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Sobre a integrabilidade de subfibrados invariantes de codimensão um de skew-products parcialmente hiperbólicos

São José do Rio Preto
2018

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Tese apresentada como parte dos requisitos para obtenção do título de Doutor em Matemática, junto ao Programa de Pós-Graduação em Matemática, Área de Concentração - Sistemas Dinâmicos, do Instituto de Biociências, Letras e Ciências Exatas da Universidade Estadual Paulista "Júlio de Mesquita Filho", Câmpus de São José do Rio Preto.

Financiadora: CAPES

Orientador: Prof. Dr. Vanderlei Minori Horita

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Abstract

In this work we prove that there is no contact partially hyperbolic skew-product $F : \mathbb{T}^2 \times \mathbb{S}^1 \to \mathbb{T}^2 \times \mathbb{S}^1$ of the form $F(p,t) = (f(p), t + \gamma(p))$, where f is an Anosov diffeomorphism and $\gamma \in C^r(\mathbb{T}^2)$ is a coboundary.

Keywords: partial hyperbolicity, Anosov diffeomorphism, skew-product, integrability, invariant subbundle, contact diffeomorphism.

Resumo

Neste trabalho mostramos que não existe skew-product de contato parcialmente hiperbólico $F : \mathbb{T}^2 \times \mathbb{S}^1 \to \mathbb{T}^2 \times \mathbb{S}^1$ da forma $F(p, t) = (f(p), t + \gamma(p))$, onde f é um difeomorfismo de Anosov e $\gamma \in C^r(\mathbb{T}^2)$ é um cobordo.

Palavras-chave: hiperbolicidade parcial, difeomorfismos de Anosov, skew-product, integrabilidade, subfibrado invariante, difeomorfismo de contacto.

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Chapter 1

Introduction

The goal of this work is to study partially hyperbolic conservative contact diffeomorphisms in $\mathbb{T}^3$. Most of the works related to contact dynamics are about *Anosov contact flows*, which are Anosov flows defined in contact manifolds such that its hyperbolic structure $E^s \oplus E^u$ is the contact structure of the manifold [1, 2, 27, 28].

In general, when the invariant center bundle of a partially hyperbolic diffeomorphism has dimension 1, the bundles E^s and E^u are not jointly integrable, that is, $E^s \oplus E^u$ is not integrable. However, nonitegrability of $E^s \oplus E^u$ does not mean that it qualifies to be a contact structure. Indeed, a contact struture is defined to be a codimension 1 subbundle of an odd-dimensional manifold that is *as far as possible from being integrable* or *maximally nonintegrable*. The geometric meaning of *maximal nonintegrability* is that there are no hypersurfaces tangent to the given subbundle, not even locally, whereas an ordinary nonintegrable subbundle may have some form of local integrability. This suggests that we can establish different types of nonintegrability. A way to establish this is through the Frobenius' Integrability Theorem that, for codimension 1 subbundles, can be stated as follows.

Theorem 1.1 (Frobenius' Integrability Theorem)**.** *Let M be an odd-dimensional manifold and $\xi \subset TM$ be a condimension 1 C^k subbundle, $k \geq 1$. If α is defining the 1-form for ξ in some open neighborhood $U \subset M$ then ξ is integrable iff $\alpha \wedge d\alpha = 0$.*

This theorem implies that one can achieve nonitegrability by providing a subbundle such that $(\alpha \wedge d\alpha)(p) \neq 0$, for some $p \in M$. On the other hand, the contact condition for ξ requires that $\alpha \wedge (d\alpha)^n \neq 0$ at every point in M, that is, $\alpha \wedge (d\alpha)^n$ is required to be a volume form in M. When $\dim(M) = 3$ it becomes clear what "*as far as possible from being integrable*" means: ξ is a contact structure if $\alpha \wedge d\alpha \neq 0$ at every point in M, which is the complete opposite of it being integrable.

For the intermediate case when ξ is nonintegrable but not a contact structure, Eliashberg and Thurston defined the concept of *confoliations* and studied stability properties of such objects [10].

For dynamically coherent partially hyperbolic diffeomorphisms, a natural possibility for the invariant contact structure is $E^s \oplus E^u$ and in dimension 3 this is actually the unique choice, i.e., $E^s \oplus E^u$ is a contact structure for any dynamically coherent partially hyperbolic contact diffeomorphisms in a 3-dimensional manifold. However, there still remains the question if such diffeomorphisms exist. For instance, in the Heisenberg manifold, which is the quotient of the Heisenberg group by a discrete subgroup and can also be viewed as a $\mathbb{S}^1$-bundle of over $\mathbb{T}^2$, all bundle automorphisms over Anosov automorphisms of $\mathbb{T}^2$ are contact diffeomorphisms with $E^s \oplus E^u$ being the invariant contact structures in Heisenberg manifold, see [26].

We consider conservative partially hyperbolic skew-products $F : \mathbb{T}^2 \times \mathbb{S}^1 \to \mathbb{T}^2 \times \mathbb{S}^1$ of the form

$$F(x,t) = (f(x), t + \gamma(x)), \tag{1.1}$$

where f is an Anosov diffeomorphism and $\gamma \in C^r(\mathbb{T}^2, \mathbb{S}^1)$. The motivation for this choice is due to the following conjecture posed by Pujals, see details in Bonatti-Wilkinson [4], concerning the classification of partially hyperbolic diffeomorphisms:

Conjecture 1.1 (Pujals)**.** In dimesion 3 any partially hyperbolic diffeomorphism is leaf conjugate to one the following models:

(i) An Anosov diffeomorphism of $\mathbb{T}^3$;

(ii) A partially hyperbolic skew-product over an Anosov diffeomorphism of $\mathbb{T}^3$ (in this case the skew-product is defined in either $\mathbb{T}^3$ or a 3-dimensional nilmanifold);

(iii) The time-1 map of an Anosov flow.

We show that for a certain class of maps $\gamma : \mathbb{T}^2 \to \mathbb{S}^1$ the skew-product (1.1) fails to be a contact diffeomorphism.

Let G be a topological group with group operation $*$. Recall that a map $\gamma : M \to G$ from a smooth manifold M into G is a *coboundary relative to the diffeomorphism* $f : M \to M$ if there is a map $\mu : M \to G$ such that

$$\gamma = \mu \circ f * \mu^{-1}.$$

The map μ is called a *transfer map* for the coboundary γ. An equation involving coboundaries where the unknown is the transfer function is called a *cohomological equation*. The existence of solutions for cohomological equations is not a difficult matter when no regularity conditions are required, however when we do require regularity of the solutions the difficulty of the problem increases dramatically. For instance, when f is an Anosov diffeomorphism, the Livšic's Theorem guarantees the existence of solutions with some regularity provided the function γ satisfies a certain *periodic orbit obstruction* and the proof of the theorem heavily relies on the dynamical properties of f. See [18], [19] and [13] for details and also [29], [30] and [14] for generalizations.

The main result of this work follows.

Theorem A. *There are no contact partially hyperbolic skew-products $F : \mathbb{T}^2 \times \mathbb{S}^1 \to \mathbb{T}^2 \times \mathbb{S}^1$ of the form $F(p,t) = (f(p), t + \gamma(p))$, where $f : \mathbb{T}^2 \to \mathbb{T}^2$ is an Anosov diffeomorphism and $\gamma \in C^r(\mathbb{T}^2; \mathbb{S}^1)$ is a coboundary.*

The main ingredients in the proof are Theorems B and C.

Theorem B. *Let M^n be a compact n-dimensional manifold, $F : M \times \mathbb{S}^1 \to M \times \mathbb{S}^1$ be a skew-product of the form $F(p,t) = (f(x), t + \gamma(p))$, where f is a Anosov C^r diffeomorphism and $\gamma \in C^r(M; \mathbb{S}^1)$, $r \geq 1$. If γ is a coboundary relative to f with transfer map $\mu \in C^1(M, \mathbb{S}^1)$ then there exists an F-invariant codimension 1 subbundle $\xi \subset T(M \times \mathbb{S}^1)$ transversal to $T\mathbb{S}^1$ such that ξ is tangent to the graph of μ.*

In dimension 3, using the notion of *characteristic foliations* (see Section 3.2) we prove, roughly speaking, that the characteristic vector field when restricted to the leaves of a transversal codimension 1 foliation tangent to a codimension 1 subbundle is locally hamiltonian.

Theorem C. *Let M^2 be a compact surface without boundary, $F : M \times \mathbb{S}^1 \to M \times \mathbb{S}^1$ be a skew-product of the form $F(p,t) = (f(x), t + \gamma(p))$, where f is a Anosov C^r diffeomorphism and $\gamma \in C^r(M; \mathbb{S}^1)$, $r \geq 1$. If γ is a coboundary relative to f with transfer map $\mu \in C^1(M; \mathbb{S}^1)$ then there exists an DF-invariant codimension 1 subbundle $\xi \subset T(M \times \mathbb{S}^1)$ transversal to $T\mathbb{S}^1$ such that the vector field X_ξ reprensenting the characteristic foliation of ξ in any hypersurface $S \subset M \times \mathbb{S}^1$ transversal to $T\mathbb{S}^1$ is locally hamiltonian.*

See Section 2.4.2 for definition and details regarding the almost complex structure $\mathbf{J}$.

Existence of dynamically coherent contact partially hyperbolic diffeomorphisms in $\mathbb{T}^3$ is an interesting matter since the contact structure must be $E^s \oplus E^u$. In particular, if there are any contact partially hyperbolic diffeomorphisms in $\mathbb{T}^3$ then all of them must have the *accessibility property* due to *Chow's Theorem*, see [22, Theorem 3.3].

Let $f : M \to M$ be a conservative partially hyperbolic diffeomorphism and $p \in M$. The *accessibility class* of p is the set $A_p \subseteq M$ of points of M that can be reached from p by a concatenation of a finite number of paths entirely contained in either stable or unstable leaves of f. We say that f has the *accessibility property* or that f is *accessible* if for some

$p \in M$ we have $A_p = M$. We say that f is *essentially accessible* if every measurable set that is union of accessible classes has either full or null measure. We say that f is *C^r-stably accessible* em $\mathrm{Diff}^r_m(M)$ if f has the accessibility property and there exists a neighborhood of f in the C^r-topology such that every diffeomorphism g in this neighborhood has the accessibility property.

The notion of accessibility is a key property to prove *ergodicity*. A conservative diffeomorphism $f : M \to M$ with invariant measure μ is said to be *ergodic* if the only f-invariant subsets have either full or null measure. We say that f is *C^r-stably ergodic* if f is ergodic and there exist a neighborhood $U \subset \mathrm{Diff}^r_m(M)$ of f in the C^r-topology such that every $g \in U$ is also ergodic.

In [22], Phug and Shub established the following conjecture:

Conjecture 1.2. Stable ergodicity is C^1-open and C^r-dense among conservative partially hyperbolic diffeomorphisms, $r > 1$.

The following conjectures in [22] highlight the relation between the two notions and provide a program to prove ergodicity:

Conjecture 1.3. If f is a C^r conservative partially hyperbolic diffeomorphism and C^r-stably accessible, $r > 1$, then f is ergodic.

Conjecture 1.4. Stable accessibility is C^r-dense among conservative partially hyperbolic diffeomorphisms, with $r > 1$.

These conjectures describe ergodicity using the notion of accessibility and since then a lot of work has been produced in attempts to prove Conjecture 1.2. In [7], Burns and Wilkinson proved the Conjecture 1.3 assuming a condition of *center bunching*, which is an assumption made on the variation of the domination of the center direction with respect to the strong stable and unstable foliations. Rodriguez-Hertz, Rodriguez-Hertz and Ures proved Conjecture 1.2 in the case of one dimensional center bundle [23].

In dimension 3 it follows that stable ergodicity is C^∞-dense among conservative partially hyperbolic diffeomorphisms (see [23]). The question about existence of non-ergodic partially hyperbolic diffeomorphims was also partially answered in [24]:

Theorem 1.2 (Hertz-Hertz-Ures)**.** *If N is a 3-dimensional nilmanifold other than $\mathbb{T}^3$, then every conservative partially hyperbolic diffeomorphism in N is ergodic.*

In the same paper they conjecture the following

Conjecture 1.5 (**Ergodic Conjecture**)**.** If a partially hyperbolic diffeomorphism of a 3-dimensional manifold is not ergodic, then there exist a topological 2-torus tangent to $E^s \oplus E^u$.

Theorem B implies that there is a codimension 1 submanifold tangent to $E^s \oplus E^u$, which is the graph of the transfer map and diffeomorphic to $\mathbb{T}^2$.

This work is organized as follows. In Chapter 2 we state all basic definitions and results needed in the proof.

In the Section 3.1.1 we begin with a proof of Theorem B in the particular case where f is an Anosov linear automorphism and γ is a map with higher regularity enough to make the solutions of the cohomological equation $\gamma = \mu \circ f - \mu$ to be at least C^2. Section 3.1.2 deals with the general case of integrability $E^s \oplus E^u$ of a skew-product $F : M \times \mathbb{S}^1 \to M \times \mathbb{S}^1$ of the form (1.1), where M is compact n-dimensional manifold. At first we suppose that F admits a codimension 1 invariant subbundle ξ transversal to $\mathbb{S}^1$ and study the local behavior of this subbundle near the submanifolds $M_t = M \times \{t\}$, $t \in \mathbb{S}^1$ (Lemma 3.1). In the case where γ is a coboundary, we prove that $E^s \oplus E^u$ is integrable (though not in the sense of Frobenius) and admits a compact leaf diffeomorphic to M.

In Section 3.2 we consider the 3-dimensional case, where M is a surface (symplectic manifold of dimension 2). Here we use the notion of *characteristic foliation*, a tool of contact geometry to analyze semi-local aspects of contact structures, to investigate the question of integrability of the hyperbolic structure. It follows that if there exists an integral F-invariant codimension 1 subbundle transversal to $\mathbb{S}^1$ then there is a local hamiltonian vector field in M such that each leaf of the foliation is locally the graph of the associated hamiltonian $\mathbb{S}^1$-maps (Theorem C).

Finally, in Section 3.3 we conclude proving that a skew-product F of $\mathbb{T}^3$ when γ is a coboundary cannot be a contact diffeomorphism (Theorem A). We give two different proofs based on Theorems B and C.

Chapter 2

Definitions and basic results

2.1 Vector bundles

Let E and M^n be C^r manifolds and $\pi : E \to M$ be a submersion. We say that the triple (E, M, π) is a C^r *regular vector bundle* over M if for each $p \in M$ the set $E_p = \pi^{-1}(p)$ is a vector space of dimension k and there exists a *local trivialization* of an open neighborhood $U \subset M$ such that $p \in U$, i.e., a diffeomorphism $\psi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ such that for any two such trivializations (U_i, ψ_i) and (U_j, ψ_j) with $U_i \cap U_j \neq \emptyset$, the map $\psi_i \circ \psi_j^{-1} : (U_i \cap U_j) \times \mathbb{R}^k \to (U_i \cap U_j) \times \mathbb{R}^k$ has the form

$$\psi_i \circ \psi_j^{-1}(p, x) = (p, \Phi_{ij}(p)x),$$

where $\Phi_{ij} : U_i \cap U_j \to \mathrm{GL}(\mathbb{R}^k)$ is C^r. A vector bundle (E, M, π) is said to be *trivial* if there exists a global trivialization of E.

The most simple examples of vector bundles are the *tangent* and *contangent bundles* of a smooth manifold M. The tangent bundle is the set

$$TM = \coprod_{p \in M} T_pM,$$

defined as the disjoint union of all tangent vector spaces of M. The contangent bundle is the set

$$T^*M = \coprod_{p \in M} T_p^*M,$$

defined as the disjoint union of all duals of the tangent vectors spaces of M.

Let (E, M, π) and $(\widetilde{E}, M, \widetilde{\pi})$ be two vector bundles over M. A *vector bundle morphism* between E and $\widetilde{E}$ is a map $h : E \to \widetilde{E}$ such that $\widetilde{\pi} \circ h = \pi$ and $h|_{\pi^{-1}(p)} : \pi^{-1}(p) \to \widetilde{\pi}^{-1}(h(p))$ is homomorphism between vector spaces, for each $p \in M$. We say that a vector bundle morphism is an *isomorphism of vector bundles* if the fibered homomorphism $h|_{\pi^{-1}(p)}$: $\pi^{-1}(p) \to \widetilde{\pi}^{-1}(h(p))$ is an isomorphism of vector spaces.

A C^r*-section* of vector bundle (E, M, π) is a map $\sigma : M \to E$ such that $\pi \circ \sigma = \mathrm{id}_M$. We write $\Gamma^r(E)$ for the set of C^r-sections of E which has a natural structure of $C^\infty(M)$-module due to the defining property of its elements. When $E = TM$ we have that $\Gamma^r(E) = \mathfrak{X}^r(M)$ is the space of C^r vector fields of M and when $E = T^*M$, $\Gamma^r(E) = \Omega_1^r(M)$ is the space of C^r 1-forms of M.

When $E = TM$ we can still define another operation of $\Gamma^r(E)$, namely the *Lie bracket* between two vector fields $X, Y \in \Gamma^r(E)$. The resulting vector field of this operation, denoted by $[X, Y]$, is a C^{r-1} vector field defined as derivation in the space of C^r function of M by

$$[X, Y](f) = X(Y(f)) - Y(X(f)), \tag{2.1}$$

for all $f \in C^r(M)$.

Given a local system of coordinates $x_1, \dots, x_n$ in M, let $X = X^i \dfrac{\partial}{\partial x_i}$ and $Y = Y^i \dfrac{\partial}{\partial x_i}$ be the represention s of X and Y in these coordinates. Then evaluating (2.1) using the local representations of X and Y at every $f \in C^r(M)$ we obtain

$$[X, Y] = \left(X^i \frac{\partial Y^j}{\partial x_i} - Y^i \frac{\partial X^j}{\partial x_i} \right) \frac{\partial}{\partial x_j},$$

which is the local representation of the Lie bracket in the coordinates $x_1, \dots, x_n$.

Remark 2.1. Here we are using the *Einstein summation convention* to write the vector fields X, Y and $[X, Y]$, where we assume that when an index appear as a subscript and superscript in the same expression then this expression is a sum of all over possible values that index can attain and we drop the summation symbol for simplicity. Thus, if we write, for example, $X = X^i \dfrac{\partial}{\partial x_i}$ we are actually considering

$$X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x_i}.$$

The C^∞-module $\mathfrak{X}^r(M)$ equipped with the Lie bracket operation is a (Lie) algebra of the tangent bundle TM.

2.1.1 Subbundles and integrability

Let (E, M, π) be a vector bundle over M. A *vector subbundle* of E is a nonempty subset $H \subseteq E$ such that the natural inclusion map $i : H \to E$ is a morphism of vector bundles between (E, M, π) and (H, M, π_H), where $\pi_H : H \to M$ is the restriction of π to the set H. This imply, for instance, that $\pi_H^{-1}(p)$ is a vector subspace of $\pi^{-1}(p)$, for any $p \in M$.

Let $\xi \subset TM$ be a subbundle. An *integral submanifold of* ξ is a submanifold $N \subseteq M$ such that $T_pN = \xi_p$, for all $p \in N$, where ξ_p denotes the fiber through the point p. We say that a subbundle $\xi \subset TM$ is *integrable* if any point $p \in M$ belongs to an integral submanifold of ξ.

The question of wether or not there exists integral submanifolds for a given subbundle and its integrability is matter of the following result of Frobenius (see [16]).

Theorem 2.1 (Frobenius' Integrability Theorem). *A subbundle $\xi \subset TM$ is integrable if and only if the subset $\Gamma^r(\xi) \subset \mathfrak{X}^r(M)$ of C^r-sections of ξ is a (Lie) subalgebra of $\mathfrak{X}^r(M)$.*

2.1.2 Foliations

Consider in $\mathbb{R}^{m+n}$ subsets of the form

$$\mathbb{R}^m_c = \left\{(x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+n}) \in \mathbb{R}^{m+n};\ x_{m+1} = c_1, \ldots, x_{m+n} = c_n\right\},$$

where $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$. The subsets $\mathbb{R}^m_c$ are affine subspaces of $\mathbb{R}^{m+n}$ that forms a *partition* of it, i.e.,

$$\mathbb{R}^{m+n} = \bigcup_{c \in \mathbb{R}^n} \mathbb{R}^m_c$$

and $\mathbb{R}^m_c \cap \mathbb{R}^m_{\tilde{c}} = \emptyset$ if $c \neq \tilde{c}$.

The family $\mathcal{F}^m = \{\mathbb{R}^m_c\}_{c \in \mathbb{R}^n}$ is a *foliation of dimension* m or *codimension* n in $\mathbb{R}^{m+n}$. Each subset $\mathbb{R}^m_c \in \mathcal{F}^m$ is a connected submanifold of $\mathbb{R}^{m+n}$ and are called *leaves* of the foliation $\mathcal{F}^m$.

Now let M^{m+n} be a smooth manifold of dimension $m+n$ and $\mathcal{F} = \{L_\alpha\}_{\alpha \in \Lambda}$ be a family of connected subset of M. We say that $\mathcal{F}$ is partition of M if

$$M = \bigcup_{\alpha \in \Lambda} L_\alpha$$

and $L_\alpha \cap L_\beta = \emptyset$ if $\alpha \neq \beta$.

Let $\mathcal{F}$ be a partition of M. An *m-foliated local chart* (U, φ) in M is a local chart of M such that $\varphi(L_\alpha \cap U) = \mathbb{R}^m_{c_\alpha}$, for some $c_\alpha \in \mathbb{R}^n$. An *m-foliated atlas* $\mathcal{A} = \{(U_i, \varphi_i);\ i \in I\}$ is an atlas of M which is exclusively comprised of m-foliated local charts of M. If M has an m-foliated atlas then we say that $\mathcal{F}$ is a *foliation of dimension* m or an *m-foliation* of M and each subset L_α is a leaf of the foliation $\mathcal{F}$. Since $L_\alpha \cap U = \varphi^{-1}(\mathbb{R}^m_{c_\alpha})$, it follows that each leaf L_α is a submanifold of dimension m in M.

Recall that every foliation $\mathcal{F}$ generates an integrable subbundle $E(\mathcal{F}) \subset TM$ whose integral submanifolds are precisely the leaves of the foliation $\mathcal{F}$. Let $\pi : \mathbb{R}^{m+n} \to \mathbb{R}^n$ be the natural projection on the last n coordinates. Then, given an m-foliated local chart (U, φ), the map $\psi = \pi \circ \varphi : U \to \mathbb{R}^n$ is a submersion such that, for each $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$, we have $\psi^{-1}(c) = \varphi^{-1}(\mathbb{R}^m_c) = L_\alpha$, for some $\alpha \in \Lambda$, and consequently, $\ker(D\psi) = TL_\alpha$. Thus, if $\mathcal{A} = \{(U_i, \varphi_i); i \in I\}$ is an m-foliated atlas of M, the set

$$E(\mathcal{F}) = \bigcup_{i \in I} \ker(D\psi_i)$$

is an integrable subbundle of TM whose integral submanifolds are the leaves of the foliation $\mathcal{F}$.

For a more general treatment and examples of the theory of fiber bundles and foliated manifolds see [3], [15], [16] and [17].

2.2 Differential forms

Let M be a C^r smooth manifold and TM and T^*M be its tangent and cotangent bundle, respectively. A C^ℓ *differential* 1-*form*, $0 \le \ell \le r$, is a section of contangent bundle, i.e., is a map $\alpha : M \to T^*M$ such that $\pi\alpha = \mathrm{id}_M$, where $\pi : T^*M \to M$ is the canonical projection map. Then, for each $p \in M$, $\alpha(p) = \alpha_p \in (T_pM)^*$ is linear functional in T_pM.

Given a local coordinate system $x_1, \ldots, x_n$ in an open set $U \subset M$, with its corresponding cotangent bundle coordinates $x_1, \ldots, x_n, dx_1, \ldots, dx_n$, the local representation of a differential 1-form α has the form

$$\alpha = \sum_{j=1}^{n} a_j dx_j,$$

where the $a_j : U \to \mathbb{R}$ are C^ℓ functions of M. It follows that the linear functionals α_p are given by

$$\alpha_p = \alpha(p) = \sum_{j=1}^{n} a_j(p)(dx_j)_p,$$

where $(dx_1)_p, \ldots, (dx_n)_p$ is the dual basis of the coordinate base vectors $\left.\dfrac{\partial}{\partial x_1}\right|_p, \ldots, \left.\dfrac{\partial}{\partial x_n}\right|_p$.

Define the *kernel* of a differential 1-form $\alpha : M \to T^*M$ as the set

$$\ker(\alpha) = \coprod_{p \in M} \ker(\alpha_p) \subset TM,$$

formed by the disjoint union of set which are the kernels of the linear functionals α_p, for each $p \in M$. In general, $\ker(\alpha)$ is not a subbundle of TM, since the fibers may have different dimension.

The points $p \in M$ such that $\alpha_p \neq 0$ are called *regular points* of α while the points such that $\alpha_p = 0$ are called *singular points*. When α does not have singular points its kernel is a subbundle of TM, since in this case $\alpha_p \neq 0$ and $\ker(\alpha_p)$ has codimension 1 in T_pM, for every $p \in M$. On the other hand, if $p \in M$ is a singular point of α, then $\ker(\alpha_p) = T_pM$ and in these points the differentiability of the trivializations cannot be guaranteed. Those types of bundles are commonly known as *singular fiber bundles*.

Let $\Omega_1^\ell(M)$ be the set of C^ℓ differential 1-forms in M. This set has a natural structure of vector space over $\mathbb{R}$ which is also a $C^\ell(M)$-module when we extend the standard product between real numbers and elements of $\Omega_1^\ell(M)$ to the space $C^\ell(M)$ of C^ℓ functions of M. If $dx_1, \ldots, dx_n$ are local generators of $\Omega_1^\ell(M)$ for some local coordinate system $x_1, \ldots, x_n$, we define the *wedge product* $\wedge$ on the generators of $\Omega_1^\ell(M)$ as

$$(dx_i \wedge dx_j)_p(u, v) = \det \begin{pmatrix} (dx_i)_p(u) & (dx_i)_p(v) \\ (dx_j)_p(u) & (dx_j)_p(v) \end{pmatrix} = u_i v_j - u_j v_i,$$

for each $p \in M$ and any two tangent vectors $u = \displaystyle\sum_{k=1}^{n} u_k \left.\frac{\partial}{\partial x_k}\right|_p$ e $v = \displaystyle\sum_{k=1}^{n} v_k \left.\frac{\partial}{\partial x_k}\right|_p$ in T_pM.

The local $C^\ell(M)$-module generated by the elements $\{dx_i \wedge dx_j\}_{i<j}$ is called the space of *differential* 2*-forms* of class C^ℓ in M and denoted by $\Lambda_2^\ell(M)$. Locally, any differential 2-form in M can be generally written as

$$\omega = \sum_{i<j} a_{ij} dx_i \wedge dx_j,$$

where the $a_{ij} : U \to \mathbb{R}$ are C^ℓ functions defined in an open set $U \subseteq M$. Thus, $\omega_p = \omega(p)$ is an alternating bilinear form in T_pM, for any $p \in M$.

In general, we define the $C^\ell(M)$-module $\Lambda_r^\ell(M)$ of the C^ℓ *differential* r*-forms*, with $r \leq \dim(M)$, as the space generated by the set of elements $\{dx_{i_1}, \ldots, dx_{i_r}\}_{i_1 < \ldots < i_r}$ defined at each $p \in M$ by the relation

$$(dx_J)_p(u_1, \ldots, u_r) = (dx_{i_1} \wedge \ldots \wedge dx_{i_r})_p(u_1, \ldots, u_r) = \det(dx_{i_j}(u_k))_{j \times k},$$

for any r tangent vectors $u_1, \ldots, u_r \in T_pM$, where $J = \{i_1 < \ldots < i_r\} \subset \{1, \ldots, n\} = I_n$. Therefore, we can locally represent an r-form ω by the expression

$$\omega = \sum_{J \subset I_n} a_J dx_J.$$

Exterior product. Let $X \in \mathfrak{X}^\ell(M)$ and $\omega \in \Lambda_r^\ell(M)$. Define the *exterior product* of X with ω as the $(r-1)$-form $i(X)\omega \in \Omega_{r-1}^\ell(M)$ given by

$$[i(X)\omega](Y_1, \ldots, Y_{r-1}) = \omega(X, Y_1, \ldots, Y_{r-1}),$$

for any vector fields $Y_1, \ldots, Y_{r-1} \in \mathfrak{X}^\ell(M)$.

Pull-back. Given a C^ℓ diffeomorphism $f : M_1 \to M_2$ and the r-form $\omega \in \Lambda_r^\ell(M_2)$, we define the *pull-back* of ω to M_1 as the r-form $f^*\omega \in \Lambda_r^\ell(M_1)$ defined by

$$(f^*\omega)_p(v_1, \ldots, v_r) = \omega_p(Df_p v_1, \ldots, Df_p v_r),$$

for any $p \in M$ and any tangent vectors $v_1, \ldots, v_r \in T_pM$.

The pull-back, as a map $f^* : \Lambda_r^\ell(M_2) \to \Lambda_r^\ell(M_1)$ between vector spaces is an $\mathbb{R}$-linear isomorphism with inverse $(f^*)^{-1} = (f^{-1})^*$.

Let M be a smooth manifold, $\omega \in \Lambda_r^\ell(M)$ and $f : M \to M$ be a diffeomorphism. We say that ω is a *coboundary with respect to* f if there exists an r-form $\mu \in \Lambda_r^s(M)$, with $s \leq \ell$, such that

$$\omega = f^*\mu - \mu.$$

Exterior derivative. We can define a natural map $d : \Lambda_r^\ell(M) \to \Lambda_{k+1}^{\ell-1}(M)$ called the *exterior derivative* that associates to each r-form $\omega \in \Lambda_r^\ell(M)$ a $(r+1)$-form $d\omega \in \Lambda_{r+1}^{\ell-1}(M)$. To define $d\omega$ locally we consider local coordinates $x_1, \ldots, x_n$ in an open neighborhood U of M such that ω is written in general form

$$\omega = \sum_J a_J dx_J,$$

where $J = \{i_1 < \ldots < i_r\} \subset \{1, \ldots, n\}$, $dx_J = dx_{i_1} \wedge \ldots \wedge dx_{i_r}$ and $a_J : U \to \mathbb{R}$ are C^ℓ functions. Then, $d\omega$ is defined as

$$d\omega = \sum_J da_J \wedge dx_J = \sum_{j,J} \frac{\partial a_J}{\partial x_j} dx_j \wedge dx_J.$$

Note that $dx_j \wedge dx_J = dx_j \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_r}$ is an $(r+1)$-form and $\dfrac{\partial a_J}{\partial x_j}$ are $C^{\ell-1}$ functions, for each $j = 1, \ldots, n$, and J. Thus, $d\omega \in \Lambda_{r+1}^{\ell-1}(M)$.

A C^ℓ differential r-form ω is said to be *closed* if $d\omega = 0$. It is said to be *exact* if there exists $\alpha \in \Lambda_{r-1}^{\ell+1}(M)$ such that $d\alpha = \omega$.

Closed and exact differential forms are related through the following lemma

Lemma 2.1. *Let* $d^{k+1} : \Lambda^{\ell+2}_{r-2}(M) \to \Lambda^{\ell+1}_{r-1}(M)$ *and* $d^k : \Lambda^{\ell+1}_{r-1}(M) \to \Lambda^{\ell}_{r}(M)$ *be the exterior derivative maps defined in the corresponding spaces of forms. Then,* $d^k \circ d^{k+1} = 0$.

In particular, this lemma shows that every exact differential form is closed. Refer to [17], [16] and [12] for more details of the theory of differential forms.

2.3 Natural isomorphism between $\Omega^{\ell}_1(M)$ and $\mathfrak{X}^{\ell}(M)$

Let (M, g) be a riemannian manifold and denote by $\mathbf{G_f}$ the *metric tensor* of g with respect to a local frame $\mathbf{f} = \{E_1, \ldots, E_n\}$ in M, i.e., the metric tensor is the $(2,0)$-tensor whose matrix reprensetantion is $\mathbf{G_f} = (g(E_i, E_j))$ in the local frame $\mathbf{f}$ and for any vector fields $X = X^i E_i$ and $Y = Y^i E_i$, we have

$$g(X, Y) = \mathbf{X}^t \mathbf{G_f} \mathbf{Y},$$

where $\mathbf{X}$ and $\mathbf{Y}$ are the matrix representations of the vector fields X and Y as column vectors and $\mathbf{X}^t$ is the transpose matrix of $\mathbf{X}$. When we are dealing with a single frame $\mathbf{f}$ we will drop the subscript on the tensor and write $\mathbf{G}$ instead of $\mathbf{G_f}$. We will see shortly that the action of $\mathbf{G}$ on single tangent vectors induces a bundle isomorphism between TM and T^*M.

Proposition 2.1. *There exist a natural bundle isomorphism* $S : TM \to T^*M$ *induced by the metric tensor.*

Proof. To exhibit the isomorphism S we will consider the *natural pairing* of sections of both TM and T^*M, which is defined by

$$\langle \alpha, X \rangle = \alpha(X),$$

for all $\alpha \in \Omega^{\ell}_1(M)$ and $X \in \mathfrak{X}^{\ell}(M)$.

It follows that the natural pairing is a C^{ℓ} smooth $(1,1)$-tensor field in M, i.e., for each $p \in M$, the map $\langle\ ,\ \rangle_p : T^*_pM \times T_pM \to \mathbb{R}$ is a $(1,1)$-tensor such that the function $p \mapsto \langle \alpha_p, X_p \rangle_p$ is C^{ℓ}, for all $\alpha \in \Omega^{\ell}_1(M)$ and $X \in \mathfrak{X}^{\ell}(M)$.

Let $p \in M$ and $X_p \in T_pM$. We define S pointwise as the linear functional $S_pX_p \in T^*_pM$ satisfying

$$\langle S_pX_p, V \rangle_p = g_p(X_p, V),$$

for all $V \in T_pM$. Let $x_1, \ldots, x_n, \dfrac{\partial}{\partial x_1}, \ldots, \dfrac{\partial}{\partial x_n}$ and $x_1, \ldots, x_n, dx_1, \ldots, dx_n$ be local coordinates on TM and T^*M, respectively, induced by the local coordinates $x_1, \ldots, x_n$ on M.

Taking $V = \left.\dfrac{\partial}{\partial x_i}\right|_p$ we have

$$\left\langle S_p X_p, \left.\frac{\partial}{\partial x_i}\right|_p \right\rangle_p = g_p\left(X_p, \left.\frac{\partial}{\partial x_i}\right|_p\right) = \sum_{j=1}^{n} g_{ij} dx_j(X_p).$$

Therefore, the matrix representation of S in the coordinate basis of TM and T^*M is $\mathbf{S}_p = (s_{ij}(p))$, where

$$s_{ij}(p) = \left\langle S_p \left.\frac{\partial}{\partial x_i}\right|_p, \left.\frac{\partial}{\partial x_j}\right|_p \right\rangle = g_{ij}(p).$$

So we have $\mathbf{S}_p = \mathbf{G}_p$ and the linear transformation that naturally identifies T_pM and T_p^*M in the basis induced by the coordinate system is the matrix representation of the metric tensor in the same basis.

Finally, we define $S : TM \to T^*M$ as $S(p, v) = (p, S_p v)$, for all $(p, v) \in TM$. This is a bundle isomorphism between TM and T^*M since S_p is a vector space isomorphism for each $p \in M$ by construction and the smoothness of the map $p \mapsto S_p v$ is inherited from the smoothness of the metric g.

The bundle isomorphism $\mathbf{S} : TM \to T^*M$ is usually denoted by $\flat : TM \to T^*M$ (flat) and its inverse by $\sharp : T^*M \to TM$ (sharp) and they are both called the *musical isomorphisms* associated to the riemannian metric g.

$\square$

2.4 Symplectic manifolds

Symplectic vector spaces. Let V be a vector space and $\omega : V \times V \to \mathbb{R}$ be a bilinear form in V. We say that $\omega : V \times V \to \mathbb{R}$ is *alternating* or *antisymmetric* if $\omega(u, v) = -\omega(v, u)$, for all $u, v \in V$. It follows immediately from the definition that $\omega(u, u) = 0$, for any $u \in V$.

We say that $\omega : V \times V \to \mathbb{R}$ is *nondegenerate* if the following condition holds: if $\omega(u, v) = 0$, for all $v \in V$, then $u = 0$.

Finally, a bilinear form $\omega : V \times V \to \mathbb{R}$ is said to be *symplectic* if it is both alternating and nondegenerate.

A *symplectic vector space* is vector space V equipped with a symplectic form ω. It follows from the definitions that a symplectic vector space must be an even dimensional vector space.

If (V^{2n}, ω) is a symplectic vector space, then the symplectic form defines a natural volume form Ω on V defined by $\Omega = \omega^n$.

Definition 2.1. Let (V^{2n}, ω) be a symplectic vector space. A *symplectic basis* for V is a basis $\mathbf{e}_1, \dots, \mathbf{e}_n, \mathbf{f}_1, \dots, \mathbf{f}_n$ of V satisfying

$$\omega(\mathbf{e}_i, \mathbf{e}_j) = \omega(\mathbf{f}_i, \mathbf{f}_j) = 0,$$

$$\omega(\mathbf{e}_i, \mathbf{f}_j) = \delta_{ij},$$

where δ_{ij} is the Kronecker's delta.

Relative to a symplectic basis, the matrix of the symplectic form ω is given by

$$\mathbf{W} = \mathbf{J}_0 = \begin{pmatrix} 0 & \mathbf{I}_n \\ -\mathbf{I}_n & 0 \end{pmatrix},$$

where $\mathbf{I}_n$ is $n \times n$ identity matrix. Therefore, for any two vectors $u, v \in V$ we have

$$\omega(u, v) = \mathbf{u}^t \mathbf{W} \mathbf{v}.$$

Definition 2.2. Let (V_1, ω_1) and (V_1, ω_2) be symplectic vector spaces and $A : V_1 \to V_2$ a linear transformation. We say that A is a *symplectic linear transformation* if, for any $u, v \in V_1$, we have $\omega_2(Au, Av) = \omega_1(u, v)$.

Fixing a symplectic basis on both V_1 and V_2 the definition of symplectic linear transformation implies that $\mathbf{A}^t \mathbf{W}_2 \mathbf{A} = \mathbf{W}_1$, where $\mathbf{A}$ is the matrix of linear transformation A with respect to the fixed basis and $\mathbf{W}_1$ and $\mathbf{W}_2$ are the matrix of the 2-forms ω_1 and ω_2, respectively, in their respective basis.

Symplectic manifolds. Let M^{2n} be an even dimesional manifold and ω be a differential 2-form in M. We say that ω is a *symplectic form* in M if ω is closed and $\omega(p) : T_pM \times T_pM \to \mathbb{R}$ is a simplectic form in the tangent space T_pM, for every $p \in M$.

A *symplectic manifold* is a pair (M, ω) where M is an even dimensional manifold and ω is a symplectic form in M. We say the symplectic manifold (M, ω) is an *exact symplectic manifold* when there exists a differential 1-form α in M such that $\omega = d\alpha$.

The existence of a symplectic form on M implies that in each tangent space we can define a volume form that varies smoothly with the point. In particular, it follows that every symplectic manifold is orientable.

The standard example of symplectic manifold is the euclidian space $\mathbb{R}^{2n}$ with the *standard symplectic form*

$$\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j,$$

where $x_1 \dots, x_n, y_1, \dots, y_n$ are global cartesian coordinates in $\mathbb{R}^{2n}$.

Let ω be a symplectic form and $(U; x_1, \dots, x_n, y_1, \dots, y_n)$ be local chart around $p \in M$. A *symplectic local chart* in M is a local chart $(U; x_1, \dots, x_n, y_1, \dots, y_n)$ such that ω is written as

$$\omega = \sum_{j=1}^{n} dx_j \wedge dy_j.$$

It follows that the set

$$\left\{ \left.\frac{\partial}{\partial x_1}\right|_p, \dots, \left.\frac{\partial}{\partial x_n}\right|_p, \left.\frac{\partial}{\partial y_1}\right|_p, \dots, \left.\frac{\partial}{\partial y_n}\right|_p \right\}$$

is symplectic basis for T_pM, for each $p \in U$.

Theorem 2.2 (Darboux's Theorem). *Let (M, ω) be a symplectic manifold. For every $p \in M$ there exists a symplectic local chart $(U, x_1, \dots, x_n, y_1, \dots, y_n)$ in M such that $p \in U$.*

In particular, this theorem implies that there are no local sympletic invariants, i.e., any symplectic manifold has the same local symplectic struture. This is actually one of the major differences between symplectic and riemannian geometry where in a riemannian manifold one has the curvature as a local invariant.

Let (M_1, ω_1) and (M_2, ω_2) be symplectic manifolds and consider a diffeomorphism $f : M_1 \to M_2$. We say that f is a *symplectic diffeomorphism* or *symplectomorphism* if

$$f^*\omega_2 = \omega_1,$$

where $f^*\omega_2$ is the pull-back of ω_2 by f. If there exists a symplectic diffeomorphism between the symplectic manifolds (M_1, ω_1) and (M_2, ω_2) we say that they are *symplectomorphic.*

Therefore, symplectic diffeomorphisms are diffeomorphisms that preserves the symplectic struture of the manifolds. When $M_1 = M_2 = M$ and $\omega_1 = \omega_2 = \omega$, it follows that ω is invariant by f or that ω is a fixed point of the pull-back map $f^* : \Omega_1^\ell(M) \to \Omega_1^\ell(M)$.

Denote by $\mathrm{Symp}^k(M)$ the set of symplectic C^k diffeomorphisms of M. It follows immediately from the definition that a symplectic diffeomorphism is conservative and that $\mathrm{Symp}^k(M)$ is subgroup of the group of conservative diffeomorphisms of M.

We refer to [20] and [21] for more details on the theory symplectic manifolds and its topology.

2.4.1 Natural isomorphism between $\Omega_1^\ell(M)$ and $\mathfrak{X}^\ell(M)$ in symplectic manifolds

We see in Section 2.1.1 that a riemannian metric induces a natural bundle isomorphism between the tangent and cotangent bundles and this construction was only possible because the metric tensor was invertible, which is equivalent to the metric be nondegenerate. Since every symplectic form is nondegenerate we can construct another bundle isomorphism between TM and T^*M using the tensor associated to the symplectic form.

Let $p \in M$ and consider the linear map $\mu_p : T_pM \to T^*M$ defined by

$$\mu_p(v) = \iota(v)\omega_p,$$

where $\iota(v)\omega_p : T_pM \to \mathbb{R}$ is the linear functional given by $\iota(v)\omega_p(u) = \omega_p(v, u)$. Since ω_p is nondegenerate, it follows easily that μ_p is invertible for every $p \in M$ and is, therefore, an isomorphism. We define the bundle isomorphism $\mu : TM \to T^*M$ as

$$\mu(p, v) = (p, \mu_p(v)).$$

This construction yields another bundle isomorphism which does not depend on a riemannian metric in the symplectic manifold. If we have both a symplectic and a riemannian structure on the manifold M we see in Section 2.4.2 that the corresponding bundle isomorphisms gives rise to a third structure in M which is compatible with the metric and the symplectic form.

2.4.2 Almost complex structures

Let V^{2n} be an even dimensional vector space. A *linear complex structure* in V is a linear isomorphism $\mathbf{J} : V \to V$ satisfying $\mathbf{J}^2 = -\mathbf{I}_{2n}$, where $\mathbf{I}_{2n} : V \to V$ is the identity.

When an $2n$-dimensional vector space admits a linear complex structure $\mathbf{J}$ we can extend the scalar multiplication in V to $\mathbb{C}$ by declaring that $iv = \mathbf{J}v$, for any $v \in V$. Therefore, for any complex number $a + bi \in \mathbb{C}$ and $v \in V$, we have

$$(a + bi)v = av + b\mathbf{J}v.$$

We also say in this case that V is a complex vector space of complex dimension n.

Any n-dimensional vector space W gives rise to a complex vector space $\widetilde{W}$ of complex dimension n. Indeed, let $\widetilde{W} = W \oplus iW$, with the usual operations $(a + ib)(u + iv) = (au - bv) + i(av + bu)$ and $(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2)$, for any $a + ib \in \mathbb{C}$

and $u + iv, u_1 + iv_1, u_2 + iv_2 \in \widetilde{W}$.

This is a complex vector space naturally isomorphic to $W \times W$ with the linear complex structure $\mathbf{J}_0 : W \times W \to W \times W$ given by

$$\mathbf{J}_0(u, v) = (-v, u).$$

The isomorphism between $\widetilde{W}$ and $W \times W$ is then given by $u + iv \mapsto (u, 0) + \mathbf{J}_0(v, 0)$.

Observe that by choosing any basis for W, the matrix representation of $\mathbf{J}_0$ in product basis for $W \times W$ is

$$\mathbf{J}_0 = \begin{pmatrix} 0 & -\mathbf{I}_n \\ \mathbf{I}_n & 0 \end{pmatrix},$$

where $\mathbf{I}_n$ is the $n \times n$ identity matrix. This linear complex structure is usually called the *standard complex structure*. When $W = \mathbb{R}^n$, we have $\widetilde{W} = \mathbb{C}^n$ and $W \times W = \mathbb{R}^{2n}$, and we see that the construction above shows how both $\mathbb{R}^{2n}$ and $\mathbb{C}^n$ are isomorphic as complex vector spaces.

Now let us consider a symplectic vector space (V, ω) with an inner product g and a linear complex structure $\mathbf{J}$ in V. We say that ω, g and $\mathbf{J}$ are *mutually compatible* or that $(\omega, g, \mathbf{J})$ is a *compatible triple* if for any vectors $u, v \in V$ the following holds

1. $\omega(u, \mathbf{J}v) = g(u, v)$;
2. $g(\mathbf{J}u, v) = \omega(u, v)$,
3. $\mathbf{J}u = S^{-1}\mu(u)$,

where S and μ are the natural isomorphisms between V and V^* induced by g and ω, respectively.

Let M be differentiable manifold. An *almost complex structure* $\mathbf{J}$ in M is a smooth choice of linear complex strutures $\mathbf{J}(p) : T_pM \to T_pM$ in each tangent space of M.

A standard result about almost complex structures is the following:

Proposition 2.2. *Every symplectic manifold with a riemmanian metric admits a compatible almost complex structure.*

The *compatibility* of the almost complex structure here is meant as that in each tangent space the inner product induced by the riemannian metric, the sympletic form and the linear complex structure coming from the almost complex structure is a compatible triple as defined above. A proof of this result can be found in Chapter 1 of [20].

2.4.3 Hamiltonian vector fields

Let (M^{2n}, ω) be a symplectic manifold and $H : M \to \mathbb{R}$ be C^{r+1} function. A *hamiltonian vector field* with associated *hamiltonian function* H is the unique vector field $X_H \in \mathfrak{X}^r(M)$ satisfying

$$\omega(X_H, v) = DH(v)$$

for every vector field $v \in \mathfrak{X}^r(M)$. It follows immediately from the definition that when $c \in \mathbb{R}$ is a regular value for H, the integral curves of the hamiltonian vector field with starting points in $H^{-1}(c)$ are entirely contained in $H^{-1}(c)$. In particular, this implies that if $H(p) = c$ then $H(\psi^t_{X_H}(p)) = c$, where $\psi^t_{X_H} : M \to M$ is the flow of X_H, i.e., the hamiltonian function is invariant under the flow of X_H.

The diffeomorphisms $\psi^t_{X_H} : M \to M$ of the hamiltonian flow are symplectic diffeomorphisms for each $t \in \mathbb{R}$ (see [20]).

Given a symplectic riemannian manifold with a compatible triple $(\omega, g, \boldsymbol{J})$ and a C^{r+1} hamiltonian function $H : M \to \mathbb{R}$, we can always find symplectic coordinates in M such that X_H is written as

$$X_H = \sum_{j=1}^{n} \left(\frac{\partial H}{\partial y_j} \frac{\partial}{\partial x_j} - \frac{\partial H}{\partial x_j} \frac{\partial}{\partial y_j} \right) = -\boldsymbol{J} \, \mathrm{grad}(H). \tag{2.2}$$

This is the usual representation of the hamiltonian vector we find most of the time in the euclidian spaces. Note that first equation in (2.2) can be written as $X_H = -\mathbf{J}_0 \, \mathrm{grad}_0(H)$, where

$$\mathbf{J}_0 = \begin{pmatrix} 0 & -\mathbf{I}_n \\ \mathbf{I}_n & 0 \end{pmatrix}$$

is the matrix of the symplectic form relative to the symplectic coordinates and

$$\mathrm{grad}_0(H) = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial x_i} \frac{\partial}{\partial x_i} + \frac{\partial H}{\partial y_i} \frac{\partial}{\partial y_i} \right).$$

But $\mathrm{grad}_0(H) = \mathbf{G} \, \mathrm{grad}(H)$, where $\mathbf{G}$ is the metric tensor and $\mathrm{grad}(H)$ is the gradient of H relative to the metric in M. Then

$$X_H = -\mathbf{J}_0 \mathbf{G} \, \mathrm{grad}(H) = (\mathbf{G}^{-1} \mathbf{J}_0)^{-1} \, \mathrm{grad}(H) = \mathbf{J}^{-1} \, \mathrm{grad}(H) = -\mathbf{J} \, \mathrm{grad}(H),$$

since $(\omega, g, \mathbf{J})$ is a compatible triple.

2.5 Contact manifolds

Let M^{2n+1} be an odd dimensional manifold and α be a differential 1-form in M. We say that α is *contact form* in M if $d\alpha|_{\ker(\alpha)}$ is nondegenerate or, equivalently, $\alpha \wedge (d\alpha)^n \neq 0$. In this case, we say that the pair (M, α) is a *contact manifold* and the subbundle $\xi = \ker(\alpha)$ is the *contact structure* in M.

The condition $\alpha \wedge (d\alpha)^n \neq 0$ implies that $\Omega = \alpha \wedge (d\alpha)^n$ is a volume form in M and therefore, contact manifolds defined in this way are all orientable manifolds. In the case of codimension 1 subbundles, which are locally the kernel of 1-forms, the Frobenius integrability condition (see Theorem 2.1) is equivalent to $\alpha \wedge d\alpha = 0$. For this reason, we say that $\alpha \wedge (d\alpha)^n \neq 0$ is a *condition of maximal nonintegrability.*

Theorem 2.3 (Darboux's Theorem)**.** *Let (M, α) be a contact manifold. For every $p \in M$ there exists a chart $(U, x_1, \dots, x_n, y_1, \dots, y_n, z)$ around p in M such that*

$$\alpha = dz - \sum_{j=1}^{n} y_j dx_j.$$

This is the contact version of Darboux's Theorem that we saw for symplectic manifolds and it has the same consequences of its sympletic counterpart: contact manifolds have local contact invariants.

Let (M, α) be a contact manifold. There exist a unique vector field R in M satifying

$$\begin{cases} \alpha(R) = 1, \\ d\alpha(R, Y) = 0, \ \forall Y \in \mathfrak{X}(M). \end{cases}$$

A vector field with these properties is called the *Reeb vector field* of the contact form α. Note that the definition of Reeb vector field is heavily dependant on the contact form and therefore, contact manifolds defined through a contact structure that are not the kernel of a global 1-form does not support such vector fields.

Let (M_1, α_1) and (M_2, α_2) be contact manifolds and $f : M_1 \to M_2$ be a diffeomorphism. We say that f is a *contact diffeomorphism* or *contactomorphism* if there exists a strictly positive C^∞ function $\lambda : M_1 \to \mathbb{R}$ such that $f^*\alpha_2 = \lambda\alpha_1$. When $\lambda \equiv 1$ we say that f is a *strict contact diffeomorphism.*

Contact diffeomorphisms preserves the contact structures of the contact manifolds for the condition $f^*\alpha_2 = \lambda\alpha_1$ implies that $Df(\ker(\alpha_1)) = \ker(\alpha_2)$. When $M_1 = M_2 = M$ and $\alpha_1 = \alpha_2 = \alpha$, this means that f leaves invariant a codimension 1 subbundle of TM, namely

$\ker(\alpha)$. Moreover, if f is a strict contact diffeomorphism then f is conservative, since it preserves the volume form $\Omega = \alpha \wedge (d\alpha)^n$.

For more details on the geometry and topology of contact manifolds see [11].

2.6 Partially hyperbolic diffeomorphisms

Hyperbolic diffeomorphisms. Let M^n be a manifold and $f : M \to M$ be a diffeomorphism of M. We say that a closed subset $\Lambda \subseteq M$ is *f-invariant* if $f(\Lambda) = \Lambda$. We also say that a splitting $T_\Lambda M = E \oplus F$ of the restricted tangent bundle to the points of Λ is *f-invariant* if $Df(E) = E$ and $Df(F) = F$.

The f-invariant subset $\Lambda \subseteq M$ is said to be *hyperbolic* if there exist an f-invariant splitting $T_\Lambda M = E^s \oplus E^u$ and constants $C > 0$, $0 < \lambda < 1$ such that

$$||Df^n v^s|| \leq C\lambda^n ||v^s|| \;\text{ e }\; ||Df^{-n} v^u|| \leq C\lambda^n ||v^u||,$$

for any $v^s \in E^s$ and $v^u \in E^u$. When $\Lambda = M$ we say that f is an *Anosov diffeomorphism.*

The bundles E^s and E^u are called the *stable* and *unstable* subbundles of f, respectively. In general, these bundle are only continuous bundles with their differentiability being guaranteed only under certain bunching conditions over f. Even so we can garantee the integrability of these bundles.

Theorem 2.4 (Stable Manifold Theorem)**.** *Let $f : M \to M$ be a C^r Anosov diffeomorphism. There exist Hölder-continuous foliations $\mathcal{W}^s$ and $\mathcal{W}^u$ in M such that $T_q\mathcal{W}^s(p) = E^s_q$ and $T_{q'}\mathcal{W}^u(p) = E^u_{q'}$, for every $p \in M$, $q \in \mathcal{W}^s(p)$ and $q' \in \mathcal{W}^u(p)$. Moreover, each leaf $\mathcal{W}^\delta(p) \subset M$ is C^r submanifold of M, for $\delta = s, u$.*

We define the *global stable* and *unstable manifolds* of f at $p \in M$ as the sets

$$\mathcal{W}^s(p) = \{q \in M;\; d(f^n(q), f^n(p)) \to 0 \text{ as } n \to \infty\},$$

$$\mathcal{W}^u(p) = \left\{q \in M;\; d(f^{-n}(q), f^{-n}(p)) \to 0 \text{ as } n \to \infty\right\}.$$

Now consider a small $\varepsilon > 0$. Define the *local stable* and *unstable sets* of f at $p \in M$ as the sets

$$\mathcal{W}^s(p,\varepsilon) = \{q \in M;\; d(f^n(q), f^n(p)) \leq \varepsilon,\; \forall n \geq 0\},$$

$$\mathcal{W}^u(p,\varepsilon) = \left\{q \in M;\; d(f^{-n}(q), f^{-n}(p)) \leq \varepsilon,\; \forall n \geq 0\right\}.$$

It is possible to show that there exist $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ the global stable and unstable manifolds of f at p can be written as

$$\mathcal{W}^s(p) = \bigcup_{n\geq 0} f^{-n}(\mathcal{W}^s(p,\varepsilon)) \quad \text{and} \quad \mathcal{W}^u(p) = \bigcup_{n\geq 0} f^{n}(\mathcal{W}^u(p,\varepsilon)).$$

See [6] for details.

Partially hyperbolic diffeomorphisms. Let M be a manifold and $f : M \to M$ be diffeomorphism. We say that f is a *partially hyperbolic diffeomorphism* if there exists an f-invariant splitting $TM = E^s \oplus E^c \oplus E^u$ of the tangent bundle satisfying the following conditions:

(i) $Df(E^\delta) = E^\delta$, para $\delta = s, c$ e u;

(ii) there are constants $C > 0$ e $0 < \lambda < 1$ such that

$$||Df_p^n v_s|| \leq C\lambda^n ||v_s|| \quad \text{e} \quad ||Df_p^{-n} v_u|| \leq C\lambda^n ||v_u||,$$

para todo $p \in M$, $v_s \in E_p^s$ e $v_u \in E_p^u$;

(iii) there are real numbers $0 < \mu_1 < 1 < \mu_2$, usually dependent on the point p, such that

$$||Df_p|_{E_p}|| < \mu_1 < ||Df_p|_{E_p^c}|| < \mu_2 < ||Df_p|_{E_p^u}||,$$

where $||Df_p|_{E_p^\delta}||$ is the norm of the linear transformation $Df_p|_{E_p^\delta} : E_p^\delta \to E_{f(p)}^\delta$. When μ_1 and μ_2 are independent of p we say that f is *absolutely partially hyperbolic.*

In general, to study such diffeomorphisms we try to discover how much properties this class of diffeomorphisms share in common with Anosov diffeomorphisms. Those are mainly, ergodicity, density of periodic orbits, transitivity and entropy.

The Stable Manifold Theorem still holds for partially hyperbolic diffeomorphisms but the central bundle E^c is not in general integrable. A partially hyperbolic diffeomorphism $f : M \to M$ is said to be *dynamically coherent* when $E^s \oplus E^c$ and $E^c \oplus E^u$ are jointly integrable. This implies, in particular, that E^c is also integrable (see [8]).

See [9] and [25] for surveys on the theory of partially hyperbolic dynamics.

Chapter 3

On Integrability of $E^s_F \oplus E^u_F$

3.1 Proof of Theorem B

3.1.1 Particular case

To establish some of the ideas let us prove Theorem B in the particular case where the Anosov diffeomorphism is linear. In this case we can use a variation of Livšic's Theorem due to Walkner in [29] to ensure the existence of solutions provided γ satisfies a certain condition.

Let $f_A : \mathbb{T}^2 \to \mathbb{T}^2$ be a linear torus automorphism induced by the linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$A(x, y) = (2x + y, x + y).$$

Notice that the matrix of this linear transformation in the canonical basis is

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

and whose eigenvalues are $\lambda_s = \dfrac{3-\sqrt{5}}{2}$ and $\lambda_u = \dfrac{3+\sqrt{5}}{2}$. If e^s and e^u are respectives eigenvectors it is easy to see that $||A^n e^s|| = \lambda_s^n ||e_s||$ and $||A^{-n} e^u|| = \lambda_u^{-n}||e_u||$, where $0 < \lambda_s < 1 < \lambda_u$. Then, taking $\lambda = \max\{\lambda_s, \lambda_u^{-1}\}$ we have that $0 < \lambda < 1$, $||A^n e_s|| \leq \lambda^n ||e_s||$ and $||A^{-n} e_u|| \leq \lambda^n ||e_u||$. Moreover, f_A is a conservative Anosov diffeomorphism of $\mathbb{T}^2$, since $\det(Df_A) = \det(A) = 1$.

Finding the eigenspaces of A using standand Linear Algebra we obtain $e_s = 2\dfrac{\partial}{\partial x} - (-1 + \sqrt{5})\dfrac{\partial}{\partial y}$ and $e_u = 2\dfrac{\partial}{\partial x} + (-1 - \sqrt{5})\dfrac{\partial}{\partial y}$ to be the generators of the stable and unstable bundles of f_A, respectively.

Let $F : \mathbb{T}^2 \times \mathbb{S}^1 \to \mathbb{T}^2 \times \mathbb{S}^1$ be a skew-product of the form

$$F(x, y, t) = (f_A(x, y), \phi_{(x,y)}(t)),$$

where $\phi_{(x,y)}(t) = t + \gamma(x, y)$, with $\gamma \in C^r(\mathbb{T}^2, \mathbb{R})$, is a C^r family of rotation of $\mathbb{S}^1$. In this case, F is partially hyperbolic diffeomorphism whose central direction is tangent to the fibers $\{(x, y)\} \times \mathbb{S}^1$, for all $(x, y) \in \mathbb{T}^2$. Since $E^s_F \oplus E^u_F$ is transversal to the fibers it is resonable to expect that it is a perturbation of $TM = E^s_{f_A} \oplus E^u_{f_A}$ in the central direction. Indeed, we show that

$$E^s_F \oplus E^u_F = \text{span}\left\{\frac{\partial}{\partial x} + u_1\frac{\partial}{\partial t}, \frac{\partial}{\partial y} + u_2\frac{\partial}{\partial t}\right\},$$

where x, y are global coordinates in $\mathbb{T}^2$, t is the global coordinate in $\mathbb{S}^1$ and u_1 and u_2 are solutions to the following smooth system of equations

$$\begin{cases} \dfrac{\partial \gamma}{\partial x} = 2u_1 \circ f_A + u_2 \circ f_A - u_1 \\ \\ \dfrac{\partial \gamma}{\partial y} = u_1 \circ f_A + u_2 \circ f_A - u_2 \end{cases} \tag{3.1}$$

Proposition 3.1. *Let $F : \mathbb{T}^2 \times \mathbb{S}^1 \to \mathbb{T}^2 \times \mathbb{S}^1$ be a partially hyperbolic skew-product given by*

$$F(x, y, t) = (f_A(x, y), \phi_{(x,y)}(t)),$$

where $f_A : \mathbb{T}^2 \to \mathbb{T}^2$ is an Anosov linear automorphism induced by a linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$ and $\phi_{(x,y)} : \mathbb{S}^1 \to \mathbb{S}^1$ is a C^r-family of rotations of the circle with parameters in $\mathbb{T}^2$. Then $E^s_F \oplus E^u_F = \text{span}\left\{\frac{\partial}{\partial x} + u_1\frac{\partial}{\partial t}, \frac{\partial}{\partial y} + u_2\frac{\partial}{\partial t}\right\}$, where u_1 and u_2 are solutions of (3.1)*.*

Proof. It suffices to show that if u_1 and u_2 are solutions of (3.1) then the codimension 1 subbundle $\xi = \text{span}\left\{\frac{\partial}{\partial x} + u_1\frac{\partial}{\partial t}, \frac{\partial}{\partial y} + u_2\frac{\partial}{\partial t}\right\}$ is F-invariant. Indeed, suppose $\frac{\partial}{\partial t} \in \xi$. Then, for each $p = (x, y) \in \mathbb{T}^2$, there must be constants $\alpha_1(p), \alpha_2(p) \in \mathbb{R}$, such that

$$\left.\frac{\partial}{\partial t}\right|_p = \alpha_1(p)\left(\left.\frac{\partial}{\partial x}\right|_p + u_1(p)\left.\frac{\partial}{\partial t}\right|_p\right) + \alpha_2(p)\left(\left.\frac{\partial}{\partial y}\right|_p + u_2(p)\left.\frac{\partial}{\partial t}\right|_p\right) =$$

$$= \alpha_1(p)\left.\frac{\partial}{\partial x}\right|_p + \alpha_2(p)\left.\frac{\partial}{\partial y}\right|_p + [\alpha_1(p)u_1(p) + \alpha_2(p)u_2(p)]\left.\frac{\partial}{\partial t}\right|_p.$$

The vector fields $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are linearly independent at every point and so it follows that

$\alpha_1(p) = \alpha_2(p) = 0$, for all $p \in \mathbb{T}^2$ and this implies that $\frac{\partial}{\partial t} = 0$, which is absurd, and since the only DF-invariant codimension 1 subbundles of $T(\mathbb{T}^2 \times \mathbb{S}^1)$ are either $E^s \oplus E^u$, $E^s \oplus E^c$ or $E^c \oplus E^u$, it will follow from the F-invariance of ξ that $\xi = E^s \oplus E^u$.

Now, suppose u_1 and u_2 satisfies (3.1) and write $X = \frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial t}$ and $Y = \frac{\partial}{\partial y} + u_2 \frac{\partial}{\partial t}$ for simplicity. Then, for each $p \in \mathbb{T}^2$,

$$DF_p X = 2 \left.\frac{\partial}{\partial x}\right|_{f_A(p)} + \left.\frac{\partial}{\partial y}\right|_{f_A(p)} + \left(\frac{\partial \gamma}{\partial x} + u_1\right) \left.\frac{\partial}{\partial t}\right|_{f_A(p)}.$$

From (3.1) we have $u_1 = 2u_1 \circ f_A + u_2 \circ f_A - \frac{\partial \gamma}{\partial x}$. Therefore, replacing u_1 in the above equation we obtain

$$DF_p X = 2 \left.\frac{\partial}{\partial x}\right|_{f_A(p)} + \left.\frac{\partial}{\partial y}\right|_{f_A(p)} + (2u_1 \circ f_A + u_2 \circ f_A) \left.\frac{\partial}{\partial t}\right|_{f_A(p)} =$$

$$= 2\left(\left.\frac{\partial}{\partial x}\right|_{f_A(p)} + (u_1 \circ f_A) \left.\frac{\partial}{\partial t}\right|_{f_A(p)} \right) + \left(\left.\frac{\partial}{\partial y}\right|_{f_A(p)} + (u_2 \circ f_A) \left.\frac{\partial}{\partial t}\right|_{f_A(p)} \right) =$$

$$= 2X(f_A(p)) + Y(f_A(p)).$$

Thus, $DF(X) \in \xi$. A similar argument shows that we also have $DF(Y) \in \xi$ and consequently ξ is DF-invariant. □

Recall from Riemannian Geometry that the *gradient* of a function γ in a riemmanian manifold M is a vector field $\mathrm{grad}(\gamma)$ intrisincally defined by the relation

$$d\gamma_p(v) = g_p(\mathrm{grad}(\gamma)_p, v), \tag{3.2}$$

for any $p \in M$ and $v \in T_pM$. Choosing local coordinates $x_1, \ldots, x_n$ in M, write

$$g = g_{ij} dx^i \otimes dx^j \quad \text{and} \quad \mathrm{grad}(\gamma) = h^i \frac{\partial}{\partial x_i}$$

using the summation convention. By (3.2) we have

$$\frac{\partial \gamma}{\partial x_k} = d\gamma\left(\frac{\partial}{\partial x_k}\right) = g_{ij} dx^i \oplus dx^j \left(h^i \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_k}\right) = g_{ik} h^i.$$

Let $\mathrm{grad}_0(\gamma) = \sum_{i=1}^{n} \frac{\partial \gamma}{\partial x_i} \frac{\partial}{\partial x_i}$. Then, by the previous equation it follows that

$$\mathrm{grad}_0(\gamma) = \mathbf{G}\, \mathrm{grad}(\gamma),$$

where $\mathbf{G} = (g_{ij})$ is the metric tensor.

If we write $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, then system (3.1) can be rewritten as

$$\operatorname{grad}_0(\gamma) = \mathbf{G}\operatorname{grad}(\gamma) = \operatorname{Jac}(f_A)^t \mathbf{u} \circ f_A - \mathbf{u}, \tag{3.3}$$

where $\mathbf{G}$ is the metric tensor of $\mathbb{T}^2$, $\operatorname{grad}(\gamma)$ is the gradient vector field with respect to the riemannian metric in $\mathbb{T}^2$ and $\operatorname{Jac}(f_A)$ is the jacobian matrix of f_A.

The equation (3.3) is an example of *twisted cohomological equation* on a manifold M with values in $\mathbb{R}^k$. In general, those equations are of the form

$$\phi = \mathbf{A}\mathbf{u} \circ f - \mathbf{u}, \tag{3.4}$$

where $\phi, \mathbf{u} \in C^\ell(M; \mathbb{R}^k)$, $f \in \operatorname{Diff}^\ell(M)$, and $A : M \to \operatorname{GL}(\mathbb{R}^k)$ is C^ℓ. When $\mathbf{A} \equiv \mathbf{Id}_k$ the equation (3.4) is called *untwisted.*

Walkden proved in [29], in the more general case of a twisted cohomological equation with values on Lie group G but with fixed automorphism $\mathbf{A} : G \to G$, that the cohomological equation (3.4) admits a Hölder-continuous solution $\mathbf{u}$ whenever ϕ is Hölder-continuous and satisfy a certain *periodic orbit obstruction* condition stated in his paper. He also improves his result in the same work establishing a cap for the regularity of solutions of the cohomological equation where, whenever f and ϕ are C^r, the solutions $\mathbf{u}$ obtained are at most $C^{r-1+\varepsilon}$, for $0 < \varepsilon < 1$. So in the case of a linear Anosov automorphism we use Walkner's result to guarantee existence of solutions of (3.3) having higher regularity.

A case where we can establish an explicit solution for equation (3.3) is when γ is a *coboundary* relative to f_A, i.e., $\gamma = \mu \circ f_A - \mu$, for some function $\mu : \mathbb{T}^2 \to \mathbb{R}$. From the Livšic's Theorem if γ is C^k we obtain a transfer function μ that is at least $C^{r-1+\text{Hölder}}$. Then, supposing $r \geq 2$, we obtain μ differentiable and in this case we have

$$d\gamma = d(\mu \circ f_A) - d\mu = f^* d\mu - d\mu.$$

Taking local coordinates x, y, t in $\mathbb{T}^2 \times \mathbb{S}^1$ the previous differential equation is written as

$$\begin{cases} \dfrac{\partial \gamma}{\partial x} = 2\dfrac{\partial \mu}{\partial x} \circ f_A + \dfrac{\partial \mu}{\partial y} \circ f_A - \dfrac{\partial \mu}{\partial x} \\[2ex] \dfrac{\partial \gamma}{\partial y} = \dfrac{\partial \mu}{\partial x} \circ f_A + \dfrac{\partial \mu}{\partial y} \circ f_A - \dfrac{\partial \mu}{\partial y} \end{cases} \tag{3.5}$$

Therefore, a solution of (3.3) is $u_1 = \dfrac{\partial \mu}{\partial x}$ and $u_2 = \dfrac{\partial \mu}{\partial y}$ and locally we have

$$E_F^s \oplus E_F^u = \operatorname{span}\left\{ \frac{\partial}{\partial x} + \frac{\partial \mu}{\partial x}\frac{\partial}{\partial t}, \frac{\partial}{\partial y} + \frac{\partial \mu}{\partial y}\frac{\partial}{\partial t} \right\},$$

which is the kernel of the differential 1-form $\alpha = dt - \dfrac{\partial \mu}{\partial x}dx - \dfrac{\partial \mu}{\partial y}dy = dt - d\mu$ since

$$\alpha\left(\frac{\partial}{\partial x} + \frac{\partial \mu}{\partial x}\frac{\partial}{\partial t}\right) = \alpha\left(\frac{\partial}{\partial y} + \frac{\partial \mu}{\partial y}\frac{\partial}{\partial t}\right) = 0.$$

In general, α is only a Hölder-continuous 1-form. However, taking r *sufficiently large*, we suppose that α is differentiable. In this case, $d\alpha = 0$ and therefore, $\alpha \wedge d\alpha = 0$, that is, $E_F^s \oplus E_F^u$ is Frobenius integrable. We will see in Theorem 3.2 that, even in the continuous case, we have the integrability of $E_F^s \oplus E_F^u$ in the sense that there exists a continuous foliation tangent to it.

3.1.2 General case

Consider a skew-product $F : M \times \mathbb{S}^1 \to M \times \mathbb{S}^1$ of the form

$$F(p,t) = (f(p), \phi_t(p)), \tag{3.6}$$

where M is an n-dimensional manifold and $f : M \to M$ is an Anosov diffeomorphism.

Let α be a 1-form in $M \times \mathbb{S}^1$, transversal to $\mathbb{S}^1$ in the sense that the tangent direction $\dfrac{\partial}{\partial t}$ to $\mathbb{S}^1$ is transversal to $\ker(\alpha)$, for all $(p,t) \in M \times \mathbb{S}^1$.

Lemma 3.1. *Let $\alpha = v_t dt + \beta_t$ be a 1-form in $M \times \mathbb{S}^1$, where $\beta_1 \in \Omega_1^\ell(M)$, and suppose that* $\ker(\alpha)$ *is F-invariant and transversal to $\mathbb{S}^1$. Then,*

$$F^*\left(\frac{\beta_t}{v_t}\right) - \phi_t' \frac{\beta_t}{v_t} = -d\phi_t, \tag{3.7}$$

where ϕ_t' is the derivative of ϕ_t with respect to t and $F^* : \Omega_1^\ell(M \times \mathbb{S}^1) \to \Omega_1^\ell(M \times \mathbb{S}^1)$ is the pull-back transformation induced by F. Conversely, if (3.7) holds, with $v_t \neq 0$ at every point, then $\ker(\alpha)$ *DF-invariant.*

Proof. Consider the splitting $T(M \times \mathbb{S}^1) \cong TM \oplus T\mathbb{S}^1$ induced by the global trivialization of the trivial bundle $M \times \mathbb{S}^1$. This splitting induces a module isomorphism between the $C^r(M \times \mathbb{S}^1, \mathbb{R})$-modules $\Omega_1^r(M \times \mathbb{S}^1)$ and $\Omega_1^r(M) \oplus \Omega_1^r(\mathbb{S}^1)$, which is realized by taking the natural projections $\pi_1 : TM \oplus T\mathbb{S}^1 \to TM$ and $\pi_2 : TM \oplus T\mathbb{S}^1 \to T\mathbb{S}^1$ and noting that the induced pullback maps $\pi_1^* : \Omega_1^r(M) \to \Omega_1^r(M \times \mathbb{S}^1)$ and $\pi_2^* : \Omega_1^r(\mathbb{S}^1) \to \Omega_1^r(M \times \mathbb{S}^1)$ satisfies $\Omega_1^r(M \times \mathbb{S}^1) = \pi_1^*\Omega_1^r(M) \oplus \pi_2^*\Omega_1^r(\mathbb{S}^1)$. Hence any 1-form in $\Omega_1^r(M)$ or $\Omega_1^r(\mathbb{S}^1)$ can be considered as a 1-form in $\Omega_1^r(M \times \mathbb{S}^1)$ via this identification.

Since $\ker(\alpha)$ is DF-invariant there exists $\lambda : M \times \mathbb{S}^1 \to \mathbb{R}$ such that $F^*\alpha = \lambda\alpha$. Then

$$F^*\alpha = \lambda\alpha \Rightarrow F^*(v_t dt + \beta_t) = \lambda(v_t dt + \beta_t) \Rightarrow$$

$$F^*(v_t dt) + F^*\beta_t = \lambda v_t dt + \lambda\beta_t \tag{3.8}$$

Note that

$$F^*(v_t dt) = (v_{\phi_t} \circ f)d\phi_t = (v_{\phi_t} \circ f)d\phi_t + (v_{\phi_t} \circ f)\phi_t' dt,$$

where $d\phi_t$ is the partial derivative of ϕ_t in the direction of TM with respect to the splitting $T(M \times \mathbb{S}^1) \cong TM \oplus T\mathbb{S}^1$. For simplicity, we will write v_{ϕ_t} instead of $v_{\phi_t} \circ f$ so that $F^*(v_t dt) = v_{\phi_t} d\phi_t + v_{\phi_t}\phi_t' dt$. Then, from this and (3.8) we have

$$v_{\phi_t} d\phi_t + v_{\phi_t}\phi_t' dt + F^*\beta_t = \lambda v_t dt + \lambda\beta_t \Rightarrow$$

$$v_{\phi_t}\phi_t' dt - \lambda v_t dt = F^*\beta_t - \lambda\beta_t + v_{\phi_t} d\phi_t \Rightarrow$$

$$(v_{\phi_t}\phi_t' - \lambda v_t)dt = F^*\beta_t - \lambda\beta_t + v_{\phi_t} d\phi_t.$$

Note that the left side of this equation is a multiple of dt while that right side is a linear combination of dx and dy, since $\beta_t \in \Omega_1^\ell(M)$. Now, since $\{dx, dy, dt\}$ is a local basis for $\Omega_1^\ell(M \times \mathbb{S}^1)$, it follows that

$$\phi_t' v_{\phi_t} - \lambda v_t = 0 \tag{3.9}$$

and

$$F^*\beta_t - \lambda\beta_t + v_{\phi_t} d\phi_t = 0 \tag{3.10}$$

by linear independency of dx, dy and dt. The transversality condition of $\ker(\alpha)$ implies that either $v_t > 0$ or $v_t < 0$ at every point, otherwise there would be point $(p, t) \in M \times \mathbb{S}^1$ such that $T_p\mathbb{S} \subset \ker(\alpha)_{(p,t)}$. Thus, $\dfrac{\beta_t}{v_t}$ is defined in all of $M \times \mathbb{S}^1$ and by (3.10)

$$F^*\beta_t - \lambda\beta = -v_{\phi_t} d\phi_t \Rightarrow -d\phi_t = \frac{F^*\beta_t}{v_{\phi_t}} - \frac{\lambda\beta_t}{v_{\phi_t}}$$

Recall that $F^* v_t = v_{\phi_t}$ and obviously $F^*(v_t)^{-1} = v_{\phi_t}^{-1}$. Also, by (3.9) we have $\dfrac{\lambda}{v_{\phi_t}} = \dfrac{\phi_t'}{v_t}$. Then, by the previous equation we obtain

$$-d\phi_t = F^*\left(\frac{\beta_t}{v_t}\right) - \phi_t'\frac{\beta_t}{v_t}.$$

Conversely, suppose (3.7) holds. Then, $F^*\left(\frac{\beta_t}{v_t}\right) = -d\phi_t + \phi_t'\frac{\beta_t}{v_t}$ and

$$F^*\left(\frac{\alpha}{v_t}\right) = F^*(dt) + F^*\left(\frac{\beta_t}{v_t}\right) = d\phi_t + \phi_t'dt + \phi_t'\frac{\beta_t}{v_t} - d\phi_t = \phi_t'dt + \phi_t'\frac{\beta_t}{v_t} \Rightarrow$$

$$F^*(v_t^{-1})F^*\alpha = \frac{\phi_t'}{v_t}\left(v_t dt + \beta_t\right) \Rightarrow$$

$$(v_{\phi_t}\circ f)^{-1}F^*\alpha = \frac{\phi_t'}{v_t}\left(v_t dt + \beta_t\right) = \lambda\alpha \Rightarrow$$

$$F^*\alpha = \frac{\phi_t' v_{\phi_t}\circ f}{v_t}\left(v_t dt + \beta_t\right) = \lambda\alpha$$

where $\lambda = v_t^{-1}\cdot\phi_t'\cdot(v_{\phi_t}\circ f)$. Therefore, $\ker(\alpha)$ is F-invariant.

□

The previous lemma assumes a differential 1-form with a specific form, which at first seems to be very restrictive. The next lemma however, shows that this is not really the case.

Lemma 3.2. *Let M^n be a compact riemannian manifold and $S\subset M$ be a codimension 1 closed submanifold of M. Then, every C^r differential 1-form α can be written as $\alpha = v_t dt + \beta_t$ in a neighborhood V_ε of S, where $v_t\in C^r(S;\mathbb{R})$ and $\beta_t\in\Omega_1^r(S)$ are families of C^r maps and families of C^r differential 1-forms of S, respectively.*

Proof. Consider a tubular neighborhood V of S in M. Since M is compact we can find $\varepsilon > 0$ such that $S\times(-\varepsilon,\varepsilon)$ is embedded into V, where we identify S with $S\times\{0\}$.

Let $\varphi : S\times(-\varepsilon,\varepsilon)\to V$ be this embedding and $\widetilde{V}_\varepsilon = \varphi(S\times(-\varepsilon,\varepsilon))$. Consider the projection $\pi:\widetilde{V}_\varepsilon\to S$ induced by the canonical projection $\pi_0 : S\times(-\varepsilon,\varepsilon)\to S$ defined by $\pi = \pi_0\circ\varphi^{-1}$. For $\varepsilon > 0$ sufficiently small we can assume that $\widetilde{V}_\varepsilon$ belongs to a cover of S by slice charts of M, i.e.,

$$S\subset\bigcup_{i\in I} W_i,$$

where $(W_i, x_1,\dots,x_{n-1}, t)$ are local coordinates with the property that $S\cap W_i = \{p\in W_i;\ t(p) = 0\}$. Note that we can always find such an ε by passing to a finite subcover of S and take $\varepsilon > 0$ to be the smallest $\varepsilon_i > 0$ such that $\widetilde{V}_{\varepsilon_i} = \varphi((S\cap W_i)\times(-\varepsilon_i,\varepsilon_i))\subset W_i$. Then, if α is a 1-form in M its restriction to the local charts $\widetilde{V}_{\varepsilon_i}$ is written as

$$i^*_{\widetilde{V}_\varepsilon}\alpha = v_t dt + \sum_{j=1}^{n-1} u_j dx_j = v_t dt + \beta_t,$$

where $i_{\widetilde{V}_\varepsilon} : \widetilde{V}_\varepsilon \to M$ is the inclusion, $v_t \in C^k(S, \mathbb{R})$ and $\beta_t = \sum_{j=1}^{n-1} u_j(\ \cdot\ , t)dx_j \in \Omega_1^k(S)$ is the local representation of β_t with coordinate functions $u_j(\ \cdot\ , t) \in C^k(\widetilde{V}_\varepsilon \cap S; \mathbb{R})$.

□

Proof of Theorem B. Let ξ be an F-invariant codimension 1 subbundle transversal to $T\mathbb{S}^1$ and consider a local defining 1-form α for ξ. Let $U \subset M \times \mathbb{S}^1$ be such that $\xi|_U = \ker(i_U^* \alpha)$, for some 1-form α, where $i_U : U \to M \times \mathbb{S}^1$ is the inclusion.

By Lemma 3.2, there exist a sufficiently small open neighborhood V_{ε,t_0} of $M \times \{t_0\}$ such that

$$i^*_{U \cap V_{\varepsilon,t_0}} \alpha = v_t dt + \beta_t,$$

where $v_t : U \cap V_{\varepsilon,t_0} \to \mathbb{R}$ and $\beta_t \in \Omega_1^\ell(M)$ are C^ℓ families with parameters in $\mathbb{S}^1$, with $\ell \leq r$.

Since ξ is transversal to $T\mathbb{S}^1$ we have either $v_t > 0$ or $v_t < 0$. Recall that

$$\phi_t = t + \gamma \ (\text{mod. } 1),$$

for some function $\gamma \in C^k(M; \mathbb{R})$. From the hypothesis of F-invariance of ξ and by Lemma 3.1 we have

$$F^* \left(\frac{\beta_t}{v_t} \right) - \frac{\beta_t}{v_t} = -d\gamma, \tag{3.11}$$

with v_t satisfying $\lambda v_t = v_{t+\gamma} \circ f$, for some C^∞ function $\lambda : M \times \mathbb{S}^1 \to \mathbb{R}^+$ due to (3.9) and the fact that $\phi_t' = 1$. In particular, $d\gamma$ is coboundary relative to F.

It follows that the 1-form $\eta = dt + \widetilde{\beta}_t$ is F-invariant, where $\widetilde{\beta}_t = v_t^{-1} \beta_t$, since

$$F^* \eta = d(t + \gamma) + F^* \widetilde{\beta}_t = dt + d\gamma + \widetilde{\beta}_t - d\gamma = dt + \widetilde{\beta}_t = \eta,$$

as consequence of (3.11).

Now suppose that γ is coboundary relative to f with transfer function μ. Since γ is C^r we obtain a transfer function μ that is at least C^1 by the classical Livšic's Theorem (see [13]). Then

$$d\gamma = f^*(d\mu) - d\mu = F^*(d\widetilde{\mu}) - d\widetilde{\mu},$$

where $\widetilde{\mu} : M \times \mathbb{S}^1 \to \mathbb{S}^1$ is defined as $\widetilde{\mu}(x, t) = \mu(x)$.

Therefore, $\widetilde{\alpha} = dt - d\widetilde{\mu}$ is also an F-invariant 1-form since

$$F^* \widetilde{\alpha} = d(t + \gamma) - F^*(d\widetilde{\mu}) = dt + F^*(d\widetilde{\mu}) - d\widetilde{\mu} - F^*(d\widetilde{\mu}) = dt - d\widetilde{\mu} = \widetilde{\alpha}.$$

We also note that $\widetilde{\alpha}$ is also transversal to $\mathbb{S}^1$ since $\widetilde{\alpha}\left(\dfrac{\partial}{\partial t}\right) = 1$ and consequently $\dfrac{\partial}{\partial t} \notin \ker(\widetilde{\alpha})$ Now F being partially hyperbolic with one dimensional central direction there is only one F-invariant codimension 1 subbunble transversal to $E^c = T\mathbb{S}^1$ and hence $\xi|_{\widetilde{U}} = \ker(\eta|_{\widetilde{U}}) = \ker(i^*_{\widetilde{U}}\widetilde{\alpha})$, where $\widetilde{U} \subset U \cap V_{\varepsilon,t_0}$ is an open set.

Define $\mu_{t_0} : M \to \mathbb{S}^1$ as

$$\mu_{t_0}(p) = t_0 + \mu(p),$$

for some $t_0 \in \mathbb{S}^1$, and $\widetilde{\mu}_{t_0} : M \times \mathbb{S}^1 \to \mathbb{S}^1$ as $\widetilde{\mu}_{t_0}(p,t) = \mu_{t_0}(p)$. Let $S_{\mu_t} = \mathrm{Graph}(\mu_t) \subset M \times \mathbb{S}^1$ be the graph of μ_t and $i_{S_{\mu_t}} : S_{\mu_t} \to M \times \mathbb{S}^1$ e the inclusion. Then, for fixed $\widetilde{t}$, we have

$$i^*_{S_{\mu_{\widetilde{t}}} \cap \widetilde{U}}\widetilde{\alpha} = d\mu_{\widetilde{t}} - d\widetilde{\mu} = d\widetilde{\mu} - d\widetilde{\mu} = 0.$$

Hence, $T_{(p,t)}S_{\mu_{\widetilde{t}}} = \ker(i^*_{\widetilde{U}}\widetilde{\alpha})_{(p,t)}$, for all $(p,t) \in S_{\mu_{\widetilde{t}}} \cap \widetilde{U}$. That is, $\xi|_{\widetilde{U}} = \ker(i^*_{\widetilde{U}}\widetilde{\alpha})$ is integrable and the family $\mathcal{F} = (S_{\mu_t})_{t\in\mathbb{S}^1}$ is a local F-invariant codimension 1 foliation.

□

Since each point $(p,t) \in M \times \mathbb{S}^1$ lies in a submanifold $M_t = M \times \{t\}$ transversal to the fibers, it follows that the previous construction can be done in any open set of $M \times \mathbb{S}^1$. Let

$$M \times \mathbb{S}^1 \subseteq \bigcup_{i\in I} V_i$$

be an open cover of $M \times \mathbb{S}^1$ formed by open sets such that $\xi|_{V_i}$ are locally integrable and α_i be the respective F-invariant 1-forms for each open set V_i. Then,

$$\xi = \bigcup_{j\in I} \ker(\alpha_j)$$

is a DF-invariant subbundle of $M\times\mathbb{S}^1$ transversal to $T\mathbb{S}^1$ such that $\xi_{(p,\mu(p))} = \ker(\alpha_j)_{(p,\mu_t(p))} = T_{(p,\mu_t(p))}S_{\mu_t}$, with $(p,\mu_t(p)) \in V_j$. This implies that S_{μ_t} is everywhere tangent to ξ consequently the foliation $\mathcal{F} = (S_{\mu_t})_{t\in\mathbb{S}^1}$ is tangent to ξ. This proves the following corollary of Theorem B.

Corollary 3.1. *If $F : M \times \mathbb{S}^1 \to M \times \mathbb{S}^1$ is a partially hyperbolic skew-product of the form $F(p,t) = (f(p), t + \gamma(p))$, where f is Anosov, γ is a coboundary and ξ is an DF-invariant codimension 1 subbundle transversal to the fibers then $\mathcal{F} = (\mathrm{Graph}(t + \gamma(p)))_{t\in\mathbb{S}^1}$ is a foliation tangent to ξ.*

Lemma 3.3. *Let M^n be an n-dimensional manifold and $F : M \times \mathbb{S}^1 \to M \times \mathbb{S}^1$ be a skew-product given by $F(p,t) = (f(p), t + \gamma(p))$, with f being an Anosov diffeomorphism and $\gamma \in C^r(M; \mathbb{S}^1)$. If $\xi \subset T(M \times \mathbb{S}^1)$ is an DF-invariant codimension 1 subbundle transversal to $T\mathbb{S}^1$ then $\xi = E^s_F \oplus E^u_F$.*

Proof. Since F is partially hyperbolic it admits the invariant subbundles E^s, E^c, E^u and their Whitney sums as well. The only F-invariant codimension 1 subbundle transversal to E^c is $E^s \oplus E^u$ and therefore if $\xi \subset T(M \times \mathbb{S}^1)$ is an DF-invariant codimension 1 subbundle transversal to $T\mathbb{S}^1 = E^c$ it follows that $\xi = E^s \oplus E^u$.

□

From Theorems B and Lemma 3.3 we obtain the following conclusion

Corollary 3.2. *Let M^n be compact n-dimensional manifold and $F : M \times \mathbb{S}^1 \to M \times \mathbb{S}^1$ be a skew-product given by $F(x,t) = (f(x), t + \gamma(x))$, where f is an Anosov diffeomorphism and $\gamma \in C^r(M;\mathbb{R})$. If γ is a coboundary relative to f with transfer map μ then $E^s_F \oplus E^u_F$ is integrable and tangent to* Graph(μ).

Corollary 3.2 implies in particular that the foliation tangent to $E^s_F \oplus E^u_F$ has at least a compact leaf. Indeed, if γ is a coboundary with transfer map μ then ξ is tangent to the graph of μ which is a compact subset of $M \times \mathbb{S}^1$ diffeomorphic to M.

3.2 Proof of Theorem C

Characteristic Foliations. Let (M^{2n+1}, α) be a $(2n+1)$-dimensional contact manifold, $S \subset M$ be codimension 1 submanifold of M and $\xi \subset TM$ be codimension 1 subbundle. Define the *characteristic foliation* of ξ in S as the foliation of S obtained by a representative of the equivalence class of vector fields that generates the one-dimensional bundle $(\xi \cap TS)^{\perp} \subset TS$, where the $\perp$ means $(d\alpha|_{TS})$-orthogonality. Given a 2-form ω in a manifold N, we say that two vector fields X and Y in N are *ω-orthogonal* if $\omega(X,Y) = 0$. In dimension 3 it happens that $\xi \cap TS = (\xi \cap TS)^{\perp}$, but this is not the case in higher dimension.

In general, the characteristic foliation has *singular points*, i.e., points where the representative vector field of $\xi \cap TS$ vanishes. These points are precisely the points where ξ is tangent to S.

Lemma 3.4. *Let $S \subset M$ be a hypersurface and $U \subset M^3$ be an open set such that $U \cap S \neq \emptyset$ and $\xi|_U = \ker(\alpha)$, for some 1-form α. Then, a vector field $X_\xi \in TS$ is a generator of the characteristic foliation if and only if $\iota_{X_\xi}\omega = i_S^*\alpha$ for some area form ω in S.*

Proof. Let $X_\xi \in TS$ be a vector field satisfying $\iota_{X_\xi}\omega = i_S^*\alpha$ for some area form ω in S. It suffices to show that $X_\xi \in \ker(\alpha)$ since this implies that $X_\xi \in TS \cap \ker(\alpha)$ and hence is a generator of the characteristic foliation. But this is true since

$$i_S^*\alpha(X_\xi) = \iota_{X_\xi}\omega(X_\xi) = \omega(X_\xi, X_\xi) = 0.$$

Conversely, let X_ξ be a vector field generating the characteristic foliation and Ω be an arbitrary area form in S. Then, the vector field Y satifying $\iota_Y\Omega = i_S^*\alpha$ is also a generator of the characteristic foliation and thus, there exists a positive C^∞ function $\lambda : M \to \mathbb{R}^+$ such that $Y = \lambda X_\xi$.

Let $\omega = \lambda\Omega$. Since $\lambda > 0$, the 2-form ω is also an area form S and

$$\iota_{X_\xi}\omega = \lambda\iota_{X_\xi}\Omega = \iota_{\lambda X_\xi}\Omega = \iota_Y\Omega = i_S^*\alpha.$$

$\square$

Let (M^2, ω) be a symplectic manifold, $F : M \times \mathbb{S}^1 \to M \times \mathbb{S}^1$ be a skew-product of the form (3.6), $M_{t_0} = M \times \{t_0\}$ and $\xi \subset TM$ a codimension 1 subbundle and consider the slice chart formed by local symplectic coordinates x, y in M and the global coordinate t in $\mathbb{S}^1$. Then, by Lemma 3.2, there exists an open set $U \subset M \times \mathbb{S}^1$ such that $U \cap M_t \neq 0$ e $\xi|_U = \ker\alpha$, with $\alpha = v_t dt + \beta_t$ e $\beta_t \in \Omega_1^k(M_t) = \Omega_1^k(M)$.

Proposition 3.2. *If $X_\xi = X_\xi^1\dfrac{\partial}{\partial x} + X_\xi^2\dfrac{\partial}{\partial y}$ is the characteristic vector field of ξ in local symplectic coordinates of M_t associated to the symplectic form ω, then $\beta_t = -X_\xi^2 dx + X_\xi^1 dy$.*

Proof. We first observe that M_t is diffeomorphic to M through $\pi_t : M_t \to M$ obtained by restricting the projection along the fibers $\pi : M \times \mathbb{S}^1 \to M$ to M_t. Indeed, the projection along the fibers of the trivial bundle $M \times \mathbb{S}^1$ is the natural projection on the first component and the graph of any smooth map is diffeomorphic to its domain through the natural projection. Since M is a symplectic manifold with symplectic form ω we pull back this symplectic form to M_t with π_t obtaining a symplectic form $\omega_t = \pi_t^*\omega$ on each M_t. Choosing symplectic local coordinates x, y on M and the global coordinate t on $\mathbb{S}^1$ we have $\omega = dx \wedge dy$ and $\pi_t(x, y, t) = (x, y)$. Then, we trivially have $\omega_t = \pi_t^*\omega = \omega$.

The symplectic form $\omega = dx \wedge dy$ is an area form in M_t and by Lemma 3.4, the characteristic vector field of ξ in M_t associated to ω satisfies $\iota_{X_\xi}\omega = i_{M_t}^*\alpha$. But, fixing $t_0 \in \mathbb{S}^1$, we have

$$i_{M_{t_0}}^*\alpha = d(t_0) + \beta_{t_0} = \beta_{t_0},$$

for any $t_0 \in \mathbb{S}^1$. Then,

$$(\beta_t)_p(v) = (i_{M_t}^*\alpha)_p(v) = (\iota_{X_\xi}\omega)_p(v) = \det\begin{pmatrix} X_\xi^1 & X_\xi^2 \\ v_1 & v_2 \end{pmatrix} = -X_\xi^2 v_1 + X_\xi^1 v_2,$$

for any $p \in M$ and $v = v_1 \left.\dfrac{\partial}{\partial x}\right|_p + v_2 \left.\dfrac{\partial}{\partial y}\right|_p \in T_pM$. Therefore $\beta_t = -X_\xi^2 dx + X_\xi^1 dy$.

□

Thus, in dimension 3 we have that every subbundle $\xi \subset T(M \times \mathbb{S}^1)$ transversal to $\mathbb{S}^1$ can be locally written as the kernel of a 1-form given by

$$\alpha = v_t dt - X_2 dx + X_1 dy, \tag{3.12}$$

where X_1 and X_2 are the components of the characteristic vector field of ξ in M_t. If $v_t \neq 0$ at every point, then we can rescale the characteristic vector field X_ξ by multiplying it by v_t, thus obtaining $\alpha = v_t \widetilde{\alpha}$, where

$$\widetilde{\alpha} = dt - X_2 dx + X_1 dy \tag{3.13}$$

and $\xi = \ker(\alpha) = \ker(\widetilde{\alpha})$. In this case, we can work (3.13) instead of (3.12).

Proof of Theorem C. Let $S \subset M \times \mathbb{S}^1$ be a surface transversal to $\mathbb{S}^1$. By the Implicit Function Theorem there exist an open set $U \subset M$ such that $S \cap \pi^{-1}(U)$ is the graph of a map $h : U \to \mathbb{S}^1$ and since $\dim(S) = \dim(M)$, it follows that the restriction of the bundle projection $\pi : M \times \mathbb{S}^1 \to M$ to S is a local diffeomorphism.

Shrinking U if necessary, let (U, x, y) be a local symplectic chart in M and define local symplectic coordinates and area form in $S \cap \pi^{-1}(U)$ as the ones induced by the pullback of the coordinates x, y and area form $\omega = dx \wedge dy$ by the bundle projection π. We shall use the same notation for both coordinates and area forms in S and M. By a simple calculation, we see that $\pi^*\omega = \omega$.

Since γ is a coboundary with transfer map μ, we know by a part of the proof of Theorem A that ξ is locally the kernel of the 1-form $\alpha = dt - d\mu$. From the defining equation $\iota_{X_\xi}\omega = i_S^*\alpha$ of a vector field representing the characteristic foliation of ξ in S, we have

$$-X_2 dx + X_1 dy = \iota_{X_\xi} = i_S^*\alpha = \left(\frac{\partial h}{\partial x} - \frac{\partial \mu}{\partial x}\right) dx + \left(\frac{\partial h}{\partial y} - \frac{\partial \mu}{\partial y}\right) dy.$$

and so $X_1 = -\dfrac{\partial(h-\mu)}{\partial y}$ and $X_2 = \dfrac{\partial(h-\mu)}{\partial x}$.

Let g be the riemannian metric in S compatible with the symplectic form ω, $\mathbf{J}$ be the almost complex structure in S and $\operatorname{grad}_0(h-\mu) = \dfrac{\partial(h-\mu)}{\partial x}\dfrac{\partial}{\partial x} + \dfrac{\partial(h-\mu)}{\partial y}\dfrac{\partial}{\partial y}$ in the local symplectic coordinates x, y, where $\operatorname{grad}_0(h-\mu)$ is the jacobian matrix of $h-\mu$.

In general, the gradient $\operatorname{grad}(h)$ of a function $h : M \to \mathbb{R}$ with respect to a riemannian metric g on a riemannian manifold (M, g) is defined by $Dh_p(v) = g_p(\operatorname{grad}(h), v)$, for all

$p \in M$ and $v \in T_pM$. By replacing v with each basis element $\dfrac{\partial}{\partial x_i}$ on local coordinates $x_1, \ldots, x_n$ in M one gets $\mathrm{grad}_0(h) = \mathbf{G}\,\mathrm{grad}(h)$.

Then, we have

$$X_\xi|_{S\cap\pi^{-1}(U)} = -\mathbf{J}_0\,\mathrm{grad}_0(h-\mu) = \mathbf{J}_0^{-1}\mathbf{G}\,\mathrm{grad}(h-\mu) = \mathbf{J}^{-1}\,\mathrm{grad}(h-\mu) = -\mathbf{J}\,\mathrm{grad}(h-\mu),$$

where $\mathbf{J}_0$ and $\mathbf{G}$ are the tensors associated to ω and g, respectively, and we used the fact that if $(\omega, g, \mathbf{J})$ is a compatible triple in M then $\mathbf{G}^{-1}\mathbf{J}_0 = \mathbf{J}$ (see Section 2.4.2).

□

3.3 Proof of Theorem A

3.3.1 First proof

Dynamical coherence poses a large restriction for partially hyperbolic diffeomorphisms to be contact diffeomorphisms. Indeed, the dynamical coherence property implies that if a partially hyperbolic diffeomorphism is a contact diffeomorphism then the invariant contact structure must be $E^s \oplus E^u$, for it is the only invariant codimension 1 subbundle of $\mathbb{T}^3$ that can be nonintegrable.

In this sense, Theorem A is a refinement of Theorem B which completely excludes skew-products of the form (1.1), with γ being a coboundary, from the contact category, whereas in the higher dimensional case this may not be true in general.

To prove Theorem A we use the next lemma, which is an improvement of Lemma 3.3 in the 3-dimensional case and follows directly from the uniqueness of F-invariant subbundles.

Lemma 3.5. *Let M^3 be an 3-dimensional manifold and $F : M \times \mathbb{S}^1 \to M \times \mathbb{S}^1$ be a skew-product given by $F(x,t) = (f(x), t + \gamma(x))$, with f being an Anosov diffeomorphism and $\gamma \in C^r(M; \mathbb{S}^1)$. If $\xi \subset T(M \times \mathbb{S}^1)$ is an F-invariant codimension 1 subbundle then one of the following possibilities holds: $\xi = E^s_F \oplus E^u_F$, $\xi = E^s_F \oplus \mathrm{span}\left\{\dfrac{\partial}{\partial t}\right\}$ or $\xi = \mathrm{span}\left\{\dfrac{\partial}{\partial t}\right\} \oplus E^u_F$.*

This is lemma is true since the only F-invariant codimension 1 subbundles are $E^s \oplus E^u$, $E^c \oplus E^u$ and $E^s \oplus E^c$.

From this lemma and [5] it follows immediately that if F is a partially hyperbolic contact skew-product of $\mathbb{T}^3$ as above then the contact structure must be $E^s_F \oplus E^u_F$, since F is dynamically coherent.

Finally, Lemma 3.5 together with Corollary 3.2 implies that the only F-invariant codimension 1 subbundles $E^s \oplus E^u$, $E^c \oplus E^u$ and $E^s \oplus E^c$ are integrable, which results in F not being a contact diffeomorphism. This proves Theorem A.

3.3.2 Second proof

We will also give a proof of Theorem A based on the notion of characteristic foliations. In this case we will need to assume that the transfer function is at least C^2, so that the vector field representing the characteristic foliation is at least C^1.

Theorem 3.1 (Theorem A)**.** *There are no contact partially hyperbolic skew-products $F : \mathbb{T}^2 \times \mathbb{S}^1 \to \mathbb{T}^2 \times \mathbb{S}^1$ of the form $F(p,t) = (f(p), t + \gamma(p))$, where $f : \mathbb{T}^2 \to \mathbb{T}^2$ is an Anosov diffeomorphism and $\gamma \in C^r(\mathbb{T}^2; \mathbb{S}^1)$ is a coboundary with transfer map $\mu \in C^2(\mathbb{T}^2, \mathbb{S}^1)$*

The Theorem 3.1 follows from following lemma, which gives a condition for subbundle to be a contact structure in the neighborhood of a surface based on the its characteristic foliation.

Lemma 3.6. *A vector field X in a surface $S \subset M^3$ represents the the characteristic foliation of some contact structure near S if, and only if, the following condition holds: $\mathrm{div}_\omega(X)(p) \neq 0$ on every point $p \in S$ for which $X(p) = 0$, where $\mathrm{div}_\omega(X)$ is the divergence of the vector field X relative to the area form ω in S.*

Proof. See Lemma 4.6.3 in [11].

□

Proof of Theorem 3.1. We first note that since ξ is transversal to $T\mathbb{S}^1$ it suffices to consider only surfaces transversal to $T\mathbb{S}^1$, for no surface tangent to $T\mathbb{S}^1$ will have points of tangency with ξ.

Let $S \subset \mathbb{T}^2 \times \mathbb{S}^1$ be a surface transversal to $\mathbb{S}^1$. Then, by Theorem C, the characteristic foliation of ξ in S is generated by some locally hamiltonian vector field X, i.e., given a local symplectic chart (U, x, y) em S we have

$$X = -\frac{\partial h}{\partial y}\frac{\partial}{\partial x} + \frac{\partial h}{\partial x}\frac{\partial}{\partial y}.$$

for some map $h : U \to \mathbb{S}^1$.

Recall that the *divergence* of a vector field X relative to an area form ω in S is a function $\mathrm{div}_\omega(X) : S \to \mathbb{R}$ defined by the equation

$$\mathcal{L}_X\omega = \mathrm{div}_\omega(X)\omega,$$

where $\mathcal{L}_X\omega$ is the Lie derivative of ω with respect to X. By *Cartan's Formula*

$$\mathcal{L}_X\omega = \iota_X d\omega + d(\iota_X\omega)$$

and the fact that $d\omega = 0$, since it is a symplectic form in S, it follows that

$$\mathrm{div}_\omega(X)\omega = d(\iota_X\omega).$$

Since X is locally hamiltonian, we have $\iota_X\omega = dh$ in any local symplectic chart. Therefore, $\mathrm{div}_\omega(X)\omega = 0$ and, consequently, $\mathrm{div}_\omega(X) = 0$ for all points in S, including those where S is tangent to ξ.

This proves that ξ cannot be a contact structure and Theorem A follows from Lemma 3.5.

$\square$

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