k}(x) \prec F_{\mathcal{C}^k}(x^*)$, i.e.

$$f_{c(\lambda)}(x) < f_{c(\lambda)}(x^*), \quad \text{for all } \lambda \in [0,1]^k.$$

Since $F_{\mathcal{C}^k}$ is invex, from Proposition 4.4.2, there exists a function $\eta : X \times X \to X$ such that

$$f_{c(\lambda_0)}(x) - f_{c(\lambda_0)}(x^*) \geq f'_{c(\lambda_0)}(x^*) \cdot \eta(x,y) = 0$$

for all $x \in X$. Therefore

$$f_{c(\lambda_0)}(x) \geq f_{c(\lambda_0)}(x^*),$$

which is a contradiction. □

A concept of convexity for interval-valued functions was introduced in [9].

Definition 4.4.4. *([9]) Let $F_{\mathcal{C}^k} : X \to \mathbb{I}$ be an differentiable interval-valued function. We say that $F_{\mathcal{C}^k}$ is convex iff*

$$F_{\mathcal{C}^k}(\gamma x + (1-\gamma)y) \stackrel{\preceq}{=} \gamma F_{\mathcal{C}^k}(x) + (1-\gamma)F_{\mathcal{C}^k}(y),$$

for all $x, y \in X$ and $\gamma \in [0,1]$.

Remark 4.4.5. *From previous definition of convexity and from constrained parametric representation of $F_{\mathcal{C}^k}$ we have that $F_{\mathcal{C}^k}$ is convex iff $f_{c(\lambda)}$ is convex for all $\lambda \in [0,1]$. Thus, from Proposition 4.4.2, all convex interval-valued functions are invex interval-valued functions.*

Therefore convexity for interval-valued functions is also a sufficient condition to assure that each stationary point is a local weak minimum.

Example 4.4.6. *Let $F_{\mathcal{C}^1}$ be an interval-valued function defined as in the Example 4.1.3. Then the constrained parametric representation is given by*

$$F_{\mathcal{C}^1}(x) = \{f_{c(\lambda)}(x) = (2\lambda+1)x^2 - 2x \ : \ \lambda \in [0,1]\}.$$

From Example 4.3.3 we have that $F_{\mathcal{C}^1}$ is differentiable and

$$SP\left(F_{\mathcal{C}^1}\right) = \left[\frac{1}{3}, 1\right].$$

Since $f_{c(\lambda)}(x) = (2\lambda+1)x^2 - 2x$ is a convex function for each $\lambda \in [0,1]$ then $F_{\mathcal{C}^1}$ is convex. So from Theorem 4.4.3 we have that $\left[\frac{1}{3}, 1\right]$ is the set of all weak minimum for $F_{\mathcal{C}^1}$.

Example 4.4.7. *We consider the interval-valued function $F_{\mathcal{C}^2} : \mathbb{R} \to \mathbb{I}$ defined as in the Example 4.1.4, that is*

$$F_{\mathcal{C}^2}(x) = [1,2]x^2 - [3,5]x.$$

In this case, the constrained parametric representation of $F_{\mathcal{C}^2}(x)$ is given by

$$F_{\mathcal{C}^2}(x) = \left\{(\lambda_1+1)x^2 - (2\lambda_2+3)x \ : \ \lambda_1, \lambda_2 \in [0,1]\right\},$$

which is differentiable. So, taking into account Proposition 4.3.2, we obtain the stationary points for $F_{\mathcal{C}^2}$ by solving

$$x = \frac{2\lambda_2+3}{2(\lambda_1+1)},$$

with $\lambda_1, \lambda_2 \in [0,1]$. Thus

$$SP\left(F_{\mathcal{C}^2}\right) = \left[\frac{3}{4}, \frac{5}{2}\right].$$

Since $F_{\mathcal{C}^2}$ is convex, from Theorem 4.4.3, we have that $\left[\frac{3}{4}, \frac{5}{2}\right]$ is the set of all weak minimum for $F_{\mathcal{C}^2}$.

Next we show that invextity is both a necessary and a sufficient condition for every stationary point being an optimal solution.

Theorem 4.4.8. *Let $F_{\mathcal{C}^k} : X \to \mathbb{I}$ be an differentiable interval-valued function. Then $F_{\mathcal{C}^k}$ is invex iff each stationary point is a local weak minimum for $F_{\mathcal{C}^k}$.*

Proof. Because of Theorem 4.4.3, it suffices to prove the converse. Let $x, y \in X$. If $0 \in F'_{\mathcal{C}^k}(y)$ we take $\eta(x,y) = 0$. Now if $0 \notin F'_{\mathcal{C}^k}(y)$ then $F'_{\mathcal{C}^k}(y) \subset \mathbb{R}^+$ or $F'_{\mathcal{C}^k}(y) \subset \mathbb{R}^-$. If $F'_{\mathcal{C}^k}(y) \subset \mathbb{R}^+$ then we take

$$\eta(x,y) = \min_{\lambda \in [0,1]^k} \frac{f_{c(\lambda)}(x) - f_{c(\lambda)}(y)}{f'_{c(\lambda)}(y)},$$

and so we have

$$\frac{f_{c(\lambda)}(x) - f_{c(\lambda)}(y)}{f'_{c(\lambda)}(y)} \geq \eta(x, y).$$

Therefore

$$f_{c(\lambda)}(x) - f_{c(\lambda)}(y) \geq \eta(x, y) f'_{c(\lambda)}(y),$$

for all $\lambda \in [0, 1]^k$. Now, if $F'_{\mathcal{C}^k}(y) \subset \mathbb{R}^-$ then we take

$$\eta(x, y) = \max_{\lambda \in [0,1]^k} \frac{f_{c(\lambda)}(x) - f_{c(\lambda)}(y)}{f'_{c(\lambda)}(y)},$$

and so we have

$$\frac{f_{c(\lambda)}(x) - f_{c(\lambda)}(y)}{f'_{c(\lambda)}(y)} \leq \eta(x, y).$$

Therefore

$$f_{c(\lambda)}(x) - f_{c(\lambda)}(y) \geq \eta(x, y) f'_{c(\lambda)}(y),$$

for all $\lambda \in [0, 1]^k$. This completes the proof. □

4.5 Interval optimization problems with inequality constraints

This section considers the following (scalar) interval optimization problem with interval inequality constraints

$$\begin{array}{lll} (\mathbf{CIO}) & \min & F_{\mathcal{C}^k}(x) \\ & \text{subject to} & G_{i,\mathcal{C}^{l_i}}(x) \overset{\preceq}{=} 0, \;\; i = 1, 2, ..., m \\ & & x \in X \end{array}$$

where $F_{\mathcal{C}^k}, G_{i,\mathcal{C}^{l_i}} : X \to \mathbb{I}$ are interval-valued functions, every $G_{i,\mathcal{C}^{l_i}}$ is a constraint of the problem (**CIO**) and X is a non null open subset of $\mathbb{R}$.

We will consider below, the conditions that must be satisfied so that a certain feasible point of the problem (**CIO**) be optimal. Such conditions, commonly known as first order conditions, involve the first order interval derivative. We also present constraint interval versions of well known optimization results.

We denote by

$$M = \{x \in X : G_{i,\mathcal{C}^{l_i}}(x) \overset{\preceq}{=} 0, \;\; i = 1, 2, ..., m\}$$

the feasible solution set of problem (**CIO**).

We define $I = \{1, 2, .., m\}$ for simplicity, and for every feasible point $x \in M$, the set of index of the active constraints:

$$I(x) = \left\{ i \in I : 0 \in G_{i,\mathcal{C}^{l_i}}(x) \right\}.$$

As in the last sections, we use the constrained parametric representation given by (4.2) for all interval expressions.

Remark 4.5.1. *We associate to the* (**CIO**) *problem its equivalent constrained parametric representation given by*

$$\begin{array}{lll} (\mathbf{CCIO}) & \min & f_{c(\lambda^k)}(x) \\ & \textit{subject to} & g_{i,c(\lambda^{l_i})}(x) \leq 0, \;\; i = 1, 2, ..., m \\ & & x \in X, \end{array}$$

where $\lambda^k \in [0,1]^k$ *is a vector with* k *components where each component is related to its respective component of the interval vector* $\mathcal{C}^k$*,* $\lambda^{l_i} \in [0,1]^l$*,* l_i*. This means that there are* l *new parameters in the constraint* i*, where* $f_{c(\lambda^k)}, g_{i,c(\lambda^{l_i})} : \mathbb{R} \to \mathbb{R}$ *are differentiable functions.*

It is clear from the Remark 2.3.2 that the coordinates of the parameters $c(\lambda^k)$ *and* $c(\lambda^{l_i})$*,* $i = 1, 2, ..., m$ *will be interdependent, as in the next example.*

Example 4.5.2. *Consider the problem*

$$\begin{array}{lll} (\mathbf{CIO1}) & \min & [0,3]x^2 + [-1,2]x + [1,4] \\ & \textit{subject to} & [0,3]x + [-2,0] \stackrel{\preceq}{=} 0 \\ & & [-1,2]x \stackrel{\preceq}{=} 0. \end{array}$$

Here

- $F_{\mathcal{C}^3}(x) = [0,3]x^2 + [-1,2]x + [1,4]$*;*
- $G_{1,\mathcal{C}^{1_1}}(x) = [0,3]x + [-2,0]$*;*
- $G_{2,\mathcal{C}^{0_2}}(x) = [-1,2]x$*.*

The interval $[0,3]$ *of the cost function and first constraint are interdependent, as well interval* $[-1,2]$ *in the cost function and constraint 2. It follows from (4.1) and Remark (2.3.2), that:*

- $f_{c(\lambda^3)}(x) = (0 + 3\lambda_1)x^2 + (-1 + 3\lambda_2)x + (1 + 3\lambda_3)$;
- $g_{1,c(\lambda^{1_1})}(x) = (0 + 3\lambda_1)x + (-2 + 2\lambda_4)$;
- $g_{2,c(\lambda^{0_2})}(x) = (-1 + 3\lambda_2)x$.

with $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in [0, 1]^4$*, consequently the constrained equivalent associated problem is*

$$\begin{array}{lll} (\mathbf{CCIO1}) & \min & (0 + 3\lambda_1)x^2 + (-1 + 3\lambda_2)x + (1 + 3\lambda_3) \\ & \textit{subject to} & (0 + 3\lambda_1)x + (-2 + 2\lambda_4) \leq 0, \\ & & (-1 + 3\lambda_2)x \leq 0, \end{array}$$

with $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in [0, 1]^4$.

4.6 Necessary conditions of interval optimality

This section, presents first order interval conditions so that a feasible point of (**CIO**) problem is optimal. For this, we will use the (**CCIO**) equivalent problem. Next, we present a geometrical characterization of local optimality for our problem (**CIO**).

Proposition 4.6.1. *Let* $F_{\mathcal{C}^k}, G_{i,\mathcal{C}^{l_i}} : X \to \mathbb{I}$, $i \in I$ *be a differentiable interval-valued functions of the (**CIO**) problem where* $F_{\mathcal{C}^k}$ *is the objective function and* $G_{i,\mathcal{C}^{l_i}}$ *are the constraints. If* $x^* \in M$ *is a local weak minimum of* $F_{\mathcal{C}^k}$ *over* M*, then the system*

$$\begin{aligned} F'_{\mathcal{C}^k}(x^*) \cdot d &\prec 0 \\ G'_{i,\mathcal{C}^{l_i}}(x^*) \cdot d &\prec 0, \ i \in I(x^*) \end{aligned} \tag{4.5}$$

has no solution $d \in \mathbb{R}$.

Proof. Let $x^* \in M$ a local weak minimum of (**CIO**). Suppose by contradiction that over M, exist a direction $d \in \mathbb{R}$ that resolves the system (4.5). This implies that it also solves the equivalent constrained parametric representation system below

$$\begin{aligned} f'_{c(\lambda^k)}(x^*) \cdot d &< 0, \ \lambda^k \in [0, 1]^k \\ g'_{i,c(\lambda^{l_i})}(x^*) \cdot d &< 0, \ i \in I(x^*) \ \lambda^{l_i} \in [0, 1]^{l_i}. \end{aligned} \tag{4.6}$$

Consequently, exist $\overline{x} \in M$, such that,

$$f_{c(\lambda^k)}(\overline{x}) < f_{c(\lambda^k)}(x^*), \ \forall \lambda^k \in [0, 1]^k.$$

Minimizing and maximizing $f_{c(\lambda^k)}(\overline{x})$ and $f_{c(\lambda^k)}(x^*)$ in λ^k,

$$F_{\mathcal{C}^k}(\overline{x}) < F_{\mathcal{C}^k}(x^*)$$

therefore $\overline{x} \in M$ contradicts the minimality of x^*. $\square$

Another important result in optimization is the Fritz John theorem that has an interval version that we present next.

Theorem 4.6.2. *Let $x^* \in M$ be a local weak minimum of (**CIO**) and $F_{\mathcal{C}^k}, G_{i,\mathcal{C}^{l_i}} : X \to \mathbb{I}$ differentiable interval-valued functions with their constraint parametric representations continuous in λ^k and λ^{l_i}, $i \in I$, respectively. Then, there exist scalars $\delta_0, \delta_i \in \mathbb{R}$, $i \in I$, not all simultaneously zero, such that:*

$$0 \in \delta_0 \cdot F'_{\mathcal{C}^k}(x^*) + \sum_{i \in I} \delta_i \cdot G'_{i,\mathcal{C}^{l_i}}(x^*); \tag{4.7}$$

$$\delta_0, \delta_i \geq 0, \ i \in I; \tag{4.8}$$

$$0 \in \delta_i \cdot G_{i,\mathcal{C}^{l_i}}(x^*), \ i \in I. \tag{4.9}$$

Proof. Let $x^* \in M$, a weak local minimum of (**CIO**) problem. Considering Remark 4.5.1, x^* is also solution of (**CCIO**) for every $(\lambda^k, \lambda^{l_1}, \lambda^{l_2}, ..., \lambda^{l_m}) \in [0,1]^{k+l_1+l_2+...+l_m}$. From the differentiability of $f_{c(\lambda^k)}$ and $g_{c(\lambda^{l_i})}$, $i \in I$, we have that there exist δ_0, δ_i, $i \in I$ not all zeros, such that

$$\delta_0 \cdot f'_{c(\lambda^k)}(x^*) + \sum_{i \in I} \delta_i \cdot g'_{i,c(\lambda^{l_i})}(x^*) = 0$$

$$\delta_0, \delta_i \geq 0, \ i \in I$$

$$\delta_i \cdot g_{i,c(\lambda^{l_i})}(x^*) = 0, \ i \in I.$$

From continuity of $f'_{c(\lambda^k)}(x^*)$ at λ^k, and the continuity of $g'_{i,c(\lambda^{l_i})}(x^*)$ at λ^{l_i}, $\forall i \in I$ we obtain,

$$\begin{aligned} 0 \in \delta_0 \cdot [\min_{\lambda^k \in [0,1]^k} f'_{c(\lambda^k)}(x^*), \max_{\lambda^k \in [0,1]^k} f'_{c(\lambda^k)}(x^*)] \\ + \sum_{i \in I} \delta_i \cdot [\min_{\lambda^k \in [0,1]^k} g'_{i,c(\lambda^{l_i})}(x^*), \max_{\lambda^k \in [0,1]^k} g'_{i,c(\lambda^{l_i})}(x^*)]; \end{aligned}$$

$$\delta_0, \delta_i \geq 0, \ \ i \in I;$$

$$0 \in \delta_i \cdot [\min_{\lambda^k \in [0,1]^k} g_{i,c(\lambda^{l_i})}(x^*), \max_{\lambda^k \in [0,1]^k} g_{i,c(\lambda^{l_i})}(x^*)], \ \ i \in I.$$

The result of the Theorem follows immediately. □

The Karush-Kuhn-Tucker conditions, which provides nonzero multiplier associated with the cost function, are obtained by imposing some constraints qualification. Next we present these types of results in an interval version involving constraint interval arithmetic.

We say that the set $\{V_i\}_{i\in I}$ of interval elements is *independent* if for every $\lambda \in [0,1]$ the set of vectors $\{v_i(\lambda)\}_{i\in I}$ is linearly independent.

Definition 4.6.3. *We say that the (**CIO**) problem satisfies the constraint qualification if the set* $\left\{G'_{i,\mathcal{C}^{l_i}}(x^*)\right\}_{i\in I(x^*)}$ *is independent.*

Theorem 4.6.4. *Let $x^* \in M$ be a weak minimum for (**CIO**), $F_{\mathcal{C}^k}, G_{i,\mathcal{C}^{l_i}} : X \to \mathbb{I}$ be differentiable interval-valued functions with their constraint parametric representations continuous in λ^k and λ^{l_i}, $i \in I$, respectively, and suppose that (**CIO**) problem satisfies the constraint qualification in x^*. Then there exist $\mu_i \in \mathbb{R}, i \in I$, such that*

$$0 \in F'_{\mathcal{C}^k}(x^*) + \sum_{i\in I} \mu_i \cdot G'_{i,\mathcal{C}^{l_i}}(x^*); \tag{4.10}$$

$$\mu_i \geq 0, \ \ i \in I; \tag{4.11}$$

$$0 \in \mu_i \cdot G_{i,\mathcal{C}^{l_i}}(x^*), \ \ i \in I. \tag{4.12}$$

Proof. From the last theorem, there exist multipliers δ_0, δ_i, $(i \in I)$ satisfying the equations (4.7), (4.8) and (4.9). If $\delta_0 = 0$, by equation (4.7) we obtain $\sum_{i\in I} \delta_i \cdot g'_{i,c(\lambda^{l_i})}(x^*) = 0$ with $\delta_i \geq 0$ and not all zero, which contradicts the constraint qualification. Now just define

$$\mu_i = \frac{\delta_i}{\delta_0}, \ \ i \in I,$$

and analogously to the last proof we obtain the desired result. □

A feasible point $x^* \in M$ for (**CIO**) problem is called a *Karush-Kuhn-Tucker point* if there exists $\mu_i \in \mathbb{R}$, $i \in I$ verifying the equations (4.10), (4.11) and (4.12).

4.7 Interval KKT - invexity and sufficient condition.

This section presents results for the class of interval KKT-invex problems. As for the standard class of optimization problems, we show that the class of interval KKT-invex problems is the largest class of interval valued problems, for which, Karush-Kuhn-Tucker conditions are necessary and sufficient for weak global optimality. The next proposition guarantees the sufficiency of Karush-Kuhn-Tucker conditions for a global weak optimality.

Proposition 4.7.1. *If $x^* \in M$ is a Karush-Kuhn-Tucker point of (**CIO**) problem, and the interval valued functions $F_{\mathcal{C}^k}, G_{i,\mathcal{C}^{l_i}} : X \to \mathbb{I}$, $i \in I$ are invex for the same $\eta : X \times X \to X$. Then x^* is a global weak minimum of (**CIO**) problem.*

Proof. Let the (**CCIO**) equivalent problem. Let $x^* \in M$ be a Karush-Kuhn-Tucker point of (**CIO**) problem. Then exist $\mu_i \in \mathbb{R}, i \in I$, such that,

$$f'_{c(\lambda^k)}(x^*) + \sum_{i\in I} \mu_i g'_{i,c(\lambda^{l_i})}(x^*) = 0;$$
$$\mu_i \geq 0,\ i \in I;$$
$$\mu_i g_{i,c(\lambda^{l_i})}(x') = 0,\ i \in I.$$

Suppose that the (**CCIO**) problem is invex, then exist $\eta : X \times X \to X$ such that,

$$\begin{aligned} f_{c(\lambda^k)}(x) - f_{c(\lambda^k)}(x^*) &\geq f'_{c(\lambda^k)}(x^*)\eta(x, x^*) \\ g_{i,c(\lambda^{l_i})}(x) - g_{i,c(\lambda^{l_i})}(x^*) &\geq g'_{i,c(\lambda^{l_i})}(x^*)\eta(x, x^*). \end{aligned}$$

Therefore,

$$\begin{aligned} f_{c(\lambda^k)}(x) - f_{c(\lambda^k)}(x^*) &\geq \left[f'_{c(\lambda^k)}(x^*) + \sum_{i\in I} \mu_i g'_{i,c(\lambda^{l_i})}(x^*) \right] \eta(x, x^*) \\ &\quad + \sum_{i\in I} \mu_i g_{i,c(\lambda^{l_i})}(x^*) - \sum_{i\in I} \mu_i g_{i,c(\lambda^{l_i})}(x) \\ &= -\sum_{i\in I} \mu_i g_{i,c(\lambda^{l_i})}(x) \geq 0. \end{aligned}$$

Thus

$$f_{c(\lambda^k)}(x) \geq f_{c(\lambda^k)}(x^*),\ \lambda_k \in [0,1]^k,$$

$F_{\mathcal{C}^k}(x^*) \prec F_{\mathcal{C}^k}(x)$ for all $x \in X$, and consequently x^* is a global weak minimum of (**CIO**) problem. □

The following definition leads us to a weaker concept than invexity, which still preserves the sufficiency of Karush-Kuhn-Tucker's conditions for optimality. The problem (**CIO**) is called *Interval Karush-Kuhn-Tucker Invex* (or IKKT-invex, for short) if there exists a function $\eta : X \times X \to X$ such that,

$$\begin{aligned} F_{\mathcal{C}^k}(x) - F_{\mathcal{C}^k}(y) &\succeq\!\!\!= F'_{\mathcal{C}^k}(y) \cdot \eta(x,y) \\ -G'_{i,\mathcal{C}^{l_i}}(y) \cdot \eta(x,y) &\succeq\!\!\!= 0,\ i \in I(y) \end{aligned}$$

for all $x, y \in X$.

We next show that, for interval KKT-invex problem the Karush-Kuhn-Tucker conditions are necessary and sufficient for optimality.

Theorem 4.7.2. *The problem (**CIO**) is IKKT-invex, iff every Karush-Kuhn-Tucker point is a global weak minimum of (**CIO**)*

Proof. The sufficiency follows immediately from Proposition 4.7.1 and the IKKT-invex definition. As regards the converse, suppose that every Karush-Kuhn-Tucker point is a weak global minimum of (**CIO**) problem. Let $x, y \in M$. If $F_{\mathcal{C}^k}(x) \prec F_{\mathcal{C}^k}(y)$, then y is not a weak minimum of $F_{\mathcal{C}^k}$, and from hypotheses y is not a Karush-Kuhn-Tucker point. It mean that there are no $\delta_0 > 0$ and $\delta_i \geq 0,\ i \in I(y)$, such that, $0 \in \delta_0 \cdot F'_{\mathcal{C}^k}(y) + \sum_{i \in I} \delta_i \cdot G'_{i,\mathcal{C}^{l_i}}(y)$. Consequently, $\delta_0 \cdot f'_{c(\lambda^k)}(x^*) + \sum_{i \in I} \delta_i \cdot g'_{i,c(\lambda^{l_i})}(x^*) \neq 0,\ \forall (\lambda^k, \lambda^{l_1}, \lambda^{l_2}, ..., \lambda^{l_m}) \in [0,1]^{k+l_1+l_2+...+l_m}$. By classical Motzkin alternative theorem, there is $v \in \mathbb{R}$ that depends on y, such that

$$\begin{aligned} f'_{c(\lambda^k)}(y) \cdot v &> 0, \\ g'_{i,c(\lambda^{l_i})}(y) \cdot v &> 0,\ i \in I(y). \end{aligned}$$

Defining,

$$\eta(x,y) = \max_{\lambda^k \in [0,1]^k} \frac{f_{c(\lambda^k)}(x) - f_{c(\lambda_k)}(y)}{f'_{c(\lambda^k)}(y)}$$

when $f'_{c(\lambda^k)}(y) < 0$, we have

$$f_{c(\lambda^k)}(x) - f_{c(\lambda^k)}(y) \geq \eta(x,y) \cdot f'_{c(\lambda^k)}(y).$$

Hence $F_{\mathcal{C}^k}(x) - F_{\mathcal{C}^k}(y) \succeq\!\!\!= F'_{\mathcal{C}^k}(y) \cdot \eta(x,y)$, and clearly, if $i \in I(y)$, then $-G'_{i,\mathcal{C}^{l_i}}(y) \cdot \eta(x,y) \succeq\!\!\!= 0$. For $f'_{c(\lambda^k)}(y) > 0$, consider

$$\eta(x,y) = \min_{\lambda^k \in [0,1]^k} \frac{f_{c(\lambda^k)}(x) - f_{c(\lambda^k)}(y)}{f'_{c(\lambda^k)}(y)}.$$

Analogously,

$$f_{c(\lambda^k)}(x) - f_{c(\lambda^k)}(y) \geq \eta(x,y) \cdot f'_{c(\lambda^k)}(y),$$

hence $F_{\mathcal{C}^k}(x) - F_{\mathcal{C}^k}(y) \succeq F'_{\mathcal{C}^k}(y) \cdot \eta(x,y)$, and clearly, if $i \in I(y)$, then $-G'_{i,\mathcal{C}^{l_i}}(y) \cdot \eta(x,y) \succeq 0$.

If $F_{\mathcal{C}^k}(x) \succeq F_{\mathcal{C}^k}(y)$ define $\eta(x,y) = 0$ and obtain immediately $F_{\mathcal{C}^k}(x) - F_{\mathcal{C}^k}(y) \succeq F'_{\mathcal{C}^k}(y) \cdot \eta(x,y)$, and $-G'_{i,\mathcal{C}^{l_i}}(y) \cdot \eta(x,y) \succeq 0$ if $i \in I(y)$. □

Example 4.7.3. *Consider the following (scalar) interval optimization problem*

$$\begin{array}{lll} (\mathbf{CIO2}) & \min & [1,3]x^2 - 2x \\ & \textit{subject to} & x - [\frac{1}{2}, \frac{3}{2}] \preceq 0, \\ & & -x \preceq 0 \end{array}$$

*Let x^* be an optimal point of our (**CIO2**) problem. From KKT conditions (Theorem 4.6.4), we can see that $x^* \in [\frac{1}{3}, \frac{1}{2}]$. Consequently, for example, if $x^* = \frac{1}{2}$ it saturates the constraint $G_{1,\mathcal{C}^{1_1}}$ ($I(x^*) = \{1\}$). Since $\{G'_{i,\mathcal{C}^{l_i}}(\frac{1}{2})\}$ is independent the (**CIO2**) problem satisfies the constraint qualification (Definition 4.6.3), and the interval KKT conditions guarantee the existence of $\mu_1, \mu_2 \in \mathbb{R}$ such that*

$$0 \in F'_{\mathcal{C}^k}(x^*) + \sum_{i \in I} \mu_i \cdot G'_{i,\mathcal{C}^{l_i}}(x^*);$$

$$\mu_i \geq 0, \ \ i \in I;$$

$$0 \in \mu_i \cdot G_{i,\mathcal{C}^{l_i}}(x^*), \ \ i \in I.$$

As x^* does not saturate $i = 2$, we have $\mu_2 = 0$.

*$F'_{\mathcal{C}^k}(\frac{1}{2}) = [-1, 1]$, $G'_{1,\mathcal{C}^{1_1}}(\frac{1}{2}) = 1$ and $G'_{2,\mathcal{C}^{0_2}}(\frac{1}{2}) = -1$, so that $\mu_1 = \frac{1}{2}$. It is also clear that our (**CIO2**) problem is IKKT-invex. Therefore from Theorem 4.7.2, x^* is a weak global minimum.*

4.8 Conclusion

This chapter was considered optimization problems without and with constraints where the parameters (coefficient) of the objective function and the constraints are intervals and used the constrained interval arithmetic recently introduced by W. Lodwick [25]. For the unconstrained problem we have introduced a new concept of stationary points for this class of interval-valued functions and given an useful tool to find these points. We have introduced the concept of minimum for this class of interval optimization problems and showed that all minimizers are necessarily stationary points. Moreover, we have introduced the concept of invexity for interval-valued functions and showed that this is a sufficient condition for a stationary point be a minimizer.

We provided both interval Fritz John and interval Karush-Kuhn-Tucker necessary conditions of optimality for the constrained problem. Then we introduced KKT-points and KKT-invex problems and derived the result which shows that the class of interval KKT-invex problems is the largest class in which interval KKT-invex points are global weak minimum.

Chapter 5

Quotient space of intervals

It is well known that the space of all closed and bounded intervals with the standard interval arithmetic is not a linear space; it is a quasilinear space [3, 34]. In particular, an interval does not have inverse element and therefore subtraction does not have many useful properties (see [14, 35]). Then, this section shows an approach trying to study with a vector space structure using quotient spaces, this study was done in [38] and the authors analyze some algebraic and topological properties of this quotient space. they introduce a concept of differentiability for equivalence classes-valued functions and then, they make a comparison with other concepts of differentiability.

The study found in [38] uses the following considerations and bibliographical references that we consider important to quote. The concepts of Hukuhara difference and generalized Hukuhara difference between two intervals [20] and Stefanini& Bede in [35], Radström's embedding theorem [33], the concept of π differentiability for interval-valued functions [6] and some properties of this derivative can be found in [6, 14], quotient spaces of fuzzy numbers [19], which are tools to developing of fuzzy mathematical analysis [32].

5.1 The quotient space of intervals

It is well known that the addition is associative, commutative and its neutral element is $\{0\}$. If $\beta = -1$, scalar multiplication gives the opposite $-A = (-1)A = \{-a : a \in A\}$ but, in general, $A +_s (-) \cdot_s A \neq \{0\}$, that is, the space $\mathbb{I}$ is not a linear space. This fact is a crucial point due the necessity of working on a linear space in order to define in a suitable sense the derivative of interval valued functions. Taking into account this problem, we will introduce a natural equivalence relation between elements of $\mathbb{I}$ which can be used to divide $\mathbb{I}$ into equivalence classes having group properties for the addition operation.

First of all, given $A = [\underline{a}, \overline{a}] \in \mathbb{I}$, then A is said symmetric if $\underline{a} = -\overline{a}$; the class of symmetric intervals of $\mathbb{I}$ will be denoted by $\mathcal{S}$. Then we have the following definition.

Definition 5.1.1 ([38]). *Let $A, B \in \mathbb{I}$, $A = [\underline{a}, \overline{a}], B = [\underline{b}, \overline{b}]$. We say that A is equivalent to B, and write $A \sim B$, if and only if $A - B \in \mathcal{S}$. Here $A - B$ denotes the standard difference defined by $A - B = [\underline{a} - \overline{b}, \overline{a} - \underline{b}]$.*

Let A be an interval given by $A = [\underline{a}, \overline{a}]$. We define the mark of A denoted by M_A, such as

$$M_A = \underline{a} + \overline{a}$$

Theorem 5.1.2. *Let $A, B \in \mathbb{I}$, $A = [\underline{a}, \overline{a}], B = [\underline{b}, \overline{b}]$. $A \sim B$ if and only if, they have the same mark. i.e.*

$$M_A = M_B.$$

Proof. $A \sim B$ if and only if $[\underline{a}, \overline{a}] - [\underline{b}, \overline{b}] = [-c, c] \in \mathcal{S}$ it mean that $\underline{a} - \overline{b} = -c$ and $\overline{a} - \underline{b} = c$ therefore, $\underline{a} + \overline{a} = \underline{b} + \overline{b}$. □

Example 5.1.3. *In this example we show some equivalent intervals.*

- $[-2, 3] \sim [-10, 11] \sim [0, 1] \sim [0.5, 0.5]$,
- $[1, 3] \sim [-1, 5] \sim [-2, 6] \sim [0, 4] \sim [2, 2]$.

The relation $\sim$ is an equivalence relation, that is, $\sim$ is reflexive, symmetric and transitive. We will denote by $\langle A \rangle$ the equivalence class containing the interval $A \in \mathbb{I}$. The set of equivalence classes will be denoted by $\mathbb{I}/\mathcal{S}$. Note that if $\langle A \rangle \in \mathbb{I}/\mathcal{S}$, then from Definition 5.1.1 it holds that

$$\begin{aligned}
\langle A \rangle &= \langle [\underline{a}, \overline{a}] \rangle \qquad (5.1)\\
&= \begin{cases} \langle [0, \underline{a} + \overline{a}] \rangle \text{ if } \underline{a} + \overline{a} \geq 0 \\ \langle [\underline{a} + \overline{a}, 0] \rangle \text{ if } \underline{a} + \overline{a} < 0 \end{cases}\\
&= \langle [\min \{0, \underline{a} + \overline{a}\} \ , \ \max \{0, \underline{a} + \overline{a}\}] \rangle .
\end{aligned}$$

In particular, $\mathcal{S} = \langle [0, 0] \rangle := \langle 0 \rangle$.

Graphically, if we consider Figure 5.1, we can see that the class representatives are on the axes that correspond to the interval space, and every class is represented by a orthogonal line to identity line contained in the interval space (proper interval space).

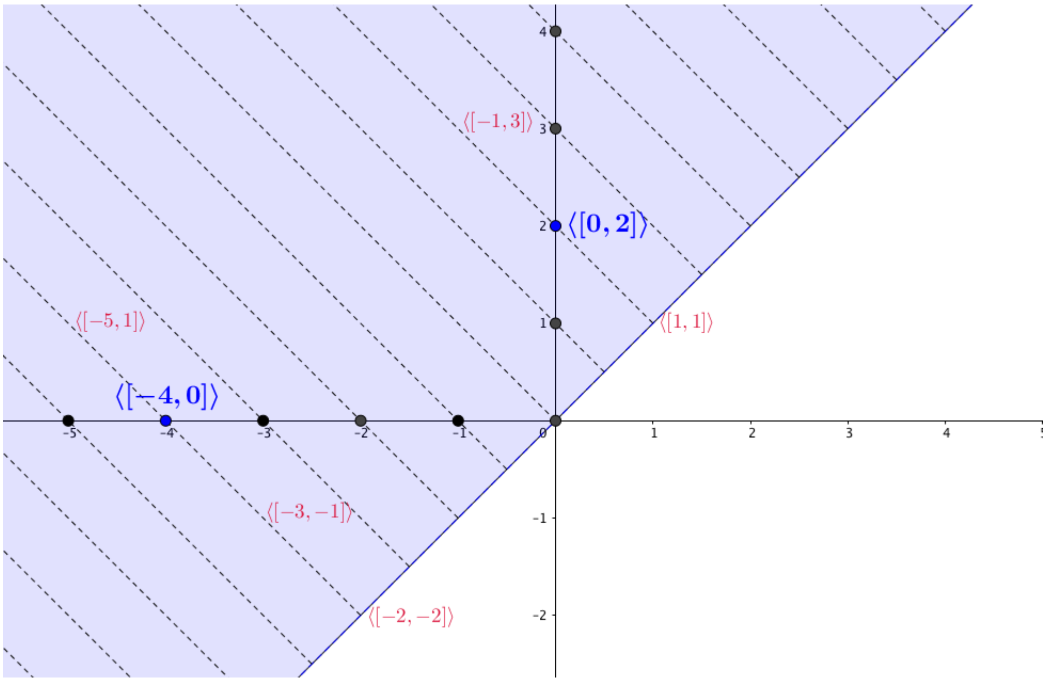

Figure 5.1: Quotient Space of Intervals

For any $\langle A\rangle, \langle B\rangle \in \mathbb{I}/\mathcal{S}$ we define the addition $\langle A\rangle + \langle B\rangle$ by

$$\langle A\rangle + \langle B\rangle = \langle A + B\rangle.$$

Proposition 5.1.4. *If* $\langle A\rangle, \langle B\rangle \in \mathbb{I}/\mathcal{S}$ *then* $M_{A+B} = M_A + M_B$

Proof. The proof is immediately from Theorem 5.1.2. □

Multiplication of an element of $\mathbb{I}/\mathcal{S}$ by a real number λ is the following:

$$\lambda \cdot \langle A\rangle = \langle \lambda \cdot A\rangle.$$

Lemma 5.1.5 ([38]). $(\mathbb{I}/\mathcal{S}, +, \cdot)$ *is a linear space.*

Proposition 5.1.6 ([38]). *Let* $\langle A\rangle, \langle B\rangle \in \mathbb{I}/\mathcal{S}$. *Then*

$$\langle A \ominus_{gH} B\rangle = \langle A\rangle - \langle B\rangle.$$

Proof. Given $A, B \in \mathbb{I}$, taking into account (2.2.2) and (5.1), we have that

$$\begin{aligned}
&\langle A \ominus_{gH} B\rangle \\
= \; & \langle\left[\min\left\{\underline{a}-\underline{b}, \overline{a}-\overline{b}\right\}, \max\left\{\underline{a}-\underline{b}, \overline{a}-\overline{b}\right\}\right]\rangle \\
= \; & \langle\left[\min\left\{0\,,\ (\underline{a}-\underline{b})+(\overline{a}-\overline{b})\right\},\right. \\
& \quad \left.\max\left\{0\,,\ (\underline{a}-\underline{b})+(\overline{a}-\overline{b})\right\}\right]\rangle \\
= \; & \langle\left[\min\left\{0\,,\ (\underline{a}-\overline{b})+(\overline{a}-\underline{b})\right\},\right. \\
& \quad \left.\max\left\{0\,,\ (\underline{a}-\overline{b})+(\overline{a}-\underline{b})\right\}\right]\rangle \\
= \; & \langle\left[\underline{a}-\overline{b}\,,\ \overline{a}-\underline{b}\right]\rangle \\
= \; & \langle A - B\rangle \\
= \; & \langle A\rangle - \langle B\rangle .
\end{aligned}$$

□

We now provide a norm $\|\cdot\|$ on the space $\mathbb{I}/\mathcal{S}$.

Definition 5.1.7 ([38]). *Let $\langle A\rangle = \langle[\underline{a}, \overline{a}]\rangle \in \mathbb{I}/\mathcal{S}$. We define the norm of $\langle A\rangle$ by*

$$\|\langle A\rangle\| = |\underline{a}+\overline{a}|.$$

Remark 5.1.8 ([38]). *$(\mathbb{I}/\mathcal{S}, \|\cdot\|)$ is a normed linear space. Moreover, we have the metric d_{sup} on $\mathbb{I}/\mathcal{S}$ defined by*

$$d_{sup}(\langle A\rangle, \langle B\rangle) = \|\langle A\rangle - \langle B\rangle\|,$$

for all $\langle A\rangle, \langle B\rangle \in \mathbb{I}/\mathcal{S}$. Notice that for $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$, $d_{sup}(\langle A\rangle, \langle B\rangle) = |(\underline{a}+\overline{a}) - (\underline{b}+\overline{b})|$.

The following properties is a immediate consequence.

Proposition 5.1.9 ([38]). *Let $\langle A\rangle, \langle B\rangle, \langle C\rangle \in \mathbb{I}/\mathcal{S}$. Then d_{sup} is translation invariant, that is,*

$$d_{sup}(\langle A\rangle + \langle C\rangle, \langle B\rangle + \langle C\rangle) = d_{sup}(\langle A\rangle, \langle B\rangle);$$

Lemma 5.1.10 ([38]). *$(\mathbb{I}/\mathcal{S}, d_{sup})$ is a complete metric space.*

5.2 Differentiability

This section analyzes the differentiability of interval class mappings $F : T \longrightarrow \mathbb{I}/\mathcal{S}$. From now on, we will denote by $T = [a, b]$ a closed interval. For this section, let $f : T \to \mathbb{I}$ be an interval-valued function. We will denote $f(t) = [\underline{f}(t), \overline{f}(t)]$, where $\underline{f}(t) \le \overline{f}(t)$, for all $t \in T$. The functions $\underline{f}$ and $\overline{f}$ are called the lower and the upper (endpoint) functions of F, respectively.

The usual definition of continuity of mappings between metric spaces will be used. We shall say that a function $F : T \longrightarrow \mathbb{I}/\mathcal{S}$ is continuous at $t_0 \in T$ if for every $\epsilon > 0$ there exists a $\delta = \delta(t_0, \epsilon) > 0$ such that

$$d_{sup}(F(t), F(t_0)) < \epsilon,$$

for all $t \in T$ with $\|t - t_0\| < \delta$.

Definition 5.2.1 ([38]). *A mapping $F : T = [a, b] \longrightarrow \mathbb{I}/\mathcal{S}$ is differentiable at $t_0 \in T$ if there exists an $F'(t_0) \in \mathbb{I}/\mathcal{S}$ such that*

$$\lim_{h\to 0} d_{sup}\left(\frac{F(t_0 + h) - F(t_0)}{h},\ F'(t_0)\right) = 0.$$

If $t_0 = a$ (or $to = b$), then we consider only $h \to 0^+$ (or $h \to 0^-$).

Theorem 5.2.2 ([38]). *Let $F : T \to \mathbb{I}/\mathcal{S}$ such that $F(t) = \langle[\underline{f}(t), \overline{f}(t)]\rangle$ for all $t \in T$. F is differentiable if and only if the mapping $g : T \to \mathbb{R}$ given by $g(t) = \underline{f}(t) + \overline{f}(t)$ is differentiable.*

Theorem 5.2.3. *If $F, G : T \longrightarrow \mathbb{I}/\mathcal{S}$ are differentiable and $\lambda \in \mathbb{R}$, then $F + G$ and λF are differentiable and $(F + G)'(t) = F'(t) + G'(t)$ and $(\lambda F)'(t) = \lambda F'(t)$ for $t \in T$.*

Proof. Let $t, t + h \in T$ with $h \neq 0$. Then it is enough to observe that

$$d_{sup}\left((F + G)'(t), F'(t) + G'(t)\right) \le d_{sup}\left(\frac{(F + G)(t + h) - (F + G)(t)}{h}, (F + G)'(t)\right)$$
$$+ d_{sup}\left(\frac{F(t + h) - F(t)}{h}, F'(t)\right) + d_{sup}\left(\frac{G(t + h) - G(t)}{h}, G'(t)\right) \longrightarrow 0,\ \text{as } h \longrightarrow 0,$$

and

$$\begin{aligned} d_{sup}\left((\lambda F)'(t), \lambda F'(t)\right) &\le d_{sup}\left(\frac{(\lambda F)(t + h) - (\lambda F)(t)}{h}, (\lambda F)'(t)\right) \\ &+ |\lambda| d_{sup}\left(\frac{F(t + h) - F(t)}{h}, F'(t)\right) \longrightarrow 0,\ \text{as } h \longrightarrow 0, \end{aligned}$$

□

The next theorem says that the concept of differentiability given in Definition 5.2.1 generalizes in some sense to gH-differentiability.

Theorem 5.2.4. *Let $f : T \longrightarrow \mathbb{I}/\mathcal{S}$ with $f(t) \in F(t)$. If f is $(k)gH$-differentiable for $k = i, ii, iii, iv$, then F is differentiable on T in the sense of Definition 5.2.1.*

Proof. Since $f(t) \in F(t)$ for all $t \in T$, we have $F(t) = \langle f(t)\rangle$. Now, we divide this proof in two parts.

(a) If f is gH-differentiable in the form (a) of the theorem 3.0.2. Then $\underline{f}$ and $\overline{f}$ are differentiable at $t_0 \in T$ and we get

$$\begin{aligned}
d_{sup}\left(\frac{F(t+h)-F(t)}{h}, \langle f'(t)\rangle\right) &= d_{sup}\left(\frac{\langle[\underline{f}(t+h),\overline{f}(t+h)]\rangle - \langle[\underline{f}(t),\overline{f}(t)]\rangle}{h}, \langle f'(t)\rangle\right)\\
&= \left|\frac{\underline{f}(t+h)+\overline{f}(t+h)-(\underline{f}(t)+\overline{f}(t))}{h} - (\underline{f}'(t)+\overline{f}'(t))\right|\\
&\leq \left|\frac{\underline{f}(t+h)-\underline{f}(t)}{h} - \underline{f}'(t)\right| + \left|\frac{\overline{f}(t+h)-\overline{f}(t)}{h} - \overline{f}'(t)\right|\\
&\to 0 \text{ as } h \to 0.
\end{aligned}$$

(b) If f is gH-differentiable in the form (b) of the theorem 3.0.2. Then $\underline{f}'_-(t_0), \underline{f}'_+(t_0), \overline{f}'_-(t_0)$ and $\overline{f}'_+(t_0)$ exists and satisfy $\underline{f}'_-(t_0) = \overline{f}'_+(t_0)$ and $\underline{f}'_+(t_0) = \overline{f}'_-(t_0)$. Then we get

$$\begin{aligned}
d_{sup}\left(\frac{F(t+h)-F(t)}{h}, \langle f'(t)\rangle\right) &= d_{sup}\left(\frac{\langle[\underline{f}(t+h),\overline{f}(t+h)]\rangle - \langle[\underline{f}(t),\overline{f}(t)]\rangle}{h}, \langle f'(t)\rangle\right)\\
&= \left|\frac{\underline{f}(t+h)+\overline{f}(t+h)-(\underline{f}(t)+\overline{f}(t))}{h} - (\underline{f}'_+(t)+\overline{f}'_+(t))\right|\\
&\leq \left|\frac{\underline{f}(t+h)-\underline{f}(t)}{h} - \underline{f}'_+(t)\right| + \left|\frac{\overline{f}(t+h)-\overline{f}(t)}{h} - \overline{f}'_+(t)\right|\\
&\to 0 \text{ as } h \to 0^+.
\end{aligned}$$

and

$$\begin{aligned}
d_{sup}\left(\frac{F(t+h)-F(t)}{h}, \langle f'(t)\rangle\right) &= d_{sup}\left(\frac{\langle[\underline{f}(t+h),\overline{f}(t+h)]\rangle - \langle[\underline{f}(t),\overline{f}(t)]\rangle}{h}, \langle f'(t)\rangle\right)\\
&= \left|\frac{\underline{f}(t+h)+\overline{f}(t+h)-(\underline{f}(t)+\overline{f}(t))}{h} - (\overline{f}'_-(t)+\underline{f}'_-(t))\right|\\
&\leq \left|\frac{\underline{f}(t+h)-\underline{f}(t)}{h} - \underline{f}'_-(t)\right| + \left|\frac{\overline{f}(t+h)-\overline{f}(t)}{h} - \overline{f}'_-(t)\right|\\
&\to 0 \text{ as } h \to 0^-.
\end{aligned}$$

□

By virtue of Definition 5.2.1 and Lemma 5.2.2, clearly we can obtain the following proposition.

Proposition 5.2.5. *The differentiability given in 5.2.1 is a homogeneous and additive operator, i.e., for differentiable functions* $F, G : T \to \mathbb{I}/\mathcal{S}$ *and for* $\alpha \in \mathbb{R}$

1. $(\alpha F)' = \alpha F'$

2. $(F + G)' = F' + G'$

Proof. The proof is immediately of Theorem 5.2.2. □

With this proposition we can analyze some examples studied in the Chapter 3.

Example 5.2.6. *Considering the interval-valued functions of the Example 3.1.1, we have* $F(x) = \langle[-|x|, |x|]\rangle = \langle[0, 0]\rangle$ *and* $G(x) = \langle[0, e^{-x}]\rangle$ *and* $(F + G)(x) = \langle[-|x|, |x| + e^{-x}]\rangle = \langle[0, e^{-x}]\rangle$*. Consequently, by Lemma 5.2.2, all this functions are differentiable in the sense of Definition, contrary to* gH*-derivative.*

5.3 Conclusion

This chapter showed a quotient space of intervals with respect to the family of symmetric intervals and this quotient space is a normed linear space. Since the space of intervals can be embed on this quotient space, the concept of differentiability for interval-valued functions have linearity properties and we made a comparison with other concepts of differentiability and showed interesting examples.

Chapter 6

Final considerations and future perspectives

The study made in this Thesis is divided into three parts, so we will make the final considerations and future perspectives for each case. About the algebra of generalized Hukuhara differentiable interval-valued functions which was presented here is a vital tool for the development of interval analysis. We can immediately mention some of the future work to be done: interval differential equations, min-max interval optimization problem, numerical methods to find zeros of interval polynomials, and all these presented results, as well as those proposed for future work can be extended to the fuzzy context.

The second part of this Thesis, which are necessary and sufficient conditions for interval optimization problems involving constrained interval arithmetic, the contributions paves the way to new research extending the theory of interval optimization in a number of ways. These include interval optimization problems with n-variables, possibly with fuzzy coefficients, calculus of variations and control optimization problems which contain in their model formulations either interval or fuzzy coefficients.

The quotient space of intervals, which is the third and last part of the Thesis, established a theory that provides an algebraic interval structure of vector space for interval spaces. This makes it possible to study mathematics theory, such as differential equations, dynamical systems, numerical analysis, optimization, control, etc., over interval spaces.

Bibliography

[1] I. Ahmad, A. Jayswal, and J. Banerjee. On interval-valued optimization problems with generalized invex functions. *Journal of Inequalities and Applications*, 2013(1):313, 2013.

[2] T. Allahviranloo and S. Gholami. Note on generalized Hukuhara differentiability of interval-valued functions and interval differential equations. *Journal of Fuzzy set Valued Analysis*, 2012:1–4, 2012.

[3] S. M. Aseev. Quasilinear operators and their application in the theory of multivalued mappings. *Trudy Matematicheskogo Instituta imeni VA Steklova*, 167:25–52, 1985.

[4] J.-P. Aubin and A. Cellina. *Differential Inclusions*. Springer Berlin Heidelberg, 1984.

[5] J.-P. Aubin and H. Frankowska. Differential inclusions. In *Set-Valued Analysis*, pages 1–27. Birkhäuser Boston, October 2008.

[6] H. Banks and M. Q. Jacobs. A differential calculus for multifunctions. *Journal of Mathematical Analysis and Applications*, 29(2):246–272, 1970.

[7] B. Bede. *Mathematics of Fuzzy Sets and Fuzzy Logic*. Springer Berlin Heidelberg, 2013.

[8] B. Bede and L. Stefanini. Generalized differentiability of fuzzy-valued functions. *Fuzzy Sets and Systems*, 230:119–141, November 2013.

[9] A. K. Bhurjee and G. Panda. Efficient solution of interval optimization problem. *Mathematical Methods of Operations Research*, 76(3):273–288, August 2012.

[10] J. Campos, E. Assunção, G. Silva, W. Lodwick, and M. Teixeira. Discrete-time interval optimal control problem. *International Journal of Control*, pages 1–7, 2017.

[11] J. R. Campos, E. Assuncao, G. N. Silva, and W. Lodwick. Biological control of caterpillar sugarcane (diatraea saccharalis) considering interval mathematical models. *BIOMATH*, 5(1):1604232, May 2016.

[12] Y. Chalco-Cano, W. Lodwick, and B. Bede. Single level constraint interval arithmetic. *Fuzzy Sets and Systems*, 257:146–168, December 2014.

[13] Y. Chalco-Cano, W. A. Lodwick, and A. Rufian-Lizana. Optimality conditions of type KKT for optimization problem with interval-valued objective function via generalized derivative. *Fuzzy Optimization and Decision Making*, 12(3):305–322, February 2013.

[14] Y. Chalco-Cano, H. Román-Flores, and M. Jiménez-Gamero. Generalized derivative and π-derivative for set-valued functions. *Information Sciences*, 181(11):2177–2188, June 2011.

[15] Y. Chalco-Cano, A. Rufián-Lizana, H. Román-Flores, and M. Jiménez-Gamero. Calculus for interval-valued functions using generalized Hukuhara derivative and applications. *Fuzzy Sets and Systems*, 219:49–67, May 2013.

[16] T. Costa, Y. Chalco-Cano, W. Lodwick, and G. Silva. Generalized interval vector spaces and interval optimization. *Information Sciences*, 311:74–85, August 2015.

[17] L. C. de Barros, R. C. Bassanezi, and W. A. Lodwick. *A First Course in Fuzzy Logic, Fuzzy Dynamical Systems, and Biomathematics*. Springer Berlin Heidelberg, 2017.

[18] D. Dubois and H. Prade. Inverse operations for fuzzy numbers. *IFAC Proceedings Volumes*, 16(13):399–404, July 1983.

[19] D. H. Hong and H. Y. Do. Additive decomposition of fuzzy quantities. *Information sciences*, 88(1-4):201–207, 1996.

[20] M. Hukuhara. Integration des applications mesurables dont la valeur est un compact convexe. *Funkcialaj Ekvacioj*, 10(3), 1967.

[21] M. Inuiguchi and Y. Kume. Goal programming problems with interval coefficients and target intervals. *European Journal of Operational Research*, 52(3):345–360, June 1991.

[22] H. Ishibuchi and H. Tanaka. Multiobjective programming in optimization of the interval objective function. *European Journal of Operational Research*, 48(2):219–225, September 1990.

[23] U. A. S. Leal. *Incerteza intervalar em otimização e controle.* PhD thesis, Universidade Estadual Paulista, 2015.

[24] L. Li, S. Liu, and J. Zhang. On interval-valued invex mappings and optimality conditions for interval-valued optimization problems. *Journal of Inequalities and Applications*, 2015(1), June 2015.

[25] W. A. Lodwick. *Constrained interval arithmetic.* Citeseer, 1999.

[26] W. A. Lodwick. Interval and fuzzy analysis: A unified approach. In *Advances in Imaging and Electron Physics*, pages 75–192. Elsevier, 2007.

[27] W. A. Lodwick and O. A. Jenkins. Constrained intervals and interval spaces. *Soft Computing*, 17(8):1393–1402, February 2013.

[28] W. A. Lodwick and P. Thipwiwatpotjana. *Flexible and Generalized Uncertainty Optimization.* Springer International Publishing, 2017.

[29] S. Markov. Calculus for interval functions of a real variable. *Computing*, 22(4):325–337, December 1979.

[30] R. E. Moore. Interval analysis. *Prince-Hall, Englewood Cliffs, NJ*, 1969.

[31] R. Osuna-Gómez, Y. Chalco-Cano, B. Hernández-Jiménez, and G. Ruiz-Garzón. Optimality conditions for generalized differentiable interval-valued functions. *Information Sciences*, 321:136–146, November 2015.

[32] D. Qiu, C. Lu, W. Zhang, Q. Zhang, and C. Mu. Basic theorems for fuzzy differential equations in the quotient space of fuzzy numbers. *Advances in Difference Equations*, 2014(1):303, 2014.

[33] H. Rådström. An embedding theorem for spaces of convex sets. *Proceedings of the American Mathematical Society*, 3(1):165–169, 1952.

[34] M. A. Rojas-Medar, M.-D. Jiménez-Gamero, Y. Chalco-Cano, and A. Viera-Brandão. Fuzzy quasilinear spaces and applications. *Fuzzy Sets and Systems*, 152(2):173–190, 2005.

[35] L. Stefanini and B. Bede. Generalized Hukuhara differentiability of interval-valued functions and interval differential equations. *Nonlinear Analysis: Theory, Methods & Applications*, 71(3-4):1311–1328, August 2009.

[36] W. L. Strother. *Continuity for multi-valued functions and some applications to topology.* PhD thesis, Tulane University of Louisiana, 1952.

[37] J. Tao and Z. Zhang. Properties of interval-valued function space under the gH-difference and their application to semi-linear interval differential equations. *Advances in Difference Equations*, 2016(1), February 2016.

[38] E. J. Villamizar-Roa, Y. Chalco-Cano, and H. Roman-Flores. Interval-valued functions in a quotient space. In *2015 Annual Conference of the North American Fuzzy Information Processing Society (NAFIPS) held jointly with 2015 5th World Conference on Soft Computing (WConSC)*. IEEE, aug 2015.

[39] H.-C. Wu. The Karush–Kuhn–Tucker optimality conditions in an optimization problem with interval-valued objective function. *European Journal of Operational Research*, 176(1):46–59, January 2007.

[40] H.-C. Wu. The Karush–Kuhn–Tucker optimality conditions in multiobjective programming problems with interval-valued objective functions. *European Journal of Operational Research*, 196(1):49–60, July 2009.