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DYNAMICAL CHIRAL SYMMETRY BREAKING: THE FERMIONIC GAP EQUATION WITH DYNAMICAL GLUON MASS AND CONFINEMENT

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"No quiero llegar a ser Físico Teórico, ni tampoco Físico Experimental. Quiero que mi formación me conduzca a ser algo más. Quiero llegar a ser... físico".
UdeA 2010

"O físico não só cria os modelos ou realiza os experimentos. O físico, depois de criar os modelos, propõe experimentos. Ou depois de realizar os experimentos, propõe modelos".
IFT-UNESP 2013

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To God, for his care and protection throughout my life.

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Abstract

Some aspects of chiral symmetry breaking for quarks in the fundamental representation are discussed in the framework of the Schwinger-Dyson equations. We study the fermionic gap equation including effects of dynamical gluon mass. Studying the bifurcation equation of this gap equation we verify that the interaction is not strong enough to generate a satisfactory dynamical quark mass. We also discuss how the introduction of a confining propagator may change this scenario as recently pointed out by Cornwall [1], so we study a "complete" gap equation composed by the one-dressed-gluon exchange term and a confining term: $M\left(p^2\right) = M_c\left(p^2\right) + M_{1g}\left(p^2\right)$. We find asymptotic solutions for this gap equation in the cases of "constant coupling" and "running coupling constant". This last case is an improvement of the constant coupling calculation of Doff, Machado and Natale [2].

Key words:: Chiral Symmetry Breaking; Dynamical Quark Mass; Schwinger-Dyson Equations; Confining Propagator.

Knowledge areas: Particle Physics; Quantum Chromodynamics.

Resumo

Alguns aspectos da quebra de simetria quiral para quarks na representação fundamental são discutidos no contexto das equações de Schwinger-Dyson. Estudamos a equação de gap fermionica incluindo o efeito de uma massa dinâmica para os gluons. Ao estudar esta equação de gap verificamos que a interação não é forte o suficiente para gerar uma massa dinâmica dos quarks compatível com os dados experimentais. Também discutimos como a introdução de um propagador confinante pode mudar este cenário, exatamente como foi proposto por Cornwall [1] recentemente, desta forma estudamos uma equação de gap "completa", composta pela troca de um gluon massivo e por um termo confinante: $M\left(p^2\right) = M_c\left(p^2\right) + M_{1g}\left(p^2\right)$. Encontramos soluções assintótica desta equação de gap nos casos de constante de acoplamento "constante" e "corredora". Este último caso corresponde a um aprimoramento do cálculo com constante de acoplamento "constante" feito por Doff, Machado e Natale [2].

Palavras chave: Quebra de simetria quiral; Massa dinâmica de quarks; Equações de Schwinger-Dyson; Propagador confinante.

Areas do conhecimento: Física de particulas; Cromodinamica quântica.

Contents

Chapter 1

Introduction

Symmetries are some of the most important properties that we use to describe nature. Symmetries are related to conservation laws and this is why we really like when a physical system is invariant under spatial translations, rotations or time translation, because this imply conservations of the linear momentum, angular momentum and energy (respectively), and this conservation laws are powerful tools used through all the physics, from classical to quantum mechanics. Despite the symmetries importance, one aspect that is more important is when the symmetries are broken. For example, it is in terms of a spontaneously symmetry breaking (SSB) that Landau described the magnetization of ferromagnetic systems [3], and this is an important model of phase transitions which has been extended to other fields like particle physics leading to the Higgs mechanism [4]. This model of spontaneously symmetry breaking is a success in the electroweak model, where it was introduced in order to describe the gauge boson and fermions mass generation. This process, where we start with a Lagrangian with massless fermions and in the end they obtain masses is called Chiral Symmetry Breaking (CSB). The CSB can occurs in two ways; spontaneously (as in the Landau model), and dynamically (as we will discuss here).

Dynamical chiral symmetry breaking (DCSB) is understood as the mechanism where the masses are created by self-interaction without introduction of scalar fields. For example, a proton is composed by three light quarks, each with a current mass of about 5MeV, but the proton has a mass about 1GeV. If we sum the mass of three light quarks, the proton would have a mass of about 15MeV, so the big question is: "From where come the proton mass?", and DCSB is the answer to this question.

DCSB has been studied also in QED and pure QCD or with the inclusion of a NJL type of interaction [5, 6, 7, 8]. These analysis are modified by the introduction of a dynamical gluons mass (m_g) which is a necessary ingredient for a complete analysis of DCSB because it existence has been confirmed via analytical and numerical calculations [9, 10, 11].

Presentation of the work

The present work is organized in the next way: In section two we briefly review some properties of Quantum Chromodynamic (QCD); we start with a historical remark where it is defined all the language used when speaking about QCD and some references for a deeper study. The line followed in the historical remark is:

$$\text{Strong interaction} \rightarrow \text{Yukawa model} \rightarrow \text{Yang-Mills theory}$$

$$\text{Experimental facts: Color, Flavor} \rightarrow \text{QCD}$$

$$\text{Phenomenology:} \left\{ \begin{array}{c} Nonrelativistic\, interacting\, potential \\ Bag\, model \end{array} \right\} \rightarrow \text{DCSB}$$

In section two, we also present the QCD Lagrangian, the Feynman rules, the Schwinger-Dyson equations, and with these tools, we obtain the gap equation, equation which leads us to study the phenomenon of DCSB. In section three we present few details about chiral symmetry breaking, just in order to point out some definitions, comment about the importance of CSB, and review of one of the first models in that direction, the Nambu-Jona-Lasinio model and finally we present an important result of dynamical mass generation which is the "Gluon mass generation".

Section four is the main part of this work, it is where we present our results on CSB in QCD. We start studying the gap equation for QCD in the one-dressed-gluon exchange case and with some approximations, we obtain results about the dynamical quark mass. For some reasons, presented in this work, we see that the inclusion of a dynamical gluon mass into the QCD gap equation does not have strength enough to generate a satisfactory dynamical quark mass [12, 13] and we can see this in the cases of constant coupling and with a running coupling constant. This scenario is modified with the inclusion of a new ingredient into the gap equation as proposed by Cornwall [1],and this ingredient is confinement in the form of a confining effective propagator (CEP), so we study a "complete" gap equation $M\left(p^2\right) = M_c\left(p^2\right) + M_{1g}\left(p^2\right)$ which mimics the phenomenological formula $V_F(r) = K_F r - 4\alpha_s/3r$ for the interaction potential between quarks. This complete GAP equation is studied in two cases, for constant coupling as in [2] and as a new ingredient, we study the inclusion of the running behaviour in the coupling constant. Finally the last sections are devoted to our conclusions, references and some calculations.

Chapter 2

Quantum Chromodynamics

2.1 Historical Remark

Since its proposal in the 30's, the strong interaction is one of the challenges of the theoretical physics for understanding the fundamental building blocks of matter and their interactions. It is proposed as the interaction responsible for the stability of the atomic nuclei, which of course, must be very "strong" in order to surpass the "strong" electromagnetic repulsion (due to the short distance, $\backsim 10fm$) between the protons present in the atomic nuclei. One of the first models in order to describe this interaction was proposed by Yukawa [14] who thought in two interacting fermions (Nucleons; Proton and/or Neutrons) which exchange a "virtual" particle in analogy with Quantum Electrodynamics (QED). Since this interaction must be of short range (is "limited" to the size of the atomic nuclei), and by the Heissenberg uncertainty principle, Yukawa proposed that this quanta must be massive and predicted the existence of a particle with a mass of about 120MeV (particle identified later with the π meson).

Heissenberg proposed that in the Yukawa model there is a symmetry [15], which Wigner called isospin [16]. Isospin reflect the fact that the strong interaction is the same between two protons or two neutrons, or between a proton and a neutron, namely, strong interaction is invariant by a rotation in the "isospin space" (or by a exchange of protons by neutrons), but this symmetry is said approximate because of the difference of mass between neutrons and

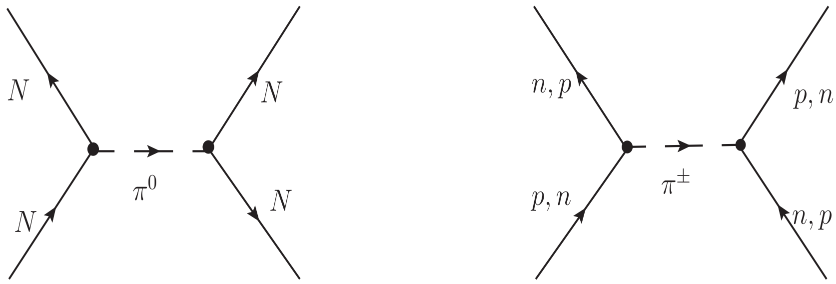


Figure 2.1: Feynman diagrams of Nucleon interaction in the Yukawa model.

protons. Isospin is associated to a symmetry group, the SU(2) group, and it was because of this that Yang and Mills [17] proposed a field theory for SU(2) in order to study strong interaction (called later Yang-Mills theory and extended to SU(N)).

In the Yang-Mills model, we have a Lagrangian which is analogous to the QED Lagrangian, but generalized to SU(2) Group:

$$\mathcal{L} = -\frac{1}{4} F_a^{\mu\nu} F_{\mu\nu}^a + \bar{\psi}_N \left(\gamma^\mu D_\mu - im\right) \psi_N \tag{2.1}$$

With $F_{\mu\nu}^a = \partial_\mu \pi_\nu^a - \partial_\nu \pi_\mu^a + g\epsilon^{abc}\pi_\mu^b \pi_\nu^c$ being the strength tensor for SU(2), g the coupling constant, $N = n$, p means "nucleon", and $D_\mu = \partial_\mu - ig\tau^a \pi_\mu^a$ the covariant derivative where τ are the Pauli's matrices (a, b, c = 1, 2, 3). The field π represent the pion in the sense that $\pi^0 = \pi^3$, $\pi^+ = \pi^1 + i\pi^2$, $\pi^- = \pi^1 - i\pi^2$.

This Lagrangian is gauge invariant if the π quantas are massless, and because of this, this model was not generally accepted at that time by the scientific community, at the point of almost being forgotten [18].

Because of the discovery of a large spectrum of new strong interacting particles (hadrons) in the 50's by the particles accelerators (Cosmotron, Bevatron, Cyclotron, etc.) it was necessary the classification of this variety of particles. This classification led Gell-Mann to postulate the existence of quarks as the fundamental constituents of the hadrons [19]. In the Gell-Mann model there are three types of quarks (u, d, s) associated to a "flavour" SU(3) symmetry group. Three quarks constitute a Barion (proton, neutron, Λ, etc.) and the Mesons (π, ρ, etc.) are constituted by a quark-antiquark pair. Besides flavor, another quantum number for the quarks was postulated, the "color" (green, blue, red), first proposed by O.W. Greenberg [20] in order to solve the statistic problem in hadrons and it was confirmed later theoretically and by experimental facts (Cancellation of anomalies, Decay of π^0 into two photons, the branching ratio $R = \sigma(e^+e^- \to hadrons)/\sigma(e^+e^- \to \mu^+\mu^-)/$, etc. [21]). The quark-quark interaction was associated to the color charge and using the Yang-Mills theory for "color" SU(3) symmetry group, was born the Quantum Chromodynamics (QCD) as the theory to understand the strong interaction [22]. In QCD the gauge bosons are called "gluons" which couple with the quarks due to the color charge, as well as couple to themselves.

QCD had a great success in the explanation of the strong interaction at high energies, especially in the experiments of Deep Inelasttic Scattering and the parton model, where the calculated scattering processes are measured experimentally with high precision [23]. Because of no evidence nor detection of free quarks, it was proposed one of the main properties of the QCD, the "confinement of quarks and gluons" i.e. quarks and gluons interact strongly in a limited region of space and the force to separate them beyond that limit must be infinity [24]. The other main QCD property is asymptotic freedom, developed by D. Gross, F. Wilczek and H. Politzer [25, 26], in order to explain the problem of the scaling in the deep inelasttic scattering. Asymptotic freedom tells us that the strong interaction is "weak" in the limit of short distances (or high momenta), and is stronger when the distance is increased, in other words, the "strong coupling constant" increases with the distance. QCD describe the asymptotic freedom, but it does not have up to now an analytical description of confinement.

Since the 70's the spectrum of mesons was extensively studied, so many different models of nonrelativistic interacting quarks bounded by a radial potential $V(r)$ were proposed. The most used one was the Cornell's potential [27, 28], which has the form:

$$V(r) = -\frac{4}{3}\frac{\alpha_s}{r} + K_F r \tag{2.2}$$

Where, K_F is the string tension and α_s is the strong coupling constant.

The first term in Cornell's potential is the coulomb-like term, which expresses the analogy between the QED and QCD and the second is the confining term, which depends linearly with the distance, so it is dominant for long distances.

Another model used to study the hadron dynamic and the DIS was the bag model in which relativistic massless quarks surrounded by a confining "bag" interact weakly except in the limiting region [29]. Both models, the nonrelativistic interacting potential and the confining bag did show a striking success [20, 29], therefore they must have some elements of truth, despite they are contradictories; in one place we have heavy nonrelativistic constituent quarks, of mass M, with little binding, and for the other we have very light relativistic quark, with current mass m, obeying the chiral symmetry constrains of current algebra (where $M \gg m$). Chiral symmetry breaking is a scheme that unify that kind of contradictions, as pointed out by B. MacKellar et.al [30] and emerges as a explanation of other kind of phenomena as we will discuss in the next sections.

2.2 General Properties

QCD is a gauge theory for SU(3) symmetry group, which is used to describe the fundamental interaction between quarks and the intermediary bosons. The QCD Lagrangian is:

$$\mathcal{L}_{QCD} = -\frac{1}{4}G_a^{\mu\nu}G^a_{\mu\nu} + \sum_{i,j}^{N_f} \bar{\psi}_i \left(\gamma^\mu D_\mu - im\right)_{ij} \psi_j \tag{2.3}$$

Where $G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s \epsilon^{abc} A^b_\mu A^c_\nu$, being the strength tensor for SU(3), g_s the strong coupling constant and $D_\mu = \partial_\mu - i g_s \lambda^a A^a_\mu$ the covariant derivative (where λ^a are the Gell-Mann matrices and a, b, c = 1, ..., 8). The field A represent the gluon and N_f is the number of flavors present in the theory.

It is impossible to define a gluon propagator from this Lagrangian without making a choice of gauge. This lead us to make use of the Fadeev-Poppov procedure to quantize the gauge fields, and with this we obtain a gauge fixing term and a ghost contribution [31]:

$$\mathcal{L}_\alpha = -\frac{1}{2\alpha}\left(\partial_\mu A^\mu\right)^2 \qquad \text{and} \qquad \mathcal{L}_{ghost} = \partial_\mu \eta^{a\dagger}\left(D^\mu_{ab}\eta^b\right)^2 \tag{2.4}$$

With these formulas equation (2.3) is transformed into the following:

$$\mathcal{L}^{eff}_{QCD} = -\frac{1}{4}G^{\mu\nu}_a G^a_{\mu\nu} - \frac{1}{2\alpha}\left(\partial_\mu A^\mu\right)^2 + \sum_{i,j}^{N_f} \bar{\psi}_i \left(\gamma^\mu D_\mu - im\right)_{ij}\psi_j + \partial_\mu \eta^{a\dagger}\left(D^\mu_{ab}\eta^b\right) \tag{2.5}$$

This Lagrangian has the following momentum space Feynman rules:

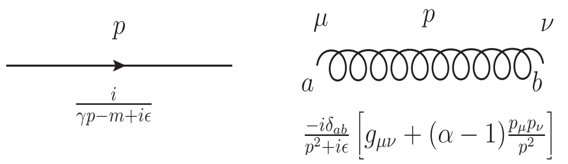


Figure 2.2: Quark and gluon bare propagators.

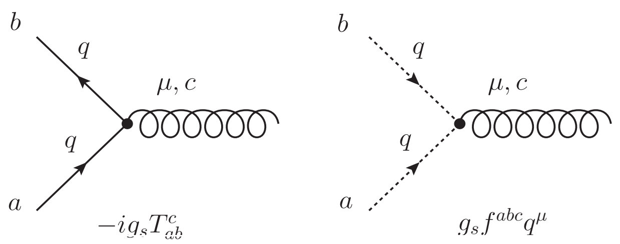
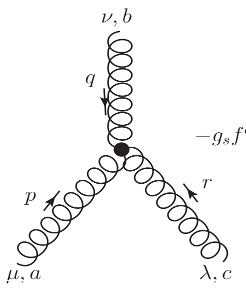


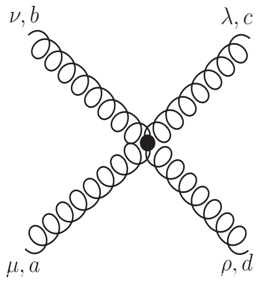
Figure 2.3: Quark-gluon and ghost-gluon vertex.



$$-g_s f^{abc}\left[(p-q)_\lambda g_{\mu\nu} + (q-r)_\mu g_{\nu\lambda} + (r-p)_\nu g_{\lambda\mu}\right]$$

$$p+q+r=0$$

Figure 2.4: Three - gluons vertex.



$$\begin{aligned} -ig_s^2\big[& f^{abe}f^{cde}(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda}) \\ & f^{ace}f^{dbe}(g_{\mu\rho}g_{\nu\lambda} - g_{\mu\nu}g_{\rho\lambda}) \\ & f^{ade}f^{bce}(g_{\mu\nu}g_{\lambda\rho} - g_{\mu\lambda}g_{\nu\rho})\big] \end{aligned}$$

Figure 2.5: Four - gluons vertex.

These rules allow us to perform perturbative calculations in the high energy regime of QCD in term of powers of g_s. This scheme is justified by the asymptotic freedom property: The running coupling constant become more and more small in the high energy domain, but in the low momentum domain it become high, so high that the perturbative expansion is not anymore justified [25].

Since dynamical chiral symmetry breaking is a low energy phenomenon, we need other tools in order to study it. The two main tools used to study the non-perturbative regime of QCD are Lattice QCD [32] and the Shwinger-Dyson equations (SDE) [33].

2.3 Schwinger Dyson Equations

As a introduction to the SDE, this is a brief deduction performed in the Abelian case (in the Euclidean space), just for simplicity. A complete analysis including the non-Abelian case can be found in [33].

The SDE are an infinite tower of integral equations that relate the n-point functions of the theory and since they do not have an exact solution (as far as we know), we have to consider some approximations or truncations in order to solve them. Historically the SDE were deduced diagrammatically, but a simple way to deduce them is introducing the generating functional for the n-point functions [34]:

$$Z[J,\xi,\bar{\xi}] = \int D\left[A,\bar{\psi},\psi\right] e^{-S\left[A,\bar{\psi},\psi;J,\eta,\bar{\eta}\right]} \tag{2.6}$$

Where the notation used is:

$$D\left[A,\bar{\psi},\psi\right] = DAD\bar{\psi}D\psi \qquad\qquad S\left[\varphi_i;\xi_i\right] = S\left[\varphi_i\right] - \varphi_i\xi_i$$

$$\varphi_i \equiv \left(A,\bar{\psi},\psi\right) \qquad\qquad \xi_i \equiv \left(J,\eta,\bar{\eta}\right)$$

$$\varphi_i\xi_i = \int d^dx \left[J_\mu(x)A_\mu(x) + \bar{\psi}(x)\eta(x) + \bar{\eta}(x)\psi(x)\right]$$

In Euclidean space we use the action:

$$S\left[\varphi_i\right] = \int d^dx \left[\frac{1}{4}\left(F_{\mu\nu}(x)\right)^2 + \frac{1}{2\alpha}\left(\partial_\mu A_\mu(x)\right)^2 + \bar{\psi}(x)\left(\gamma.D+m\right)\psi(x)\right] \tag{2.7}$$

Where:

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) \qquad \text{and} \qquad D_\mu = \partial_\mu - ieA_\mu \tag{2.8}$$

For an arbitrary function and a closed path (c.p.) $\int_{c.p.} dx \frac{d}{dx} f(x) = 0$, and since in the definition of $Z\left[\varphi_i\right]$ in the Euclidean Space, we perform the path integral over closed paths of infinity leght [34], we have:

$$\int D\varphi_i \frac{\delta}{\delta \varphi_i} e^{-S[\varphi_i]+\varphi_i \xi_i} = 0$$

$$\left(\frac{\delta S}{\delta \varphi_i} \left[\frac{\delta}{\delta \xi_i} \right] - \xi_i \right) Z\left[\xi_i\right] = 0 \tag{2.9}$$

These are the Schwinger-Dyson equations in Euclidean space.

Now, consider the equation (2.9) for $\varphi(x) = A(x)$ and $\xi(x) = J(x)$:

$$\left(\frac{\delta S}{\delta A_\mu(x)} \left[\frac{\delta}{\delta J_\mu}, \frac{-\delta}{\delta \eta}, \frac{\delta}{\delta \bar{\eta}} \right] - J_\mu(x) \right) Z\left[J, \eta, \bar{\eta}\right] = 0$$

The derivative over the action stand:

$$\frac{\delta S}{\delta A_\mu(x)} = -\triangle^{-1}_{\mu\nu}\left(\alpha\right) A_\nu\left(x\right) - ie\bar{\psi}(x)\gamma_\mu \psi(x)$$

Where:

$$-\triangle^{-1}_{\mu\nu}\left(\alpha\right) = \delta_{\mu\nu}\partial^2 - \left(1 - \frac{1}{\alpha}\right)\partial_\nu \partial_\mu$$

In terms of the generating functional of the full connected Green's functions W and the classical action Γ, which are defined as:

$$Z\left[\xi_i\right] = \exp^{-w[\xi_i]} \qquad and \qquad \Gamma\left[\varphi_i^{cl}\right] = w\left[\xi_i\right] + \varphi_i^{cl}\xi_i \tag{2.10}$$

We obtain the equation [*]:

$$-\triangle^{-1}_{\mu\nu}\delta(x-y) - ie\int d^d u d^d w \, Tr\left[\gamma_\mu S(x,u)\,\Gamma_\nu(y;u,w)\,S(w,x)\right] = D^{-1}_{\mu\nu}(x,y) \tag{2.11}$$

[*] Here we have set $\xi_i = 0 = \varphi_i^{cl}$ and with this, all the terms $\delta_\xi W$ cancel out i.e. the connected one point Green's functions are zero

Where we have used the relation:

$$-\frac{\delta^2 w}{\delta\eta_A(x)\delta\bar{\eta}_B(z)} = \left(\frac{\delta^2\Gamma}{\delta\psi_B(z)\delta\bar{\psi}_A(x)}\right)^{-1} \tag{2.12}$$

And the definitions:

$$\frac{\delta^2\Gamma}{\delta A_\nu(y)\delta A_\mu(x)}|_{A=\psi=\bar{\psi}=0} = D^{-1}_{\mu\nu}(x,\, y) = (\text{Full photon propagator}) \tag{2.13}$$

$$\frac{\delta^2\Gamma}{\delta\psi(y)\delta\bar{\psi}(x)}|_{A=\psi=\bar{\psi}=0} = S^{-1}(x,\, y) = (\text{Full electron propagator}) \tag{2.14}$$

$$\frac{\delta^3\Gamma}{\delta A_\nu(z)\delta\psi(y)\delta\bar{\psi}(x)}|_{A=\psi=\bar{\psi}=0} = \Gamma_\nu(z;\, x,\, y) = (\text{Full vertex function}) \tag{2.15}$$

In momentum space the equation (2.11) is written as:

$$D_{\mu\nu}(q)^{-1} = D^{(0)}_{\mu\nu}(q)^{-1} - \Pi_{\mu\nu}(q)$$

Where we have defined the photon polarization tensor (Figure 2.6):

$$\Pi_{\mu\nu}(p) = -ie\int\frac{d^dk}{(2\pi)^d}\, tr\left[\gamma_\mu S(k)\,\Gamma_\nu(k,\, p-k)\, S(p-k)\right]$$

Using the Lorentz structure defined by the Ward-Takahashi identities we have:

$$D_{\mu\nu}(q^2) = \frac{d(q^2)}{q^2}\left[\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right] + \frac{\alpha}{q^2}\frac{q_\mu q_\nu}{q^2} \tag{2.16}$$

Where $d(q^2) = 1/\left[1+\Pi(q^2)\right]$ represents the photon wave function renormalization and α the gauge-fixing parameter.

Equation (2.16) represents the complete photon propagator in the Euclidean space.

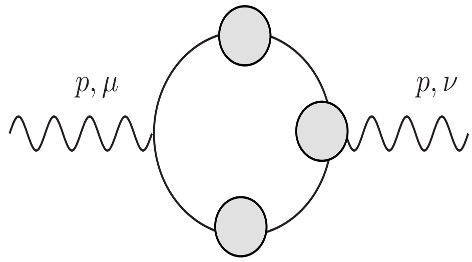


Figure 2.6: Photon polarization tensor.

A similar procedure but with the spinor fields lead us to obtain the DSE for the quark propagator. For spinors, equation (2.9) is:

$$\left(\frac{\delta S}{\delta\bar{\psi}(x)}\left[\frac{\delta}{\delta J}, \frac{-\delta}{\delta\eta}, \frac{\delta}{\delta\bar{\eta}}\right] - \eta(x)\right) Z\left[J, \eta, \bar{\eta}\right] = 0$$

Where:

$$\frac{\delta}{\delta\bar{\psi}(x)} S = \left(\partial - ieA(x) + m\right)\psi(x)$$

Again we use the connected Green function and the classical action and use the relations used before (equations (2.10) to (2.15)):

$$(\gamma.\partial + m)S(x,\, y) + ie\int d^d w\left[d^d u d^d z\, D_{\mu\nu}(x,\, z)\gamma_\mu S(x,\, u)\Gamma_\nu(z;\, u,\, w)\right] S(w,\, y) - \delta(x,\, y) = 0$$

Or, in momentum space:

$$S^{-1}(p) = S_0^{-1}(p) + \Sigma(p) \tag{2.17}$$

Where $\Sigma(p)$ is the quark self-energy (Figure 2.7:

$$\Sigma(p) = ie \int \frac{d^d q}{(2\pi)^d} \gamma_\mu D_{\mu\nu}(p-q) S(q) \Gamma_\nu(p-q,\, q)$$

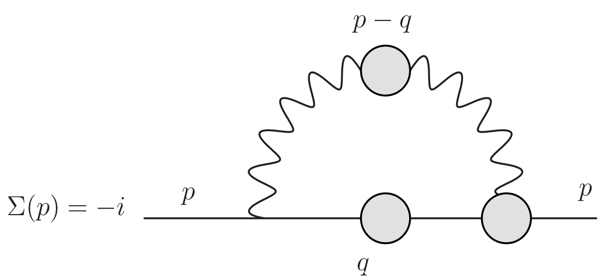


Figure 2.7: Electron self-energy.

2.4 QED gap Equation

The equation (2.17) has the diagrammatic form:

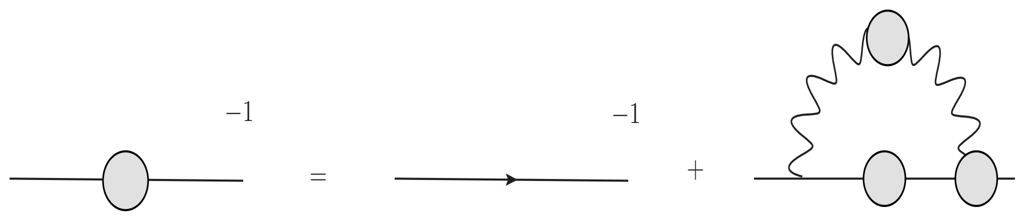


Figure 2.8: SDE for the electron propagator.

This equation is our starting point to deduce the formal expression for the gap equation.

We know that the propagator has the general form:

$$S(p) = \frac{1}{\gamma.pA(p) + B(p)} = \frac{iZ(p)}{\gamma.p + iM(p)} \tag{2.18}$$

With this, the bare propagator and the self energy we obtain the relations:

$$\frac{M(p^2)}{Z(p^2)} = m + \frac{tr\left[\sum(p)\right]}{tr\left[1\right]} \tag{2.19}$$

$$\frac{1}{Z(p^2)} = 1 + \frac{i\,tr\left[p\sum[p]\right]}{p^2 tr\left[1\right]} \tag{2.20}$$

In the Landau gauge ($\alpha = 0$) the gluon propagator is considerably simplified and it is well justified to use the so called "rainbow approximation" [35]:

$$\Gamma_\beta(q, p-q) \equiv ie\gamma_\beta$$

$$D_{\alpha\beta}(p-q) \equiv \frac{g\left[(p-q)^2\right]}{(p-q)^2}\left[\delta_{\alpha\beta} - \frac{(p-q)_\alpha(p-q)_\beta}{(p-q)^2}\right]$$

After taking the trace the equations (2.19) and (2.20) become:

$$\frac{M(p^2)}{Z(p^2)} = m + (d-1)e^2 \int \frac{d^dq}{(2\pi)^d} \frac{\bar{g}^2\left[(p-q)^2\right]}{(p-q)^2} \frac{M(q^2)}{q^2 + M^2(q^2)} Z(q^2) \tag{2.21}$$

$$\frac{1}{Z(p^2)} = 1 + ie^2 \int \frac{d^dq}{(2\pi)^d} \frac{Z(q^2)\bar{g}^2\left[(p-q)^2\right]}{p^2\left[q^2 + M^2(q^2)\right]} \left[\frac{(d-3)p\cdot q}{(p-q)^2} + \frac{2p\cdot(p-q)q\cdot(p-q)}{(p-q)^4}\right] \tag{2.22}$$

The main difficulty with the equation (2.22) is the angular dependence present in the effective charge. This difficulty is avoided by considering the Landau-Abrikosov-Khalatnikov (LAK) approximation [36]:

$$\bar{g}^2[(p-k)^2] \approx \bar{g}^2(p^2)\theta\left(p^2-k^2\right) + \bar{g}^2(k^2)\theta\left(k^2-p^2\right) \tag{2.23}$$

After the LAK approximation the angular integration only affect the last bracket. It can be shown (see Appendix A) that this angular integral is zero in four dimensions:

$$I_d = \int_0^\pi d\theta \sin^{d-2}\theta \left[\frac{(d-3)p\cdot q(p-q)^2 + 2p\cdot(p-q)q\cdot(p-q)}{(p-q)^4}\right] = 0 \tag{2.24}$$

With this, the equations (2.21) and (2.22) reduce to:

$$\frac{1}{Z(q^2)} = 1$$

$$M(p^2) = m_0 + e^2 \int \frac{d^4k}{(2\pi)^4} \frac{3d[(p-k)^2]}{(p-k)^2} \frac{M(k^2)}{k^2+M^2(k^2)} \tag{2.25}$$

And this is the Fermionic gap equation in the Abelian case.

Chapter 3

Chiral Symmetry

Massless fermions have a defined chirality. Lagrangians with massless fermions are invariant under the "chiral transformation":

$$\psi(x) \rightarrow e^{i\theta\gamma_5}\psi(x)$$

Where θ is a parameter and $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$.

When a mass term appears in the Lagrangian via some mechanism (spontaneous or dynamical), this term mix the two chiral components of the spinor, so that the Lagrangian is not any more invariant under the chiral transformation. In this case we say that occurred a Chiral Symmetry Breaking.

3.1 Chiral Symmetry Breaking

As we said before, the proton mass and the meson spectroscopy show evidences of some mechanism of mass generation and since it is present in the hadrons, it must be related to the strong interaction. Dynamical chiral symmetry breaking could explain the significant difference between the pions and the other hadrons. It is well know that when a continuous symmetry of the Lagrangian is broken, appear in the theory some massless particles called the Goldstone bosons [37]. In the picture of DCSB the pions are the Goldstone bosons of the theory and they are massive because the chiral symmetry is not an exact symmetry, and this fact is illustrated by the well known Gell-Mann-Oakes-Renner equation [38] that show the relation between the pions mass and the bare quark mass:

$$m_q \left\langle \bar{\psi}\psi \right\rangle = f_\pi^2 m_\pi^2$$

Where $\left\langle \bar{\psi}\psi \right\rangle$ is the quark condensate, m_π the pion mass, m_q the quark bare mass and f_π is the pion decay constant.

3.2 Nambu-Jona-Lasinio Model

The Nambu-Jona-Lasinio is a strong interaction model with a four fermion interaction. The importance of this model is that it contains the main symmetries related to the strong interaction i.e. Isospin (in $SU(2)$) and chiral symmetry. The main success of the model is the possibility of describe the dynamical chiral symmetry breaking and to be able to reproduce the Goldberger-Treiman and the Gell-Mann-Oakes-Renner relations (as pointed out in [39]).

The NJL action is:

$$S_{NJL}[\bar{\psi}\psi] = \int d^d x \left[\bar{\psi}(x) \left(\gamma.\partial + m\right) \psi(x) + \frac{1}{2} G_0 \left(\bar{\psi}(x)\psi(x)\right)^2 \right]$$

With this, we use the equation (2.9), and:

$$\frac{\delta S_{NJL}}{\delta \bar{\psi}(x)} = \left(\gamma.\partial + m\right) \psi(x) + G_0 \left(\bar{\psi}(x)\psi(x)\right) \psi(x)$$

To obtain:

$$(\gamma.\partial + m)\, S(x,\, y) + \frac{\delta^4 W}{\delta\eta(x)\delta\bar{\eta}(x)\delta\eta(y)\delta\bar{\eta}(x)} = \delta(x-y)$$

This equation can be written as (in momentum space):

$$S^{-1}(p) = S_0^{-1}(p) + \Sigma_{NJL}(0)$$

Where:

$$\Sigma_{NJL}(0) = G_0 \int \frac{d^d q}{(2\pi)^d} Tr\left[S(q)\right]$$

Using equation (2.18) and taking the trace, we obtain equations similar to (2.19) and (2.20):

$$\frac{M(p^2)}{Z(p^2)} = m + G_0 N_f \int \frac{d^d q}{(2\pi)^d} \frac{M(q^2)}{q^2 + M^2(q^2)} Z(q^2) \tag{3.1}$$

$$\frac{1}{Z(p^2)} = 1 - G_0 N_f \int \frac{d^d q}{(2\pi)^d} \frac{Z(q^2)\,(p\cdot q)}{p^2\left[q^2 + M^2(q^2)\right]} \tag{3.2}$$

In four dimensions we have the integral $\int_0^\pi d\theta \sin^2\theta \cos\theta = 0$ in the $Z(p^2)$ term, so that:

$$Z(p^2) = 1$$

$$M(p^2) = m + G_0 N_f \int \frac{d^4 q}{(2\pi)^4} \frac{M(q^2)}{q^2 + M^2(q^2)}$$

And this is the gap equation for the NJL model.

This gap equation presents an important result, in the chiral limit i.e. when $m = 0$ the dynamical quark mass has a nonzero value for any p. This is an explicit form of the dynamical symmetry breaking phenomenon.

3.3 Dynamical Gluon Mass [40]

In the early eighties, working in the Landau gauge and Euclidean space, Cornwall obtained a gauge invariant solution for the gluon propagator that behaved as $1/\left(k^2+m^2(k^2)\right)$ [41]. In this case, as $k^2 \to 0$, the function $m^2(k^2)$ was interpreted as a dynamical gluon mass with the limit $m^2(k^2 \to 0) = m_g^2$. This solution, which became known as a "confined solution", reproduces the expected perturbative behavior of the gluon propagator at large k^2 because $m^2(k^2 \to \infty) = 0$.

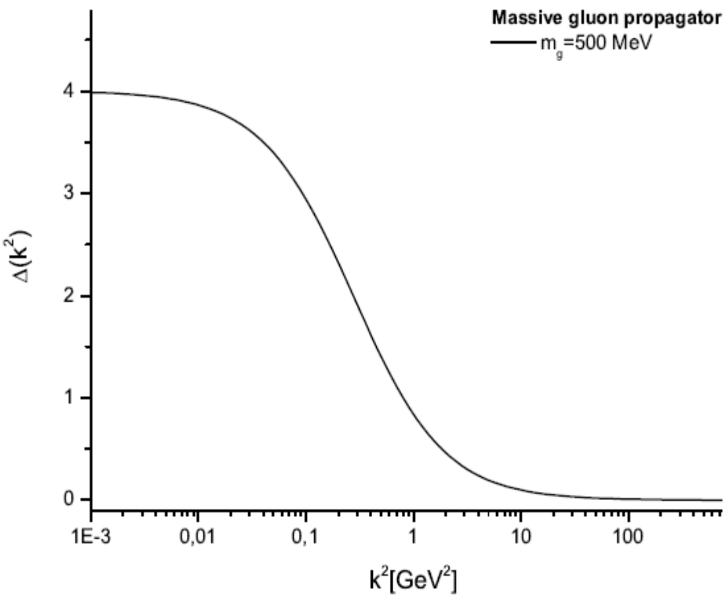


Figure 3.1: The massive gluon propagator with $n_f = 3$ and $\Lambda_{QCD} = 300\text{MeV}$ [2].

A great step ahead in this phenomenon has also been provided by the QCD lattice simulations in Landau gauge, which strongly support the existence of an infrared finite gluon propagator [9, 10, 11] (Figure 3.1). This is interesting enough, because indicates the appearance of a dynamical mass scale for the gluon, which imply in the existence of a non-trivial QCD infrared fixed point, i.e. the freezing of the coupling constant at the origin of momenta [42].

Cornwall indicated that a dynamically generated gluon mass induces vortex solutions in the theory and these are responsible for the quark confinement [41]. It is not surprising that DSE for the QCD propagators led to different solutions throughout the years as long as they were solved with different truncations and approximations, as, for instance, the choice of the trilinear gluon vertex plays a crucial role in the solution. It was a great step forward in the understanding of QCD the fact that lattice and the solutions of Schwinger-Dyson equations show now this consistent result.

Chapter 4

Dynamical Quark Mass

4.1 One-Dressed-Gluon Exchange

Here we consider the gap equation (2.25) for QCD in the massive one-dressed-gluon exchange case including a dynamical gluon mass:

$$M_{1g}(p^2) = C_2 \int \frac{d^4k}{(2\pi)^4} \frac{\bar{g}^2[(p-k)^2]}{\left[(p-k)^2 + m_g^2\right]} \frac{3M(k^2)}{[k^2 + M^2(k^2)]} \tag{4.1}$$

Where the effective charge is given by [41]:

$$\bar{g}^2(k^2) = \frac{1}{b \ln\left[\frac{k^2+4m_g^2}{\Lambda_{QCD}^2}\right]} \tag{4.2}$$

Where, C_2 is the quark Casimir eigenvalue ($C_2 = 4/3$ for quarks in the fundamental representation), $b = \frac{11N-2n_f}{48\pi^2}$ is the one-loop coefficient in the beta-function for the $SU(N)$ group with n_f flavors, m_g is the dynamical gluon mass (here we neglect the running of the gluon mass) and Λ_{QCD}^2 is the QCD mass scale.

To solve (4.1) we use the LAK approximation (2.23) and the angle approximation [44]:

$$\int d\Omega_4 \frac{1}{(p-k)^2 + m_g^2} \approx 2\pi^2 \left[\frac{\theta\left(p^2-k^2\right)}{p^2+m_g^2} + \frac{\theta\left(k^2-p^2\right)}{k^2+m_g^2}\right]$$

So we have:

$$M_{1g}(p^2) = \frac{3C_2}{16\pi^2}\int dk^2\left[\frac{\theta\left(p^2-k^2\right)}{p^2+m_g^2}\bar{g}^2(p^2)+\frac{\theta\left(k^2-p^2\right)}{k^2+m_g^2}\bar{g}^2(k^2)\right]\frac{k^2}{[k^2+M^2(k^2)]}M_{1g}(k^2) \tag{4.3}$$

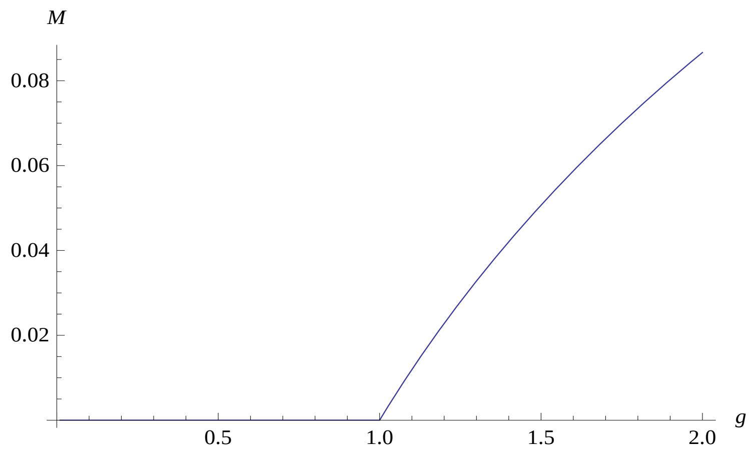


Figure 4.1: Bifurcation of the running quark mass.

This integral equation admit trivial solutions $M(p^2) = 0$ and (by assumption) a nontrivial one (Figure 4.1) and when we deal with the high momentum region in the neighborhood of the critical point g_0, it is sufficient to consider the linearized version of (4.3) [45, 46] to find the critical behavior:

$$M_{1g}(x) = \lambda\int_0^{\Lambda^2} dy\left[\frac{\theta(x-y)\bar{g}^2(x)}{\left(x+m_g^2\right)}+\frac{\theta(y-x)\bar{g}^2(y)}{\left(y+m_g^2\right)}\right]M_{1g}(y) \tag{4.4}$$

Where we have introduced an UV cutoff Λ and we used the substitutions:

$$x = p^2; \qquad y = k^2; \qquad \lambda = \tfrac{3C_2}{16\pi^2} \tag{4.5}$$

4.1.1 Constant Coupling

When considering coupling constant we write $\bar{g}^2(x) \approx \bar{g}^2(0) \equiv g_0$ and we define: $\lambda_0 = 3g_0C_2/16\pi^2$. With this approximation, equation (4.4) stand:

$$M_{1g}(x) = \lambda_0 \int_0^{\Lambda^2} dy \left[\frac{\theta(x-y)}{\left(x+m_g^2\right)} + \frac{\theta(y-x)}{\left(y+m_g^2\right)} \right] M_{1g}(y)$$

This equation can be easily transformed into the differential equation:

$$\left(x+m_g^2\right)^2 M''_{1g}(x) + 2\left(x+m_g^2\right) M'{}_{1g}(x) + \lambda_0 M_{1g}(x) = 0$$

Which admit two asymptotic solutions:

$$M_{1g}^{\pm}(x) = \left(x+m_g^2\right)^{\alpha_\pm}$$

Where: $\alpha_\pm = -(1/2) \pm \sqrt{(1/4) - \lambda_0}$.

If we have a trivial coupling, $\lambda_0 = 0$ and we do not have CSB. The classical analysis state that for some value of λ_0 we start having CSB and this value is related to the value in which the square turns to an imaginary number [1, 47]. This give us the condition for nontrivial solution $\lambda_0 \geq 1/4$:

$$\alpha_s(0) = \frac{\bar{g}(0)}{4\pi} \geq \frac{\pi}{3C_2} \geq 0.8$$

This value is in agreement with those found by Atkinson and Johnson [48] and Chang et.al [49] by different analysis. The coupling constant of equation (4.2) at zero momentum does not go up to this value (Figure 4.2), so our equation does not satisfy the minimum condition for DCSB existence.

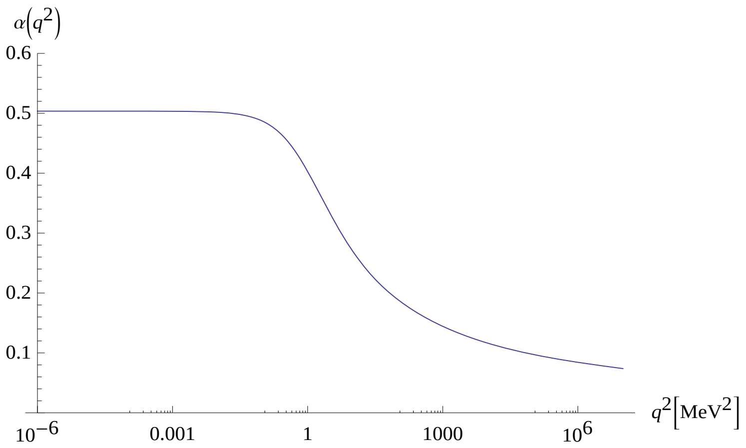


Figure 4.2: Running coupling constant for QCD with dynamical gluon mass (equation (4.2) with $m_g = 2\Lambda_{QCD} \approx 600MeV$).

4.1.2 Running Coupling Constant

When we consider the running of the coupling constant we obtain the integral equation:

$$M_{1g}(x) = a_0 \int_0^{\Lambda^2} dy K_1(x,y) M_{1g}(y) \tag{4.6}$$

With the Kernel:

$$K_1(x,y) = \left[\frac{\theta(x-y)}{\left(x+m_g^2\right) \ln\left(\frac{x+4m_g^2}{\Lambda_{QCD}^2}\right)} + \frac{\theta(y-x)}{\left(y+m_g^2\right) \ln\left(\frac{y+4m_g^2}{\Lambda_{QCD}^2}\right)} \right]$$

Where we have used the equation (4.2) and $a_0 = 3C_2/16b\pi^2$ is the Lane constant [50].

Equation (4.6) is a Fredholm equation of the form: $\varphi(x) = \lambda \int_a^b dy K(x,y)\varphi(y)$. The condition for obtain non-trivial solutions is given by [51]:

$$|\lambda| \geq \frac{1}{\sqrt{\int_a^b \int_a^b K^2(x,y) dx dy}} \tag{4.7}$$

Which is translated in our case to:

$$\frac{3C_2}{16b\pi^2} \geq \frac{1}{\sqrt{\int_a^b\int_a^b \left[\frac{\theta(x-y)}{(x+m_g^2)\ln\left(\frac{x+4m_g^2}{\Lambda_{QCD}^2}\right)} + \frac{\theta(y-x)}{(y+m_g^2)\ln\left(\frac{y+4m_g^2}{\Lambda_{QCD}^2}\right)}\right]^2 dxdy}}$$

From this equation we obtain a condition for DCSB:

$$f(m_g) = \frac{3C_2}{16b\pi^2}\sqrt{\int_a^b\int_a^b \left[\frac{\theta(x-y)}{(x+m_g^2)\ln\left(\frac{x+4m_g^2}{\Lambda_{QCD}^2}\right)} + \frac{\theta(y-x)}{(y+m_g^2)\ln\left(\frac{y+4m_g^2}{\Lambda_{QCD}^2}\right)}\right]^2 dxdy} \geq 1$$

When this condition is satisfied, we obtain nontrivial solution for the asymptotic behavior of the gap equation for the one-dressed-gluon exchange, and this occur for a value of m_g of about 150MeV (Figure 4.3), value which is quite far from the known value of $m_g \approx 600MeV$ [52].

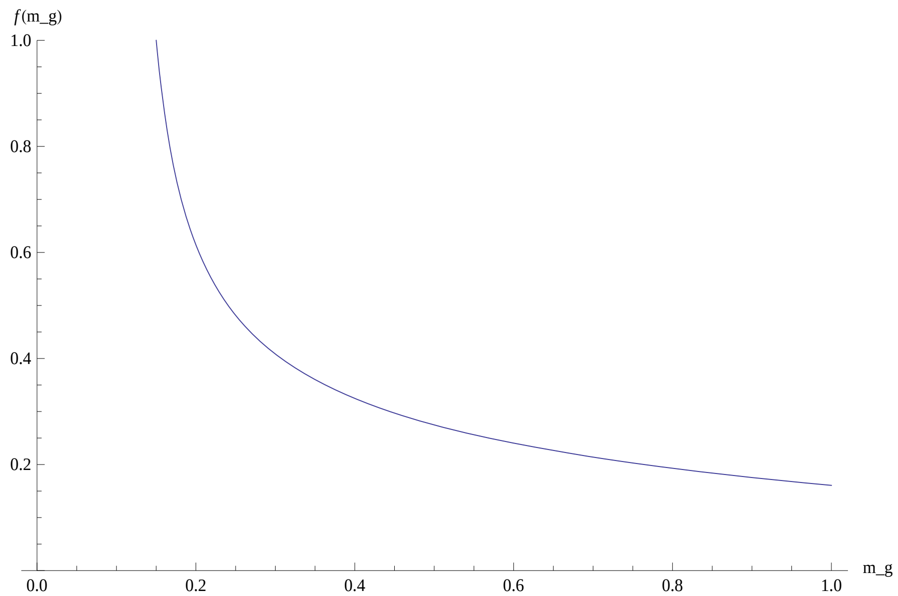


Figure 4.3: Condition on m_g for DCSB in ODGE case.

4.2 One-Dressed-Gluon Exchange + Confinement

Since the inclusion of dynamically massive gluons into the GE does not lead to the expected DCSB for quarks, it is clear that something is missing in the equation (4.1). In a recent paper [1], Cornwall has proposed the inclusion of a "Confining Effective Propagator" of the form $D^{\mu\nu}_{eff} \equiv \delta^{\mu\nu} D_{eff}(k)$, where:

$$D_{eff}(k) = \frac{8\pi K_F}{(k^2+m^2)^2} \tag{4.8}$$

From which we have the GE:

$$\begin{aligned} M_c(p) &= \int \frac{d^4k}{(2\pi)^4} D_{eff}(p-k) \frac{4M(k^2)}{[k^2+M^2(k^2)]} \\ &= \int \frac{d^4k}{(2\pi)^4} \frac{8\pi K_F}{(k^2+m^2)^2} \frac{4M(k^2)}{[k^2+M^2(k^2)]} \end{aligned} \tag{4.9}$$

Where, K_F is the string tension and m is a parameter related to the dynamical quark mass which must be present due to entropic reasons [1].

One of the main reasons to justify the inclusion of a term like (4.8) is the fact that it can reproduce the linear term present in the phenomenological potential for quarkonium (equation (2.2)) Because the potential between static quark charges is related to the Fourier transform of the time-time component of the full gluon propagator, by the equation:

$$V_F(r) = -\frac{2C_2}{\pi} \int d^3\mathbf{q}\alpha_s(\mathbf{q}^2)\Delta_{00}(\mathbf{q})\, exp^{i\mathbf{q}\cdot\mathbf{r}}$$

Where $\alpha_s(\mathbf{q}^2)$ is the running coupling constant and $\Delta_{00}(\mathbf{q})$ the zero-zero component of the gluon propagator in the momentum configuration.

It has been shown that the linear term of the potential can not be obtained from the gluonic propagator obtained in the lattice QCD or in the gluonic SDE [27], only a propagator which behaves $\propto \frac{1}{q^4}$ can reproduce that linear term, and since there are no fundamental fields in nature with such a propagator, this D_{eff} can be understood as a confining effect (it is not a propagator in the usual sence) that should be introduced explicitly in the gap equation in order to generate DCSB [1]. So, from D_{eff} we can obtain the linear term, and the other term $\propto \frac{1}{r}$ can be obtained from the one-gluon exchange sector, this fact can be expressed in the gap equation by supposing a complete gap equation:

$$M(p^2) = M_c(p^2) + M_{1g}(p^2)$$

$$M(p^2) = \int \frac{d^4k}{(2\pi)^4} \left\{ \frac{32\pi K_F}{[(p-k)^2+m^2]^2} + \frac{3C_2\bar{g}^2[(p-k)^2]}{[(p-k)^2+m_g^2]} \right\} \frac{M(k^2)}{k^2+M^2(k^2)} \tag{4.10}$$

To study the critical behavior of equation (4.10) we start considering the linearized version and we perform some of the approximations used before, so that our complete gap equation stand:

$$M(x) = \frac{1}{\pi} \int_0^{\Lambda_M} dy \left\{ \theta(x-y) \left[\frac{A_1}{(x+\kappa)^2} + \frac{A_2(x)}{(x+\epsilon)} \right] + \theta(y-x) \left[\frac{A_1}{(y+\kappa)^2} + \frac{A_2(y)}{(y+\epsilon)} \right] \right\} M(y) \tag{4.11}$$

Where we have defined the variables: $x = p^2/M^2$, $y = k^2/M^2$, $\kappa = m^2/M^2$, $\epsilon = m_g^2/M^2$, $a_1 = 2K_F/\pi M^2 = A_1/\pi$, $a_2(x) = 3C_2\bar{g}^2(x)/16\pi^2 = A_2(x)/\pi$, $M = M(0)$ and we have introduced an ultraviolet (UV) cutoff $\Lambda_M = \Lambda^2/M^2$.

With this, we have the linear-integral equation:

$$M(x) = \frac{1}{\pi} \int_0^{\Lambda_M} dy K(x,y) M(y)$$

With the Kernel:

$$K(x,y) = \left\{\theta(x-y)\left[\frac{A_1}{(x+\kappa)^2} + \frac{A_2(x)}{(x+\epsilon)}\right] + \theta(y-x)\left[\frac{A_1}{(y+\kappa)^2} + \frac{A_2(y)}{(y+\epsilon)}\right]\right\}$$

With this new Kernel, we can check the condition (4.7):

$$f(m,\, M) = \frac{1}{\pi}\sqrt{\int_a^b \int_a^b K^2(x,y)dxdy} \geq 1$$

This condition mark a set of points in which we have non-trivial solutions (Figure 4.4) in terms of m and M.

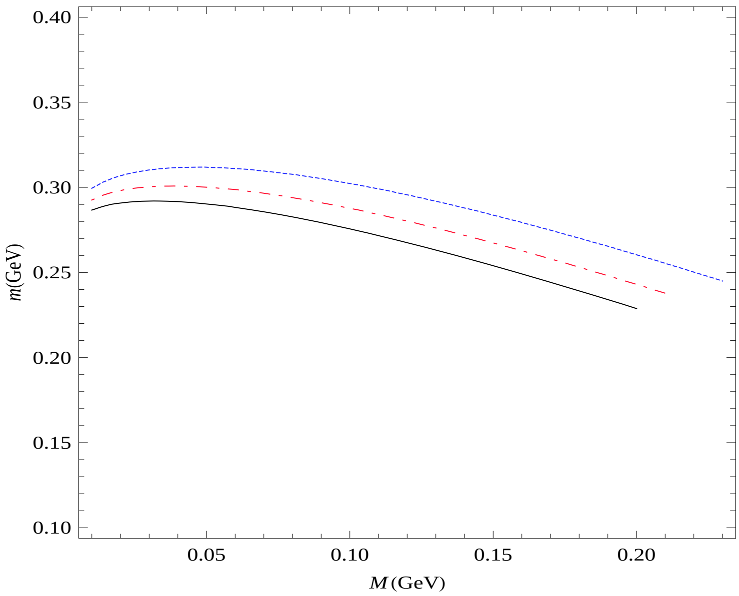


Figure 4.4: Critical condition for $\Lambda_{QCD} = 300MeV$ and $K_F = 0.18GeV^2$. The curves were obtained for $m_g = 600, 650 \text{and} 700MeV$ [2]. Bellow these lines is the region where we have DCSB.

4.2.1 Constant Coupling and Asymptotic Behavior

When considering constant coupling we have the integral equation:

$$f(x) = \int_0^{\Lambda_M} dy \left\{ \theta(x-y) \left[\frac{a_1}{(x+\kappa)^2} + \frac{a_2}{(x+\epsilon)} \right] + \theta(y-x) \left[\frac{a_1}{(y+\kappa)^2} + \frac{a_2}{(y+\epsilon)} \right] \right\} f(y) \quad (4.12)$$

Where: $a_2 = a_2(0) = 3C_2 g_0/16\pi^2$ and $f(x) = M(x)/M$.

The equation (4.12) can be transformed into a differential equation. Using the θ functions we replace the integral $\int_0^{\Lambda_M} dy()$ by $\int_0^x dy() + \int_x^{\Lambda_M} dy()$, in order to obtain*:

$$f(x) = \left[\frac{a_1 + a_2(x+1)}{(x+1)^2} \right] \int_0^x f(y)dy + \int_x^{\Lambda_M} \left[\frac{a_1 + a_2(y+1)}{(y+1)^2} \right] f(y)dy$$

and:

$$f'(x) = - \left[\frac{2a_1 + a_2(x+1)}{(x+1)^3} \right] \int_0^x f(y)dy$$

With these two equations we obtain the UV boundary condition:

$$f'(\Lambda_M) = -\frac{1}{\Lambda_M} f(\Lambda_M) = 0 \qquad \text{(UV BC)}$$

So, from our integral equation, we obtain the differential equation:

$$(x+1)^3 \left[2a_1 + a_2(x+1)\right] f''(x) + 2(x+1)^2 \left[3a_1 + a_2(x+1)\right] f'(x) + \\ + \left[2a_1 + a_2(x+1)\right]^2 f(x) = 0 \quad (4.13)$$

With the boundary conditions:

$$f(x)\Big|_{x\to 0} = 1 \qquad \text{and} \qquad f'(x)\Big|_{x\to\Lambda_M} = 0$$

*Here we consider $\kappa = 1 = \epsilon$ just in order to simplify the calculations and because in the UV behavior it does not affect our analysis

An asymptotic solution of (4.13) (for $x \to \infty$) has an convergent power serie of the form:

$$f(x) = \sum_{s=0}^{\infty} c_s (x+1)^{-(s+\alpha)}$$

The solutions of (4.13) are:

$$f_+(x) = (x+1)^{-\gamma_+} \sum_{s=0}^{\infty} \frac{c_s^+}{(x+1)^s} \qquad \text{and} \qquad f_-(x) = (x+1)^{\gamma_-} \sum_{s=0}^{\infty} \frac{c_s^-}{(x+1)^s}$$

Where: $\gamma_\pm = \frac{\omega \pm 1}{2}$; $\omega = \sqrt{1-4a_2}$

and:

$$\frac{c_1^\pm}{c_0^\pm} = -\frac{2a_1\left[\pm\gamma_\pm\left(\pm\gamma_\pm - 2\right) + 2a_2\right]}{a_2\left[\pm\gamma_\pm\left(\pm\gamma_\pm + 1\right) + a_2\right]} = c^\pm$$

The general solution is a linear combination of these ones, this is (up to first order):

$$f(x) \approx b_1(x+1)^{-\gamma_+}\left(1 + \frac{c^+}{x+1}\right) + b_2(x+1)^{\gamma_-}\left(1 + \frac{c^-}{x+1}\right) \tag{4.14}$$

Classically these solutions are called the regular and irregular because of their behavior in the UV [53], so that: $f(x) = f_{reg}(x) + f_{irreg}(x)$. Where:

$$f_{reg}(x) = b_1(x+1)^{-\gamma_+}\left(1 + \frac{c^+}{x+1}\right)$$
$$f_{irreg}(x) = b_2(x+1)^{\gamma_-}\left(1 + \frac{c^-}{x+1}\right)$$

Applying the UV boundary condition we find an interesting result:

$$\frac{b_1}{b_2} = \left[\frac{\gamma_-}{\gamma_+}\right] x^{\omega}_{x\to\Lambda_M}$$

If this ratio is infinite, b_2 must be zero (because $b_1 \to \infty$ does not make sense) and we say that the irregular solution is suppressed (or the regular is dominant). Nevertheless, we can see that if we restrict our cutoff Λ^2 to be finite, and in particular of the order of M^2, this scenario could change and it could be interesting for technicolor theories as pointed out by Doff, Machado and Natale [2].

In our case we suppose an infinite cut-off so our general solution exist only with the regular part. Applying the IR boundary conditions we have:

$$f(x) \approx \frac{(x+1)^{-\gamma_+}}{(1+c^+)}\left(1+\frac{c^+}{x+1}\right)$$

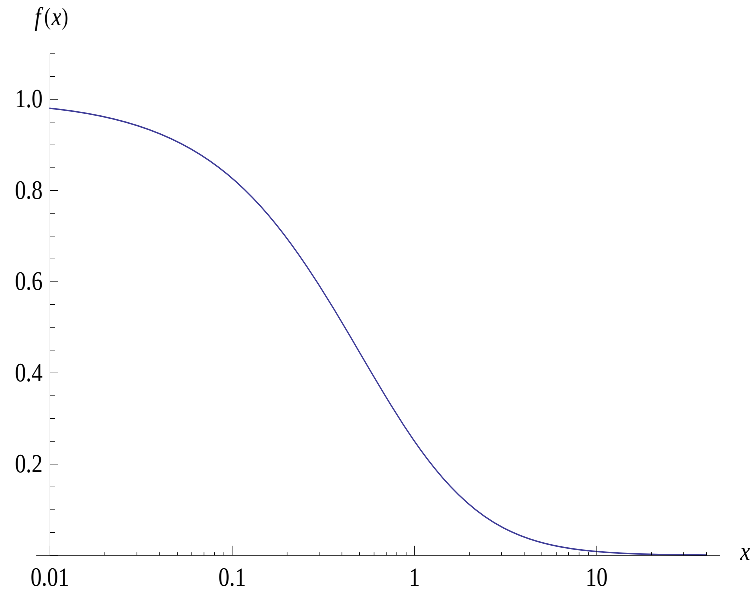


Figure 4.5: Nontrivial solution for the fermionic gap equation - constant coupling (for $m_g = 2\Lambda_{QCD} = 600MeV$, $K_F = 0.18GeV^2$, $\alpha_s(0) = 0.5$ and $M = 600MeV$).

4.2.2 Running Coupling Constant

We will now solve the equation (4.10) taking into account the running of the coupling constant (using equation (4.2)):

$$f(x) = \int_0^{\Lambda_M} dy \left\{ \theta(x-y)\left[\frac{a_1}{(x+\kappa)^2} + \frac{a}{(x+\epsilon)\ln\left[\frac{x+1}{L^2}\right]}\right] + \right.$$

$$+\,\theta(y-x)\left[\frac{a_1}{(y+\kappa)^2}+\frac{a}{(y+\epsilon)\ln\left[\frac{y+1}{L^2}\right]}\right]\Bigg\}\,f(y)$$

Where $a = 3c_2/16b\pi^2$ is the Lane constant [50], $L^2 = \Lambda^2_{QCD}/M^2$ and we have used the approximation†: $\ln\left[\frac{x+4}{L^2}\right] \approx \ln\left[\frac{x+1}{L^2}\right]$.

This integral equation can be transformed into a differential one using the substitution:

$$z = \ln\frac{x+1}{L^2} = z_\Lambda + \ln\frac{x+1}{\Lambda_M}$$

Where:

$$x = 0 \rightarrow z = z_0 = \ln\frac{1}{L^2} = z_\Lambda + \ln\left(\frac{1}{\Lambda_M}\right)$$

$$x = \Lambda_M \rightarrow z \approx z_\Lambda = \ln\frac{\Lambda_M}{L^2} = \ln\left(\frac{\Lambda^2}{\Lambda^2_{QCD}}\right)$$

So we obtain the integral equation (with $a_3 = a_1/L^2$):

$$f(z) = \int_{z_0}^{z_\Lambda} dw\left\{\left[\frac{a_3}{e^{2z}}+\frac{a}{ze^z}\right]\theta(z-w)+\left[\frac{a_3}{e^{2w}}+\frac{a}{we^w}\right]\theta(w-z)\right\}e^w f(w) \tag{4.15}$$

Using the θ functions we have:

$$f(z) = \int_{z_0}^{z} dw\left[a_3e^{-2z}+\frac{ae^{-z}}{z}\right]e^w\,f(w) + \int_{z}^{z_\Lambda} dw\left[a_3e^{-2w}+\frac{ae^{-w}}{w}\right]e^w\,f(w)$$

$$f'(z) = -\left[2a_3e^{-z}+a\left(\frac{1+z}{z^2}\right)\right]e^{-z}\int_{z_0}^{z} e^w\,f(w)dw$$

Which imply the UV condition:

†This is well justified since we are interested in the asymptotic behavior

$$f'(z \to z_\Lambda) = -f(z \to z_\Lambda) = 0$$

And from (4.15) we obtain the differential equation:

$$\left[2a_3 z^4 e^{-z} + az^2(1+z)\right] f''(z) + \left[4a_3 z^4 e^{-z} + a(z^3 + 2z^2 + 2z)\right] f'(z) +$$

$$+ \left[2a_3 z^2 e^{-z} + a(1+z)\right]^2 f(z) = 0$$

With the boundary conditions:

$$f(z)\Big|_{z\to 0} = 1 \qquad \text{and} \qquad f'(z)\Big|_{z\to z_\Lambda} = 0$$

In the UV regime the exponential term is suppressed so it is enough to study the differential equation:

$$(z^3 + z^2)f''(z) + (z^3 + 2z^2 + 2z)f'(z) + a(z+1)^2 f(z) = 0 \tag{4.16}$$

To solve this differential equation we will use the expansion method [8], which basically consist in suppose a solution of the form:

$$f(z) = e^{\beta z} \sum_{n=0}^{\infty} b_n z^{-(n+\alpha)}$$

Replacing this solution into the equation we find two values for β and for α:

$$\left.\begin{array}{ccc} \beta = 0 & \to & \alpha = a \\ \beta = -1 & \to & \alpha = 1 - a \end{array}\right\} \tag{4.17}$$

Finally, we obtain two solutions:

$$f_1(z) = e^{-z} z^{-1+a} \sum_{n=0}^{\infty} R_n z^{-n} \qquad \text{and} \qquad f_2(z) = z^{-a} \sum_{n=0}^{\infty} I_n z^{-n}$$

Where:

$$I_1 = (a + a^2) I_0 \qquad \text{and} \qquad R_1 = (a - a^2) R_0$$

The general solution (up to first order and in terms of x) stand:

$$f(x) \approx b_1 \frac{1}{\left(\frac{x+1}{L^2}\right)} \ln^{-1+a}\left(\frac{x+1}{L^2}\right) + b_2 \ln^{-a}\left(\frac{x+1}{L^2}\right)$$

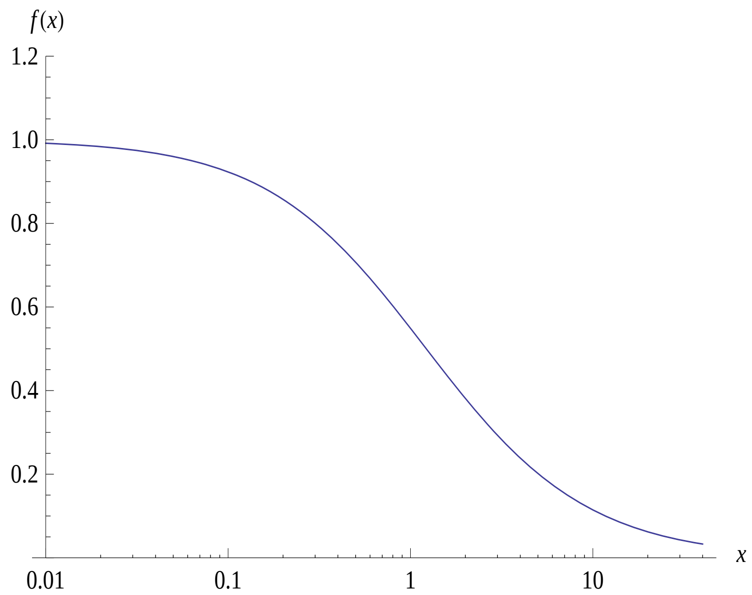


Figure 4.6: Nontrivial solution for the fermionic gap equation - running coupling constant ($m_g = 2\Lambda_{QCD} = 600MeV$, $K_F = 0.18GeV^2$).

Applying the boundary conditions we find:

$$\frac{b_1}{b_2}\Big|_{x \to \Lambda_M} = \frac{a\left(\frac{x+1}{L^2}\right)}{(a-1)\ln^{-1+2a}\left(\frac{x+1}{L^2}\right) - \ln^{2a}\left(\frac{x+1}{L^2}\right)} \to \infty$$

So, again we only obtain the regular solution (applying the IR condition):

$$f(x) = \ln^{1-a}\left(L^{-2}\right)\frac{\ln^{-1+a}\left(\frac{x+1}{L^2}\right)}{x+1}$$

With this result we improve the calculation of reference [2] where the coupling was assumed to be constant. The main modification found here is the logarithmic form of the solutions which matches with the asymptotic solutions found by Lane and Politzer some years ago [53]. This logaritmic dependence make the solution a little more damped as ilustrated in the figure 4.7.

Respect with the analysis of what solution is dominant, here we found that the regular is the dominant, so there are no changes in the analysis performed in the constant coupling case. However this result may change if the confining effect is reduced to a momentum region below M^2 as pointed out by Doff, Machado and Natale [54].

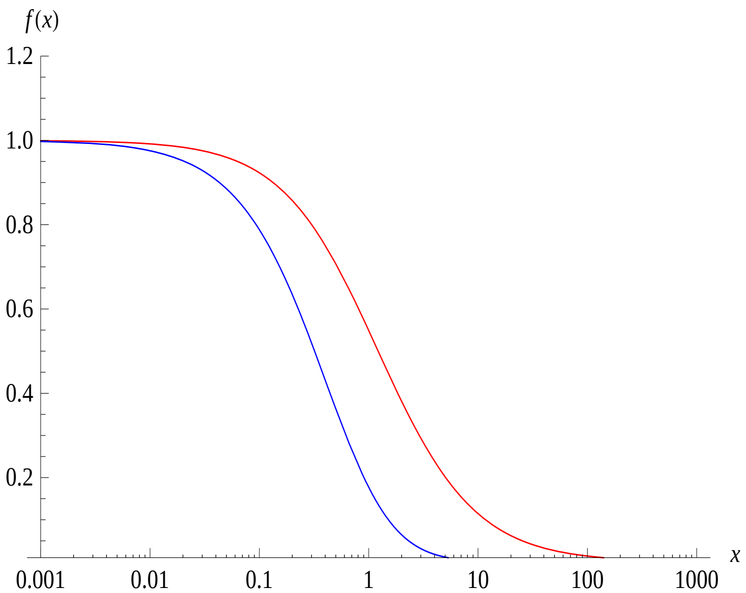


Figure 4.7: Nontrivial solution for the fermionic gap equation: Constant coupling and running coupling constant cases.

Chapter 5

Summary and Conclusions

We study dynamical chiral symmetry breaking in QCD using a gap equation with the inclusion of a confining effective propagator. In order to be consistent with lattice results, the inclusion of a dynamical gluon mass into the gap equation is also necessary. Considering only this last effect we do not obtain a phenomenologically satisfactory amount of chiral symmetry breaking, because the interaction strength diminishes as a consequence of the dynamical gluon mass, and no non-trivial solutions of the gap equation can be found for the expected values of the gluon mass.

Including confinement into the gap equation, as proposed by Cornwall, we can obtain a satisfactory dynamical quark mass. We found solutions for the complete gap equation (with massive and confining gluons) in two cases: Considering a constant coupling and taking into account the running coupling constant. In the first case we reproduced the calculations performed by Doff, Machado and Natale [2]. In the second case we included the effect of the running coupling, and although new logarithmic factors of momenta appear, we have not found fundamental differences from the previous results.

For further studies we propose the analysis of the complete gap equation including a cutoff proportional to the parameter m (the confinement scale). This cutoff can be included in the confinement part or in the complete gap equation, so we could separate the different physical regions. This analysis could lead us to different solutions and new phenomenological consequences as discused in [54].

Appendix A

Some calculations

A.1 Proof of equation (2.24):

This equation can be written like this:

$$I_d = \int_0^\pi d\theta \sin^{d-2}\theta \left[\frac{2(d-2)(p\cdot q)^2}{(p-q)^4} + \frac{(d-1)(p^2+q^2)(p\cdot q)}{(p-q)^4} + \frac{2p^2q^2}{(p-q)^4} \right]$$

Or:

$$I_d = 2(d-2)I_2^2(x,y;d) + (d-1)(x+y)\sqrt{xy}I_2^1(x,y;d) - 2xyI_2^0(x,y;d)$$

Where: $p^2 = x$, $q^2 = y$, and:

$$I_a^b(x,y;d) = \int_0^\pi d\theta \frac{\sin^{d-2}\theta \cos^b\theta}{\left(x+y-2\sqrt{xy}\cos\theta\right)^a}$$

In order to demonstrate our goal we will use a trick. Let us define the function:

$$L_n^p(\theta) = \frac{\sin^{p+1}\theta}{(a+b\cos\theta)^{n-1}}$$

Whose derivative can be written as:

$$\frac{d}{d\theta}L_n^p(\theta) = \frac{\sin^p\theta}{(a+b\cos\theta)^{n-1}}\left[(p-n+2)\,b\cos^2\theta + a(p+1)\cos\theta + b(n-1)\right]$$

and now we integrate:

$$\int_0^\pi d\theta\frac{d}{d\theta}L_n^p(\theta) = (p-n+2)\,b\int_0^\pi d\theta\frac{\sin^p\theta\cos^2\theta}{(a+b\cos\theta)^n} + a(p+1)\int_0^\pi d\theta\frac{\sin^p\theta\cos\theta}{(a+b\cos\theta)^n} + \\ + b(n-1)\int_0^\pi d\theta\frac{\sin^p\theta}{(a+b\cos\theta)^n} = 0$$

If we set: $n = 2 = p$, $a = x+y$ and $b = -2\sqrt{xy}$, we obtain:

$$4xy\int_0^\pi d\theta\frac{\sin^2\theta\cos^2\theta}{\left(x+y-2\sqrt{xy}\cos\theta\right)^2} - 3(x+y)\sqrt{xy}\int_0^\pi d\theta\frac{\sin^2\theta\cos\theta}{\left(x+y-2\sqrt{xy}\cos\theta\right)^2} + \\ + 2xy\int_0^\pi d\theta\frac{\sin^2\theta}{\left(x+y-2\sqrt{xy}\cos\theta\right)^2} = 0$$

Or:

$$I_{d=4} = 0$$

A.2 Proof of equation (4.13):

Here we start in the equation (4.12):

$$f(x) = \frac{1}{\pi}\int_0^{\Lambda_M} dy\left\{\theta(x-y)\left[\frac{a_1}{(x+\kappa)^2} + \frac{a_2}{(x+\epsilon)}\right] + \right. \\ \left. + \theta(y-x)\left[\frac{a_1}{(y+\kappa)^2} + \frac{a_2}{(y+\epsilon)}\right]\right\} f(y)$$

To transform this equation into a differential equation, using the θ function properties we replace the integral $\int_0^{\Lambda_M} dy()$ by $\int_0^{x} dy() + \int_x^{\Lambda_M} dy()$ in order to obtain:

$$f(x)=\int_0^x\left[\frac{a_1}{(x+1)^2}+\frac{a_2}{(x+1)}\right]f(y)dy+\int_x^{\Lambda_M}\left[\frac{a_1}{(y+1)^2}+\frac{a_2}{(y+1)}\right]f(y)dy$$

$$f(x)=\int_0^x\left[\frac{a_1+a_2(x+1)}{(x+1)^2}\right]f(y)dy+\int_x^{\Lambda_M}\left[\frac{a_1+a_2(y+1)}{(y+1)^2}\right]f(y)dy$$

Now, we rewrite the first term like:

$$I_1=\int_0^x\left[\frac{a_1+a_2(x+1)}{(x+1)^2}-\frac{a_1+a_2(y+1)}{(y+1)^2}+\frac{a_1+a_2(y+1)}{(y+1)^2}\right]f(y)dy$$

So, returning to $f(x)$:

$$f(x)=\int_0^x\left[\frac{a_1+a_2(x+1)}{(x+1)^2}-\frac{a_1+a_2(y+1)}{(y+1)^2}\right]f(y)dy+\int_0^{\Lambda_M}\left[\frac{a_1+a_2(y+1)}{(y+1)^2}\right]f(y)dy$$

The first term may by rewritten as:

$$\begin{aligned}I_3&=\int_0^x\left\{\int_y^x\frac{d}{dz}\left[\frac{a_1+a_2(z+1)}{(z+1)^2}\right]dz\right\}f(y)dy\\&=\int_0^x\int_y^x\left[\frac{a_2(z+1)^2-2\left[a_1+a_2(z+1)\right](z+1)}{(z+1)^4}\right]f(y)dzdy\\&=-\int_0^x\int_y^x\left[\frac{2a_1+a_2(z+1)}{(z+1)^3}\right]f(y)dzdy\end{aligned}\tag{A.1}$$

Exchanging the limits of integration:

$$I_3=-\int_0^x\int_0^z\left[\frac{2a_1+a_2(z+1)}{(z+1)^3}\right]f(y)dydz$$

And returning to $f(x)$:

$$f(x) = \int_0^x \int_0^z \left[\frac{2a_1 + a_2(z+1)}{(z+1)^3}\right] f(y)dydz + I_0$$

$$f(x) = \int_0^x \left[\frac{2a_1 + a_2(z+1)}{(z+1)^3}\right] \int_0^z f(y)dydz + I_0$$

If we derive:

$$f'(x) = -\frac{2a_1 + a_2(x+1)}{(x+1)^3} \int_0^x f(y)dy$$

Now, isolating the integral:

$$\frac{(x+1)^3}{2a_1 + a_2(x+1)} f'(x) = -\int_0^x f(y)dy$$

and derivating again with respect to x:

$$\frac{3(x+1)^2\left[a_2(x+1) + 2a_1\right] - a_2(x+1)^3}{\left[2a_1 + a_2(x+1)\right]^2} f'(x) + \frac{(x+1)^3}{2a_1 + a_2(x+1)} f''(x) = -f(x)$$

we finally obtain the second-order differential equation:

$$(x+1)^3\left[2a_1 + a_2(x+1)\right] f''(x) + (x+1)^2\left[6a_1 + 2a_2(x+1)\right] f'(x) +$$

$$+\left[2a_1 + a_2(x+1)\right]^2 f(x) = 0$$

With the boundary conditions:

$$f(x)\Big|_{x\to 0} = 1 \qquad \textit{and} \qquad f'(x)\Big|_{x\to \Lambda_M} = 0$$

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