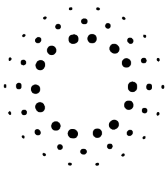

IFT - UNESP
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DISSERTAÇÃO DE MESTRADO IFT–D.005/25

Shannon Mutual Information of Localized One-Particle States in Scalar Field Theory

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May, 2025

Izquierdo Inga, Willy Antony
I99s Shannon mutual information of localized one-particle states in scalar field theory / Willy Antony Izquierdo Inga. – São Paulo, 2025
131 f: il. color.

Dissertação (mestrado) – Universidade Estadual Paulista (Unesp), Instituto de Física Teórica (IFT), São Paulo
Orientador: Gastão Inácio Krein
Coorientador: David Rosa Junior

1. Teoria quântica de campos. 2. Informação quântica. 3. Partículas (Física nuclear). I. Título

Sistema de geração automática de fichas catalográficas da Unesp. Biblioteca do Instituto de Física Teórica (IFT), São Paulo. Dados fornecidos pelo autor(a).

SHANNON MUTUAL INFORMATION OF LOCALIZED ONE-PARTICLE STATES IN SCALAR FIELD THEORY

Dissertação de Mestrado apresentada ao Instituto de Física Teórica do Câmpus de São Paulo, da Universidade Estadual Paulista "Júlio de Mesquita Filho", como parte dos requisitos para obtenção do título Mestre em Ciências, área: Física.

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São Paulo, 02 de junho de 2025.

To my family, with all my love and gratitude. This work is dedicated to you, who have always been my source of inspiration, support, and strength.

Acknowledgements

I would like to express my sincere gratitude to my advisor, Professor Gastão Inácio Krein, for his invaluable guidance, unwavering support, and insightful discussions throughout the course of this master's thesis. His expertise and encouragement were instrumental in shaping this research.

I am also deeply grateful to my co-advisor, Dr. David Rosa Junior, for his continuous support, constructive feedback, and for sharing his profound knowledge, which significantly contributed to the development of this work.

I extend my heartfelt thanks to CAPES for the financial support provided, which was crucial for the realization of this research.

Finally, I wish to express my deepest appreciation to my family for their unconditional love, patience, and encouragement during this challenging and rewarding journey. Their support has been a constant source of motivation.

"Sic transit gloria mundi
How doth the busy bee,
Dum vivimus vivamus,
I stay mine enemy!"

Emily Dickinson, *Springfield Daily Republican*

Resumo

Nesta dissertação, investigamos a informação mútua de Shannon para um campo escalar quântico sem massa livre em $d+1$ dimensões, focando nas correlações entre duas regiões espaciais complementares. Nossa abordagem central envolveu o cálculo dessa medida de informação clássica diretamente dos elementos diagonais da matriz densidade expressa na base de configuração do campo. Essa metodologia forneceu um arcabouço para quantificar as correlações clássicas que persistem ou emergem após processos de decoerência quântica ou medição, que efetivamente diagonalizam a matriz densidade do sistema.

Empregamos o formalismo funcional da representação de Schrödinger para calcular as entropias de Shannon associadas às regiões complementares, a partir das quais derivamos a informação mútua. Para lidar com as integrais funcionais, utilizamos tanto a aproximação de ponto de sela quanto o truque das réplicas. Computamos a informação mútua de Shannon para dois casos: o estado de vácuo e estados de partícula-única localizados. Para o estado de vácuo, nossos resultados reproduziram aqueles encontrados na literatura existente. Para estados localizados, descritos por soluções de pacote de ondas da equação de Schrödinger relativística, a informação mútua de Shannon pode ser expressa como a informação mútua do vácuo mais termos dependentes do perfil do pacote de ondas.

Como uma aplicação do formalismo desenvolvido, calculamos a informação mútua de Rényi-2 (também conhecida como entropia de colisão) em uma dimensão espacial, considerando regiões definidas como semirretas e empregando um pacote de ondas lorentziano. Encontramos que a presença de uma excitação geralmente aumenta a informação compartilhada em comparação com o caso do vácuo. Esse excesso de informação é maximizado quando a partícula está localizada próximo à fronteira e diminui à medida que a partícula se afasta, com a taxa de decaimento dependendo da largura do pacote de ondas.

No geral, nossas descobertas contribuem para a compreensão das correlações clássicas em teoria quântica de campos e oferecem uma perspectiva sobre o conteúdo de informação que é diretamente relevante para cenários envolvendo decoerência e medição. Este trabalho fornece um ponto de vista complementar aos cálculos tradicionais de entropia de emaranhamento, concentrando-se na informação clássica acessível codificada nas probabilidades de configuração do campo.

Palavras Chaves: Teoria Quântica de Campos; Informação Mútua de Shannon; Decoerência; Emaranhamento; Replica Trick; Representação de Schrödinger; Pacotes de Ondas.

Áreas do conhecimento: Física de Partículas e Campos; Informação Quântica.

Abstract

In this dissertation, we investigated the Shannon mutual information for a free quantum massless scalar field in $d+1$ dimensions, focusing on the correlations between two complementary spatial regions. Our central approach involved calculating this classical information measure directly from the diagonal elements of the density matrix expressed in the field configuration basis. This methodology provided a framework for quantifying the classical correlations that persist or emerge after quantum decoherence or measurement processes, which effectively diagonalize the system's density matrix.

We employed the Schrödinger picture functional formalism to compute the Shannon entropies associated with the complementary regions, from which we derived the mutual information. To handle the functional integrals, we utilized both the saddle-point approximation and the replica trick. We computed the Shannon mutual information for two cases: the vacuum state and localized single-particle states. For the vacuum state, our results reproduced those found in the existing literature. For localized states, described by wave packet solutions of the relativistic Schrödinger equation, the Shannon mutual information can be expressed as the vacuum mutual information plus terms dependent on the wave packet profile.

As an application of the developed formalism, we calculated the Rényi-2 mutual information (also known as the collision entropy) in one spatial dimension, considering regions defined as half-lines and employing a Lorentzian wave packet. We found that the presence of an excitation generally increases the shared information compared to the vacuum case. This excess information is maximized when the particle is localized near the boundary and diminished as the particle moves away, with the decay rate depending on the wave packet's width.

Overall, our findings contribute to the understanding of classical correlations in quantum field theory and offer a perspective on the information content that is directly relevant to scenarios involving decoherence and measurement. This work provides a complementary viewpoint to traditional entanglement entropy calculations by focusing on the accessible, classical information encoded in field configuration probabilities.

Keywords: Quantum Field Theory; Shannon Mutual Information; Decoherence; Entanglement; Replica Trick; Schrödinger Picture; Wave Packets.

Knowledge Area: Physics of Particles and Fields; Quantum Information.

Contents

Chapter 1

Introduction and Motivation

Quantum Field Theory (QFT) stands as the paramount theoretical framework unifying quantum mechanics and special relativity, providing the language to describe the fundamental constituents of matter and their interactions. It governs phenomena ranging from high-energy particle collisions to the subtle properties of condensed matter systems and the quantum nature of the early universe [1]. A central theme within QFT, inherited from quantum mechanics but endowed with new complexities, is the study of correlations between different parts of a physical system. Among these, quantum entanglement represents the most profound departure from classical intuition, describing non-local connections where the state of a whole system is determined, yet the states of its individual parts remain fundamentally undefined until measured [2, 3].

Understanding entanglement in QFT, particularly between spatially distinct regions, is not merely an academic exercis [4, 5]. It holds deep implications for fundamental physics, touching upon the nature of locality, the information paradox in black hole physics [6], the structure of the quantum vacuum [7], and the characterization of phases and phase transitions in quantum matter [8]. Quantifying entanglement in this context typically involves calculating the entanglement entropy, often defined via the von Neumann entropy of the reduced density matrix associated with a subregion. For a bipartite system in a pure state $|\Psi\rangle$, divided into subsystems A and B, the reduced density matrix of subsystem A is $\rho_A = \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|)$. The entanglement entropy $S_A = -\mathrm{Tr}_A(\rho_A \ln \rho_A)$ quantifies the entanglement between A and B. While elegant, the calculation of entanglement entropy in QFT is notoriously challenging, often plagued by ultraviolet divergences related to the short-distance correlations across the boundary separating the regions, requiring regularization and renormalization techniques[4, 5, 9, 10, 11].

Furthermore, the idealized scenario of isolated quantum systems, for which concepts like von Neumann entropy are typically defined for pure states, rarely holds true in reality [12]. Quantum systems inevitably interact with their surrounding environment—be it a thermal bath, stray particles, or measurement devices.

This interaction leads to decoherence, a process crucial for understanding how the classical world emerges from the quantum substrate. Decoherence describes the dynamical process by which a system loses its quantum coherence—the ability to exist in superpositions of states—due to becoming entangled with the numerous, often unobserved, degrees of freedom of its environment [13].

The effect of decoherence is often modeled by examining the evolution of the system's density matrix, ρ_S. Initially, the system might be in a pure state, perhaps a superposition $|\psi\rangle = \sum_i c_i|i\rangle$. Interaction with the environment $|E\rangle$ leads to an entangled state of the composite system, $|\Psi_{SE}\rangle = \sum_i c_i|i\rangle|E_i\rangle$. If the environmental states $|E_i\rangle$ corresponding to different system states $|i\rangle$ become approximately orthogonal ($\langle E_i|E_j\rangle \approx \delta_{ij}$), tracing out the environmental degrees of freedom results in a reduced density matrix for the system:

$$\rho'_S = \mathrm{Tr}_E(|\Psi_{SE}\rangle\langle\Psi_{SE}|) = \sum_{i,j} c_i c_j^* |i\rangle\langle j|\langle E_j|E_i\rangle \approx \sum_i |c_i|^2 |i\rangle\langle i|.$$

The off-diagonal elements, representing quantum coherences ($\langle i|\rho'_S|j\rangle$ for $i \neq j$), rapidly decay, while the diagonal elements, $|c_i|^2 = \langle i|\rho'_S|i\rangle$, persist. These diagonal elements represent the classical probabilities of finding the system in the states $|i\rangle$. The basis $\{|i\rangle\}$ in which the density matrix becomes diagonal (or nearly diagonal) is called the pointer basis, determined dynamically by the nature of the system-environment interaction [14]. Decoherence thus effectively transforms a quantum superposition into a classical statistical mixture, explaining the absence of macroscopic superpositions and providing a mechanism for the apparent "collapse" of the wave function during measurement, although debates continue on whether it fully resolves the measurement problem [15]. The timescale for this process, the decoherence time, is typically extremely short for macroscopic systems, but it is also the primary bottleneck for realizing fault-tolerant quantum computation, as it destroys the delicate quantum states used as qubits [16].

The interaction leading to decoherence, whether through environmental coupling or an explicit measurement process, effectively converts the initial quantum entanglement into classical correlations between the measured outcomes or the system's constituent parts. A natural question arises: how can we quantify the information content, specifically the correlations, that remain in the system after decoherence has occurred or when we focus solely on the classical aspects encoded in the diagonal probabilities?

To quantify these resulting classical correlations, we turn to classical infor-

mation theory. The Shannon entropy, $H(X) = -\sum_i p(x_i) \log p(x_i)$, measures the uncertainty or information content associated with a classical random variable X having probability distribution $p(x_i)$. The Shannon mutual information between two random variables X and Y, defined as $I(X;Y) = H(X) + H(Y) - H(X,Y)$, quantifies the amount of information X provides about Y (and vice versa), measuring their statistical dependence. It represents the purely classical correlations shared between X and Y [17, 18].

This thesis undertakes the calculation of the Shannon mutual information for a quantum massless scalar field $\phi(x)$ in $d+1$ dimensions, focusing on the correlations between two complementary spatial regions Ω and $\overline{\Omega}$ at a fixed time. The key aspect of our approach is that we compute the Shannon mutual information based *explicitly on the diagonal elements* of the density matrix in the field configuration basis. The state of the quantum field in the Schrödinger picture is described by a wave functional $\Psi[\phi]$, where $\phi(x)$ is a classical field configuration. The probability of finding the field in configuration $\phi(x)$ is given by $P[\phi] = |\Psi[\phi]|^2$. This $P[\phi]$ corresponds to the diagonal elements of the density matrix $\rho = |\Psi\rangle\langle\Psi|$ in the field basis $|\phi\rangle$. By restricting the field configurations to regions Ω and $\overline{\Omega}$, we obtain reduced probability distributions $P_\Omega[g_\Omega]$ and $P_{\overline{\Omega}}[g_{\overline{\Omega}}]$ by integrating out the complementary region's configurations. The Shannon entropies $S[\Omega] = -\int [Dg_\Omega] P_\Omega[g_\Omega] \ln P_\Omega[g_\Omega]$ and similarly for $S[\overline{\Omega}]$ and $S[\Omega \cup \overline{\Omega}]$ are then computed, allowing the determination of the Shannon mutual information $I_S(\Omega, \overline{\Omega}) = S[\Omega] + S[\overline{\Omega}] - S[\Omega \cup \overline{\Omega}]$.

The relevance of this specific calculation to decoherence lies precisely in its focus on the diagonal density matrix elements (field configuration probabilities) and the use of Shannon's measure. It quantifies the classical correlations inherent in the probability distribution $P[\phi]$. When decoherence occurs, driving the system towards a state diagonal in the pointer basis (which, for field theories interacting with localized probes, might naturally be related to the field configuration basis), the off-diagonal quantum coherences are lost. The Shannon mutual information $I_S(\Omega, \overline{\Omega})$ calculated here captures the amount of shared information that persists in the classical probabilities associated with field configurations in Ω and $\overline{\Omega}$ after this loss of quantum coherence. It provides a measure of how much of the original quantum information, possibly including entanglement, is converted into accessible classical correlations by the decoherence process. As stated in [12], this classical mutual information $I(A:B)$ is equivalent to the quantum mutual information $I(\rho_{AB}^{diag})$ of the post-measurement (or post-decoherence) state ρ_{AB}^{diag},

which is diagonal.

Therefore, by calculating $I_S(\Omega, \overline{\Omega})$ for the massless scalar field—both in the vacuum state and in localized excited states—this work aims to explore the structure and flow of information in a QFT system from an information-theoretic perspective that directly connects to the outcomes of decoherence or measurement. We analyze how these classical correlations depend on the nature of the quantum state (vacuum vs. excitation) and potentially on the geometric configuration of the regions Ω and $\overline{\Omega}$. This provides a complementary viewpoint to traditional entanglement entropy calculations, focusing on the measurable, classical information accessible after interaction with an environment or apparatus has projected the system onto a set of classical alternatives. The massless scalar field serves as a simple, yet non-trivial, arena for these investigations, representing a fundamental building block of QFT. The calculations leverage the Schrödinger picture functional formalism and employ techniques like the saddle-point approximation and the replica trick to handle the functional integrals involved in computing the Shannon entropies.

This thesis is organized as follows:

- **Chapter 2:** Reviews the fundamental concepts from both classical and quantum information theory necessary for the subsequent analysis. This includes Shannon entropy, mutual information, relative entropy, Rényi entropy, quantum states, entanglement, von Neumann entropy, quantum mutual information, and provides a detailed discussion of decoherence mechanisms and their implications [17, 18, 19, 13, 14].

- **Chapter 3:** Introduces the formalism for a quantum scalar field in the Schrödinger picture. It details the derivation of the ground state wave functional using the path integral and saddle-point approximation, defines creation and annihilation operators, computes the reduced probability distributions for subregions, and calculates the Shannon entropies and mutual information for the vacuum state using the replica trick [1, 9].

- **Chapter 4:** Extends the analysis to localized single-particle excited states of the scalar field. The chapter derives the wave functional for such states, discusses suitable wave packet solutions, computes the reduced probability distributions, and calculates the Shannon entropies and mutual information using the replica trick for these excited states [20, 21].

- **Chapter 5:** Applies the formalism developed to calculate the Rényi entropy (specifically for $n = 2$) in the simplified case of one spatial dimension with regions defined as half-lines, utilizing the derived wave packets.
- **Conclusion:** Summarizes the main results of the thesis, discusses their implications for understanding correlations and the effects of decoherence in QFT, and outlines perspectives for future research.
- **Appendices:** Contain detailed mathematical derivations supporting the main text, including properties of the fractional Laplacian operator [22], Green functions, normalization conditions, and the calculation of inverse operators.

This structure ensures a logical progression from foundational concepts to specific calculations, culminating in a broader discussion of the results' significance.

Chapter 2

Classical and Quantum Information

Quantum information theory merges principles of quantum mechanics and information science to analyze how information is stored, processed, and transmitted in quantum systems. It provides powerful tools to quantify quantum entanglement, a phenomenon where subsystems exhibit correlations that defy classical explanations. These tools are essential for understanding the behavior of entanglement as well as the correlations between the degrees of freedom of the scalar field across two complementary regions explored in the thesis [18].

We begin summarizing key classical information theory concepts like Shannon entropy, mutual information, relative entropy, and Rényi entropy. These measures quantify uncertainty, dependence, and differences between probability distributions, serving as precursors to their quantum counterparts used in quantum field theory (QFT) by introducing entropy measures, such as von Neumann and Rényi entropies, which quantify uncertainty and correlations in quantum systems. We then explore quantum entanglement, focusing on its quantification via entanglement entropy and mutual information.

2.1 Classical Information

Classical information theory, pioneered by Claude Shannon in the mid-20th century [17], provides a mathematical framework for quantifying information, uncertainty, and communication in classical systems. Its concepts help quantify uncertainty and correlations, essential for understanding quantum entanglement in quantum field theory (QFT).

2.1.1 Shannon Entropy

Shannon entropy measures the uncertainty or information content of a discrete random variable. For a random variable X with possible outcomes $\{x_1, x_2, \ldots, x_n\}$

and probability mass function $p(x)$, the Shannon entropy $H(X)$ is:

$$H(X) = -\sum_{i=1}^{n} p(x_i) \log_2 p(x_i), \tag{2.1}$$

where the logarithm is base 2 (yielding units in bits), and by convention, $0 \log_2 0 = 0$. It has the following properties:

- **Non-negativity:** $H(X) \geq 0$, with equality if one outcome has probability 1 (deterministic case).
- **Maximum value:** For a finite set of n outcomes, $H(X) \leq \log_2 n$, achieved when all outcomes are equally likely $p(x_i) = 1/n$.
- **Additivity:** For independent random variables X and Y, the joint entropy is $H(X, Y) = H(X) + H(Y)$.
- **Concavity:** $H(X)$ is a concave function of the probability distribution $p(x)$.

Entropy represents the average number of bits needed to encode the outcome of X optimally or the uncertainty about X. For example, a fair coin ($p(\text{heads}) = p(\text{tails}) = 0.5$) has $H(X) = \log_2 2 = 1$ bit, reflecting maximum uncertainty for two outcomes.

2.1.2 Mutual Information

Mutual information quantifies the dependence between two random variables, measuring how much information one provides about the other. For discrete random variables X and Y with joint probability $p(x, y)$ and marginals $p(x)$, $p(y)$, the mutual information $I(X; Y)$ is:

$$I(X; Y) = \sum_{x,y} p(x, y) \log_2 \left(\frac{p(x, y)}{p(x)p(y)} \right). \tag{2.2}$$

Alternatively, it can be expressed using entropies:

$$I(X; Y) = H(X) + H(Y) - H(X, Y), \tag{2.3}$$

where $H(X, Y) = -\sum_{x,y} p(x, y) \log_2 p(x, y)$ is the joint entropy. It has the following properties:

- **Non-negativity:** $I(X;Y) \geq 0$, with equality if and only if X and Y are independent.
- **Symmetry:** $I(X;Y) = I(Y;X)$.
- **Relation to conditional entropy:** $I(X;Y) = H(X) - H(X|Y)$, where $H(X|Y)$ is the conditional entropy, measuring uncertainty in X given Y.
- **Upper bound:** $I(X;Y) \leq \min(H(X), H(Y))$.

Mutual information represents the reduction in uncertainty about X upon learning Y. For example, if X and Y are identical (perfectly correlated), $I(X;Y) = H(X)$; if independent, $I(X;Y) = 0$.

2.1.3 Relative Entropy

Relative entropy, or Kullback-Leibler divergence, measures the difference between two probability distributions. For distributions P and Q over the same set, the relative entropy $D(P|Q)$ is:

$$D(P|Q) = \sum_x P(x) \log_2 \left(\frac{P(x)}{Q(x)} \right). \tag{2.4}$$

It has the following properties:

- **Non-negativity:** $D(P|Q) \geq 0$, with equality if and only if $P = Q$.
- **Asymmetry:** $D(P|Q) \neq D(Q|P)$, so it is not a true distance metric.
- **Additivity for independent distributions:** For product distributions, $D(P_1P_2|Q_1Q_2) = D(P_1|Q_1) + D(P_2|Q_2)$.

Relative entropy quantifies how much additional information is needed to encode samples from P using a code optimized for Q.

2.1.4 Rényi Entropy

Rényi entropy generalizes Shannon entropy, parameterized by $\alpha \geq 0$, $\alpha \neq 1$. For a random variable X, it is:

$$H_\alpha(X) = \frac{1}{1-\alpha} \log_2 \left(\sum_x p(x)^\alpha \right). \tag{2.5}$$

As $\alpha \to 1$, $H_\alpha(X) \to H(X)$. It has the following properties:

- **Non-negativity:** $H_\alpha(X) \geq 0$.
- **Monotonicity:** $H_\alpha(X) \geq H_\beta(X)$ for $\alpha < \beta$.
- **Special cases:** $H_0(X) = \log_2 |\{x : p(x) > 0\}|$ (log of support size); $H_2(X) = -\log_2 \sum_x p(x)^2$ (collision entropy).

Rényi entropy provides a family of entropy measures, emphasizing different aspects of the probability distribution. For example, $H_2(X)$ is related to the probability of two independent samples being identical.

2.2 Quantum Information

Quantum information theory provides a framework to quantify information and correlations in quantum systems, offering insights into the structure of quantum fields and their entanglement properties.

2.2.1 Quantum Entanglement

Quantum entanglement is a hallmark of quantum mechanics, describing correlations between subsystems that cannot be explained classically. For a multipartite system with n subsystems, the total Hilbert space is the tensor product of individual Hilbert spaces:

$$H = \otimes_{l=1}^{n} H_l. \tag{2.6}$$

By the superposition principle, a general pure state in this space can be written as:

$$|\psi\rangle = \sum_{i_1,\ldots,i_n} c_{i_1,\ldots,i_n} |i_1\rangle \otimes \cdots \otimes |i_n\rangle, \tag{2.7}$$

where $|i_l\rangle$ are basis states for subsystem l, and $c_{i_1,\ldots,i_n}$ are complex coefficients. The state is entangled if it cannot be expressed as a product of individual subsystem states:

$$|\psi\rangle \neq |\psi_1\rangle \otimes \cdots \otimes |\psi_n\rangle. \tag{2.8}$$

A classic example is the Bell state:

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \tag{2.9}$$

where measurements on one qubit affect the other instantaneously.

For mixed states, described by a density matrix ρ, the state is entangled if it cannot be written as a convex combination of product states:

$$\rho \neq \sum_i p_i \rho_1^i \otimes \cdots \otimes \rho_n^i, \tag{2.10}$$

where $p_i \geq 0$, $\sum_i p_i = 1$, and ρ_l^i are density matrices for subsystem l. This definition captures the non-separability of quantum states and is central to studying entanglement in QFT systems [2].

2.2.2 Entanglement Entropy

To quantify the entanglement of a system, we can follow the axiomatic approach according to [2] or [3], where certain functions that satisfy basic entanglement properties are defined. These functions can then be used to attempt to quantify the amount of entanglement present in a given multipartite state. Specifically for a pure bipartite state of a composite system, the von Neumann entropy of one subsystem serves as a valid measure of entanglement, as outlined in [3]. This quantity, often referred to as *entanglement entropy*, satisfies the necessary axioms for an entanglement measure.

In particular, let us analyze the entanglement in a bipartite system. In order to do this, let's suppose the system is in a pure state $|\Psi\rangle$ whose density matrix for the Hilbert space $\mathcal{H}_{tot}$ is given by

$$\rho_{tot} = |\Psi\rangle \langle\Psi| . \tag{2.11}$$

We then divide the total system into two subsystems A and $B = \overline{A}$ complementary to each other as in Fig. 2.1.

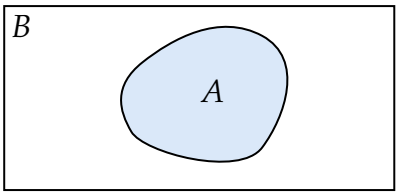


Figure 2.1

Thus, the total Hilbert space is given by a direct product of two Hilbert spaces

of the subsystems,

$$\mathcal{H}_{tot} = \mathcal{H}_A \otimes \mathcal{H}_B. \tag{2.12}$$

Let $\{|i\rangle_B, i = 1, 2, ...\}$ be an orthonormal basis in $\mathcal{H}_B$ and define the reduced density matrix ρ_A of the system A by taking the partial trace over the system B,

$$\rho_A = \mathrm{Tr}_B(\rho_{tot}) = \sum_i {}_B\langle i|\, \rho_{tot}\, |i\rangle_B . \tag{2.13}$$

Now, let's consider a pure state $|\Psi\rangle$ in a general form

$$|\Psi\rangle = \sum_{i,j} c_{i,j}\, |i\rangle_A \otimes |j\rangle_B , \tag{2.14}$$

where $|i\rangle_A$ and $|j\rangle_B$ are orthonormal bases for $\mathcal{H}_A = \{|i\rangle_A, i = 1, ..., d_A\}$ and $\mathcal{H}_B = \{|j\rangle_B, i = 1, ..., d_B\}$, respectively, and the coeffient $c_{i,j}$ is a $d_A \times d_B$ matrix with complex entries. Thus, when c_{ij} factorizes, $c_{ij} = c_i^A c_j^B$, the state $|\Psi\rangle$ is called a separable state (a pure product state) and can be transformed into the product form $|\Psi\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle$, where $|\Psi_A\rangle = \sum_i c_i^A\, |i\rangle_A$ and $|\Psi_B\rangle = \sum_j c_j^B\, |j\rangle_B$.

In this context, we can define the entanglement entropy of the subsystem A, by the von Neumann entropy of the reduced density matrix ρ_A,

$$S_A = -\mathrm{Tr}_A[\rho_A \ln \rho_A]. \tag{2.15}$$

Note that a separable state has zero entanglement entropy $S_A = 0$ because it is composed of pure states.

We can simplify (2.14) by changing the bases into the Schmidt decomposition form:

$$|\Psi\rangle = \sum_{k=1}^{\min(d_A, d_B)} \sqrt{p_k}\, |\psi_k\rangle_A \otimes |\psi_k\rangle_B , \tag{2.16}$$

where p_k are non-negative real numbers satisfying

$$\sum_k p_k = 1, \tag{2.17}$$

and $|\psi_k\rangle_{A,B}$ are new orthonormal bases for the subsystems A and B. The new orthonormal bases $|\psi_k\rangle_{A.B}$ are the unitary transformation of the $|i\rangle_{A,B}$:

$$|\psi_k\rangle_A = \sum_i U_{ik}\, |i\rangle_A \quad \text{and} \quad |\psi_k\rangle_B = \sum_\mu V_{k\mu}\, |\mu\rangle_B . \tag{2.18}$$

The Schmidt decomposition (2.16) is particularly nice as it yields a mixed state density matrix with the probability distribution $\{p_k\}$ for the reduced density matrix:

$$\rho_A = \sum_{i=1}^{d_B} {}_B\langle\psi_i|\Psi\rangle\langle\Psi|\psi_i\rangle_B = \sum_{k=1}^{\min(d_A,d_B)} p_k |\psi_k\rangle\langle\psi_k| \tag{2.19}$$

In the Schmidt decomposition, the entanglement entropy

$$S_A = -\sum_{k=1}^{\min(d_A,d_B)} p_k \ln p_k \tag{2.20}$$

is nothing but the Shannon entropy of the probability distribution $\{p_k\}$.

An important property of the entanglement entropy of a pure bipartite state is that its value is the same for both subsystems A and its complement $B = \overline{A}$:

$$S_A = S_B \tag{2.21}$$

This follows from the symmetry of the Schmidt decomposition under the exchange of A and B.

For example, in the Bell state $|\Phi^+\rangle$, $\rho_A = \frac{1}{2}I$, with eigenvalues $1/2, 1/2$, so:

$$S_A = -2\left(\frac{1}{2}\ln\frac{1}{2}\right) = \ln 2, \tag{2.22}$$

indicating maximal entanglement for two qubits [3].

Finally, since the entropy is a central concept in information theory, measuring uncertainty or information content. In quantum mechanics, the von Neumann entropy generalizes the Shannon entropy to quantum states. For a pure state $\rho = |\psi\rangle\langle\psi|$, $S(\rho) = 0$, indicating complete knowledge. For mixed states, $S(\rho) > 0$, reflecting uncertainty. The von Neumann entropy has key properties:

- **Positivity:** $S(\rho) \geq 0$, with equality for pure states.
- **Unitary invariance:** $S(U\rho U^\dagger) = S(\rho)$ for unitary U.
- **Subadditivity:** For a bipartite system AB, $S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B)$.
- **Strong subadditivity:** For subsystems A, B, C, $S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AB}) + S(\rho_{BC})$.

These properties make von Neumann entropy a cornerstone for studying quantum correlations, as noted in standard texts like Nielsen-Chuang [19].

Area Law and the Ryu-Takayanagi Conjecture

A fundamental property of entanglement entropy in quantum field theories is the *area law*, which states that for a given spatial region, the leading term of the entanglement entropy is proportional to the area of the boundary of that region, not its volume. This is a surprising result, as one might naively expect the entropy to be an extensive quantity proportional to the volume [23].

The area law can be expressed as (see eq. (3) [4]):

$$S_A = g_{d-1}\frac{\partial A}{\epsilon^{d-1}} + \cdots + g_1\frac{\partial A}{\epsilon^{-1}} + g_0 \partial A \ln(\epsilon) + S_A^0 \tag{2.23}$$

where:

- S_A is the entanglement entropy of a subregion A and S_A^0 its finite part.
- ∂A is the boundary of region A.
- ϵ is a short distance cutoff.
- g_i are local and extensive functions on the boundary ∂A , which are homogeneous of degree i.

The area law reflects the locality of interactions in quantum field theory: most entanglement comes from modes localized near the boundary ∂A. Degrees of freedom deep inside A or $\overline{A}$ are largely uncorrelated with the other region, contributing little to entanglement entropy.

A powerful tool for understanding and calculating entanglement entropy in certain quantum field theories is the *Ryu-Takayanagi conjecture*, which arises from the AdS/CFT correspondence (holographic principle). This conjecture provides a simple and elegant geometric formula for the entanglement entropy of a region A in a d-dimensional conformal field theory (CFT) that has a gravitational dual in a $(d+1)$-dimensional Anti-de Sitter (AdS) spacetime.

The Ryu-Takayanagi formula is (see eq.(1.5) [24]):

$$S_A = \frac{\text{Area}(\gamma_A)}{4G_N^{(d+2)}} \tag{2.24}$$

where:

- γ_A is the d dimensional static minimal surface in AdS_{d+2} whose boundary is given by ∂A.

- $G_N^{(d+2)}$ is the $d+2$-dimensional Newton constant.

This formula remarkably connects a quantum information quantity (entanglement entropy) to a purely geometric quantity (the area of a minimal surface). A key feature of the Ryu-Takayanagi conjecture is that it naturally reproduces the area law. The minimal surface γ_A extends into the bulk, and its area is dominated by the region near the boundary of A, which is where the divergence arises, correctly capturing the area-law scaling.

2.2.3 Relations Between Entanglement Measures

Among several measures of quantum entanglement we list a few, relative entropy, mutual information, and Rényi entropy, that are often used in the applications to QFT and discuss their connections to each other.

Relative Entropy

Relative entropy quantifies the distinguishability between two quantum states described by density matrices ρ and σ:

$$S(\rho\|\sigma) = \mathrm{Tr}[\rho(\ln\rho - \ln\sigma)]. \tag{2.25}$$

It is non-negative, with $S(\rho|\sigma) = 0$ if and only if $\rho = \sigma$, and serves as a measure of the "distance" between states, though it is not a true metric due to asymmetry. In QFT, relative entropy is used to compare the joint state of subsystems with their separable counterparts, as discussed below.

Mutual Information

Mutual information measures the total correlations (classical and quantum) between two subsystems A and B:

$$I(A,B) = S_A + S_B - S_{A\cup B}, \tag{2.26}$$

where $S_{A\cup B} = S(\rho_{A\cup B})$ is the von Neumann entropy of the joint system. For a pure state, $\rho_{A\cup B} = |\Psi\rangle\langle\Psi|$, so $S_{A\cup B} = 0$, and:

$$I(A,B) = S_A + S_B = 2S_A, \tag{2.27}$$

since $S_A = S_B$. In QFT, mutual information quantifies correlations between spatial regions Ω and $\overline{\Omega}$, making it a central quantity in the thesis's calculations.

The mutual information can be written by the relative entropy as

$$I(A,B) = S(\rho_{A\cup B} \| \rho_A \otimes \rho_B). \tag{2.28}$$

This relation shows that the mutual information between two subsystems quantifies how much the state $\rho_{A\cup B}$ for the union $A \cup B$ differs from the separable state $\rho_A \otimes \rho_B$ and can be a good measure of global correlations over spatially disconnected regions in QFT.

Rényi Entropy

The Rényi entropy is a one-parameter generalization of entanglement entropy labeled by an integer n, called the replica parameter

$$S_n(A) = \frac{1}{1-n} \ln \mathrm{Tr}_A \left(\rho_A^n\right). \tag{2.29}$$

In the $n \to 1$ limit with the normalization $\mathrm{Tr}_A(\rho_A) = 1$. The Rényi entropy is reduced to entanglement entropy,

$$\begin{aligned} S_A &= -\lim_{n\to 1} \frac{\ln \mathrm{Tr}_A \left(\rho_A^n\right)}{n-1} \\ &= -\lim_{n\to 1} \partial_n \ln \mathrm{Tr}_A \left(\rho_A^n\right). \end{aligned} \tag{2.30}$$

Although the Rényi entropy is defined only for an integer n, the analytic continuation is assumed in taking the $n \to 1$ limit. This method is called the ***replica trick*** that is often employed for the entanglement entropy calculation in QFT.

2.2.4 Additional Quantum Information Concepts

Contrasting classical to quantum behavior is one of the main challenges in the field of quantum information theory. In this way, it becomes important to introduce some concepts that help us differentiate them.

Relative Entropy of Coherence

Quantum coherence, the ability of a quantum system to exist in superpositions, is a key resource in quantum information processing, complementing entanglement. It is quantified relative to a fixed basis, often the computational or physical basis of the system.

For a quantum state ρ, the relative entropy of coherence is defined as:

$$C_{\text{rel}}(\rho) = S(\rho \| \rho_{\text{diag}}) = S(\rho_{\text{diag}}) - S(\rho), \tag{2.31}$$

where $\rho_{\text{diag}} = \sum_i \langle i|\rho|i\rangle |i\rangle\langle i|$ is the diagonal part of ρ in the basis $|i\rangle$, and $S(\rho) = -\operatorname{Tr}(\rho \ln \rho)$ is the von Neumann entropy. Here, $S(\rho_{\text{diag}}) = -\sum_i p_i \ln p_i$, with $p_i = \langle i|\rho|i\rangle$, is the Shannon entropy of the diagonal elements. It has the following properties:

- **Non-negativity:** $C_{\text{rel}}(\rho) \geq 0$, with equality if ρ is incoherent (diagonal in the basis).
- **Monotonicity:** It does not increase under incoherent operations, which preserve the diagonal structure.
- **Additivity:** For tensor product states, $C_{\text{rel}}(\rho \otimes \sigma) = C_{\text{rel}}(\rho) + C_{\text{rel}}(\sigma)$.

These properties, established in Baumgratz et al., 2014 [25], make C_{rel} a valid coherence measure. For example, for a pure state $|\psi\rangle = \sum_i c_i |i\rangle$, $S(\rho) = 0$, and $C_{\text{rel}}(\rho) = -\sum_i |c_i|^2 \ln |c_i|^2$, reflecting the coherence in the basis.

Coherence is relevant for understanding quantum advantages in QFT systems, where superpositions may influence correlation structures. The relative entropy of coherence provides a framework to quantify this resource, potentially applicable to discretized QFT models or quantum information tasks.

Maximal Classical Mutual Information

Classical correlations extractable from ρ_{AB} refer to the correlations between two quantum subsystems A and B, described by the bipartite quantum state ρ_{AB}, that can be revealed or quantified by performing local measurements on A and B. These correlations are "classical" because they manifest as statistical dependencies in the probability distributions of measurement outcomes, resembling correlations in classical probability theory, as opposed to quantum correlations like entanglement that depend on the quantum state's superposition or nonlocality.

The maximal mutual information is a key concept in quantum information theory that quantifies the maximum amount of classical correlation that can be extracted from a bipartite quantum state ρ_{AB} through local measurements on subsystems A and B. It represents the classical portion of the total correlations present in the state, making it essential for understanding quantum discord and the distinction between classical and quantum correlations.

The maximal mutual information, often denoted $C_S(\rho_{AB})$, is defined as:

$$C_S(\rho_{AB}) = \max_{\Pi_A^i, \Pi_B^j} I(\Pi_A \otimes \Pi_B[\rho_{AB}]), \tag{2.32}$$

where:

- $\{\Pi_A^i\}$ and $\{\Pi_B^j\}$ are sets of local measurements (typically projective measurements or Positive Operator-Valued Measures, POVMs) on subsystems A and B, with outcomes labeled i and j.
- $\Pi_A \otimes \Pi_B[\rho_{AB}]$ produces a joint probability distribution $p(i,j) = \mathrm{Tr}[(\Pi_A^i \otimes \Pi_B^j)\rho_{AB}]$.
- $I(\Pi_A \otimes \Pi_B[\rho_{AB}]) = H(p_A) + H(p_B) - H(p_{AB})$ is the classical mutual information, where:
- $p_A(i) = \sum_j p(i,j)$ and $p_B(j) = \sum_i p(i,j)$ are the marginal probability distributions.
- $H(p) = -\sum p_k \log p_k$ is the Shannon entropy of the probability distribution p.

The maximization is performed over all possible local measurement bases to achieve the highest classical correlation.

This quantity measures the maximum statistical dependence between the measurement outcomes of A and B, reflecting the classical correlations extractable from the quantum state.

Examples

- **Bell State:** For the Bell state $\rho_{AB} = |\psi\rangle\langle\psi|$, where $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, measuring both qubits in the σ_z basis yields:

$$p(0,0) = p(1,1) = \frac{1}{2}, \;\; p_A(0) = p_A(1) = p_B(0) = p_B(1) = \frac{1}{2},$$
$$H(p_A) = H(p_B) = 1, \;\; H(p_{AB}) = 1, \;\; I = 1 + 1 - 1 = 1.$$

This is optimal due to the state's symmetry, so $C_S(\rho_{AB}) = 1$ bit, indicating 1 bit of classical correlation is extractable. Measurements in other aligned bases (e.g., $\sigma_x \otimes \sigma_x$) yield the same result.

- **Classical State:** For $\rho_{AB} = \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11|$, measuring in the computational basis gives:

$$p(0,0) = p(1,1) = \frac{1}{2}, \;\; I = 1.$$

This is the maximum, so $C_S(\rho_{AB}) = 1$ bit, reflecting the purely classical correlation in the state.

- **Bell-Diagonal State:** For $\rho_{AB} = \frac{1}{4}(I \otimes I + \sum_{k=1}^{3} c_k \sigma_k \otimes \sigma_k)$, the maximal mutual information is given by:

$$C_S(\rho_{AB}) = \sum_{i=0}^{1} \frac{1 + (-1)^i \kappa}{2} \log \left(\frac{1 + (-1)^i \kappa}{2} \right),$$

where $\kappa = \max(|c_1|, |c_2|, |c_3|)$. The optimal measurement is in the Pauli basis corresponding to the largest $|c_k|$.

Optimization of Measurements

To achieve $C_S(\rho_{AB})$, the local measurements must be optimized to maximize $I(\Pi_A \otimes \Pi_B[\rho_{AB}])$. This involves:

- **Parameterizing Measurements:** For qubits, measurements are parameterized by Bloch sphere angles (θ, ϕ), defining projectors along a direction $\vec{n}$.
- **Analytical Solutions:** For specific states like Bell-diagonal states, the optimal basis is determined by the state's correlation tensor (e.g., the Pauli operator with the largest coefficient).
- **Numerical Optimization:** For general states, use techniques like gradient descent, simulated annealing, or grid search over measurement parameters to find the maximum I.
- **Symmetry Exploitation:** Use the state's symmetries to reduce the parameter space, e.g., for Bell-diagonal states, restricting to Pauli bases.

Symmetric Quantum Discord

Symmetric quantum discord quantifies the nonclassical correlations in ρ_{AB} that cannot be captured by classical measurement outcomes, even with optimal local measurements. These correlations include, but are not limited to, entanglement, and may exist in separable states, making discord a broader measure of quantumness than entanglement. The symmetric nature makes it particularly relevant for systems where the roles of A and B are interchangeable, such as in certain quantum communication protocols or symmetric physical systems [26].

Symmetric quantum discord, denoted $Q_S(\rho_{AB})$, is defined as the difference between the total correlations, measured by the quantum mutual information, and the maximum classical correlations extractable through local measurements on both subsystems:

$$Q_S(\rho_{AB}) = I(\rho_{AB}) - C_S(\rho_{AB}), \tag{2.33}$$

where:

- $I(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$ is the quantum mutual information, with $S(\rho) = -\mathrm{Tr}(\rho \log \rho)$ being the von Neumann entropy, and $\rho_A = \mathrm{Tr}B(\rho AB)), (\rho_B = \mathrm{Tr}A(\rho AB)$ as the reduced density matrices.
- $C_S(\rho_{AB}) = \max_{\Pi_A^i, \Pi_B^j} I(\Pi_A \otimes \Pi_B[\rho_{AB}])$ is the maximal mutual information, where: Π_A^i and Π_B^j are local projective measurements (or POVMs) on A and B.
- $p(i,j) = \mathrm{Tr}[(\Pi_A^i \otimes \Pi_B^j)\rho_{AB}]$ is the joint probability distribution.
- $I(\Pi_A \otimes \Pi_B[\rho_{AB}]) = H(p_A) + H(p_B) - H(p_{AB})$ is the classical mutual information, with $H(p) = -\sum p_k \log p_k$ as the Shannon entropy, and marginals $p_A(i) = \sum_j p(i,j)$, $p_B(j) = \sum_i p(i,j)$.

The maximization in $C_S(\rho_{AB})$ is over all possible local measurement bases to extract the highest classical correlation.

This definition ensures symmetry by optimizing measurements over both subsystems equally, unlike asymmetric discord $D(A|B) = I(A:B) - J(A|B)$, which maximizes classical correlations over measurements on one subsystem (e.g., B), [27, 28].

Examples

- **Bell State:**For $\rho_{AB} = |\psi\rangle\langle\psi|$, where $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

- **Quantum mutual information:** $S(\rho_{AB}) = 0$ (pure state), $\rho_A = \rho_B = \frac{1}{2}I$, $S(\rho_A) = S(\rho_B) = 1$, so $I(\rho_{AB}) = 1 + 1 - 0 = 2$.
- **Maximal mutual information:** Measuring in $\sigma_z \otimes \sigma_z$, $p(0,0) = p(1,1) = \frac{1}{2}$, $H(p_A) = H(p_B) = H(p_{AB}) = 1$, so $C_S(\rho_{AB}) = 1 + 1 - 1 = 1$. This is optimal.
- **Symmetric quantum discord:** $Q_S(\rho_{AB}) = 2 - 1 = 1$ bit, reflecting the strong quantum correlations due to entanglement.
- **Classical State:**For $\rho_{AB} = \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11|$:
- $S(\rho_{AB}) = 1$, $\rho_A = \rho_B = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$, $S(\rho_A) = S(\rho_B) = 1$, so $I(\rho_{AB}) = 1 + 1 - 1 = 1$.
- Measuring in the computational basis, $p(0,0) = p(1,1) = \frac{1}{2}$, $C_S(\rho_{AB}) = 1$.
- $Q_S(\rho_{AB}) = 1 - 1 = 0$, indicating no quantum correlations, as expected for a classical state.
- **Bell-Diagonal State:**For $\rho_{AB} = \frac{1}{4}(I \otimes I + \sum_{k=1}^{3} c_k \sigma_k \otimes \sigma_k)$:
- $I(\rho_{AB})$ depends on the eigenvalues of ρ_{AB}.
- $C_S(\rho_{AB}) = \sum_{i=0}^{1} \frac{1+(-1)^i \kappa}{2} \log\left(\frac{1+(-1)^i \kappa}{2}\right)$, where $\kappa = \max(|c_1|, |c_2|, |c_3|)$, achieved by measuring in the Pauli basis of the largest $|c_k|$.
- $Q_S(\rho_{AB}) = I(\rho_{AB}) - C_S(\rho_{AB})$. For $c_1 = c_2 = c_3 = 1$, $C_S = 1$, and $I(\rho_{AB}) \approx 2$, so $Q_S \approx 1$.

2.2.5 Decoherence in Quantum Systems

Decoherence is a fundamental process in quantum mechanics that describes the loss of quantum coherence in a system due to its interaction with an external environment. It is a primary mechanism by which quantum systems transition from exhibiting quantum behaviors, such as superposition and entanglement, to classical behaviors, effectively explaining the emergence of classicality in a quantum world. Decoherence plays a critical role in quantum information theory, quantum computing, and the study of quantum-to-classical transitions [13].

Decoherence occurs when a quantum system, initially in a coherent superposition or entangled state, interacts with an environment, leading to the suppression

of off-diagonal elements in the system's density matrix in a preferred basis. This process does not involve energy dissipation (as in relaxation) but rather the entanglement of the system with environmental degrees of freedom, causing the system's quantum information to become distributed and inaccessible locally.

Mathematically, consider a quantum system S with density matrix ρ_S, initially in a pure state, such as a superposition $\alpha|0\rangle + \beta|1\rangle$. The system interacts with an environment E, initially in state ρ_E, under a total Hamiltonian $H = H_S + H_E + H_{SE}$, where H_{SE} is the system-environment interaction term. The joint system evolves unitarily via the operator $U = e^{-iHt/\hbar}$, but the reduced density matrix of the system, obtained by tracing out the environment, is:

$$\rho_S' = \mathrm{Tr}_E[U(\rho_S \otimes \rho_E)U^\dagger] \tag{2.34}$$

Decoherence manifests as the rapid decay of off-diagonal elements in ρ_S' in a specific basis, called the pointer basis, determined by the interaction H_{SE}. For a qubit, the density matrix:

$$\rho_S = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \beta\alpha^* & |\beta|^2 \end{pmatrix}, \tag{2.35}$$

may evolve to:

$$\rho_S' \approx \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{pmatrix}, \tag{2.36}$$

if the environment distinguishes the basis states $|0\rangle$ and $|1\rangle$, indicating a loss of coherence.

Physical Significance

Decoherence is crucial for understanding several phenomena:

- **Quantum-to-Classical Transition:** It explains why macroscopic systems exhibit classical behavior, as their interactions with large environments suppress quantum superpositions, favoring pointer states that are robust to environmental monitoring [29].
- **Measurement Problem:** Decoherence provides a framework for the apparent collapse of the wave function during measurement. By entangling the system with a measuring apparatus (an environment), it selects definite outcomes without requiring a physical collapse, though some argue it doesn't fully resolve the measurement problem [15].

- **Quantum Computing:** Decoherence is the primary obstacle to maintaining quantum states in qubits, causing errors in superpositions and entanglement. The coherence time, determines how long a qubit can perform computations before losing its quantum information [16].

Mechanisms of Decoherence

Decoherence arises from various interactions between the system and its environment, each described through different formalisms:

- **Density-Matrix Approach:** The most common description, where off-diagonal elements of the system's density matrix decay due to entanglement with the environment. For a qubit, if the environment states $|E_0\rangle$ and $|E_1\rangle$ become orthogonal ($\langle E_0|E_1\rangle \approx 0$), the off-diagonal terms vanish, converting a pure state to a mixed state.(see eq. (5) of [14]).

- **Loss of Interference:** Decoherence suppresses quantum interference terms, which are responsible for phenomena like interference patterns in the double-slit experiment. The probability of transitioning between states includes interference terms before decoherence:

$$\text{prob}_{\text{before}}(\psi \to \phi) = |\langle\psi|\phi\rangle|^2 = \sum_i |\psi_i^* \phi_i|^2 + \sum_{i \neq j} \psi_i^* \psi_j \phi_j^* \phi_i. \tag{2.37}$$

  After decoherence, these terms vanish:

$$\text{prob}_{\text{after}}(\psi \to \phi) \approx \sum_i |\psi_i^* \phi_i|^2. \tag{2.38}$$

- **Dirac Notation:** The system's state evolves as $|\psi\rangle|E\rangle \to \sum_i \psi_i |i\rangle|E_i\rangle$. Orthogonality of environmental states ($\langle E_i|E_j\rangle \approx \delta_{ij}$) causes decoherence by eliminating superposition.

- **Operator-Sum Representation:** Decoherence is modeled as a non-unitary process using Kraus operators $\hat{W}_k$:

$$\rho_S(t) = \sum_l \hat{W}_k(t) \rho_S(0) \hat{W}_k^\dagger(t), \quad \sum_l \hat{W}_k^\dagger(t) \hat{W}_k(t) = \hat{I}_S, \tag{2.39}$$

  where $\hat{W}_k(t) \equiv \sqrt{p_{i_k}} = \langle E_{j_k}| U(t) |E_{j_k}\rangle$ represent environmental effects (see the eq. (11) of [14]).

- **Semigroup Approach:** The system's dynamics are described by a master equation, such as the Lindblad equation, which includes a decoherence term $L_D[\rho_S]$:

$$\dot{\rho}_S(t) = -\frac{i}{\hbar}[\tilde{H}_S, \rho_S(t)] + L_D[\rho_S(t)]. \tag{2.40}$$

  The Lindblad term models noise, causing non-unitary evolution[1].

These mechanisms highlight that decoherence is driven by the environment's ability to "monitor" the system, entangling it with external degrees of freedom and selecting a preferred basis.

Examples

- **Double-Slit Experiment:**In the double-slit experiment, an electron in a superposition of passing through two slits produces an interference pattern. If stray particles interact with the electron, entangling its path with the environment, the interference terms vanish, and the pattern becomes classical, resembling particles passing through one slit or the other .

- **Schrödinger's Cat:**The hypothetical superposition of a cat being "alive" and "dead" decoheres rapidly due to interactions with the environment (e.g., air molecules, photons). The pointer states ("alive" or "dead") are stable, making the superposition unobservable, and the cat appears in a definite state.

- **Qubit in a Quantum Computer:**A qubit in a superposition $\alpha|0\rangle + \beta|1\rangle$ may decohere due to electromagnetic noise or thermal vibrations. For a superconducting qubit, the density matrix's off-diagonal elements decay within microseconds, turning the state into a classical mixture, disrupting quantum computations [16].

- **Entangled Qubits:**A Bell state like $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ under dephasing becomes a classical mixture, reducing its symmetric quantum discord to zero as all correlations become extractable classically.

[1]The Lindblad equation can be used to describe the generation of entanglement between two initially unentangled qubits. (see [30]).

Relation to Quantum Discord

Decoherence directly impacts quantum correlations, such as those measured by symmetric quantum discord ($Q_S(\rho_{AB}) = I(\rho_{AB}) - C_S(\rho_{AB})$). As decoherence entangles the system with the environment, it reduces the quantum mutual information $I(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$, while classical correlations $C_S(\rho_{AB})$ may increase as the state becomes more classical. For example, a Bell state under dephasing transitions to a classical mixture, reducing Q_S to zero, as all correlations become extractable classically. This interplay highlights decoherence's role in diminishing the quantumness of correlations, affecting tasks like quantum cryptography and computing.

The Effect of Decoherence on Entangled Systems

Research consistently indicates that when two entangled systems undergo decoherence, typically through independent interactions with their respective environments, their entanglement is generally lost. This occurs because each system becomes entangled with its environment, effectively transferring the original entanglement to the environment. For instance, consider two qubits, A and B, initially in an entangled state. If A interacts with environment EA and B with EB, the total state evolves to include correlations with EA and EB. Tracing out the environments (a standard procedure in quantum mechanics to find the reduced density matrix of A and B) often results in a mixed state for A and B, which is typically separable, meaning no entanglement remains.

This process is well-documented in quantum information theory. For example, studies on decoherence channels, such as amplitude damping or phase damping, show that entanglement measures like concurrence decrease over time as decoherence parameters increase. A notable phenomenon, *entanglement sudden death*, describes cases where entanglement vanishes completely after a finite time due to environmental interactions, as discussed in [31, 32, 33].

While the general trend is loss of entanglement, there are exceptions under controlled conditions. The Nature Physics article *Protecting entanglement from decoherence using weak measurement and quantum measurement reversal* [34], proposes and demonstrates a scheme using weak measurements and quantum measurement reversal to protect entanglement, effectively circumventing entanglement sudden death. This suggests that, with specific interventions, entanglement can survive decoherence, but such methods are not part of the natural, uncontrolled

decoherence process.

In conclusion, the decoherence of two entangled systems generally implies the loss of their entanglement, driven by the systems becoming entangled with their environments. While exceptions exist under controlled conditions with protective techniques, the standard scenario without intervention leads to entanglement decay.

Chapter 3

Quantum Scalar Field Theory and Entanglement Entropy in the Schrödinger Picture

Quantum Field Theory (QFT) provides a framework for describing systems with infinite degrees of freedom, such as quantum scalar fields. While the Heisenberg picture is commonly used, the Schrödinger picture offers a powerful alternative for studying quantum correlations, particularly entanglement entropy. In the Schrödinger picture, quantum states evolve over time as wave functionals, while operators remain time-independent. This formalism is especially suited for computing entanglement entropy, as it allows direct manipulation of field configurations to derive probability distributions and reduced density matrices. This chapter combines the technical derivations of the ground state wave functional with the entanglement entropy calculations for the vacuum state of a free real scalar field. We present a comprehensive treatment of quantum scalar field theory in the Schrödinger representation, covering the ground state wave functional, annihilation and creation operators, probability distributions, and their application to entanglement entropy using the replica trick. The material is tailored to support the thesis's focus on quantifying entanglement between spatial regions $\Omega \subset \mathbb{R}^d$ and its complement $\overline{\Omega}$. We will use the units $c = \hbar = 1$.

3.1 Quantum Scalar Field Theory: Basics

A free real scalar field $\varphi(x)$ in $d+1$-dimensional Minkowski spacetime is described by the action

$$S[\varphi] = \frac{1}{2} \int d^{d+1}x \left(\partial_\mu \varphi \partial^\mu \varphi - m^2 \varphi^2 \right), \tag{3.1}$$

where m is the mass of the field, and the metric convention is $\eta_{\mu\nu} = \text{diag}(1, -1, \ldots, -1)$. The Euler-Lagrange equation derived from this action is the Klein-Gordon equation:

$$(\partial_\mu \partial^\mu + m^2)\varphi(x) = 0. \tag{3.2}$$

In the canonical quantization approach, the field $\varphi(x)$ and its conjugate momentum $\pi(x) = \partial_0 \varphi(x)$ are promoted to operators $\hat{\phi}$ and $\hat{\pi}$ respectively, satisfying the equal-time commutation relations:

$$[\hat{\phi}(t, \mathbf{x}), \hat{\pi}(t, \mathbf{y})] = i\delta^{(d)}(\mathbf{x} - \mathbf{y}), \;\; [\hat{\phi}(t, \mathbf{x}), \hat{\phi}(t, \mathbf{y})] = [\hat{\pi}(t, \mathbf{x}), \hat{\pi}(t, \mathbf{y})] = 0. \tag{3.3}$$

The Hamiltonian is obtained via the Legendre transform:

$$H = \int d^d\mathbf{x} \left[\frac{1}{2}\hat{\pi}^2 + \frac{1}{2}(\nabla\hat{\phi})^2 + \frac{1}{2}m^2\hat{\phi}^2 \right]. \tag{3.4}$$

The quantum field $\hat{\phi}(x)$ that solves the Klein-Gordon equation can be expanded as:

$$\hat{\varphi}(x) = \frac{1}{(2\pi)^{d/2}} \int \frac{d^d\mathbf{k}}{\sqrt{2\omega_\mathbf{k}}} \left(a(\mathbf{k}) e^{-ik\cdot x} + a^\dagger(\mathbf{k}) e^{ik\cdot x} \right)_{k_0 = \omega_\mathbf{k}}, \tag{3.5}$$

where $\omega_\mathbf{k} \equiv \sqrt{m^2 + \mathbf{k}^2}$ and the operators $a_\mathbf{k}$ and $a^\dagger_\mathbf{k}$ satisfy the commutation relations:

$$\begin{aligned} \left[a(\mathbf{k}), a^\dagger(\mathbf{k}') \right] &= \delta(\mathbf{k} - \mathbf{k}'), \\ [a(\mathbf{k}), a(\mathbf{k}')] &= \left[a^\dagger(\mathbf{k}), a^\dagger(\mathbf{k}') \right] = 0. \end{aligned} \tag{3.6}$$

The Hamiltonian operator, can be rewritten in terms of $a_\mathbf{k}$ and $a^\dagger_\mathbf{k}$ as:

$$H = \frac{1}{2} \int d^d\mathbf{k}\, \omega_\mathbf{k} \left(a^\dagger(\mathbf{k}) a(\mathbf{k}) + a(\mathbf{k}) a^\dagger(\mathbf{k}) \right). \tag{3.7}$$

In this way, the vacuum state is defined by the action of the annihilation operator $a(\mathbf{k})$:

$$a(\mathbf{k}) \, |0\rangle = 0. \tag{3.8}$$

In the Heisenberg picture, the field operators evolve, and the state is fixed. In contrast, in the Schrödinger picture, the operators $\hat{\phi}(\mathbf{x})$ and $\hat{\pi}(\mathbf{x})$ are time-independent, and the state evolves according to the Schrödinger equation:

$$i\frac{\partial}{\partial t}|\Psi(t)\rangle = H|\Psi(t)\rangle. \tag{3.9}$$

Since the field $\hat{\phi}(\mathbf{x})$ is defined over a continuous spatial domain, the quantum state in the Schrödinger picture is described by a wave functional $\Psi[\phi, t]$, where $\phi(\mathbf{x})$ is a classical field configuration, and $\Psi[\phi, t] = \langle \phi | \Psi(t) \rangle$ is the probability amplitude for the field to be in the configuration $\phi(\mathbf{x})$ at time t.

3.2 Functional Formalism in the Schrödinger Picture

In the Schrödinger picture, the field operator $\hat{\phi}(\mathbf{x})$ acts as a multiplication operator, and the conjugate momentum $\hat{\pi}(\mathbf{x})$ is represented as a functional derivative:

$$\hat{\phi}(\mathbf{x})\Psi[\phi] = \phi(\mathbf{x})\Psi[\phi], \quad \hat{\pi}(\mathbf{x})\Psi[\phi] = -i\frac{\delta}{\delta\phi(\mathbf{x})}\Psi[\phi]. \tag{3.10}$$

These operators satisfy the commutation relations:

$$[\hat{\phi}(\mathbf{x}), \hat{\pi}(\mathbf{y})] = i\delta^{(d)}(\mathbf{x} - \mathbf{y}). \tag{3.11}$$

The Hamiltonian in the Schrödinger picture becomes:

$$H = \int d^d\mathbf{x} \left[-\frac{1}{2}\frac{\delta^2}{\delta\phi(\mathbf{x})^2} + \frac{1}{2}(\nabla\phi(\mathbf{x}))^2 + \frac{1}{2}m^2\phi(\mathbf{x})^2 \right]. \tag{3.12}$$

The time-dependent Schrödinger equation for the wave functional is:

$$i\frac{\partial}{\partial t}\Psi[\phi, t] = \int d^d\mathbf{x} \left[-\frac{1}{2}\frac{\delta^2}{\delta\phi(\mathbf{x})^2} + \frac{1}{2}(\nabla\phi(\mathbf{x}))^2 + \frac{1}{2}m^2\phi(\mathbf{x})^2 \right] \Psi[\phi, t]. \tag{3.13}$$

For a stationary state, such as the ground state, we solve the time-independent Schrödinger equation:

$$H\Psi[\phi] = E\Psi[\phi]. \tag{3.14}$$

3.3 Ground State Wave Functional

The ground state wave functional $\Psi_0[\phi]$ for the free scalar field can be derived by solving the time-independent Schrodinger equation (3.14), for the lowest energy state E_0. This approach directly determines the functional form of $\Psi_0[\phi]$ using the Hamiltonian for the scalar field in the Schrodinger picture, given by (3.12).

We seek the ground state wave functional $\Psi_0[\phi]$. For a free field theory, which describes a system of uncoupled harmonic oscillators in momentum space, the

ground state is expected to be of Gaussian form. We therefore make the ansatz

$$\Psi_0[\phi] = \mathcal{N} \exp\left(-\frac{1}{2}\int d^d\mathbf{x} \int d^d\mathbf{y}\phi(\mathbf{x})g(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\right), \tag{3.15}$$

where $\mathcal{N}$ is a normalization constant and $g(\mathbf{x},\mathbf{y})$ is a symmetric kernel, $g(\mathbf{x},\mathbf{y}) = g(\mathbf{y},\mathbf{x})$, to be determined. This can be written more compactly by defining an operator O such that $[O\phi](\mathbf{x}) = \int d^d\mathbf{y}\, g(\mathbf{x},\mathbf{y})\phi(\mathbf{y})$. The operator O is assumed to be linear, positive, and self-adjoint. The ansatz becomes

$$\Psi_0[\phi] = \mathcal{N} \exp\left(-\frac{1}{2}\int d^d\mathbf{x}\, \phi(\mathbf{x})[O\phi](\mathbf{x})\right). \tag{3.16}$$

To apply the Hamiltonian to this ansatz, we need the functional derivatives of $\Psi_0[\phi]$

$$\frac{\delta\Psi_0[\phi]}{\delta\phi(\mathbf{x})} = -\left(\int d^d\mathbf{y}\, g(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\right)\Psi_0[\phi] = -[O\phi](\mathbf{x})\Psi_0[\phi]. \tag{3.17}$$

The second functional derivative is

$$\frac{\delta^2\Psi_0[\phi]}{\delta\phi(\mathbf{x})^2} = \frac{\delta}{\delta\phi(\mathbf{x})}(-[O\phi](\mathbf{x})\Psi_0[\phi]) = -g(\mathbf{x},\mathbf{x})\Psi_0[\phi] + ([O\phi](\mathbf{x}))^2\Psi_0[\phi]. \tag{3.18}$$

Here, $g(\mathbf{x},\mathbf{x})$ is a shorthand for the kernel $g(\mathbf{x},\mathbf{y})$ evaluated at $\mathbf{y} = \mathbf{x}$, it relates to the trace of the operator O when integrated.

Now, we evaluate the action of each term in the Hamiltonian (3.12) on $\Psi_0[\phi]$:

The potential energy term

$$\left(\frac{1}{2}\int d^d\mathbf{x}\, m^2\phi(\mathbf{x})^2\right)\Psi_0[\phi] = \left(\frac{1}{2}\int d^d\mathbf{x}\, \phi(\mathbf{x})(m^2\phi)(\mathbf{x})\right)\Psi_0[\phi]. \tag{3.19}$$

The gradient term, after an integration by parts (assuming $\phi(\mathbf{x})$ vanishes at spatial infinity)

$$\left(\frac{1}{2}\int d^d\mathbf{x}\, (\nabla\phi(\mathbf{x}))^2\right)\Psi_0[\phi] = \left(\frac{1}{2}\int d^d\mathbf{x}\, \phi(\mathbf{x})(-\nabla^2\phi)(\mathbf{x})\right)\Psi_0[\phi]. \tag{3.20}$$

The functional kinetic term

$$\int d^d\mathbf{x}\left(-\frac{1}{2}\frac{\delta^2}{\delta\phi(\mathbf{x})^2}\right)\Psi_0[\phi] = -\frac{1}{2}\int d^d\mathbf{x}\left[-g(\mathbf{x},\mathbf{x}) + ([O\phi](\mathbf{x}))^2\right]\Psi_0[\phi] \\ = \left(\frac{1}{2}\int d^d\mathbf{x}\, g(\mathbf{x},\mathbf{x}) - \frac{1}{2}\int d^d\mathbf{x}\,([O\phi](\mathbf{x}))^2\right)\Psi_0[\phi]. \tag{3.21}$$

The term $\int d^d\mathbf{x}\, g(\mathbf{x},\mathbf{x})$ is the trace of the operator O, denoted $\mathrm{Tr}(O)$. The term $\int d^d\mathbf{x}\,([O\phi](\mathbf{x}))^2$ can be written as

$$\int d^d\mathbf{x}\,\phi(\mathbf{x})\left[O^2\phi\right](\mathbf{x}) \equiv \int d^d\mathbf{x}\,\phi(\mathbf{x})\int d^d\mathbf{y}\,\phi(\mathbf{y})\int d^d\mathbf{z}\, g(\mathbf{x},\mathbf{z})g(\mathbf{z},\mathbf{y}). \tag{3.22}$$

Substituting these into the Schrodinger equation $H\Psi_0[\phi] = E_0\Psi_0[\phi]$

$$E_0\Psi_0[\phi] = \left[\frac{1}{2}\mathrm{Tr}(O) - \frac{1}{2}\int d^d\mathbf{x}\,\phi(\mathbf{x})[O^2\phi](\mathbf{x}) + \frac{1}{2}\int d^d\mathbf{x}\,\phi(\mathbf{x})[-\nabla^2\phi](\mathbf{x})\right]\Psi_0[\phi] \\ + \frac{1}{2}\left[\int d^d\mathbf{x}\,\phi(\mathbf{x})[m^2\phi](\mathbf{x})\right]\Psi_0[\phi]. \tag{3.23}$$

For this equation to hold for all field configurations $\phi(\mathbf{x})$, the terms quadratic in $\phi(\mathbf{x})$ on the right-hand side must sum to zero when acting on $\Psi_0[\phi]$ (as E_0 is a constant), and the constant term must be equal to E_0.

Equating the ϕ-dependent quadratic forms

$$-\frac{1}{2}\int d^d\mathbf{x}\phi(\mathbf{x})\left[O^2\phi\right](\mathbf{x}) + \frac{1}{2}\int d^d\mathbf{x}\phi(\mathbf{x})\left[-\nabla^2\phi\right](\mathbf{x}) + \frac{1}{2}\int d^d\mathbf{x}\phi(\mathbf{x})\left[m^2\phi\right](\mathbf{x}) = 0. \tag{3.24}$$

This implies

$$\int d^d\mathbf{x}\,\phi(\mathbf{x})\left[O^2\phi\right](\mathbf{x}) = \int d^d\mathbf{x}\,\phi(\mathbf{x})(-\nabla^2 + m^2)\phi(\mathbf{x}). \tag{3.25}$$

Thus, the operator O^2 must be equal to $(-\nabla^2 + m^2)$. Therefore, we can take

$$O = \sqrt{-\nabla^2 + m^2}. \tag{3.26}$$

This operator is the fractional Laplacian operator discussed in Appendix A, for $m = 0$.

The ground state energy E_0 is then given by the ϕ-independent term

$$E_0 = \frac{1}{2}\mathrm{Tr}(O) = \frac{1}{2}\mathrm{Tr}\left(\sqrt{-\nabla^2 + m^2}\right). \tag{3.27}$$

This energy is the sum of the zero-point energies of all the normal modes of the field, $\frac{1}{2}\sum_{\mathbf{k}} \omega_{\mathbf{k}}$, where $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$. This sum is divergent and requires regularization.

The ground state wave functional is therefore:

$$\Psi_0[\phi] = \mathcal{N} \exp\left(-\frac{1}{2}\int d^d\mathbf{x}\, \phi(\mathbf{x})\sqrt{-\nabla^2 + m^2}\phi(\mathbf{x})\right). \tag{3.28}$$

Finally, the normalization constant $\mathcal{N}$ is determined by the condition $\int [D\phi]|\Psi_0[\phi]|^2 = 1$:

$$\mathcal{N}^2 \int [D\phi] \exp\left(-\int d^d\mathbf{x}\, \phi(\mathbf{x})[O\phi](\mathbf{x})\right) = 1. \tag{3.29}$$

The Gaussian functional integral is given by $[\det O]^{-1/2}$, up to constants related to the definition of the functional measure

$$\int [D\phi] \exp\left(-\int d^d\mathbf{x}\, \phi(\mathbf{x})[O\phi](\mathbf{x})\right) = [\det O]^{-1/2}. \tag{3.30}$$

Thus,

$$\mathcal{N}^2[\det O]^{-1/2} = 1 \Longrightarrow \mathcal{N} = [\det O]^{1/4}. \tag{3.31}$$

So the normalized ground state wave functional is

$$\Psi_0[\phi] = [\det O]^{1/4} \exp\left(-\frac{1}{2}\int d^d\mathbf{x}\, \phi(\mathbf{x})[O\phi](\mathbf{x})\right), \tag{3.32}$$

where $O \equiv \sqrt{-\nabla^2 + m^2}$.

Due to the technical challenges involved in obtaining the kernel integral representation of the fractional operator with mass, we choose to work with the massless fractional Laplacian operator $O \equiv \sqrt{-\nabla^2}$ (see (A.18)). The properties of this fractional operator are discussed in more detail in Appendix A.

3.4 Annihilation and Creation Operators

In the Schrödinger picture, annihilation and creation operators are defined as functional operators acting on the wave functional. The annihilation operator $\hat{a}(\mathbf{x})$ is constructed to satisfy $\hat{a}(\mathbf{x})\Psi_0[\phi] = 0$, ensuring that the ground state is annihilated. We define:

$$\begin{aligned}\hat{a}(\mathbf{x}) \equiv & \frac{1}{\sqrt{\gamma}}\mathcal{O}^{-1/2}\left[\mathcal{O}\phi(\mathbf{x}) + i\pi(\mathbf{x})\right] \\ = & \frac{1}{\sqrt{\gamma}}\left[\mathcal{O}^{1/2}\phi(\mathbf{x}) + \mathcal{O}^{-1/2}\frac{\delta}{\delta\phi(\mathbf{x})}\right],\end{aligned} \tag{3.33}$$

where $\mathcal{O}^{1/2}$ is the fractional operator with Fourier transform $\omega_{\mathbf{k}}^{1/2}$, and γ is a normalization constant. The creation operator is the Hermitian conjugate:

$$\begin{aligned}\hat{a}^\dagger(\mathbf{x}) \equiv & \frac{1}{\sqrt{\gamma}}\mathcal{O}^{-1/2}\left[\mathcal{O}\phi(\mathbf{x}) - i\pi(\mathbf{x})\right] \\ = & \frac{1}{\sqrt{\gamma}}\left[\mathcal{O}^{1/2}\phi(\mathbf{x}) - \mathcal{O}^{-1/2}\frac{\delta}{\delta\phi(\mathbf{x})}\right].\end{aligned} \tag{3.34}$$

We can verify that the wave functional $\Psi_0[\phi]$ (3.32) satisfies the equation

$$\hat{a}(\mathbf{x})\Psi_0[\phi] = \frac{1}{\sqrt{\gamma}}\left[\mathcal{O}^{1/2}\phi(\mathbf{x}) + \mathcal{O}^{-1/2}\frac{\delta}{\delta\phi(\mathbf{x})}\right]\Psi_0[\phi] = 0. \tag{3.35}$$

For this, we must take into account that

$$\begin{aligned}\frac{\delta\Psi_0[\phi]}{\delta\phi(\mathbf{x})} = & -\frac{1}{2}\mathcal{N}\exp\left(-\frac{1}{2}\int d^d\mathbf{x}'\,\phi(\mathbf{x}')\mathcal{O}\phi(\mathbf{x}')\right) \\ & \times\int d^d\mathbf{x}'\left[\frac{\delta\phi(\mathbf{x}')}{\delta\phi(\mathbf{x})}\,\mathcal{O}\phi(\mathbf{x}') + \phi(\mathbf{x}')\mathcal{O}\frac{\delta\phi(\mathbf{x}')}{\delta\phi(\mathbf{x})}\right] \\ = & -\Psi_0[\phi]\left[\int d^d\mathbf{x}'\delta(\mathbf{x}'-\mathbf{x})\,\mathcal{O}\phi(\mathbf{x}')\right] \\ = & -\Psi_0[\phi]\mathcal{O}\phi(\mathbf{x}).\end{aligned} \tag{3.36}$$

In this way, we can verify that the wave functional (3.32) satisfies this equation. The commutation relations are:

$$[\hat{a}(\mathbf{x}), \hat{a}^\dagger(\mathbf{y})] = \delta^{(d)}(\mathbf{x}-\mathbf{y}), \tag{3.37}$$

with $\gamma = 2$ to ensure proper normalization. These operators define the vacuum and are essential for constructing excited states and computing correlation functions.

3.5 Reduced Probabilities and the Reduced Density Matrix

In Quantum Field Theory (QFT), the reduced probability density provides insight into the field configurations within a spatial region Ω, independent of the complementary region $\overline{\Omega}$. For the ground state of a free scalar field, we compute the probability of observing a field configuration $g_\Omega(\mathbf{x})$ on Ω, when measuring the field ϕ on Ω, irrespective of the field value on $\overline{\Omega}$. These regions are depicted in Fig. 3.1.

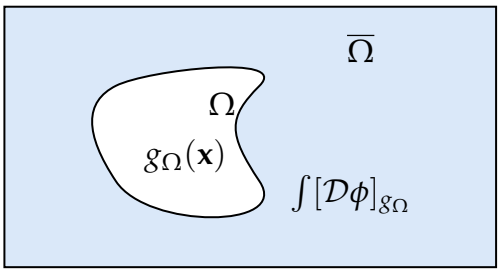


Figure 3.1

The reduced probability density is defined as:

$$P_{0\Omega}[g_\Omega] = \int [\mathcal{D}\phi]_{g_\Omega} P_0[\phi], \tag{3.38}$$

where $P_0[\phi]$ is the vacuum state probability distribution, and the measure $[\mathcal{D}\phi]_{g_\Omega}$ integrates over all field configurations $\phi(\mathbf{x})$ satisfying:

$$\phi(\mathbf{x}) = g_\Omega(\mathbf{x}), \quad \mathbf{x} \in \Omega. \tag{3.39}$$

The ground state probability distribution is given by:

$$P_0[\phi] = |\Psi_0[\phi]|^2 = \left[\det\left(\mu^{-1}\mathcal{O}\right)\right]^{1/2} \exp\left(-\int d^d\mathbf{x}\, \phi(\mathbf{x})[\mathcal{O}\phi](\mathbf{x})\right), \tag{3.40}$$

where $\mathcal{O} = \sqrt{-\nabla^2}$ is the fractional operator, and μ is a dimensional regular-

ization constant. This Gaussian distribution reflects the quadratic nature of the Hamiltonian and is normalized:

$$\int [\mathcal{D}\phi]\ P_0[\phi] = 1. \tag{3.41}$$

To evaluate the functional integral in (3.38), we employ the saddle point approximation. We define the functional:

$$\mathcal{W}[\phi] \equiv \int d^d\mathbf{x}'\ \phi(\mathbf{x}')[\mathcal{O}\phi](\mathbf{x}') \tag{3.42}$$

and perform a Taylor series expansion of the functional $\mathcal{W}[\phi]$ around the field configuration ϕ'. The expansion is given by

$$\begin{aligned}\mathcal{W}[\phi] =& \mathcal{W}[\phi'] + \int d^d\mathbf{x}\ \delta\phi(\mathbf{x})\ \frac{\delta\mathcal{W}[\phi]}{\delta\phi(\mathbf{x})}\bigg|_{\phi=\phi'} \\ &+ \frac{1}{2}\int d^d\mathbf{x}'\ \delta\phi(\mathbf{x}') \int d^d\mathbf{x}\ \delta\phi(\mathbf{x})\ \frac{\delta^2\mathcal{W}[\phi]}{\delta\phi(\mathbf{x}')\delta\phi(\mathbf{x})}\bigg|_{\phi=\phi'} + \cdots,\end{aligned} \tag{3.43}$$

where $\delta\phi = \phi - \phi'$. Since $\mathcal{W}[\phi]$ is a quadratic functional, all terms beyond the second order vanish, simplifying the expression to

$$\begin{aligned}\mathcal{W}[\phi] =& \mathcal{W}[\phi'] + \int d^d\mathbf{x}\ \delta\phi(\mathbf{x})\ \frac{\delta\mathcal{W}[\phi]}{\delta\phi(\mathbf{x})}\bigg|_{\phi=\phi'} \\ &+ \frac{1}{2}\int d^d\mathbf{x}'\ \delta\phi(\mathbf{x}') \int d^d\mathbf{x}\ \delta\phi(\mathbf{x})\ \frac{\delta^2\mathcal{W}[\phi]}{\delta\phi(\mathbf{x}')\delta\phi(\mathbf{x})}\bigg|_{\phi=\phi'}.\end{aligned} \tag{3.44}$$

For convenience, we choose $\phi' = \phi_c$, where ϕ_c is the stationary point defined by

$$\frac{\delta\mathcal{W}[\phi]}{\delta\phi(\mathbf{x})}\bigg|_{\phi=\phi_c} = 0. \tag{3.45}$$

With this choice, the functional reduces to

$$\mathcal{W}[\phi] = \mathcal{W}[\phi_c] + \frac{1}{2}\int d^d\mathbf{x}'\ \delta\phi(\mathbf{x}') \int d^d\mathbf{x}\ \delta\phi(\mathbf{x})\ \mathcal{O}(\mathbf{x}',\mathbf{x}), \tag{3.46}$$

where the kernel $\mathcal{O}(\mathbf{x}',\mathbf{x})$ is given by

$$\mathcal{O}(\mathbf{x}',\mathbf{x}) = \frac{\delta^2\mathcal{W}[\phi]}{\delta\phi(\mathbf{x}')\delta\phi(\mathbf{x})}\bigg|_{\phi=\phi_c}. \tag{3.47}$$

Now, to compute the stationary point ϕ_c, we take the functional derivative

$$\begin{aligned}\frac{\delta}{\delta\phi(\mathbf{x})}\mathcal{W}[\phi] =& \int d^d\mathbf{x}' \, \frac{\delta\phi(\mathbf{x}')}{\delta\phi(\mathbf{x})}[\mathcal{O}\phi](\mathbf{x}') + \int d^d\mathbf{x}' \, \phi(\mathbf{x}')\frac{\delta}{\delta\phi(\mathbf{x})}[\mathcal{O}\phi](\mathbf{x}') \\ =& \int d^d\mathbf{x}' \, \delta(\mathbf{x}'-\mathbf{x})\mathcal{O}\phi(\mathbf{x}') + \int d^d\mathbf{x}' \, \phi(\mathbf{x}')\mathcal{O}\delta(\mathbf{x}'-\mathbf{x}) \\ =& 2\mathcal{O}\phi(\mathbf{x}). \end{aligned} \tag{3.48}$$

Setting the variation to zero yields the nonlocal Poisson problem for the saddle point field $\phi_c := \phi_\Omega$:

$$\begin{aligned}[\mathcal{O}\phi_\Omega](\mathbf{x}) &= 0, \qquad \mathbf{x} \in \overline{\Omega} \\ \phi_\Omega(\mathbf{x}) &= g_\Omega(\mathbf{x}), \quad \mathbf{x} \in \Omega, \end{aligned} \tag{3.49}$$

The solution to this problem is:

$$\phi_\Omega(\mathbf{x}) = \begin{cases} \int_\Omega d^d\mathbf{y}\mathcal{P}(\mathbf{x},\mathbf{y})g_\Omega(\mathbf{y}), & \mathbf{x} \in \overline{\Omega} \\ g_\Omega(\mathbf{x}), & \mathbf{x} \in \Omega. \end{cases} \tag{3.50}$$

where $\mathcal{P}(\mathbf{x},\mathbf{y})$ is the Poisson kernel, defined in terms of the kernel $K(\mathbf{x}-\mathbf{y})$ as specified in equation (A.47).

To proceed, we perform a change of variables, $\phi(\mathbf{x}) = \phi_\Omega(\mathbf{x}) + \varepsilon(\mathbf{x})$, where $\varepsilon(\mathbf{x}) = \delta\phi(\mathbf{x})$ represents fluctuations around the saddle point. Substituting into the probability distribution, we obtain:

$$P_{0\Omega}[g_\Omega] = P_0[\phi_\Omega]\int [\mathcal{D}\varepsilon]_{\overline{\Omega}} \exp\left(-\int_{\overline{\Omega}} d^d\mathbf{x} \, \varepsilon(\mathbf{x})[\mathcal{O}\varepsilon](\mathbf{x})\right), \tag{3.51}$$

The measure $[\mathcal{D}\varepsilon]_\Omega$ integrates over fields $\varepsilon(\mathbf{x})$ satisfying:

$$\varepsilon(\mathbf{x}) = 0, \;\; \mathbf{x} \in \Omega. \tag{3.52}$$

Evaluating the Gaussian integral over $\varepsilon(\mathbf{x})$, we find:

$$\begin{aligned} P_{0\Omega}[g_\Omega] &= \left[\det_{\overline{\Omega}}\left(\mu^{-1}\mathcal{O}\right)\right]^{-1/2} P_0[\phi_\Omega] \\ &= \left[\frac{\det\left(\mu^{-1}\mathcal{O}\right)}{\det_{\overline{\Omega}}\left(\mu^{-1}\mathcal{O}\right)}\right]^{1/2} \exp\left(-\int d^d\mathbf{x} \, \phi_\Omega(\mathbf{x})[\mathcal{O}\phi_\Omega](\mathbf{x})\right). \end{aligned} \tag{3.53}$$

This expression quantifies the reduced probability density, incorporating the determinant ratio to account for the functional integration over the constrained region Ω.

Similarly, we obtain

$$\begin{aligned} P_{0\overline{\Omega}}[g_{\overline{\Omega}}] &= \left[\det_{\Omega}\left(\mu^{-1}\mathcal{O}\right)\right]^{-1/2} P_0[\phi_{\overline{\Omega}}] \\ &= \left[\frac{\det\left(\mu^{-1}\mathcal{O}\right)}{\det_{\Omega}\left(\mu^{-1}\mathcal{O}\right)}\right]^{1/2} \exp\left(-\int d^d\mathbf{x}\, \phi_{\overline{\Omega}}(\mathbf{x})[\mathcal{O}\phi_{\overline{\Omega}}](\mathbf{x})\right). \end{aligned} \tag{3.54}$$

3.6 Replica Trick for the Shannon Entropy for the Vacuum

The Shannon entropy quantifies the uncertainty in the field configurations of the vacuum state restricted to a spatial region $\Omega \subset \mathbb{R}^d$ for a free scalar field in Quantum Field Theory (QFT). For the ground state, the reduced probability distribution $P_{0\Omega}[g_\Omega]$ describes the likelihood of measuring field configurations $g_\Omega(\mathbf{x})$ for $\mathbf{x} \in \Omega$. The Shannon entropy over this region is defined as:

$$S_0[\Omega] = -\int [\mathcal{D}g_\Omega]\, P_{0\Omega}[g_\Omega]\, \ln P_{0\Omega}[g_\Omega]. \tag{3.55}$$

The reduced probability distribution is given by (3.53):

$$P_{0\Omega}[g_\Omega] = \left[\frac{\det\left(\mu^{-1}\mathcal{O}\right)}{\det_{\overline{\Omega}}\left(\mu^{-1}\mathcal{O}\right)}\right]^{1/2} \exp\left(-\int d^d\mathbf{x}\phi_\Omega(\mathbf{x})[\mathcal{O}\phi_\Omega](\mathbf{x})\right). \tag{3.56}$$

The functional integral in (3.55) is challenging due to the logarithmic term, so we employ the replica trick to compute it. The replica trick involves evaluating the replicated partition function

$$Z_{0n}^{\Omega} = \int [\mathcal{D}g_\Omega](P_{0\Omega}[g_\Omega])^n \tag{3.57}$$

and obtaining the Shannon entropy via

$$S_0[\Omega] = -\lim_{n\to 1}\frac{\partial}{\partial n}\ln Z_{0n}^{\Omega} = -\frac{\partial}{\partial n}\ln\left[\int [\mathcal{D}g_\Omega](P_{0\Omega}[g_\Omega])^n\right]\Bigg|_{n=1}. \tag{3.58}$$

To compute Z^{Ω}_{0n}, we express $(P_{0\Omega}[g_\Omega])^n$:

$$(P_{0\Omega}[g_\Omega])^n = \left[\frac{\det\left(\mu^{-1}\mathcal{O}\right)}{\det_{\overline{\Omega}}\left(\mu^{-1}\mathcal{O}\right)}\right]^{n/2} \exp\left(-n\int d^d\mathbf{x}\phi_\Omega(\mathbf{x})[\mathcal{O}\phi_\Omega](\mathbf{x})\right). \tag{3.59}$$

Thus, the replicated partition function becomes

$$Z^{\Omega}_{0n} = \left[\frac{\det\left(\mu^{-1}\mathcal{O}\right)}{\det_{\overline{\Omega}}\left(\mu^{-1}\mathcal{O}\right)}\right]^{n/2} \int[\mathcal{D}g_\Omega]\exp\left(-n\int d^d\mathbf{x}\phi_\Omega(\mathbf{x})[\mathcal{O}\phi_\Omega](\mathbf{x})\right). \tag{3.60}$$

To calculate this integral, we will replace $\mathcal{O}\to n\mathcal{O}$ in $P_0[\phi]$ and $P_{0\Omega}[g_\Omega]$

$$P^n_0[\phi] = \left[\det\left(\mu^{-1}n\mathcal{O}\right)\right]^{1/2}\exp\left(-\int d^d\mathbf{x}\,\phi(\mathbf{x})[n\mathcal{O}\phi](\mathbf{x})\right) \tag{3.61}$$

$$P^n_{0\Omega}[g_\Omega] = \left[\frac{\det\left(\mu^{-1}n\mathcal{O}\right)}{\det_{\overline{\Omega}}\left(\mu^{-1}n\mathcal{O}\right)}\right]^{1/2}\exp\left(-\int d^d\mathbf{x}\,\phi_\Omega(\mathbf{x})[n\mathcal{O}\phi_\Omega](\mathbf{x})\right), \tag{3.62}$$

which are normalized,

$$\int[\mathcal{D}\phi]P^n_0[\phi] = 1, \qquad \int[\mathcal{D}g_\Omega]P^n_{0\Omega}[g_\Omega] = 1. \tag{3.63}$$

The second condition is obtained from the first by using the splitting

$$\int[\mathcal{D}\phi] = \int[\mathcal{D}g_\Omega]\int[\mathcal{D}\phi]_{g_\Omega}. \tag{3.64}$$

Thus, considering (3.63), we have that

$$\begin{aligned} 1 &= \int[\mathcal{D}g_\Omega]P^n_{0\Omega}[g_\Omega] \\ &= \left[\frac{\det\left(\mu^{-1}n\mathcal{O}\right)}{\det_{\overline{\Omega}}\left(\mu^{-1}n\mathcal{O}\right)}\right]^{1/2}\int[\mathcal{D}g_\Omega]\exp\left(-\int d^d\mathbf{x}\,\phi_\Omega(\mathbf{x})[n\mathcal{O}\phi_\Omega](\mathbf{x})\right). \end{aligned} \tag{3.65}$$

Therefore

$$\int[\mathcal{D}g_\Omega]\exp\left(-\int d^d\mathbf{x}\,\phi_\Omega(\mathbf{x})[n\mathcal{O}\phi_\Omega](\mathbf{x})\right) = \left[\frac{\det_{\overline{\Omega}}\left(\mu^{-1}n\mathcal{O}\right)}{\det\left(\mu^{-1}n\mathcal{O}\right)}\right]^{1/2}. \tag{3.66}$$

In this way, the partition replicated function (3.60) will be given by

$$Z_{0n}^{\Omega} = \left[\frac{\det\left(\mu^{-1}\mathcal{O}\right)}{\det_{\overline{\Omega}}\left(\mu^{-1}\mathcal{O}\right)}\right]^{n/2}\left[\frac{\det_{\overline{\Omega}}\left(\mu^{-1}n\mathcal{O}\right)}{\det\left(\mu^{-1}n\mathcal{O}\right)}\right]^{1/2}. \tag{3.67}$$

Therefore the Shannon entropy $S_0[\Omega]$ will be

$$S_0[\Omega] = -\partial_n \ln\left\{\left[\frac{\det\left(\mu^{-1}\mathcal{O}\right)}{\det_{\overline{\Omega}}\left(\mu^{-1}\mathcal{O}\right)}\right]^{n/2}\left[\frac{\det_{\overline{\Omega}}\left(\mu^{-1}n\mathcal{O}\right)}{\det\left(\mu^{-1}n\mathcal{O}\right)}\right]^{1/2}\right\}\Bigg|_{n=1}. \tag{3.68}$$

Similarly

$$S_0[\overline{\Omega}] = -\partial_n \ln\left\{\left[\frac{\det\left(\mu^{-1}\mathcal{O}\right)}{\det_{\Omega}\left(\mu^{-1}\mathcal{O}\right)}\right]^{n/2}\left[\frac{\det_{\Omega}\left(\mu^{-1}n\mathcal{O}\right)}{\det\left(\mu^{-1}n\mathcal{O}\right)}\right]^{1/2}\right\}\Bigg|_{n=1}. \tag{3.69}$$

Also, we can calculate the entropy $S_0[\overline{\Omega}\cup\Omega]$ in the vacuum state. In this way, we have

$$S_0[\overline{\Omega}\cup\Omega] = -\,\partial_n \ln\left\{\int[\mathcal{D}\phi]\;\left(P_0[\phi]\right)^n\right\}\Bigg|_{n=1}. \tag{3.70}$$

where if consider (3.40)

$$P_0[\phi] = \left[\det\left(\mu^{-1}\mathcal{O}\right)\right]^{1/2}\exp\left(-\int d^dx'\;\phi(\mathbf{x}')[\mathcal{O}\phi](\mathbf{x}')\right). \tag{3.71}$$

We obtain

$$\begin{aligned} S_0[\overline{\Omega}\cup\Omega] &= -\,\partial_n \ln\left\{\left[\det\left(\mu^{-1}\mathcal{O}\right)\right]^{n/2}\int[\mathcal{D}\phi]\;\exp\left(-n\int d^dx'\;\phi(\mathbf{x}')[\mathcal{O}\phi](\mathbf{x}')\right)\right\}\Bigg|_{n=1} \\ &= -\,\partial_n \ln\left\{\left[\det\left(\mu^{-1}\mathcal{O}\right)\right]^{n/2}\left[\det\left(n\mu^{-1}\mathcal{O}\right)\right]^{-1/2}\right\}\Bigg|_{n=1} \\ &= -\,\partial_n \ln\left\{\frac{\left[\det\left(\mu^{-1}\mathcal{O}\right)\right]^{n/2}}{\left[\det\left(n\mu^{-1}\mathcal{O}\right)\right]^{1/2}}\right\}\Bigg|_{n=1}. \end{aligned} \tag{3.72}$$

It is important to point out that an alternative to the replica trick method to compute functional averages of the logarithm of a function is the distributional zeta function method, introduced by Svaiter and Svaiter in Ref. [35]. This method has been used in the context of quenched disordered systems in statistical mechanics, where the average of interest is the free energy, i.e. the logarithm of the partition function, with the average being over a probability distribution of the disorder [36,

37, 38, 39, 40]. In our case, the Shannon entropy $S_0[\Omega]$, defined in Eq. (3.55), is the average of $\ln P_{0\Omega}[g_\Omega]$ computed with the probability distribution $P_{0\Omega}[g_\Omega]$. The distributional zeta function, for this case, is defined as

$$\Phi(s) := \int [\mathcal{D}g_\Omega] \left(P_{0\Omega}[g_\Omega]\right)^{1-s}. \tag{3.73}$$

The Shannon entropy $S_0[\Omega]$ is obtained as

$$S_0[\Omega] = -\frac{d}{ds}\Phi(s)\Big|_{s=0}. \tag{3.74}$$

The problem then reduces to computing $\Phi(s)$. Such an approach was never pursued in this kind of problem. As such, we employ the replica trick method in this work. In addition, the replica trick allows for the direct calculation of the Rényi entropy, which will be useful for simplifying future calculations in applications.

3.7 Mutual Information for the Vacuum State

The mutual information $I_0(\Omega, \overline{\Omega})$ quantifies the correlations between field configurations in the complementary spatial regions Ω and $\overline{\Omega}$ for the vacuum state of a free scalar field in Quantum Field Theory (QFT). Unlike conventional QFT approaches that define mutual information using von Neumann (entanglement) entropy, this thesis employs Shannon entropies. The mutual information is defined as:

$$I_0(\Omega, \overline{\Omega}) = S_0[\Omega] + S_0[\overline{\Omega}] - S_0[\Omega \cup \overline{\Omega}]. \tag{3.75}$$

where $S_0[\,\cdot\,]$ represents the Shannon entropy of the field configuration probability distribution in the specified region.

Considering the entropies given by (3.68), (3.69), and (3.72)

$$S_0[\Omega] = S_0[\Omega \cup \overline{\Omega}] - \frac{1}{2}\partial_n \ln\left\{\frac{\det_{\overline{\Omega}}\left(n\mu^{-1}\mathcal{O}\right)}{\left[\det_{\overline{\Omega}}\left(\mu^{-1}\mathcal{O}\right)\right]^n}\right\}\Bigg|_{n=1}, \tag{3.76}$$

$$S_0[\overline{\Omega}] = S_0[\Omega \cup \overline{\Omega}] - \frac{1}{2}\partial_n \ln\left\{\frac{\det_{\Omega}\left(n\mu^{-1}\mathcal{O}\right)}{\left[\det_{\Omega}\left(\mu^{-1}\mathcal{O}\right)\right]^n}\right\}\Bigg|_{n=1}, \tag{3.77}$$

$$S_0[\Omega \cup \overline{\Omega}] = -\frac{1}{2}\partial_n \ln\left\{\frac{\left[\det\left(\mu^{-1}\mathcal{O}\right)\right]^n}{\det\left(n\mu^{-1}\mathcal{O}\right)}\right\}\Bigg|_{n=1}. \tag{3.78}$$

The mutual information will be

$$\begin{aligned} I_0(\Omega,\overline{\Omega}) =& S_0[\Omega\cup\overline{\Omega}] - \frac{1}{2}\partial_n \ln\left\{\frac{\left[\det_{\overline{\Omega}}\left(n\mu^{-1}\mathcal{O}\right)\right]\left[\det_{\Omega}\left(n\mu^{-1}\mathcal{O}\right)\right]}{\left[\det_{\Omega}\left(\mu^{-1}\mathcal{O}\right)\det_{\overline{\Omega}}\left(\mu^{-1}\mathcal{O}\right)\right]^n}\right\}\Bigg|_{n=1} \\ =& \frac{1}{2}\ln\left\{\frac{\det_{\Omega}\left(\mu^{-1}\mathcal{O}\right)\det_{\overline{\Omega}}\left(\mu^{-1}\mathcal{O}\right)}{\det\left(\mu^{-1}\mathcal{O}\right)}\right\} \\ & - \frac{1}{2}\partial_n \ln\left\{\frac{\det_{\Omega}\left(n\mu^{-1}\mathcal{O}\right)\det_{\overline{\Omega}}\left(n\mu^{-1}\mathcal{O}\right)}{\det\left(n\mu^{-1}\mathcal{O}\right)}\right\}\Bigg|_{n=1}. \end{aligned} \tag{3.79}$$

This result is consistent with findings by Junior and Oxman [9], as noted in their equation (34), confirming the validity of the approach. The ratio $\frac{\det_{\Omega}\left(\mu^{-1}\mathcal{O}\right)\det_{\overline{\Omega}}\left(\mu^{-1}\mathcal{O}\right)}{\det\left(\mu^{-1}\mathcal{O}\right)}$ can be computed using techniques from [9], such as heat kernel methods or numerical approaches for specific geometries of Ω.

This mutual information serves as a baseline for comparing correlations in the vacuum state with those in excited states, as explored in later sections.

Chapter 4

Shannon Entropy of the Localized State

In the realm of quantum field theory, the analysis of a single-particle state extends beyond its mere existence to understanding the information it encodes and how that information is distributed across space. While the vacuum state provides a baseline for correlations inherent to the quantum field itself, a localized single-particle state introduces a localized excitation, fundamentally altering the information landscape. Calculating the Shannon entropy for such a state allows us to quantify the uncertainty in the field configuration within a specific region, revealing how the presence of the particle modifies the underlying vacuum fluctuations.

Furthermore, the Shannon mutual information between two regions in the presence of the particle becomes a crucial measure. It quantifies the classical correlations that are directly accessible through measurement, providing a window into how much of the particle's quantum information is converted into classical correlations due to decoherence or measurement processes. This approach is particularly relevant as it focuses on the diagonal elements of the density matrix, which are precisely the elements that survive decoherence. By comparing the mutual information of the single-particle state to that of the vacuum, we can isolate the contribution of the particle to the shared information between the regions, offering insights into the nature of locality and information flow in the presence of quantum excitations. This provides a complementary perspective to traditional entanglement entropy calculations, focusing on the classical information content that is ultimately accessible in realistic experimental scenarios.

This chapter derives the wave functional and probability distribution for the excited state, ensuring proper normalization. We compute reduced probabilities for spatial regions using the saddle point method and calculate the Shannon and Rényi entropies via the replica trick. The mutual information between complementary regions is then determined, highlighting the effects of excitations compared to the vacuum state. Advanced mathematical techniques, including fractional Laplacian operators and Gaussian integrals, are utilized, with detailed derivations

provided in the appendices.

4.1 Localized States

In quantum field theory, a single-particle state with definite momentum $\mathbf{k}$ is typically represented as:

$$|\mathbf{k}\rangle = a^\dagger(\mathbf{k})|0\rangle, \tag{4.1}$$

where $a^\dagger(\mathbf{k})$ is the creation operator for a particle with momentum $\mathbf{k}$, and $|0\rangle$ is the vacuum state. This state corresponds to a plane-wave state, delocalized across space.

This work focuses on *localized states* [20], which describe particles confined to a finite spatial region, in contrast to plane-wave states. These states are superpositions of momentum eigenstates, weighted by a time-dependent wave packet profile $g(t, \mathbf{k})$. A generic single-particle state is thus:

$$|\Psi(t)\rangle = \int d^d\mathbf{k}\, g(t, \mathbf{k}) a^\dagger(\mathbf{k})|0\rangle, \tag{4.2}$$

where the integral is over d-dimensional momentum space, and the particle's energy is:

$$\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}, \tag{4.3}$$

with m being the particle's mass. The state $|\Psi(t)\rangle$ evolves according to the Schrödinger equation:

$$i\frac{\partial|\Psi(t)\rangle}{\partial t} = H|\Psi(t)\rangle, \tag{4.4}$$

where H is the Hamiltonian of the free scalar field. The wave packet profile $g(t, \mathbf{k})$ satisfies:

$$i\frac{\partial g(t, \mathbf{k})}{\partial t} = \omega_{\mathbf{k}} g(t, \mathbf{k}), \tag{4.5}$$

with the solution:

$$g(t, \mathbf{k}) = e^{-i\omega_{\mathbf{k}} t} g(\mathbf{k}), \tag{4.6}$$

where $g(\mathbf{k})$ is the initial profile at $t = 0$.

To express the state in position space, the position-space creation operator is defined as:

$$a^\dagger(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}} \int d^d\mathbf{k}\, a^\dagger(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{4.7}$$

which is the same operator defined in (3.34).

The position-space wave function $f(t,\mathbf{x})$ is the Fourier transform of $g(t,\mathbf{k})$:

$$f(t,\mathbf{x}) = \frac{1}{(2\pi)^{d/2}} \int d^d\mathbf{k}\, g(t,\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{4.8}$$

or, using the time-evolved profile:

$$f(t,\mathbf{x}) = \frac{1}{(2\pi)^{d/2}} \int d^d\mathbf{k}\, g(\mathbf{k}) e^{-i\omega_{\mathbf{k}}t} e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{4.9}$$

This wave function satisfies the relativistic Klein-Gordon equation:

$$i\frac{\partial f(t,\mathbf{x})}{\partial t} = \sqrt{-\nabla^2 + m^2} f(t,\mathbf{x}). \tag{4.10}$$

The single-particle state can be reexpressed in position space as:

$$|\Psi(t)\rangle = \int d^d\mathbf{x}\, f(t,\mathbf{x}) a^\dagger(\mathbf{x})|0\rangle. \tag{4.11}$$

This form represents the state as a superposition of states created at different positions $\mathbf{x}$, weighted by $f(t,\mathbf{x})$, providing an intuitive picture of the particle's localization. Also, the scalar product becomes

$$\langle\Psi|\Psi\rangle = \int d^d\mathbf{k}\, g^*(t,\mathbf{k}) g(t,\mathbf{k}) = \int d^d\mathbf{x}\, f^*(t,\mathbf{x}) f(t,\mathbf{x}). \tag{4.12}$$

4.1.1 Wave Functional for the Localized State

The wave functional for the excited state is $\Psi_1[\phi] = \langle\phi|\Psi\rangle$, where $|\phi\rangle$ is the field basis state satisfying $\hat{\phi}(\mathbf{x})|\phi\rangle = \phi(\mathbf{x})|\phi\rangle$. Using the position-space expression for $|\Psi\rangle$, we have:

$$\Psi_1[\phi] = \int d^d\mathbf{x}\, f(t,\mathbf{x}) \langle\phi|a^\dagger(\mathbf{x})|0\rangle. \tag{4.13}$$

Applying $\hat{a}^\dagger(\mathbf{x})$ to $\Psi_0[\phi]$, and noting that $\pi(\mathbf{x})\Psi_0[\phi] = i\mathcal{O}\phi\Psi_0[\phi]$, we get:

$$\hat{a}^\dagger(\mathbf{x})\Psi_0[\phi] = \frac{1}{\sqrt{\gamma}}\mathcal{O}^{-1/2}[\mathcal{O}\phi(\mathbf{x}) - i\pi(\mathbf{x})]\Psi_0[\phi] = \frac{2}{\sqrt{\gamma}}\left[\mathcal{O}^{1/2}\phi\right](\mathbf{x})\Psi_0[\phi], \tag{4.14}$$

since $\mathcal{O}^{-1/2}[\mathcal{O}\phi](\mathbf{x}) = \mathcal{O}^{1/2}\phi(\mathbf{x})$. Thus, the wave functional is:

$$\Psi_1[\phi] = \frac{2}{\sqrt{\gamma}} \int d^d\mathbf{x}\, f(t,\mathbf{x}) \mathcal{O}^{1/2}\phi(\mathbf{x})\Psi_0[\phi]. \tag{4.15}$$

This expression for the wave functional of the excited state is in accordance with [1] (see his eq.(10.30)).

4.1.2 Probability Distribution and Normalization

The probability distribution for the excited state is:

$$P_1[\phi] = |\Psi_1[\phi]|^2 = \frac{4}{\gamma}\left|\int d^d\mathbf{x}\, f(t,\mathbf{x})\mathcal{O}^{1/2}\phi(\mathbf{x})\right|^2 P_0[\phi], \tag{4.16}$$

where $P_0[\phi] = \left[\det\left(\mu^{-1}\mathcal{O}\right)\right]^{1/2}\exp\left(-\int d^d\mathbf{x}\phi(\mathbf{x})[\mathcal{O}\phi](\mathbf{x})\right)$. Expanding the quadratic term:

$$\begin{aligned}\left|\int d^d\mathbf{x}\, f(t,\mathbf{x})\mathcal{O}^{\frac{1}{2}}\phi\right|^2 &= \int d^d\mathbf{x}d^d\mathbf{y}\, f^*(t,\mathbf{x})f(t,\mathbf{y})\mathcal{O}^{\frac{1}{2}}\phi(\mathbf{x})\mathcal{O}^{\frac{1}{2}}\phi(\mathbf{y}) \\ &= \int d^d\mathbf{x}d^d\mathbf{y}\left[R^*(t,\mathbf{x})R(t,\mathbf{y}) + I^*(t,\mathbf{x})I(t,\mathbf{y})\right]\mathcal{O}^{\frac{1}{2}}\phi(\mathbf{x})\mathcal{O}^{\frac{1}{2}}\phi(\mathbf{y})\end{aligned} \tag{4.17}$$

where $R(t,\mathbf{x})$ and $I(t,\mathbf{x})$ are the real and imaginary parts, respectively, of the wave packet $f(t,\mathbf{x})$.i.e. $f = R + iI$. Thus, we can rewrite the probability distribution in the following form:

$$\begin{aligned}P_1[\phi] =& \frac{4}{\gamma}\left[\det\left(\mu^{-1}\mathcal{O}\right)\right]^{1/2} \\ &\times\left(\frac{\delta^2}{\delta\alpha^2} - \frac{\delta^2}{\delta\beta^2}\right)\exp\left(-\int d^d\mathbf{x}\left(\phi(\mathbf{x})[\mathcal{O}\phi](\mathbf{x}) - f_{\alpha,\beta}(t,\mathbf{x})\mathcal{O}^{\frac{1}{2}}\phi(\mathbf{x})\right)\right)\Bigg|_{\alpha=\beta=0},\end{aligned} \tag{4.18}$$

where we have considered the following notation $f_{\alpha,\beta} = \alpha R + i\beta I$.

To normalize, we require:

$$\int [\mathcal{D}\phi]P_1[\phi] = 1. \tag{4.19}$$

So, to compute the integral $\int [\mathcal{D}\phi] P_1[\phi]$, we can use the fact

$$\int [\mathcal{D}\phi] \exp\left(-\frac{1}{2}\int d^d\mathbf{x}\,(\phi(\mathbf{x})[A\phi](\mathbf{x}) - 2\rho(\mathbf{x})\phi(\mathbf{x}))\right) = [\det(A)]^{-1/2} \exp\left(\frac{1}{2}\int d^d\mathbf{x}\,\rho(\mathbf{x})\left[A^{-1}\rho\right](\mathbf{x})\right), \tag{4.20}$$

with $A \equiv 2\mathcal{O}$ and $\rho \equiv \alpha\mathcal{O}^{1/2} f(t,\mathbf{x})$. This yields

$$\begin{aligned}\int [\mathcal{D}\phi] P_1[\phi] =& \frac{4}{\gamma}\left(\frac{\delta^2}{\delta\alpha^2} - \frac{\delta^2}{\delta\beta^2}\right) \exp\left(\frac{1}{4}\int d^d\mathbf{x}\, f^2_{\alpha,\beta}(\mathbf{x},t)\right)\Big|_{\alpha=\beta=0} \\ =& \frac{4}{\gamma}\left(\frac{1}{2}\int d^d\mathbf{x}\, R^2(\mathbf{x},t) + \frac{1}{2}\int d^3\mathbf{x}\, I^2(\mathbf{x},t)\right) \\ =& \frac{2}{\gamma}\left[\int d^d\mathbf{x}\,|f(\mathbf{x},t)|^2\right].\end{aligned} \tag{4.21}$$

Therefore, for the normalization condition to be satisfied, it must be fulfilled that

$$\frac{2}{\gamma}\int d^d\mathbf{x}\,|f(\mathbf{x},t)|^2 = 1. \tag{4.22}$$

Thus, the constant γ will be

$$\gamma = 2\int d^d\mathbf{x}\,|f(\mathbf{x},t)|^2 = 2. \tag{4.23}$$

4.2 Wave Packet Solutions of the Relativistic Shrödinger Equation

This section explores the wave packet function $f(t,\mathbf{x})$, a fundamental tool for characterizing the excited states of a quantum scalar field. In this sense we will see that there exist two solutions of (4.10) that minimize the position-momentum uncertainty and the position-velocity uncertainty.

We know that the wave packet function, that follow (4.10), is given by

$$f(t,\mathbf{x}) = \frac{1}{(2\pi)^{d/2}}\int d^d\mathbf{k}\, g(t,\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{4.24}$$

where

$$g(t,\mathbf{k}) = e^{-i\omega_{\mathbf{k}}t}g(\mathbf{k}), \qquad \omega_{\mathbf{k}} = \sqrt{m^2+\mathbf{k}^2} \tag{4.25}$$

and

$$\int d^d\mathbf{k}\ |g(\mathbf{k})|^2 = \int d^d\mathbf{x}\ |f(0,\mathbf{x})|^2 = 1. \tag{4.26}$$

Thus, in general, we have

$$\begin{aligned} f(t,\mathbf{x}) =& \frac{1}{(2\pi)^{d/2}}\int d^d\mathbf{k}\, g(\mathbf{k})e^{-i\omega_{\mathbf{k}}t+i\mathbf{k}\cdot\mathbf{x}} \\ =& \frac{1}{(2\pi)^{d/2}}\int d^d\mathbf{k}\ \left(\frac{1}{(2\pi)^{d/2}}\int d^d\mathbf{x}' f(0,\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}'}\right) e^{-i\omega_{\mathbf{k}}t+i\mathbf{k}\cdot\mathbf{x}} \\ =& \int d^d\mathbf{x}'\ \langle \mathbf{x},t|\mathbf{x}',0\rangle\, f(0,\mathbf{x}'), \end{aligned} \tag{4.27}$$

where $\langle \mathbf{x},t|\mathbf{x}',0\rangle$ called the Feynman kernel is given by

$$\langle \mathbf{x},t|\mathbf{x}',0\rangle = \frac{1}{(2\pi)^d}\int d^d\mathbf{k}\exp\left(-i\omega_{\mathbf{k}}t + i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')\right). \tag{4.28}$$

Therefore, we can see that to determine the evolution of one arbitrary wave packet $f(0,\mathbf{x})$ is necessary to determine the Feynman kernel (4.28).

In the analysis of localized one-particle states, we will specifically consider wave packet solutions of the relativistic Schrödinger equation in the case of a massless scalar field in one spatial dimension. This choice is motivated by both mathematical tractability and utility as an illustrative example. The one-dimensional massless case simplifies the dispersion relation to $\omega_k = |k|$, enabling explicit analytical calculations of functional integrals involving the fractional Laplacian, with well-defined Green's functions and kernels (see Appendix A and C).

4.2.1 Minimal Position-Momentum Uncertainty Wave Packet

It is well known that the wave packet with minimal position-momentum uncertainty is given by the Gaussian function. Thus, for the one-dimensional case

$$g(k) = Ae^{-\sigma k^2}, \tag{4.29}$$

where A is a normalization constant given by

$$A^2\int_{-\infty}^{\infty} dk\, e^{-2\sigma k^2} = 1, \tag{4.30}$$

so

$$A = \left(\frac{2\sigma}{\pi}\right)^{1/4}. \tag{4.31}$$

Now, considering the scalar field without mass, this implies that $\omega_k = |k|$. Then, from (4.24), for the probability amplitude, we have

$$\begin{aligned} f(t,x) =& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, g(k) e^{-i\omega_k t + ikx} \\ =& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, \left(Ae^{-\sigma k^2}\right) e^{-i|k|t + ikx} \\ =& \frac{A}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, e^{-\sigma k^2 + ikx - i|k|t} \\ =& \frac{A}{\sqrt{2\pi}} \left[\int_{0}^{\infty} dk\, e^{-\sigma k^2 + ik(x-t)} + \int_{-\infty}^{0} dk\, e^{-\sigma k^2 + ik(x+t)}\right] \\ =& \frac{A}{\sqrt{2\pi}} \left[\int_{0}^{\infty} dk\, e^{-\sigma k^2 + ik(x-t)} + \int_{0}^{\infty} dk\, e^{-\sigma k^2 - ik(x+t)}\right]. \end{aligned} \tag{4.32}$$

At this point, we can use the integral given in [41](3.322.3)

$$\int_0^\infty \exp\left(-\frac{x^2}{4\beta} - \gamma x\right) dx = \sqrt{\pi\beta} \exp\left(\beta\gamma^2\right)\left(1 - \Phi\left(\gamma\sqrt{\beta}\right)\right), \qquad \mathbb{R}e(\beta) > 0$$

where the function $\Phi(x)$ is the Error function

$$\mathrm{Erf}(x) = \Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt. \tag{4.33}$$

Therefore the equation (4.32) will be

$$f(t,x) = \frac{1}{2}\left(\frac{1}{2\pi\sigma}\right)^{1/4} \left[e^{-\frac{(x-t)^2}{4\sigma}}\left(1 + i\Phi\left(\frac{x-t}{2\sqrt{\sigma}}\right)\right) + e^{-\frac{(x+t)^2}{4\sigma}}\left(1 - i\Phi\left(\frac{x+t}{2\sqrt{\sigma}}\right)\right)\right], \tag{4.34}$$

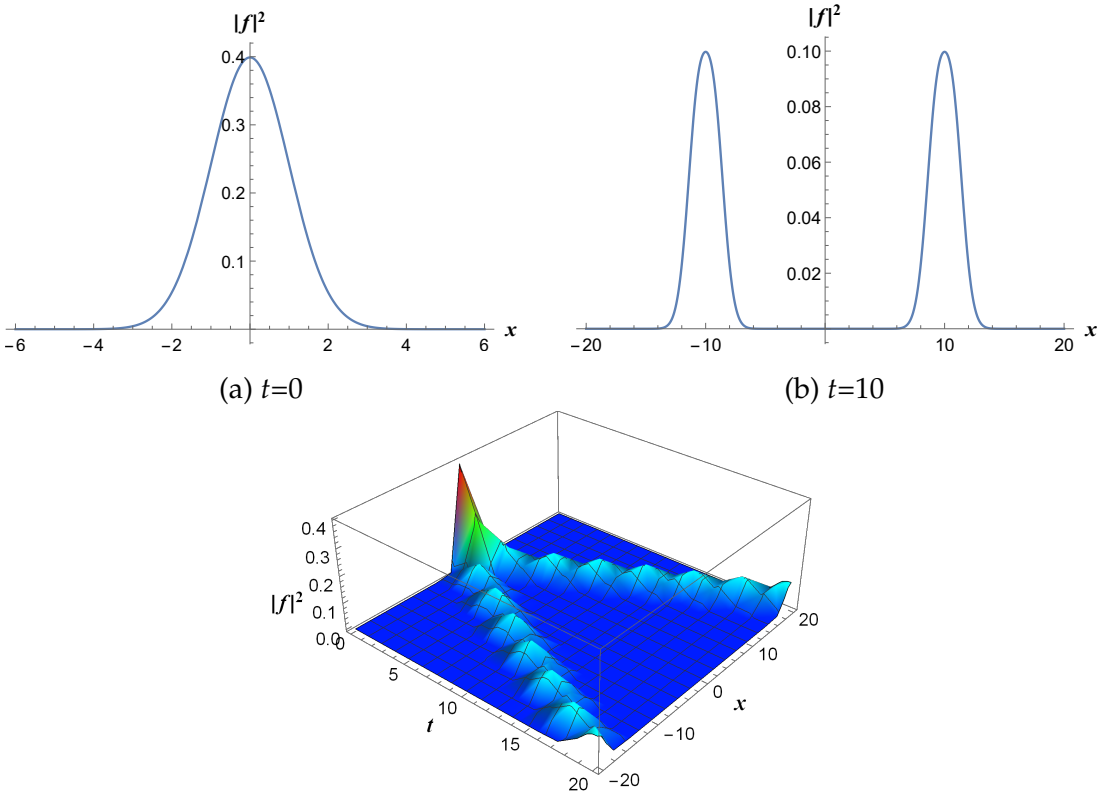
where we use the property of the Error function $\Phi(iz) = i\Phi(z)$. We can observe the evolution with time of this Gaussian wave packet in the Figure 4.1.

We can separate the wave packet (4.34) into its real and imaginary parts

$$f(t,x) = R(t,x) + iI(t,x), \tag{4.35}$$

where

$$R(t,x) = \frac{1}{2}\frac{A}{\sqrt{2\sigma}}\left[e^{-\frac{(x-t)^2}{4\sigma}} + e^{-\frac{(x+t)^2}{4\sigma}}\right] \tag{4.36}$$



(a) t=0 (b) t=10

(c)

Figure 4.1: Figures (a), (b), and (c) show the evolution with time t of the Gaussian wave packet (4.34) with width $\sigma = 1$.

and

$$I(t,x) = \frac{1}{2}\frac{A}{\sqrt{2\sigma}}\left[e^{-\frac{(x-t)^2}{4\sigma}}\Phi\left(\frac{x-t}{2\sqrt{\sigma}}\right) - e^{-\frac{(x+t)^2}{4\sigma}}\Phi\left(\frac{x+t}{2\sqrt{\sigma}}\right)\right]. \tag{4.37}$$

Now, we try to calculate the expression $\left[\mathcal{O}^{1/2}f\right](t,x)$ which help us to calculate the expressions $a_f(t,x)$ given by (B.15) and $b_f(t)$ given by (B.16).

Calculation of $\left[\mathcal{O}^{1/2}f\right](t,x)$

From the definition of the fractional Laplacian via the Fourier transform, we have that

$$\left[\mathcal{O}^{1/2}f\right](t,x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, e^{ikx}\sqrt{|k|}g(t,k) \tag{4.38}$$

So, considering (4.25) and (4.29), we have

$$\begin{aligned}\left[\mathcal{O}^{1/2}f\right](t,x) =& \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, e^{ikx}\sqrt{|k|}\left(Ae^{-\sigma k^2-i|k|t}\right)\\ =& \frac{A}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\,\sqrt{|k|}e^{-\sigma k^2-i|k|t+ikx}\\ =& \frac{A}{\sqrt{2\pi}}\left[\int_0^{\infty} dk\,\sqrt{k}e^{-\sigma k^2+ik(x-t)}+\int_{-\infty}^{0} dk\,\sqrt{-k}e^{-\sigma k^2+ik(x+t)}\right]\\ =& \frac{A}{\sqrt{2\pi}}\left[\int_0^{\infty} dk\,\sqrt{k}e^{-\sigma k^2+ik(x-t)}+\int_0^{\infty} dk\,\sqrt{k}e^{-\sigma k^2-ik(x+t)}\right] \quad (4.39)\end{aligned}$$

To solve this, we use the integral given in [41](3.462.1)

$$\int_0^{\infty} x^{\nu-1}e^{-\beta x^2-\gamma x}dx = (2\beta)^{-\nu/2}\Gamma(\nu)\exp\left(\frac{\gamma^2}{8\beta}\right)D_{-\nu}\left(\frac{\gamma}{\sqrt{2\beta}}\right),\ \mathrm{Re}\,\beta>0,\ \mathrm{Re}\,\nu>0. \quad (4.40)$$

Thus, considering $\nu \equiv 3/2$, $\beta \equiv \sigma$ and $\gamma \equiv \{-i(x-t), i(x+t)\}$

$$\begin{aligned}\left[\mathcal{O}^{1/2}f\right](t,x) =& \frac{1}{2}\frac{A}{\sqrt{2\pi}}(2\sigma)^{-3/4}\left[\exp\left(-\frac{(x-t)^2}{8\sigma}\right)D_{-3/2}\left(-i\frac{x-t}{\sqrt{2\sigma}}\right)\right.\\ &\left.+\exp\left(-\frac{(x+t)^2}{8\sigma}\right)D_{-3/2}\left(i\frac{x+t}{\sqrt{2\sigma}}\right)\right] \quad (4.41)\end{aligned}$$

since $A = (2\sigma/\pi)^{1/4}$

$$\begin{aligned}\left[\mathcal{O}^{1/2}f\right](t,x) =& \frac{1}{4}\frac{1}{\sqrt{\sigma}(\pi)^{3/4}}\left[\exp\left(-\frac{(x-t)^2}{8\sigma}\right)D_{-3/2}\left(-i\frac{x-t}{\sqrt{2\sigma}}\right)\right.\\ &\left.+\exp\left(-\frac{(x+t)^2}{8\sigma}\right)D_{-3/2}\left(i\frac{x+t}{\sqrt{2\sigma}}\right)\right], \quad (4.42)\end{aligned}$$

where the function $D_\nu(z)$ is the Parabolic Cylinder Function.

4.2.2 Minimal Position-Velocity Uncertainty Wave Packet

The position-velocity uncertainty relation can be derived using the Cauchy-Schwarz inequality applied to the operators $A = x - \langle x\rangle$ and $B = v - \langle v\rangle$, where the velocity is defined as the group velocity $v = \partial_k\omega_k$. With this, the relationship is given by

$$\Delta x\Delta v \geq \frac{1}{2}|\langle[x,v]\rangle| \quad (4.43)$$

The commutator gives us

$$[x, v] = [i\partial_k, \partial_k \omega_k] = i\partial_k^2 \omega_k \tag{4.44}$$

Thus, the position-velocity uncertainty relation will be

$$\Delta x \Delta v \geq \frac{1}{2}\langle \partial_k^2 \omega_k \rangle. \tag{4.45}$$

Now, to ensure that this uncertainty relation is minimal, the Cauchy inequality must be an equality, which is satisfied when $A = \lambda B$ for some constant λ. Thus, to find a wave packet with minimal position-velocity uncertainty, the following condition must be met:

$$(x - \langle x \rangle) g(k) = \lambda (v - \langle v \rangle) g(k), \tag{4.46}$$

where the velocity is defined as the group velocity $v = \partial_k \omega_k$ and λ can be determined from the uncertainty relation (4.45), being given by

$$|\lambda| = \frac{\langle \partial_k^2 \omega_k \rangle}{2(\Delta v)^2} \tag{4.47}$$

Taking into account the fact that $x = i\partial_k$, we have

$$(i\partial_k - \langle x \rangle) g(k) = \lambda (\partial_k \omega_k - \langle v \rangle) g(k) \tag{4.48}$$

rearrange

$$\partial_k g(k) + i\lambda \partial_k \omega_k g(k) - i(\lambda \langle v \rangle - \langle x \rangle) g(k) = 0. \tag{4.49}$$

This is a first-order differential equation. Let's define $\alpha = i\lambda$ and $\beta = \alpha \langle v \rangle - i\langle x \rangle$, so

$$\frac{dg}{g} = -(\alpha \partial_k \omega_k - \beta) dk. \tag{4.50}$$

Integrating, we have

$$g(k) = A e^{-\alpha \omega_k + \beta k} \tag{4.51}$$

where A is a normalization constant.

To a wave packet centered at $x = 0$, from [21] we have the expectation values of a variety of operators for relativistic wave packets, a minimal position-velocity

uncertainty product

$$\begin{aligned}
\langle x\rangle =&0, \quad \langle x^2\rangle = \alpha^2-\beta^2-\frac{2\alpha\sqrt{\alpha^2-\beta^2}}{K_1}\int_\alpha^\infty d\alpha' K_0\left(2m\sqrt{\alpha'^2-\beta^2}\right)\\
\langle v\rangle =&\frac{\beta}{\alpha}, \quad \langle v^2\rangle = 1-\frac{2\sqrt{\alpha^2-\beta^2}}{\alpha K_1}\int_\alpha^\infty d\alpha'\, K_0\left(2m\sqrt{\alpha'^2-\beta^2}\right)\\
\langle p\rangle =&\frac{\beta}{\alpha^2-\beta^2}\left(1+m\sqrt{\alpha^2-\beta^2}\frac{K_0}{K_1}\right),\\
\langle p^2\rangle =&\frac{m^2\beta^2}{\alpha^2-\beta^2}+\frac{\alpha^2+3\beta^2}{2\left(\alpha^2-\beta^2\right)}\left(1+m\sqrt{\alpha^2-\beta^2}\frac{K_0}{K_1}\right)\\
\langle \omega_k\rangle =&\frac{\alpha}{\alpha^2-\beta^2}\left(1+m\sqrt{\alpha^2-\beta^2}\frac{K_0}{K_1}\right)-\frac{1}{2\alpha}.
\end{aligned}$$

Here K_0 and K_1 are modified Bessel functions of degree zero and one

$$K_0 = K_0\left(2m\sqrt{\alpha'^2-\beta^2}\right), \qquad K_1 = K_1\left(2m\sqrt{\alpha'^2-\beta^2}\right). \tag{4.52}$$

The massless limit

For the case of the massless particle $\omega_k = |k|$, the wave packet (4.51) will be

$$g(k) = A\exp(-\alpha|k|+\beta k) \tag{4.53}$$

From the normalization condition, we have

$$\int_{-\infty}^{\infty} dk|g(k)|^2 = |A|^2\left[\int_0^\infty dk\, e^{-2(\mathrm{Re}(\alpha)-\mathrm{Re}(\beta))k}-\int_0^\infty dk\, e^{-2(\mathrm{Re}(\alpha)+\mathrm{Re}(\beta))k}\right]. \tag{4.54}$$

This implies that

$$|A|^2 = \frac{\mathrm{Re}(\alpha)^2-\mathrm{Re}(\beta)^2}{\mathrm{Re}(\alpha)} \tag{4.55}$$

with the convergence condition $|\,\mathrm{Re}(\beta)| \le \mathrm{Re}(\alpha)$.

To the wave packet centered in $x = 0$, again from [21], we have

$$\begin{aligned}
\langle x\rangle =& 0, \quad \langle x^2\rangle = (\Delta x)^2 = \alpha^2 - \beta^2, \\
\langle v\rangle =& \frac{\beta}{\alpha}, \quad \langle v^2\rangle = 1, \quad (\Delta v)^2 = 1 - \frac{\beta^2}{\alpha^2}, \\
\langle p\rangle =& \frac{\beta}{\alpha^2-\beta^2}, \quad \langle p^2\rangle = \frac{\alpha^2+3\beta^2}{2\left(\alpha^2-\beta^2\right)^2}, \quad (\Delta p)^2 = \frac{\alpha^2+\beta^2}{2\left(\alpha^2-\beta^2\right)^2}, \\
\langle \omega_k\rangle =& \frac{\alpha^2+\beta^2}{2\alpha\left(\alpha^2-\beta^2\right)}, \quad \langle \omega_k^2\rangle = \frac{\alpha^2+3\beta^2}{2\left(\alpha^2-\beta^2\right)^2}, \quad (\Delta\omega_k)^2 = \frac{\alpha^4+4\alpha^2\beta^2-\beta^4}{4\alpha^2\left(\alpha^2-\beta^2\right)^2}.
\end{aligned} \tag{4.56}$$

It is important to highlight that, although massless particles travel at the speed of light, the condition $|\langle v\rangle| \leq 1$ generally holds. This arises because a particle can simultaneously propagate both leftward and rightward with a nonzero probability amplitude. Only in the limit $|\beta| \to \alpha$ does the particle become entirely left- or right-moving, leading to $|\langle v\rangle| \to 1$, $\Delta v \to 0$, and a non-spreading wave packet. Also, for this massless case, we note that $\partial_k^2 \omega_k = 0$, which simplifies the uncertainty relation (4.45) to:

$$\Delta x \Delta v \geq 0 \tag{4.57}$$

and, from the position and velocity uncertainties of (4.56)

$$\Delta x \Delta v = \frac{\alpha^2 - \beta^2}{\alpha} \tag{4.58}$$

We see that the condition for minimum uncertainty in position-velocity is given by the equality $|\beta| = \alpha$. This implies that the normalization constant (4.56) is null, $A = 0$, meaning that there is no wave packet with minimum position-velocity uncertainty for the massless particle.

On the other hand, although the wave packet (4.53) does not minimize the position-velocity uncertainty relation, it remains a valid wave packet for representing the state of a massless relativistic particle. Due to its simple form in configuration space, it will be quite useful for future calculations of the entanglement entropy of an excited state.

The form in the configuration space of the wave packet (4.53) is given by its

Fourier transform (4.24) and (4.25)

$$\begin{aligned} f(t,x) =& \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} dk\, g(t,k) e^{ikx} \\ =& \frac{A}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, e^{-i|k|(t-i\alpha)} e^{ik(x-i\beta)} \\ =& \frac{A}{\sqrt{2\pi}} \left[\int_0^{\infty} dk\, e^{-ik[(t-i\alpha)+(x-i\beta)]} + \int_0^{\infty} dk\, e^{-ik[(t-i\alpha)-(x-i\beta)]} \right]. \end{aligned} \tag{4.59}$$

Using the fact

$$\begin{aligned} \int_0^{\infty} e^{-ik(a-ib)}\, dk =& i\, \frac{e^{-ika} e^{-kb}}{a-ib} \Bigg|_{k=0}^{k=\infty} \\ =& -\frac{i}{a-ib}, \quad \mathrm{Re}\, b > 0. \end{aligned} \tag{4.60}$$

Then

$$\int_0^{\infty} dk\, e^{-ik[(t-i\alpha)+(x-i\beta)]} = -\frac{i}{(t-i\alpha)+(x-i\beta)}, \quad \mathrm{Re}\,\alpha + \mathrm{Re}\,\beta > 0 \tag{4.61}$$

and

$$\int_0^{\infty} dk\, e^{-ik[(t-i\alpha)-(x-i\beta)]} = -\frac{i}{(t-i\alpha)-(x-i\beta)}, \quad \mathrm{Re}\,\alpha - \mathrm{Re}\,\beta > 0. \tag{4.62}$$

Combining these results, we have

$$f(t,x) = -i\frac{A}{\sqrt{2\pi}} \left[\frac{1}{(t-i\alpha)+(x-i\beta)} + \frac{1}{(t-i\alpha)-(x-i\beta)} \right]. \tag{4.63}$$

Computing the sum:

$$\frac{1}{(t-i\alpha)+(x-i\beta)} + \frac{1}{(t-i\alpha)-(x-i\beta)} = \frac{2(t-i\alpha)}{(t-i\alpha)^2-(x-i\beta)^2}. \tag{4.64}$$

Thus:

$$f(t,x) = 2i\sqrt{\frac{\mathrm{Re}\,\alpha^2 - \mathrm{Re}\,\beta^2}{2\pi\,\mathrm{Re}\,\alpha}} \frac{t-i\alpha}{(x-i\beta)^2-(t-i\alpha)^2}, \quad |\mathrm{Re}\,\beta| < \mathrm{Re}\,\alpha \tag{4.65}$$

Considering $t = 0$ and $\mathrm{Re}\,\beta = 0, \mathrm{Im}\,\alpha = 0$, we get a more simple expression of

this wave packet

$$f(0,x) = \sqrt{\frac{\alpha}{2\pi}} \frac{2\alpha}{(x + \operatorname{Im}\beta)^2 + \alpha^2} \tag{4.66}$$

where we can see that this expression corresponds to the normalized Lorentzian function, being α the width parameter and $\langle x \rangle = -\operatorname{Im}\beta$. Considering that α and β are reals, the behavior of the wave packet (4.65) is showed by the Figures 4.2 and 4.3, where we can see that in $t = 0$ start with its maximum value in $x = 0$ and then when $t > 0$ evolves according to the parameter β. If the parameter β is zero, then the particle has the same probability of being on the right side as on the left side. However, if the parameter β is different from zero, then the particle will have a higher probability of being on the left side if $\beta < 0$, or a higher probability of being on the right side if $\beta > 0$.

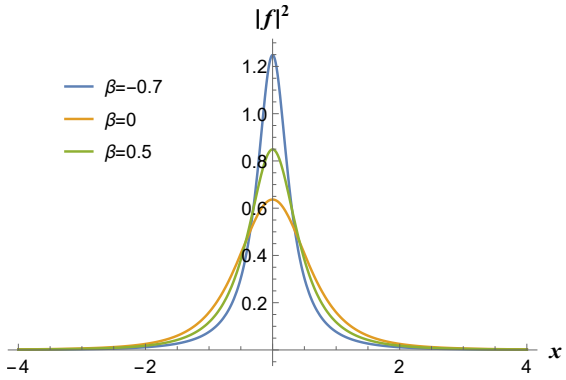


Figure 4.2: $\alpha = 1$, $t = 0$

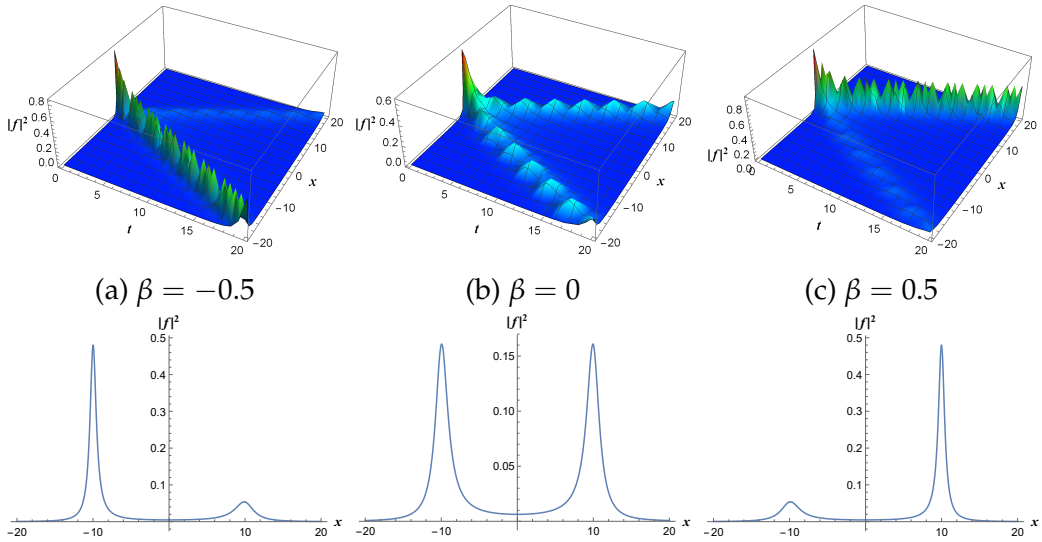
Calculation of $\mathcal{O}^{1/2} f(t,x)$

From the definition of fractional Laplacian via the Fourier transform we have that

$$\left[\mathcal{O}^{1/2} f\right](t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, e^{ikx} \sqrt{|k|}\, g(t,k)$$

So, considering (4.53), we have

$$\begin{aligned}\left[\mathcal{O}^{1/2} f\right](t,x) =& \frac{A}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, \sqrt{|k|} e^{-i|k|(t-i\alpha)} e^{ik(x-i\beta)} \\ =& \frac{A}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} dk\, \sqrt{-k} e^{ik[(x-i\beta)+(t-i\alpha)]} + \int_{0}^{\infty} dk\, \sqrt{k} e^{ik[(x-i\beta)-(t-i\alpha)]}\right]\end{aligned} \tag{4.67}$$



(a) $\beta = -0.5$ (b) $\beta = 0$ (c) $\beta = 0.5$

(d) $\beta = -0.5$, $t = 10$ (e) $\beta = 0$, $t = 10$ (f) $\beta = 0.5$, $t = 10$

Figure 4.3: All the images show us the evolution with time of the Lorentzian wave packet with $\alpha = 1$ for different values of β parameter.

change the variable $-k \to k$ for the first integral

$$\left[\mathcal{O}^{1/2} f\right](t,x) = \frac{A}{\sqrt{2\pi}}\left[\int_0^\infty dk\, \sqrt{k} e^{-ik[(x-i\beta)+(t-i\alpha)]} + \int_0^\infty dk\, \sqrt{k} e^{ik[(x-i\beta)-(t-i\alpha)]}\right] \tag{4.68}$$

Using eq.(3.326.2) given by Gradshteyn and Ryzhik [41]

$$\int_0^\infty dx\, x^m \exp\left(-\beta x^n\right) = \frac{\Gamma(\gamma)}{n\beta^\gamma}, \quad \gamma = \frac{m+1}{n}, \quad [\mathrm{Re}\,\beta > 0, \mathrm{Re}\, m > 0, \mathrm{Re}\, n > 0]. \tag{4.69}$$

Considering that $m = 1/2$, $n = 1$ and $\beta = a - ib$ with $a > 0$, we have

$$\begin{aligned}\int_0^\infty dk \sqrt{k} e^{ikb} e^{-ka} =& \frac{\Gamma(3/2)}{(a-ib)^{3/2}} \\ =& \frac{\sqrt{\pi}}{2}\frac{1}{(a-ib)^{3/2}}.\end{aligned} \tag{4.70}$$

Considering the fact $z^{s/r} = \left(|z| e^{i\arg(z)}\right)^{s/r} = |z|^{s/r} e^{i\arg(z) s/r}$. for the denominator $z = a - ib$, we have that $|z|^2 = a^2 + b^2$, $\arg(z) = \arctan\left(-\frac{b}{a}\right)$, then

$z^{3/2} = \left(a^2+b^2\right)^{3/4} e^{i\arctan\left(-\frac{b}{a}\right)\frac{3}{2}}$. Thus, the integral comes out

$$\int_0^\infty dk\sqrt{k}e^{ikb}e^{-ka} = \frac{\sqrt{\pi}}{2\left(a^2+b^2\right)^{3/4}}e^{i\arctan\left(\frac{b}{a}\right)\frac{3}{2}}, \quad a>0. \tag{4.71}$$

Inserting the last result in (4.68), as long as $|\operatorname{Re}\beta| < \operatorname{Re}\alpha$, we obtain

$$\left[\mathcal{O}^{1/2}f\right](t,x) = \frac{1}{2}\sqrt{\frac{\operatorname{Re}\alpha^2-\operatorname{Re}\beta^2}{2\operatorname{Re}\alpha}}\left[\frac{1}{[\alpha+\beta+i(x+t)]^{3/2}}+\frac{1}{[\alpha-\beta-i(x-t)]^{3/2}}\right] \tag{4.72}$$

$$= \frac{1}{2}\sqrt{\frac{\operatorname{Re}\alpha^2-\operatorname{Re}\beta^2}{2\operatorname{Re}\alpha}}\left[\frac{\exp\left(-i\arctan\left(\frac{x+\operatorname{Im}\beta+t+\operatorname{Im}\alpha}{\operatorname{Re}\alpha+\operatorname{Re}\beta}\right)\frac{3}{2}\right)}{\left[(\operatorname{Re}\alpha+\operatorname{Re}\beta)^2+(x+\operatorname{Im}\beta+t+\operatorname{Im}\alpha)^2\right]^{3/4}} + \frac{\exp\left(i\arctan\left(\frac{x+\operatorname{Im}\beta-t-\operatorname{Im}\alpha}{\operatorname{Re}\alpha-\operatorname{Re}\beta}\right)\frac{3}{2}\right)}{\left[(\operatorname{Re}\alpha-\operatorname{Re}\beta)^2+(x+\operatorname{Im}\beta-t-\operatorname{Im}\alpha)^2\right]^{3/4}}\right], \tag{4.73}$$

for the more simple case with $t=0$, $\operatorname{Im}\beta = -\langle x\rangle$ and $\operatorname{Re}\beta = \operatorname{Im}\alpha = 0$

$$\left[\mathcal{O}^{1/2}f\right](0,x) = \frac{1}{2}\sqrt{\frac{\alpha}{2}}\left[\frac{1}{[\alpha+i(x-\langle x\rangle)]^{3/2}}+\frac{1}{[\alpha-i(x-\langle x\rangle)]^{3/2}}\right], \alpha>0 \tag{4.74}$$

$$= \sqrt{\frac{\alpha}{2}}\frac{1}{\left[\alpha^2+(x-\langle x\rangle)^2\right]^{3/4}}\cos\left(\frac{3}{2}\arctan\left(\frac{x-\langle x\rangle}{\alpha}\right)\right), \alpha>0 \tag{4.75}$$

4.3 Reduced Probabilities

To calculate the Shannon entropy over the region Ω we need to calculate the reduced probability density of getting $g_\Omega(\mathbf{x})$, $\mathbf{x}\in\Omega$, when measuring the field ϕ on Ω, irrespective of the field value on $\overline{\Omega}$

$$P_{1\Omega}[g_\Omega] = \int [\mathcal{D}\phi]_{g_\Omega} P_1[\phi] \tag{4.76}$$

where the measure $[\mathcal{D}\phi]_{g_\Omega}$ integrates over the fields $\phi(x)$ with the condition

$$\phi(\mathbf{x}) = g_\Omega(\mathbf{x}), \quad \mathbf{x}\in\Omega. \tag{4.77}$$

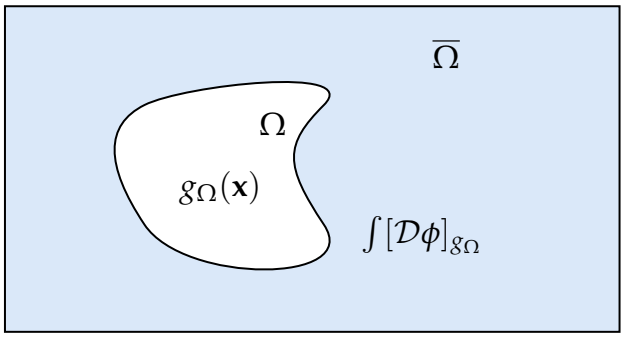


Figure 4.4

For the excited state $\Psi_1[\phi]$, the probability density is given by (4.88)

$$P_1[\phi] = \frac{4}{\gamma}\left[\det\left(\mu^{-1}\mathcal{O}\right)\right]^{1/2} \times \left(\frac{\delta^2}{\delta\alpha^2} - \frac{\delta^2}{\delta\beta^2}\right) \exp\left(-\int d^d\mathbf{x}\left(\phi(\mathbf{x})[\mathcal{O}\phi](\mathbf{x}) - f_{\alpha,\beta}(t,\mathbf{x})\mathcal{O}^{\frac{1}{2}}\phi(\mathbf{x})\right)\right)\Big|_{\alpha=\beta=0}. \tag{4.78}$$

Then, to integrate (4.76), we use the saddle point approximation, then we need to find the stationary point of $\mathcal{W}[\phi, f] \equiv \int d^d\mathbf{x}\ \left(\phi(\mathbf{x})[\mathcal{O}\phi](\mathbf{x}) - f_{\alpha,\beta}(t,\mathbf{x})\left[\mathcal{O}^{1/2}\phi\right](\mathbf{x})\right)$

$$\begin{aligned}\frac{\delta}{\delta\phi(\mathbf{x})}\mathcal{W}[\phi,f] =& \int d^d\mathbf{x}'\ \frac{\delta\phi(\mathbf{x}')}{\delta\phi(\mathbf{x})}[\mathcal{O}\phi](\mathbf{x}') + \int d^d\mathbf{x}'\ \phi(\mathbf{x}')\frac{\delta}{\delta\phi(\mathbf{x})}[\mathcal{O}\phi](\mathbf{x}') \\ &- \int d^d\mathbf{x}' f_{\alpha,\beta}(t,\mathbf{x}')\frac{\delta}{\delta\phi(\mathbf{x})}\left[\mathcal{O}^{1/2}\phi\right](\mathbf{x}') \\ =& \int d^d\mathbf{x}'\ \delta(\mathbf{x}'-\mathbf{x})\mathcal{O}\phi(\mathbf{x}') + \int d^d\mathbf{x}'\ \phi(\mathbf{x}')\mathcal{O}\delta(\mathbf{x}'-\mathbf{x}) \\ &- \int d^d\mathbf{x}' f_{\alpha,\beta}(t,\mathbf{x})\mathcal{O}^{1/2}\delta(\mathbf{x}'-\mathbf{x}) \\ =& 2\mathcal{O}\phi(\mathbf{x}) - \mathcal{O}^{1/2}f_{\alpha,\beta}(t,\mathbf{x})\end{aligned} \tag{4.79}$$

Then, we arrive at a unique saddle point, which satisfies the nonlocal Poisson problem,

$$\begin{aligned}[\mathcal{O}\phi_\Omega](\mathbf{x}) &= \frac{1}{2}\mathcal{O}^{1/2}f_{\alpha,\beta}(t,\mathbf{x}), && \mathbf{x}\in\overline{\Omega} \\ \phi_\Omega(\mathbf{x}) &= g_\Omega(\mathbf{x}), && \mathbf{x}\in\Omega.\end{aligned} \tag{4.80}$$

The change of variables $\phi(\mathbf{x}) = \phi_\Omega(\mathbf{x}) + \varepsilon(\mathbf{x})$ leads to

$$P_{1\Omega}[g_\Omega] = P_1[\phi_\Omega] \int [\mathcal{D}\varepsilon]_{\overline{\Omega}} \exp\left(- \int_{\overline{\Omega}} d^d\mathbf{x}\, \varepsilon(\mathbf{x}) [\mathcal{O}\varepsilon](\mathbf{x}) \right), \tag{4.81}$$

where the measure $[\mathcal{D}\varepsilon]_{\overline{\Omega}}$ integrates over the fields $\varepsilon(\mathbf{x})$ such that

$$\varepsilon(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega. \tag{4.82}$$

We get

$$P_{1\Omega}[g_\Omega] = \left[\det_{\overline{\Omega}} \left(\mu^{-1}\mathcal{O} \right) \right]^{-1/2} P_1[\phi_\Omega]. \tag{4.83}$$

Applying the property (A.9) we have

$$\begin{aligned} &\int d^d\mathbf{x} \left(\phi_\Omega(\mathbf{x}) [\mathcal{O}\phi_\Omega](\mathbf{x}) - f_{\alpha,\beta}(t,\mathbf{x}) \left[\mathcal{O}^{1/2}\phi_\Omega \right](\mathbf{x}) \right) \\ &\qquad = \int d^d\mathbf{x} \left(\phi_\Omega(\mathbf{x}) - \left[\mathcal{O}^{-1/2} f_{\alpha,\beta} \right](t,\mathbf{x}) \right) [\mathcal{O}\phi_\Omega](\mathbf{x}), \end{aligned} \tag{4.84}$$

so

$$\begin{aligned} P_{1\Omega}[g_\Omega] =& \frac{4}{\gamma} \left[\frac{\det\left(\mu^{-1}\mathcal{O}\right)}{\det_{\overline{\Omega}}\left(\mu^{-1}\mathcal{O}\right)} \right]^{1/2} \\ &\times \left(\frac{\delta^2}{\delta\alpha^2} - \frac{\delta^2}{\delta\beta^2} \right) \exp\left(- \int d^d\mathbf{x} \left(\phi_\Omega(\mathbf{x}) - \mathcal{O}^{-\frac{1}{2}} f_{\alpha,\beta}(t,\mathbf{x}) \right) [\mathcal{O}\phi_\Omega](\mathbf{x}) \right) \Big|_{\alpha=\beta=0} \end{aligned} \tag{4.85}$$

In a similar manner, we obtain

$$\begin{aligned} P_{1\overline{\Omega}}[g_{\overline{\Omega}}] =& \frac{4}{\gamma} \left[\frac{\det\left(\mu^{-1}\mathcal{O}\right)}{\det_{\Omega}\left(\mu^{-1}\mathcal{O}\right)} \right]^{1/2} \\ &\times \left(\frac{\delta^2}{\delta\alpha^2} - \frac{\delta^2}{\delta\beta^2} \right) \exp\left(- \int d^d\mathbf{x} \left(\phi_{\overline{\Omega}}(\mathbf{x}) - \mathcal{O}^{-\frac{1}{2}} f_{\alpha,\beta}(t,\mathbf{x}) \right) [\mathcal{O}\phi_{\overline{\Omega}}](\mathbf{x}) \right) \Big|_{\alpha=\beta=0} \end{aligned} \tag{4.86}$$

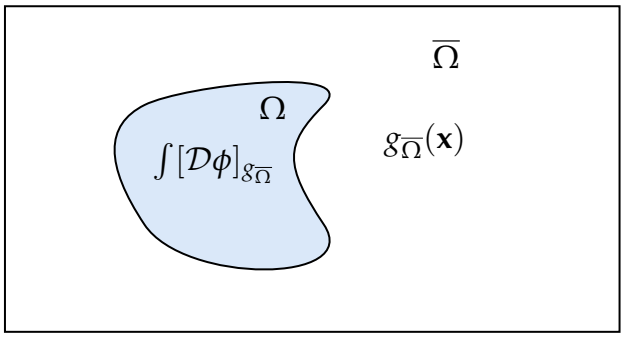


Figure 4.5

4.4 Shannon Mutual Information in the Excited State Without Replica Trick

Now, consider the mutual information

$$I(\Omega, \overline{\Omega}) = \int [\mathcal{D}\phi] P[\phi] \ln \left(\frac{P[\phi]}{P_{\Omega}[g_{\Omega}] P_{\overline{\Omega}}[g_{\overline{\Omega}}]} \right) \tag{4.87}$$

with

$$\begin{aligned} P_1[\phi] =& \frac{4}{\gamma} \left[\det\left(\mu^{-1} \mathcal{O} \right) \right]^{1/2} \\ & \times \left(\frac{\delta^2}{\delta \alpha^2} - \frac{\delta^2}{\delta \beta^2} \right) \exp \left(- \int d^d \mathbf{x} \left(\phi(\mathbf{x}) [\mathcal{O}\phi](\mathbf{x}) - f_{\alpha,\beta}(t, \mathbf{x}) \mathcal{O}^{\frac{1}{2}} \phi(\mathbf{x}) \right) \right) \Big|_{\alpha=\beta=0} . \end{aligned} \tag{4.88}$$

$$P_{1\Omega}[g_{\Omega}] = \left[\det_{\overline{\Omega}} \left(\mu^{-1} \mathcal{O} \right) \right]^{-1/2} P_1[\phi_{\Omega}] \tag{4.89}$$

$$P_{1\overline{\Omega}}[g_{\overline{\Omega}}] = \left[\det_{\Omega} \left(\mu^{-1} \mathcal{O} \right) \right]^{-1/2} P_1[\phi_{\overline{\Omega}}] \tag{4.90}$$

For the excited state, we have

$$
\begin{aligned}
I_1(\Omega,\overline{\Omega}) &= \int [\mathcal{D}\phi] P_1[\phi] \ln\left(\frac{P_1[\phi]}{P_{1\Omega}[g_\Omega] P_{1\overline{\Omega}}[g_{\overline{\Omega}}]}\right) \\
&= \int [\mathcal{D}\phi] P_1[\phi] \ln\left(\frac{P_1[\phi]}{\left[\det_{\overline{\Omega}}\left(\mu^{-1}\mathcal{O}\right)\right]^{-\frac{1}{2}} P_1[\phi_\Omega]\left[\det_{\Omega}\left(\mu^{-1}\mathcal{O}\right)\right]^{-\frac{1}{2}} P_1[\phi_{\overline{\Omega}}]}\right) \\
&= \frac{1}{2}\ln\left(\det_{\overline{\Omega}}\left(\mu^{-1}\mathcal{O}\right)\det_{\Omega}\left(\mu^{-1}\mathcal{O}\right)\right) + \int [\mathcal{D}\phi] P_1[\phi] \ln\left(\frac{P_1[\phi]}{P_1[\phi_\Omega] P_1[\phi_{\overline{\Omega}}]}\right)
\end{aligned}
\tag{4.91}
$$

where we use the normalized condition

$$
\int [\mathcal{D}\phi] P_1[\phi] = 1 \tag{4.92}
$$

Now, we compute the expression in the Logarithm

$$
\begin{aligned}
\ln\left(\frac{P_1[\phi]}{P_1[\phi_\Omega] P_1[\phi_{\overline{\Omega}}]}\right) =& \ln\left(\frac{\gamma}{4}\right) - \frac{1}{2}\ln\left(\det\left(\mu^{-1}\mathcal{O}\right)\right) \\
&+ 2\ln\left|\frac{\int d^d\mathbf{z} f(t,\mathbf{z})\left[\mathcal{O}^{1/2}\phi\right](\mathbf{z})}{\left(\int d^d\mathbf{x} f(t,\mathbf{x})\mathcal{O}^{\frac{1}{2}}\phi_\Omega(\mathbf{x})\right)\left(\int d^d\mathbf{y} f(t,\mathbf{y})\mathcal{O}^{\frac{1}{2}}\phi_{\overline{\Omega}}(\mathbf{y})\right)}\right| \\
&- \int d^d\mathbf{x}\, \phi(\mathbf{x})[\mathcal{O}\phi](\mathbf{x}) + \int d^d\mathbf{x}\, \phi_\Omega(\mathbf{x})[\mathcal{O}\phi_\Omega](\mathbf{x}) \\
&+ \int d^d\mathbf{x}\, \phi_{\overline{\Omega}}(\mathbf{x})[\mathcal{O}\phi_{\overline{\Omega}}](\mathbf{x})
\end{aligned}
\tag{4.93}
$$

Considering that the mutual information produced by the ground state, which is given by

$$
\begin{aligned}
I_0(\Omega,\overline{\Omega}) =& \frac{1}{2}\ln\left(\frac{\det_{\overline{\Omega}}\left(\mu^{-1}\mathcal{O}\right)\det_{\Omega}\left(\mu^{-1}\mathcal{O}\right)}{\det\left(\mu^{-1}\mathcal{O}\right)}\right) \\
&+ \int [\mathcal{D}\phi] P_0[\phi] \int d^d\mathbf{x}\Big(\phi_\Omega(\mathbf{x})\mathcal{O}\phi_\Omega(\mathbf{x}) + \phi_{\overline{\Omega}}(\mathbf{x})\mathcal{O}\phi_{\overline{\Omega}}(\mathbf{x}) - \phi(\mathbf{x})\mathcal{O}\phi(\mathbf{x})\Big)
\end{aligned}
\tag{4.94}
$$

Then, the mutual information produced by the excited state will be

$$\begin{aligned} I_1(\Omega,\overline{\Omega}) =& I_0(\Omega,\overline{\Omega}) + \ln\left(\frac{\gamma}{4}\right) \\ &+ 2\int [\mathcal{D}\phi] P_1[\phi] \ln\left|\frac{\int d^d\mathbf{z} f(t,\mathbf{z})\left[\mathcal{O}^{1/2}\phi\right](\mathbf{z})}{\int d^d\mathbf{x} f(t,\mathbf{x})\mathcal{O}^{1/2}\phi_\Omega(\mathbf{x}) \int d^d\mathbf{y} f(t,\mathbf{y})\mathcal{O}^{1/2}\phi_{\overline{\Omega}}(\mathbf{y})}\right| \\ &+ \int [\mathcal{D}\phi]\Big(P_1[\phi] - P_0[\phi]\Big) \int d^d\mathbf{x}\Big(\phi_\Omega(\mathbf{x})[\mathcal{O}\phi_\Omega](\mathbf{x}) + \phi_{\overline{\Omega}}(\mathbf{x})[\mathcal{O}\phi_{\overline{\Omega}}](\mathbf{x}) \\ &- \phi(\mathbf{x})[\mathcal{O}\phi](\mathbf{x})\Big) \end{aligned} \tag{4.95}$$

4.5 Replica Trick for the Shannon Entropy

Since the integral of the third term of (4.95) contains a logarithm, we do not know how to compute the mutual information, such that it will be independent of the field. For this reason, instead of the usual definition for the entanglement entropy, we will use the Renyi entropy.

4.5.1 Calculation of Shannon Entropies over the Regions Ω and $\overline{\Omega}$

The Shannon entropy of the excited state in the subregion Ω will be

$$S_1[\Omega] = -\,\partial_n \ln Z_{1n}^{\Omega}\Big|_{n=1} \tag{4.96}$$

where

$$Z_{1n}^{\Omega} = \int [\mathcal{D}g_\Omega]\,\left(P_{1\Omega}[g_\Omega]\right)^n \tag{4.97}$$

and

$$\int [\mathcal{D}\phi] = \int [\mathcal{D}g_\Omega] \int [\mathcal{D}\phi]_{g_\Omega} \tag{4.98}$$

with

$$P_{1\Omega}[g_\Omega] = \frac{4}{\gamma}\left[\frac{\det\left(\mu^{-1}\mathcal{O}\right)}{\det_{\overline{\Omega}}\left(\mu^{-1}\mathcal{O}\right)}\right]^{1/2}\left(\frac{\delta^2}{\delta\alpha^2} - \frac{\delta^2}{\delta\beta^2}\right)\exp\left(W[\phi_\Omega, f_{\alpha,\beta}]\right)\Big|_{\alpha=\beta=0}, \tag{4.99}$$

and for the entropy of the excited state in the subregion $\overline{\Omega}$ we have to consider

$$P_{1\overline{\Omega}}[g_{\overline{\Omega}}] = \frac{4}{\gamma}\left[\frac{\det\left(\mu^{-1}\mathcal{O}\right)}{\det_{\Omega}\left(\mu^{-1}\mathcal{O}\right)}\right]^{1/2}\left(\frac{\delta^2}{\delta\alpha^2}-\frac{\delta^2}{\delta\beta^2}\right)\exp\left(W[\phi_{\overline{\Omega}},f_{\alpha,\beta}]\right)\Big|_{\alpha=\beta=0} \tag{4.100}$$

Thus, we have the following result

$$Z^{\Omega}_{1n} = \frac{2^{2n}}{\gamma^n}\left[\frac{\det\left(\mu^{-1}\mathcal{O}\right)}{\det_{\overline{\Omega}}\left(\mu^{-1}\mathcal{O}\right)}\right]^{n/2}\int[\mathcal{D}g_{\Omega}]\left[\left(\frac{\delta^2}{\delta\alpha^2}-\frac{\delta^2}{\delta\beta^2}\right)\exp\left(W[\phi_{\Omega},f_{\alpha\beta}]\right)\Big|_{\alpha=\beta=0}\right]^n \tag{4.101}$$

The variations with respect α and β of $W[\phi_{\Omega},f_{\alpha,\beta}]$ are obtained in (B.24). Thus, the reduced probability $P_{1\Omega}[g_{\Omega}]$ (4.99) will be

$$P_{1\Omega}[g_{\Omega}] = \frac{4}{\gamma}\left[\frac{\det\left(\mu^{-1}\mathcal{O}\right)}{\det_{\overline{\Omega}}\left(\mu^{-1}\mathcal{O}\right)}\right]^{1/2}\left[\left|\int_{\Omega}d^d\mathbf{z}\;g(\mathbf{z})a_f(\mathbf{z})\right|^2+2b_R+2b_I\right]\exp(W[g_{\Omega}]). \tag{4.102}$$

Therefore, the entanglement entropy of the excited state in the subregion Ω (4.101) will be

$$S_1[\Omega] = -\,\partial_n \ln Z^{\Omega}_{1(n)}\Big|_{n=1} \tag{4.103}$$

where

$$Z^{\Omega}_{1(n)} = \frac{4^n}{\gamma^n}\left[\frac{\det\left(\mu^{-1}\mathcal{O}\right)}{\det_{\overline{\Omega}}\left(\mu^{-1}\mathcal{O}\right)}\right]^{\frac{n}{2}}\int[\mathcal{D}g_{\Omega}]\left[\left|\int_{\Omega}d^d\mathbf{z}g(\mathbf{z})a_f(\mathbf{z})\right|^2+2b_R+2b_I\right]^n e^{nW[g_{\Omega}]} \tag{4.104}$$

Renyi entropy for $n = 2$

At this point, given the complexity of calculating Shannon entropy (4.103), it becomes more convenient to first compute the Rényi entropy for $n = 2$ and then generalize to any n. Thus, from (4.103), for the Rényi entropy with $n = 2$, we must have:

$$S^{(2)}_1[\Omega] = -\ln Z^{\Omega}_{1,(2)} \tag{4.105}$$

where

$$Z^{\Omega}_{1,(2)} = \frac{16}{\gamma^2}\left[\frac{\det\left(\mu^{-1}\mathcal{O}\right)}{\det_{\overline{\Omega}}\left(\mu^{-1}\mathcal{O}\right)}\right]\int[\mathcal{D}g_{\Omega}]\left[\left|\int_{\Omega} d^d\mathbf{z}\, g(\mathbf{z})a_f(\mathbf{z})\right|^4 + 4(b_R+b_I)\left|\int_{\Omega} d^d\mathbf{z}\, g(\mathbf{z})a_f(\mathbf{z})\right|^2 + 4(b_R+b_I)^2\right]\exp(2W[g_{\Omega}]) \tag{4.106}$$

Now, to calculate the integral $\int[\mathcal{D}g_{\Omega}]$ in order to obtain the Renyi entropy, let us define the generating functional

$$\begin{aligned} Z[J] &= \int[\mathcal{D}g_{\Omega}]e^{nW[g_{\Omega}]+\int_{\Omega} d^d\mathbf{x}\, g_{\Omega}(\mathbf{x})J(\mathbf{x})} \\ &= \int[\mathcal{D}g_{\Omega}]\exp\left(-\frac{n}{2}\int_{\Omega} d^d\mathbf{x}\, g_{\Omega}(\mathbf{x})\int_{\Omega} d^d\mathbf{z}\, g(\mathbf{z})\hat{o}(\mathbf{x},\mathbf{z}) + \int_{\Omega} d^d\mathbf{x}\, g_{\Omega}(\mathbf{x})J(\mathbf{x})\right), \end{aligned} \tag{4.107}$$

where we have taken into account the expression for $W[g_{\Omega}]$ given in (D.9), setting $\alpha = \beta = 0$

$$W[\phi_{\Omega}] = -\frac{1}{2}\int_{\Omega} d^d\mathbf{x}\, g_{\Omega}(\mathbf{x})\int_{\Omega} d^d\mathbf{z}\, g(\mathbf{z})\hat{o}(\mathbf{x},\mathbf{z}). \tag{4.108}$$

We use the fact again

$$\int[\mathcal{D}\phi]\exp\left(-\frac{1}{2}\int d^d\mathbf{x}\,\left(\phi(\mathbf{x})[A\phi](\mathbf{x}) - 2\rho(\mathbf{x})\phi(\mathbf{x})\right)\right) = [\det(A)]^{-1/2}\exp\left(\frac{1}{2}\int d^d\mathbf{x}\,\rho(\mathbf{x})\left[A^{-1}\rho\right](\mathbf{x})\right) \tag{4.109}$$

considering that $A \equiv no$ and $\rho \equiv J$. Then, we obtain that

$$\begin{aligned} Z[J] &= [\det(n\hat{o})]^{-1/2}\exp\left(\frac{1}{2n}\int_{\Omega} d^d\mathbf{x}\, J(\mathbf{x})\left[o^{-1}J\right](\mathbf{x})\right) \\ &= [\det(n\hat{o})]^{-1/2}\exp\left(\frac{1}{2n}\int_{\Omega} d^d\mathbf{x}\, J(\mathbf{x})\int_{\Omega} d^d\mathbf{y}\, J(\mathbf{y})\,\hat{o}^{-1}(\mathbf{x},\mathbf{y})\right) \end{aligned} \tag{4.110}$$

Notice that the integral of the product of any number of g_{Ω} with the Gaussian

factor can be written in terms of derivatives of $Z[J]$, which are known:

$$\int [\mathcal{D}g_\Omega] g_\Omega(\mathbf{x}_1)\cdots g_\Omega(\mathbf{x}_n)\exp\left(-\frac{n}{2}\int_\Omega d^d\mathbf{x} g_\Omega(\mathbf{x})\int_\Omega d^d\mathbf{z} g(\mathbf{z})\hat{o}(\mathbf{x},\mathbf{z})\right) = \frac{\delta Z[J]}{\delta J(\mathbf{x}_1)\cdots\delta J(\mathbf{x}_n)}\bigg|_{J=0} \tag{4.111}$$

From equation (4.105), we see that it is necessary to determine the second and fourth variations of the generating functional $Z[J]$. Thus the first derivative will be

$$\frac{\delta Z[J]}{\delta J(\mathbf{x}_1)} = [\det(o)]^{-1/2}\int_\Omega d^d\mathbf{y}\, J(\mathbf{y})\hat{o}_s^{-1}(\mathbf{x}_1,\mathbf{y})\exp\left(\frac{1}{2n}\int_\Omega d^d\mathbf{x}\, J(\mathbf{x})\left[o^{-1}J\right](\mathbf{x})\right) \tag{4.112}$$

where we have defined

$$\frac{1}{2}\left(\hat{o}^{-1}(\mathbf{x}_1,\mathbf{x}_2)+\hat{o}^{-1}(\mathbf{x}_2,\mathbf{x}_1)\right)\equiv\hat{o}_s^{-1}(\mathbf{x}_2,\mathbf{x}_1) \tag{4.113}$$

The second derivative will be

$$\begin{aligned}\frac{\delta Z[J]}{\delta J(\mathbf{x}_1)\delta J(\mathbf{x}_2)} =& [\det(\hat{o})]^{-1/2}\,\hat{o}_s^{-1}(\mathbf{x}_1,\mathbf{x}_2)\exp\left(\frac{1}{2}\int_\Omega d^d\mathbf{x}\, J(\mathbf{x})\left[o^{-1}J\right](\mathbf{x})\right)\\ &+[\det(\hat{o})]^{-1/2}\int_\Omega d^d\mathbf{y}\, J(\mathbf{y})\hat{o}_s^{-1}(\mathbf{x}_1,\mathbf{y})\int_\Omega d^d\mathbf{y}\, J(\mathbf{y})\hat{o}_s^{-1}(\mathbf{x}_2,\mathbf{y})\\ &\times\exp\left(\frac{1}{2}\int_\Omega d^d\mathbf{x}\, J(\mathbf{x})\left[o^{-1}J\right](\mathbf{x})\right)\end{aligned} \tag{4.114}$$

then

$$\frac{\delta Z[J]}{\delta J(\mathbf{x}_1)\delta J(\mathbf{x}_2)}\bigg|_{J=0} = [\det(\hat{o})]^{-1/2}\,\hat{o}_s^{-1}(\mathbf{x}_1,\mathbf{x}_2) \tag{4.115}$$

Finally, the fourth variation will be

$$\begin{aligned}\frac{\delta Z[J]}{\delta J(\mathbf{x}_1)\delta J(\mathbf{x}_2)\delta J(\mathbf{x}_3)\delta J(\mathbf{x}_4)}\bigg|_{J=0} =& [\det(\hat{o})]^{-1/2}\,\Big[\hat{o}_s^{-1}(\mathbf{x}_1,\mathbf{x}_2)\,\hat{o}_s^{-1}(\mathbf{x}_3,\mathbf{x}_4)\\ &+\hat{o}_s^{-1}(\mathbf{x}_1,\mathbf{x}_3)\hat{o}_s^{-1}(\mathbf{x}_2,\mathbf{x}_4)+\hat{o}_s^{-1}(\mathbf{x}_1,\mathbf{x}_4)\hat{o}_s^{-1}(\mathbf{x}_2,\mathbf{x}_3)\Big]\end{aligned} \tag{4.116}$$

Take into account (4.115), (4.116), and the result of the normalization condition for the vacuum (D.5) with the change $\hat{o}\leftrightarrow 2\hat{o}$. From (4.106) we will have

$$
\begin{aligned}
Z^{\Omega}_{1,(2)} =& \frac{16}{\gamma^2}\left[\frac{\det\left(\mu^{-1}\mathcal{O}\right)}{\det_{\overline{\Omega}}\left(\mu^{-1}\mathcal{O}\right)}\right]\left[\frac{\det_{\overline{\Omega}}\left(\mu^{-1}2\mathcal{O}\right)}{\det\left(\mu^{-1}2\mathcal{O}\right)}\right]^{\frac{1}{2}} \\
&\times\left[2\left(\frac{1}{2}\int_{\Omega} d^d\mathbf{x}_1 a_f(\mathbf{x}_1)\left[o_s^{-1}a_{f^*}\right](\mathbf{x}_1)\right)^2+\left|\frac{1}{2}\int_{\Omega} d^d\mathbf{x} a_f(\mathbf{x})\left[o_s^{-1}a_f\right](\mathbf{x})\right|^2\right. \\
&\left.+4(b_R+b_I)\frac{1}{2}\int_{\Omega} d^d\mathbf{x}\, a_f(\mathbf{x})\left[o_s^{-1}a_{f^*}\right](\mathbf{x})+4(b_R+b_I)^2\right]. \qquad (4.117)
\end{aligned}
$$

Considering the result of the normalization condition for the excited state (D.19) we have

$$
\begin{aligned}
Z^{\Omega}_{1,(2)} =& \frac{16}{\gamma^2}\left[\frac{\det\left(\mu^{-1}\mathcal{O}\right)}{\det_{\overline{\Omega}}\left(\mu^{-1}\mathcal{O}\right)}\right]\left[\frac{\det_{\overline{\Omega}}\left(\mu^{-1}2\mathcal{O}\right)}{\det\left(\mu^{-1}2\mathcal{O}\right)}\right]^{\frac{1}{2}} \\
&\times\left[\frac{1}{2}\left(\frac{\gamma}{4}-2(b_R+b_I)\right)^2+\frac{1}{4}\left|\int_{\Omega} d^d\mathbf{x}\, a_f(\mathbf{x})\left[o_s^{-1}a_f\right](\mathbf{x})\right|^2\right. \\
&\left.+2(b_R+b_I)\left(\frac{\gamma}{4}-2(b_R+b_I)\right)+4(b_R+b_I)^2\right] \\
=& \frac{16}{\gamma^2}\left[\frac{\det\left(\mu^{-1}\mathcal{O}\right)}{\det_{\overline{\Omega}}\left(\mu^{-1}\mathcal{O}\right)}\right]\left[\frac{\det_{\overline{\Omega}}\left(\mu^{-1}2\mathcal{O}\right)}{\det\left(\mu^{-1}2\mathcal{O}\right)}\right]^{\frac{1}{2}} \\
&\times\left[\frac{1}{4}\left|\int_{\Omega} d^d\mathbf{x}\, a_f(\mathbf{x})\left[o_s^{-1}a_f\right](\mathbf{x})\right|^2+\frac{\gamma^2}{32}+2\left(b_R+b_I\right)^2\right] \qquad (4.118)
\end{aligned}
$$

Now, if we take into account the expression for the Shannon entropy of the vacuum state (3.68) we can note that

$$
S_1^{(2)}[\Omega] = -\ln Z^{\Omega}_{1,(2)} = S_0^{(2)}[\Omega] - \ln \tilde{F}^{(2)}_{\Omega}(t) \qquad (4.119)
$$

where

$$
\tilde{F}^{(2)}_{\Omega}(t) = \frac{4}{\gamma^2}\left|\int_{\Omega} d^d\mathbf{x}\, a_f(\mathbf{x})\left[o_s^{-1}a_f\right](\mathbf{x})\right|^2+\frac{1}{2}+\frac{32}{\gamma^2}\left(b_R+b_I\right)^2 \qquad (4.120)
$$

To obtain the Rényi entropy with $n = 2$ over the region $\overline{\Omega}$, we can follow the same procedure, which will result in an expression similar to the Rényi entropy

over the region Ω, but with the exchange of regions $\Omega \leftrightarrow \overline{\Omega}$

$$S_1^{(2)}[\overline{\Omega}] = S_0^{(2)}[\overline{\Omega}] - \ln \tilde{F}_{\overline{\Omega}}^{(2)}(t) \tag{4.121}$$

where

$$\tilde{F}_{\overline{\Omega}}^{(2)}(t) = \frac{4}{\gamma^2}\left|\int_{\overline{\Omega}} d^d\mathbf{x}\, \overline{a}_f(t,\mathbf{x})\left[\overline{o}_s^{-1}\overline{a}_f\right](,\mathbf{x})\right|^2 + \frac{1}{2} + \frac{32}{\gamma^2}\left(\overline{b}_R(t) + \overline{b}_I(t)\right)^2 \tag{4.122}$$

with

$$\frac{1}{2}\hat{\overline{o}}(\mathbf{x},\mathbf{z}) \equiv [\mathcal{O}_{\overline{\Omega}}\delta](\mathbf{x}-\mathbf{z}) + \delta(\mathbf{x}-\mathbf{z})\int_\Omega d^d\mathbf{y}\, K(\mathbf{x}-\mathbf{y}) - \int_\Omega d^d\mathbf{y} K(\mathbf{x}-\mathbf{y})\overline{\mathcal{P}}(\mathbf{z},\mathbf{y}) \tag{4.123}$$

$$\begin{aligned}\overline{a}_f(t,\mathbf{z}) \equiv &\left[\mathcal{O}^{1/2}f\right](t,\mathbf{z}) + \frac{1}{2}\int_\Omega d^d\mathbf{y} K(\mathbf{z}-\mathbf{y})\overline{\mathcal{G}}\left[\mathcal{O}^{1/2}f\right](t,\mathbf{y}) \\ &+ \frac{1}{2}\int_\Omega d^d\mathbf{x}\overline{\mathcal{P}}(\mathbf{z},\mathbf{x})\left[\mathcal{O}^{1/2}f\right](t,\mathbf{x})\end{aligned} \tag{4.124}$$

and

$$\overline{b}_f(t) \equiv \frac{1}{4}\int_\Omega d^d\mathbf{x}\,\overline{\mathcal{G}}\left[\mathcal{O}^{1/2}f\right](t,\mathbf{x})\left[\mathcal{O}^{1/2}f\right](t,\mathbf{x}) \tag{4.125}$$

Renyi entropy for any n

For this general case, from (4.103) we can see that it is necessary to determine the $2m$-th variation of the generating functional $Z[J]$. Inspecting the expressions for the second (4.115) and fourth (4.116) variations of $Z[J]$, we can state that the $2m$-th variation of $Z[J]$ will be composed of products of m disjoint operators $\hat{o}_s^{-1}$

$$\left.\frac{\delta Z[J]}{\delta J(\mathbf{x}_1)\cdots\delta J(\mathbf{x}_{2m})}\right|_{J=0} = \frac{1}{n^m}[\det(o)]^{-1/2}\sum_{\text{permut}}\hat{o}_s^{-1}(\mathbf{x}_{k_1},\mathbf{x}_{k_2})\cdots\hat{o}_s^{-1}(\mathbf{x}_{k_{2m-1}},\mathbf{x}_{k_{2m}}). \tag{4.126}$$

The sum over the permutations of the end points $\mathbf{x}_k$ is to be formed in such a way that each pair $(\mathbf{x}_i,\mathbf{x}_j)$ enters only once and the pairs are ordered, $i > j$, to avoid double counting, being the total number of terms in (4.126) equal to $(2m-1)!!$.

Inserting this expression in (4.104) and using Newton's binomial theorem, we

have

$$
\begin{aligned}
Z^{\Omega}_{1(n)} =& \frac{4^n}{\gamma^n}\left[\frac{\det\left(\mu^{-1}\mathcal{O}\right)}{\det_{\overline{\Omega}}\left(\mu^{-1}\mathcal{O}\right)}\right]^{\frac{n}{2}} \int [\mathcal{D}g_{\Omega}] \left[\left|\int_{\Omega} d^d\mathbf{z} g(\mathbf{z}) a_f(\mathbf{z})\right|^2 + 2b_R + 2b_I\right]^n e^{nW[g_{\Omega}]} \\
=& \frac{2^{2n}}{\gamma^n}\left[\frac{\det\left(\mu^{-1}\mathcal{O}\right)}{\det_{\overline{\Omega}}\left(\mu^{-1}\mathcal{O}\right)}\right]^{\frac{n}{2}} \sum_{m=1}^{n} 2^{n-m}\binom{n}{m}(b_R+b_I)^{n-m} \\
& \times \int_{\Omega} d^d\mathbf{z}_1 \cdots d^d\mathbf{z}_{2m}\, a_f(\mathbf{z}_1)\cdots a_f(\mathbf{z}_m) a_{f^*}(\mathbf{z}_{m+1})\cdots a_{f^*}(\mathbf{z}_{2m}) \\
& \times \left.\frac{\delta Z[J]}{\delta J(\mathbf{z}_1)\cdots\delta J(\mathbf{z}_{2m})}\right|_{J=0} \\
=& \frac{2^{2n}}{\gamma^n}\left[\frac{\det\left(\mu^{-1}\mathcal{O}\right)}{\det_{\overline{\Omega}}\left(\mu^{-1}\mathcal{O}\right)}\right]^{\frac{n}{2}} \left[\frac{\det_{\overline{\Omega}}\left(\mu^{-1}n\mathcal{O}\right)}{\det\left(\mu^{-1}n\mathcal{O}\right)}\right]^{1/2} \sum_{m=1}^{n} 2^{n-m}\binom{n}{m}(b_R+b_I)^{n-m}\frac{1}{n^m} \\
& \times \sum_{\text{permut}} \int_{\Omega} d^d\mathbf{z}_1 \cdots d^d\mathbf{z}_{2m}\, a_f(\mathbf{z}_1)\cdots a_f(\mathbf{z}_m) a_{f^*}(\mathbf{z}_{m+1})\cdots a_{f^*}(\mathbf{z}_{2m}) \\
& \times \hat{o}_s^{-1}(\mathbf{z}_{k_1},\mathbf{z}_{k_2})\cdots\hat{o}_s^{-1}(\mathbf{z}_{k_{2m-1}},\mathbf{z}_{k_{2m}})
\end{aligned}
\tag{4.127}
$$

We can note that

$$
\begin{aligned}
& \sum_{\text{permut}} \int_{\Omega} d^d\mathbf{z}_1 \cdots d^d\mathbf{z}_{2m}\, a_f(\mathbf{z}_1)\cdots a_f(\mathbf{z}_m) a_{f^*}(\mathbf{z}_{m+1})\cdots a_{f^*}(\mathbf{z}_{2m}) \\
& \qquad \times \hat{o}_s^{-1}(\mathbf{z}_{k_1},\mathbf{z}_{k_2})\cdots\hat{o}_s^{-1}(\mathbf{z}_{k_{2m-1}},\mathbf{z}_{k_{2m}}) \\
& = \sum_{2r+l=m} C_{rl} \left|\int_{\Omega} d^d\mathbf{x}\, a_f(\mathbf{x}) \left[o_s^{-1} a_f\right](\mathbf{x})\right|^{2r} \left(\int_{\Omega} d^d\mathbf{x}\, a_f(\mathbf{x}) \left[o_s^{-1} a_{f^*}\right](\mathbf{x})\right)^l
\end{aligned}
\tag{4.128}
$$

where C_{rl} is a coefficient that satisfies

$$
\sum_{2r+l=m} C_{rl} = (2m-1)!! \tag{4.129}
$$

and we noting by direct evaluation that $C_{01} = 1$, $C_{02} = 2$, $C_{10} = 1$, $C_{30} = 6$ and

$C_{11} = 9$. Then, inserting the general term (4.128) in (4.127), we obtain

$$Z^{\Omega}_{1(n)} = \frac{2^{3n}}{\gamma^n}\left[\frac{\det\left(\mu^{-1}\mathcal{O}\right)}{\det_{\overline{\Omega}}\left(\mu^{-1}\mathcal{O}\right)}\right]^{\frac{n}{2}}\left[\frac{\det_{\overline{\Omega}}\left(\mu^{-1}n\mathcal{O}\right)}{\det\left(\mu^{-1}n\mathcal{O}\right)}\right]^{\frac{1}{2}}\sum_{m=1}^{n}\left(\frac{1}{2n}\right)^m\binom{n}{m}(b_R+b_I)^{n-m} \times \sum_{2r+l=m} C_{rl}\left|\int_{\Omega} d^d\mathbf{x}\, a_f(\mathbf{x})\left[o_s^{-1}a_f\right](\mathbf{x})\right|^{2r}\left(\int_{\Omega} d^d\mathbf{x}\, a_f(\mathbf{x})\left[o_s^{-1}a_{f^*}\right](\mathbf{x})\right)^{l} \tag{4.130}$$

From the normalization condition for the excited state (D.19)

$$\int_{\Omega} d^d\mathbf{x}\, a_f(\mathbf{x})\left[o_s^{-1}a_{f^*}\right](\mathbf{x}) = \frac{\gamma}{4} - 2(b_R+b_I) \tag{4.131}$$

and apply again Newton's binomial theorem

$$Z^{\Omega}_{1(n)} = \frac{2^{3n}}{\gamma^n}\left[\frac{\det\left(\mu^{-1}\mathcal{O}\right)}{\det_{\overline{\Omega}}\left(\mu^{-1}\mathcal{O}\right)}\right]^{\frac{n}{2}}\left[\frac{\det_{\overline{\Omega}}\left(\mu^{-1}n\mathcal{O}\right)}{\det\left(\mu^{-1}n\mathcal{O}\right)}\right]^{\frac{1}{2}}\sum_{m=1}^{n}\left(\frac{1}{2n}\right)^m\binom{n}{m}(b_R+b_I)^{n-m} \times \sum_{2r+l=m} C_{rl}\left|\int_{\Omega} d^d\mathbf{x}\, a_f(\mathbf{x})\left[o_s^{-1}a_f\right](\mathbf{x})\right|^{2r}\sum_{k=1}^{l}(-1)^k\binom{l}{k}(b_R+b_I)^k\left(\frac{\gamma}{4}\right)^{l-k} \tag{4.132}$$

Now, if we take into account the expression for the Shannon entropy of the vacuum state (3.68) we can note that the Shannon entropy of the excited state over the region Ω must be given by

$$S_1[\Omega] = -\,\partial_n Z^{\Omega}_{1(n)}\Big|_{n=1} = S_0[\Omega] - \partial_n \ln\left\{\tilde{F}^{(n)}_{\Omega}(t)\right\}\Big|_{n=1} \tag{4.133}$$

where

$$\tilde{F}^{(n)}_{\Omega}(t) = \sum_{m=1}^{n}\binom{n}{m}\sum_{2r+l=m} C_{rl}\sum_{k=1}^{l}(-1)^k\binom{l}{k}\left|\int_{\Omega} d^d\mathbf{x}\, a_f(\mathbf{x})\left[o_s^{-1}a_f\right](\mathbf{x})\right|^{2r} \times (b_R+b_I)^{n-m+k}\frac{2^{3n+2k-m-2l}}{n^m\gamma^{n+k-l}} \tag{4.134}$$

In this way, we can see that to determine the Shannon entropy of the excited state is necessary determine b_R, b_I and $\int_{\Omega} d^d\mathbf{x}\, a_f(\mathbf{x})\left[o_s^{-1}a_f\right](\mathbf{x})$.

In a similar way we can obtain the Shannon entropy of the excited state over

the region $\overline{\Omega}$

$$S_1[\overline{\Omega}] = S_0[\overline{\Omega}] - \partial_n \ln \left\{ \tilde{F}^{(n)}_{\overline{\Omega}}(t) \right\} \Big|_{n=1} \tag{4.135}$$

where

$$\tilde{F}^{(n)}_{\overline{\Omega}}(t) = \sum_{m=1}^{n} \binom{n}{m} \sum_{2r+l=m} C_{rl} \sum_{k=1}^{l} (-1)^k \binom{l}{k} \left| \int_{\overline{\Omega}} d^d\mathbf{x}\, \overline{a}_f(\mathbf{x}) \left[\overline{o}_s^{-1} \overline{a}_f \right](\mathbf{x}) \right|^{2r} \times (\overline{b}_R + \overline{b}_I)^{n-m+k} \frac{2^{3n+2k-m-2l}}{n^m \gamma^{n+k-l}} \tag{4.136}$$

4.5.2 Calculation of Shannon Entropy over the Region $\Omega \cup \overline{\Omega}$

The Shannon entropy over the region $\Omega \cup \overline{\Omega}$ is given by

$$S_1[\Omega \cup \overline{\Omega}] = -\partial_n Z^{\Omega\cup\overline{\Omega}}_{1(n)} \Big|_{n=1}, \tag{4.137}$$

where

$$Z^{\Omega\cup\overline{\Omega}}_{1(n)} = \int [\mathcal{D}\phi]\, (P_1[\phi])^n \tag{4.138}$$

and

$$P_1[\phi] = \frac{4}{\gamma} \left[\det\left(\mu^{-1}\mathcal{O}\right) \right]^{\frac{1}{2}} \exp\left(- \int d^d\mathbf{x} \phi(\mathbf{x}) [\mathcal{O}\phi](\mathbf{x}) \right) \left| \int d^d\mathbf{y} f(t,\mathbf{y}) \left[\mathcal{O}^{\frac{1}{2}} \phi \right](\mathbf{y}) \right|^2 \tag{4.139}$$

The calculation of Shannon entropy over the region $\Omega \cup \overline{\Omega}$ can be performed in two different but equivalent ways. The first method consists of expressing the replicated partition function $Z^{\Omega\cup\overline{\Omega}}_{1(n)}$ in terms of variations of α and β using (4.139).

$$\begin{aligned} Z^{\Omega\cup\overline{\Omega}}_{1(n)} &= \left(\frac{4}{n\gamma}\right)^n \left[\det\left(\mu^{-1}\mathcal{O}\right) \right]^{\frac{n}{2}} \int [\mathcal{D}\phi] \exp\left(- \int d^d\mathbf{x}\, n\phi(\mathbf{x}) [\mathcal{O}\phi](\mathbf{x}) \right) \\ &\quad \times \left| \int d^d\mathbf{y} f(t,\mathbf{y}) \left[n^{\frac{1}{2}} \mathcal{O}^{\frac{1}{2}} \phi \right](\mathbf{y}) \right|^{2n} \\ &= \frac{2^{2n}}{(n\gamma)^n} \left[\det\left(\mu^{-1}\mathcal{O}\right) \right]^{\frac{n}{2}} \left(\frac{\delta^2}{\delta\alpha^2} - \frac{\delta^2}{\delta\beta^2} \right)^n \int [\mathcal{D}\phi] \exp\Bigg(- \int d^d\mathbf{x} \Big(n\phi(\mathbf{x}) \mathcal{O}\phi(\mathbf{x}) \\ &\quad - f_{\alpha,\beta}(t,\mathbf{x}) n^{\frac{1}{2}} \mathcal{O}^{\frac{1}{2}} \phi(\mathbf{x}) \Big) \Bigg) \Bigg|_{\alpha=\beta=0} \end{aligned} \tag{4.140}$$

using the property (A.9) of the fractional Laplacian and take into account the fact

$$\int [\mathcal{D}\phi] \exp\left(-\frac{1}{2}\int d^d\mathbf{x}\,(\phi(\mathbf{x})[A\phi](\mathbf{x}) - 2\rho(\mathbf{x})\phi(\mathbf{x}))\right) = [\det(A)]^{-1/2}\exp\left(\frac{1}{2}\int d^d\mathbf{x}\,\rho(\mathbf{x})\left[A^{-1}\rho\right](\mathbf{x})\right), \tag{4.141}$$

with $A \equiv 2n\mathcal{O}$ and $\rho \equiv n^{\frac{1}{2}}\mathcal{O}^{\frac{1}{2}} f_{\alpha,\beta}$. Then, we obtain that

$$\begin{aligned} Z_{1(n)}^{\Omega\cup\overline{\Omega}} =& \frac{2^{2n}}{(n\gamma)^n}\left[\det\left(\mu^{-1}\mathcal{O}\right)\right]^{n/2}\left[\det\left(n\mu^{-1}\mathcal{O}\right)\right]^{-1/2} \\ &\times\left(\frac{\delta^2}{\delta\alpha^2}-\frac{\delta^2}{\delta\beta^2}\right)^n \exp\left(\frac{1}{4}\int d^d\mathbf{x}\, O^{1/2} f_{\alpha,\beta}(\mathbf{x},t)\mathcal{O}^{-1/2} f_{\alpha,\beta}(\mathbf{x},t)\right)\Bigg|_{\alpha=\beta=0} \\ =& \frac{2^{2n}}{(n\gamma)^n}\frac{\left[\det\left(\mu^{-1}\mathcal{O}\right)\right]^{n/2}}{\left[\det\left(n\mu^{-1}\mathcal{O}\right)\right]^{1/2}}\left(\frac{\delta^2}{\delta\alpha^2}-\frac{\delta^2}{\delta\beta^2}\right)^n \exp\left(\frac{1}{4}\int d^d\mathbf{x}\, f_{\alpha,\beta}^2(\mathbf{x},t)\right)\Bigg|_{\alpha=\beta=0} \end{aligned} \tag{4.142}$$

Specifically, we will first determine the Rényi entropy for $n = 2$ and then generalize it to any natural n.

Renyi Entropy Over the Region $\Omega\cup\overline{\Omega}$ for $n = 2$

From (4.142), for the replicated partition function $Z_{1(n)}^{\Omega\cup\overline{\Omega}}$ with $n = 2$ we must have

$$\begin{aligned} Z_{1(2)}^{\Omega\cup\overline{\Omega}} =& \frac{4}{\gamma^2}\left[\det\left(\mu^{-1}\mathcal{O}\right)\right]\left(\frac{\delta^2}{\delta\alpha^2}-\frac{\delta^2}{\delta\beta^2}\right)^2\int[\mathcal{D}\phi]\exp\Bigg(-\int d^d\mathbf{x}\Big(\phi(\mathbf{x})[2\mathcal{O}\phi](\mathbf{x}) \\ &-\sqrt{2}f_{\alpha\beta}(t,\mathbf{x})\mathcal{O}^{\frac{1}{2}}\phi(\mathbf{x})\Big)\Bigg)\Bigg|_{\alpha=\beta=0} \end{aligned} \tag{4.143}$$

using the fact that

$$\int [\mathcal{D}\phi] \exp\left(-\frac{1}{2}\int d^d\mathbf{x}\,(\phi(\mathbf{x})[A\phi](\mathbf{x}) - 2\rho(\mathbf{x})\phi(\mathbf{x}))\right) = [\det(A)]^{-1/2}\exp\left(\frac{1}{2}\int d^d\mathbf{x}\,\rho(\mathbf{x})\left[A^{-1}\rho\right](\mathbf{x})\right), \tag{4.144}$$

considering that $A \equiv 4\mathcal{O}$ and $\rho \equiv \sqrt{2}\mathcal{O}^{1/2} f_{\alpha,\beta}$.

$$\begin{aligned} Z_{1(2)}^{\Omega\cup\overline{\Omega}} &= \left(\frac{2}{\gamma}\right)^2 \left[\det\left(\mu^{-1}\mathcal{O}\right)\right]\ \left[\det\left(\mu^{-1}2\mathcal{O}\right)\right]^{-1/2} \\ &\quad\times\left(\frac{\delta^2}{\delta\alpha^2} - \frac{\delta^2}{\delta\beta^2}\right)^2 \exp\left(\frac{1}{4}\int d^d\mathbf{x}\left[\mathcal{O}^{\frac{1}{2}} f_{\alpha,\beta}\right](\mathbf{x},t)\mathcal{O}^{-1}\left[\mathcal{O}^{\frac{1}{2}} f_{\alpha,\beta}\right](\mathbf{x},t)\right)\Bigg|_{\alpha=\beta=0} \\ &= \left(\frac{2}{\gamma}\right)^2 \frac{\left[\det\left(\mu^{-1}\mathcal{O}\right)\right]}{\left[\det\left(\mu^{-1}2\mathcal{O}\right)\right]^{1/2}}\ \left(\frac{\delta^2}{\delta\alpha^2} - \frac{\delta^2}{\delta\beta^2}\right)^2 \exp\left(\frac{1}{4}\int d^d\mathbf{x}\, f_{\alpha,\beta}^2(\mathbf{x},t)\right)\Bigg|_{\alpha=\beta=0} \end{aligned} \tag{4.145}$$

Now, by a direct calculation we can verify that

$$\begin{aligned} \left(\frac{\delta^2}{\delta\alpha^2} - \frac{\delta^2}{\delta\beta^2}\right)^2 e^{S_{\alpha\beta}}\Bigg|_{\alpha=\beta=0} &= 3\left(\delta_\alpha^2 S\right)^2 + 3\left(\delta_\beta^2 S\right)^2 - 2\delta_\alpha^2 S\delta_\beta^2 S - 4\left(\delta_{\alpha\beta}^2\right)^2 \\ &= \left[\delta_\alpha^2 S + \delta_\beta^2 S + 2\delta_{\alpha\beta}^2\right]\left[\delta_\alpha^2 S + \delta_\beta^2 S - 2\delta_{\alpha\beta}^2\right] + 2\left[\delta_\alpha^2 S - \delta_\beta^2 S\right]^2 \end{aligned} \tag{4.146}$$

with the condition $S_{\alpha\beta} = \delta_\alpha S = \delta_\beta S = 0$ when $\alpha = \beta = 0$.

Also, if we consider that

$$\begin{aligned} S_{\alpha\beta} &= \frac{1}{4}\int d^d\mathbf{x}\, f_{\alpha,\beta}^2(\mathbf{x},t) \\ &= \frac{1}{4}\int d^d\mathbf{x}\,\alpha^2 R^2(\mathbf{x},t) - \frac{1}{4}\int d^d\mathbf{x}\,\beta^2 I^2(\mathbf{x},t) + \frac{i}{2}\int d^d\mathbf{x}\,\alpha\beta I(\mathbf{x},t)R(\mathbf{x},t) \end{aligned} \tag{4.147}$$

We can verify that

$$\frac{1}{2}\int d^d\mathbf{x}\, f^2(\mathbf{x},t) = \delta_\alpha^2 S + \delta_\beta^2 S + 2\delta_{\alpha\beta}^2 S \tag{4.148}$$

$$\frac{1}{2}\int d^d\mathbf{x}\, f^{*2}(\mathbf{x},t) = \delta_\alpha^2 S + \delta_\beta^2 S - 2\delta_{\alpha\beta}^2 S \tag{4.149}$$

$$\frac{\gamma}{4} = \delta_\alpha^2 S - \delta_\beta^2 S \tag{4.150}$$

this yields that

$$\left(\frac{\delta^2}{\delta\alpha^2} - \frac{\delta^2}{\delta\beta^2}\right)^2 e^{S_{\alpha\beta}}\bigg|_{\alpha=\beta=0} = \left|\frac{1}{2}\int d^d\mathbf{x}\, f^2(\mathbf{x},t)\right|^2 + 2\left(\frac{\gamma}{4}\right)^2 \tag{4.151}$$

and therefore the replicated partition function $Z_{1(2)}^{\Omega\cup\overline{\Omega}}$ will be

$$Z_{1(2)}^{\Omega\cup\overline{\Omega}} = \left(\frac{2}{\gamma}\right)^2 \frac{\left[\det\left(\mu^{-1}\mathcal{O}\right)\right]}{\left[\det\left(\mu^{-1}2\mathcal{O}\right)\right]^{1/2}} \left[\left|\frac{1}{2}\int d^d\mathbf{x}\, f^2(\mathbf{x},t)\right|^2 + 2\left(\frac{\gamma}{4}\right)^2\right] \tag{4.152}$$

and if we consider the expression of the Shannon entropy for the vacuum over the region $\Omega\cup\overline{\Omega}$ (3.72). Then

$$S_1[\Omega\cup\overline{\Omega}] = -\ln Z_{1(2)}^{\Omega\cup\overline{\Omega}} = S_0[\Omega\cup\overline{\Omega}] - \ln \tilde{F}_{\Omega\cup\overline{\Omega}}^{(2)}(t) \tag{4.153}$$

where

$$\tilde{F}_{\Omega}^{(2)}(t) = \left(\frac{2}{\gamma}\right)^2 \left[\left|\frac{1}{2}\int d^d\mathbf{x}\, f^2(\mathbf{x},t)\right|^2 + 2\left(\frac{\gamma}{4}\right)^2\right] \tag{4.154}$$

We can obtain this same expression through the second method, which consists of using the generating functional $Z[J]$

$$\begin{aligned} Z[J] &\equiv \int [\mathcal{D}g_\Omega] e^{nW[\phi]+\int d^d\mathbf{x}\,\phi(\mathbf{x})J(\mathbf{x})} \\ &= \int [\mathcal{D}g_\Omega] \exp\left(-n\int d^d\mathbf{x}\,\phi(\mathbf{x})[\mathcal{O}\phi](\mathbf{x}) + \int d^d\mathbf{x}\,\phi(\mathbf{x})J(\mathbf{x})\right) \end{aligned} \tag{4.155}$$

where if we use

$$\begin{aligned} &\int [\mathcal{D}\phi] \exp\left(-\frac{1}{2}\int d^d\mathbf{x}\,\left(\phi(\mathbf{x})[A\phi](\mathbf{x}) - 2\rho(\mathbf{x})\phi(\mathbf{x})\right)\right) \\ &\qquad = [\det(A)]^{-1/2} \exp\left(\frac{1}{2}\int d^d\mathbf{x}\,\rho(\mathbf{x})\left[A^{-1}\rho\right](\mathbf{x})\right), \end{aligned} \tag{4.156}$$

considering that $A \equiv 2n\mathcal{O}$ and $\rho \equiv J$. Then, we obtain that

$$Z[J] = \left[\det\left(\mu^{-1}n\mathcal{O}\right)\right]^{-1/2} \exp\left(\frac{1}{4n}\int d^d\mathbf{x}\, J(\mathbf{x})\left[\mathcal{O}^{-1}J\right](\mathbf{x})\right) \tag{4.157}$$

and considering its respective variations with respect to J we have

$$\begin{aligned} Z_{1(2)}^{\Omega\cup\overline{\Omega}} =& \frac{16}{\gamma^2}\left[\det\left(\mu^{-1}\mathcal{O}\right)\right]\int[\mathcal{D}\phi]\exp\left(-2\int d^d\mathbf{x}\phi(\mathbf{x})\mathcal{O}\phi(\mathbf{x})\right)\left|\int d^d\mathbf{y} f(t,\mathbf{y})\mathcal{O}^{\frac{1}{2}}\phi(\mathbf{y})\right|^4 \\ =& \frac{16}{\gamma^2}\left[\det\left(\mu^{-1}\mathcal{O}\right)\right]\int d^d\mathbf{x}_1 d^d\mathbf{x}_2 d^d\mathbf{x}_3 d^d\mathbf{x}_4\, \mathcal{O}^{\frac{1}{2}}f(t,\mathbf{x}_1)\mathcal{O}^{\frac{1}{2}}f(t,\mathbf{x}_2)\mathcal{O}^{\frac{1}{2}}f^*(t,\mathbf{x}_3) \\ & \times \mathcal{O}^{\frac{1}{2}}f^*(t,\mathbf{x}_4)\int[\mathcal{D}\phi]\phi(\mathbf{x}_1)\phi(\mathbf{x}_2)\phi(\mathbf{x}_3)\phi(\mathbf{x}_4)\exp\left(-n\int d^d\mathbf{x}\phi(\mathbf{x})\mathcal{O}\phi(\mathbf{x})\right) \\ =& \frac{16}{\gamma^2}\left[\det\left(\mu^{-1}\mathcal{O}\right)\right]\int d^d\mathbf{x}_1 d^d\mathbf{x}_2 d^d\mathbf{x}_3 d^d\mathbf{x}_4 \mathcal{O}^{\frac{1}{2}}f(t,\mathbf{x}_1)\mathcal{O}^{\frac{1}{2}}f(t,\mathbf{x}_2) \\ & \times \mathcal{O}^{\frac{1}{2}}f^*(t,\mathbf{x}_3)\mathcal{O}^{\frac{1}{2}}f^*(t,\mathbf{x}_4)\left.\frac{\delta Z[J]}{\delta J(\mathbf{x}_1)\cdots\delta J(\mathbf{x}_4)}\right|_{J=0} \end{aligned} \tag{4.158}$$

Now, if we take into account

$$\begin{aligned} \left.\frac{\delta Z[J]}{\delta J(\mathbf{x}_1)\delta J(\mathbf{x}_2)J(\mathbf{x}_3)J(\mathbf{x}_4)}\right|_{J=0} =& \frac{1}{4^2}\left[\det\left(\mu^{-1}2\mathcal{O}\right)\right]^{-\frac{1}{2}}\left[\mathcal{O}^{-1}(\mathbf{x}_1,\mathbf{x}_2)\,\mathcal{O}^{-1}(\mathbf{x}_3,\mathbf{x}_4)\right. \\ & \left.+\mathcal{O}^{-1}(\mathbf{x}_1,\mathbf{x}_3)\mathcal{O}^{-1}(\mathbf{x}_2,\mathbf{x}_4)+\mathcal{O}^{-1}(\mathbf{x}_1,\mathbf{x}_4)\,\mathcal{O}^{-1}(\mathbf{x}_2,\mathbf{x}_3)\right] \end{aligned} \tag{4.159}$$

Thus, we obtain that

$$Z_{1(2)}^{\Omega\cup\overline{\Omega}} = \left(\frac{2}{\gamma}\right)^2 \frac{\left[\det\left(\mu^{-1}\mathcal{O}\right)\right]}{\left[\det\left(\mu^{-1}2\mathcal{O}\right)\right]^{1/2}}\left[\left|\frac{1}{2}\int d^d\mathbf{x}\, f^2(\mathbf{x},t)\right|^2 + 2\left(\frac{\gamma}{4}\right)^2\right], \tag{4.160}$$

which, when compared with the result obtained in (4.152), shows that they are exactly the same expressions, thus confirming the equivalence of both methods in calculating Shannon entropy over the region $\Omega\cup\overline{\Omega}$.

Renyi Entropy Over the Region $\Omega\cup\overline{\Omega}$ for any n

For this general case, we proceed from (4.142), where we need to expand all variations with respect to α and β. To this end, we first apply Newton's binomial

theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}. \tag{4.161}$$

Then

$$Z_{1(n)}^{\Omega\cup\overline{\Omega}} = \left(\frac{4}{n\gamma}\right)^n \frac{\left[\det\left(\mu^{-1}\mathcal{O}\right)\right]^{n/2}}{\left[\det\left(n\mu^{-1}\mathcal{O}\right)\right]^{1/2}} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(\frac{\delta^2}{\delta\alpha^2}\right)^{n-k} \left(\frac{\delta^2}{\delta\beta^2}\right)^k e^{\mathcal{S}_{\alpha\beta}}\Big|_{\alpha=\beta=0}, \tag{4.162}$$

where we have denoted $\mathcal{S}_{\alpha\beta} = \frac{1}{4}\int d^d\mathbf{x}\, f^2_{\alpha,\beta}(\mathbf{x},t)$.

Now, considering the Faa di Bruno's formula for the exponential

$$\begin{aligned}\frac{d^n}{dx^n} e^{f(x)} =& e^{f(x)} \sum \frac{n!}{k_1!\cdots k_n!} \left(\frac{f'(x)}{1!}\right)^{k_1} \cdots \left(\frac{f^{(n)}(x)}{n!}\right)^{k_n} \\ \equiv& e^{f(x)} B_n\left(f'(x), ..., f^{(n)}(x)\right)\end{aligned} \tag{4.163}$$

where the sum is over all the partitions of n, i.e. values of k_1,...,k_n such that

$$k_1 + 2k_2 + \cdots + nk_n = n \tag{4.164}$$

Thus for the variations with respect to α and β of $e^{\mathcal{S}_{\alpha\beta}}$, we have

$$\begin{aligned}\left(\frac{\delta^2}{\delta\alpha^2} - \frac{\delta^2}{\delta\beta^2}\right)^n e^{\mathcal{S}_{\alpha\beta}}\Big|_{\alpha=\beta=0} =& \sum_{k=0}^{n} (-1)^k \frac{n!}{(n-k)!k!} \sum_{l=0}^{k} \frac{2k!}{(2k-2l)!l!} \frac{(2n-2k)!}{(\lambda_{kl})!} \left(\frac{1}{2!}\right)^{\lambda_{kl}+l} \\ &\times \left(\delta_\alpha^2\mathcal{S}\right)^{\lambda_{kl}} \left(\delta_\alpha\delta_\beta\mathcal{S}\right)^{2k-2l} \left(\delta_\beta^2\mathcal{S}\right)^l\end{aligned} \tag{4.165}$$

with $\lambda_{kl} = n - 2k + l \in \mathbb{Z}^{+0}$, where we have take into account $S_{\alpha\beta} = \delta_\alpha S = \delta_\beta S = 0$ when $\alpha = \beta = 0$. Therefore, the Shannon entropy for the excited state over the region $\Omega \cup \overline{\Omega}$ will be

$$Z_{1(n)}^{\Omega\cup\overline{\Omega}} = \left(\frac{4}{n\gamma}\right)^n \frac{\left[\det\left(\mu^{-1}\mathcal{O}\right)\right]^{n/2}}{\left[\det\left(n\mu^{-1}\mathcal{O}\right)\right]^{1/2}} \sum_{k=0}^{n}(-1)^k \frac{n!}{(n-k)!k!} \sum_{l=0}^{k} \frac{2k!}{(2k-2l)!l!} \frac{(2n-2k)!}{(\lambda_{kl})!} \times \left(\frac{1}{2!}\right)^{\lambda_{kl}+l} \left(\delta_\alpha^2\mathcal{S}\right)^{\lambda_{kl}/2} (\delta_\alpha\delta_\beta\mathcal{S})^{2k-2l} \left(\delta_\beta^2\mathcal{S}\right)^l \tag{4.166}$$

being

$$\delta_\alpha^2 S = \frac{1}{2}\int d^d\mathbf{x}\, R^2(\mathbf{x},t) \tag{4.167}$$

$$\delta_\beta^2 S = -\frac{1}{2}\int d^d\mathbf{x}\, I^2(\mathbf{x},t) \tag{4.168}$$

$$\delta_\alpha\delta_\beta S = \frac{i}{2}\int d^d\mathbf{x}\, I(\mathbf{x},t)R(\mathbf{x},t) \tag{4.169}$$

and if we again take into account the Shannon entropy for the vacuum over the region $\Omega\cup\overline{\Omega}$ (3.72), we can express

$$S_1[\Omega\cup\overline{\Omega}] = -\,\partial_n \ln Z_{1(n)}^{\Omega\cup\overline{\Omega}}\Big|_{n=1} = S_0[\Omega\cup\overline{\Omega}] - \partial_n \ln\left\{\tilde{F}_{\Omega\cup\overline{\Omega}}^{(n)}(t)\right\}\Big|_{n=1} \tag{4.170}$$

where

$$\tilde{F}_{\Omega\cup\overline{\Omega}}^{(n)}(t) = \left(\frac{4}{n\gamma}\right)^n \sum_{k=0}^{n}(-1)^k \frac{n!}{(n-k)!k!} \sum_{l=0}^{k} \frac{2k!}{(2k-2l)!l!} \frac{(2n-2k)!}{(\lambda_{kl})!} \left(\frac{1}{2!}\right)^{\lambda_{kl}+l} \left(\delta_\alpha^2\mathcal{S}\right)^{\lambda_{kl}/2} (\delta_\alpha\delta_\beta\mathcal{S})^{2k-2l} \left(\delta_\beta^2\mathcal{S}\right)^l. \tag{4.171}$$

On the other hand, using the generating functional method defined in (4.155),

we can obtain an equivalent expression. Therefore

$$\begin{aligned}
Z_{1(n)}^{\Omega\cup\overline{\Omega}} =& \frac{4^n}{\gamma^n}\left[\det\left(\mu^{-1}\mathcal{O}\right)\right]^{\frac{n}{2}}\int[\mathcal{D}\phi]\exp\left(-n\int d^d\mathbf{x}\phi(\mathbf{x})\mathcal{O}\phi(\mathbf{x})\right)\left|\int d^d\mathbf{x}f(t,\mathbf{x})\mathcal{O}^{\frac{1}{2}}\phi(\mathbf{x})\right|^{2n}\\
=& \frac{4^n}{\gamma^n}\left[\det\left(\mu^{-1}\mathcal{O}\right)\right]^{\frac{n}{2}}\int d^d\mathbf{x}_1\cdots d^d\mathbf{x}_{2n}\mathcal{O}^{\frac{1}{2}}f(t,\mathbf{x}_1)\cdots\mathcal{O}^{\frac{1}{2}}f(t,\mathbf{x}_n)\\
&\times\mathcal{O}^{\frac{1}{2}}f^*(t,\mathbf{x}_{n+1})\cdots\mathcal{O}^{\frac{1}{2}}f^*(t,\mathbf{x}_{2n})\int[\mathcal{D}\phi]\phi(\mathbf{x}_1)\cdots\phi(\mathbf{x}_{2n})e^{-n\int d^d\mathbf{x}\phi(\mathbf{x})\mathcal{O}\phi(\mathbf{x})}\\
=& \frac{4^n}{\gamma^n}\left[\det\left(\mu^{-1}\mathcal{O}\right)\right]^{\frac{n}{2}}\int d^d\mathbf{x}_1\cdots d^d\mathbf{x}_{2n}\left[\mathcal{O}^{\frac{1}{2}}f\right](t,\mathbf{x}_1)\cdots\left[\mathcal{O}^{\frac{1}{2}}f\right](t,\mathbf{x}_n)\\
&\times\mathcal{O}^{\frac{1}{2}}f^*(t,\mathbf{x}_{n+1})\cdots\mathcal{O}^{\frac{1}{2}}f(t,\mathbf{x}_{2n})\left.\frac{\delta Z[J]}{\delta J(\mathbf{x}_1)\cdots\delta J(\mathbf{x}_{2n})}\right|_{J=0}.
\end{aligned}\tag{4.172}$$

Now, if we take into account

$$\begin{aligned}
\left.\frac{\delta Z[J]}{\delta J(\mathbf{x}_1)\cdots\delta J(\mathbf{x}_{2m})}\right|_{J=0} =& \left[\det\left(\mu^{-1}n\mathcal{O}\right)\right]^{-\frac{1}{2}}\frac{1}{(2n)^m}\\
&\times\sum_{\text{permut}}\mathcal{O}^{-1}(\mathbf{x}_{k_1},\mathbf{x}_{k_2})\cdots\mathcal{O}^{-1}(\mathbf{x}_{k_{2m-1}},\mathbf{x}_{k_{2m}})
\end{aligned}\tag{4.173}$$

We can note that

$$\begin{aligned}
&\frac{1}{(2n)^n}\sum_{\text{permut}}\int_\Omega d^d\mathbf{x}_1\cdots d^d\mathbf{x}_{2n}\mathcal{O}^{\frac{1}{2}}f(t,\mathbf{x}_1)\cdots\mathcal{O}^{\frac{1}{2}}f(t,\mathbf{x}_n)\\
&\qquad\times\mathcal{O}^{\frac{1}{2}}f(t,\mathbf{x}_{n+1})\cdots O^{\frac{1}{2}}f(t,x_{2n})\mathcal{O}^{-1}(\mathbf{x}_{k_1},\mathbf{x}_{k_2})\cdots\mathcal{O}^{-1}(\mathbf{x}_{k_{2n-1}},\mathbf{x}_{k_{2n}})\\
&=\frac{1}{(2n)^n}\sum_{2r+l=n}C_{rl}\left|\int d^d\mathbf{x}\mathcal{O}^{\frac{1}{2}}f(t,\mathbf{x})\mathcal{O}^{-\frac{1}{2}}f(t,\mathbf{x})\right|^{2r}\left(\int d^d\mathbf{x}\mathcal{O}^{\frac{1}{2}}f(t,\mathbf{x})\mathcal{O}^{-\frac{1}{2}}f^*(t,\mathbf{x})\right)^l\\
&=\frac{1}{(2n)^n}\sum_{2r+l=n}C_{rl}\left|\int d^d\mathbf{x}\,f^2(t,\mathbf{x})\right|^{2r}\left(\frac{\gamma}{2}\right)^l
\end{aligned}\tag{4.174}$$

Thus, we obtain that

$$Z_{1(n)}^{\Omega\cup\overline{\Omega}} = \left(\frac{2}{n\gamma}\right)^n\frac{\left[\det\left(\mu^{-1}\mathcal{O}\right)\right]^{n/2}}{\left[\det\left(\mu^{-1}n\mathcal{O}\right)\right]^{1/2}}\sum_{2r+l=n}C_{rl}\left|\int d^d\mathbf{x}f^2(t,\mathbf{x})\right|^{2r}\left(\frac{\gamma}{2}\right)^l\tag{4.175}$$

and using again the Shannon entropy for the vacuum over the region $\Omega\cup\overline{\Omega}$ (3.72), we obtain

$$S_1[\Omega \cup \overline{\Omega}] = -\,\partial_n \ln Z_{1(n)}^{\Omega\cup\overline{\Omega}}\Big|_{n=1} = S_0[\Omega \cup \overline{\Omega}] - \partial_n \ln \left\{\tilde{F}^{(n)}_{\Omega\cup\overline{\Omega}}(t)\right\}\Big|_{n=1} \tag{4.176}$$

where

$$\tilde{F}^{(n)}_{\Omega\cup\overline{\Omega}}(t) = \left(\frac{2}{n\gamma}\right)^n \sum_{2r+l=n} C_{rl} \left|\int d^d\mathbf{x}\, f^2(t,\mathbf{x})\right|^{2r} \left(\frac{\gamma}{2}\right)^l. \tag{4.177}$$

It is interesting to observe that comparing the two equivalent forms of $\tilde{F}^{(n)}_{\Omega\cup\overline{\Omega}}(t)$, (4.171) and (4.177), results in the following identity:

$$\sum_{2r+l=n} C_{rl} \left|\int d^d\mathbf{x} f^2(t,\mathbf{x})\right|^{2r} \left(\frac{\gamma}{2}\right)^l = \sum_{k=0}^{n} (-1)^k \frac{n!}{(n-k)!k!} \sum_{l=0}^{k} \frac{2k!}{(2k-2l)!l!} \frac{(2n-2k)!}{(\lambda_{kl})!} \times \left(\frac{1}{2}\right)^{\lambda_{kl}+l} \left(\delta_\alpha^2 \mathcal{S}\right)^{\lambda_{kl}} (\delta_{\alpha\beta}\mathcal{S})^{2k-2l} \left(\delta_\beta^2 \mathcal{S}\right)^l \tag{4.178}$$

where $\lambda_{kl} = n - 2k + l \in \mathbb{Z}^{+0}$, $\delta_\alpha^2 S = \int d^d\mathbf{x}\, R^2$, $\delta_\beta^2 S = -\int d^d\mathbf{x}\, I^2$ and $\delta_{\alpha\beta} S = i\int d^d\mathbf{x}\, IR$. This identity can be seen as a factorization of the real and imaginary parts of the probability amplitude $f(t,\mathbf{x})$.

Finally, let us analyze the case when $f(t,\mathbf{x})$ is purely real for all t, i.e., $I(t,\mathbf{x}) = 0$ and $R(t,\mathbf{x}) = f(t,\mathbf{x}), \forall t$. According to the expression for $\tilde{F}^{(n)}_{\Omega\cup\overline{\Omega}}(t)$ in (4.171), we realize that the only nonzero terms in the summation occur when $k = l = 0$, resulting in the following:

$$\begin{aligned}
\tilde{F}^{(n)}_{\Omega\cup\overline{\Omega}}(t) &= \left(\frac{4}{n\gamma}\right)^n \frac{(2n)!}{n!} \left(\frac{1}{2}\right)^n \left(\delta_\alpha^2 \mathcal{S}\right)^n \\
&= \left(\frac{2}{n\gamma}\right)^n \frac{(2n)!}{n!} \left(\frac{1}{2}\right)^n \left(\int d^d\mathbf{x}\, f^2(t,\mathbf{x})\right)^n \\
&= \left(\frac{2}{n\gamma}\right)^n (2n-1)!! \left(\frac{\gamma}{2}\right)^n \\
&= \frac{1}{n^n}(2n-1)!!
\end{aligned} \tag{4.179}$$

where we can see that this quantity is a constant independent of time. Also, the

Shannon entropy will be

$$S_1[\Omega \cup \overline{\Omega}] = S_0[\Omega \cup \overline{\Omega}] - \partial_n \ln \left\{ \frac{(2n)!}{n!} \left(\frac{1}{2n} \right)^n \right\} \Big|_{n=1}. \tag{4.180}$$

To calculate the expression $\partial_n \ln \left\{ \frac{2n!}{n!} \left(\frac{1}{2n} \right)^n \right\} \Big|_{n=1}$, we need to compute the partial derivative with respect to n of the given logarithmic expression and evaluate it at $n = 1$. Since n appears in factorials, which are typically defined for non-negative integers, we extend the factorial to real numbers using the gamma function, $\Gamma(n+1) = n!$, to handle differentiation.

First, simplify the expression inside the logarithm:

$$\frac{2n!}{n!} = \frac{\Gamma(2n+1)}{\Gamma(n+1)}, \quad \left(\frac{1}{2n} \right)^n = (2n)^{-n}. \tag{4.181}$$

Thus, the expression becomes

$$\frac{\Gamma(2n+1)}{\Gamma(n+1)} (2n)^{-n}. \tag{4.182}$$

The logarithm is

$$\ln \left\{ \frac{\Gamma(2n+1)}{\Gamma(n+1)} (2n)^{-n} \right\} = \ln \Gamma(2n+1) - \ln \Gamma(n+1) - n \ln(2n). \tag{4.183}$$

We need to compute

$$\partial_n \left[\ln \Gamma(2n+1) - \ln \Gamma(n+1) - n \ln(2n) \right]. \tag{4.184}$$

Differentiate each term

1. For $\ln \Gamma(2n+1)$: Using the chain rule, let $u = 2n+1$, so $\ln \Gamma(2n+1) = \ln \Gamma(u)$, and $\frac{du}{dn} = 2$. The derivative is

$$\partial_n \ln \Gamma(2n+1) = \psi(2n+1) \cdot 2, \tag{4.185}$$

   where $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ is the digamma function.

2. For $-\ln \Gamma(n+1)$: Let $v = n+1$, so $\ln \Gamma(n+1) = \ln \Gamma(v)$, and $\frac{dv}{dn} = 1$. The derivative is

$$\partial_n [-\ln \Gamma(n+1)] = -\psi(n+1) \cdot 1 = -\psi(n+1). \tag{4.186}$$

3. For $-n\ln(2n)$: Use the product rule on $-n\ln(2n)$. Let $f=-n$, $g=\ln(2n)$, so
$$f'=-1,\quad g'=\frac{1}{2n}\cdot 2=\frac{1}{n}. \tag{4.187}$$
Thus
$$\partial_n[-n\ln(2n)]=-1\cdot\ln(2n)+(-n)\cdot\frac{1}{n}=-\ln(2n)-1. \tag{4.188}$$

Combining these, the derivative is
$$\partial_n \ln\left\{\frac{(2n)!}{n!}\left(\frac{1}{2n}\right)^n\right\}=2\psi(2n+1)-\psi(n+1)-\ln(2n)-1. \tag{4.189}$$

Now evaluate at $n=1$
$$\begin{aligned}\partial_n \ln\left\{\frac{(2n)!}{n!}\left(\frac{1}{2n}\right)^n\right\}\Big|_{n=1} =&2\psi(2\cdot 1+1)-\psi(1+1)-\ln(2\cdot 1)-1\\ =&2\psi(3)-\psi(2)-\ln 2-1.\end{aligned} \tag{4.190}$$

Use the recurrence formula $\psi(x+1)=\psi(x)+\frac{1}{x}$ and the known values of the digamma function

- $\psi(2)=\psi(1+1)=\psi(1)+\frac{1}{1}=-\gamma_E+1$, where γ_E is the Euler-Mascheroni constant.
- $\psi(3)=\psi(2+1)=\psi(2)+\frac{1}{2}=-\gamma_E+1+\frac{1}{2}=-\gamma_E+\frac{3}{2}$.

So
$$2\psi(3)=2\left(-\gamma_E+\frac{3}{2}\right)=-2\gamma_E+3, \tag{4.191}$$
$$-\psi(2)=-(-\gamma_E+1)=\gamma_E-1. \tag{4.192}$$

Thus
$$2\psi(3)-\psi(2)=(-2\gamma_E+3)+(\gamma_E-1)=-\gamma_E+2. \tag{4.193}$$

Finally
$$(-\gamma_E+2)-\ln 2-1=-\gamma_E+1-\ln 2. \tag{4.194}$$

The final answer is
$$\boxed{\partial_n \ln\left\{\frac{(2n)!}{n!}\left(\frac{1}{2n}\right)^n\right\}\Big|_{n=1}=1-\gamma_E-\ln 2} \tag{4.195}$$

Inserting this result in (4.180), we obtain

$$S_1[\Omega \cup \overline{\Omega}] = S_0[\Omega \cup \overline{\Omega}] + \gamma_E - 1 + \ln 2. \tag{4.196}$$

This result is quite puzzling due to the independence of the mutual information of the excited state with respect to the probability amplitude $f(\mathbf{x}, t)$, recalling that to obtain this result it has been assumed that $f(\mathbf{x}, t)$ is real for all t. However, it is worth noting that considering the relativistic Schrödinger equation for $f(\mathbf{x}, t)$ (4.10), this assumption implies that the amplitude $f(\mathbf{x}, t)$ is zero. Therefore, to obtain a correct result, it is necessary to consider that the amplitude $f(\mathbf{x}, t)$ has both a real and an imaginary part.

To corroborate this fact, we start from equation (4.10)

$$i\frac{\partial f(t, \mathbf{x})}{\partial t} = [\mathcal{O}f](t, \mathbf{x}) \tag{4.197}$$

along with its conjugate

$$-i\frac{\partial f^*(t, \mathbf{x})}{\partial t} = [\mathcal{O}f^*](t, \mathbf{x}) \tag{4.198}$$

the sum of both equations results in

$$-\frac{\partial I(t, \mathbf{x})}{\partial t} = [\mathcal{O}R](t, \mathbf{x}) \tag{4.199}$$

therefore, if $I(\mathbf{x}, t)$ is zero for all t, then $R(\mathbf{x}, t)$ will also be zero.

4.6 Shannon Mutual Information in the Excited State With Replica Trick

According to the definition of mutual information

$$I(\Omega, \overline{\Omega}) = S[\Omega] + S[\overline{\Omega}] - S[\Omega \cup \overline{\Omega}].$$

Taking into account the expressions for the Shannon entropies over the regions Ω, $\overline{\Omega}$, and $\Omega \cup \overline{\Omega}$, given by (4.133), (4.135), and (4.176) respectively, we obtain that the Shannon mutual information is given by:

$$I_1(\Omega,\overline{\Omega}) = I_0(\Omega,\overline{\Omega}) - \partial_n \ln\left\{\frac{\tilde{F}^{(n)}_{\Omega}(t)\tilde{F}^{(n)}_{\overline{\Omega}}(t)}{\tilde{F}^{(n)}_{\Omega\cup\overline{\Omega}}(t)}\right\}\Bigg|_{n=1}, \tag{4.200}$$

where $\tilde{F}^{(n)}_{\Omega}(t)$, $\tilde{F}^{(n)}_{\overline{\Omega}}(t)$ and $\tilde{F}^{(n)}_{\Omega\cup\overline{\Omega}}(t)$ are given by (4.134), (4.136) and (4.177) respectively.

The investigation into the Shannon mutual information for a localized single-particle excited state reveals how the presence of a particle fundamentally alters the information landscape of a quantum field compared to the vacuum. The results presented provide significant insights into the nature of classical correlations in quantum field theory, which are particularly relevant for understanding scenarios involving measurement and decoherence.

The central result of this section is the expression for the Shannon mutual information in the excited state, $I_1(\Omega,\overline{\Omega})$, derived using the replica trick. The final expression is elegantly presented in relation to the mutual information of the vacuum state, $I_0(\Omega,\overline{\Omega})$.

The equation (4.200) shows that the mutual information in the excited state is the vacuum mutual information plus a correction term. This correction term is dependent on the replica parameter n and a set of functions, $\tilde{F}^{(n)}$, which encapsulate the properties of the excited state via its wave packet profile, $f(t,x)$. This explicitly connects the change in shared information to the presence and characteristics of the localized particle.

Directly calculating the Shannon mutual information from the above expression is complicated by the need to take a derivative with respect to the replica parameter n and then evaluate it at $n = 1$. To obtain concrete physical insights, the analysis cleverly pivots to a more tractable case: the Rényi mutual information for $n = 2$ (also known as the collision entropy). This avoids the difficult $n \to 1$ limit and allows for direct algebraic calculation. For $n = 2$, considering the Renyi entropies (4.119), (4.121), and (4.153) the mutual information is given by

$$I^{(2)}_1(\Omega,\overline{\Omega}) = I^{(2)}_0(\Omega,\overline{\Omega}) - \ln\left\{\frac{\tilde{F}^{(2)}_{\Omega}(t)\tilde{F}^{(2)}_{\overline{\Omega}}(t)}{\tilde{F}^{(2)}_{\Omega\cup\overline{\Omega}}(t)}\right\}, \tag{4.201}$$

where $\tilde{F}^{(2)}_{\Omega}(t)$, $\tilde{F}^{(2)}_{\overline{\Omega}}(t)$ and $\tilde{F}^{(2)}_{\Omega\cup\overline{\Omega}}(t)$ are given by (4.120), (4.122) and (4.154) respectively.

$$\tilde{F}^{(2)}_{\Omega}(t) = \frac{4}{\gamma^2}\left|\int_{\Omega} d^d\mathbf{x}\, a_f(\mathbf{x})\left[o_s^{-1}a_f\right](\mathbf{x})\right|^2 + \frac{1}{2} + \frac{32}{\gamma^2}\left(b_R + b_I\right)^2 \tag{4.202}$$

$$\tilde{F}^{(2)}_{\overline{\Omega}}(t) = \frac{4}{\gamma^2}\left|\int_{\overline{\Omega}} d^d\mathbf{x}\, \overline{a}_f(\mathbf{x})\left[\overline{o}_s^{-1}\overline{a}_f\right](\mathbf{x})\right|^2 + \frac{1}{2} + \frac{32}{\gamma^2}\left(\overline{b}_R + \overline{b}_I\right)^2 \tag{4.203}$$

$$\tilde{F}^{(2)}_{\Omega\cup\overline{\Omega}}(t) = \left(\frac{2}{\gamma}\right)^2\left[\left|\frac{1}{2}\int d^d\mathbf{x}\, f^2(\mathbf{x},t)\right|^2 + 2\left(\frac{\gamma}{4}\right)^2\right] \tag{4.204}$$

The functions $\tilde{F}^{(2)}$ are explicitly defined in terms of the particle's wave packet, making numerical investigation possible.

Chapter 5

Renyi Entropy For the Excited State in One Dimension

The preceding chapter established a comprehensive and general framework for the Shannon mutual information of a localized excited state in $d+1$ dimensions. However, the final expression for the Shannon mutual information, $I_1(\Omega, \overline{\Omega})$, relies on the analytically challenging task of calculating the functions $\tilde{F}^{(n)}_{\Omega}(t)$, $\tilde{F}^{(n)}_{\overline{\Omega}}(t)$, and $\tilde{F}^{(n)}_{\Omega\cup\overline{\Omega}}(t)$ for a general replica parameter n and then evaluating the derivative at $n = 1$.

To move from this formal expression to a tangible application and extract physical insights, we will focus on a more tractable, yet physically significant, scenario. In this chapter, we will therefore specialize our analysis in two key ways.

First, instead of the Shannon entropy, we will compute the Rényi entropy for the specific case of $n = 2$, also known as the collision entropy. This choice circumvents the difficult $n \to 1$ limit, allowing for a direct algebraic calculation of the mutual information, as expressed in equation (4.201). Second, we will restrict the analysis to a one-dimensional spatial setting, where the complementary regions Ω and $\overline{\Omega}$ are defined as half-lines $(-\infty, 0]$ and $[0, \infty)$, respectively. This simplification of both the dimensionality and the geometry of the boundary makes the necessary functional integrals and operator inversions manageable.

By employing the Lorentzian wave packet solutions derived earlier, this approach will allow us to perform explicit calculations and numerically investigate how the particle's presence—specifically its location and spatial width—modifies the shared information between the two regions compared to the vacuum.

5.1 Renyi entropy for the regions $\Omega \equiv \langle -\infty, 0]$ and $\overline{\Omega} \equiv [0, \infty\rangle$

For this case, we take into account the wave packet given by (4.65). Now, first of all, we calculate the function $a_f(t,z)$ given by (B.15)

$$\begin{aligned} a_f(t,z) &= \left[\mathcal{O}^{1/2} f\right](t,z) + \frac{1}{2}\int_0^\infty dy\, K(z-y)\mathcal{G}\left[\mathcal{O}^{1/2} f\right](t,y) \\ &\quad + \frac{1}{2}\int_0^\infty dx\, \mathcal{P}(z,x)\left[\mathcal{O}^{1/2} f\right](t,x) \\ &= \left[\mathcal{O}^{1/2} f\right](t,z) + \frac{1}{2}\int_0^\infty dx \left[\mathcal{O}^{1/2} f\right](t,x)\left[\int_0^\infty dy K(z-y)G(y,x) + \mathcal{P}(z,x)\right] \\ &= \left[\mathcal{O}^{1/2} f\right](t,z) + \int_0^\infty dx\, \left[\mathcal{O}^{1/2} f\right](t,x)\mathcal{P}(z,x) \end{aligned} \tag{5.1}$$

where we take into account the relation (A.100), then considering the Poisson kernel (A.65) and the result (4.72), for $\operatorname{Im}\alpha = \operatorname{Re}\beta = 0$, $\operatorname{Re}\alpha \equiv \alpha$ and $\operatorname{Im}\beta = -\langle x\rangle$, we have

$$\begin{aligned} a_f(z) =& \sqrt{\frac{\alpha}{2}}\frac{1}{(\alpha + i(t+z-\langle x\rangle))^{3/2}}\left(\frac{1}{2} + \frac{1}{\pi}\operatorname{arcsec}\left(\sqrt{-\frac{t-\langle x\rangle - i\alpha}{z}}\right)\right) \\ &+ \sqrt{\frac{\alpha}{2}}\frac{1}{(\alpha - i(z-\langle x\rangle - t))^{3/2}}\left(\frac{1}{2} + \frac{1}{\pi}\operatorname{arcsec}\left(\sqrt{\frac{t+\langle x\rangle - i\alpha}{z}}\right)\right) \\ &- \frac{1}{\pi}\sqrt{\frac{\alpha}{2}}\frac{1}{\sqrt{-z}}\left[\frac{\sqrt{i}}{\alpha - i(z-\langle x\rangle - t)} + \frac{\sqrt{-i}}{\alpha + \beta + i(t+z-\langle x\rangle)}\right] \end{aligned} \tag{5.2}$$

Secondly, we calculate numerically the function $b_f(t)$ given by (B.16)

$$b[f](t) = \frac{1}{4}\int_0^\infty dx \int_0^\infty dy\, G(x,y)\left[\mathcal{O}^{1/2} f\right](t,y)\left[\mathcal{O}^{1/2} f\right](t,x)$$

Thus, the Renyi entropy with $n = 2$ over the region $\Omega \equiv \langle -\infty, 0]$, will be obtained from the function $\tilde{F}_\Omega^{(2)}(t)$, which is given by

$$\tilde{F}_\Omega^{(2)}(t) = \left|\int_{-\infty}^0 dx\, a_f(t,x)\left[o_s^{-1} a_f\right](t,x)\right|^2 + \frac{1}{2} + 8(b_R(t) + b_I(t))^2.$$

Renyi entropy over the region $\Omega \equiv \langle -\infty, 0]$ for $t = 0$

For this case, it is more convenient to set $\mathrm{Re}\,\beta = \mathrm{Im}\,\alpha = 0$ in the expression for the Lorentzian wave packet (4.65). This simplifies the analysis, as these conditions ensure that the wave packet $f(t, \mathbf{x})$ at $t = 0$ is purely real. Consequently, to compute $\tilde{F}^{(2)}_{\Omega}(t)$, we do not need to evaluate the integral involving the inverse operator o_s^{-1}; instead, we only need to apply the normalization condition (D.15). Thus, the Rényi entropy for the region $\Omega \equiv (-\infty, 0]$ is obtained as follows

$$\begin{aligned} S_1[\Omega] &= S_0[\Omega] - \ln\left(\tilde{F}^{(2)}_{\Omega}(0)\right) \\ &= S_0[\Omega] - \ln\left(12 b_R^2(0) - 2 b_R(0) + \frac{3}{4}\right). \end{aligned} \tag{5.3}$$

Numerically, we obtain a graphic for the expression $-\ln\left(\tilde{F}^{(2)}_{\Omega}(0)\right)$: Figure 5.1 illustrates the correction $\Delta S^{(2)}(\Omega) := -\ln\left(\tilde{F}^{(2)}_{\Omega}(0)\right)$ to the Rényi-2 entropy for the region $\Omega = \langle -\infty, 0]$ as a function of the particle's initial position $\langle x \rangle$. When the particle is localized well within Ω (i.e., $\langle x \rangle \ll 0$), the correction $\Delta S^{(2)}(\Omega)$ is approximately constant and positive, with its magnitude dependent on the wave packet's width α. This constant increase relative to the vacuum state's entropy indicates that the presence of the localized excitation introduces additional uncertainty in the field configurations within Ω. The particle, described by a Lorentzian wave packet, enriches the set of probable field states, leading to a higher collision entropy. Physically, this reflects the perturbation of the quantum field by the excitation, which increases the diversity of measurement outcomes in Ω. The dependence on α suggests that broader wave packets (larger α) may distribute the particle's influence over a larger spatial region, potentially reducing the entropy correction compared to narrower wave packets, which concentrate the excitation's effect.

As the particle's center approaches the boundary at $x = 0$ (i.e., $\langle x \rangle \to 0$), $\Delta S^{(2)}(\Omega)$ reaches a maximum value. This peak indicates that the particle's proximity to the boundary enhances the information content in Ω, likely due to the wave packet's overlap with the boundary, which amplifies correlations across the interface between Ω and $\overline{\Omega}$. The maximum entropy correction near the boundary suggests that the field configurations in Ω are most diverse when the particle is positioned such that its wave packet significantly interacts with both regions, maximizing the uncertainty in measurement outcomes.

When the particle moves into $\overline{\Omega}$ (i.e., $\langle x \rangle > 0$) and is far from the boundary (i.e.,

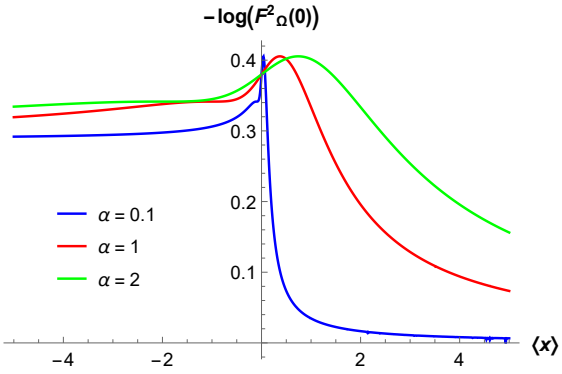


Figure 5.1: $t = 0$

$\langle x \rangle \gg 0$), $\Delta S^{(2)}(\Omega)$ decreases to zero. This behavior is expected, as the excitation is now primarily located in the complementary region $\overline{\Omega}$, and its influence on the field configurations in Ω becomes negligible. In this regime, the reduced density matrix for Ω approaches that of the vacuum state, resulting in a Rényi-2 entropy indistinguishable from the vacuum case.

Renyi entropy over the region $\overline{\Omega} \equiv [0, \infty\rangle$ for $t = 0$

Again, in this case, we set $\mathrm{Re}\,\beta = \mathrm{Im}\,\alpha = 0$ for the wave packet $f(t, x)$. Also, considering the following quantities:

$$\overline{\mathcal{P}}(z, y) = \frac{1}{\pi}\sqrt{\frac{|y|}{z}}\frac{1}{|x - z|}, \qquad z \geq 0, y \leq 0. \tag{5.4}$$

$$G(|x|, |y|) = \overline{G}(x, y) = \begin{cases} \frac{2}{\pi}\,\mathrm{arctanh}\left(\sqrt{\frac{y}{x}}\right), & x < y \\ \frac{2}{\pi}\,\mathrm{arctanh}\left(\sqrt{\frac{x}{y}}\right), & x > y \end{cases}, \quad x < 0, y < 0 \tag{5.5}$$

$$\overline{a}[f](t, z) = \left[\mathcal{O}^{1/2} f\right](t, z) + \int_{-\infty}^{0} dx \, \left[\mathcal{O}^{1/2} f\right](t, x)\overline{\mathcal{P}}(z, x), \qquad z > 0 \tag{5.6}$$

$$\begin{aligned} \overline{b}[f](t) =& \frac{1}{4}\int_{-\infty}^{0} dx \int_{-\infty}^{0} dy\, \overline{G}(x, y) \left[\mathcal{O}^{1/2} f\right](t, y) \left[\mathcal{O}^{1/2} f\right](t, x) \\ =& \frac{1}{4}\int_{0}^{\infty} dx \int_{0}^{\infty} dy\, G(x, y) \left[\mathcal{O}^{1/2} f\right](t, -y) \left[\mathcal{O}^{1/2} f\right](t, -x) \end{aligned} \tag{5.7}$$

$$\overline{o}_s^{-1}[\overline{a}[f]](t, x) = -\frac{1}{2\pi}\left(\int_x^{\infty} dz\, \overline{a}[f](t, z) \ln\left(1 - \frac{x}{z}\right) + \int_0^x dz\, \overline{a}[f](t, z) \ln\left(\frac{x}{z} - 1\right)\right) \tag{5.8}$$

and

$$\tilde{F}^{(2)}_{\overline{\Omega}}(t) = \left| \int_0^\infty dx\, \bar{a}[f](t,x) \bar{o}_s^{-1}[\bar{a}[f]](t,x) \right|^2 + \frac{1}{2} + 8(\bar{b}[R](t) + \bar{b}[I](t))^2. \tag{5.9}$$

With all these quantities, we can calculate the Renyi entropy over the region $\overline{\Omega} \equiv [0,\infty\rangle$. However for the more simplest case, when $t = 0$, we use again the normalization condition (D.15). Thus, the Rényi entropy for the region $\overline{\Omega} \equiv [0,\infty\rangle$ is obtained as follows

$$\begin{aligned} S_1[\overline{\Omega}] &= S_0[\overline{\Omega}] - \ln\left(\tilde{F}^{(2)}_{\overline{\Omega}}(0)\right) \\ &= S_0[\overline{\Omega}] - \ln\left(12\bar{b}_R^2(0) - 2\bar{b}_R(0) + \frac{3}{4}\right). \end{aligned} \tag{5.10}$$

Numerically, we generate a plot for the expression $-\ln\left(\tilde{F}^{(2)}_{\overline{\Omega}}(0)\right)$. Figure 5.1 shows the correction $\Delta S^{(2)}(\overline{\Omega})$ for the region $\overline{\Omega} = [0,\infty\rangle$ as a function of $\langle x\rangle$, exhibiting complementary behavior to Figure 5.2. When the particle is localized deep within $\overline{\Omega}$ (i.e., $\langle x\rangle \gg 0$), $\Delta S^{(2)}(\overline{\Omega})$ is approximately constant and positive, with its magnitude again dependent on α. This constant correction reflects the increased uncertainty in the field configurations within $\overline{\Omega}$ due to the presence of the excitation, analogous to the effect observed in Ω. The dependence on the wave packet's width α indicates that the spatial extent of the excitation modulates the entropy increase, with narrower wave packets leading to a more pronounced effect due to their localized impact.

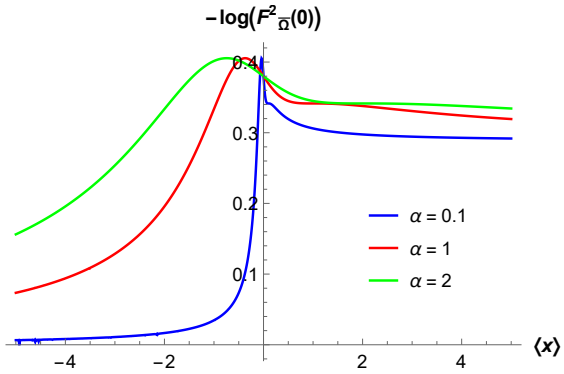


Figure 5.2: $t = 0$

As the particle approaches the boundary from within $\overline{\Omega}$ (i.e., $\langle x\rangle \to 0$), $\Delta S^{(2)}(\overline{\Omega})$ reaches a maximum, mirroring the behavior in Ω. This peak occurs because the wave packet's proximity to the boundary enhances the field's informational com-

plexity in $\overline{\Omega}$, as the particle's influence extends across the boundary, increasing the diversity of field configurations. The maximum entropy correction near $x = 0$ underscores the role of boundary interactions in amplifying correlations between the regions.

When the particle is located in Ω (i.e., $\langle x \rangle < 0$) and moves far from the boundary (i.e., $\langle x \rangle \ll 0$), $\Delta S^{(2)}(\overline{\Omega})$ decreases to zero. This indicates that the excitation's influence on $\overline{\Omega}$ becomes negligible, and the Rényi-2 entropy for $\overline{\Omega}$ reverts to the vacuum value, as the field configurations in $\overline{\Omega}$ are unaffected by the particle in Ω.

Physical Insights and Implications

The behavior of $\Delta S^{(2)}(\Omega)$ and $\Delta S^{(2)}(\overline{\Omega})$ as functions of $\langle x \rangle$ provides significant insights into the spatial distribution of classical information in the quantum field due to a localized excitation.

1. **Constant Enhancement Inside the Region:** The approximately constant entropy correction when the particle is well within one region (Ω or $\overline{\Omega}$) reflects the excitation's role in consistently increasing the field's informational complexity, with the magnitude tuned by the wave packet's width α.

2. **Effect of the Wave Packet Width:** Narrower wave packets, which localize the particle more precisely, tend to produce larger entropy corrections due to their concentrated effect on the field (more certainty in its position, less uncertainty needed to describe it), while broader wave packets distribute this effect over a larger region, potentially reducing the correction.

3. **Boundary Peak and Decay Outside the Region:** The maximum entropy correction near the boundary at $x = 0$ highlights the critical role of the interface between Ω and $\overline{\Omega}$ in mediating correlations. When the particle is near the boundary, its wave packet overlaps both regions, leading to enhanced correlations and a peak in the Rényi-2 entropy. This suggests that the boundary acts as a conduit for information sharing, maximizing the uncertainty in field configurations in both regions simultaneously. The decay of the entropy correction to zero as the particle moves far into the complementary region underscores the localized nature of the excitation's influence, with the field reverting to the vacuum state in the unaffected region.

These results are particularly relevant in the context of decoherence and measurement in QFT. The Rényi-2 entropy captures the classical information encoded in the diagonal elements of the density matrix, which persist after quantum coherence is lost due to environmental interactions or measurements. The enhanced entropy near the boundary suggests that localized excitations can serve as probes of the field's correlation structure, with their position and spatial extent tuning the distribution of information across spatial regions. This complements traditional entanglement entropy studies by focusing on the accessible classical correlations, providing a bridge between quantum and classical descriptions of the system. The dependence on α further implies that experimental control over the wave packet's width could be used to manipulate the information content, offering potential applications in quantum information processing and the study of decoherence-induced classicality in QFT systems.

5.2 Renyi mutual information between the regions Ω and $\overline{\Omega}$

With the results obtained in the last section, we can obtain the Renyi mutual information between the regions Ω and $\overline{\Omega}$. First of all, considering the more simplest case $\mathrm{Re}\,\beta = \mathrm{Im}\,\alpha = t = 0$, from (4.154) or (4.179) we can see that $\tilde{F}^{(2)}_{\Omega\cup\overline{\Omega}}(0) = \frac{3}{4}$.Thus, we numerically generate a plot of the expression

$$-\ln\left(\frac{\tilde{F}^{(2)}\Omega(0)\tilde{F}^{(2)}\overline{\Omega}(0)}{\tilde{F}^{(2)}_{\Omega\cup\overline{\Omega}}(0)}\right). \tag{5.11}$$

Figure 5.3 shows that, in general, the amount of information shared between the regions Ω and $\overline{\Omega}$ in the excited state exceeds that of the vacuum state. This excess reaches its maximum when the particle is located at the origin, and decays rapidly as the particle moves away from it.

Physical Insights from the One-Dimensional Case

The numerical results lead to several key physical conclusions:

- **Increased Shared Information:** In general, the presence of a localized particle *increases the amount of shared information* between the two regions compared to the vacuum state. This suggests that the excitation introduces new

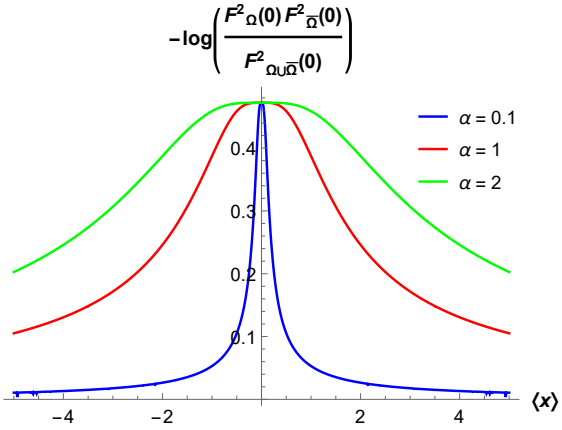


Figure 5.3: $t = 0$

correlations that are accessible as classical information.

- **Dependence on Particle Position:** The increase in mutual information is *maximized when the particle is located at the boundary* between the regions $(x = 0)$. This peak indicates that the classical correlations between Ω and $\bar{\Omega}$ are maximized when the particle's wave packet is symmetrically distributed across the boundary, leading to the greatest shared information between the regions. The independence of the peak value from σ suggests a universal feature of the field's correlation structure at the boundary, where the particle's influence is equally split between Ω and $\bar{\Omega}$. As the particle is moved away from the boundary into either region, this excess information decays rapidly. This decay indicates that the classical correlations between Ω and $\bar{\Omega}$ diminish as the excitation becomes localized deep within one region, reducing the shared information to the vacuum level.

- **Influence of Wave Packet Width:** The rate at which the mutual information decays as the particle moves from the boundary is *dependent on the width of the wave packet*. A narrowly localized wave packet leads to a sharp, fast decay, while a broader, more spatially extended wave packet results in a slower decay. This implies that a more delocalized particle can mediate correlations over a larger distance, although the maximum information shared is less than that of a particle situated directly at the boundary.

An important technical point noted in the analysis is that for the relativistic Schrödinger equation, a purely real wave packet $f(t, x)$ would be identically zero, which is unphysical. Therefore, for a correct and physically meaningful result, the wave packet must have both real and imaginary parts.

In summary, the results of this section successfully demonstrate that the Shannon mutual information, calculated via the replica trick, is a sensitive probe of a quantum field's state. The shift to the $n = 2$ Rényi entropy provides a practical method for extracting clear, physical insights, showing that a particle excitation enhances classical correlations in a way that depends critically on its position and spatial distribution.

Chapter 6

Conclusions and perspectives

This thesis embarked on an investigation into the Shannon mutual information for a quantum massless scalar field, specifically focusing on the correlations between two complementary spatial regions, Ω and $\overline{\Omega}$. A key characteristic of this work is the explicit calculation of this classical information measure from the diagonal elements of the density matrix in the field configuration basis. This approach offers a direct avenue to quantify the classical correlations that persist or emerge after quantum decoherence or measurement processes, which effectively diagonalize the system's density matrix in a particular basis, such as the field configuration basis. The overarching goal was to explore the structure and flow of information within a Quantum Field Theory (QFT) framework, providing insights that are complementary to traditional entanglement entropy calculations by focusing on accessible classical information.

To achieve this, we employed the Schrödinger picture functional formalism. The probability of field configurations, $P[\phi] = |\Psi[\phi]|^2$, served as the foundation for defining Shannon entropies for the regions Ω, $\overline{\Omega}$, and their union $\Omega \cup \overline{\Omega}$. The Shannon mutual information was then determined as $I_S(\Omega, \overline{\Omega}) = S[\Omega] + S[\overline{\Omega}] - S[\Omega \cup \overline{\Omega}]$. The calculations leveraged techniques such as the saddle-point approximation and the replica trick to manage the functional integrals inherent in computing these Shannon entropies.

The investigation was carried out for two distinct scenarios: the vacuum state and a localized single-particle excited state of the massless scalar field.

In Chapter 3, we laid the groundwork by introducing the formalism for a quantum scalar field in the Schrödinger picture. We derived the ground state wave functional using the path integral and saddle-point approximation. Following this, we defined creation and annihilation operators and proceeded to compute the reduced probability distributions for subregions. The Shannon entropies and, consequently, the Shannon mutual information for the vacuum state were then calculated using the replica trick. The expression for the mutual information in

the vacuum state was found to be

$$I_0(\Omega,\overline{\Omega}) = \frac{1}{2}\ln\left(\frac{\det_\Omega(\mu^{-1}O)\,\det_{\overline{\Omega}}(\mu^{-1}O)}{\det(\mu^{-1}O)}\right) - \frac{1}{2}\,\frac{\partial}{\partial n}\ln\left(\frac{\det_\Omega(n\mu^{-1}O)\,\det_{\overline{\Omega}}(n\mu^{-1}O)}{\det(n\mu^{-1}O)}\right)\Bigg|_{n=1}$$

, a result consistent with previous findings in the literature.

Chapter 4 extended this analysis to localized single-particle excited states. We began by deriving the wave functional for these states,$\Psi_1[\phi] = \frac{2}{\sqrt{\gamma}}\int d^dx f(t,x)O^{1/2}\phi(x)\Psi_0[\phi]$, and discussed suitable wave packet solutions, including those minimizing position-momentum and position-velocity uncertainty. The probability distribution for the excited state was established as $P_1[\phi] = \frac{4}{\gamma}\left|\int d^dx f(t,x)O^{1/2}\phi(x)\right|^2 P_0[\phi]$. Subsequently, the reduced probability distributions for these excited states were computed using the saddle-point method. The Shannon entropies and mutual information were then determined using the replica trick. The Shannon mutual information for the excited state was expressed in relation to the vacuum mutual information and correction terms dependent on the wave packet profile $f(t,x)$ and the replica parameter n:

$$I_1(\Omega,\overline{\Omega}) = I_0(\Omega,\overline{\Omega}) - \frac{\partial}{\partial n}\ln\left(\frac{\tilde{F}^{(n)}_\Omega(t)\tilde{F}^{(n)}_{\overline{\Omega}}(t)}{\tilde{F}^{(n)}_{\Omega\cup\overline{\Omega}}(t)}\right)\Bigg|_{n=1}.$$

In Chapter 5, we specialized the formalism to calculate the Rényi entropy (for $n = 2$) in one spatial dimension, considering regions defined as half-lines and utilizing the derived wave packets, particularly the Lorentzian wave packet. Numerical results for the Rényi mutual information for $n = 2$ indicated that the information shared between the two half-lines in an excited state generally exceeds that of the vacuum state. This excess information peaks when the particle is localized near the boundary between the regions and decays as the particle moves away, with the decay rate influenced by the width of the wave packet. This suggests that more spatially extended wave packets share more information across the boundary, though less than a sharply localized particle at the boundary.

The findings of this thesis contribute to the understanding of how classical correlations, as quantified by Shannon mutual information, arise and are structured in a simple QFT model. By focusing on the diagonal elements of the density matrix, we have provided a framework that directly connects to scenarios involving decoherence or measurement, where off-diagonal quantum coherences are diminished. The comparison between the vacuum and excited states reveals how the presence of a quantum excitation alters these classical correlations, offering insights into the interplay between quantum states and the extractable classical information they encode.

6.1 Perspectives for Future Research:

The work presented in this thesis opens several avenues for future investigation:

- **Different Quantum States:** The analysis could be extended to other types of quantum states, such as multi-particle states, coherent states, or thermal states, to explore how different quantum features influence Shannon mutual information.
- **Interacting Field Theories:** While this thesis focused on a massless scalar field (a free theory), a significant extension would be to consider interacting QFTs. This would introduce much richer dynamics and correlation structures, though the calculations would become considerably more complex.
- **Higher Dimensions and Different Geometries:** The specific calculations in Chapter 5 were for one spatial dimension and half-line regions. Exploring higher spatial dimensions ($d > 1$) and different geometries for the regions Ω and $\overline{\Omega}$ (e.g., spheres, strips) would provide a more comprehensive understanding of the geometric dependence of Shannon mutual information.
- **Dynamical Scenarios:** This work primarily considered static configurations or snapshots in time (fixed-time field configurations). Investigating the time evolution of Shannon mutual information, for instance, during a quench or other non-equilibrium processes, could reveal interesting dynamics of information flow.
- **Connection to Other Information Measures:** A deeper exploration of the relationship between the Shannon mutual information calculated here and other quantum information measures, such as entanglement of formation, quantum discord (particularly the symmetric quantum discord discussed in Chapter 2), or relative entropy of coherence, could provide a more complete picture of the information content in QFT systems and how it transforms under decoherence.
- **Specific Wave Packet Analysis:** The results in Chapter 5 for the Lorentzian wave packet showed interesting dependencies on localization and width. A more systematic study with different wave packet shapes and their evolution, particularly the minimal position-momentum uncertainty (Gaussian) wave packet, could provide further insights into how particle localization impacts classical correlations. The observation that a purely real wave packet $f(t, x)$ implies $f(t, x) = 0$ for the relativistic Schrödinger equation warrants careful handling of wave packet reality conditions in future work.
- **Further Development of the Poisson Problem Formalism:** Appendix A details an iterative approach to solving the non-homogeneous Poisson problem for the

fractional Laplacian, leading to series representations for the Poisson kernel (A.47) and Green's function (A.86). This formalism, developed to handle the specific requirements of the thesis calculations, could be further investigated. A detailed comparison of this method, particularly its convergence properties and computational efficiency, with established techniques in potential theory, such as those described in "Foundations of modern potential theory" by N. S. Landkof [42], could be undertaken. Such a study might reveal advantages or specific regimes where the iterative approach of Appendix A offers new insights or practical benefits for solving fractional differential equations, especially for the boundary conditions and geometries relevant to QFT entanglement problems. Exploring the applicability of this specific construction of Green's functions and Poisson kernels to other physical problems involving fractional operators could also be a fruitful research direction.

In summary, this thesis has provided a detailed calculation of Shannon mutual information for a massless scalar field, offering a perspective on classical correlations within a quantum field theory context. The results and the formalism developed herein pave the way for further explorations into the rich interplay between quantum field theory, information theory, and the process of decoherence.

Appendix A

Fractional Laplacian Operator

In this appendix we have considered some formalisms about the fractional Laplacian operator [43].

The Fourier transform of any function $u(\mathbf{x}) \in L^1(\mathbb{R}^n)$[1] can set as

$$u(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}} \int d^d\mathbf{k}\, \hat{u}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \tag{A.1}$$

and respectively

$$\hat{u}(\mathbf{k}) = \frac{1}{(2\pi)^{d/2}} \int d^d\mathbf{x}\, u(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} \tag{A.2}$$

We say that $\hat{f}$ is the Fourier transform of f in a distributional sense, for f that satisfies $\int d^d\mathbf{x} \frac{|f(\mathbf{x})|}{1+|\mathbf{x}|^p} < \infty$ for some $p \in \mathbb{N}$ if, for any $\varphi \in \mathcal{S}\left(\mathbb{R}^d\right)$[2] We have that

$$\int d^d\mathbf{x}\, \hat{f}(\mathbf{x})\varphi(\mathbf{x}) = \int d^d\mathbf{x}\, f(\mathbf{x})\hat{u}(\mathbf{x}) \tag{A.3}$$

In a simpler manner, using the Fourier transform, let us define the fractional Laplacian operator $\mathcal{O}^{2s} \equiv \left(m^2 - \nabla^2\right)^s$ as

$$[\mathcal{O}^{2s}u](\mathbf{x}) = \frac{1}{(2\pi)^{d/2}} \int d^d\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}} \omega_{\mathbf{k}}^{2s} \hat{u}(\mathbf{k}) \tag{A.4}$$

with $\omega_{\mathbf{k}} = \sqrt{m^2 + \mathbf{k}^2}$. Also

[1] The space $L^1(\mathbb{R})$ is defined as the set of all Lebesgue measurable functions $f : \mathbb{R} \to \mathbb{C}$ (or $\mathbb{R}$) for which the integral of the absolute value of the function is finite. Formally, it can be expressed as: $L^1(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{C} \mid \int_{\mathbb{R}} |f(x)|\, dx < \infty \right\}$.

[2] The Schwartz space, denoted as $\mathcal{S}\left(\mathbb{R}^d\right)$, is a space of functions that are smooth (infinitely differentiable) and rapidly decreasing. Formally, it can be expressed as:

$$\mathcal{S}\left(\mathbb{R}^d\right) := \left\{ f \in C^\infty\left(\mathbb{R}^d\right) \text{ s.t. } \forall \alpha, \beta \in \mathbb{N}_0^d,\ \sup_{\mathbf{x}\in\mathbb{R}^d} \left| \mathbf{x}^\beta D^\alpha f(\mathbf{x}) \right| < \infty \right\}$$

$$\begin{aligned}[\mathcal{O}^{2s}u](\mathbf{x}) =& \frac{1}{(2\pi)^{d/2}}\int d^d\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}}\omega_{\mathbf{k}}^{2s}\left[\frac{1}{(2\pi)^{d/2}}\int d^d\mathbf{x}\, u(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}\right]\\ =& \frac{1}{(2\pi)^{d/2}}\int d^d\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}}\left[\frac{1}{(2\pi)^{d/2}}\int d^d\mathbf{x}\, u(\mathbf{x})\omega_{\mathbf{k}}^{2s}e^{-i\mathbf{k}\cdot\mathbf{x}}\right]\end{aligned} \tag{A.5}$$

Thus, the inverse Fourier transform of $[\mathcal{O}^{2s}u](\mathbf{x})$ will be

$$\widehat{[\mathcal{O}^{2s}u]}(\mathbf{k}) = \frac{1}{(2\pi)^{d/2}}\int d^d\mathbf{x}\, u(\mathbf{x})\omega_{\mathbf{k}}^{2s}e^{-i\mathbf{k}\cdot\mathbf{x}} \tag{A.6}$$

We can verify the following composite property

$$\left[\mathcal{O}^{2s'}\left[\mathcal{O}^{2s''}u\right]\right](\mathbf{x}) = [\mathcal{O}^{2(s'+s'')}u](\mathbf{x}) \tag{A.7}$$

since

$$\begin{aligned}\left[\mathcal{O}^{2s'}\left[\mathcal{O}^{2s''}u\right]\right](\mathbf{x}) =& \frac{1}{(2\pi)^{d/2}}\int d^d\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}}\omega_{\mathbf{k}}^{2s'}\widehat{[\mathcal{O}^{2s''}u]}(\mathbf{k})\\ =& \frac{1}{(2\pi)^{d/2}}\int d^d\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}}\omega_{\mathbf{k}}^{2s'}\left[\frac{1}{(2\pi)^{d/2}}\int d^d\mathbf{x}'\, u(\mathbf{x}')\omega_{\mathbf{k}}^{2s''}e^{-i\mathbf{k}\cdot\mathbf{x}'}\right]\\ =& \frac{1}{(2\pi)^{d/2}}\int d^d\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}}\omega_{\mathbf{k}}^{2(s'+s'')}\left[\frac{1}{(2\pi)^{d/2}}\int d^d\mathbf{x}'\, u(\mathbf{x}')e^{-i\mathbf{k}\cdot\mathbf{x}'}\right]\\ =& \frac{1}{(2\pi)^{d/2}}\int d^d\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}}\omega_{\mathbf{k}}^{2(s'+s'')}\hat{u}(\mathbf{k})\\ =& [\mathcal{O}^{2(s'+s'')}u](\mathbf{x})\end{aligned} \tag{A.8}$$

Also, we can verify

$$\int d^d\mathbf{x}\, f(\mathbf{x})\left[\mathcal{O}^{2s}g\right](\mathbf{x}) = \int d^d\mathbf{x}\, \left[\mathcal{O}^{2s}f\right](\mathbf{x})g(\mathbf{x}) \tag{A.9}$$

since

$$\begin{aligned}\int d^d\mathbf{x}\, f(\mathbf{x})\left[\mathcal{O}^{2s}g\right](\mathbf{x}) &= \int d^d\mathbf{x}\, f(\mathbf{x})\frac{1}{(2\pi)^{d/2}}\int d^d\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}}\omega_{\mathbf{k}}^{2s}\hat{g}(\mathbf{k})\\ &= \int d^d\mathbf{k}'\hat{f}(\mathbf{k}')\int d^d\mathbf{k}\, \omega_{\mathbf{k}}^{2s}\hat{g}(\mathbf{k})\frac{1}{(2\pi)^d}\int d^d\mathbf{x}\, e^{i(\mathbf{k}'+\mathbf{k})\cdot\mathbf{x}}\\ &= \int d^d\mathbf{k}'\hat{f}(\mathbf{k}')\int d^d\mathbf{k}\, \omega_{\mathbf{k}}^{2s}\hat{g}(\mathbf{k})\delta(\mathbf{k}'+\mathbf{k})\\ &= \int d^d\mathbf{k}'\hat{f}(\mathbf{k}')\omega_{\mathbf{k}'}^{2s}\int d^d\mathbf{k}\, \hat{g}(\mathbf{k})\delta(\mathbf{k}'+\mathbf{k})\\ &= \int d^d\mathbf{x}\, \left[\mathcal{O}^{2s}f\right](\mathbf{x})g(\mathbf{x})\end{aligned} \tag{A.10}$$

Formally, the fractional Laplacian of $u(\mathbf{x}) \in \mathcal{S}\left(\mathbb{R}^d\right)$ is defined as

$$\begin{aligned}\left(-\nabla^2\right)^s u(\mathbf{x}) &:= C(d,s) P.V. \int d^d\mathbf{y} \frac{u(\mathbf{x})-u(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+2s}} \\ &= C(d,s) \lim_{\epsilon\to 0} \int_{\mathbb{R}^d/B_\epsilon(\mathbf{x})} d^d\mathbf{y} \frac{u(\mathbf{x})-u(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+2s}}\end{aligned} \tag{A.11}$$

where $s \in <0,1>$, and $C(d,s)$ is a constant which is give by

$$C(d,s) = \left(\int d^d\zeta \frac{1-\cos\zeta_1}{|\mathbf{1}|^{d+2s}}\right)^{-1} = \frac{2^{2s} s\Gamma\left(\frac{d}{2}+s\right)}{\pi^{\frac{d}{2}}\Gamma(1-s)} \tag{A.12}$$

being ζ_1 the first coordinate of ζ.

If we consider $m=0$ in (A.4), we can probe that the definitions of fractional Laplacian (A.4) and (A.11) are equivalent. To this purpose we can see that (A.11) take the following equivalent form

$$\left(-\nabla^2\right)^s u(\mathbf{x}) = -\frac{1}{2} C(d,s) \int d^d\mathbf{y} \frac{u(\mathbf{x}+\mathbf{y})+u(\mathbf{x}-\mathbf{y})-2u(\mathbf{x})}{|\mathbf{y}|^{d+2s}}. \tag{A.13}$$

Using the Fourier transform (A.1), we can obtain

$$\begin{aligned}\left(-\nabla^2\right)^s u(\mathbf{x}) &= -\frac{1}{2} C(d,s) \int d^d\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}} \hat{u}(\mathbf{k}) \int d^d\mathbf{y} \frac{e^{i\mathbf{k}\cdot\mathbf{y}}+e^{-i\mathbf{k}\cdot\mathbf{y}}-2}{|\mathbf{y}|^{d+2s}} \\ &= \int d^d\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}} \hat{u}(\mathbf{k}) \left[C(d,s) \int d^d\mathbf{y} \frac{1-\cos(\mathbf{k}\cdot\mathbf{y})}{|\mathbf{y}|^{d+2s}}\right]\end{aligned} \tag{A.14}$$

denoting

$$\mathcal{J}^s(\mathbf{k}) = \int d^d\mathbf{y} \frac{1-\cos(\mathbf{k}\cdot\mathbf{y})}{|\mathbf{y}|^{d+2s}} \tag{A.15}$$

we can prove that it is rotationally invariant

$$\mathcal{J}^s(\mathbf{k}) = \mathcal{J}^s(|\mathbf{k}|e_1) \tag{A.16}$$

Thus making the change of variable $\mathbf{\imath} = |\mathbf{k}|\mathbf{y}$

$$\begin{aligned}\mathcal{J}^s(|\mathbf{k}|e_1) &= \int d^d\mathbf{y} \frac{1-\cos(|\mathbf{k}|y_1)}{|\mathbf{y}|^{d+2s}} \\ &= \frac{1}{|\mathbf{k}|^d} \int d^d\mathbf{\imath} \frac{1-\cos(\zeta_1)}{|\mathbf{\imath}|^{d+2s}} \\ &= C^{-1}(d,s)|\mathbf{k}|^{2s}\end{aligned} \tag{A.17}$$

Therefore (A.11) is equivalent to (A.4) with $m = 0$.

To simplify the notations, we denote the fractional Laplacian as

$$[\mathcal{O}^{2s}u](\mathbf{x}) = \int d^d\mathbf{y} K^s(\mathbf{x}-\mathbf{y})[u(\mathbf{x}) - u(\mathbf{y})], \tag{A.18}$$

where $m = 0$, i.e. $\mathcal{O} := \sqrt{-\nabla^2}$, and

$$K^s(\mathbf{x}-\mathbf{y}) := C(d,s)\frac{1}{|\mathbf{x}-\mathbf{y}|^{d+2s}} \tag{A.19}$$

A.1 Inverse of Fractional Laplacian Operator

In terms of the Fourier transform, the inverse fractional Laplacian can be defined as [44]:

$$[\mathcal{O}^{-2s}u](\mathbf{x}) = \frac{1}{(2\pi)^{d/2}}\int d^d\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}}\omega_{\mathbf{k}}^{-2s}\hat{u}(\mathbf{k}) \tag{A.20}$$

Let verify that

$$[\mathcal{O}^{-2s}[\mathcal{O}^{2s}u]](\mathbf{x}) = u(\mathbf{x}) \tag{A.21}$$

In fact

$$\begin{aligned}
[\mathcal{O}^{-2s}[\mathcal{O}^{2s}u]](\mathbf{x}) =& \frac{1}{(2\pi)^{d/2}}\int d^d\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}}|\mathbf{k}|^{-2s}\widehat{[\mathcal{O}^{2s}u]}(\mathbf{k}) \\
=& \frac{1}{(2\pi)^{d/2}}\int d^d\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}}|\mathbf{k}|^{-2s}\left[\frac{1}{(2\pi)^{d/2}}\int d^d\mathbf{x}'\, u(\mathbf{x}')|\mathbf{k}|^{2s}e^{-i\mathbf{k}\cdot\mathbf{x}'}\right] \\
=& \int d^d\mathbf{x}'\, u(\mathbf{x}')\left[\frac{1}{(2\pi)^d}\int d^d\mathbf{k}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\right] \\
=& \int d^d\mathbf{x}'\, u(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}') \\
=& u(\mathbf{x})
\end{aligned} \tag{A.22}$$

respectively it can be verify that $[\mathcal{O}^{2s}[\mathcal{O}^{-2s}u]](\mathbf{x}) = u(\mathbf{x})$.

Also, we can expresess this operator in the integral way. For this puspose, symple, we must consider the Fourier transform of u in (A.20) and then integrate in $\mathbf{k}$. Thus

$$\begin{aligned}
[\mathcal{O}^{-2s}u](\mathbf{x}) =& \frac{1}{(2\pi)^d}\int d^d\mathbf{k}\int d^d\mathbf{x}'\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\omega_{\mathbf{k}}^{-2s}u(\mathbf{x}') \\
=& \frac{1}{(2\pi)^d}\int d^d\mathbf{x}' u(\mathbf{x}')\int d^d\mathbf{k}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\omega_{\mathbf{k}}^{-2s}.
\end{aligned} \tag{A.23}$$

For the integral in $\mathbf{k}$, we obtain

$$\begin{aligned}\int d^d\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}}\omega_{\mathbf{k}}^{-2s} =&\Omega_{d-2}\int_0^\infty d\kappa\, \kappa^{d-1}\omega_\kappa^{-2s}\int_0^\pi d\theta\, \sin^{d-2}\theta e^{i\kappa r\cos\theta}\\ =&\Omega_{d-2}\int_0^\infty d\kappa\, \kappa^{d-1}\omega_\kappa^{-2s}\left[\sqrt{\pi}\left(\frac{2}{\kappa r}\right)^{\frac{d}{2}-1}\Gamma\left(\frac{d-1}{2}\right)J_{\frac{d}{2}-1}(\kappa r)\right]\\ =&\sqrt{\pi}\Omega_{d-2}\Gamma\left(\frac{d-1}{2}\right)\left(\frac{2}{r}\right)^{\frac{d}{2}-1}\int_0^\infty d\kappa\, \kappa^{d/2}\omega_\kappa^{-2s}J_{\frac{d}{2}-1}(\kappa r)\\ =&\sqrt{\pi}\Omega_{d-2}\Gamma\left(\frac{d-1}{2}\right)\left(\frac{2}{r}\right)^{\frac{d}{2}-1}\int_0^\infty d\kappa\, \kappa^{d/2}\left(m^2+\kappa^2\right)^{-s}J_{\frac{d}{2}-1}(\kappa r)\end{aligned}\tag{A.24}$$

For $m=0$

$$\int d^d\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}}|\mathbf{k}|^{-2s} = \sqrt{\pi}\Omega_{d-2}\Gamma\left(\frac{d-1}{2}\right)\left(\frac{2}{r}\right)^{\frac{d}{2}-1}\int_0^\infty d\kappa\, \kappa^{d/2-2s}J_{\frac{d}{2}-1}(\kappa r)\tag{A.25}$$

Considering the fact

$$\int_0^\infty dx\, x^\mu J_\nu(ax) = 2^\mu a^{-\mu-1}\frac{\Gamma\left(\frac{1}{2}+\frac{\nu}{2}+\frac{\mu}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{\nu}{2}-\frac{\mu}{2}\right)},\quad -\operatorname{Re}\nu-1<\operatorname{Re}\mu<\frac{1}{2}+\operatorname{Re}\nu,\ a>0,\tag{A.26}$$

we get

$$\begin{aligned}\int d^d\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}}|\mathbf{k}|^{-2s} =&\pi^{\frac{d}{2}}2^{d-2s}r^{2s-d}\frac{\Gamma\left(\frac{d}{2}-s\right)}{\Gamma(s)}\\ =&(2\pi)^{\frac{d}{2}}\frac{2^{\frac{d}{2}-s}\Gamma\left(\frac{d}{2}-s\right)}{2^s\Gamma(s)}\left(\frac{1}{|\mathbf{x}|}\right)^{d-2s}\end{aligned}\tag{A.27}$$

Denoting $C(d,-s)=\dfrac{2^{\frac{d}{2}-s}\Gamma\left(\frac{d}{2}-s\right)}{(2\pi)^{\frac{d}{2}}2^s\Gamma(s)}$, we have that

$$\int d^d\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}}|\mathbf{k}|^{-2s} = (2\pi)^d C(d,-s)\frac{1}{|\mathbf{x}|^{d-2s}}\tag{A.28}$$

Therefore, inserting this result in (A.28) when $m=0$

$$\begin{aligned}[\mathcal{O}^{-2s}u](\mathbf{x}) =&C(d,-s)\int d^d\mathbf{y}\,\frac{u(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d-2s}}\\ =&C(d,-s)\int d^d\mathbf{y}\,\frac{u(\mathbf{x}-\mathbf{y})}{|\mathbf{y}|^{d-2s}}.\end{aligned}\tag{A.29}$$

We can note that this integral form of the inverse fractional Laplacian is valid for $s \in \langle 0, \frac{d}{2} \rangle$. So we see that for one dimensional case this integral for only is valid for $s \in \langle 0, \frac{1}{2} \rangle$.

A.2 Non-homogeneous Poisson problem

The non-local Poisson problem is given by

$$\begin{aligned} [\mathcal{O}u](\mathbf{x}) &= f(\mathbf{x}) \qquad \mathbf{x} \in \overline{\Omega} \\ u(\mathbf{x}) &= g(\mathbf{x}), \qquad \mathbf{x} \in \Omega \end{aligned} \tag{A.30}$$

Using the Kernel definition of the fractional Laplacian operator, when $\mathbf{x} \in \overline{\Omega}$, we obtain

$$\begin{aligned} [\mathcal{O}u](\mathbf{x}) =& P.V. \int_{\overline{\Omega}} d^d\mathbf{z}\, K(\mathbf{x}-\mathbf{z})[u(\mathbf{x}) - u(\mathbf{z})] + \int_{\Omega} d^d\mathbf{y}\, K(\mathbf{x}-\mathbf{y})[u(\mathbf{x}) - g(\mathbf{y})] \\ =& P.V. \int_{\overline{\Omega}} d^d\mathbf{z}\, K(\mathbf{x}-\mathbf{z})[u(\mathbf{x}) - u(\mathbf{z})] + u(\mathbf{x}) \int_{\Omega} d^d\mathbf{y}\, K(\mathbf{x}-\mathbf{y}) \\ & - \int_{\Omega} d^d\mathbf{y}\, K(\mathbf{x}-\mathbf{y}) g(\mathbf{y}) \end{aligned} \tag{A.31}$$

where we have separated the integral into one part where the kernel K is finite and another where it is infinite. In general we can express the solution for the non-local Poisson problem as

$$u(\mathbf{x}) = \int_{\Omega} d^d\mathbf{y}\, \delta(\mathbf{x}-\mathbf{y}) g(\mathbf{x}) + \int_{\overline{\Omega}} d^d\mathbf{y}\, \delta(\mathbf{x}-\mathbf{y})\, u(\mathbf{y}) \tag{A.32}$$

So, when $\mathbf{x} \in \overline{\Omega}$,

$$u(\mathbf{x}) = \int_{\overline{\Omega}} d^d\mathbf{y}\, \delta(\mathbf{x}-\mathbf{y})\, u(\mathbf{y}), \qquad \mathbf{x} \in \overline{\Omega} \tag{A.33}$$

Inserting this last expression in (A.31)

$$\begin{aligned} [\mathcal{O}u](\mathbf{x}) =& P.V. \int_{\overline{\Omega}} d^d\mathbf{z}\, K(\mathbf{x}-\mathbf{z}) \left[\int_{\overline{\Omega}} d^d\mathbf{y}\, \delta(\mathbf{x}-\mathbf{y})\, u(\mathbf{y}) - \int_{\overline{\Omega}} d^d\mathbf{y}\, \delta(\mathbf{z}-\mathbf{y})\, u(\mathbf{y}) \right] \\ & + u(\mathbf{x}) \int_{\Omega} d^d\mathbf{y}\, K(\mathbf{x}-\mathbf{y}) - \int_{\Omega} d^d\mathbf{y}\, K(\mathbf{x}-\mathbf{y}) g(\mathbf{y}) \\ =& \int_{\overline{\Omega}} d^d\mathbf{y}\, u(\mathbf{y})\, P.V. \int_{\overline{\Omega}} d^d\mathbf{z}\, K(\mathbf{x}-\mathbf{z})[\, \delta(\mathbf{x}-\mathbf{y}) - \delta(\mathbf{z}-\mathbf{y})\,] \\ & + u(\mathbf{x}) \int_{\Omega} d^d\mathbf{y}\, K(\mathbf{x}-\mathbf{y}) - \int_{\Omega} d^d\mathbf{y}\, K(\mathbf{x}-\mathbf{y}) g(\mathbf{y}). \end{aligned} \tag{A.34}$$

Defining the fractional Laplacian of the Dirac delta over the region $\overline{\Omega}$ as

$$[\mathcal{O}_{\overline{\Omega}}\delta](\mathbf{x}-\mathbf{y}) \equiv P.V. \int_{\overline{\Omega}} d^d\mathbf{z}\, K(\mathbf{x}-\mathbf{z})[\delta(\mathbf{x}-\mathbf{y}) - \delta(\mathbf{z}-\mathbf{y})] \tag{A.35}$$

Thus,

$$[\mathcal{O}u](\mathbf{x}) = \int_{\overline{\Omega}} d^d\mathbf{y}\, u(\mathbf{y})[\mathcal{O}_{\overline{\Omega}}\delta](\mathbf{x}-\mathbf{y}) + u(\mathbf{x}) \int_{\Omega} d^d\mathbf{y}\, K(\mathbf{x}-\mathbf{y}) - \int_{\Omega} d^d\mathbf{y}\, K(\mathbf{x}-\mathbf{y})g(\mathbf{y}) \tag{A.36}$$

Therefore in the equation (A.30), when $\mathbf{x} \in \overline{\Omega}$, we have

$$\begin{aligned} f(\mathbf{x}) &= [\mathcal{O}u](\mathbf{x}) \\ &= \int_{\overline{\Omega}} d^d\mathbf{y}\, u(\mathbf{y})[\mathcal{O}_{\overline{\Omega}}\delta](\mathbf{x}-\mathbf{y}) + u(\mathbf{x}) \int_{\Omega} d^d\mathbf{y}\, K(\mathbf{x}-\mathbf{y}) - \int_{\Omega} d^d\mathbf{y}\, K(\mathbf{x}-\mathbf{y})g(\mathbf{y}) \end{aligned} \tag{A.37}$$

this yield

$$u(\mathbf{x}) = \frac{1}{\int_{\Omega} d^d\mathbf{z}\, K(\mathbf{x}-\mathbf{z})} \left[f(\mathbf{x}) + \int_{\Omega} d^d\mathbf{y}\, K(\mathbf{x}-\mathbf{y})g(\mathbf{y}) - \int_{\overline{\Omega}} d^d\mathbf{y}\, u(\mathbf{y})[\mathcal{O}_{\overline{\Omega}}\delta](\mathbf{x}-\mathbf{y}) \right], \tag{A.38}$$

following an iterative process, we have

$$u(\mathbf{x}) = \mathcal{G}[f](\mathbf{x}) + u^{(0)}(\mathbf{x}) + \int_{\overline{\Omega}} \prod_{n=1}^{\infty} d^d\mathbf{y}_n \frac{[\mathcal{O}_{\overline{\Omega}}\delta](\mathbf{y}_{n-1}-\mathbf{y}_n)}{\int_{\Omega} d^d\mathbf{y} K(\mathbf{y}_{n-1}-\mathbf{y})} u(\mathbf{y}_n). \tag{A.39}$$

Therefore, if the correction term is very small, we can obtain the solution

$$\begin{aligned} u(\mathbf{x}) &= u^{(0)}(\mathbf{x}) + \mathcal{G}[f](\mathbf{x}), \qquad \mathbf{x} \in \overline{\Omega} \\ u(\mathbf{x}) &= g(\mathbf{x}), \qquad\qquad\qquad\quad \mathbf{x} \in \Omega \end{aligned} \tag{A.40}$$

being

$$u^{(0)}(\mathbf{x}) = \int_{\Omega} d^d\mathbf{y}\, \mathcal{P}(\mathbf{y},\mathbf{x})g(\mathbf{y}) \quad , \mathbf{x} \in \overline{\Omega} \tag{A.41}$$

We define the Poisson kernel with $\mathbf{x} \in \overline{\Omega}$ and $\mathbf{y} \in \Omega$

$$\begin{aligned} \mathcal{P}(\mathbf{y},\mathbf{x}) = &\frac{K(\mathbf{x}-\mathbf{y})}{\int_{\Omega} d^d\mathbf{z} K(\mathbf{x}-\mathbf{z})} + \sum_{n=1}^{\infty} (-1)^n \int_{\overline{\Omega}} d^d\mathbf{y}_1 \cdots d^d\mathbf{y}_n \frac{K(\mathbf{y}_n-\mathbf{y})}{\int_{\Omega} d^3\mathbf{z} K(\mathbf{y}_n-\mathbf{z})} \\ &\times \frac{[\mathcal{O}_{\overline{\Omega}}\delta](\mathbf{y}_{n-1}-\mathbf{y}_n)}{\int_{\Omega} d^d\mathbf{z}\, K(\mathbf{y}_{n-1}-\mathbf{z})} \cdots \frac{[\mathcal{O}_{\overline{\Omega}}\delta](\mathbf{x}-\mathbf{y}_1)}{\int_{\Omega} d^d\mathbf{z}\, K(\mathbf{x}-\mathbf{z})} \end{aligned} \tag{A.42}$$

and the non-homogeneous solution

$$\begin{aligned} \mathcal{G}[f](\mathbf{x}) = &\frac{f(\mathbf{x})}{\int_{\Omega} d^d\mathbf{z} K(\mathbf{x}-\mathbf{z})} + \sum_{n=1}^{\infty} (-1)^n \int_{\overline{\Omega}} d^d\mathbf{y}_1 \cdots d^d\mathbf{y}_n \frac{f(\mathbf{y}_n)}{\int_{\Omega} d^d\mathbf{z} K(\mathbf{y}_n-\mathbf{z})} \\ &\times \frac{[\mathcal{O}_{\overline{\Omega}}\delta](\mathbf{y}_{n-1}-\mathbf{y}_n)}{\int_{\Omega} d^d\mathbf{z} K(\mathbf{y}_{n-1}-\mathbf{z})} \cdots \frac{[\mathcal{O}_{\overline{\Omega}}\delta](\mathbf{x}-\mathbf{y}_1)}{\int_{\Omega} d^d\mathbf{z} K(\mathbf{x}-\mathbf{z})}. \end{aligned} \tag{A.43}$$

Defining

$$\mathbb{K}(\mathbf{x},\mathbf{y}) \equiv \frac{[\mathcal{O}_{\overline{\Omega}}\delta](\mathbf{x}-\mathbf{y})}{\int_\Omega d^d\mathbf{z}\, K(\mathbf{x}-\mathbf{z})} \tag{A.44}$$

$$\mathcal{P}_0(\mathbf{y},\mathbf{x}) \equiv \frac{K(\mathbf{x}-\mathbf{y})}{\int_\Omega d^d\mathbf{z} K(\mathbf{x}-\mathbf{z})} \tag{A.45}$$

$$\mathcal{G}_0[f](\mathbf{x}) \equiv \frac{f(\mathbf{x})}{\int_\Omega d^d\mathbf{z} K(\mathbf{x}-\mathbf{z})} \tag{A.46}$$

We can simplify the Poisson kernel with $\mathbf{x} \in \overline{\Omega}$ and $\mathbf{y} \in \Omega$

$$\mathcal{P}(\mathbf{y},\mathbf{x}) = \mathcal{P}_0(\mathbf{y},\mathbf{x}) + \sum_{n=1}^{\infty}(-1)^n \int_{\overline{\Omega}} d^d\mathbf{y}_1 \cdots d^d\mathbf{y}_n \mathcal{P}_0(\mathbf{y},\mathbf{y}_n)\mathbb{K}(\mathbf{y}_{n-1},\mathbf{y}_n)\cdots\mathbb{K}(\mathbf{y}_1,\mathbf{y}_2)\mathbb{K}(\mathbf{x},\mathbf{y}_1) \tag{A.47}$$

and the non-homogeneous solution

$$\mathcal{G}[f](\mathbf{x}) = \mathcal{G}_0[f](\mathbf{x}) + \sum_{n=1}^{\infty}(-1)^n \int_{\overline{\Omega}} d^d\mathbf{y}_1 \cdots d^d\mathbf{y}_n \mathcal{G}_0[f](\mathbf{y}_n)\mathbb{K}(\mathbf{y}_{n-1},\mathbf{y}_n)\cdots\mathbb{K}(\mathbf{y}_1,\mathbf{y}_2)\mathbb{K}(\mathbf{x},\mathbf{y}_1) \tag{A.48}$$

A.3 Poisson Kernel

We define the Poisson kernel with $\mathbf{x} \in \overline{\Omega}$ and $\mathbf{y} \in \Omega$ as

$$\mathcal{P}(\mathbf{y},\mathbf{x}) = \mathcal{P}_0(\mathbf{y},\mathbf{x}) + \sum_{n=1}^{\infty}(-1)^n \int_{\overline{\Omega}} d^d\mathbf{y}_1 \cdots d^d\mathbf{y}_n \mathcal{P}_0(\mathbf{y},\mathbf{y}_n)\mathbb{K}(\mathbf{y}_{n-1},\mathbf{y}_n)\cdots\mathbb{K}(\mathbf{y}_1,\mathbf{y}_2)\mathbb{K}(\mathbf{x},\mathbf{y}_1) \tag{A.49}$$

If we integrate the variable $\mathbf{y}$ over Ω and considering the fact $\int_\Omega d^d\mathbf{y}\, \mathcal{P}_0(\mathbf{y},\mathbf{x}) = 1$, we obtain

$$\begin{aligned}\int_\Omega d^d\mathbf{y}\, \mathcal{P}(\mathbf{y},\mathbf{x}) = &\int_\Omega d^d\mathbf{y}\, \mathcal{P}_0(\mathbf{y},\mathbf{x}) \\ &+ \sum_{n=1}^{\infty}(-1)^n \int_{\overline{\Omega}} d^d\mathbf{y}_1 \cdots d^d\mathbf{y}_n \int_\Omega d^d\mathbf{y}\, \mathcal{P}_0(\mathbf{y},\mathbf{y}_n)\mathbb{K}(\mathbf{y}_{n-1},\mathbf{y}_n)\cdots\mathbb{K}(\mathbf{x},\mathbf{y}_1) \\ =&1 + \sum_{n=1}^{\infty}(-1)^n \int_{\overline{\Omega}} d^d\mathbf{y}_1 \cdots d^d\mathbf{y}_n \mathbb{K}(\mathbf{y}_{n-1},\mathbf{y}_n)\cdots\mathbb{K}(\mathbf{y}_1,\mathbf{y}_2)\mathbb{K}(\mathbf{x},\mathbf{y}_1)\end{aligned} \tag{A.50}$$

Now taking into account the definitions (A.44) and (A.35), we get

$$\begin{aligned}\int_{\overline{\Omega}} d^d\mathbf{y}_n\, \mathbb{K}(\mathbf{y}_{n-1},\mathbf{y}_n) &= \int_{\overline{\Omega}} d^d\mathbf{y}_n \frac{[\mathcal{O}_{\overline{\Omega}}\delta](\mathbf{y}_{n-1}-\mathbf{y}_n)}{\int_\Omega d^d\mathbf{z}\, K(\mathbf{y}_{n-1}-\mathbf{z})}\\ &= P.V.\int_{\overline{\Omega}} d^d\mathbf{z}\frac{K(\mathbf{y}_{n-1}-\mathbf{z})-K(\mathbf{y}_{n-1}-\mathbf{z})}{\int_\Omega d^d\mathbf{z}\, K(\mathbf{y}_{n-1}-\mathbf{z})}\\ &= 0 \end{aligned} \tag{A.51}$$

Therefore we conclude that

$$\int_\Omega d^d\mathbf{y}\, \mathcal{P}(\mathbf{y},\mathbf{x}) = 1 \tag{A.52}$$

This result is in accord with Lemma A.5 of Claudia Bucur [22].
Also, note that the Poisson kernel follows the recurrence relation

$$\mathcal{P}_n(\mathbf{y},\mathbf{x}) = \mathcal{P}_0(\mathbf{y},\mathbf{x}) - \int_{\overline{\Omega}} d^d\mathbf{v}\, \mathcal{P}_{n-1}(\mathbf{y},\mathbf{v})\mathbb{K}(\mathbf{x},\mathbf{v}) \tag{A.53}$$

We can expand this recurrence relation, which yields an equivalent form

$$\begin{aligned}\mathcal{P}_n(\mathbf{y},\mathbf{x}) =& \mathcal{P}_0(\mathbf{y},\mathbf{x}) - \int_{\overline{\Omega}} d^d\mathbf{v}\, \mathcal{P}_{n-1}(\mathbf{y},\mathbf{v})\frac{[\mathcal{O}_{\overline{\Omega}}\delta](\mathbf{x}-\mathbf{v})}{\int_\Omega d^d\mathbf{z} K(\mathbf{x}-\mathbf{z})}\\ =& \mathcal{P}_0(\mathbf{y},\mathbf{x}) - \frac{1}{\int_\Omega d^d\mathbf{z} K(\mathbf{x}-\mathbf{z})}\lim_{\epsilon\to 0}\Bigg[\mathcal{P}_{n-1}(\mathbf{y},\mathbf{x})\int_{\overline{\Omega}/B_\epsilon(\mathbf{x})} d^d\mathbf{v}\, K(\mathbf{x}-\mathbf{v})\\ &- \int_{\overline{\Omega}/B_\epsilon(\mathbf{x})} d^d\mathbf{v}\, K(\mathbf{x}-\mathbf{v})\mathcal{P}_{n-1}(\mathbf{y},\mathbf{v})\Bigg] \end{aligned} \tag{A.54}$$

A.3.1 Poisson Kernel of a Ball

According to Claudia Bucur [22] eq.(1.14) the Poisson kernel for any $\mathbf{x}\in B_r$ and any $\mathbf{y}\in\mathbb{R}^n/B_r$, is given by

$$P_r(\mathbf{y},\mathbf{x}) = c(d,s)\left(\frac{r^2-|\mathbf{x}|^2}{|\mathbf{y}|^2-r^2}\right)^s \frac{1}{|\mathbf{x}-\mathbf{y}|^d}, \tag{A.55}$$

where d is the dimension, $s\in\langle 0,1\rangle$ is the degree of the fractional Laplacian and $c(d,s)$ is a normalization constant.

In particular for the one-dimensional case $d=1$ and $s=1/2$, we will have

$$\mathcal{P}_l(y,x) = \frac{1}{\pi}\left(\frac{l^2-|x|^2}{|y|^2-l^2}\right)^{1/2}\frac{1}{|x-y|},\quad x\in\langle -l,l\rangle, y\in\langle-\infty,-l]\cup[l,\infty\rangle. \tag{A.56}$$

where we noted that $\Omega\equiv\langle-\infty,-l]\cup[l,\infty\rangle$ and $\overline{\Omega}\equiv\langle -l,l\rangle$.

Taking in account that

$$K(x-y) = \frac{1}{\pi}\frac{1}{|x-y|^2} \qquad x, y \in \mathbb{R}. \tag{A.57}$$

We can determine the following quantities:

$$\mathcal{P}_0(y,x) = \frac{l^2-x^2}{2l}\frac{1}{(x-y)^2}, \quad x \in \langle -l, l\rangle, y \in \langle -\infty, -l] \cup [l, \infty\rangle. \tag{A.58}$$

$$\int_\Omega dz\, K(x-z) = \frac{1}{\pi}\frac{2l}{l^2-x^2} \tag{A.59}$$

$$\int_{\overline{\Omega}/B_\epsilon(x)} dv\, K(x-v) = \frac{2}{\pi}\left[\frac{1}{\epsilon} - \frac{l}{l^2-x^2}\right] \tag{A.60}$$

Also

$$\mathcal{P}_l(y,x)\int_{\overline{\Omega}/B_\epsilon(x)} dv\, K(x-v) = \frac{1}{\pi}\left(\frac{l^2-|x|^2}{|y|^2-l^2}\right)^{1/2}\frac{1}{|x-y|}\frac{2}{\pi}\left[\frac{1}{\epsilon} - \frac{l}{l^2-x^2}\right] \tag{A.61}$$

and

$$\begin{aligned}\int_{\overline{\Omega}/B_\epsilon(x)} dv K(x-v)\mathcal{P}_l(y,v) &= \int_{-l}^{x-\epsilon} dv K(x-v)\mathcal{P}_l(y,v) + \int_{x+\epsilon}^{l} dv K(x-v)\mathcal{P}_l(y,v) \\ &= -\frac{1}{\pi}\frac{1}{(y-x)^2} + \frac{1}{\pi^2}\frac{2}{\epsilon}\left(\frac{l^2-x^2}{y^2-l^2}\right)^{1/2}\frac{1}{|x-y|}\end{aligned} \tag{A.62}$$

with this quantities we can verify the concurrence relation

$$\begin{aligned}\mathcal{P}_l(y,x) =& \mathcal{P}_0(y,x) - \frac{1}{\int_\Omega dz\, K(x-z)}\lim_{\epsilon\to 0}\left[\mathcal{P}_l(y,x)\int_{\overline{\Omega}/B_\epsilon(x)} dv\, K(x-v)\right. \\ &\left. - \int_{\overline{\Omega}/B_\epsilon(x)} dv\, K(x-v)\mathcal{P}_l(y,v)\right] \\ =& \mathcal{P}_l(y,x)\end{aligned} \tag{A.63}$$

A.3.2 Poisson Kernel for the Half-space

According to Abatangelo et al. [45] or Bogdan [46] eq.(3.40) the Poisson kernel for the half-space is given by

$$\mathcal{P}(\mathbf{y},\mathbf{x}) = c(d,s)\left(\frac{x_d}{|y_d|}\right)^s \frac{1}{|\mathbf{x}-\mathbf{y}|}, \qquad \mathbf{x} \in \mathbb{R}^d_+,\ \mathbf{y} \in \mathbb{R}^d/\mathbb{R}^d_+, \tag{A.64}$$

where $\mathbb{R}^d_+ \equiv \{\mathbf{y} = \{y_1, ..., y_d\} \in \mathbb{R}^d; y_d > 0\}$.

In particular for the one-dimensional case $d = 1$ and $s = 1/2$, we will have

$$\mathcal{P}(y,x) = \frac{1}{\pi}\left(\frac{x}{|y|}\right)^{1/2}\frac{1}{|x-y|}, \qquad y \leq 0, x \geq 0. \tag{A.65}$$

where we noted that $\Omega \equiv \langle -\infty, 0]$ and $\overline{\Omega} \equiv [0, \infty\rangle$.

Taking in account that

$$K(x-y) = \frac{1}{\pi}\frac{1}{|x-y|^2} \qquad x, y \in \mathbb{R}. \tag{A.66}$$

We can determine the following quantities:

$$\mathcal{P}_0(y,x) = \frac{x}{(x-y)^2}, \qquad y \leq 0, x \geq 0 \tag{A.67}$$

$$\int_\Omega dz\, K(x-z) = \frac{1}{\pi}\frac{1}{x} \tag{A.68}$$

$$\int_{\overline{\Omega}/B_\epsilon(x)} dv\, K(x-v) = \frac{1}{\pi}\left[\frac{2}{\epsilon} - \frac{1}{x}\right] \tag{A.69}$$

Also

$$\begin{aligned}\mathcal{P}_l(y,x)\int_{\overline{\Omega}/B_\epsilon(x)} dv\, K(x-v) =& \frac{1}{\pi}\left(\frac{x}{|y|}\right)^{1/2}\frac{1}{|x-y|}\frac{1}{\pi}\left[\frac{2}{\epsilon} - \frac{l}{x}\right] \\ =& \frac{2}{\epsilon}\frac{1}{\pi^2}\left(\frac{x}{|y|}\right)^{1/2}\frac{1}{|x-y|} - \frac{2}{\pi^2}\frac{1}{\sqrt{x|y|}}\frac{1}{|x-y|}\end{aligned} \tag{A.70}$$

and

$$\begin{aligned}\int_{\overline{\Omega}/B_\epsilon(x)} dv\, K(x-v)\mathcal{P}(y,v) =& \int_0^{x-\epsilon} dv\, K(x-v)\mathcal{P}(y,v) + \int_{x+\epsilon}^{\infty} dv\, K(x-v)\mathcal{P}(y,v) \\ =& \frac{1}{\pi^2}\frac{1}{\sqrt{|y|}}\left[\int_0^{x-\epsilon} dv\, \frac{\sqrt{v}}{(x-v)^2|v-y|}\right. \\ & \left. + \int_{x+\epsilon}^{\infty} dv\, \frac{\sqrt{v}}{(x-v)^2|v-y|}\right] \\ =& \frac{1}{\pi^2}\frac{1}{\sqrt{|y|}}\left[-\pi\frac{\sqrt{|y|}}{(x-y)^2} + 2\frac{\sqrt{x}}{|x-y|\epsilon}\right]\end{aligned} \tag{A.71}$$

with this quantities we can verify the concurrence relation

$$\begin{aligned}\mathcal{P}(y,x) =&\mathcal{P}_0(y,x) - \frac{1}{\int_\Omega dz\, K(x-z)} \lim_{\epsilon\to 0}\Bigg[\mathcal{P}(y,x)\int_{\overline{\Omega}/B_\epsilon(x)} dv\, K(x-v) \\ &- \int_{\overline{\Omega}/B_\epsilon(x)} dv\, K(x-v)\mathcal{P}(y,v)\Bigg] \\ =&\mathcal{P}(y,x)\end{aligned} \tag{A.72}$$

A.4 Green Function

From the non-homogeneous Poisson problem (A.30), if we consider $g(\mathbf{x}) = 0$, the solution will be

$$\begin{aligned} u(\mathbf{x}) =&\mathcal{G}[f](\mathbf{x}), && \mathbf{x}\in\overline{\Omega} \\ u(\mathbf{x}) =&0, && \mathbf{x}\in\Omega. \end{aligned} \tag{A.73}$$

Also, from the solution for a spherical region of the paper of Claudia Bucur [22], we can generalize this solution to any region $\overline{\Omega}$. To this purpose we consider Theorem 2.8 of Claudia Bucur [22], to define

$$h(\mathbf{x}) = \int_{\mathbb{R}^n} d^d\mathbf{y}\, G_0(\mathbf{x}-\mathbf{y}) f(\mathbf{y}), \quad \forall\, \mathbf{x}\in\mathbb{R}^n \tag{A.74}$$

where

$$[\mathcal{O}G_0](\mathbf{x}) = \delta(\mathbf{x}) \qquad \forall\, \mathbf{x}\in\mathbb{R}^n. \tag{A.75}$$

Also, from this theorem, we know that

$$[\mathcal{O}h](\mathbf{x}) = f(\mathbf{x}), \quad \forall\, \mathbf{x}\in\mathbb{R}^n \tag{A.76}$$

Then, defining the new function

$$v(\mathbf{x}) \equiv u(\mathbf{x}) - h(\mathbf{x}), \quad \forall\, \mathbf{x}\in\mathbb{R}^n. \tag{A.77}$$

Applying the fractional Laplacian operator to this new function and considering the non-homogeneous Poisson problem (A.30) with $g(\mathbf{x}) = 0$, we obtain

$$\begin{aligned} [\mathcal{O}v](\mathbf{x}) =&0, && \mathbf{x}\in\overline{\Omega} \\ v(\mathbf{x}) =&-h(\mathbf{x}), && \mathbf{x}\in\Omega. \end{aligned} \tag{A.78}$$

Taking into account the homogeneous solution of (A.41)

$$v(\mathbf{x}) = -\int_{\Omega} d^d\mathbf{y}\, \mathcal{P}(\mathbf{y},\mathbf{x})h(\mathbf{y}), \quad \mathbf{x} \in \overline{\Omega}. \tag{A.79}$$

Therefore

$$u(\mathbf{x}) = h(\mathbf{x}) - \int_{\Omega} d^d\mathbf{y}\, \mathcal{P}(\mathbf{y},\mathbf{x})h(\mathbf{y}), \quad \mathbf{x} \in \overline{\Omega}. \tag{A.80}$$

From the definition (A.75) of G_0 if $\mathbf{y} \in \Omega$

$$\begin{aligned} [\mathcal{O}G_0](\mathbf{x}-\mathbf{y}) =& 0, \qquad \mathbf{x} \in \overline{\Omega} \\ G_0(\mathbf{x}-\mathbf{y}) =& G_0(\mathbf{x}-\mathbf{y}), \qquad \mathbf{x} \in \Omega. \end{aligned} \tag{A.81}$$

Again from the homogeneous solution of (A.30)

$$G_0(\mathbf{x}-\mathbf{y}) = \int_{\Omega} d^d\mathbf{z}\, \mathcal{P}(\mathbf{z},\mathbf{x})G_0(\mathbf{z}-\mathbf{y}), \quad \mathbf{y} \in \Omega, \mathbf{x} \in \overline{\Omega} \tag{A.82}$$

This result is in accord with the Lemma A.8 found in [22].

Inserting the expression (A.74) in (A.80) we have

$$\begin{aligned} u(\mathbf{x}) &= \int_{\mathbb{R}^n} d^d\mathbf{y}\, G_0(\mathbf{x}-\mathbf{y})f(\mathbf{y}) - \int_{\Omega} d^d\mathbf{z}\, \mathcal{P}(\mathbf{z},\mathbf{x}) \int_{\mathbb{R}^n} d^d\mathbf{y}\, G_0(\mathbf{z}-\mathbf{y})f(\mathbf{y}) \\ &= \int_{\overline{\Omega}} d^d\mathbf{y}\, G_0(\mathbf{x}-\mathbf{y})f(\mathbf{y}) + \int_{\Omega} d^d\mathbf{y}\, G_0(\mathbf{x}-\mathbf{y})f(\mathbf{y}) \\ &\quad - \int_{\Omega} d^d\mathbf{z}\, \mathcal{P}(\mathbf{z},\mathbf{x}) \left[\int_{\overline{\Omega}} d^d\mathbf{y}\, G_0(\mathbf{z}-\mathbf{y})f(\mathbf{y}) + \int_{\Omega} d^d\mathbf{y}\, G_0(\mathbf{z}-\mathbf{y})f(\mathbf{y}) \right] \\ &= \int_{\overline{\Omega}} d^d\mathbf{y}\, f(\mathbf{y}) \left[G_0(\mathbf{x}-\mathbf{y}) - \int_{\Omega} d^d\mathbf{z}\, \mathcal{P}(\mathbf{z},\mathbf{x})G_0(\mathbf{z}-\mathbf{y}) \right] \\ &\quad + \int_{\Omega} d^d\mathbf{y}\, f(\mathbf{y}) \left[G_0(\mathbf{x}-\mathbf{y}) - \int_{\Omega} d^d\mathbf{z}\, \mathcal{P}(\mathbf{z},\mathbf{x})G_0(\mathbf{z}-\mathbf{y}) \right] \end{aligned} \tag{A.83}$$

Considering the result (A.82)

$$\begin{aligned} u(\mathbf{x}) &= \int_{\overline{\Omega}} d^d\mathbf{y}\, f(\mathbf{y}) \left[G_0(\mathbf{x}-\mathbf{y}) - \int_{\Omega} d^d\mathbf{z}\, \mathcal{P}(\mathbf{z},\mathbf{x})G_0(\mathbf{z}-\mathbf{y}) \right] \\ &\quad + \int_{\Omega} d^d\mathbf{y}\, f(\mathbf{y})[G_0(\mathbf{x}-\mathbf{y}) - G_0(\mathbf{x}-\mathbf{y})] \\ &= \int_{\overline{\Omega}} d^d\mathbf{y}\, f(\mathbf{y}) \left[G_0(\mathbf{x}-\mathbf{y}) - \int_{\Omega} d^d\mathbf{z}\, \mathcal{P}(\mathbf{z},\mathbf{x})G_0(\mathbf{z}-\mathbf{y}) \right] \end{aligned} \tag{A.84}$$

Thus, when $g(\mathbf{x}) = 0$, the solution of (A.30) become

$$\begin{aligned} u(\mathbf{x}) =& \int_{\overline{\Omega}} d^d\mathbf{y}\, G(\mathbf{x},\mathbf{y})f(\mathbf{y}), \quad \mathbf{x} \in \overline{\Omega} \\ u(\mathbf{x}) =& 0, \qquad \mathbf{x} \in \Omega \end{aligned} \tag{A.85}$$

where for any $\mathbf{x}, \mathbf{y} \in \overline{\Omega}$ and $\mathbf{x} \neq \mathbf{y}$

$$G(\mathbf{x}, \mathbf{y}) \equiv G_0(\mathbf{x} - \mathbf{y}) - \int_\Omega d^d\mathbf{z}\, G_0(\mathbf{y} - \mathbf{z})\mathcal{P}(\mathbf{z}, \mathbf{x}) \tag{A.86}$$

From the unity of solution we state that

$$\mathcal{G}[f](\mathbf{x}) = \int_{\overline{\Omega}} d^d\mathbf{y}\, G(\mathbf{x},\mathbf{y}) f(\mathbf{y}) \tag{A.87}$$

It is worth mentioning that this result can be obtained directly using the non-homogeneous solution of the Poisson problem (A.48) and the definition of the Poisson kernel (A.47).

A.4.1 Green Function for the Ball

The Green function for the ball is given by eq.(1.16) of [22]. For any $\mathbf{x}, \mathbf{z} \in B_r$ and $\mathbf{x} \neq \mathbf{z}$

$$G(\mathbf{x}, \mathbf{z}) = G_0(\mathbf{x} - \mathbf{y}) - \int_{\mathbb{R}^d / B_r} d^d\mathbf{y}\, G_0(\mathbf{z} - \mathbf{y}) P_r(\mathbf{y}, \mathbf{x}) \tag{A.88}$$

explicitly we have

$$G(\mathbf{x}, \mathbf{z}) = \kappa(d, s)|\mathbf{z} - \mathbf{x}|^{2s-d} \int_0^{r_0(\mathbf{x},\mathbf{z})} dt \frac{t^{s-1}}{(t+1)^{n/2}}, \qquad d \neq 2s \tag{A.89}$$

$$G(x, z) = \frac{1}{\pi} \ln \left(\frac{r^2 - xz + \sqrt{(r^2 - x^2)\,(r^2 - z^2)}}{r|x - z|} \right) \qquad d = 2s \tag{A.90}$$

where $\kappa(d, s)$ is a constant that depends only on d and s. Also

$$r_0(\mathbf{x}, \mathbf{z}) = \frac{\left(r^2 - |\mathbf{x}|^2\right)\left(r^2 - |\mathbf{z}|^2\right)}{r^2|\mathbf{x} - \mathbf{z}|^2} \tag{A.91}$$

A.4.2 Green Function for the Half-space

We know that the Green function is given by

$$G(\mathbf{x}, \mathbf{y}) = G_0(\mathbf{x} - \mathbf{y}) - \int_\Omega d^d\mathbf{z} G_0(\mathbf{y} - \mathbf{z})\mathcal{P}(\mathbf{z}, \mathbf{x}), \quad \mathbf{x}, \mathbf{y} \in \overline{\Omega},\ \mathbf{x} \neq \mathbf{y} \tag{A.92}$$

In the one-dimensional case considering that $\overline{\Omega} \equiv [0, \infty\rangle$ and $\Omega \equiv \langle -\infty, 0]$. Also from the result given by [22] eq.(1.13), for any $x \in \mathbb{R}/\{0\}$

$$G_0(x) = -\frac{1}{\pi} \ln |x| \tag{A.93}$$

Thus, taken in account that $x > 0, y > 0$ we have

$$\begin{aligned}\int_{-\infty}^{0} dz\, G_0(y-z)\mathcal{P}(z,x) &= -\frac{1}{\pi^2}\int_{-\infty}^{0} dz\, \ln|y-z| \left(\frac{x}{|z|}\right)^{1/2} \frac{1}{|x-z|} \\ &= -\frac{1}{\pi^2}\int_{-\infty}^{0} dz\, \sqrt{-\frac{x}{z}}\frac{\ln(y-z)}{(x-z)} \\ &= -\frac{1}{\pi^2}\left[2\pi \operatorname{arctanh}\left(\sqrt{\frac{y}{x}}\right) + \pi \ln|x-y| + i\pi^2\Theta(y-x)\right] \\ &= -\frac{2}{\pi}\operatorname{arctanh}\left(\sqrt{\frac{y}{x}}\right) - \frac{1}{\pi}\ln|x-y| - i\Theta(y-x) \end{aligned} \tag{A.94}$$

Then

$$\begin{aligned} G(x,y) &= -\frac{1}{\pi}\ln|x-y| - \left[-\frac{2}{\pi}\operatorname{arctanh}\left(\sqrt{\frac{y}{x}}\right) - \frac{1}{\pi}\ln|x-y| - i\Theta(y-x)\right] \\ &= -\frac{1}{\pi}\ln|x-y| + \frac{2}{\pi}\operatorname{arctanh}\left(\sqrt{\frac{y}{x}}\right) + \frac{1}{\pi}\ln|x-y| + i\Theta(y-x) \\ &= \frac{2}{\pi}\operatorname{arctanh}\left(\sqrt{\frac{y}{x}}\right) + i\Theta(y-x) \end{aligned} \tag{A.95}$$

We can express the Green function in the following form

$$G(x,y) = \begin{cases} \frac{2}{\pi}\operatorname{arctanh}\left(\sqrt{\frac{y}{x}}\right), & x > y \\ \frac{2}{\pi}\operatorname{arctanh}\left(\sqrt{\frac{x}{y}}\right), & x < y \end{cases} \tag{A.96}$$

Finally, with the following integrals

$$\int_0^x dy\, \frac{1}{(z-y)^2}\operatorname{arctanh}\left(\sqrt{\frac{y}{x}}\right) = \frac{x\operatorname{arctanh}\left(\sqrt{\frac{x}{|z|}}\right)}{\sqrt{xz}(x+|z|)} \tag{A.97}$$

and

$$\int_x^\infty dy\, \frac{1}{(z-y)^2}\operatorname{arctanh}\left(\sqrt{\frac{x}{y}}\right) = \frac{x\left(\pi - 2\operatorname{arctanh}\left(\sqrt{\frac{x}{|z|}}\right)\right)}{2\sqrt{x|z|}(x+|z|)} \tag{A.98}$$

We can obtain

$$\begin{aligned}\int_0^\infty dy\, K(z-y)G(y,x) =& \frac{1}{\pi}\int_0^\infty dy\, \frac{G(y,x)}{(z-y)^2} \\ =& \frac{2}{\pi^2}\left[\frac{x\,\operatorname{arctanh}\left(\sqrt{\frac{x}{|z|}}\right)}{\sqrt{x|z|}(x+|z|)} + \frac{x\left(\pi - 2\operatorname{arctanh}\left(\sqrt{\frac{x}{|z|}}\right)\right)}{2\sqrt{x|z|}(x+|z|)}\right] \\ =& \frac{1}{\pi}\sqrt{\frac{x}{|z|}}\frac{1}{|x-z|}\end{aligned} \tag{A.99}$$

and considering the Poisson kernel for the halfline (A.65) we get

$$\int_0^\infty dy\, K(z-y)G(y,x) = \mathcal{P}(z,x) \qquad z \leq 0,\ x \geq 0. \tag{A.100}$$

This last result, show us other relation between the Green function and the Poisson kernel for the halfline case.

Appendix B

Important Calculations

In this appendix, we are going to develop some relevant calculations for this thesis.

B.1 $W[\phi_\Omega, f_{\alpha,\beta}]$ in terms of g_Ω

To calculate the integral in $[\mathcal{D}g_\Omega]$ of (4.101), we need to express $W[\phi_\Omega, f_{\alpha,\beta}] \equiv -\int d^d\mathbf{x}\,\left(\phi_\Omega(\mathbf{x})[\mathcal{O}\phi_\Omega](\mathbf{x}) - f_{\alpha,\beta}(t,\mathbf{x})\left[\mathcal{O}^{1/2}\phi_\Omega\right](\mathbf{x})\right)$ in terms of g_Ω. Thus

$$\begin{aligned} W[\phi, f_{\alpha,\beta}] &= -\int_\Omega d^d\mathbf{x}\,\left(\phi_\Omega(\mathbf{x})[\mathcal{O}\phi_\Omega](\mathbf{x}) - \left[\mathcal{O}^{1/2}f_{\alpha,\beta}\right](t,\mathbf{x})\phi_\Omega(\mathbf{x})\right) \\ &\quad -\int_{\overline{\Omega}} d^d\mathbf{x}\,\left(\phi_\Omega(\mathbf{x})[\mathcal{O}\phi_\Omega](\mathbf{x}) - \left[\mathcal{O}^{1/2}f_{\alpha,\beta}\right](t,\mathbf{x})\phi_\Omega(\mathbf{x})\right) \\ &= -\int_\Omega d^d\mathbf{x}\,\left(g_\Omega(\mathbf{x})[\mathcal{O}\phi_\Omega](\mathbf{x}) - \left[\mathcal{O}^{1/2}f_{\alpha,\beta}\right](t,\mathbf{x})g_\Omega(\mathbf{x})\right) \\ &\quad +\frac{1}{2}\int_{\overline{\Omega}} d^d\mathbf{x}\,\phi_\Omega(\mathbf{x})\mathcal{O}^{1/2}f_{\alpha,\beta}(t,\mathbf{x}) \end{aligned} \tag{B.1}$$

where in the last equality we taking on account that ϕ_Ω satisfies the nonlocal Poisson problem,

$$\begin{aligned} [\mathcal{O}\phi_\Omega](\mathbf{x}) &= \frac{1}{2}\mathcal{O}^{1/2}f_{\alpha,\beta}(t,\mathbf{x}), \qquad & \mathbf{x}\in\overline{\Omega} \\ \phi_\Omega(\mathbf{x}) &= g_\Omega(\mathbf{x}), & \mathbf{x}\in\Omega \end{aligned} \tag{B.2}$$

with solution

$$\phi_\Omega(\mathbf{x}) = \phi_\Omega^{(0)}(\mathbf{x}) + \frac{1}{2}\mathcal{G}\left[\mathcal{O}^{1/2}f_{\alpha,\beta}\right](t,\mathbf{x}), \qquad \mathbf{x}\in\overline{\Omega} \tag{B.3}$$

being

$$\phi_\Omega^{(0)}(\mathbf{x}) = \int_\Omega d^d\mathbf{y}\,\mathcal{P}(\mathbf{y},\mathbf{x})g_\Omega(\mathbf{y}), \quad \mathbf{x}\in\overline{\Omega}. \tag{B.4}$$

Now, using the fact for $\mathbf{x}\in\Omega$

$$[\mathcal{O}\phi_\Omega](\mathbf{x}) = [\mathcal{O}g_\Omega](\mathbf{x}) + \int_{\overline{\Omega}} d^d\mathbf{y}K(\mathbf{x}-\mathbf{y})[g_\Omega(\mathbf{y}) - \phi_\Omega(\mathbf{y})]. \tag{B.5}$$

Then

$$[\mathcal{O}\phi_\Omega](\mathbf{x}) = [\mathcal{O}g_\Omega](\mathbf{x}) + \int_{\overline{\Omega}} d^d\mathbf{y} K(\mathbf{x}-\mathbf{y}) \left[g_\Omega(\mathbf{y}) - \phi_\Omega^{(0)}(\mathbf{y}) - \frac{1}{2}\mathcal{G}\left[\mathcal{O}^{1/2} f_{\alpha,\beta}\right](t,\mathbf{y}) \right]. \quad \text{(B.6)}$$

Therefore, inserting this result in (B.1), we have

$$\begin{aligned} W[\phi_\Omega, f_{\alpha\beta}] = &- \int_\Omega d^d\mathbf{x}\, g_\Omega(\mathbf{x})[\mathcal{O}g_\Omega](\mathbf{x}) + \int_\Omega d^d\mathbf{x} g_\Omega(\mathbf{x}) \left[\mathcal{O}^{1/2} f_{\alpha,\beta}\right](t,\mathbf{x}) \\ &- \int_\Omega d^d\mathbf{x} g_\Omega(\mathbf{x}) \int_{\overline{\Omega}} d^d\mathbf{y} K(\mathbf{x}-\mathbf{y}) \left[g_\Omega(\mathbf{y}) - \phi_\Omega^{(0)}(\mathbf{y}) - \frac{1}{2}\mathcal{G}\left[\mathcal{O}^{\frac{1}{2}} f_{\alpha\beta}\right](t,\mathbf{y}) \right] \\ &+ \frac{1}{2}\int_{\overline{\Omega}} d^d\mathbf{x} \left(\phi_\Omega^{(0)}(\mathbf{x}) + \frac{1}{2}\mathcal{G}\left[\mathcal{O}^{\frac{1}{2}} f_{\alpha,\beta}\right](t,\mathbf{x}) \right) \left[\mathcal{O}^{1/2} f_{\alpha,\beta}\right](t,\mathbf{x}). \end{aligned} \quad \text{(B.7)}$$

Now, considering the following notation

$$[\mathcal{O}_\Omega u](\mathbf{x}) \equiv P.V. \int_\Omega d^d\mathbf{y}\, K(\mathbf{x}-\mathbf{y})[u(\mathbf{x}) - u(\mathbf{y})], \qquad \mathbf{x} \in \Omega. \quad \text{(B.8)}$$

Also when $\mathbf{x} \in \Omega$

$$[\mathcal{O}g_\Omega](\mathbf{x}) = [\mathcal{O}_\Omega g_\Omega](\mathbf{x}) + g_\Omega(\mathbf{x}) \int_{\overline{\Omega}} d^d\mathbf{y}\, K(\mathbf{x}-\mathbf{y}) - \int_{\overline{\Omega}} d^d\mathbf{y}\, K(\mathbf{x}-\mathbf{y}) g_\Omega(\mathbf{y}). \quad \text{(B.9)}$$

Then

$$\begin{aligned} [\mathcal{O}g_\Omega](\mathbf{x}) + \int_{\overline{\Omega}} d^d\mathbf{y}\, K(\mathbf{x}-\mathbf{y}) g_\Omega(\mathbf{y}) =& [\mathcal{O}_\Omega g_\Omega](\mathbf{x}) + g_\Omega(\mathbf{x}) \int_{\overline{\Omega}} d^d\mathbf{y}\, K(\mathbf{x}-\mathbf{y}) \\ =& \int_\Omega d^d\mathbf{z}\, g_\Omega(\mathbf{z}) A(\mathbf{x},\mathbf{z}) \end{aligned} \quad \text{(B.10)}$$

where

$$A(\mathbf{x},\mathbf{z}) \equiv [\mathcal{O}_\Omega \delta](\mathbf{x}-\mathbf{z}) + \delta(\mathbf{x}-\mathbf{z}) \int_{\overline{\Omega}} d^d\mathbf{y}\, K(\mathbf{x}-\mathbf{y}). \quad \text{(B.11)}$$

Thus

$$\begin{aligned} W[\phi_\Omega, f_{\alpha,\beta}] = &- \int_\Omega d^d\mathbf{x}\, g_\Omega(\mathbf{x}) \int_\Omega d^d\mathbf{z}\, g(\mathbf{z}) A(\mathbf{x},\mathbf{z}) + \int_\Omega d^d\mathbf{x}\, g_\Omega(\mathbf{x}) \left[\mathcal{O}^{1/2} f_{\alpha,\beta}\right](t,\mathbf{x}) \\ &+ \int_\Omega d^d\mathbf{x}\, g_\Omega(\mathbf{x}) \int_{\overline{\Omega}} d^d\mathbf{y} K(\mathbf{x}-\mathbf{y}) \left[\phi_\Omega^{(0)}(\mathbf{y}) + \frac{1}{2}\mathcal{G}\left[\mathcal{O}^{1/2} f_{\alpha,\beta}\right](t,\mathbf{y}) \right] \\ &+ \frac{1}{2}\int_{\overline{\Omega}} d^d\mathbf{x} \left(\phi_\Omega^{(0)}(\mathbf{x}) + \frac{1}{2}\mathcal{G}\left[\mathcal{O}^{1/2} f_{\alpha,\beta}\right](t,\mathbf{x}) \right) \left[\mathcal{O}^{1/2} f_{\alpha,\beta}\right](t,\mathbf{x}). \end{aligned} \quad \text{(B.12)}$$

To further simplify the function $W[\phi_\Omega, f_{\alpha,\beta}]$, we can expand all the products and regroup terms according to the orders of g_Ω. This gives us the following result

$$W[\phi_\Omega, f_{\alpha,\beta}] = -\frac{1}{2}\int_\Omega d^d\mathbf{x}\, g_\Omega(\mathbf{x}) \int_\Omega d^d\mathbf{z}\, g_\Omega(\mathbf{z}) \hat{o}(\mathbf{x},\mathbf{z}) + \int_\Omega d^d\mathbf{z}\, g_\Omega(\mathbf{z}) a_{f_{\alpha\beta}}(\mathbf{z}) + b_{f_{\alpha\beta}} \quad \text{(B.13)}$$

where we have defined the following operators

$$\begin{aligned}\frac{1}{2}\hat{o}(\mathbf{x},\mathbf{z}) \equiv & A(\mathbf{x},\mathbf{z}) - \int_{\overline{\Omega}} d^d\mathbf{y} K(\mathbf{x}-\mathbf{y})\mathcal{P}(\mathbf{z},\mathbf{y}) \\ = & [\mathcal{O}_\Omega \delta](\mathbf{x}-\mathbf{z}) + \delta(\mathbf{x}-\mathbf{z}) \int_{\overline{\Omega}} d^d\mathbf{y}\, K(\mathbf{x}-\mathbf{y}) - \int_{\overline{\Omega}} d^d\mathbf{y} K(\mathbf{x}-\mathbf{y})\mathcal{P}(\mathbf{z},\mathbf{y}) \end{aligned} \tag{B.14}$$

$$\begin{aligned}a_f(t,\mathbf{z}) \equiv & \left[\mathcal{O}^{1/2} f\right](t,\mathbf{z}) + \frac{1}{2}\int_{\overline{\Omega}} d^d\mathbf{y} K(\mathbf{z}-\mathbf{y}) \mathcal{G}\left[\mathcal{O}^{1/2} f\right](t,\mathbf{y}) \\ & + \frac{1}{2}\int_{\overline{\Omega}} d^d\mathbf{x} \mathcal{P}(\mathbf{z},\mathbf{x})\left[\mathcal{O}^{1/2} f\right](t,\mathbf{x}) \end{aligned} \tag{B.15}$$

and

$$b_f(t) \equiv \frac{1}{4}\int_{\overline{\Omega}} d^d\mathbf{x}\, \mathcal{G}\left[\mathcal{O}^{1/2} f\right](t,\mathbf{x})\left[\mathcal{O}^{1/2} f\right](t,\mathbf{x}) \tag{B.16}$$

B.2 variation of $W[\phi_\Omega, f_{\alpha,\beta}]$ respect to α and β

To obtain the variations of $W[\phi_\Omega, f_{\alpha,\beta}]$ with respect α and β, in first place we must obtain the variation of $a_{f_{\alpha\beta}}(\mathbf{z})$ and $b_{f_{\alpha\beta}}$ with respect to α and β

$$\left.\frac{\delta}{\delta\alpha} a_{f_{\alpha\beta}}(\mathbf{z})\right|_{\alpha=\beta=0} = a_R(\mathbf{z}) \tag{B.17}$$

$$\left.\frac{\delta}{\delta\beta} a_{f_{\alpha\beta}}(\mathbf{z})\right|_{\alpha=\beta=0} = i a_I(\mathbf{z}) \tag{B.18}$$

Also

$$\left.\frac{\delta}{\delta\alpha} b_{f_{\alpha\beta}}\right|_{\alpha=\beta=0} = \left.\frac{\delta}{\delta\beta} b_{f_{\alpha\beta}}\right|_{\alpha=\beta=0} = 0 \tag{B.19}$$

and finally

$$\left.\frac{\delta^2}{\delta\alpha^2} b_{f_{\alpha\beta}}\right|_{\alpha=\beta=0} = 2 b_R \tag{B.20}$$

$$\left.\frac{\delta^2}{\delta\beta^2} b_{f_{\alpha\beta}}\right|_{\alpha=\beta=0} = -2 b_I \tag{B.21}$$

With this results we can know the variations of $W[\phi_\Omega, f_{\alpha,\beta}]$ with respect α and β

$$\begin{aligned}
\frac{\delta}{\delta\alpha}W[\phi_\Omega, f_{\alpha,\beta}]\bigg|_{\alpha=\beta=0} &= \int_\Omega d^d\mathbf{z}\, g_\Omega(\mathbf{z})a_R(\mathbf{z}) \\
\frac{\delta}{\delta\beta}W[\phi_\Omega, f_{\alpha,\beta}]\bigg|_{\alpha=\beta=0} &= i\int_\Omega d^d\mathbf{z}\, g_\Omega(\mathbf{z})a_I(\mathbf{z}) \\
\frac{\delta^2}{\delta\alpha^2}W[\phi_\Omega, f_{\alpha,\beta}]\bigg|_{\alpha=\beta=0} &= 2b_R \\
\frac{\delta^2}{\delta\alpha^2}W[\phi_\Omega, f_{\alpha,\beta}]\bigg|_{\alpha=\beta=0} &= -2b_I
\end{aligned} \tag{B.22}$$

Also, we know that

$$\left(\frac{\delta^2}{\delta\alpha^2} - \frac{\delta^2}{\delta\beta^2}\right)\exp(\mathcal{S}_{\alpha,\beta}) = \exp(\mathcal{S}_{\alpha,\beta})\left[(\delta_\alpha\mathcal{S})^2 - (\delta_\beta\mathcal{S})^2 - \delta_\alpha^2\mathcal{S} + \delta_\beta^2\mathcal{S}\right] \tag{B.23}$$

Thus,

$$\begin{aligned}
&\left(\frac{\delta^2}{\delta\alpha^2} - \frac{\delta^2}{\delta\beta^2}\right)\exp(W[\phi_\Omega, f_{\alpha,\beta}])\bigg|_{\alpha=\beta=0} \\
&\qquad = \exp(W[g_\Omega])\left[\left(\frac{\delta}{\delta\alpha}W[\phi_\Omega, f_{\alpha,\beta}]\bigg|_{\alpha=\beta=0}\right)^2 - \left(\frac{\delta}{\delta\beta}W[\phi_\Omega, f_{\alpha,\beta}]\bigg|_{\alpha=\beta=0}\right)^2\right. \\
&\qquad\qquad \left. + \frac{\delta^2}{\delta\alpha^2}W[\phi_\Omega, f_{\alpha,\beta}] - \frac{\delta^2}{\delta\beta^2}W[\phi_\Omega, f_{\alpha,\beta}]\right] \\
&\qquad = \exp(W[g_\Omega])\left[\left|\int_\Omega d^d\mathbf{z}\, g(\mathbf{z})a_f(\mathbf{z})\right|^2 + 2b_R + 2b_I\right]
\end{aligned} \tag{B.24}$$

where we have taken into account

$$W[\phi_\Omega, f_{\alpha,\beta}]\bigg|_{\alpha=\beta=0} \equiv W[g_\Omega] = -\frac{1}{2}\int_\Omega d^d\mathbf{x}\, g_\Omega(\mathbf{x})\int_\Omega d^d\mathbf{z}\, g_\Omega(\mathbf{z})\hat{o}(\mathbf{x},\mathbf{z}) \tag{B.25}$$

Appendix C

Inverse Operator o_s^{-1}

In this appendix, we are going to obtain the inverse of the integral operator $\frac{1}{2}[o_s f](\mathbf{x})$ whose kernel is defined by (4.113) and (B.14):

$$[o_s f](\mathbf{x}) = \int_\Omega d^d\mathbf{z}\, o_s(\mathbf{x},\mathbf{z})\, f(\mathbf{z}), \qquad \mathbf{x} \in \Omega \tag{C.1}$$

where

$$\frac{1}{2}o_s(\mathbf{x},\mathbf{z}) = [\mathcal{O}_\Omega \delta](\mathbf{x}-\mathbf{z}) + \delta(\mathbf{x}-\mathbf{z}) \int_{\overline{\Omega}} d^d\mathbf{y}\, K(\mathbf{x}-\mathbf{y}) - \xi_s(\mathbf{x},\mathbf{z}), \qquad \mathbf{x},\mathbf{z} \in \Omega \tag{C.2}$$

being

$$\xi_s(\mathbf{x},\mathbf{z}) \equiv \frac{1}{2}\left[\int_{\overline{\Omega}} d^d\mathbf{y} K(\mathbf{x}-\mathbf{y})\mathcal{P}(\mathbf{z},\mathbf{y}) + \int_{\overline{\Omega}} d^d\mathbf{y} K(\mathbf{z}-\mathbf{y})\mathcal{P}(\mathbf{x},\mathbf{y})\right] \tag{C.3}$$

The inverse of this integral operator should satisfy

$$o_s^{-1}[o_s f](\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x} \in \Omega. \tag{C.4}$$

To obtain this inverse, we can use the Fourier transform, which simplifies the problem, especially for convolution-type operators, by converting the problem into multiplication. Thus, the Fourier transform of the integral operator is given by

$$\widehat{[o_s f]}(\mathbf{k}) = \frac{1}{(2\pi)^{d/2}} \int d^d\mathbf{x}\, [o_s f](\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}}. \tag{C.5}$$

Since the operator is defined over Ω, the standard Fourier transform over $\mathbb{R}^d$ doesn't directly apply without modification. We can extend the domain of $f(\mathbf{x}) = 0$ for $\overline{\Omega}$ and extend $o_s(\mathbf{x},\mathbf{z})$ appropriately (e.g., zero outside the region), then use the full Fourier transform.

Define $f(\mathbf{x}) = 0$ for $\mathbf{x} \notin \Omega$, so:

$$[o_s f](\mathbf{x}) = \int d^d\mathbf{z}\, o_s(\mathbf{x},\mathbf{z}) f(\mathbf{z})\, d\mathbf{z}, \tag{C.6}$$

where $o_s(\mathbf{x}, \mathbf{z})$ is the original kernel for $\mathbf{z} \in \Omega$ and possibly redefined (e.g., zero) elsewhere. The Fourier transform of the output is:

$$\widehat{[o_s f]}(\mathbf{k}) = \frac{1}{(2\pi)^{d/2}} \int d^d\mathbf{x} \left[\int d^d\mathbf{z}\, o_s(\mathbf{x}, \mathbf{z}) f(\mathbf{z}) \right] e^{-i\mathbf{k}\cdot\mathbf{x}}. \tag{C.7}$$

Interchanging integrals (assuming integrability):

$$\widehat{[o_s f]}(\mathbf{k}) = \frac{1}{(2\pi)^{d/2}} \int d^d\mathbf{z}\, f(\mathbf{z}) \left[\int d^d\mathbf{x}\, o_s(\mathbf{x}, \mathbf{z}) e^{-i\mathbf{k}\cdot\mathbf{x}} \right]. \tag{C.8}$$

Since $f(\mathbf{z}) = 0$ for $\mathbf{z} \notin \Omega$ and $o_s(\mathbf{x}, \mathbf{z}) = 0$ for $\mathbf{x}, \mathbf{z} \notin \Omega$:

$$\widehat{[o_s f]}(\mathbf{k}) = \int_\Omega d\mathbf{z}\, f(\mathbf{z}) \hat{o}_s(\mathbf{k}, \mathbf{z}) \tag{C.9}$$

where:

$$\hat{o}_s(\mathbf{k}, \mathbf{z}) = \frac{1}{(2\pi)^{d/2}} \int_\Omega d^d\mathbf{x}\, o_s(\mathbf{x}, \mathbf{z}) e^{-i\mathbf{k}\cdot\mathbf{x}}. \tag{C.10}$$

Therefore, we can find a inverse kernel $\hat{o}_s^{-1}(\mathbf{z}, \mathbf{k})$ making

$$\int d\mathbf{k}\, \hat{o}_s^{-1}(\mathbf{z}', \mathbf{k}) \hat{o}_s(\mathbf{k}, \mathbf{z}) = \delta(\mathbf{z}' - \mathbf{z}) \tag{C.11}$$

this imply that

$$\int d\mathbf{k}\, \hat{o}_s^{-1}(\mathbf{z}', \mathbf{k}) \widehat{[o_s f]}(\mathbf{k}) = f(\mathbf{z}') \tag{C.12}$$

If the kernel of the integral operator is a convolution kernel, i.e., $o_s(\mathbf{x}, \mathbf{z}) = o_s(\mathbf{x} - \mathbf{z})$, then the problem of finding its inverse can be reduced to a simple arithmetic problem, since in (C.10) we can make a change of variable $\mathbf{y} = \mathbf{x} - \mathbf{z}$

$$\hat{o}_s(\mathbf{k}, \mathbf{z}) = e^{i\mathbf{k}\cdot\mathbf{z}} \hat{o}_s(\mathbf{k}) = e^{-i\mathbf{k}\cdot\mathbf{z}} \left[\frac{1}{(2\pi)^{d/2}} \int_\Omega d^d\mathbf{y}\, o_s(\mathbf{y}) e^{-i\mathbf{k}\cdot\mathbf{y}} \right]. \tag{C.13}$$

Inserting this expression in (C.11)

$$\int d\mathbf{k}\, \hat{o}_s^{-1}(\mathbf{z}', \mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{z}} \hat{o}_s(\mathbf{k}) = \delta(\mathbf{z}' - \mathbf{z}). \tag{C.14}$$

We obtain the inverse kernel as

$$\hat{o}_s^{-1}(\mathbf{z}', \mathbf{k}) = \frac{1}{(2\pi)^d} \frac{1}{\hat{o}_s(\mathbf{k})} e^{i\mathbf{k}\cdot\mathbf{z}'} \tag{C.15}$$

also from the expression (C.9)

$$\widehat{[o_s f]}(\mathbf{k}) = \int_\Omega d\mathbf{z}\, f(\mathbf{z}) e^{i\mathbf{k}\cdot\mathbf{z}} \hat{o}_s(\mathbf{k}) = (2\pi)^{d/2} \hat{f}(\mathbf{k}) \hat{o}_s(\mathbf{k}) \tag{C.16}$$

and using (C.15), we can verify (C.12).

In general the kernel $o_s(\mathbf{x}, \mathbf{z})$ defined in (C.2) is not a convolution kernel due to the presence of the kernel $\xi_s(\mathbf{x}, \mathbf{z})$. Thus generally, numerical methods must be used to obtain the inverse of the integral operator. However, we can simplify the problem to the one-dimensional case on the half-line by applying an appropriate integral transformation (not necessarily the Fourier transform) and thus obtain the inverse of the integral operator analytically.

C.1 Inverse operator defined on the half-line

To simplify the problem, specifically we consider $\Omega = \langle -\infty, 0]$ and $\overline{\Omega} = [0, \infty\rangle$, so the integral operator (C.1), becomes

$$[o_s f](x) = \int_{-\infty}^{0} dz\, o_s(x, z)\, f(z), \qquad x \leq 0 \tag{C.17}$$

Is more convenient define new variables $u = -x$ and $v = -z$. Then substitute in (C.17) we have

$$\begin{aligned}[o_s f](-u) &= \int_{\infty}^{0} (-dv)\, o_s(-u, -v) f(-v), \qquad u \geq 0 \\ &= \int_{0}^{\infty} dv\, o_s(-u, -v) f(-v), \qquad u \geq 0 \end{aligned} \tag{C.18}$$

Redefine

$$[o_s g](u) \equiv [o_s f](-u) \tag{C.19}$$

$$\tilde{o}_s(u, v) \equiv o_s(-u, -v) \tag{C.20}$$

$$g(v) \equiv f(-v) \tag{C.21}$$

So

$$[o_s g](u) = \int_{0}^{\infty} dv\, \tilde{o}_s(u, v) g(v), \qquad u \geq 0 \tag{C.22}$$

Before proceeding to compute the inverse of the integral operator o_s, we must first determine its specific form for this one-dimensional case, keeping in mind that

$$K(x - y) = \frac{1}{\pi} \frac{1}{|x - y|^2} \qquad x, y \in \mathbb{R} \tag{C.23}$$

$$\mathcal{P}(y, v) = \frac{1}{\pi} \left(\frac{v}{|y|} \right)^{1/2} \frac{1}{|v - y|}, \qquad y \leq 0, v \geq 0. \tag{C.24}$$

Thus, from (C.2) we have

$$\frac{1}{2}o_s(x,z) = [\mathcal{O}_\Omega\delta](x-z) + \delta(x-z)\int_0^\infty dy\, K(x-y) - \xi_s(x,z), \qquad x \le 0, z \le 0$$

being

$$\xi_s(x,z) \equiv \frac{1}{2}\left[\int_0^\infty dy\, K(x-y)\mathcal{P}(z,y) + \int_0^\infty dy\, K(z-y)\mathcal{P}(x,y)\right] \tag{C.25}$$

Considering (C.23) and (C.24) we have

$$\begin{aligned}\int_0^\infty dy\, K(x-y) =& \frac{1}{\pi}\int_0^\infty dy\, \frac{1}{(x-y)^2} \\ =& \frac{1}{\pi}\frac{1}{x-y}\Big|_{y=0}^{y=\infty} \\ =& -\frac{1}{\pi}\frac{1}{x}\end{aligned} \tag{C.26}$$

Also,

$$\begin{aligned}\int_0^\infty dy\, K(x-y)\mathcal{P}(z,y) =& \frac{1}{\pi^2}\frac{1}{\sqrt{|z|}}\left[\int_0^\infty dy\, \frac{\sqrt{y}}{(x-y)^2|y-z|}\right] \\ =& \frac{1}{\pi^2}\frac{1}{\sqrt{|z|}}\left[\int_0^\infty dy\, \frac{\sqrt{y}}{(y+|x|)^2(y+|z|)}\right] \\ =& \frac{1}{\pi^2}\frac{1}{\sqrt{|z|}}\left[\frac{\pi}{2\sqrt{|x|}}\frac{\left(\sqrt{|x|}-\sqrt{|z|}\right)^2}{(x-z)^2}\right]\end{aligned} \tag{C.27}$$

therefore the kernel $\xi_s(x,z)$ will be

$$\xi_s(x,z) = \frac{1}{2\pi}\frac{1}{\sqrt{|xz|}}\frac{1}{\left(\sqrt{|x|}+\sqrt{|z|}\right)^2} \tag{C.28}$$

Therefore considering the change of variable $u = -x$ and $v = -z$

$$\frac{1}{2}\tilde{o}_s(u,v) = [\mathcal{O}_\Omega\delta](-u+v) + \frac{1}{\pi}\frac{1}{u}\delta(v-u) - \xi_s(u,v), \qquad u \ge 0, v \ge 0 \tag{C.29}$$

so, we need to find $\tilde{o}_s^{-1}(u,v)$ such that

$$\int_0^\infty du\, \tilde{o}_s^{-1}(v',u)\tilde{o}_s(u,v) = \delta(v'-v) \tag{C.30}$$

with which it can be verified that

$$\int_0^\infty du\, \tilde{o}_s^{-1}(v', u)[o_s g](u) = g(v') \tag{C.31}$$

To simplify the problem, we can use the Fourier transform, but it is more convenient for us to use the Mellin transform [47, 48]. The Mellin transform of a complex-valued function f defined on $[0, \infty\rangle$ is the function $\mathcal{M}f$ of complex variable k given by

$$\mathcal{M}f(k) = \int_0^\infty du\, u^{k-1} f(u) \tag{C.32}$$

being its inverse transform

$$\mathcal{M}^{-1}\{\mathcal{M}f\}(u) = f(u) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dk\, u^{-k}\mathcal{M}f(k) \tag{C.33}$$

The Mellin transform over the variable u of the kernel $\tilde{o}_s$ will be

$$\begin{aligned}\mathcal{M}\tilde{o}_s(k, v) &= \int_0^\infty du\, u^{k-1}\tilde{o}_s(u, v) \\ &= \int_0^\infty du\, u^{k-1}[\mathcal{O}_\Omega\delta](-u+v) + \frac{1}{\pi}v^{k-2} - \int_0^\infty du\, u^{k-1}\xi_s(u, v)\end{aligned} \tag{C.34}$$

The first term of the right side of the last equality is given by

$$\begin{aligned}\int_0^\infty duu^{k-1}\mathcal{O}_\Omega\delta(-u+v) = \int_0^\infty duu^{k-1}\lim_{\epsilon\to 0}\Bigg[&\int_{-\infty}^{-u-\epsilon} dyK(y+u)[\delta(v-u) - \delta(y+v)] \\ &+ \int_{-u+\epsilon}^{0} dyK(y+u)[\delta(v-u) - \delta(y+v)]\Bigg]\end{aligned} \tag{C.35}$$

Making a change of sign in the integral of dy, we have that

$$\begin{aligned}\int_0^\infty duu^{k-1}\mathcal{O}_\Omega\delta(-u+v) &= \int_0^\infty duu^{k-1}\lim_{\epsilon\to 0}\int_{\overline{\Omega}/B_\epsilon(u)} dyK(u-y)[\delta(v-u) - \delta(v-y)] \\ &= \lim_{\epsilon\to 0}\Bigg[\int_0^{v-\epsilon} dy\, K(v-y)\left(v^{k-1} - y^{k-1}\right) \\ &\quad + \int_{v+\epsilon}^\infty dy\, K(v-y)\left(v^{k-1} - y^{k-1}\right)\Bigg]\end{aligned} \tag{C.36}$$

Computing the integrals considering (C.23), we obtain

$$\int_0^\infty du\, u^{k-1}[\mathcal{O}_\Omega\delta](-u+v) = \frac{1}{\pi}v^{k-2}[-1 + \pi(k-1)\cot(\pi k)]. \tag{C.37}$$

Now the third term of the right side of (C.34) will be

$$\begin{aligned}\int_0^\infty du\, u^{k-1}\xi_s(u,v) =& \frac{1}{2\pi}\int_0^\infty du\, \frac{1}{\sqrt{uv}}\frac{1}{\left(\sqrt{u}+\sqrt{v}\right)^2}u^{k-1}\\ =& 2(k-1)v^{k-2}\csc(2\pi k),\quad \frac{1}{2}<\mathrm{Re}(k)<\frac{3}{2}\end{aligned} \tag{C.38}$$

Inserting (C.38) and (C.37) in (C.34) we obtain the Mellin transformation of the kernel $\tilde{o}_s$

$$\frac{1}{2}\mathcal{M}\tilde{o}_s(k,v) = v^{k-2}(1-k)\tan(\pi k),\qquad \frac{1}{2}<\mathrm{Re}(k)<\frac{3}{2} \tag{C.39}$$

Thus, the Mellin transformation of the integral operator $[o_s g](u)$ (11) will be

$$\mathcal{M}[o_s g](k) = 2\int_0^\infty dv\, g(v)\left[v^{k-2}(1-k)\tan(\pi k)\right],\ \frac{1}{2}<\mathrm{Re}(k)<\frac{3}{2} \tag{C.40}$$

To find the inverse of the kernel $\tilde{o}_s$ we consider the Parseval-Plancherel's theorem

$$\begin{aligned}\int_0^\infty dx\, f_1(x)f_2(x) =& \int_0^\infty dx\, \left[\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dk\mathcal{M}f_1(k)x^{-k}\right]f_2(x)\\ =& \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dk\, \mathcal{M}f_1(k)\int_0^\infty dx\, x^{-k}f_2(x)\\ =& \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dk_1\mathcal{M}f_1(k)\mathcal{M}f_2(1-k)\end{aligned} \tag{C.41}$$

where we have used the definition of the Mellin transformation (C.32) and its inverse (C.33). Thus from (C.30) we have

$$\begin{aligned}\delta(v-v') =& \int_0^\infty du\, \tilde{o}_s^{-1}(v',u)\tilde{o}_s(u,v)\\ =& \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dk\, \mathcal{M}\tilde{o}_s^{-1}(v',1-k)\mathcal{M}\tilde{o}_s(k,v)\end{aligned} \tag{C.42}$$

Now, inserting (C.39) in (C.42)

$$\begin{aligned}&\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dk\, \mathcal{M}\tilde{o}_s^{-1}(v',1-k)\mathcal{M}\tilde{o}_s(k,v)\\ &\qquad = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dk\, \mathcal{M}\tilde{o}_s^{-1}(v',1-k)2v^{k-2}(1-k)\tan(\pi k)\end{aligned} \tag{C.43}$$

making a change of variable $k = s + 1$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dk\ \mathcal{M}\tilde{o}_s^{-1}(v', 1-k)\mathcal{M}\tilde{o}_s(k, v) \\ = -\frac{1}{2\pi i}\int_{c-1-i\infty}^{c-1+i\infty} ds\ \mathcal{M}\tilde{o}_s^{-1}(v', -s)2v^{s-1}s\tan(\pi s) \tag{C.44}$$

and taking into account the fact

$$\delta(x-a) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} dk\ x^{-k}\ a^{k-1} \tag{C.45}$$

we have that

$$\mathcal{M}\tilde{o}_s^{-1}(v', k) = -\frac{v'^k}{2k\tan(\pi k)}. \tag{C.46}$$

Finally

$$\begin{aligned}\tilde{o}_s^{-1}(v', u) =& -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dk\ u^{-k}\frac{v'^k}{2k\tan(\pi k)} \\ =& \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dk\ \left[-\frac{1}{2k\tan(\pi k)}\right] x^{-k}\end{aligned} \tag{C.47}$$

we recognize this as an inverse Mellin transform, where $x = \frac{u}{v'} > 0$ (since $t, u > 0$ on $[0, \infty\rangle$), and the integrand $f(k) = -\frac{1}{2k\tan(\pi k)}$ is weighted by x^k. The contour is a vertical line in the complex plane with $\mathrm{Re}(k) = c$, where $\frac{1}{2} < c < \frac{3}{2}$, defining the strip of analyticity. Let's compute this using contour integration and residue theory.

$$\begin{aligned}\mathcal{M}^{-1}\left\{-\frac{1}{2k\tan(\pi k)}\right\}(x) =& \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} -\frac{1}{2k\tan(\pi k)}x^{-k}dk \\ =& -\frac{1}{2\pi}\left[(1-\theta(u-v'))\ln\left(\frac{v'}{u}-1\right)\right. \\ & \left.+\theta(u-v')\ln\left(1-\frac{v'}{u}\right)\right]\end{aligned} \tag{C.48}$$

If $v' < u$

$$\mathcal{M}^{-1}\left\{-\frac{1}{2k\tan(\pi k)}\right\}(x) = -\frac{1}{2\pi}\ln\left(1-\frac{v'}{u}\right). \tag{C.49}$$

If $v' > u$

$$\mathcal{M}^{-1}\left\{-\frac{1}{2k\tan(\pi k)}\right\}(x) = -\frac{1}{2\pi}\ln\left(\frac{v'}{u}-1\right). \tag{C.50}$$

Now, considering (C.31) we have that

$$\begin{aligned}\left[o_s^{-1}g\right](v') &= \int_0^\infty du\, \tilde{o}_s^{-1}(v',u)g(u) \\ &= -\frac{1}{2\pi}\left[\int_0^{v'} du\, g(u)\ln\left(\frac{v'}{u}-1\right)+\int_{v'}^\infty du\, g(u)\ln\left(1-\frac{v'}{u}\right)\right] \end{aligned} \tag{C.51}$$

making the change of variable $v' = -x$ and $u = -z$, let's redefine

$$\left[o_s^{-1}f\right](x) \equiv \left[o_s^{-1}g\right](-x) \tag{C.52}$$

$$f(z) \equiv g(-z) \tag{C.53}$$

Thus, we obtain the inverse of the integral operator $[o_s f](x)$

$$\left[o_s^{-1}f\right](x) = -\frac{1}{2\pi}\left[\int_{-\infty}^{x} dz\, f(z)\ln\left(1-\frac{x}{z}\right)+\int_x^0 dz\, f(z)\ln\left(\frac{x}{z}-1\right)\right] \tag{C.54}$$

Appendix D

Normalization Condition

It is interesting to analyze the normalization condition for the reduced probability distributions of the vacuum and excited states. This can help us derive relationships between expressions that are difficult to compute.

D.1 Normalization Condition for the Vacuum State

The normalization condition for the reduced probability distribution for the vacuum state is given by

$$\int [\mathcal{D}g_\Omega] P_{0\Omega}[g_\Omega] = 1, \tag{D.1}$$

where the distribution probability is given by (3.53)

$$P_{0\Omega}[g_\Omega] = \left[\frac{\det\left(\mu^{-1}\mathcal{O}\right)}{\det_{\overline{\Omega}}\left(\mu^{-1}\mathcal{O}\right)}\right]^{1/2} \exp(W[g_\Omega]) \tag{D.2}$$

being

$$W[g_\Omega] = -\frac{1}{2}\int_\Omega d^3\mathbf{x}\, g_\Omega(\mathbf{x}) \int_\Omega d^d\mathbf{z}\, g_\Omega(\mathbf{z})\hat{o}(\mathbf{x},\mathbf{z}) \tag{D.3}$$

and the operator $\hat{o}(\mathbf{x},\mathbf{z})$ is defined in (B.14).

In this way

$$\begin{aligned}\int [\mathcal{D}g_\Omega] P_{0\Omega}[g_\Omega] &= \left[\frac{\det\left(\mu^{-1}\mathcal{O}\right)}{\det_{\overline{\Omega}}\left(\mu^{-1}\mathcal{O}\right)}\right]^{1/2} \int [\mathcal{D}g_\Omega] e^{-\frac{1}{2}\int_\Omega d^d\mathbf{x}\, g_\Omega(\mathbf{x}) \int_\Omega d^d\mathbf{z}\, g_\Omega(\mathbf{z})\hat{o}(\mathbf{x},\mathbf{z})} \\ &= \left[\frac{\det\left(\mu^{-1}\mathcal{O}\right)}{\det_{\overline{\Omega}}\left(\mu^{-1}\mathcal{O}\right)}\right]^{1/2} [\det(\hat{o})]^{-1/2}\end{aligned} \tag{D.4}$$

where in the last equality we use the result of a Gaussian integral. Thus

$$\det(\hat{o}) = \left[\frac{\det\left(\mu^{-1}\mathcal{O}\right)}{\det_{\overline{\Omega}}\left(\mu^{-1}\mathcal{O}\right)}\right] \tag{D.5}$$

D.2 Normalization Condition for the Excited State

The normalization condition for the reduced probability distribution for the excited state is given by

$$\int [\mathcal{D}g_\Omega] P_{1\Omega}[g_\Omega] = 1, \tag{D.6}$$

where the distribution probability is given by (4.85) or (4.102)

$$P_{1\Omega}[g_\Omega] = \frac{4}{\gamma} \frac{\left[\det\left(\mu^{-1}\mathcal{O}\right)\right]^{1/2}}{\left[\det_{\overline{\Omega}}\left(\mu^{-1}\mathcal{O}\right)\right]^{1/2}} \left(\frac{\delta^2}{\delta\alpha^2} - \frac{\delta^2}{\delta\beta^2}\right) \exp\left(W[\phi_\Omega, f_{\alpha\beta}]\right)\Big|_{\alpha=\beta=0} \tag{D.7}$$

$$= \frac{4}{\gamma} \left[\frac{\det\left(\mu^{-1}\mathcal{O}\right)}{\det_{\overline{\Omega}}\left(\mu^{-1}\mathcal{O}\right)}\right]^{1/2} \left[\left|\int_\Omega d^d\mathbf{z}\, g_\Omega(\mathbf{z}) a_f(\mathbf{z})\right|^2 + 2b_R + 2b_I\right] \exp(W[g_\Omega]) \tag{D.8}$$

where

$$W[\phi_\Omega, f_{\alpha,\beta}] = -\frac{1}{2}\int_\Omega d^d\mathbf{x}\, g_\Omega(\mathbf{x}) \int_\Omega d^d\mathbf{z}\, g(\mathbf{z}) \hat{o}(\mathbf{x},\mathbf{z}) + \int_\Omega d^d\mathbf{z}\, g(\mathbf{z}) a_{f_{\alpha\beta}}(\mathbf{z}) + b_{f_{\alpha\beta}} \tag{D.9}$$

being a_f and b_f defined in (B.15) and (B.16) respectively.

To perform the integration of $P_{1\Omega}[g_\Omega]$ over $\int[\mathcal{D}g_\Omega]$, we can proceed in two completely equivalent ways. The first approach consists of directly integrating the Gaussian exponential from (D.7) and then considering the variations with respect to α and β.

$$\int [\mathcal{D}g_\Omega] P_{1\Omega}[g_\Omega] = \frac{4}{\gamma} \left[\frac{\det\left(\mu^{-1}\mathcal{O}\right)}{\det_{\overline{\Omega}}\left(\mu^{-1}\mathcal{O}\right)}\right]^{\frac{1}{2}} \left(\frac{\delta^2}{\delta\alpha^2} - \frac{\delta^2}{\delta\beta^2}\right) \int [\mathcal{D}g_\Omega] \exp\left(W[\phi_\Omega, f_{\alpha,\beta}]\right)\Big|_{\alpha=\beta=0} \tag{D.10}$$

using the fact that, see eq (12.45) of Greiner

$$\int [\mathcal{D}\phi] \exp\left(-\frac{1}{2}\int d^d\mathbf{x}\, \left(\phi(\mathbf{x})[A\phi](\mathbf{x}) - 2\rho(\mathbf{x})\phi(\mathbf{x})\right)\right)$$
$$= [\det(A)]^{-1/2} \exp\left(\frac{1}{2}\int d^d\mathbf{x}\, \rho(\mathbf{x}) \left[A^{-1}\rho\right](\mathbf{x})\right), \tag{D.11}$$

considering that $A \equiv o$ and $\rho \equiv a_{\alpha\beta}$. Then, we obtain that

$$\int [\mathcal{D}g_\Omega] P_{1\Omega}[g_\Omega] = \frac{4}{\gamma} \left[\frac{\det\left(\mu^{-1}\mathcal{O}\right)}{\det_{\overline{\Omega}}\left(\mu^{-1}\mathcal{O}\right)}\right]^{1/2} [\det(o)]^{-1/2}$$
$$\times \left(\frac{\delta^2}{\delta\alpha^2} - \frac{\delta^2}{\delta\beta^2}\right) \exp\left(\frac{1}{2}\int_\Omega d^d\mathbf{x} a_{\alpha\beta}(\mathbf{x}) \left[o^{-1}a_{\alpha\beta}\right](\mathbf{x}) + b_{f_{\alpha\beta}}\right)\Big|_{\alpha=\beta=0} \tag{D.12}$$

From the normalization condition for the vacuum (D.5), this expression can be simplified to

$$\int [\mathcal{D}g_\Omega] P_{1\Omega}[g_\Omega] = \frac{4}{\gamma}\left(\frac{\delta^2}{\delta\alpha^2} - \frac{\delta^2}{\delta\beta^2}\right) \exp\left(\frac{1}{2}\int_\Omega d^d\mathbf{x}\, a_{\alpha\beta}(\mathbf{x})\left[o^{-1}a_{\alpha\beta}\right](\mathbf{x}) + b_{f_{\alpha\beta}}\right)\Big|_{\alpha=\beta=0}. \tag{D.13}$$

Using the variations given in (B.17), (B.18), (B.19), (B.20), (B.21), and (B.23), we will obtain

$$\begin{aligned}\int [\mathcal{D}g_\Omega] P_{1\Omega}[g_\Omega] =& \frac{4}{\gamma}\left[\int_\Omega d^d\mathbf{x}\, a_R(\mathbf{x})\left[o^{-1}a_R\right](\mathbf{x}) + 2b_R\right. \\ &\left. + \int_\Omega d^d\mathbf{x}\, a_I(\mathbf{x})\left[o^{-1}a_I\right](\mathbf{x}) + 2b_I\right]\end{aligned} \tag{D.14}$$

Therefore

$$\int_\Omega d^d\mathbf{x}\, a_R(\mathbf{x})\left[o^{-1}a_R\right](\mathbf{x}) + \int_\Omega d^d\mathbf{x}\, a_I(\mathbf{x})\left[o^{-1}a_I\right](\mathbf{x}) = \frac{\gamma}{4} - 2(b_R(\mathbf{x}) + b_I(\mathbf{x})) \tag{D.15}$$

The second approach to performing the integration of $P_{1\Omega}[g_\Omega]$ over $\int [\mathcal{D}g_\Omega]$ is by using the generating functional $Z[J]$ defined in (4.107)

$$\begin{aligned}Z[J] =& \int [\mathcal{D}g_\Omega] \exp\left(-\frac{1}{2}\int_\Omega d^d\mathbf{x} g_\Omega(\mathbf{x}) \int_\Omega d^d\mathbf{z}\, g(\mathbf{z})\hat{o}(\mathbf{x},\mathbf{z}) + \int_\Omega d^d\mathbf{x}\, g_\Omega(\mathbf{x}) J(\mathbf{x})\right) \\ =& [\det_\Omega(o)]^{-1/2} \exp\left(\frac{1}{2}\int_\Omega d^d\mathbf{x}\, J(\mathbf{x}) \int_\Omega d^d\mathbf{y}\, J(\mathbf{y})\, \hat{o}^{-1}(\mathbf{x},\mathbf{y})\right)\end{aligned} \tag{D.16}$$

In this way we can express the integral over $\int [\mathcal{D}g_\Omega]$ of (D.8) in funtion of this generating functional $Z[J]$ as

$$\begin{aligned}\int \mathcal{D}g_\Omega P_{1\Omega}[g_\Omega] =& \frac{4}{\gamma}\left[\frac{\det(\mu^{-1}\mathcal{O})}{\det_{\overline{\Omega}}(\mu^{-1}\mathcal{O})}\right]^{\frac{1}{2}} \int [\mathcal{D}g_\Omega]\left[\left|\int_\Omega d^d\mathbf{z} g_\Omega(\mathbf{z}) a_f(\mathbf{z})\right|^2 + 2b_R + 2b_I\right] \\ &\times \exp(W[g_\Omega]) \\ =& \frac{4}{\gamma}\left[\frac{\det(\mu^{-1}\mathcal{O})}{\det_{\overline{\Omega}}(\mu^{-1}\mathcal{O})}\right]^{\frac{1}{2}} \left[\int_\Omega d^d\mathbf{x}_1 \int_\Omega d^d\mathbf{x}_2 a_f(\mathbf{x}_1) a_{f^*}(\mathbf{x}_2)\, \frac{\delta Z[J]}{\delta J(\mathbf{x}_1)\delta J(\mathbf{x}_2)}\Big|_{J=0}\right. \\ &\left. + 2(b_R + b_I)[\det(\hat{o})]^{-1/2}\right]\end{aligned} \tag{D.17}$$

Now, from (4.115) we have

$$\frac{\delta Z[J]}{\delta J(\mathbf{x}_1)\delta J(\mathbf{x}_2)}\Big|_{J=0} = [\det(\hat{o})]^{-1/2}\, \hat{o}_s^{-1}(\mathbf{x}_1,\mathbf{x}_2), \tag{D.18}$$

where $\hat{o}_s^{-1}(\mathbf{x}_2, \mathbf{x}_1) \equiv \frac{1}{2}\left(\hat{o}^{-1}(\mathbf{x}_1, \mathbf{x}_2) + \hat{o}^{-1}(\mathbf{x}_2, \mathbf{x}_1)\right)$.

Inserting this expression in (D.17) and taking into account (D.5), we have

$$\int [\mathcal{D}g_\Omega] P_{1\Omega}[g_\Omega] = \frac{4}{\gamma}\left[\int_\Omega d^d\mathbf{x}_1 \int_\Omega d^d\mathbf{x}_2\, a_f(\mathbf{x}_1) a_{f^*}(\mathbf{x}_2)\, \hat{o}_s^{-1}(\mathbf{x}_1, \mathbf{x}_2) + 2(b_R + b_I)\right]. \quad \text{(D.19)}$$

Thus, we find that

$$\int_\Omega d^d\mathbf{x}\, a_f(\mathbf{x}) \left[o_s^{-1} a_{f^*}\right](\mathbf{x}) = \frac{\gamma}{4} - 2(b_R + b_I), \quad \text{(D.20)}$$

which is completely equivalent to (D.15).

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