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**“JÚLIO DE MESQUITA FILHO”
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**Study of the dynamics around celestial bodies using analytical and semi-analytical
techniques**

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Josué Cardoso dos Santos

Estudo da dinâmica ao redor de corpos celestes utilizando técnicas analíticas e semianalíticas

Tese de Doutorado apresentado ao Conselho de Curso da Pós-Graduação em Física da Faculdade de Engenharia do Campus de Guaratinguetá, Universidade Estadual Paulista, como parte dos requisitos para obtenção do diploma de Doutor em Física.

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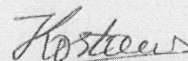
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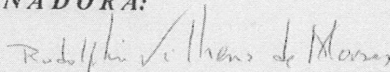
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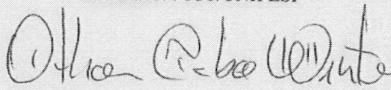


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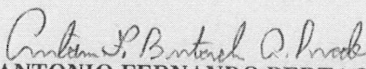
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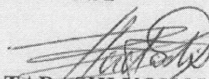
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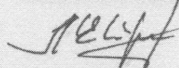
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Ao Deus triúno, Pai, Filho e Espírito Santo, Provedores de vida e
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“The greatest want of the world is the want of men — men who will not be bought or sold, men who in their inmost souls are true and honest, men who do not fear to call sin by its right name, men whose conscience is as true to duty as the needle to the pole, men who will stand for the right though the heavens fall.”

“A maior necessidade do mundo é a de homens — homens que se não comprem nem se vendam; homens que no íntimo da alma sejam verdadeiros e honestos; homens que não temam chamar o pecado pelo seu nome exato; homens, cuja consciência seja tão fiel ao dever como a bússola o é ao pólo; homens que permaneçam firmes pelo que é reto, ainda que caiam os céus.”

Ellen Gould White

NOTE

The present thesis is written in accordance with ABNT (Associação Brasileira de Normas Técnicas) standard, following the Brazilian law. The author thanks Dr. Ricardo Aparecido de Moraes for providing the template of the document and UNESP Library's staff for reviewing its final form.

ABSTRACT

Nowadays, despite the technological development experienced by science in general, a fact especially evident by the available powerful computer machines, the analytical and semi-analytical methods to study different space problems are still of great importance in the fields of astrodynamics and celestial mechanics. From the physical understanding of the motion of celestial bodies to the planing and designing of space missions, the use of mathematical models to deal with a very large number of contemporary problems plays a fundamental role in the progress of human knowledge. In this context, the present thesis presents the use of different mathematical techniques to deal with different various and current problems in astrodynamics and celestial mechanics. The studies developed throughout this work are applicable to both areas. The topics studied are the following ones: (1) The development of disturbing potentials using the double-averaging process, in order to be included in the Lagrange planetary which are numerically integrated to study features of orbits around Mercury and the Galilean moon Callisto; (2) The use of different perturbation integrals, techniques to identify and map different perturbations present in a planetary system, with focus on the analysis of systems of Giant planets with their massive moons; (3) The use of the concept of intermediary Hamiltonian and the use of a canonical transformation called elimination of the parallax, both to deal with binary systems in the context of the roto-orbital dynamics, this one as an approach of the full-two body problem; (4) An updated analysis of Gauss variational equations to study quasi-satellite orbits around the Martian moon Phobos and with analytical predictions made after obtaining linear and averaged equations of motions. Therefore, this thesis intend not only to provide important analysis and results for each specific problem which it deals with along its pages, but also seeks to highlighting the merit and current relevance of different analytical and semi-analytical methods to be used in the fields of astrodynamics and celestial mechanics. Additionally, the author also hopes to offer an outcome of diverse interesting ideas and methods to be explored in future investigations in these research fields.

KEYWORDS: Analytical and semi-analytical methods, Perturbation theory, Orbital dynamics, Rotational dynamics, Space missions, Quasi-satellite orbits, Binary systems

RESUMO

Na atualidade, a despeito do desenvolvimento tecnológico experimentado pela ciência em geral, algo especialmente evidenciado por poderosas máquinas computacionais disponíveis, os métodos analíticos e semianalíticos para o estudo de diferentes problemas espaciais ainda são de grande importância nos campos de astrodinâmica e mecânica celeste. Desde a compreensão física do movimento de corpos celestes até ao planejamento e projeto de missões espaciais, o uso de modelos matemáticos para lidar com um grande número de problemas contemporâneos desempenha um papel fundamental no progresso do conhecimento humano. Neste contexto, a presente tese apresenta o uso de diferentes técnicas matemáticas para lidar com diversos e atuais problemas em astrodinâmica e mecânica celeste. Os estudos desenvolvidos ao longo deste trabalho são aplicáveis à ambas as áreas. Os tópicos estudados são os seguintes: (1) O desenvolvimento de potenciais perturbadores usando o processo de dupla média, de forma a serem incluídos nas equações planetárias de Lagrange que são integradas numericamente para estudar características de órbitas ao redor de Mercúrio e da lua galileana Calisto; (2) A utilização de diferentes integrais de perturbação, técnicas para identificar e mapear diferentes perturbações presentes em um sistema planetário, com foco na análise de sistemas de planetas gigantes com suas luas massivas; (3) A utilização do conceito de hamiltoniana intermediária e o uso de uma transformação canônica chamada eliminação da paralaxe, ambos para lidar com sistemas binários no contexto da dinâmica roto-orbital, essa sendo uma aproximação do problema completo de dois corpos; (3) Uma análise atualizada de equações variacionais de Gauss para o estudo de órbitas quasi-satélite ao redor da lua marciana Fobos e com previsões analíticas realizadas após serem obtidas equações de movimento linearizadas e com média. Portanto, esta tese pretende não somente prover importantes análises e resultados para cada problema específico com os quais a mesma lida ao longo de suas páginas, mas também procura destacar o mérito e relevância atual de diferentes métodos analíticos e semianalíticos a serem utilizados nos campos de astrodinâmica e mecânica celeste. Adicionalmente, o autor também espera oferecer um produto de variadas ideias e métodos a serem explorados em futuras investigações nesses campos de pesquisa.

PALAVRAS-CHAVE: Métodos analíticos e semianalíticos, Teoria de Perturbação, Dinâmica orbital, Dinâmica rotacional, Missões espaciais, Órbitas quasi-satélite, Sistemas binários

LIST OF FIGURES

Figure 1 -	Geometry of a Keplerian orbit	35
Figure 2 -	Geometrical scheme of the dynamical system	36
Figure 3 -	Lifetime map (e_0 vs i_0) considering the effects of $R_2+J_2+C_{22}$. Initial conditions: $a_0 = 2510.3$ km, $g_0 = 270^\circ$, $h_0 = 90^\circ$ and $e_p = 0.0074$. The vertical bar in the right gives the value of the lifetimes in hundreds of days for an integration of 1000 days. Each unity corresponds to a hundred of days.	46
Figure 4 -	Propagation of the z -component of the orbital angular momentum $H_z = \sqrt{1 - e^2} \cos(i)$. Initial conditions: $a_0 = 2510.3$ km, $g_0 = 270^\circ$, $h_0 = 90^\circ$ and $e_p = 0.0074$. Effects due to R_2 . Numerical propagation for 6000 days.	47
Figure 5 -	Propagation of the eccentricity (e), inclination (i) and argument of the periapsis (g) considering different values for the initial inclination i_0 . Initial conditions: $a_0 = 2510.3$ km, $e_0 = 0.01$, $g_0 = 270^\circ$, $h_0 = 90^\circ$ and $e_p = 0.0074$. Effects due to R_2 . Numerical propagation for 6000 days.	48
Figure 6 -	Propagation of the components of the eccentricity vector considering different values for i_0 and with initial values $a_0 = 2510.3$ km, $e_0 = 0.01$, $h_0 = 90^\circ$ and $e_p = 0.0074$. Effects of R_2 . Simulations for $g_0 = 270^\circ$ and numerical integration for 100 years.	49
Figure 7 -	Lifetime maps (i_0 vs a_0) as a function of semi-major axis and inclination considering the effects of R_2 and $R_2+J_2+C_{22}$. It is the real value for the eccentricity of Jupiter orbiting Callisto. Initial conditions: $e_0 = 0.01$, $g_0 = 270^\circ$, $h_0 = 90^\circ$ and $e_p = 0.0074$. The vertical bar in the right gives the value of the lifetimes in hundreds of days for an integration of 1000 days. Each unity corresponds to one hundred days.	50
Figure 8 -	Lifetime maps (g_0 vs h_0), (i_0 vs g_0) and (e_0 vs i_0) for $g_0 = 180^\circ$, $e_p = 0.0074$ and considering the effects of $R_2+J_2+C_{22}$. The vertical bar in the right gives the value of the lifetimes in hundreds of days for an integration of 1000 days. Each unity corresponds to one hundred days.	51

Figure 9 -	Lifetime maps (i_0 vs h_0) for $e_p = 0.0074$ and considering the effects of $R_2+J_2+C_{22}$. Simulations for $g_0 = 270^\circ$ and 180° . The vertical bar in the right gives the value of the lifetimes in hundreds of days for an integration of 1000 days. Each unity corresponds to one hundred days.	52
Figure 10 -	Propagation of the inclination (i) considering different values for the initial inclination i_0 . Initial conditions: $a_0 = 2510.3$ km, $i_0 = 90^\circ$, $g_0 = 270^\circ$ and $h_0 = 90^\circ$. Effects due to R_2 .	53
Figure 11 -	Propagation of the eccentricity (e) considering different values for the initial eccentricity e_0 . Initial conditions: $a_0 = 2510.3$ km, $i_0 = 90^\circ$, $h_0 = 90^\circ$ and $e_p = 0.0074$. Effects due to R_2 . Simulations for $g_0 = 270^\circ$ and 180° .	54
Figure 12 -	Propagation of the eccentricity (e) considering different values for the initial inclination i_0 . Initial conditions: $a_0 = 2510.3$ km, $e_0 = 0.01$, $h_0 = 90^\circ$ and $e_p = 0.0074$. Effects due to R_2 . Simulations for $g_0 = 270^\circ$ and 180° .	55
Figure 13 -	Lifetime maps (i_0 vs a_0) for $e_p = 0.1, 0.2$ and 0.3 . Disturbing effects: $R_2+J_2+C_{22}$. Initial conditions: $e_0 = 0.01$, $g_0 = 270^\circ$ and $h_0 = 90^\circ$. The vertical bar in the right gives the value of the lifetimes in hundreds of days for an integration of 1000 days. Each unity corresponds to one hundred days.	56
Figure 14 -	Lifetime maps (i_0 vs a_0) for $e_p = 0.4, 0.5$ and 0.6 . Disturbing effects: $R_2+J_2+C_{22}$. Initial conditions: $e_0 = 0.01$, $g_0 = 270^\circ$ and $h_0 = 90^\circ$. The vertical bar in the right gives the value of the lifetimes in hundreds of days for an integration of 1000 days. Each unity corresponds to one hundred days.	57
Figure 15 -	Lifetime maps (i_0 vs a_0) for $e_p = 0.0074, 0.2$ and 0.3 . Disturbing effects: $R_2+J_2+C_{22}$. Initial conditions: $e_0 = 0.01$, $g_0 = 180^\circ$ and $h_0 = 90^\circ$. The vertical bar in the right gives the value of the lifetimes in hundreds of days for an integration of 1000 days. Each unity corresponds to one hundred days.	58
Figure 16 -	Lifetime maps (i_0 vs a_0) for $e_p = 0.4, 0.5$ and 0.6 . Disturbing effects: $R_2+J_2+C_{22}$. Initial conditions: $e_0 = 0.01$, $g_0 = 180^\circ$ and $h_0 = 90^\circ$. The vertical bar in the right gives the value of the lifetimes in hundreds of days for an integration of 1000 days. Each unity corresponds to a hundred of days.	59
Figure 17 -	Propagation of the components of the eccentricity vector considering different values for i_0 and with initial values $a_0 = 2510.3$ km, $e_0 = 0.01$, $h_0 = 90^\circ$ and $e_p = 0.0074$. Effects of $R_2+J_2+C_{22}$. Simulations for $g_0 = 270^\circ$ and numerical integration for 100 years.	61

Figure 18 - Propagation of the components of the eccentricity vector considering different values for i_0 and with initial values $a_0 = 2510.3$ km, $e_0 = 0.01$, $h_0 = 90^\circ$ and $e_p = 0.0074$. Effects of $R_2+J_2+C_{22}$. Simulations for $g_0 = 180^\circ$ and numerical integration for 100 years.	62
Figure 19 - Representation of Orbit time vs i_0 considering different values for a_0 and for $e_0 = 0.01$ and $e_p = 0.0074$. Effects of $R_2+J_2+C_{22}$. Simulations for $g_0 = 270^\circ$ and 180°	63
Figure 20 - Representation of Orbit time vs i_0 considering different values for e_0 and for $a_0 = 2510.3$ km and $e_p = 0.0074$. Effects of $R_2+J_2+C_{22}$. Simulations for $g_0 = 270^\circ$ and 180°	64
Figure 21 - Geometrical scheme of the dynamical system with a solar sail	70
Figure 22 - Concept of solar sail propulsion	70
Figure 23 - Scheme of a perfect sail (flat and 100% reflective)	71
Figure 24 - Final eccentricity map e versus g . Initial Conditions: $a = 2740.0$ km, $i = 90^\circ$, $h = 90^\circ$, $\beta = 0$. Perturbations: (a) $J_2 + C_{22} + R_{23b}$ and (b) $J_2 + J_3 + C_{22} + R_{23b}$	76
Figure 25 - Orbit time map e versus g . Initial Conditions: $a = 2740.0$ km, $i = 90^\circ$, $h = 90^\circ$, $\beta = 0$. Perturbations: (a) $J_2 + C_{22} + R_{23b}$ and (b) $J_2 + J_3 + C_{22} + R_{23b}$	76
Figure 26 - Representation of final eccentricity vs g_0 and orbit time versus g_0 considering different values for e_0 . Initial Conditions: $a = 2740.0$ km, $i = 90^\circ$, $h = 90^\circ$, $\beta = 0$. Perturbations: $J_2 + J_3 + C_{22} + R_{23b}$	77
Figure 27 - Representation of final eccentricity vs a_0 and final eccentricity versus orbit time considering different values for e_0 . Initial Conditions: $i = 90^\circ$, $g = 270^\circ$, $h = 90^\circ$, $\beta = 0$. Figure (b) also adopts $a = 2740.0$ km. Perturbations: $J_2 + J_3 + C_{22} + R_{23b}$	77
Figure 28 - Integration of e versus g . Initial Conditions: $a = 2700.0$ km, $i = 90^\circ$, $g = 270^\circ$, $h = 90^\circ$, $\beta = 0$. Perturbations: (a) $J_2 + J_3 + C_{22} + R_{23b}$ and (b) $J_2 + J_3 + C_{22} + R_{23b}$ (zoom).	78
Figure 29 - Final eccentricity map varying the parameter β and the semi-major axis a . Initial Condition: $e = 0.1$, $i = 90^\circ$, $g = 270^\circ$, $h = 90^\circ$. Perturbations: $J_2 + J_3 + C_{22} + R_{23b+SRP}$. (a) 4000 km $< a < 7000$ km, (b) 5000 km $< a < 5900$ km (zoom)	79
Figure 30 - Sample integration of e and $R_p = a(1 - e)$ for high-altitude eccentric orbits assuming different values for $\beta = \{0, 0.4, 0.8\}$. Initial Conditions: $a = 5500$ km, $e = 0.2$, $i = 90^\circ$, $g = 270^\circ$, $h = 90^\circ$. Perturbations: $J_2 + J_3 + C_{22} + R_{23b+SRP}$	79
Figure 31 - Integration of e versus g . Initial Conditions: $a = 6000$ km, $i = 90^\circ$, $g = 270^\circ$, $h = 90^\circ$. Perturbations: $J_2 + J_3 + C_{22} + R_{23b+SRP}$. (a) $\beta = 0$, (b) $\beta = 0.1$, (c) $\beta = 0.3$, (d) $\beta = 0.5$	80

Figure 32 - Lifetime map for a low-altitude orbit considering the final eccentricity of the orbit after 500 days. Initial Condition: $e = 0.01$, $g = 270^\circ$, $h = 90^\circ$, $\beta = 0$. Perturbations: $J_2 + J_3 + R2_{3b}$	82
Figure 33 - Lifetime map for a low-altitude orbit considering the lifetime of the orbit for an integration of 500 days. Initial Condition: $e = 0.01$, $g = 270^\circ$, $h = 90^\circ$, $\beta = 0$. Perturbations: $J_2 + J_3 + R2_{3b}$	82
Figure 34 - Lifetime map for a low-altitude orbit considering the final eccentricity of the orbit after 500 days. Initial Conditions: $e = 0.01$, $g = 270^\circ$, $h = 90^\circ$, $\beta = 0$. Perturbations: $J_2 + J_3 + C_{22} + R2_{3b}$	83
Figure 35 - Lifetime map for a low-altitude orbit considering the lifetime of the orbit for an integration of 500 days. Initial Condition: $e = 0.01$, $g = 270^\circ$, $h = 90^\circ$, $\beta = 0$. Perturbations: $J_2 + J_3 + C_{22} + R2_{3b}$	83
Figure 36 - Diagram e versus t (days) for different perturbations acting in an orbit. Time of integration of 500 days. Initial Condition: $a = 2490$ km, $e = 0.01$, $i = 90^\circ$, $g = 270^\circ$, $h = 90^\circ$, $\beta = 0$	84
Figure 37 - Diagram e versus t (days) for different values of inclination. Time of integration of 500 days. Initial Conditions: $a = 2490$ km, $e = 0.01$, $g = 270^\circ$, $h = 90^\circ$, $\beta = 0$. Perturbations: $J_2 + J_3 + C_{22} + R2_{3b}$	84
Figure 38 - Geometrical scheme of the dynamical system	90
Figure 39 - Perturbation Integrals as functions of a (km) and i (radians) of the spacecraft and considering the effects of the J_2 of Jupiter + 4 Galilean moons. Initial conditions: $e = 0.1$, $\Omega = \frac{\pi}{4}$ rad, $\omega = \frac{\pi}{4}$ rad	93
Figure 40 - Perturbation Integrals as functions of a (km) and i (radians) of the spacecraft and considering the effects of the J_2 of Saturn + 5 Saturnian moons. Initial conditions: $e = 0.1$, $\Omega = \frac{\pi}{4}$ rad, $\omega = \frac{\pi}{4}$ rad	94
Figure 41 - Perturbation Integrals as functions of e and i (radians) of the spacecraft and considering the effects of the J_2 of Jupiter + 4 Galilean moons. Initial conditions: $a = 1.200.000$ km, $\Omega = \frac{\pi}{4}$ rad, $\omega = \frac{2\pi}{3}$ rad	95
Figure 42 - Perturbation Integrals as functions of e and i (radians) of the spacecraft and considering the effects of the J_2 of Saturn + 5 Saturnian moons. Initial conditions: $a = 1.200.000$ km, $\Omega = \frac{\pi}{4}$ rad, $\omega = \frac{2\pi}{3}$ rad	96
Figure 43 - Perturbation Integrals as functions of e and a (km) of the spacecraft and considering the effects of the J_2 of Jupiter + 4 Galilean moons. Initial conditions: $i = 0$ rad, $\Omega = \frac{\pi}{4}$ rad, $\omega = \frac{\pi}{2}$ rad	97
Figure 44 - Perturbation Integrals as functions of e and a (km) of the spacecraft and considering the effects of the J_2 of Saturn + 5 Saturnian moons. Initial conditions: $i = 0$ rad, $\Omega = \frac{\pi}{4}$ rad, $\omega = \frac{2\pi}{3}$ rad	98
Figure 45 - Perturbation Integrals as functions of a (km) and i (radians) of the spacecraft and considering the effects of the J_2 of Neptune + Triton. Initial conditions: $e = 0.1$, $\Omega = \frac{\pi}{4}$ rad, $\omega = \frac{\pi}{4}$ rad	99

Figure 46 - Perturbation Integrals as functions of e and i (radians) of the spacecraft and considering the effects of the J_2 of Neptune + Triton. Initial conditions: $a = 350.000$ km, $\Omega = \frac{\pi}{4}$ rad, $\omega = \frac{\pi}{2}$ rad	100
Figure 47 - Perturbation Integrals as functions of e and a (km) of the spacecraft and considering the effects of the J_2 of Neptune + Triton. Initial conditions: $i = 0$ rad, $\Omega = \frac{\pi}{4}$ rad, $\omega = \frac{\pi}{2}$ rad	101
Figure 48 - Perturbation Integrals as functions of a (km) and i (radians) of the spacecraft and considering the effects of the J_2 of Neptune + Triton. Initial conditions: $e = 0.1$, $\omega = 335.570$ deg, $\Omega = 66.142$ deg	102
Figure 49 - PI_1 as a function of a (km) of the spacecraft considering different disturbing effects in a system with Saturn and 5 moons (Enceladus, Tethys, Rhea, Dione and Titan): (a) J_2 of Saturn; (b) moons; and (c) J_2 of Saturn + 5 Saturnian moons.	103
Figure 50 - PI_2 as a function of the (a) inclination i (radians) and (b) semi-major axis a (km) of the spacecraft considering the disturbing effects due to the J_2 of Jupiter + 4 Galilean moons.	104
Figure 51 - PI_2 as a function of the (a) inclination i (radians) and (b) semi-major axis a (km) of the spacecraft considering the disturbing effects due to the J_2 of Saturn + 5 Saturnian moons.	105
Figure 52 - Geometry of the Full Two-Body Problem.	107
Figure 53 - Restricted Model for the Two-Body Problem.	108
Figure 54 - Orbital reference system containing Hill-Whittaker variables and polar nodal variables associated with orbital motion.	118
Figure 55 - Rotational reference system containing Andoyer variables and polar nodal variables associated with rotational motion.	119
Figure 56 - Roto-orbital canonical variables and reference planes. Adapted from Ferrer; Lara (2012)	121
Figure 57 - Sample propagation considering the set 0 of initial conditions ($\sigma = 0.1^\circ$). Notation: $dO(x) \equiv x - x_0$, $DO(x) \equiv 1 - x/x_0$ and $pow(x) \equiv 10^x$. Angles in radians and remaining values in internal units.	133
Figure 58 - Sample propagation considering the set 1 of initial conditions ($\sigma = 1.0^\circ$). Notation: $dO(x) \equiv x - x_0$, $DO(x) \equiv 1 - x/x_0$ and $pow(x) \equiv 10^x$. Angles in radians and remaining values in internal units.	135
Figure 59 - Sample propagation considering the set 2 of initial conditions ($\sigma = 45.1^\circ$). Notation: $dO(x) \equiv x - x_0$, $DO(x) \equiv 1 - x/x_0$ and $pow(x) \equiv 10^x$. Angles in radians and remaining values in internal units.	136
Figure 60 - Sample propagation considering the set 3 of initial conditions ($\sigma = 90.1^\circ$). Notation: $dO(x) \equiv x - x_0$, $DO(x) \equiv 1 - x/x_0$ and $pow(x) \equiv 10^x$. Angles in radians and remaining values in internal units.	137

Figure 61 - Sample propagation considering the set 1 of initial conditions ($\sigma = 1.0^\circ$). Different shapes are assumed for an oblate rigid body with fixed $C = 7.0 \times 10^{-4}$ and $q_s =$ (a) 1.014492754 and (b) 1.794871795. Notation: $dO(x) \equiv x - x_0$, $DO(x) \equiv 1 - x/x_0$ and $pow(x) \equiv 10^x$. Angles in radians and remaining values in internal units.	138
Figure 62 - Sample propagation for Θ and considering three different orbital eccentricities of the set 2 ($\sigma = 45.1^\circ$). Different shapes are assumed for an oblate rigid body with fixed $C = 7.0 \times 10^{-4}$ and $q_s =$ (a) 1.014492754 and (b) 1.794871795. Notation: $dO(x) \equiv x - x_0$, $DO(x) \equiv 1 - x/x_0$ and $pow(x) \equiv 10^x$. Angles in radians and remaining values in internal units.	139
Figure 63 - Sample propagation of Δ for 500 orbits and with initial condition $\sigma = 45.1^\circ$. Notation: $DO(x) \equiv 1 - x/x_0$ and $pow(x) \equiv 10^x$. Angles in radians and remaining values in internal units.	141
Figure 64 - Sample propagation for Δ and considering three different orbital eccentricities of the set 2 ($\sigma = 45.1^\circ$). Different shapes are assumed for an oblate rigid body with fixed $C = 7.0 \times 10^{-4}$ and $q_s =$ (a) 1.014492754 and (b) 1.794871795. Notation: $dO(x) \equiv x - x_0$, $DO(x) \equiv 1 - x/x_0$ and $pow(x) \equiv 10^x$. Angles in radians and remaining values in internal units.	142
Figure 65 - Sample propagation for orbital eccentricity $e = 0.5$, shape parameter $q_s = 1.794871795$ and $\sigma = 45.1^\circ$. Notation: $dO(x) \equiv x - x_0$, $DO(x) \equiv 1 - x/x_0$ and $pow(x) \equiv 10^x$. Angles in radians and remaining values in internal units.	143
Figure 66 - Sample propagation showing deviations between the short-period correction in $\mathcal{I}_p$ and the original Hamiltonian. Shape parameters fixed as $A = 6.9 \times 10^{-4}$ and $C = 7.0 \times 10^{-4}$. Attitude angles $\sigma = 1.0^\circ$, 45.1° and 90.1° . Notation: $d(x) \equiv x_{int} - x_{or}$, $D(x) \equiv 1 - x_{or}/x_{int}$ and $pow(x) \equiv 10^x$. Values x_{or} and x_{int} refer to Original Hamiltonian and Intermediary $\mathcal{I}_p$, respectively. Angles in radians and remaining values in internal units.	144
Figure 67 - Sample propagation of 4.5 orbits showing deviations between the short-period correction in $\mathcal{I}_p$ and the original Hamiltonian. Shape parameters fixed as $A = 3.9 \times 10^{-4}$ and $C = 7.0 \times 10^{-4}$. Attitude angles $\sigma = 1.0^\circ$, 45.1° and 90.1° . Notation: $d(x) \equiv x_{int} - x_{or}$, $D(x) \equiv 1 - x_{or}/x_{int}$ and $pow(x) \equiv 10^x$. Values x_{or} and x_{int} refer to Original Hamiltonian and Intermediary $\mathcal{I}_p$, respectively. Angles in radians and remaining values in internal units.	145

Figure 68 - Error maps for the short-period correction in $\mathcal{I}_p$ with respect to the original Hamiltonian. Sample propagation of 50 orbits with initial radial distance at the orbit apoapsis $r_0 = 1.3$. Notation: $d(x) \equiv x_{int} - x_{or}$, $D(x) \equiv 1 - x_{or}/x_{int}$ and $pow(x) \equiv 10^x$. Values x_{or} and x_{int} refer to Original Hamiltonian and Intermediary $\mathcal{I}_p$, respectively. Other values of eccentricities presented similar results. Angles in radians and remaining values in internal units.	146
Figure 69 - Error maps for the short-period correction in $\mathcal{I}_p$ with respect to the original Hamiltonian. Sample propagation of 50 orbits with initial radial distance at the orbit apoapsis $r_0 = 1.3$. Notation: $d(x) \equiv x_{int} - x_{or}$, $D(x) \equiv 1 - x_{or}/x_{int}$ and $pow(x) \equiv 10^x$. Values x_{or} and x_{int} refer to Original Hamiltonian and Intermediary $\mathcal{I}_p$, respectively. Other values of eccentricities presented similar results. Angles in radians and remaining values in internal units.	147
Figure 70 - Error maps for the short-period correction in $\mathcal{I}_p$ with respect to the original Hamiltonian. Sample propagation of 50 orbits. Notation: $d(x) \equiv x_{int} - x_{or}$, $D(x) \equiv 1 - x_{or}/x_{int}$ and $pow(x) \equiv 10^x$. Values x_{or} and x_{int} refer to Original Hamiltonian and Intermediary $\mathcal{I}_p$, respectively. Other values of eccentricities presented similar results. Angles in radians and remaining values in internal units.	148
Figure 71 - Sketch of the ellipsoid representing Phobos as viewed from the North Pole looking down on the equator. The minor axis is presented as a solid line. The interior and exterior dashed lines are, respectively, the region of influence and the Hill sphere containing the Lagrange points L_1 and L_2	155
Figure 72 - Zero-velocity curves and Roche lobe of Phobos	155
Figure 73 - Scheme of a QSO in Mars-Phobos system	156
Figure 74 - Initial position of a spacecraft in a planar QSO at perimartem with $\delta_x = \delta_y = 0$	167
Figure 75 - Quasi-QSO orbit integrated over 100 Phobos revolution via the LM (blue), GVEs (red), and EHP (black)	172
Figure 76 - Time histories of the in-plane osculating orbit elements A , α , δ_x , δ_y obtained with the LM model (blue), GVEs (red), and the EHP (black)	173
Figure 77 - Time histories of the osculating orbit elements K_5 , K_6 , B , β obtained with the LM model (blue), GVEs (red), and the EHP (black)	173
Figure 78 - Position error between GVEs and EHP	174
Figure 79 - Position error between LM and EHP	174
Figure 80 - Comparison between the numerical value of n_{QSO} and its theoretical prediction $1 + \dot{\bar{\alpha}}$	176

Figure 81 - Geometry of the gravitational force over a mass element dm	199
Figure 82 - Phase space errors varying A and σ	203
Figure 83 - Phase space errors varying e and σ	204
Figure 84 - Phase space errors varying e and A for $\sigma = 1.0^\circ$	204
Figure 85 - Phase space errors varying e and A for $\sigma = 45.1^\circ$	205
Figure 86 - Phase space errors varying e and A for $\sigma = 90.1^\circ$	205
Figure 87 - Phase space errors varying r and σ	206
Figure 88 - Phase space errors varying r and e for $\sigma = 1.0^\circ$	206
Figure 89 - Phase space errors varying r and e for $\sigma = 45.1^\circ$	207
Figure 90 - Phase space errors varying r and e for $\sigma = 90.1^\circ$	207

LIST OF TABLES

Table 1 - Orbital inclinations of some planetary moons	38
Table 2 - Values of J_2 , C_{22} and mean radius for Callisto	45
Table 3 - Jupiter's data around Callisto	45
Table 4 - Shape parameters for Mercury	75
Table 5 - Gravity coefficients for Jupiter, Saturn and Neptune	91
Table 6 - Physical and orbital data for objects considered in simulations	91
Table 7 - Phobos' data around Mars	154
Table 8 - Phobos' physical data (Ellipsoidal shape)	154

CONTENTS

1	INTRODUCTION	22
1.1	MOTIVATION AND HISTORICAL OVERVIEW	22
1.2	OUTLINE OF THE WORK	23
1.3	BIBLIOGRAPHICAL PRODUCTION	25
1.3.1	Journal Papers	25
1.3.2	Conference Papers	26
1.3.3	Book Chapters	27
1.3.4	Oral Presentations in Scientific Meetings	28
1.3.5	Abstracts published in Proceedings of Scientific Meetings	29
1.3.6	Awards and Recognitions	31
2	DOUBLY-AVERAGED MODELS APPLIED TO THE DYNAMICS OF A CALLISTO ORBITER	32
2.1	DISTURBING POTENTIALS	34
2.1.1	Doubly-averaged third-body perturbation	36
2.1.2	Perturbations due to central body non-uniform shape	40
2.2	EQUATIONS OF MOTION	43
2.3	RESULTS	43
2.4	DISCUSSION	65
3	SOME CHARACTERISTICS OF A SPACECRAFT ORBITING MER- CURY	67
3.1	EQUATIONS OF MOTION AND DISTURBING FORCES	69
3.1.1	Perturbations due to the third body and the solar radiation pressure	71
3.1.2	Non-uniform distribution of mass of Mercury	72
3.2	RESULTS	74
3.2.1	Low-altitude orbit without solar radiation pressure	75
3.2.2	High-altitude orbits with solar radiation pressure	78
3.2.3	Low-altitude orbits, another approach without solar radiation pressure	81
3.3	DISCUSSION	84
4	SEARCH FOR LESS DISTURBED REGIONS FOR SPACECRAFTS IN PLANETARY SYSTEMS	86
4.1	THEORETICAL FUNDAMENTS	87
4.1.1	The Integral Approaches and concepts	87
4.1.2	Methods	88

4.1.3	Equations of motion	89
4.2	NUMERICAL SIMULATIONS	90
4.3	DISCUSSION	106
5	ROTO-ORBITAL DYNAMICS USING INTERMEDIARIES	107
5.1	UNPERTURBED ORBITAL AND ROTATIONAL MODELS	109
5.1.1	Keplerian motion using Hill-Whittaker variables	110
5.1.2	Free-rigid body motion	113
5.2	ROTO-ORBITAL DYNAMICS	118
5.2.1	Roto-orbital canonical variables	118
5.2.2	Roto-orbital Hamiltonian	120
5.3	INTERMEDIARIES	123
5.4	ELIMINATION OF THE PARALLAX	125
5.5	ROTO-ORBITAL INTERMEDIARIES AND EQUATIONS OF MOTION	127
5.6	NUMERICAL SIMULATIONS AND ANALYSIS	131
5.6.1	Setting models and test parameters	131
5.6.2	Comparisons of different dynamical models for different physical parameters	134
5.6.3	Computation of errors	143
5.7	DISCUSSION	149
6	VARIATIONAL EQUATIONS FOR FORMATION FLIGHT WITH PHOBOS	152
6.1	THE CHALLENGE OF ORBITING PHOBOS	153
6.2	GENERAL SOLUTION OF THE KEPLERIAN RELATIVE MOTION PROBLEM	157
6.3	GAUSS VARIATIONAL EQUATIONS	164
6.4	GEOMETRICAL INTERPRETATION AND NEW DYNAMICAL EQUATIONS	165
6.5	THE EFFECTS OF PHOBOS'S GRAVITY AND LINEARIZED MODEL	170
6.6	AVERAGED EQUATIONS	174
6.7	DISCUSSION	178
7	FINAL COMMENTS	180
	BIBLIOGRAPHY	181
	APPENDIX A: DISCUSSION OF ATTITUDE EFFECTS OVER THE ORBITAL MOTION IN BINARY SYSTEMS	199
	APPENDIX B: PHASE SPACE ANALYSIS OF ROTO-ORBITAL INTERMEDIARIES	203

1 INTRODUCTION

This thesis presents the use of different mathematical techniques to deal with different problems in astrodynamics and celestial mechanics. The main topics of the work are: 1) The use of doubly-averaged potentials to be included in Lagrange planetary equations to study features of orbits around Mercury and Callisto; 2) The use of Perturbation Integrals to map the perturbations present in planetary systems; 3) The use of Intermediaries and elimination of the parallax on the roto-translational problem; 4) An updated complete Gauss variational equations to study quasi-satellite orbits around Phobos, with analytical predictions made using linearized and averaged equations.

1.1 MOTIVATION AND HISTORICAL OVERVIEW

Nowadays, numerical techniques are widely used to study a wide range of problems in celestial mechanics and astrodynamics, with applications in several fields, including the classical orbital and rotational motions of celestial objects. The large development of these techniques was especially possible due to the developments of faster computers, providing support for studies with applications as the long-term dynamics of astronomical bodies, spacecrafts or several space debris (see Efstathiou et al. (1985), Everhart (1985), Chambers (1999), Murray; Dermott (2000), Rosengren et al. (2015) and their references therein), the mapping to find stability criteria (LARA; SCHEERES, 2002) and the recovering of short period effects to design science orbits in strongly perturbed astrodynamics environments (WERNER; SCHEERES, 1997; SCHEERES, 2012). However, these numerical methods have the disadvantages of requiring its redesign for different bodies, i.e., one does not provide a model useful for applications to different systems and the fast way to identify general dynamical structures for a specific or generic class a objects.

In the other hand, the general technological development experienced by science in last decades, has been improving also the quality and precision of astronomical and astrodynamical measurements, which include data about orbital and rotational parameters of large number of celestial bodies. Surprisingly for those who expected the numerical methods to be replacing the analytical techniques in majority of the studies in celestial mechanics and astrodynamics, removing in many aspects the importance of these first one in a technological era, these improvements helped to bring a renewed interest about reviewing and updating classical theoretical models for the orbital and rotational motion of systems. The analytical and semi-analytical (a combination of analytical and numerical

methods) methods appeared with the potential to provide fast identification of several and useful motions for general physical systems that respect similar mathematical structures in their equations of motion. A good example can be cited regarding the development of analytical models to describe the inhomogeneous gravitational field of bodies with non-uniform distribution of mass, where the resulting potential can be used for the investigation of particular classes of useful orbital and rotational motions applicable to space missions, as well the study of dynamical structures of motions around different irregular objects, for instance the roto-orbital coupling between two asteroids, a planet and its moon. The Hill problem also provides another good example of the current importance of such studies, once its non-dimensionalized formulation makes it useful for applications in different classes of problems, since Sun-Earth-Moon motion (HILL, 1879) up to the study of spacecrafts in formation flight around a planet or small body (YAMANAKA; ARKERSEN, 2002; GIL; SCHWARTZ, 2010; BEZROUK; PARKER, 2017). On applying the classical perturbation theories to such problems, these systems can be further investigated in with significant precision, providing physical insights about the problem a very fundamental qualitative analysis of the system with mathematical predictions.

In this context, this thesis is written with the purpose to apply and highlight some important analytical and semi-analytical techniques which, nowadays, play an important role on new discoveries and advances of space science. The focus is directed on the use of techniques to deal with problems in both areas, astrodynamics and celestial mechanics, potentially benefiting space mission designing and planing, as well the understanding of the physics of celestial bodies. More details about each technique and the problem where they are applied for in this thesis, can be seen in the respective chapter.

1.2 OUTLINE OF THE WORK

This section presents details about each chapter of the thesis, providing the reader with an overview of its content.

Chapter 1: Introduction, presentation of the work developed throughout the thesis and academic production.

Chapter 2: This chapter studies the lifetime of orbits around a moon that is in orbit around its mother planet. In the context of the inner restricted three-body problem, the dynamical model considered in the present study uses the double-averaged dynamics of a spacecraft moving around a moon under the gravitational pulling of a disturbing third body in an elliptical orbit. The non-uniform distribution of mass of the moon is also considered. Applications are performed using numerical experiments for the Callisto-spacecraft-Jupiter system, and lifetime maps for different values of the eccentricity of the disturbing body (Jupiter) are presented, in order to investigate the role of this parameter in these maps. The idea is to simulate a system with the same physical parameters of the Jupiter-Callisto system, but with larger eccentricities. These maps are also useful for validation and improvements in the results available in the literature, such as to find

conditions to extend the available time for a massless orbiting body to be in highly inclined orbits under gravitational disturbances coming from the other bodies of the system.

Chapter 3: This chapter deals with a study that takes into account the non-sphericity of the central body, the perturbation of the third body and the solar radiation pressure to analyze the behavior of the orbit of a spacecraft when it has a solar sail around Mercury. We present an approach where we plot maps to analyze frozen orbits with longer lifetimes around Mercury. A set of initial conditions, which may contribute with the scientific missions planned to visit the planet Mercury in the next few years, are presented. Frozen orbits were found, i.e orbits with smaller variation of the orbital elements. It is also presented an approach to analyze the effect of the non-sphericity of Mercury on the motion of the spacecraft. In addition, the J_2 and J_3 zonal terms are also considered, as well as the C_{22} sectoral term.

Chapter 4: The design of scientific orbits for applications in space missions require a deep understanding about how the perturbations over the spacecraft will affect its dynamics. For such missions, the information about the energy transfered/removed to/from the spacecraft's orbit due to perturbative forces is very desired. In this context, this chapter presents the use of techniques which can provide measurements of these energies and perturbation maps of the regions that can be considered for use in different missions. The study proposes to make a detailed study of the orbits, in particular identifying the ones showing largest gains or losses of energy due to the presence of the planetary moons. The results are useful to help to find regions in these planetary systems that gives larger modifications in the energy of the spacecraft.

Chapter 5: The purpose of this chapter is to test two alternative models to study the roto-translational motion of an axisymmetric rigid body, considering this body under the influence of a central gravitational field. A Hamiltonian formalism is considered based on the total angular momentum and the canonical variables associated. The motion is shaped as a perturbation of the Keplerian motion plus the free-rotation of a rigid body. After using the reduction of the nodes, the concept of intermediary is used to propose two simplified Hamiltonians to analytically solve the motion, originally non-integrable. Parameters are introduced to visualize possibilities of applications to problems such as binary asteroids and artificial satellite attitude propagation. Numerical experiments compare the two intermediaries with respect to the original model. Several values for attitude, shape, orbital eccentricity and distance are considered for a systems with a slow-rotating axisymmetric rigid body in an eccentric relative orbit, i.e., with significant disturbing effects. In general, both models present good precision after hundreds of revolutions, with some advantage for the intermediary where the elimination of the parallax is applied. This research is an outcome of a period (December/2015 - November/2016) working as a research scholar at the University of Colorado Boulder (USA) under the supervisions of Prof. Sebastián Ferrer and Prof. Daniel J. Scheeres.

Chapter 6: This chapter deals with the application of perturbation theory to study

the astrodynamics of a spacecraft around a Martian moon. The astrodynamics of the Mars-Phobos system is a subject that has been challenging many scientists, once this system presents peculiar characteristics which do not permit long-term natural orbits around Phobos. Given its mass and relative position with respect to Mars, Phobos do not present enough gravity force to maintain a body orbiting itself. In this context, the present work intends to use perturbation theory to study QSOs (quasi-satellite orbits) around Phobos. Considering a system perturbed by Phobos, the model reformulated using alternative orbital elements allows the identification of perturbed long-term natural orbits in the relative motion with this Martian moon. These results are significant, once they are able to provide support for the plans of JAXA's Martian Moons Exploration Mission (MMX) to visit the system in mid of 2020's. This research is an outcome of a period (September/2017 - November/2017) working as a research intern at the Institute of Space and Astronautical Sciences of the Japan Aerospace Exploration Agency (ISAS/JAXA) under the supervision of Prof. Yasuhiro Kawakatsu.

It is worth of note that throughout each chapter, the results which are published/submitted/in preparation are cited below the figures. At the end of each chapter, the author indicates where that specific investigation was published/submitted to/presented in. Speaking more specifically: most of the material in chapters 2, 3, 4 is already published in scientific literature and the main results in chapters 5 and 6 were submitted to scientific journals; finally, the chapter 4 also shows results presented only in a scientific conference. When the material is not fitting one of these previous categories, its source is referred using only the word "Author".

1.3 BIBLIOGRAPHICAL PRODUCTION

The list of articles for journals and conferences produced as an outcome of the PhD program during the period of March of 2014 (the beginning of the PhD program) and September of 2018 (thesis final version) is presented in this section. In order to complete the presentation of the academic production of the candidate, a list of talks and abstracts of scientific meetings is also presented, as well the main prizes and recognitions which were received during this period. The complete list follows below:

1.3.1 Journal Papers

1. Baresi, N.; Dell'Elce, L.; **Cardoso dos Santos, J.**; Kawakatsu, Y., "**On the long-term evolution of mid-altitude quasi-QSO orbits around Phobos**", *Nonlinear Dynamics*, **Submitted**.

2. **Cardoso dos Santos, J.**; Ferrer, S.; Scheeres, D. J., "**Study of the roto-translational**

dynamics using Intermediaries: Numerical experiments", *Celestial Mechanics and Dynamical Astronomy*, Submitted.

3. Sarli, B. V.; Zabalaga, M. A. C.; Telgie, A. L.; **Cardoso dos Santos, J.**; Mesquita, B. D. R.; Roman-Gonzalez, A.; Ojeda, O.; Cuentas, N. I. V.; Aguilar, A., "**Review on Space Activities in South America**", *Journal of Aeronautical History*, 2018.

4. Venigalla, C.; Baresi, N.; Aziz, J.; Bercovici, B.; Motta, G. B.; Brack, D.; Bury, L.; **Cardoso dos Santos, J.**; Dahir, A.; Davis, A.; De Smet, S.; Fulton, J.; Parrish, N.; Pellegrino, M.; Van Wal, S., "**The Near-Earth Asteroid Characterization and Observation (NEACO) Mission**", *Advances in the Astronautical Sciences*, v. 162, 34 p., 2018.

5. **Cardoso dos Santos, J.**; Carvalho, J. P. S.; Prado, A. F. B. A.; Vilhena de Moraes, R., "**Lifetime of orbits around moons in elliptical orbits using a double-averaged model: applications to a Callisto orbiter**", *Astrophysics and Space Science*, 362: 227, 2017.

6. Carvalho, J. P. S.; **Cardoso dos Santos, J.**; Prado, A. F. B. A.; Vilhena de Moraes, R., "**Some characteristics of a spacecraft around Mercury**", *Computational and Applied Mathematics*, p. 1-15, 2017.

7. Prado, A. F. B. A.; **Cardoso dos Santos, J.**; Carvalho, J. P. S.; Vilhena de Moraes, R., "**Study of Perturbation Integrals applied to the dynamics of spacecrafts around Galilean moons**", *Advances in the Astronautical Sciences*, v. 158, p. 1565-1583, 2016.

8. **Cardoso dos Santos, J.**; Carvalho, J. P. S.; Vilhena de Moraes, R.; Prado, A. F. B. A., "**Semi-analytical spacecraft dynamics around planetary moons**", *Advances in the Astronautical Sciences*, v. 156, p. 569-584, 2016.

9. **Cardoso dos Santos, J.**; Carvalho, J. P. S.; Prado, A. F. B. A.; Vilhena de Moraes, R., "**Searching for less perturbed elliptical orbits around Europa**", *Journal of Physics. Conference Series (Print)*, v. 641, p.012011, 2015.

1.3.2 Conference Papers

1. Baresi, N.; Dell'Elce, L.; **Cardoso dos Santos, J.**; Kawakatsu, Y., "**Orbit Maintenance of Quasi-satellite Trajectories via Mean Relative Orbit Elements**", *International Astronautical Congress*, Bremen, Germany, 2018.

2. Carvalho, J. P. S.; **Cardoso dos Santos, J.** "**Analysis of the Critical Inclination in Orbit of Planetary Moons**", *Proceeding Series of the Brazilian Society of Computa-*

tional and Applied Mathematics, São José dos Rio Preto, Brazil, 2017.

3. Venigalla, C.; Baresi, N.; Aziz, J.; Bercovici, B.; Motta, G. B.; Brack, D.; Bury, L.; **Cardoso dos Santos, J.**; Dahir, A.; Davis, A.; De Smet, S.; Fulton, J.; Parrish, N.; Pellegrino, M.; Van Wal, S., "**The Near-Earth Asteroid Characterization and Observation (NEACO) Mission**", *Astrodynamics Specialist Conference*, Stevenson, Washington, 2017.

4. **Cardoso dos Santos, J.**; Prado, A. F. B. A.; Carvalho, J. P. S.; Vilhena de Moraes, R., "**Search for less disturbed regions for spacecrafts in planetary systems**", *SPACE 2016 - Astrodynamics Specialist Conference*, Long Beach, California. San Diego: Univelt, Inc., 2016.

5. Prado, A. F. B. A.; **Cardoso dos Santos, J.**; Carvalho, J. P. S.; Vilhena de Moraes, R., "**Study of Perturbation Integrals applied to the dynamics of spacecrafts around Galilean moons**", *27th Space Flight Mechanics Meeting*, Napa, California. San Diego: Univelt, Inc., 2016.

6. **Cardoso dos Santos, J.**; Carvalho, J. P. S.; Vilhena de Moraes, R.; Prado, A. F. B. A., "**Dynamics of orbits for space missions taking into account a disturbing body in an elliptical-inclined orbit: Applications to planetary moons**", *International Astronautical Congress*, Jerusalem, Israel, 2015.

7. Sarli, B. V.; Zabalaga, M. A. C.; Telgie, A. L.; **Cardoso dos Santos, J.**; Mesquita, B. D. R.; Jimenez, D. M.; Roman-Gonzalez, A.; Cruz, G. V. D. E. L.; Cuentas, N. I. V.; Perazzo, F., "**South American Space Era**", *International Astronautical Congress*, Jerusalem, Israel, 2015.

8. **Cardoso dos Santos, J.**; Carvalho, J. P. S.; Prado, A. F. B. A.; Vilhena de Moraes, R., "**Semi-analytical spacecraft dynamics around planetary moons**", *Astrodynamics Specialist Conference*, Vail, Colorado, 2015.

1.3.3 Book Chapters

1. Sarli, B. V.; Zabalaga, M. A. C.; Telgie, A. L.; **Cardoso dos Santos, J.**; Mesquita, B. D. R.; Roman-Gonzalez, A.; Ojeda, O.; Cuentas, N. I. V.; Aguilar, A. "**South American Space Era**", *History of Rocketry and Astronautics - AAS History Series.*, 1st ed., San Diego - California: Univelt, v. 47, p. 133-154, 2017.

1.3.4 Oral Presentations in Scientific Meetings

1. **Cardoso dos Santos, J.**, Title: *"Some characteristics of orbits around Mercury"*, **IX Taller de Ciencias Planetarias**, La Plata, Argentina, 2018.
2. **Cardoso dos Santos, J.**, Title: *"SGAC: Beyond a Network"*, **3rd South American Space Generation Workshop**, São José dos Campos, Brazil, 2017.
3. **Cardoso dos Santos, J.**, Title: *"Study of the roto-translational dynamics using Intermediaries: Numerical experiments"*, **VII International Meeting on Celestial Mechanics (CELMEC)**, Viterbo, Italy, 2017.
4. **Cardoso dos Santos, J.**, Title: *"Study of the roto-translational dynamics using Intermediaries: Numerical experiments"*, **SDSM 2017 - Satellite Dynamics and Space Missions: Theory and Applications of Celestial Mechanics**, Viterbo, Italy, 2017.
5. **Cardoso dos Santos, J.**, Title: *"Search for orbits in the Jupiter system to explore the Galilean moons"*, **67th International Astronautical Congress**, Guadalajara. Mexico, 2016.
6. **Cardoso dos Santos, J.**, Title: *"Search for less disturbed regions for spacecrafts in planetary systems"*, **SPACE 2016 - Astrodynamics Specialist Conference**, Long Beach, California, 2016.
7. **Cardoso dos Santos, J.**, Title: *"Study of Perturbation Integrals applied to the dynamics of spacecrafts around Galilean moons"*, **27th Space Flight Mechanics Meeting**, Napa, California, 2016.
8. **Cardoso dos Santos, J.**, Title: *"Orbital evolution of ejecta and debris caused by the impactor of Hayabusa 2"*, **COSPAR Capacity Building Workshop**, Guaratinguetá, Brazil, 2015.
9. **Cardoso dos Santos, J.**, Title: *"Dynamics of orbits for space missions taking into account a disturbing body in an elliptical-inclined orbit: Applications to planetary moons"*, **IAF International Student Workshop**, Israel Institute of Technology, Haifa, Israel, 2015.
10. **Cardoso dos Santos, J.**, Title: *"Dynamics of orbits for space missions taking into account a disturbing body in an elliptical-inclined orbit: Applications to planetary moons"*, **International Astronautical Congress**, Jerusalem, Israel, 2015.

11. **Cardoso dos Santos, J.**, Title: "*Semi-analytical spacecraft dynamics around planetary moons*", **Astrodynamics Specialist Conference**, Vail, Colorado, 2015.

12. **Cardoso dos Santos, J.**, Title: "*SGAC:Beyond a Network*", **XVII Brazilian Colloquium on Orbital Dynamics**, Águas de Lindóia, Brazil, 2014.

1.3.5 Abstracts published in Proceedings of Scientific Meetings

1. **Cardoso dos Santos, J.**; Carvalho, J. P. S.; Prado, A. F. B. A.; Vilhena de Moraes, R., "**Perturbation effects over a Mercury orbiter**", *International Astronautical Congress*, 2018, Bremen, Germany. Proceedings of the International Astronautical Congress, IAC, 2018.

2. Baresi, N.; **Cardoso dos Santos, J.**; Dell'Elce, L.; Kawakatsu, Y., "**Orbit maintenance of mid-altitude quasi-satellite orbits via mean relative orbit elements**", *28th JAXA Workshop on Astrodynamics and Flight Mechanics*, 2018, Sagamihara-chi, Japan. JAXA Repository.

3. Carvalho, J. P. S.; **Cardoso dos Santos, J.**; Prado, A. F. B. A.; Vilhena de Moraes, R., "**Some characteristics of orbits around Mercury**", *IX Taller de Ciencias Planetarias*, 2018, La Plata, Argentina. Libro de Resúmenes / Libro de Resumos, 2018.

4. **Cardoso dos Santos, J.**; Ferrer, S.; Scheeres, D. J., "**Study of the roto-translational dynamics using intermediaries: numerical experiments**", *The Seventh International Meeting on Celestial Mechanics (CELMEC)*, 2017, San Martino al Cimino, Italy. CELMEC Abstracts, 2017.

5. **Cardoso dos Santos, J.**; Ferrer, S.; Scheeres, D. J., "**Study of the roto-translational dynamics using intermediaries: numerical experiments**", *SDSM 2017 - Satellite Dynamics and Space Missions: Theory and Applications of Celestial Mechanics*, 2017, San Martino al Cimino, Italy. SDSM Abstracts, 2017.

6. **Cardoso dos Santos, J.**; Prado, A. F. B. A.; Carvalho, J. P. S.; Vilhena de Moraes, R., "**Search for orbits in the Jupiter system to explore the Galilean moons**", *International Astronautical Congress*, 2016, Guadalajara, Mexico. Proceedings of the International Astronautical Congress, IAC, 2016.

7. **Cardoso dos Santos, J.**; Carvalho, J. P. S.; Vilhena de Moraes, R.; Prado, A. F. B. A., "**Searching less perturbed elliptical orbits around Europa**", *XVII Colóquio Brasileiro de Dinâmica Orbital*, 2014, Águas de Lindóia, Brazil. XVII Colóquio Brasileiro de Dinâmica Orbital-Programa e Caderno de Resumos, 2014.

8. Vilhena de Moraes, R.; Carvalho, J. P. S.; Prado, A. F. B. A.; **Cardoso dos Santos, J.**, **"Manutenção de órbitas congeladas ao redor de satélites planetários"**, *XVII Colóquio Brasileiro de Dinâmica Orbital*, 2014, Águas de Lindóia, Brazil. XVII Colóquio Brasileiro de Dinamica Orbital-Programa e Caderno de Resumos, 2014.

9. Mourão, D. C.; Costa, L. F. C. C.; Carvalho, J. P. S.; Vilhena de Moraes, R.; **Cardoso dos Santos, J.**, **"Numerical study of orbits around Europa"**, *XVII Colóquio Brasileiro de Dinâmica Orbital*, 2014, Águas de Lindóia, Brazil. XVII Colóquio Brasileiro de Dinamica Orbital-Programa e Caderno de Resumos, 2014.

10. **Cardoso dos Santos, J.**; Carvalho, J. P. S.; Vilhena de Moraes, R., **"The dynamics of a Callisto orbiter under the influence of disturbing body"**, *Small Bodies Dynamics 2014*, 2014, Ubatuba, Brazil. Small Bodies Dynamics: Abstracts, 2014. p. 22-23.

11. **Cardoso dos Santos, J.**; Carvalho, J. P. S.; Vilhena de Moraes, R., **"Stable low-altitude polar orbits around Europa and Ganymede considering a disturbing body in an elliptical orbit"**, *Small Bodies Dynamics 2014*, 2014, Ubatuba, Brazil. Small Bodies Dynamics: Abstracts, 2014. p. 22.

12. **Cardoso dos Santos, J.**; Carvalho, J. P. S.; Vilhena de Moraes, R.; Prado, A. F. B. A., **"Analytical dynamics models for space missions around minor bodies"**, *IAU General Assembly*, 2015, Honolulu, USA. IAU General Assembly, Meeting 29. Washington: American Astronomical Society, 2015. p. 2252968.

13. **Cardoso dos Santos, J.**; Carvalho, J. P. S.; Vilhena de Moraes, R.; Prado, A. F. B. A., **"Space missions orbits around small worlds"**, *IAU General Assembly*, 2015, Honolulu, USA. IAU General Assembly, Meeting 29. Washington: American Astronomical Society, 2015. p. 2248240.

14. **Cardoso dos Santos, J.**; Vilhena de Moraes, R.; Carvalho, J. P. S., **"Stability of orbits around planetary satellites considering a disturbing body in an elliptical orbit: Applications to Europa and Ganymede"**, *40th COSPAR Scientific Assembly*, 2014, Moscow, Russia. Annals of COSPAR, 2014.

15. Vilhena de Moraes, R.; Carvalho, J. P. S.; Prado, A. F. B. A.; **Cardoso dos Santos, J.**, **"Mapping elliptical orbits around Europa"**, *40th COSPAR Scientific Assembly*, 2014, Moscow, Russia. Annals of COSPAR, 2014.

16. Mourão, D. C.; Carvalho, J. P. S.; Vilhena de Moraes, R.; **Cardoso dos Santos,**

J.; Costa, L. F. C. C., "**Numerical study of orbits around Europa**", *40th COSPAR Scientific Assembly*, 2014, Moscow, Russia. *Annals of COSPAR*, 2014.

17. Cardoso dos Santos, J.; Carvalho, J. P. S.; Vilhena de Moraes, R., "**Stable low-altitude orbits around Ganymede considering a disturbing body in a circular orbit**", *XIV Latin American Regional IAU Meeting*, 2014, Florianópolis. *Revista Mexicana de Astronomia y Astrofisica Conference Series*, 2014. v. 44. p. 130-130.

1.3.6 Awards and Recognitions

1. Travel Grant - Taller de Ciencias Planetarias/Meeting on Planetary Sciences, *Universidad Nacional de La Plata, Argentina, 2018.*

2. SGAC Member of the Month - February, *Space Generation Advisory Council, Austria/United Nations, 2018.*

3. SDSM 2017 Fellowship, *The Italian Society of Celestial Mechanics and Astrodynamics (SIMCA), Italy, 2017.*

4. COSPAR Capacity Building Workshop Fellowship, *Committee on Space Research (COSPAR), France, 2017.*

5. 1st place in the Student Competition (member of the CU Boulder/UNESP Winner Team) **of the 2017 Astrodynamics Specialist Conference,** *American Astronautical Society / American Institute of Astronautics and Aeronautics (AAS/AIAA), USA, 2017.*

6. IAC - Top 3 Paper in Astrodynamics (Interactive presentation section), *International Astronautical Federation, 2015.*

7. Research Internships Abroad (BEPE) Fellowship, *São Paulo Research Foundation (FAPESP), 2015.*

8. Indication as SGAC National Point of Contact for Brazil, *Space Generation Advisory Council, Austria/United Nations, 2015.*

9. PhD Fellowship, *São Paulo Research Foundation (FAPESP), 2014.*

2 DOUBLY-AVERAGED MODELS APPLIED TO THE DYNAMICS OF A CALLISTO ORBITER

The present chapter deals with the problem of using a semi-analytical approach to be applied on a study of lifetime maps. These last ones are intended to be used for a qualitative study of a dynamical system that represents third-body effects in the context of the restricted three-body problem with an inner massless body in an eccentric orbit. The study also intends to add the effects of the gravitational field due to a central body with ellipsoidal shape (J_2 and C_{22}), i.e., with a non-uniform gravitational potential, in order to improve the analysis and make this investigation helpful for further analysis such as space missions planning. For practical purposes, an application of the model which represents the physical forces mentioned previously is considered to study the motion of a spacecraft that is assumed to be an inner body that is orbiting Callisto in low altitude orbits (~ 100 km). Callisto is the Jupiter's fourth Galilean moon, and it is used as the central body of our system. The spacecraft orbiting this moon suffers the gravitational pulling of Jupiter, considered as the external (outer) disturbing body of the system. Indeed, since Jupiter is a very massive disturbing body, the Kozai-Lidov effects appear for orbits with high inclinations (PRADO, 2003), which is a dynamical behavior first discussed in the works of Kozai (1962) and Lidov (1962) in the framework of the quadrupolar secular restricted three-body problem considering a massless particle as the inner body of the system. This particle is assumed to be orbiting a massive central body under the gravitational attraction of a massive external third body.

It is important to mention that the quadrupolar secular restricted three-body problem presents two cases, which are the inner and the outer restricted case. In the inner restricted case, which corresponds to the dynamical architecture of the present study, the conservation of the normal component of the angular momentum enables the inner particle to periodically exchange its eccentricity with inclination, which is known as the Kozai-Lidov mechanism. The "inner" restricted three-body problem, as it is sometimes called (FARAGO; LASKAR, 2010), is a precise approach for a dynamical system when the inner body has a small mass in comparison with the other two bodies of the system (the central and the external/outer ones). This restricted model is no longer justified when in the presence of systems containing higher values for the mass ratios between their three bodies, which requires non-restricted models. For instance, it is possible to mention applications in the investigation of triple stars and triple asteroid systems (NAOZ et al., 2011; FANG et al., 2011; PRADO, 2014; ARAUJO; WINTER; PRADO, 2015; CAR-

VALHO et al., 2016A,B; NAOZ, 2016). Despite not being the model considered in the present study, it is worth to mention the opposite configuration, the one with a massless body orbiting a massive inner binary system and which is denoted by "outer" restricted problem (FERRER; OSACAR, 1994; BOUÉ; LASKAR, 2009; FARAGO; LASKAR, 2010; NAOZ et al., 2017). This model can be used when modeling the dynamics of many interesting problems, such as a spacecraft orbiting a binary planetary or an asteroid system or even a planet orbiting a binary star system (GIULIATTI WINTER et al., 2013; NAOZ, 2016).

Indeed, third-body perturbations have been object of several studies for many decades. Kozai (1959) obtained analytical expressions for the principal secular and long period terms of the disturbing function due to the lunisolar gravitational attractions. Musen; Bailie; Upton (1965) expanded this work including new terms in the disturbing function. Blitzler (1959) estimated lunisolar disturbances due to secular terms using methods of classical mechanics. Using the equatorial elements of the Moon, Kaula (1962) obtained the general terms of the disturbing function for the case of the lunisolar perturbations. Giacaglia (1973) calculated the secular long and short-period terms of the disturbing body, expressed in closed forms to eliminate the dependence from the true anomalies of the perturbed and perturbing bodies. Hough (1981) developed a study considering the effects of lunisolar perturbations on spacecraft orbits close to the critical inclinations with respect to the geopotential ($i = 63.4^\circ$ and $i = 116.6^\circ$).

Later, Broucke (2003) developed a second-order theory based on double-averaging techniques, extended by Prado (2003) to the fourth-order. Other works that are worth to be checked are Scheeres; Guman; Villac (2001), Domingos; Vilhena De Moraes; Prado (2008) and Liu; Baoyin; Ma (2012). They have considered this issue in order to develop approximated expressions for the third-body potential in order to get models that can represent full models, which would require numerical integrations with good level of accuracy. These more recent works have usually taken advantage of averaging techniques to study the third-body perturbations. Averaging methods have some advantages over full methods. They are usually faster in terms of computer time and effort. A more important point is that they are much less sensitive to initial conditions of the fast variables, as the initial true anomalies (f and f_p) of the spacecraft and third perturbing body. This approximation is valid when large number of revolutions are made by the spacecraft. For instance, for a given semi-major axis $a = 2510.3$ km (orbit altitude ~ 100 km), the spacecraft's orbital period around Callisto is around 2.59 hours, while Jupiter's orbital period around its moon is about 400.53 hours, i.e., near 154.64 times slower.

It is also important to cite the idea of developing lifetime maps to analyze models for the third-body perturbation, as done in Gomes; Domingos (2016) to investigate the models presented in Domingos; Vilhena De Moraes; Prado (2008) and Domingos; Prado; Gomes (2014). In this work, Gomes; Domingos (2016) analyzed orbits around the Moon. A similar idea is used in Carvalho et al. (2018) to study some characteristics of orbits

around the planet Mercury.

Regarding applications in modern space exploration, a system of planetary moons that has one of the largest number of celestial bodies with potential to be visited by space missions is the system of Jupiter, which has the Galilean moons, among others. A proposed mission that could visit some of its moons requires orbits having low-altitude and high inclination to better map their gravity fields and surfaces. Recently, projects from National Aeronautics Space Administration (NASA) and European Space Agency (ESA) have been under development ¹. The scientific objectives proposed for Europa and Ganymede, and that have been extended to Callisto, are: 1) characterization of the ocean layers and detection of subsurface water reservoirs; 2) topographical and geological analysis of the surface; 3) study of the physical properties of the icy crusts; 4) characterization of the internal mass distribution; 5) dynamics and evolution of the interiors; 6) investigation of Ganymede's tenuous atmosphere; 7) study of Ganymede's intrinsic magnetic field and its interactions with the Jovian magnetosphere. The three moons are believed to contain internal liquid water oceans, and so they can be considered central to understanding the habitability or not of icy worlds.

In this context, the present study proposes to use a model that considers the eccentricity of the disturbing body, which can be used to improve the results available in the literature by means of investigating the role of the eccentricity of the disturbing body in the lifetime of low-altitude orbits around planetary moons. Different values (real and hypothetical ones) for the eccentricity of Jupiter are adopted in order to study in more detail the importance of this parameter. It means that hypothetical systems will be created, using the same physical parameters from the Jupiter-Callisto system, but with hypothetical values for the eccentricity. It is also important to remember that the classical Keplerian/elliptical orbits are also disturbed by effects coming from the fact that the celestial bodies are not perfect spheres and that they do not have homogeneous distributions of mass. In order to improve the analysis, the effects due to the terms J_2 and C_{22} of the non-uniform gravitational potential of Callisto are also considered.

The basic structure of this chapter is shown next. In section 2.1 of the disturbing potentials is made; in section 2.2 the equations of motion are obtained; in section 2.3 it is described and analyzed the results of the numerical simulations, and, finally, in section 2.4, a discussion about the results is presented along with some conclusions.

2.1 DISTURBING POTENTIALS

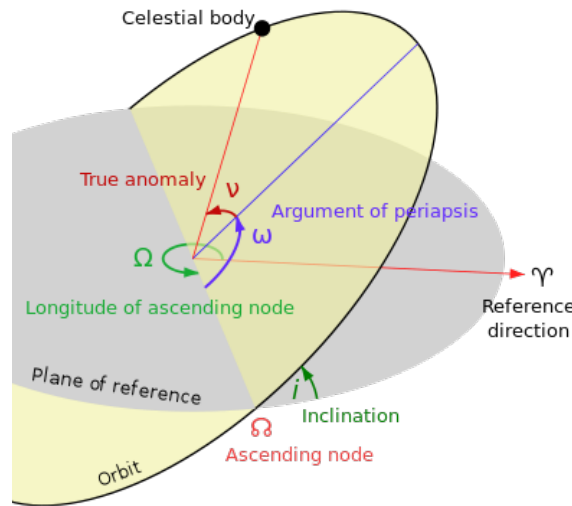
It is a classical result in astronomy that when there are no disturbing effects acting over the motion of a celestial body, it will follow a conic path around the center of mass of the system defined with another attracting body. These types of motions called orbit can be described by variables called orbital elements. A geometrical representation of

¹See: <http://sci.esa.int/juice/54994-juice-definition-study-report/> and <https://www.nasa.gov/europa/>

these variables for the case of an elliptical trajectory appears in figure 1. Indeed, the motion represented in this figure is called Keplerian motion in honour of the famous German astronomer Johannes Kepler (1571-1630) which formulated in one of its three famous laws about the planetary motion that *"The orbit of a planet is an ellipse with the Sun at one of the two foci"*. In fact, the center of mass of the system defined by the Sun and a planet is very close to Sun's center of mass (due to its much greater mass) and it is significantly close² to one of the foci of the ellipse defined by the motion. Indeed, the motion of two masses attracting each other, denoted by two-body problem, also admits parabolic and hyperbolic paths as possible solutions. However, these ones are out of the scope of the present thesis where we focus the study of perturbed Keplerian orbits. A more detailed discussion about the unperturbed orbital motion is made in chapter 5 using canonical variables.

In the case of the presence of other forces in the dynamical system (for instance the forces coming from other gravitational potentials), the motion of a small natural celestial body or a spacecraft can be described by the Lagrange planetary equations (KOVALEVSKY, 1967), as a function of the semi-major axis (a), eccentricity (e), inclination (i), argument of the periaapsis (g), longitude of the ascending node (h), mean anomaly (M), mean motion (n) and the disturbing potential R . By convenience of notation, we define $\omega \equiv g$, $\Omega \equiv h$ and $M \equiv \ell$.

Figure 1: Geometry of a Keplerian orbit



Source: Wikipedia (2018)

The six orbital elements are geometrically represented in figure 1. But, differently of what is presented in this figure and stated in the previous paragraph, we adopt to work with the variable M instead of f (denoted by true anomaly) presented in the figure. Both angles describe the motion of the celestial in its trajectory around one of the foci of the ellipse. However, the choice of working with one in favor of the other is made by two reasons: (1)

²A small difference happens especially due to Jupiter's mass attraction

we plan to include different disturbing potentials in the total disturbing potential R which appears in the planetary equations. However, each individual disturbing potential to be included is planned to be averaged using an integration in terms of the mean anomaly; (2) the fact that despite the angle f has a straight visual interpretation, it does not present an explicit time dependence, requiring the use of other classical relations of the celestial mechanics to determine the orbit motion in terms of time (see more details in Brouwer; Clemence (1961) and Murray; Dermott (2000)).

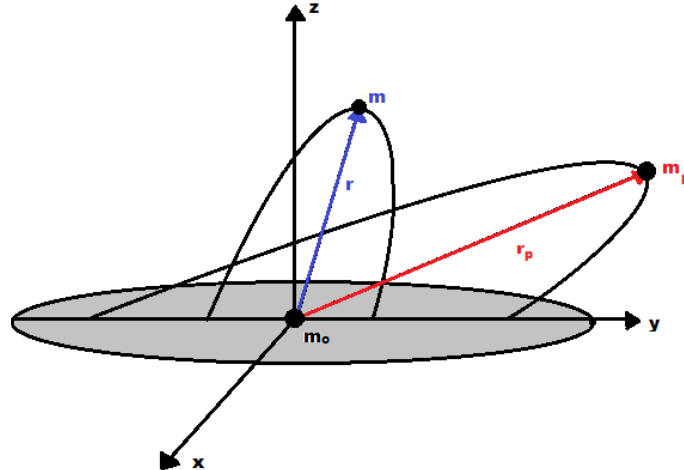
In the present study, the total disturbing potential R is considered as composed by three terms that have the main roles in disturbing the orbital motion of a celestial body: R_2 which is the perturbation due to a third body present in the system under study, as the case of a spacecraft orbiting the moon and that receives a perturbation from the Earth and the Sun; R_{J_2} , which is the term that represents the flattening at the poles of the central body of the system; $R_{C_{22}}$ which is the term that denotes the elliptical shape of the equator of the central body.

It is necessary to obtain these equations of motion for the present study, so it is required to obtain analytical expressions for the disturbing forces.

2.1.1 Doubly-averaged third-body perturbation

The first force considered is the one responsible for the disturbances caused by a third body in an inclined and eccentric orbit. The spacecraft (mass m) is considered to be orbiting a moon (mass m_0), which is assumed to be around a planet, with a mass m_p . The mean motion n appears in the Kepler's third law: $G(m_0 + m_p) = \mu_p = n_p^2 a_p^3$, where G is the gravitational constant assumed to be equal to $6.67259 \times 10^{-11} \text{ kg}^{-1} \text{ m}^3 \text{ s}^{-2}$. It is assumed that the disturbing body (planet) is in an elliptical and inclined orbit around the central body (moon) and its orbital elements are denoted by the subscript 'p': a_p, e_p, g_p, h_p, i_p and f_p . A geometrical representation of the system can be visualized in figure 2

Figure 2: Geometrical scheme of the dynamical system



Source: Cardoso dos Santos et al. (2016a)

The disturbing function due to a third body is given by (MURRAY; DERMOTT, 2000):

$$R = \frac{G(m_0 + m_p)}{|\mathbf{r}_p - \mathbf{r}|} = \frac{\mu_p}{r_p} \sum_{k=0}^{\infty} \left(\frac{r}{r_p}\right)^k P_k(\cos(S)) \quad (1)$$

where r and r_p are the spacecraft and the disturbing body orbital distances from the central body, respectively.

In this investigation, the quadrupolar model is used, once it is a satisfactory approach to the desired purposes of this study (PRADO, 2003). It means that expanding this disturbing potential up to second order ($k = 2$) in the Legendre polynomials P_k , it is possible to obtain a simplified dynamical system. Thus, it is necessary to calculate the term of the potential denominated R_2 , which is related to $P_2(\cos(S))$. Thus, we have

$$R_2 = \frac{\xi^2 \mu_p}{a_p} \left(\frac{r}{a}\right)^2 \left(\frac{a_p}{r_p}\right)^3 P_2(\cos(S)) \quad (2)$$

so that

$$P_2(\cos(S)) = \frac{3 \cos^2(S) - 1}{2} \quad (3)$$

where S is the angle between the position vectors $\mathbf{r}$ e $\mathbf{r}_p$.

Having this information and taking into account the geometry of the problem observed in figure 2, it is possible to obtain the relation which is a scalar product used to know the value of S :

$$\cos(S) = \frac{\mathbf{r} \cdot \mathbf{r}_p}{r r_p} \quad (4)$$

Once the angle S is given by the difference of the true longitudes of the spacecraft and the third (disturbing) body, ϖ_p and ϖ , respectively, we have:

$$S = (f_p - \varpi_p) - (f - \varpi) \quad (5)$$

By mean of both relations and after rearranging the terms, it is possible to obtain the relation between the angle S and the spacecraft's true anomaly f , which is the angle measured from one focus of the orbit between the reference direction adopted for the periapsis and the position vector of the body in its orbit. Thus, $\cos(S)$ can be obtained as

$$\cos(S) = \alpha \cos(f) + \gamma \sin(f) \quad (6)$$

where α and γ are auxiliary variables which change according to different assumptions made for the motion of the spacecraft and the third body, namely values for orbital eccentricity and/or inclination. The fast angle f due to the motion of the spacecraft, which produces short-period effects in this motion, appear explicitly in this expression (6).

Table 1: Orbital inclinations of some planetary moons

Planet	Planetary moons	Inclination (deg)
Jupiter	Europa	0.466
Jupiter	Ganymede	0.177
Jupiter	Callisto	0.192
Saturn	Iapetus	8.298
Uranus	Miranda	4.338
Neptune	Triton	156.865

Source: Author. Data collected from Nasa/SSD (2018)

For a dynamical system considering the both bodies in eccentric and inclined orbits, the coefficients α and γ assume the following values (LIU; BAOYIN; MA, 2012):

$$\alpha = \alpha_1 \cos(g_p + f_p) + \alpha_2 \sin(g_p + f_p) \quad (7)$$

$$\gamma = \gamma_1 \sin(g_p + f_p) - \gamma_2 \cos(g_p + f_p) \quad (8)$$

where the fast angle f_p due to the motion of the disturbing body appear explicitly in expressions (7) and (8). Slow angles appear in expressions

$$\alpha_1 = \cos(g) \cos(h - h_p) - \sin(g) \cos(i) \sin(h - h_p) \quad (9)$$

$$\begin{aligned} \alpha_2 = & \sin(g) \sin(i) \sin(i_p) + \cos(g) \cos(i_p) \sin(h - h_p) \\ & + \sin(g) \cos(i) \cos(i_p) \cos(h - h_p) \end{aligned} \quad (10)$$

$$\begin{aligned} \gamma_1 = & \cos(g) \sin(i) \sin(i_p) - \sin(g) \cos(i_p) \sin(h - h_p) \\ & + \cos(g) \cos(i) \cos(i_p) \cos(h - h_p) \end{aligned} \quad (11)$$

$$\gamma_2 = \sin(g) \cos(h - h_p) + \cos(g) \cos(i) \sin(h - h_p) \quad (12)$$

It is possible to eliminate the fast angles of the problems (short-period terms), *e.g.*, the mean anomalies of the bodies to get a simplified doubly-averaged model with potential to provide qualitative analysis of the system. This averaged system avoids sensibilities in initial conditions for the orbital position of the two orbiting bodies, along with the significant gain of computer processing time. The disturbing function of the problem is averaged independently over the mean anomalies of the spacecraft and the third body. It is a good model when the spacecraft makes large number of revolutions. The standard

definition for average used in this work is

$$\langle F \rangle = \frac{1}{2\pi} \int_0^{2\pi} (F) d\ell \quad (13)$$

where ℓ is the mean anomaly of the spacecraft.

The development of the equations is carried out in closed form to avoid expansions in eccentricity and inclination. Also, as the disturbing potential is a function of the true anomaly (see expressions (2) and (6)), it is necessary to apply a change of variables to represent the expression (13) in terms of the true anomaly. In order to satisfy these purposes, it is necessary to perform algebraic manipulations where we use known equations from celestial mechanics, namely equations (14) and (15).

$$\frac{a}{r} = \frac{(1 + e \cos(f))}{(1 - e^2)} \quad (14)$$

$$d\ell = \frac{1}{\sqrt{1-e^2}} \frac{r^2}{a^2} df \quad (15)$$

and then, the averaging technique is applied as a function of the true anomaly using equation (15). Again, this is done to eliminate the short-period terms.

After averaging over the mean anomalies (YOKOYAMA, 1999) of the spacecraft (ℓ) and the third body (ℓ_p), the disturbing potential takes the following form

$$\langle \langle R_2 \rangle \rangle = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} (R_2) d\ell d\ell_p \quad (16)$$

then, it is possible to get the following model (TRESACO; ELIPE; CARVALHO, 2016; CARVALHO, 2016):

$$\langle \langle R_2 \rangle \rangle = -\frac{15}{16} n_p^2 a_p^2 (A + B + C + D + E + F + G + H) \quad (17)$$

with

$$A = \frac{1}{4} e^2 (\cos(i) - 1)^2 (\cos^2(i_p) - 1) \cos(2g - 2h + 2h_p) \quad (18)$$

$$B = \frac{1}{4} e^2 (\cos(i) - 1)^2 (\cos^2(i_p) - 1) \cos(2g + 2h - 2h_p) \quad (19)$$

$$C = \frac{e^2}{2} \sin(i) \sin(i_p) (\cos(i) - 1) \cos(2g - h + h_p) \quad (20)$$

$$D = \frac{e^2}{2} \sin(i) \sin(i_p) (\cos(i) + 1) \cos(2g + h - h_p) \quad (21)$$

$$E = -\frac{3}{10} \left(e^2 + \frac{2}{3} \right) (\cos^2(i) - 1) (\cos^2(i_p) - 1) \times \cos(2h - 2h_p) \quad (22)$$

$$F = -\frac{3}{10} \left(e^2 + \frac{2}{3} \right) \sin(2i) \sin(2i_p) \cos(2h - 2h_p) \quad (23)$$

$$G = \frac{1}{2} e^2 \left[(\cos^2(i) - 1) (\cos^2(i_p) + 1) + 2 \sin^2(i) \sin^2(i_p) \right] \cos(2g) \quad (24)$$

$$H = -\frac{3}{10} \left(e^2 + \frac{2}{3} \right) \left[(\cos^2(i_p) + 1) \cos^2(i) + (\cos^2(i_p) - \frac{5}{3}) + 2 \sin^2(i) \sin^2(i_p) \right] \quad (25)$$

where the subscript p denotes variables referred to the disturbing body (Jupiter in our numerical experiments) and n_p is the mean motion of this body. The original disturbing functions are given in terms of the true anomalies. To compute the double integral in the averaging process, which is presented in terms of mean anomalies, the following known relations were used (see Brouwer; Clemence (1961))

$$\sin(f) = \frac{\sqrt{1 - e^2} \sin(E)}{1 - e \cos(E)} \quad (26)$$

$$\cos(f) = \frac{\cos(E) - e}{1 - e \cos(E)} \quad (27)$$

$$r = a (1 - e \cos(E)) \quad (28)$$

$$d\ell = (1 - e \cos(E)) dE \quad (29)$$

with E denoting the eccentric anomaly of the spacecraft. Analogous orbital parameters carrying the subscript p represent parameters referred to the third body.

2.1.2 Perturbations due to central body non-uniform shape

Since the orbits required for scientific missions around a planetary moon are desired to be low altitude (~ 100 km) ones and also near-circular and near-polar, the present

study concentrates in this type of orbits. It means that the present study needs to take into account the perturbation due to the non-uniform distribution of mass of the central body. The effects due to planetary moon flattening at the poles (J_2) and due to the elliptic shape of its equator (C_{22}) are included, as performed in previous works in the literature (SCHEERES; GUMAN; VILLAC, 2001; PASKOWITZ; SCHEERES, 2006; LARA; RUSSELL, 2006; CARVALHO et al., 2012A,B; VILHENA DE MORAES et al., 2016).

Considering the equatorial plane of the central body (here a planetary moon) as the reference plane, the disturbing potential related to its distribution of mass can be written in the form:

$$U_B = -\frac{\mu}{r} \left[\sum_{n=2}^{\infty} \left(\frac{R_p}{r} \right)^n J_n P_n(\sin(\phi)) - \left(\frac{R_B}{r} \right)^2 C_{22} P_{22}(\sin(\phi)) \cos(2\lambda) \right] \quad (30)$$

where μ is the gravitational constant of the body, R_B is the equatorial radius of the body, P_n are the Legendre polynomials, P_{nm} the associated Legendre polynomials, the angle ϕ is the latitude of the orbit with respect to the equator of the body, the angle λ is the longitude measured from the direction of the longest axis of the planet. Using spherical trigonometry, it is also possible to show that $\sin(\phi) = \sin(i) \sin(f + g)$.

In the present study, we take into account the disturbance due to the flattening at the poles of the central body, so that the sum in expression (30) is made up $n = 2$. Thus, the Legendre polynomial for the zonal term J_2 and the Legendre associated function for the sectoral term C_{22} can be written in the form (GIACAGLIA; MURPHY; FELSENTREGER, 1970)

$$P_2(\sin(\phi)) = \frac{1}{2}(3s^2 \sin^2(f + g) - 1) \quad (31)$$

$$P_{22}(\sin(\phi)) \cos(2\lambda) = 6(\xi^2 \cos^2(f) + \chi^2 \sin^2(f)) + \xi\chi \sin(2f) - 3(1 - s^2 \sin^2(f + g)) \quad (32)$$

with shortcuts $\xi = \cos(g) \cos(h) - c \sin(g) \sin(h)$, $\chi = -\sin(g) \cos(h) - c \cos(g) \sin(h)$, $s = \sin(i)$ and $c = \cos(i)$.

Now, it is possible to write the potential given in equation (30) as a function of the orbital elements. For this purpose, one can use equation (31) for the zonal term J_2 and the relation $\mu = n^2 a^3$, such that

$$U_{20} = -\frac{1}{2} \frac{a^3}{r^3} \epsilon n^2 (3s^2 \sin^2(f + g) - 1) \quad (33)$$

where the parameter ϵ is given by $\epsilon = J_2 R_B^2$.

Analogously, it is possible to take equation (32) for the sectoral perturbation (GI-

ACAGLIA; MURPHY; FELSENTRER, 1970). Thus,

$$U_{22} = \frac{a^3}{r^3} \delta n^2 (6\xi^2 \cos^2(f) + 6\chi^2 (\sin(f))^2 + 12\xi\chi \sin(2f) - 3 + 3s^2 \sin^2(f + g)) \quad (34)$$

where the parameter δ is given by $\delta = C_{22} R_B^2$.

Since f is the true anomaly of the artificial satellite, one can make the choice of working with a long-period dynamical model, as done for the third-body potential. It requires averaged disturbing potentials, so the concept of averaging method (see equation (13)) is again applied for the models presented in equations (33) and (34) to obtain a single-averaged model where the short-period terms are eliminated. This fact accelerates the integration of the equations of motion and provides a qualitative analysis of the system by avoiding sensibility related to the initial conditions. The development of the equations is carried out in closed form to avoid expansions in eccentricity and inclination. Thus, it is necessary to perform algebraic manipulations where known equations from celestial mechanics are used, namely equations (14) and (15).

After applying the averaging process in equation (33) and after some algebraic manipulations, the averaged disturbing potential due to the flattening at the poles becomes

$$\langle R_{J_2} \rangle = -\frac{1}{4} \frac{\epsilon n^2}{(1 - e^2)^{3/2}} (3 \sin^2(i) - 2) \quad (35)$$

where n is the mean motion of the spacecraft (see also Cardoso dos Santos et al. (2015)).

Using equation (34), the variables ξ and χ to develop the potential due to the equatorial ellipticity of the central body, we get

$$\langle R_{C_{22}} \rangle = -\frac{3}{2} \frac{\delta n^2}{(1 - e^2)^{3/2}} (\cos^2(i) - 1) \cos(2h) \quad (36)$$

In equation (36) the longitude of the ascending node h is replaced by the expression $h = h_0 - \omega t$ given by Meyer; Buglia; Desai (1994), Giacaglia (1973) and Sehnal (1960) (see also Carvalho et al. (2018)), where t is the time and ω is the body's rotation speed. This expression $h = h_0 - \omega t$ (see figure A1 in Meyer; Buglia; Desai (1994)) relates the longitude of the ascending node of the orbit with respect to Callisto (h) to the inertial longitude of the ascending node (h_0).

A brief presentation of the perturbations will appear again in next chapter 3, where it will included models for the central body's pear-shape irregular distribution of mass and the solar radiation pressure. Both models will be added to deal with the dynamics of a solar sail around Mercury .

2.2 EQUATIONS OF MOTION

Case the averaging technique is applied a second time in terms of the mean anomaly of the third body (ℓ_p) to expressions (35) and (36), we will get $\langle\langle R_{J_2} \rangle\rangle = \langle R_{J_2} \rangle$ and $\langle\langle R_{C_{22}} \rangle\rangle = \langle R_{C_{22}} \rangle$. It happens because the variable ℓ_p does not appear in expressions (35) and (36).

Thus, after obtaining the potentials for modeling the physical system of our interest, the total disturbing potential

$$R = \langle\langle R_2 \rangle\rangle + \langle R_{J_2} \rangle + \langle R_{C_{22}} \rangle = \langle\langle R_2 + R_{J_2} + R_{C_{22}} \rangle\rangle \quad (37)$$

This total disturbing potential given by expression (37) is than replaced on the classical Lagrange planetary equations (KOVALEVSKY, 1967) in order to numerically integrate the motion of the spacecraft under these gravitational effects. Those equations are shown below

$$\frac{da}{dt} = \frac{2}{na} \frac{\partial R}{\partial \ell} = 0 \quad (38)$$

$$\frac{de}{dt} = \frac{-\sqrt{1-e^2}}{na^2e} \frac{\partial R}{\partial g} + \frac{1-e^2}{na^2e} \frac{\partial R}{\partial \ell} = \frac{-\sqrt{1-e^2}}{na^2e} \frac{\partial R}{\partial g} \quad (39)$$

$$\frac{di}{dt} = \frac{1}{na^2\sqrt{1-e^2}\sin(i)} \left(\frac{\partial R}{\partial g} \cos(i) - \frac{\partial R}{\partial h} \right) \quad (40)$$

$$\frac{dh}{dt} = \frac{1}{na^2\sqrt{1-e^2}\sin(i)} \frac{\partial R}{\partial i} \quad (41)$$

$$\frac{dg}{dt} = \frac{\sqrt{1-e^2}}{na^2e} \frac{\partial R}{\partial e} - \frac{\cos(i)}{na^2\sqrt{1-e^2}\sin(i)} \frac{\partial R}{\partial i} \quad (42)$$

where the derivative $\frac{\partial R}{\partial \ell}$ is null due to the elimination of the mean anomaly ℓ in the averaging process.

2.3 RESULTS

The present study intends to analyze the magnitude of the effects of the third body's eccentricity over a massless spacecraft orbiting a central body. The effects of J_2 and C_{22} are considered in this analysis and different values for the orbital eccentricity of the third body are assumed, such that the results are able to provide some insights about the dynamics of a spacecraft around a body with elliptical shape orbiting a massive body. The Callisto-Jupiter system is used in the present study and the ideas developed here can be

extended to other systems, such as moons around massive planets, asteroids (e.g. Near-Earth Asteroids, also known as NEAs) and comets with significant close approaches with respect to the Sun. The purpose of the present investigation is attained by making lifetime maps for different values of the eccentricity of the disturbing body.

This procedure helps the search for low-altitude near-polar natural orbits around Callisto (as well for other bodies) that survive for long periods of time, which we assume in the present work as lifetimes longer than one year. This search for orbits with longer lifetimes is made by mapping different initial conditions for the orbital elements due to a spacecraft orbiting the moon. Such orbits are affected by disturbances due to the non-sphericity of the central body and especially by the Kozai-Lidov mechanism (KOZAI, 1962; LIDOV, 1962) coming from the third body (here assumed to be Jupiter). This is a known effect that allows a body to periodically exchange its inclination with eccentricity, which creates a significant decrease in the lifetimes to values of tens or a few hundreds of days (PRADO, 2003) for the case of low altitude orbits (~ 100 km). Although the effect coming from the third body makes the eccentricity to grow, the disturbing effects coming from the irregular shape of the central body (J_2 and C_{22}) tends to counterbalance the third body disturbances at some level. However, the term C_{22} sometimes can contribute with the third body's effect by counterbalancing the effect of J_2 (see figure 7). The final impact on the dynamics presents some increase in the orbit lifetimes in the presence of significant values for these higher gravitational inhomogeneities, especially considering J_2 . Besides, as seen in previous work (see Carvalho; Vilhena De Moraes; Prado (2010) and references therein), the presence of coupling effects due to the C_{22} perturbation may imply in lowering the periapsis altitude due to the periodic oscillations produced in the eccentricity. In order to provide useful information for mission-planning purposes, it is necessary to incorporate the more relevant disturbing effects in the dynamical model, which in the present investigation justifies the presence of effects due to J_2 and C_{22} , with their high and same order values.

Previous works available in the literature (SCHEERES; GUMAN; VILLAC, 2001; BROUCKE, 2003; PRADO, 2003) usually consider a circular planar doubly-averaged model for the third-body perturbation, which is a good approximation for the Galilean moons and other moons of the Solar System. However, it is possible to have some insights about the dynamics under the effects of a disturbing body in eccentric orbit by means of creating hypothetical systems with different values for its orbital eccentricity (e_p), in order to measure how strong is the impact of this parameter in the lifetimes of the orbits of a spacecraft around the central body, assumed to have an ellipsoidal shape. Therefore, selecting hypothetical different eccentricities for Jupiter's orbit, and generating these lifetime maps for a spacecraft orbiting Callisto considering these different scenarios, is also an important opportunity to test the present doubly-averaged model that considers the eccentricity and the inclination of the disturbing body. Due to the purpose of having a model which can be also useful for orbiters around NEAS and comets, especially having in mind these last

Table 2: Values of J_2 , C_{22} and mean radius for Callisto

J_2	C_{22}	Mean radius (km)
3.27×10^{-5}	1.02×10^{-5}	2410.3

Source: Author. Data collected from Anderson et al. (2001)

Table 3: Jupiter's data around Callisto

a_p (km)	e_p	g_p (deg)	h_p (deg)	i_p (deg)	n_p (deg/day)
1882700	0.0074	52.643	63.552	0.192	21.5710728

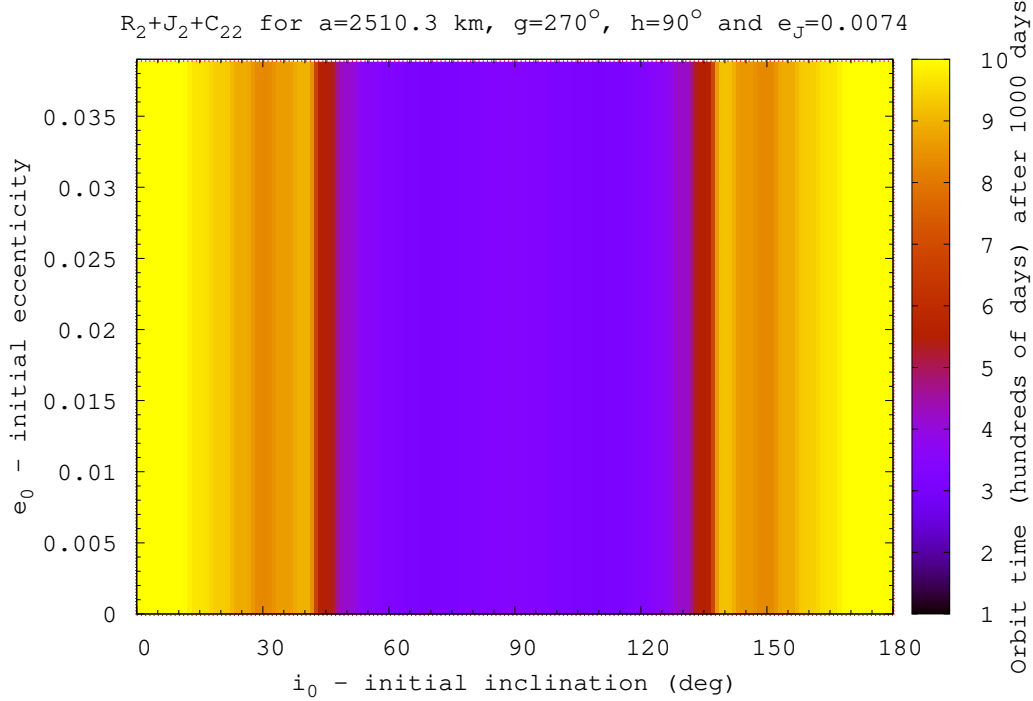
Source: Author. Data collected from Nasa/SSD (2018)

ones as objects with rather eccentric orbits under the massive Sun's gravitational field, the investigation of the role of disturber's inclination (i_p) is left for a next communication where other discussions including critical inclination and other important issues can be more properly treated, and also with applications to different celestial bodies. As Callisto's motion around Jupiter is almost planar, and the same happens with the majority of the moons with higher interest for future space exploration, the discussion about the role of the disturber's inclination by mean of generating lifetime maps is, presently, delayed.

In order to test this model for the third-body perturbation, it was integrated and mapped orbits obtained under different initial conditions for the inclination (i), varying between 30° and 150° , and semi-major axis (a) varying between 2460.3 km and 2510.3 km. For the mean radius of Callisto $R_B = 2410.3$ km (ANDERSON et al., 2001), these values of semi-major axis ($a = 2460.3$ km and $a = 2510.3$ km) correspond to average altitudes of 50 km and 100 km, respectively. The values for the gravity parameters (J_2 and C_{22}) can be seen in Table 2. The orbital parameters for the Callisto-Jupiter system are presented in Table 3. The initial conditions for other orbital elements of the spacecraft orbit are $e = 0.01$, $g = 270^\circ$ and $h = 90^\circ$. In the following figures, where one reads e_J denotes e_p for the case where the third body is Jupiter.

The figure 3 shows a first test of lifetime maps considering the models presently introduced, where the value chosen for the orbital eccentricity of the spacecraft (at least in the range shown in this figure due to the low-altitude assumed) does not impact the lifetimes as significantly as the choice of a value of the inclination when it is close to a polar orbit ($i = 90^\circ$). The range available for the spacecraft's orbital eccentricity is small due to the fact that the spacecraft is only about 100 km of altitude (with critical eccentricity $e_{max} \sim 0.0398$), which makes the values of the lifetime very sensitive in such dynamics. In this scenario, it creates fast collisions with the surface of Callisto under small increases of the perturbations over the orbital eccentricity. Therefore, figure 3 presents results for a lifetime map as a function of initial eccentricity and initial inclination. It is possible to note in the results the expected effect of the lifetimes decreasing with the increasing of the value of the initial eccentricity. The Kozai-Lidov effect is present in decreasing the lifetime of orbits closer to 90° of inclination (darker colors are close to this angle). An

Figure 3: Lifetime map (e_0 vs i_0) considering the effects of $R_2+J_2+C_{22}$. Initial conditions: $a_0 = 2510.3$ km, $g_0 = 270^\circ$, $h_0 = 90^\circ$ and $e_p = 0.0074$. The vertical bar in the right gives the value of the lifetimes in hundreds of days for an integration of 1000 days. Each unity corresponds to a hundred of days.

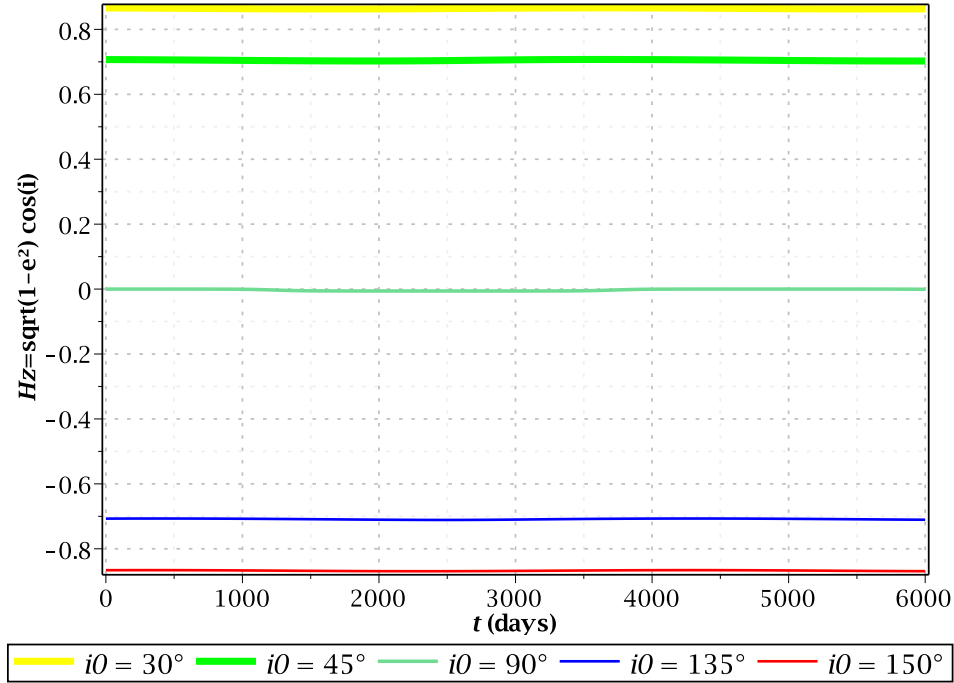


Source: Cardoso dos Santos et al. (2017)

extra note is to explain that the orbits do not survive for initial eccentricities higher than $e = 0.039$ due to the fact that this limit value of eccentricity corresponds to a distance of the periapsis equal to the mean radius of Callisto, therefore, satisfying the collision condition.

The conservation of $H_z = \sqrt{1 - e^2} \cos(i)$, i.e., the component of the spacecraft's orbital angular momentum, which is parallel to the Jupiter's angular momentum, is shown in figure 4. Indeed, under the regime of this restricted dynamical model, the conservation of this quantity allows the connection of the two parameters, orbital eccentricity and inclination, as being "exchanged" or "traded", one parameter by the other. Therefore, for spacecrafts with orbits which are near-circular and highly inclined, it is possible to occur that they become very eccentric. Thus, this growth of the eccentricity allows: 1) Impact with other bodies of the system; 2) Eccentricity increasing until the orbit reaches enough energy to escape from the system. For the more general case, where the spacecraft is replaced by natural celestial bodies orbiting others under a third-body perturbation, it is possible to occur the destruction of this celestial body due to the tidal forces, once the increasing eccentricity will make the celestial body's periapsis to pass/cross inside the Roche limit related to the more massive central body. Another important issue to mention is that the exchange of eccentricity and inclination creates oscillations which generates impacts in lower altitude orbits, since they are closer to the central body's surface. These

Figure 4: Propagation of the z -component of the orbital angular momentum $H_z = \sqrt{1 - e^2} \cos(i)$. Initial conditions: $a_0 = 2510.3$ km, $g_0 = 270^\circ$, $h_0 = 90^\circ$ and $e_p = 0.0074$. Effects due to R_2 . Numerical propagation for 6000 days.

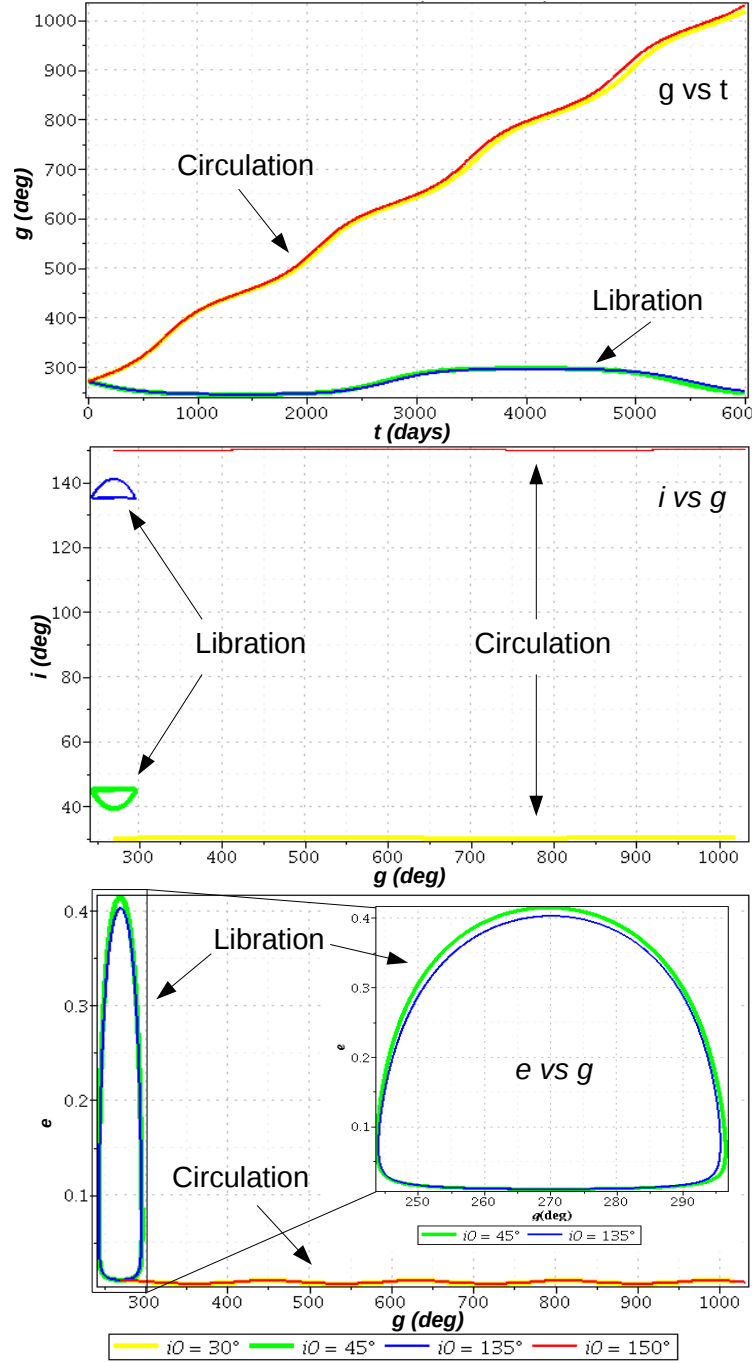


Source: Cardoso dos Santos et al. (2017)

oscillations can be seen in figure 5. The plots presented in this figure are in accordance with another expected effect due to the Kozai-Lidov mechanism, which is to cause a libration of the argument of the periaapsis g around angles $g = 90^\circ$ or $g = 270^\circ$. Figure 6 presents the components of the eccentricity vector integrated for a period of 100 years. In fact, the plots in figure 5 help to interpret the results in figure 6, with high inclined orbits (i between 45° and 135°) presenting a libration of the argument of periaapsis in the approximate interval of $g = 243^\circ$ and $g = 297^\circ$. This behavior is a result of the Kozai-Lidov mechanism, which makes g librates when orbits are highly inclined. Orbits with low-inclination present the argument of periaapsis circulating, which explain the circles and ellipses covering the full range of 360° in the second and third plots in figure 6.

The following test was made for models considering different perturbing forces on the disturbing potential: (1) R_2 ; (2) $R_2 + J_2$; (3) $R_2 + J_2 + C_{22}$. Figure 7 presents the results for lifetime maps considering these three scenarios. From scenario (1), with perturbation coming from the third body, until scenario (3) with the sum of the effects due to the non-uniform distribution of mass of Callisto. The result for the scenario (2) present a map that is similar to those referred to cases (1) and (3), however with some significant differences in the range of the inclination that produce longer lifetimes. As expected, J_2 acts as a natural protecting mechanism by smoothing the disturbing effects coming from the third body. This test was made considering the real value of the eccentricity for the Jupiter-

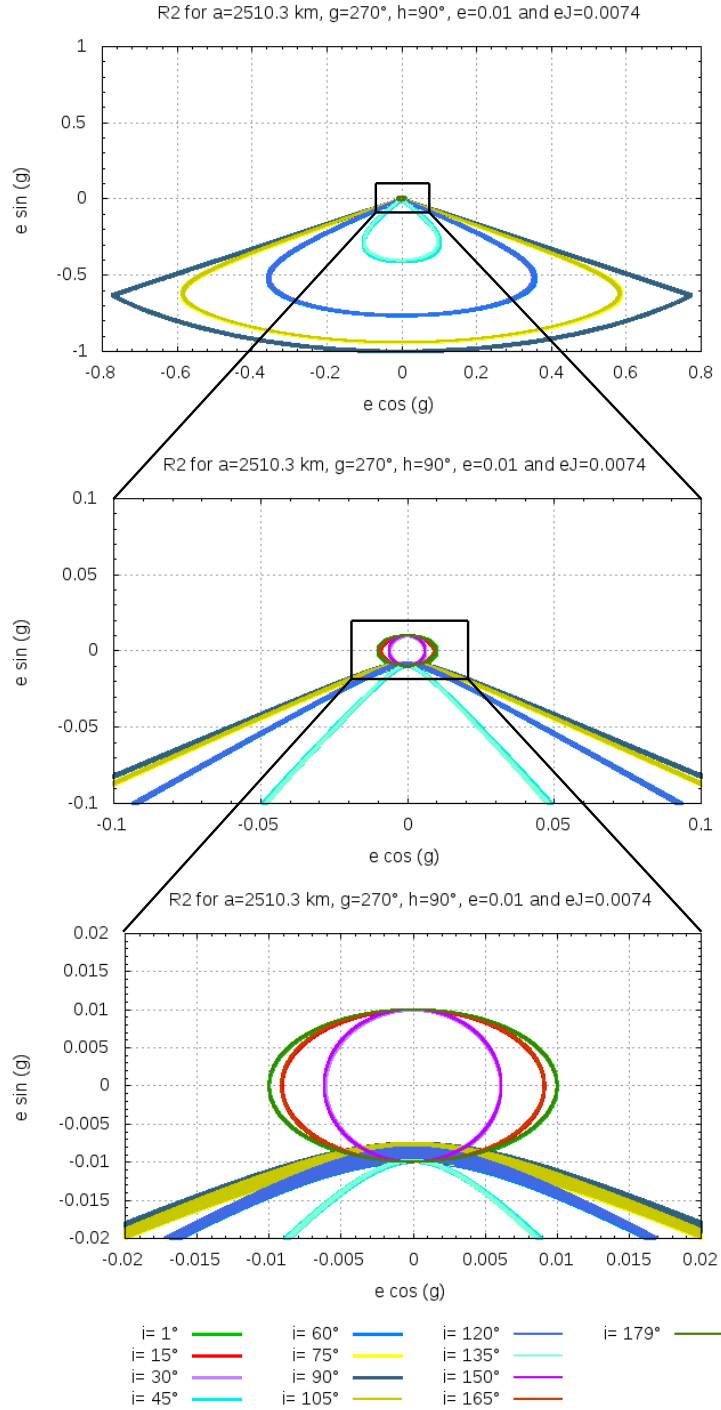
Figure 5: Propagation of the eccentricity (e), inclination (i) and argument of the periapsis (g) considering different values for the initial inclination i_0 . Initial conditions: $a_0 = 2510.3$ km, $e_0 = 0.01$, $g_0 = 270^\circ$, $h_0 = 90^\circ$ and $e_p = 0.0074$. Effects due to R_2 . Numerical propagation for 6000 days.



Source: Cardoso dos Santos et al. (2017)

Callisto system ($e_p = 0.0074$). The results show the third-body disturbance as the main perturbation in this system, especially due to the mass of the disturbing body Jupiter and the smaller magnitude of the effects due to the gravity terms J_2 and C_{22} of Callisto. The presence of extra perturbations in the system, as the case of terms like J_2 , have a role in

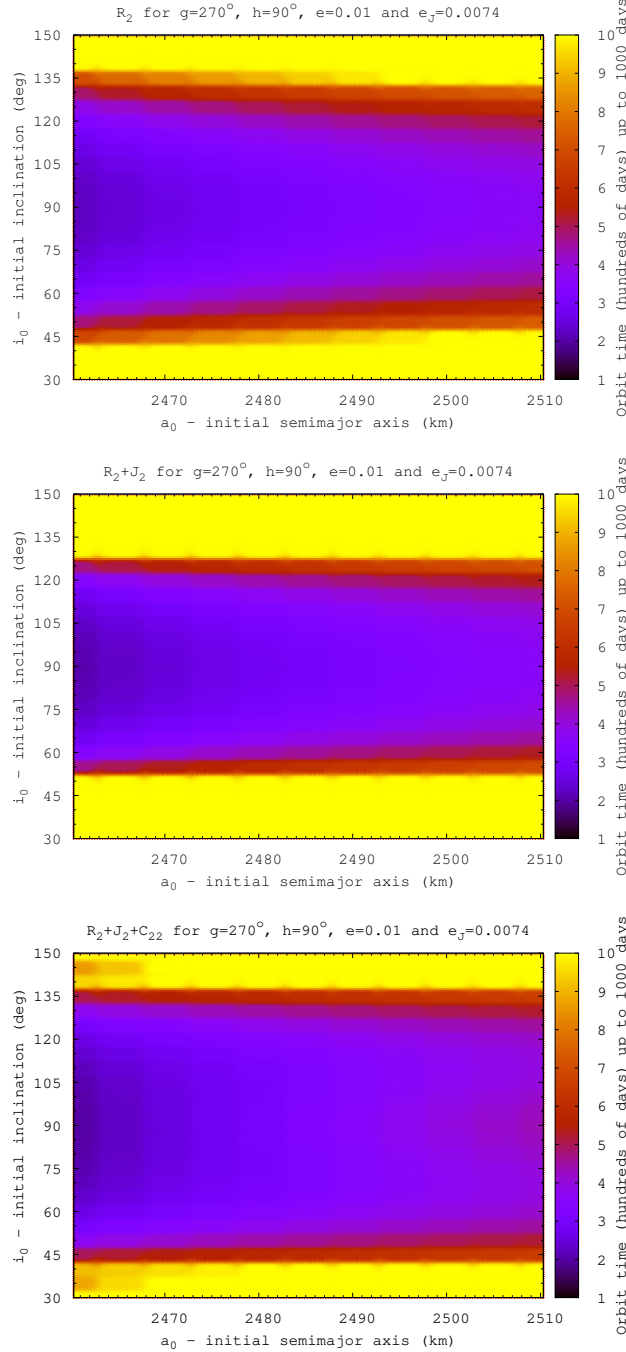
Figure 6: Propagation of the components of the eccentricity vector considering different values for i_0 and with initial values $a_0 = 2510.3$ km, $e_0 = 0.01$, $h_0 = 90^\circ$ and $e_p = 0.0074$. Effects of R_2 . Simulations for $g_0 = 270^\circ$ and numerical integration for 100 years.



Source: Cardoso dos Santos et al. (2017)

precessing the argument of the periapsis and the longitude of the ascending node of the spacecraft's orbit. However, it does not affect the lifetimes of the orbit as in this doubly-averaged model, since they are a function of the orbital eccentricity. The gravity term C_{22} also acts as a protecting mechanism once it is coupled with other orbital perturbations.

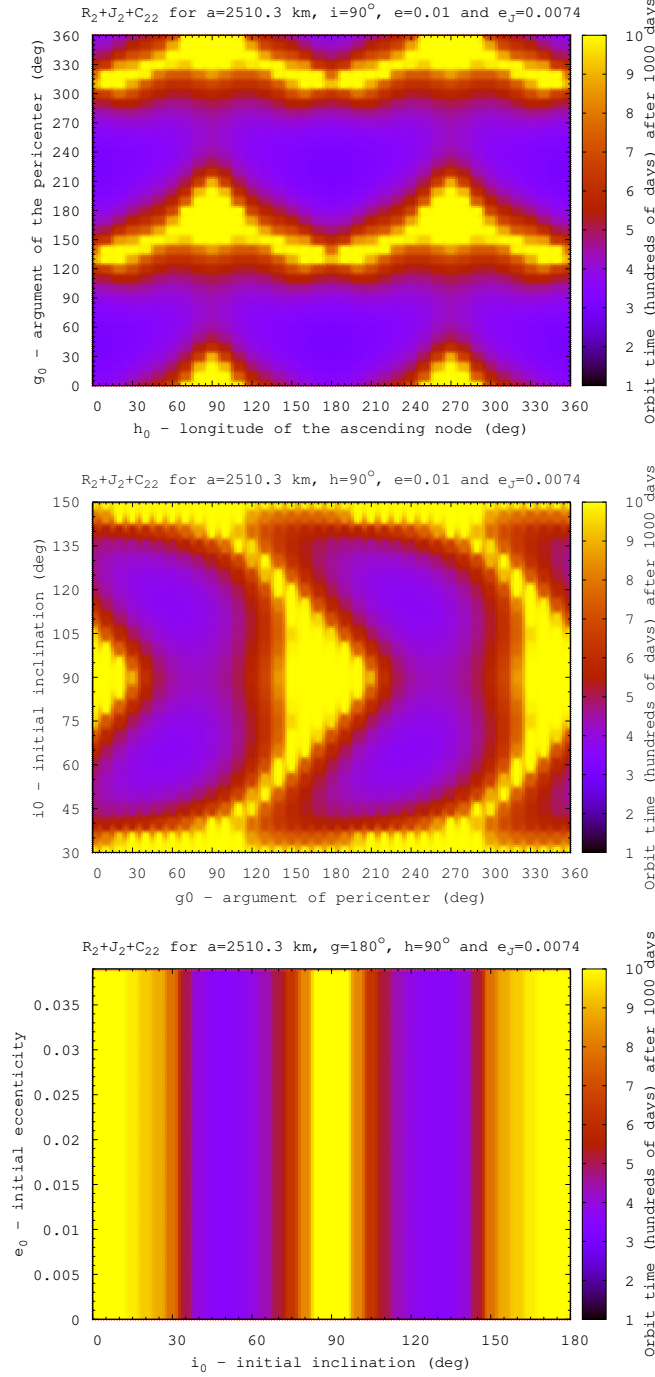
Figure 7: Lifetime maps (i_0 vs a_0) as a function of semi-major axis and inclination considering the effects of R_2 and $R_2+J_2+C_{22}$. It is the real value for the eccentricity of Jupiter orbiting Callisto. Initial conditions: $e_0 = 0.01$, $g_0 = 270^\circ$, $h_0 = 90^\circ$ and $e_p = 0.0074$. The vertical bar in the right gives the value of the lifetimes in hundreds of days for an integration of 1000 days. Each unity corresponds to one hundred days.



Source: Cardoso dos Santos et al. (2017)

As C_{22} is one of the perturbations considered in our dynamics, figures 8 and 9 are tests created to map lifetimes varying the initial value of the longitude of the ascending node h_0 . In addition, intervals with values for argument of periapsis (g_0), inclination (i_0) and

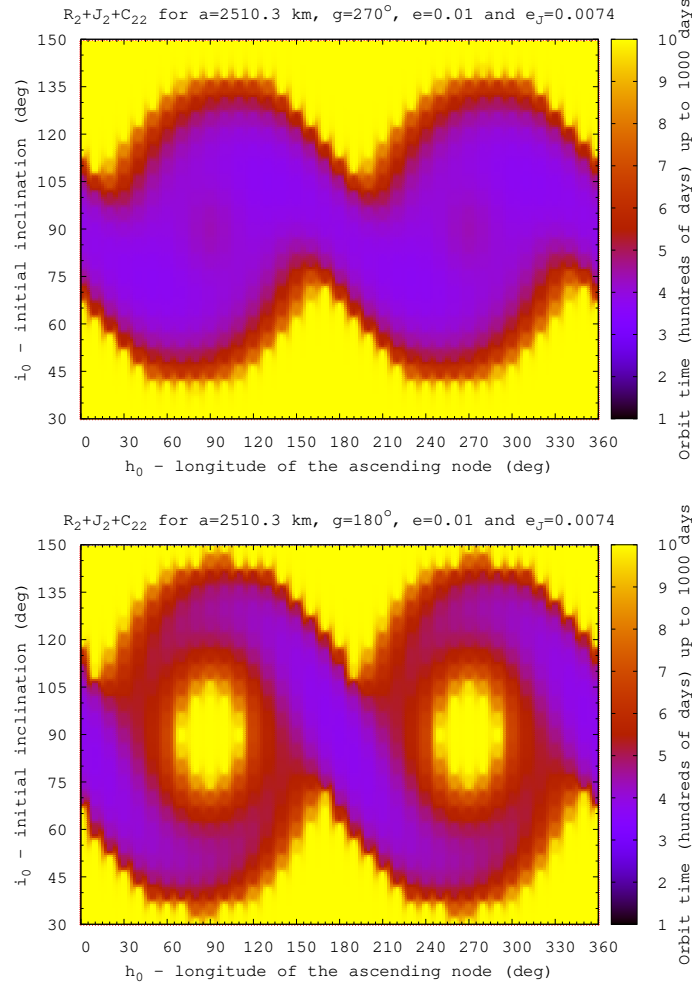
Figure 8: Lifetime maps (g_0 vs h_0), (i_0 vs g_0) and (e_0 vs i_0) for $g_0 = 180^\circ$, $e_p = 0.0074$ and considering the effects of $R_2+J_2+C_{22}$. The vertical bar in the right gives the value of the lifetimes in hundreds of days for an integration of 1000 days. Each unity corresponds to one hundred days.



Source: Cardoso dos Santos et al. (2017)

eccentricity (e_0) are mapped. It clearly shows values for this orbital parameter that can increase the lifetimes of the orbits. See, for example, that $h_0 = 90^\circ$ is able to produce a small increase in the polar orbits lifetime, e.g., $i_0 = 90^\circ$. In fact, the choice of $h_0 = 90^\circ$ in

Figure 9: Lifetime maps (i_0 vs h_0) for $e_p = 0.0074$ and considering the effects of $R_2+J_2+C_{22}$. Simulations for $g_0 = 270^\circ$ and 180° . The vertical bar in the right gives the value of the lifetimes in hundreds of days for an integration of 1000 days. Each unity corresponds to one hundred days.

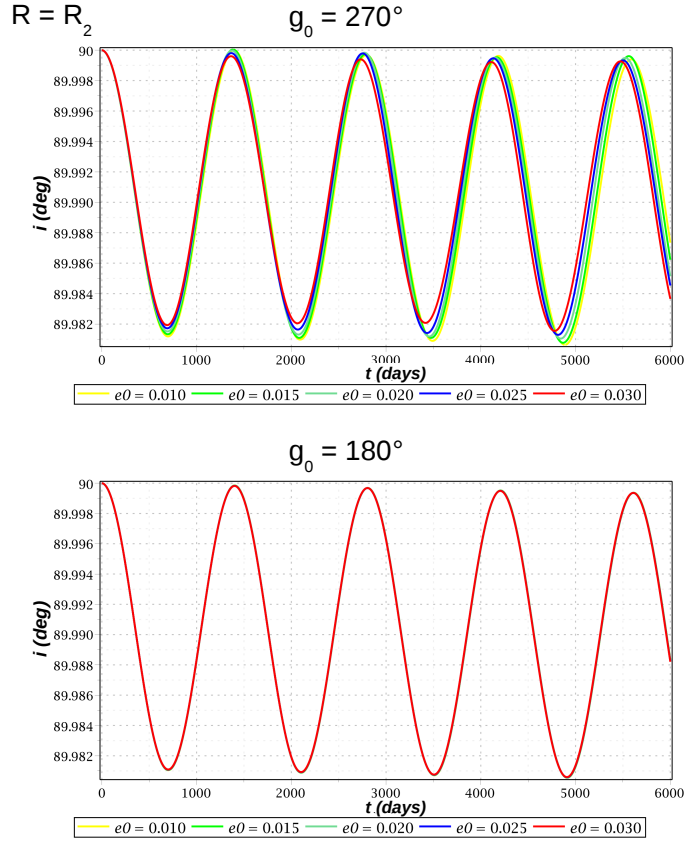


Source: Cardoso dos Santos et al. (2017)

the present numerical simulations was made based on this result. Therefore, convenient choices for the angles of the problem, as noted for the inclination and for the argument of periapsis of a spacecraft orbiting Callisto, are very important regarding extending the lifetimes of a space mission that intends to orbit this body.

Once orbits with high inclination are desired for space mission applications and the effect due to C_{22} is coupled in our dynamics, figure 9 shows tests made by creating lifetime maps considering variations in the initial value of the longitude of the ascending node h_0 and inclination i_0 . This figure clearly shows, concerning the initial values for h_0 , that this parameter can increase the lifetimes of the orbits. For instance, $h_0 = 90^\circ$ is able to produce a small increase in a polar orbit. In fact, the choice for $h_0 = 90^\circ$ in the present numerical simulations was made based on this result. Therefore, convenient choices for the angles of the problem of a spacecraft orbiting Callisto are very important regarding extending the

Figure 10: Propagation of the inclination (i) considering different values for the initial inclination i_0 . Initial conditions: $a_0 = 2510.3$ km, $i_0 = 90^\circ$, $g_0 = 270^\circ$ and $h_0 = 90^\circ$. Effects due to R_2 .

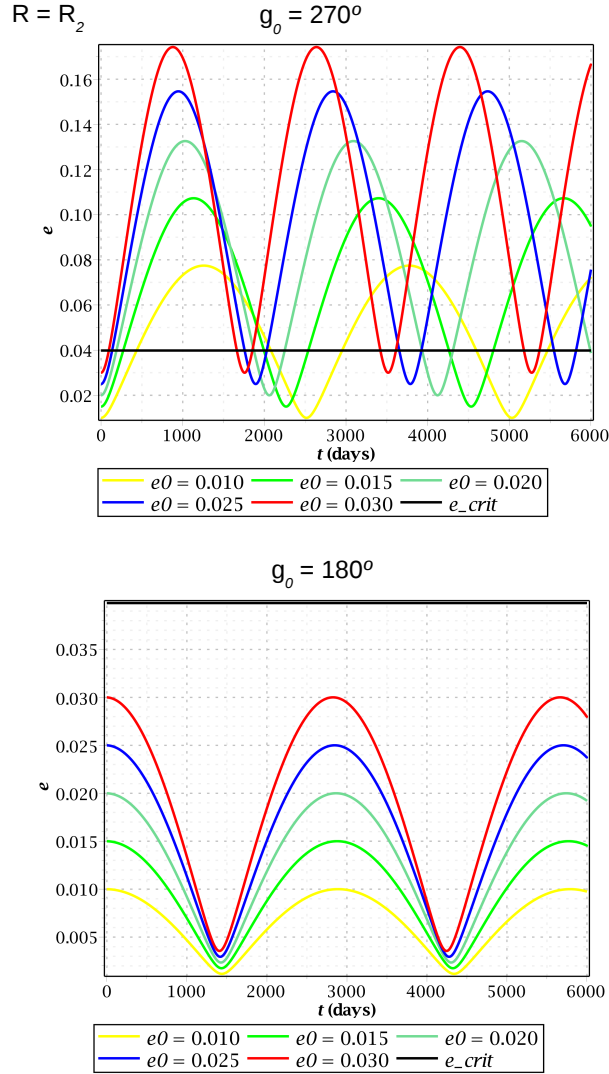


Source: Cardoso dos Santos et al. (2017)

lifetime of a space mission that intends to orbit this body. The first plot presented in figure 8 justifies the choice for $h_0 = 90^\circ$ as the value that produces higher values of lifetimes in high inclined orbits, the orbits of main interest for the present investigation. The second plot in the same figure shows values with respect to the relation $g_0 = k\pi$ radians (with k an integer) as better choices to increase the lifetime when one considers polar orbits. These differences are highlighted in figure 9, where it is presented lifetime maps as a function of the initial inclination i_0 and initial longitude of the node of the spacecraft's orbit h_0 . It also considers the cases for $g_0 = 270^\circ$ and 180° . This result appears similar to the ones presented in Condoleo et al. (2016) considering the dynamics of a spacecraft orbiting Callisto under the disturbing potentials R_2 and J_2 .

Here, figures 10 to 12 represent some orbits integrated for longer periods of time to provide some information about the long-term dynamics. The results in these figures are similar to those presented in figures 1, 2 and 4 of Prado (2003). But, these ones considered a Lunar orbiter. Figure 10 confirms the results presented in figure 3, where the choice of different initial values for the eccentricity does not impact the evolution of the inclination. Similar results appear when other values for inclination are considered and for this reason,

Figure 11: Propagation of the eccentricity (e) considering different values for the initial eccentricity e_0 . Initial conditions: $a_0 = 2510.3$ km, $i_0 = 90^\circ$, $h_0 = 90^\circ$ and $e_p = 0.0074$. Effects due to R_2 . Simulations for $g_0 = 270^\circ$ and 180° .

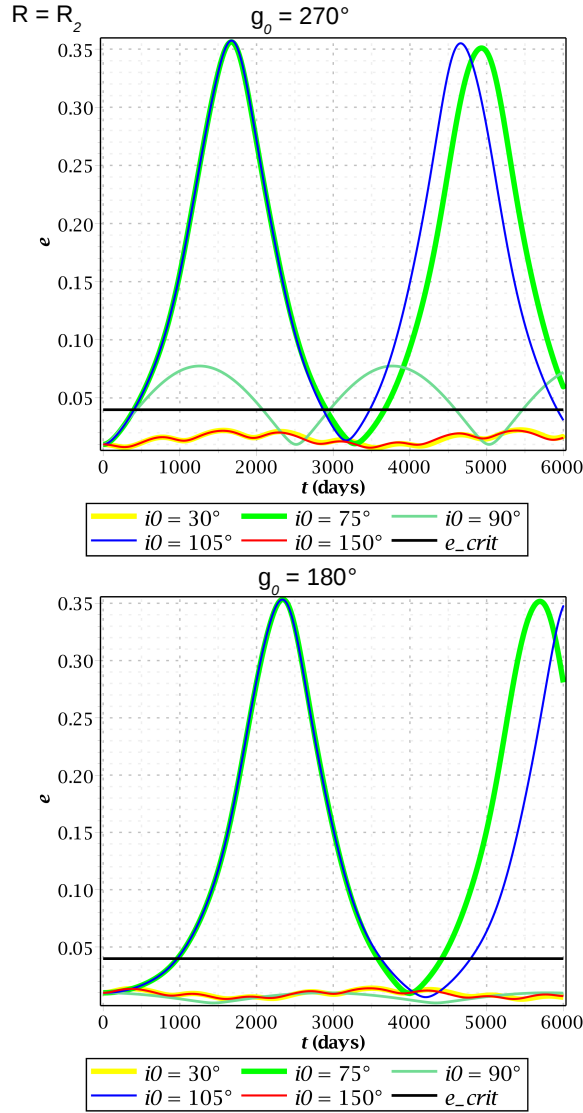


Source: Cardoso dos Santos et al. (2017)

they are omitted here. It is worth to mention the result presented for $g_0 = 270^\circ$ (upper plot) in figure 11, where it is possible to verify that the more circular the initial orbit is, the longer are the lifetimes presented. These figures 10 to 12 present the expected exchange of eccentricity and inclination, which corresponds to the previously mentioned conservation of the z -component of the orbital angular momentum $H_z = \sqrt{1 - e^2} \cos(i)$ (see figure 4). As observed in the amplitude of the oscillations presented in these figures, the small changes in one parameter will impact on the existence of small changes in the other as well.

Next, the effects of the eccentricity are measured. To perform this task, all the parameters of the Jupiter-Callisto system are kept constant, except for the eccentricity of the

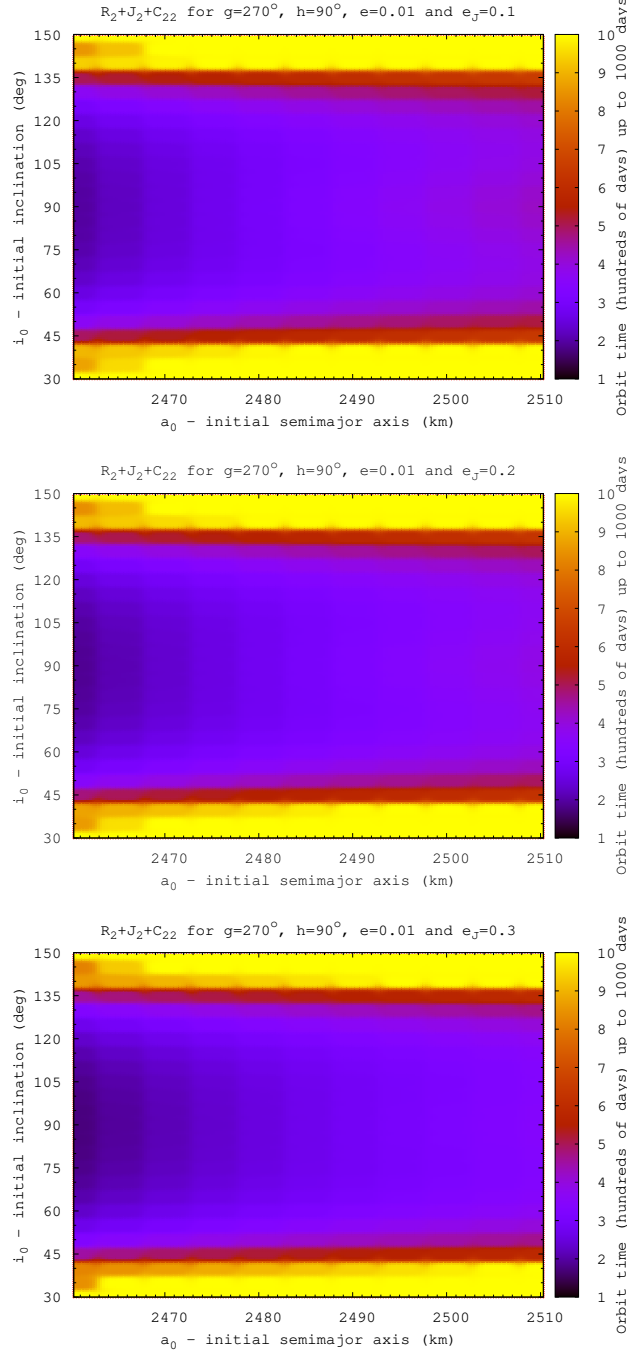
Figure 12: Propagation of the eccentricity (e) considering different values for the initial inclination i_0 . Initial conditions: $a_0 = 2510.3$ km, $e_0 = 0.01$, $h_0 = 90^\circ$ and $e_p = 0.0074$. Effects due to R_2 . Simulations for $g_0 = 270^\circ$ and 180° .



Source: Cardoso dos Santos et al. (2017)

disturbing body, which is Jupiter for this problem. It means that hypothetical systems are used to measure the effects of the eccentricity, without varying the other parameters, such that it is possible to be sure that all the modifications in the results are coming from the eccentricity of the primaries. Figures 13 and 14 show a graphical general view of the lifetimes. It also contribute to evaluate how significant is this protecting mechanism due to the gravity terms (J_2 and C_{22}), especially in the presence of a very massive third body (as the case of Jupiter). We choose different values for the eccentricity of the disturbing body $e_p = \{0.1; 0.2; 0.3; 0.4; 0.5 \text{ and } 0.6\}$ in order to create Callisto-Jupiter like systems which are able to show how much the harmonics are able to help on increasing lifetimes under extreme third-body perturbations. These results are presented in figures 13 and 14.

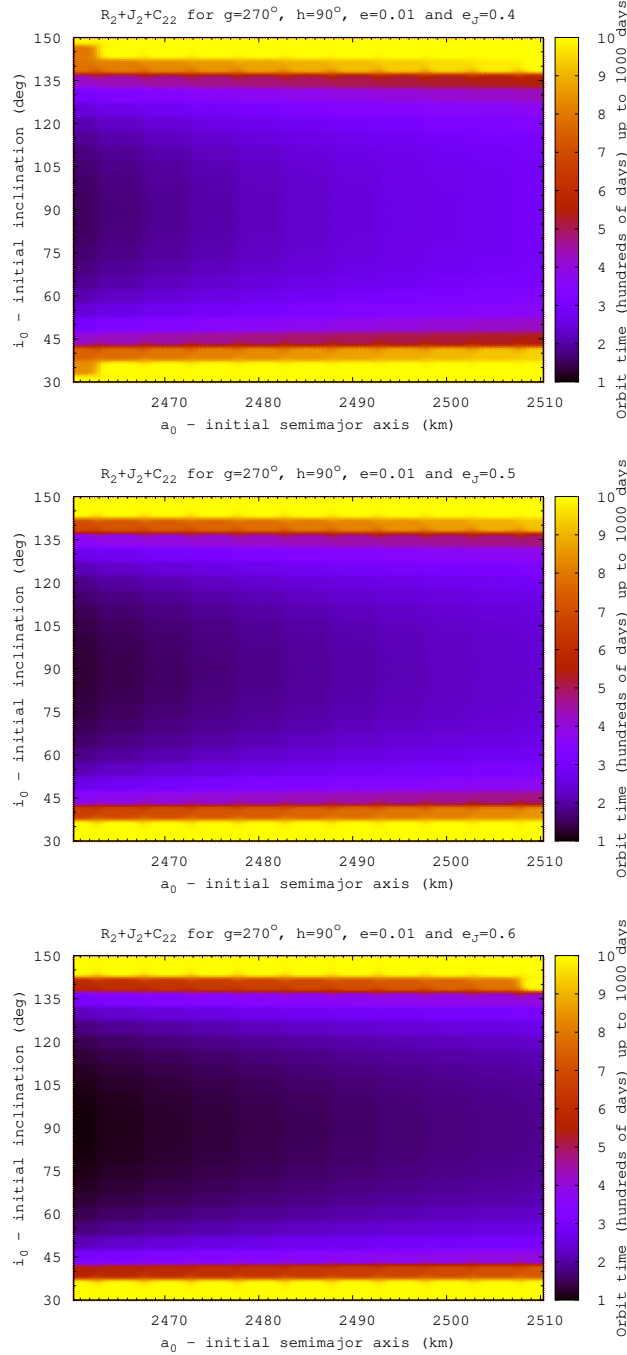
Figure 13: Lifetime maps (i_0 vs a_0) for $e_p = 0.1, 0.2$ and 0.3 . Disturbing effects: $R_2 + J_2 + C_{22}$. Initial conditions: $e_0 = 0.01$, $g_0 = 270^\circ$ and $h_0 = 90^\circ$. The vertical bar in the right gives the value of the lifetimes in hundreds of days for an integration of 1000 days. Each unity corresponds to one hundred days.



Source: Cardoso dos Santos et al. (2017)

It is worth of note the lifetime maps presented in figure 13 very similar to the ones shown in figure 7. It indicates that the significant magnitude ($\sim 10^{-5}$) of the gravity terms J_2 and C_{22} is strong enough to naturally protect the orbit against a increase of the lifetime with a very massive disturbing body in a significantly eccentric orbit. Due to the fact that more

Figure 14: Lifetime maps (i_0 vs a_0) for $e_p = 0.4, 0.5$ and 0.6 . Disturbing effects: $R_2 + J_2 + C_{22}$. Initial conditions: $e_0 = 0.01$, $g_0 = 270^\circ$ and $h_0 = 90^\circ$. The vertical bar in the right gives the value of the lifetimes in hundreds of days for an integration of 1000 days. Each unity corresponds to one hundred days.

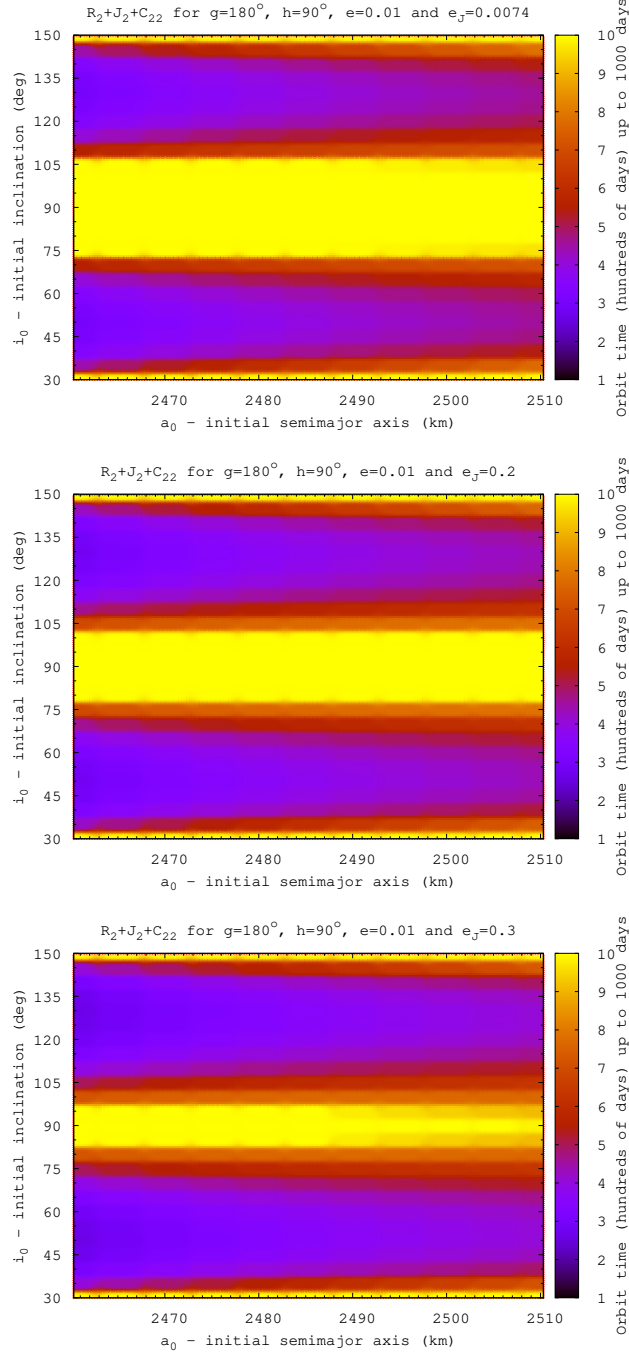


Source: Cardoso dos Santos et al. (2017)

eccentric orbits make the spacecraft to pass closer to the disturbing body, and vice versa, it is expected an increase of the disturbing effects due to its gravitational attraction. Naturally, as observed in the lifetime maps presented in figures 13 to 14, when the eccentricity e_p increases, the disturbing effects become stronger and the lifetime of the orbits tend to

decrease.

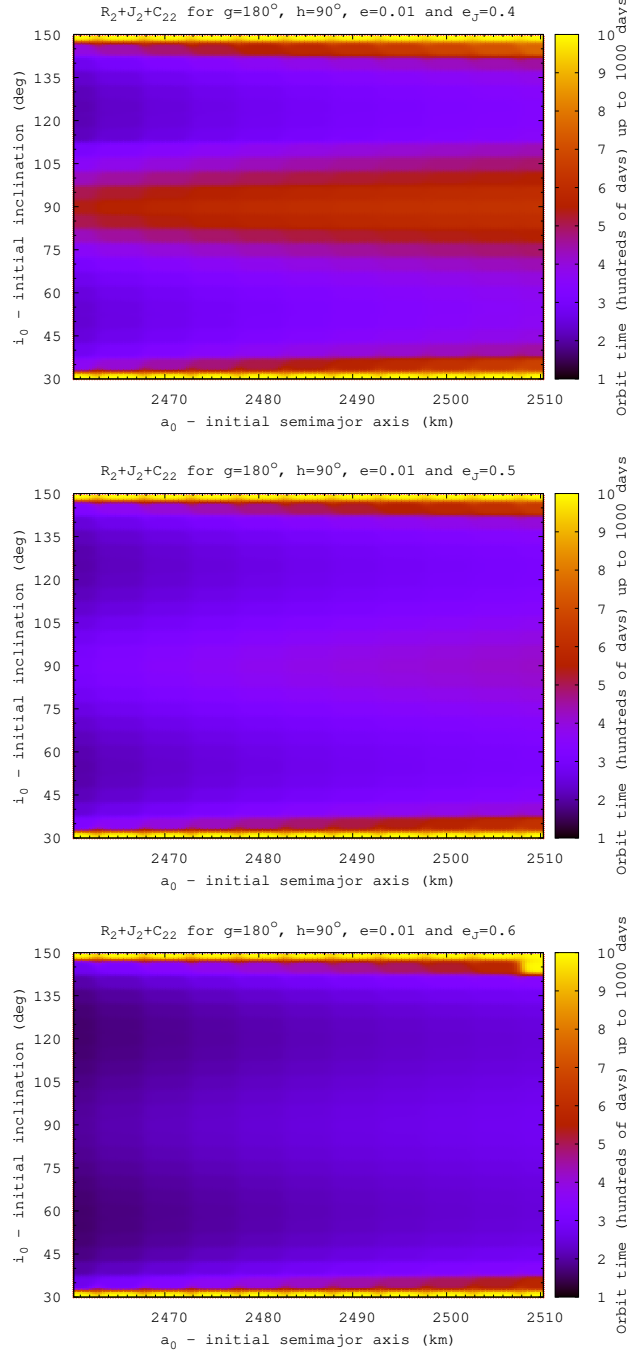
Figure 15: Lifetime maps (i_0 vs a_0) for $e_p = 0.0074, 0.2$ and 0.3 . Disturbing effects: $R_2 + J_2 + C_{22}$. Initial conditions: $e_0 = 0.01$, $g_0 = 180^\circ$ and $h_0 = 90^\circ$. The vertical bar in the right gives the value of the lifetimes in hundreds of days for an integration of 1000 days. Each unity corresponds to one hundred days.



Source: Cardoso dos Santos et al. (2017)

A first look at the results confirms the presence of the expected Kozai-Lidov effects in reducing the lifetime of orbits with high values for the inclination. These results are compatible with the ones shown in Prado (2003), that made a similar study for the partic-

Figure 16: Lifetime maps (i_0 vs a_0) for $e_p = 0.4, 0.5$ and 0.6 . Disturbing effects: $R_2 + J_2 + C_{22}$. Initial conditions: $e_0 = 0.01$, $g_0 = 180^\circ$ and $h_0 = 90^\circ$. The vertical bar in the right gives the value of the lifetimes in hundreds of days for an integration of 1000 days. Each unity corresponds to a hundred of days.



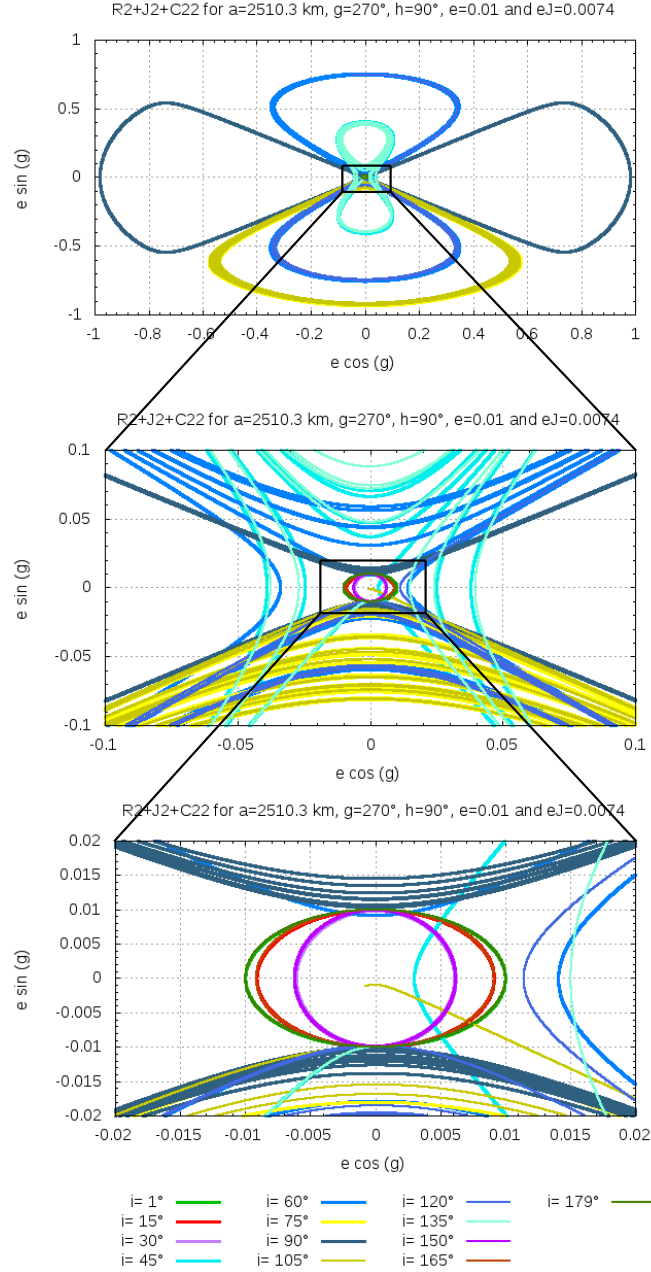
Source: Cardoso dos Santos et al. (2017)

ular value for the semi-major axis of four radius of the moon considering only the term R_2 . It is a high value of altitude for the orbit in comparison with the ones presented in the present study, which are in a range of values between 50 km and 100 km. Looking at the results obtained using the real value for the eccentricity of Callisto (figure 7), it is

noted that orbits with initial semi-major axis a_0 above 2480 km usually present lifetimes of about 300 days or more. Below this value, the lifetime depends more significantly on the inclination of the orbit. Figure 7 shows a general view of these lifetimes. In general, for the cases with orbital inclination between $\sim 42^\circ$ and $\sim 137^\circ$, there are collisions before the time reached this limit. Those results are very important when planning missions to minor bodies of the Solar System (planetary moons, asteroids, comets) under the massive third-body perturbation of the planets or Sun, since polar orbits are good candidates for surface observations, mapping and characterization. In order to attain these scientific goals, high inclined orbits offer the best coverage. The results show the eccentricity of the primaries impacting in reducing the lifetimes of orbits, in particular in the ones closer to the moon. The physical interpretation is not difficult. Larger eccentricities of the primaries allow shorter distances for the perturbing body, so the effects are larger when the moon is at the periapsis. Of course the planet is far away during the passage of the moon near the apoapsis, which reduces the perturbation. So, an analysis is required in order to study the balance of those effects, and the results show that there is an expected increase in the perturbation with the eccentricity e_p , reducing the lifetimes of the orbits. The reductions are not significant for $e_p = 0.1$ due to the protecting mechanism of J_2 and C_{22} , and also due to the fact that Callisto is the more distant object among the Galilean moons. This is not a problem in this analysis, because new reductions are noted in those minimum lifetimes when the eccentricity of the primaries is 0.2, especially for a_0 above 2500 km in the second plot of figure 13. For values of $e_p = 0.4$ and above, the plots are very different, with blue color dominating the results. It means that much smaller lifetimes (figure 14). These results are expected, but a quantification is important, as done in the present study.

Indeed, deeper exploration of these orbital parameters can be made using some values studied in Condoleo et al. (2016), where the authors explore some conditions to freeze orbits around planetary satellites with a disturbing potential taking into account the effects of the third body and of the flattening at the poles (J_2). Despite the present study takes into account a more general model with the presence of C_{22} , the values presented in the work of Condoleo and fellows can be used as good initial guide on finding initial conditions which produce longer lifetimes, despite the existence of some extra disturbing potentials. For instance, one of the conditions that the authors find to freeze a polar orbit is to set $g_0 = 0^\circ$ or $g_0 = 180^\circ$, i.e., these values freeze polar orbits when the potential $R_2 + J_2$ is taken into account. This is the same result found in figures 8 and 9, now with the additional effects due to C_{22} . Therefore, some numerical explorations can be made by reproducing some previous results considering $g_0 = 180^\circ$ instead of $g_0 = 270^\circ$. An evidence of this effect of selecting this angle to increase the lifetime of orbits can be observed by analyzing figures 15 to 16. In these cases $g_0 = 180^\circ$ is able to protect significantly the surviving time of the polar orbits, despite of the increase of the disturbance coming from the third body (Jupiter). It is highlighted when a comparison of the results shown in figures 15 and 16 is made with the results shown in figures 13 and 14. The lifetime map i_0 vs a_0 considering

Figure 17: Propagation of the components of the eccentricity vector considering different values for i_0 and with initial values $a_0 = 2510.3$ km, $e_0 = 0.01$, $h_0 = 90^\circ$ and $e_p = 0.0074$. Effects of $R_2 + J_2 + C_{22}$. Simulations for $g_0 = 270^\circ$ and numerical integration for 100 years.

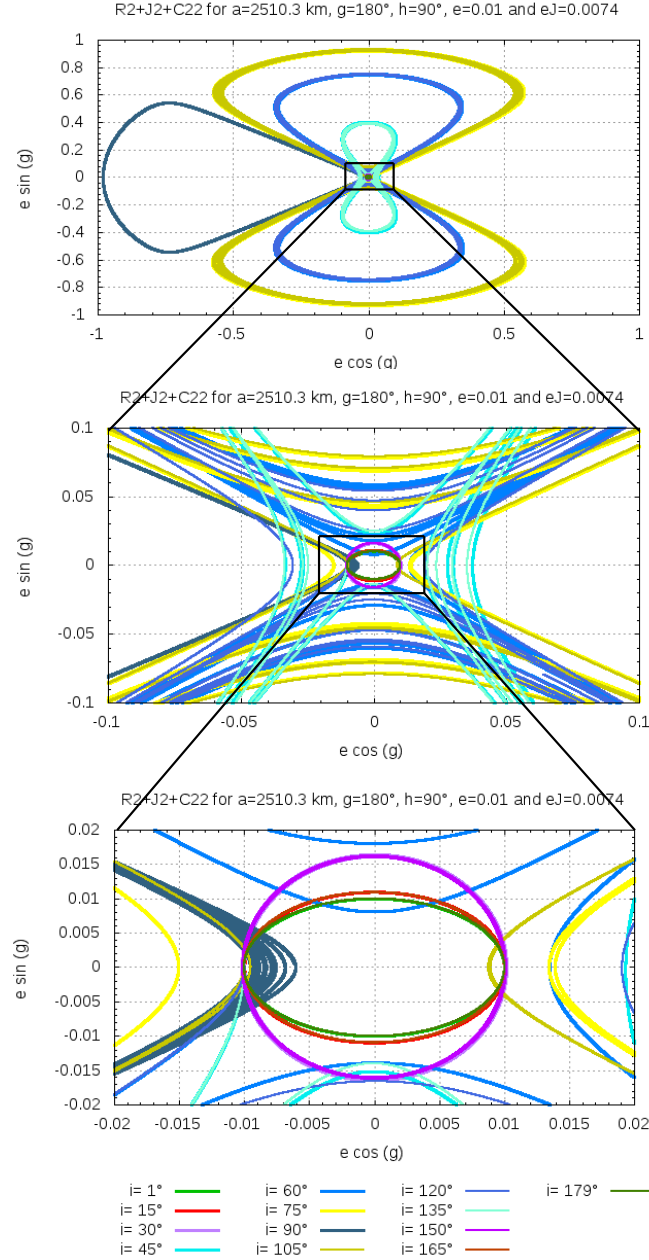


Source: Cardoso dos Santos et al. (2017)

$e_p = 0.01$ is omitted in figure 15 due to similarity with the lifetime map for the real value $e_p = 0.0074$ (first plot in figure 15), as happened with the results for $g_0 = 270^\circ$.

Figures 17 and 18 present the components of the eccentricity vector considering the disturbing potential $R_2 + J_2 + C_{22}$. Different scenarios ($g_0 = 270^\circ$ and $g_0 = 180^\circ$) are assumed and similar analysis can be done, as presented in figure 6. It is also important to highlight such figures as useful plots to identify the existence of libration and circulation

Figure 18: Propagation of the components of the eccentricity vector considering different values for i_0 and with initial values $a_0 = 2510.3$ km, $e_0 = 0.01$, $h_0 = 90^\circ$ and $e_p = 0.0074$. Effects of $R_2+J_2+C_{22}$. Simulations for $g_0 = 180^\circ$ and numerical integration for 100 years.

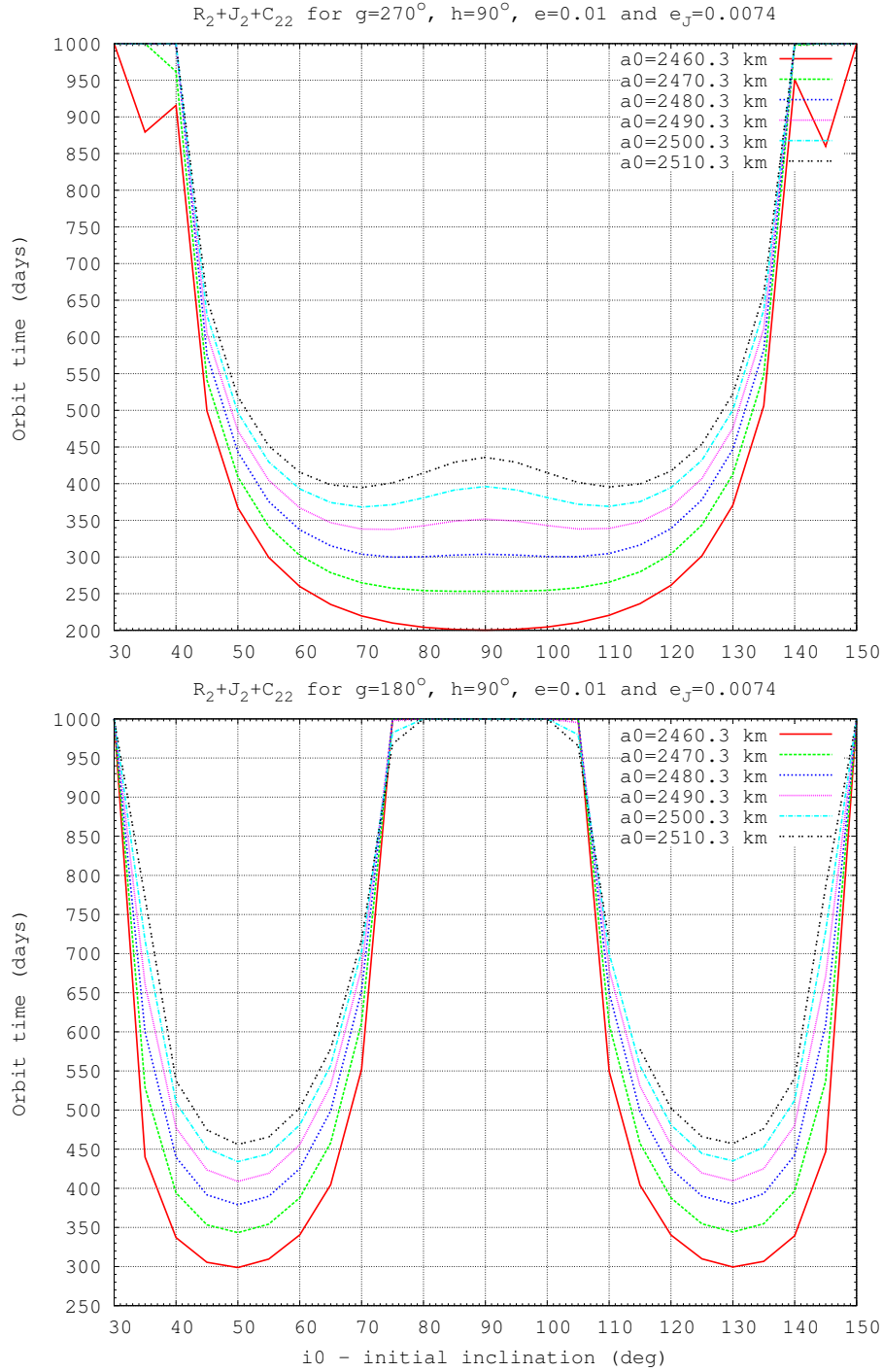


Source: Cardoso dos Santos et al. (2017)

motions.

Finally, figures 19 and 20 show the orbit time as a function of the inclination, for different values of initial semi-major axis a_0 and initial eccentricity e_0 . Results considering again scenarios with $g_0 = 270^\circ$ and $g_0 = 180^\circ$ confirm the relevance of the present discussion, especially regarding an appropriate choice of the orbital parameters capable to extend the lifetimes of orbits. In all the cases, despite the selection of different initial

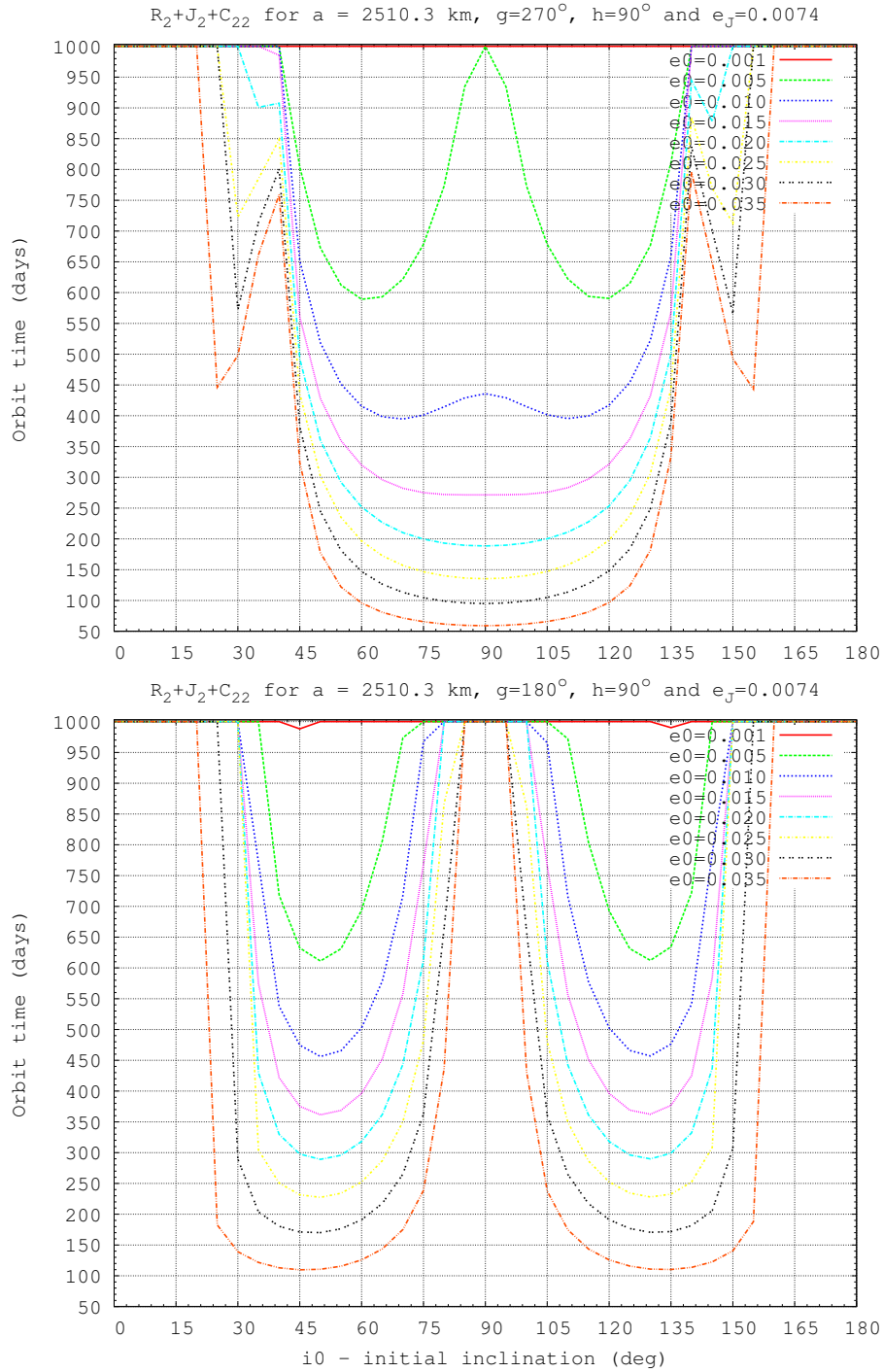
Figure 19: Representation of Orbit time vs i_0 considering different values for a_0 and for $e_0 = 0.01$ and $e_p = 0.0074$. Effects of $R_2+J_2+C_{22}$. Simulations for $g_0 = 270^\circ$ and 180° .



Source: Cardoso dos Santos et al. (2017)

values for semi-major axis a_0 and eccentricity e_0 , the choice for $g_0 = 180^\circ$ was able to extend from tens up to hundreds of days the lifetimes of the orbits. The "picks" appearing for lower inclinations in these figures are due to the use of discretized data on plotting the curves. Considering the scenario for an Europa orbiter with a dynamical model under the disturbances of the averaged models for R_2 and J_2 , one can check similar plots in figures

Figure 20: Representation of Orbit time vs i_0 considering different values for e_0 and for $a_0 = 2510.3$ km and $e_p = 0.0074$. Effects of $R_2+J_2+C_{22}$. Simulations for $g_0 = 270^\circ$ and 180° .



Source: Cardoso dos Santos et al. (2017)

8 and 9 of Scheeres; Guman; Villac (2001). In this work, the authors choose to vary the initial argument of periapsis g_0 instead of varying the initial semi-major axis a_0 or the initial eccentricity e_0 as an alternative method to find values to extend the lifetime of a Galilean moon orbiter.

2.4 DISCUSSION

In the context of the inner restricted three-body problem, the present study investigated the lifetime of orbits for a spacecraft traveling around the moon Callisto, which is in an eccentric and almost planar orbit around Jupiter. This system is chosen as a model for the application of similar studies to other celestial bodies, especially minor bodies of the Solar System, which are potential candidates to receive a space mission in the future. One of the goals of the present study was to investigate the impact of the massive disturbing body's orbital eccentricity in the lifetime of a spacecraft's orbit around a body with an ellipsoidal shape. The mathematical model used here considers the effects of the third body in elliptical and inclined orbits and the irregular shape of the central body by using the terms J_2 and C_{22} .

For the simulations, the system Jupiter-Callisto is used to give the basic physical data of masses, distances and shapes of the bodies, but the eccentricity is set with different values to show the importance of this parameter on more disturbed dynamics, such as the case of comets that are in rather eccentric orbits around the Sun. During the investigation, values for some orbital elements were determined to increase the lifetime of orbits around Callisto. These results are especially important concerning high-inclined orbits, the desired ones for global mapping and characterization. However, as these orbits usually present shorter lifetimes due to the Kozai-Lidov mechanism, the present investigation has a significant importance, because helps to find parameters which are able to extend these lifetimes, increasing the available time for space mission operations around these polar and quasi-polar regions. The Kozai-Lidov effects clearly and strongly appear in the systems simulated, showing the importance of finding the combination of parameters ($g_0 = 180^\circ$ and $h_0 = 90^\circ$) that was able to extend the lifetimes of the polar orbits to higher values. Considering orbits with inclinations above 90° , it is noted some symmetries (which are perfectly noted in some cases) in the results around the polar orbit, which means that orbits with inclination of $90^\circ + x^\circ$ and $90^\circ - x^\circ$ have very similar lifetimes.

Regarding the effects of the irregular shape of the moon, they are not very large for this system. It happens because Callisto has a near spherical shape, with $J_2 = 3.27 \times 10^{-5}$ and $C_{22} = 1.02 \times 10^{-5}$. In a short-time orbit, the effects due to these gravity terms is compared to the model considering only the third-body perturbation. Hence, the effects will be much stronger for bodies with more irregular shapes.

As expected, when considering the effects of hypothetical higher values for the orbital eccentricity of the massive third-body (Jupiter), the more eccentric the orbit, the larger impact this body will produce on decreasing the lifetime of the orbit. On the other hand, the more circular is the orbit, less affected it will be by the third-body perturbation.

Therefore, the general contribution of the study presented in this chapter is to show the lifetimes of orbits around moons in eccentric orbits, generating maps that can be used for mission designers. It confirms the lifetime maps as an efficient tool to map and find the initial conditions that extend the surviving time of the orbits, especially those highly dis-

turbed and with high values of inclination. Of course, along with this tool of the lifetime maps, other techniques can be used. For instance, more details about the mathematical and physical aspects that lead to these dynamical behaviors can be discussed. These aspects can be explored in a future communication. Similar ideas can also be extended to applications in planetary physics. A good example of an interesting system to be investigated using the present models and techniques is the system composed by the dwarf planet Haumea, two small moons and a recently discovered ring (ORTIZ et al., 2017).

A final comment refers to the fact that the work developed in this chapter is an extension of study developed by Carvalho (2011), with the use of lifetime maps and the introduction of the orbital inclination of the third body in the equations of motion.

The study presented in this chapter was published in Cardoso dos Santos et al. (2017).

3 SOME CHARACTERISTICS OF A SPACECRAFT ORBITING MERCURY

The study of the long-term dynamics of a spacecraft around Mercury was already extensively developed in the literature (see for instance Tresaco; Elipe; Carvalho (2016), Carvalho (2016), Tresaco et al. (2018) and references therein). The computation of the main effects, which include Mercury's irregular shape, other bodies' attraction and solar radiation pressure (SRP), has been extensively discussed. However, nowadays, with the improvement of quality in the data obtained in the physical parameters of this system, and with the renewed interest in sending space missions to visit this rocky planet, an update of these previous analysis is necessary. Notable first missions to explore Mercury were Mariner 10 (years of 1974 and 1975) and MESSENGER (year of 2013), which had the goal to map the surface of the planet. Previous studies as Lara et al. (2010) and Ma; Li (2013) have used Mariner 10's based data to study. The data provided by the mission MESSENGER is especially important, once it is an updated version of the data obtained in the mid of 70's, and can naturally support the next planned mission to visit Mercury: BepiColombo¹, an European Space Agency (ESA)'s mission planned to be launch in 2018 and with orbit insertion scheduled to be in 2025. This mission is being developed in partnership with the Japan Aerospace Exploration Agency (JAXA), and it comprises two main spacecrafts. Recent works published by Mazarico et al. (2014) and Stark; Oberst; Hussmann (2015) are among the works which provide important updates on Mercury's gravitational field, a necessary information for orbit analysis around this body, and with potential application to BepiColombo mission design and planning. At the same time, due to the special environment where BepiColombo is going to be flying over, a special kind of technology has brought attention of the scientific community, turning that it is very important to be considered in such studies: solar sails.

Solar sails are a type of propulsion which uses photons of the radiation emanated by the Sun as a free energy source to move a spacecraft in its trajectory. As known from Quantum Theory, the quantum packages of energy which compose the light, and that we call photons, present a linear momentum which can be transfered by collision. The idea behind the solar sails is exactly to use this characteristic to transfer the momenta of the photons due to Sun to the sail, generating a resulting acceleration. Therefore, this kind of technology is projected to be composed by large mirrors with low masses, so that they can interact with the largest possible amount of photons, meanwhile do not having to much inertia to be moved, thus providing higher values of acceleration for the spacecraft.

¹Bepi Colombo mission official website: <http://sci.esa.int/bepicolombo/>

Indeed, the gain of kinetic energy happens due to the gain of momentum after a process of reflection of the photons that collide with the sail's surface. In theory, these photons transfer their energy to solar sails (MCINNES, 1999), and on doing so, the spacecraft will move. Among the reasons which brought the interest of scientific community to this kind of propulsion system is that it is, as mentioned before, a free energy source, with the influence of the radiation pressure of the Sun being able to produce an additional force in the dynamics. This additional force can be controlled for useful applications as interplanetary trajectory, orbital maneuvering, orbit and attitude (space orientation) stabilization to name a few. In fact, due to the reflectance present in the materials by means of which the spacecrafts are built, this force must be taken into account, since it is able to change the behavior of the orbits of the objects. As mentioned, it can also change the attitude, *i.e.*, the spacecraft's pointing angle (WIE, 2004A,B; KIKUCHI et al., 2017), which is however not the focus of the investigation presented in this chapter. Here, it is presented a semi-analytical approach to study dynamical features of a spacecraft orbiting Mercury, sometimes with its propulsion system using SRP producing more significant effects (high-altitude orbits). The model of the forces acting on the spacecraft is developed similarly to what was done in the previous study for a Callisto orbiter (see also Cardoso dos Santos et al. (2017)), with the inclusion of the solar radiation force as an additional force in the dynamical system. In this study, we present an analysis to investigate orbits around Mercury using the same techniques used to study the dynamics of a spacecraft around Callisto. However, here there is the inclusion of SRP effects. The investigation intends to search for less disturbed orbits and, potentially to identify frozen ones (ELIPE; LARA, 2003; CARVALHO; VILHENA DE MORAES; PRADO, 2010). Frozen orbits are the ones which present almost (in practical) constant (in theory) orbital elements in averaged terms. The forces included in the analysis are the ones related to Mercury's irregular shape (J_2 , J_3 and C_{22}), the gravitational pull of the Sun (the third-body perturbation) and SRP. In order to have access to information the dynamical properties of a propulsion system using SRP and also for a detailed revision of the literature, the reader is directed to the works of Faria (2009), Fu; Sperber; Eke (2016), Tresaco; Elipe; Carvalho (2016) and the references therein.

Once the problem is built in the context of the restricted three body problem, the spacecraft is assumed to be the "massless" particle, thus creating an inner restricted three body problem (see the introduction of the chapter 2) with a solar sail orbiting Mercury as a central body. In the context of the dynamical modeling of the problem and the use of perturbation maps for the present study, the works of Gomes; Domingos (2016) and Cardoso dos Santos et al. (2017) present studies about the lifetime of orbits around planetary moons which are orbiting its mother planet in an elliptic motion. The mathematical modeling of the problem is similar to the one used in chapter 2, where the disturbing potentials are expanded up to the second order, not only for the irregular shape of central body, but also for the disturbing body, here assumed to be the Sun in a relative elliptical orbit around

Mercury, assumed to be the central body. However, the effects due to the inclusion of Mercury's pear-shape distribution of mass term J_3 are also taken into account, once this coefficient is only one order of magnitude below J_2 and C_{22} . The effects due to SRP are also included when the spacecraft has a large area and small mass. When this condition is satisfied, the perturbation due to SRP becomes significant and must be taken into account in the dynamics.

Therefore, the present chapter is organized as follows: the section 3.1 presents the equations of motion and a discussion about the disturbing forces taken into account in the study. After these forces are included in the dynamical system, the section 3.2 presents different types of analysis, providing regions with small perturbations in the orbital elements, which include sets of initial conditions for frozen orbits around Mercury, orbits which can potentially contributing for the planning of a space mission to orbit Mercury.

3.1 EQUATIONS OF MOTION AND DISTURBING FORCES

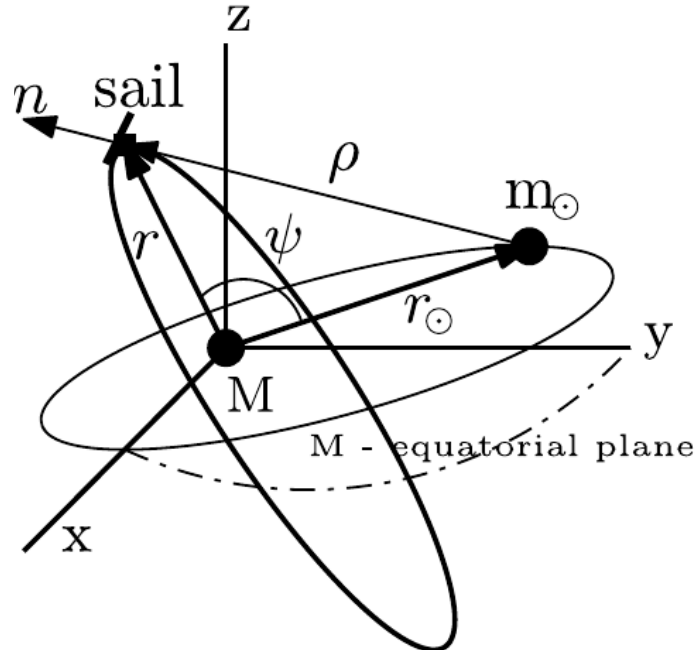
As mentioned before, our assumptions about the dynamics takes into consideration a solar sail orbiting the planet Mercury, meanwhile the Sun is assumed to be an external disturbing body orbiting Mercury with an elliptical and inclined trajectory. In the works of Tresaco; Elipe; Carvalho (2016), Cardoso dos Santos et al. (2017) and Tresaco et al. (2018), as well in the sections 2.1 and 2.2 of this thesis, the modeling of the disturbing potentials and equations of motion is presented in details. Thus, we consider $Oxyz$ as reference frame which is centered in Mercury's center and whose equatorial plane is the reference plane Oxy . The x -axis lies along the line of intersection between the equatorial plane Oxy and the orbital plane of the disturbing body, here assumed to be the Sun, and whose orbital elements to represent its elliptic and inclined path Mercury are given by: semi-major axis $a_\odot$, eccentricity $e_\odot$ and inclination $i_\odot$. The solar sail's orbit motion is described by the orbital elements: semi-major axis a , eccentricity e , inclination i , right ascension of the ascending node h , argument of the periapsis g and mean motion n . The geometrical representation of the problem can be seen in figure 21, where the orbital motion of a sail around a central body is disturbed by the gravitational pull of a third body present in the dynamical system. The dynamical model to describe this motion is given by expression 43 and it is composed by three terms: the gravitational pulls due to Mercury (the central body whose irregular shape is taken into account) and the Sun (third body), with the additional acceleration of the due to the solar radiation pressure force.

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_M + \ddot{\mathbf{r}}_{3b} + \ddot{\mathbf{r}}_{SRP} = \nabla U_M(\mathbf{r}) - \mu_\odot \left(\frac{\mathbf{r} - \mathbf{r}_\odot}{\|\mathbf{r} - \mathbf{r}_\odot\|^3} + \frac{\mathbf{r}_\odot}{\|\mathbf{r}_\odot\|^3} \right) + \beta \mu_\odot \frac{\mathbf{r} - \mathbf{r}_\odot}{\|\mathbf{r} - \mathbf{r}_\odot\|^3} \quad (43)$$

where $\mathbf{r}$ is the position vector of the sail with respect to Mercury, $\mu_\odot$ is the gravitational parameter of the Sun, and $\mathbf{r}_\odot$ is the position vector of the Sun with respect to Mercury.

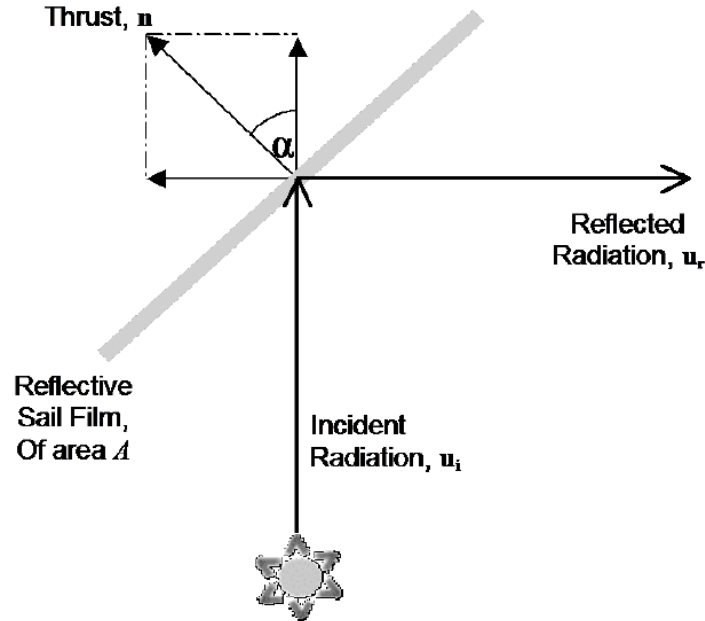
The figures 22 and 23 provide a geometrical representation of the concepts used for solar sailing. The parameter $\beta < 1$ is the SRP parameter associated to the solar sail.

Figure 21: Geometrical scheme of the dynamical system with a solar sail



Source: Tresaco; Elipe; Carvalho (2016), p. 2

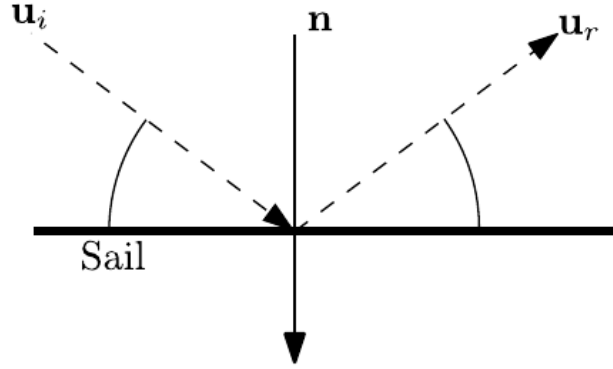
Figure 22: Concept of solar sail propulsion



Source: Macdonald; Hughes (2004)

The dimensionless sail loading parameter β is defined as the ratio between the accelerations due to SRP and Sun's gravitational attraction (MCINNEN, 1999). This definition is helpful to provide insights on finding frozen orbits as a function of the parameter β , requiring to investigate the variables that need to be assumed in the dynamics in order to make the perturbations to mutually cancel each other. Among these variables to be properly selected are the physical parameters of the sail, once the parameter is defined as

Figure 23: Scheme of a perfect sail (flat and 100% reflective)



Source: Tresaco; Elipe; Carvalho (2016), p. 2

(McINNES, 1999)

$$\beta = \frac{L_{\odot} A}{2\pi m G M_{\odot} c}, \quad (44)$$

were $M_{\odot}$ and $L_{\odot}$ are, respectively, the Sun's mass and luminosity, G is the universal gravitational constant and c is the speed of light. The solar sail's physical parameters are m and A , which correspond to its mass and area, respectively,

In this study, the motion is integrated using Lagrange planetary equations. As these equations work with a general disturbing potential composed by the sum of the disturbing potentials due to each force present in the analysis, the associated potentials due to these forces need to be obtained as done in the previous chapter about the dynamics of a Callisto orbiter.

3.1.1 Perturbations due to the third body and the solar radiation pressure

The present study intends to use the merit of analytical developments and techniques, so that the doubly-averaged motion of the solar sail is studied to reduce the degrees of freedom of the system by eliminating the short-period terms and also avoiding sensibilities to initial conditions. The qualitative long-term averaged dynamics is proven to be pretty efficient in the desired applications for the present study. The procedure to perform the double average over the dynamics is similar to the one presented in chapter 2, and depending an integration for one period over the sail's eccentric anomaly, followed by a second similar integration in terms of the Sun's eccentric over one orbital period. The changes in variables that are necessary in the integration process are also discussed in chapter 2. Thus, after the first average, the third-body perturbation of the Sun is

$$\langle R_{3b} \rangle = \frac{3a^2 n_{\odot}^2}{4} \left(\frac{a_{\odot}}{r_{\odot}} \right)^3 (\alpha^2(1 + 4e^2) + \gamma^2(1 - e^2) - (\frac{2}{3} + e^2)) \quad (45)$$

with Sun's mean motion given by $n_\odot$. When the same process is applied to the SRP perturbation, we obtain the following expression for SRP single-averaged perturbation

$$\langle R_{SRP} \rangle = -\beta \frac{3a^2 n_\odot^2}{4} \left(\frac{a_\odot}{r_\odot} \right)^3 (\alpha^2(1+4e^2) + \gamma^2(1-e^2) - (\frac{2}{3} + e^2) + \frac{2\alpha e r_\odot}{a}) \quad (46)$$

where the parameters α and γ are presented in Tresaco; Elife; Carvalho (2016) (see also equations (7) and (8)). Now second average is to be applied with respect to Sun's motion to both expressions. In order to perform it, we average over the true anomaly of the Sun ($f_\odot$), in equations (45) and (46). Thus, we obtain the doubly-averaged disturbing potential due to the effects of SRP+Sun gravitational attraction (CARVALHO, 2016)

$$\begin{aligned} \langle \langle R_{3b+SRP} \rangle \rangle = & \frac{15}{32} n_\odot^2 (\beta - 1) a^2 (1/2 e^2 (\cos(i_\odot) - 1) \times \\ & (\cos(i_\odot) + 1) (\cos(i) - 1)^2 \cos(2h - 2h_\odot - 2g) + 1/2 e^2 (\cos(i_\odot) - 1) (\cos(i_\odot) + 1) \times \\ & (\cos(i) + 1)^2 \cos(2h - 2h_\odot + 2g) + 2 \sin(i) \sin(i_\odot) e^2 \cos(i_\odot) (\cos(i) - 1) \times \\ & \cos(-2g + h - h_\odot) + 2 \sin(i) \sin(i_\odot) e^2 \cos(i_\odot) (\cos(i) + 1) \cos(2g + h - h_\odot) - \\ & 3/5 (\cos(i) + 1) (\cos(i_\odot) + 1) (\cos(i_\odot) - 1) (\cos(i) - 1) (e^2 + 2/3) \cos(2h - 2h_\odot) - \\ & \frac{12}{5} \sin(i) \cos(i_\odot) \cos(i) (e^2 + 2/3) \sin(i_\odot) \cos(h - h_\odot) + (((\cos(i))^2 - 1) (\cos(i_\odot))^2 + \\ & 2 (\sin(i))^2 (\sin(i_\odot))^2 + (\cos(i))^2 - 1) e^2 \cos(2g) - 3/5 (e^2 + 2/3) (((\cos(i))^2 + 1) \times \\ & (\cos(i_\odot))^2 - 5/3 + 2 (\sin(i))^2 (\sin(i_\odot))^2 + (\cos(i))^2)) (1 - e_\odot^2)^{-3/2} \end{aligned} \quad (47)$$

where the presence of the orbit inclination ($i_\odot = 3.3835^\circ$) and longitude of the ascending node ($h_\odot = 48.3396^\circ$) of the Sun appear in equation (47) due to the initial assumptions about its motion being elliptical and inclined. Here, these two variables related to the out-of-plane motion still appear, despite the average that removed the solar argument of the periapsis ($g_\odot$). It happens because the averaged is applied over a variable defined only in the equatorial reference plane Oxy , thus eliminating the angles defined over it, the argument of the periapsis and the true anomaly.

3.1.2 Non-uniform distribution of mass of Mercury

Now, it is necessary to obtain the expressions for the perturbations due to Mercury's irregular shape to be included in the analysis of the sail dynamics, the simplified model presented in expression (30) is gain used (see also Tresaco et al. (2018), Carvalho; de Moraes; Prado (2013) and Cardoso dos Santos et al. (2017)). In the present study, the potential due to the irregular shape of Mercury is the same used in chapter 2 to describe the irregular shape of Callisto and with the inclusion of the pear-shape and third-order term denoted as J_3 term. Thus, assuming Mercury's equatorial plane as the reference plane, this potential

is given by

$$U_M = -\frac{\mu}{r} \left[\sum_{n=2}^3 \left(\frac{R_M}{r} \right)^n J_n P_n(\sin \phi) - \left(\frac{R_M}{r} \right)^2 C_{22} P_{22}(\sin \phi) \cos(2\lambda) \right], \quad (48)$$

where μ is the gravitational constant of the Mercury, R_M is its mean radius, the angle ϕ is the latitude of the orbit with respect to Mercury's equator and λ is the longitude measured from the direction of the longest axis of Mercury. It is worth of mention that λ presents an explicit time dependence (GIACAGLIA; MURPHY; FELSENTREGER, 1970). We also have that P_n and P_{nm} are the Legendre polynomials and associated Legendre polynomials, respectively.

Using spherical trigonometry, it is possible to obtain the relation $\sin \phi = \sin i \sin(f + g)$. The Legendre polynomials and associated functions for the zonal (J_2) and sectoral (C_{22}) terms are then written in the form of the expressions (31) and (32) in chapter 2. The expression for J_3 can be written as (GIACAGLIA; MURPHY; FELSENTREGER, 1970; RAHOMA; ABD EL-SALAM, 2014)

$$P_3(\sin \phi) = \frac{5}{2}(s^3 \sin^3(f + g)) - \frac{3}{2}s \sin(f + g) \quad (49)$$

where again we defined $\xi \equiv \cos g \cos \Omega - c \sin g \sin \Omega$, $\chi \equiv -\sin g \cos \Omega - c \cos g \sin \Omega$, $s \equiv \sin i$, and $c \equiv \cos i$. Similarly to was done for expressions (33) and (34), we write, using the relation $\mu = n^2 a^3$, the potential due to expression (49):

$$U_{30} = -\frac{1}{2} \frac{a^4}{r^4} \frac{n^2}{a} \epsilon_1 (5s^3(\sin(f + g))^3 - 3s \sin(f + g)) \quad (50)$$

where $\epsilon_1 = J_3 R_M^3$.

The other perturbations due to the non-sphericity of Mercury given by expressions (33) and (34) (subsection 2.1.2) are rewritten below, and with proper subscripts to facilitate the reader:

$$U_{20} = -\frac{1}{2} \frac{a^3}{r^3} \epsilon n^2 (3s^2(\sin(f + g))^2 - 1) \quad (51)$$

$$U_{22} = \frac{a^3}{r^3} \delta n^2 (6\xi^2(\cos(f))^2 + 6\chi^2(\sin(f))^2 + 12\xi\chi \sin(2f) - 3 + 3s^2(\sin(f + g))^2) \quad (52)$$

where $\epsilon = J_2 R_M^2$ and $\delta = C_{22} R_M^2$. To write the long-period disturbing potential, we use the averaged technique presented in expression (13) ($\langle F \rangle = \frac{1}{2\pi} \int_0^{2\pi} F d\ell$) again.

After applying the average and some algebraic manipulations, the single-averaged model is obtained in the form

$$\langle R_{J3} \rangle = -\frac{3}{8} \frac{\epsilon \epsilon_1 n^2}{a(1-e^2)^{5/2}} (5s^2 - 4)s \sin(g) \quad (53)$$

As seen in expressions (35) and (36) the averaged potentials related to the other perturbations due to the non-spherical shape of Mercury are given by

$$\langle R_{J_2} \rangle = -\frac{1}{4} \frac{\epsilon n^2}{(1-e^2)^{3/2}} (3s^2 - 2) \quad (54)$$

$$\langle R_{C22} \rangle = -\frac{3}{2} \frac{\delta n^2}{(1-e^2)^{3/2}} (c^2 - 1) \cos(2h) \quad (55)$$

It is also necessary to express Mercury's rotation in the development of the disturbing potential, so that it is expressed by the variability of the ascending node h , a parameter represented as dependent on Mercury's angular velocity w . This dependence is expressed according to the expression given by Sehnal (SEHNAL, 1960; GIACAGLIA; MURPHY; FELSENTREGER, 1970; MEYER; BUGLIA; DESAI, 1994) as: $h = h_0 - wt$, where t is the time and $w = 6.1385^\circ/\text{day}$ (STARK; OBERST; HUSSMANN, 2015) is Mercury's angular velocity. Therefore, we can express the expression (55) as

$$\langle R_{C22} \rangle = -\frac{3}{2} \frac{\delta n^2}{(1-e^2)^{3/2}} (c^2 - 1) \cos(2h - 2wt) \quad (56)$$

Finally, the disturbing potential considered in this study can be written as a sum of the equations (47), (54), (53) and (56), finally assuming the form

$$R = \langle \langle R_{J_2} + R_{J_3} + R_{C22} + R_{3b+SRP} \rangle \rangle \quad (57)$$

which is similar to expression (37), however with the inclusion of two extra source of forces in the system: solar radiation pressure and the Mercury's pear-shape distribution of mass (J_3).

3.2 RESULTS

In this section it is presented a number of results which provide useful information about the dynamics of a solar sail around Mercury. The perturbations due to a third body, due to Mercury's non-spherical shape and SRP are assumed in the system by mean of the use of the disturbing potential presented in the expression (57). This expression is replaced in the Lagrange planetary equations (see section 2.2), a system of nonlinear and non-autonomous (due to C_{22} term), which are again numerically integrated using a proper codes developed using the software Maple. Thus, the time variation of the averaged orbital elements is obtained. An emphasis is given on finding less disturbed orbits, and possibly also frozen ones, which are the ones defined as orbits whose orbital parameters are constant or almost constant, namely the values for eccentricity, argument of the periaapsis and inclination. It is available in the literature a study presenting necessary and sufficient conditions to use a solar sail to minimize the effects due to other forces assumed in the dynamical model (OLIVEIRA; PRADO, 2015). Different orbits are analyzed using

Table 4: Shape parameters for Mercury

$C_{2,0}$	$C_{3,0}$	C_{22}	Mean radius (km)
-2.25045×10^{-5}	-4.76589×10^{-6}	1.24538×10^{-5}	2440.0

Source: Author. Data collected from Mazarico et al. (2014)

a similar procedure than the one adopted in the chapter 2. Special focus is given to the study of low-altitude orbits, specially those with periaapsis altitude between 200 km and 300 km, the desired ones to be used by one of the satellites of the BepiColombo mission. Thus, finding less disturbed orbits in this region can help on saving operational costs of this and other future missions to Mercury. High-altitude orbits also are investigated in order to test the effects of the parameter β .

It is important to remind that the relative orbit of Sun around Mercury is assumed to be elliptical ($e_{\odot} = 0.2056$) and inclined ($i_{\odot} = 3.3835^{\circ}$), meanwhile presenting a semi-major axis of $a_{\odot} = 5.7900 \times 10^7$, an argument of periaapsis of $g_{\odot} = 29.12^{\circ}$ and a longitude of the ascending node as $h_{\odot} = 48.34^{\circ}$ (see Stark; Oberst; Hussmann (2015) and Nasa/SSD (2018)). The table 4 present some physical parameters of Mercury as the normalized harmonics coefficients and its mean radius. Mazarico et al. (2014) provides the most updated data available regarding the spherical harmonic coefficients and it is worth of mention that the present study is one of first investigations to use these values for orbit analysis. Finally, we highlight the conversion adopted for the harmonic coefficients using the expression $J_n = -C_{n,0}$.

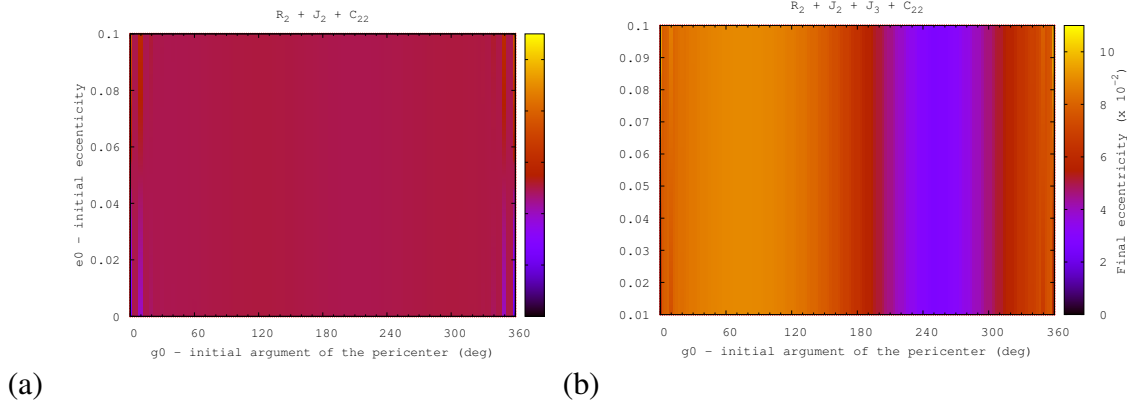
3.2.1 Low-altitude orbit without solar radiation pressure

After numerical tests, it was verified that, for low-altitude orbits (average altitude about 300 km around Mercury), the effects due to SRP are not significant in comparison with other perturbations. Therefore, the parameter β is chosen zero for the analysis made for this region. In such altitudes, the non-sphericity of Mercury and the gravitational pull due to the Sun produce more significant disturbances than SRP, so that it is possible to treat the solar sail as an usual artificial satellite/spacecraft. However, the solar sail is included when dealing with high-altitude orbits. The first results are presented bellow.

Figure 24 shows that, for a Mercury orbiter, surprisingly, the effect due to the coefficient J_3 is very significant, so that it is able to act as the known protecting effects due to the coupling of J_2 with C_{22} (CARVALHO; CARDOSO DOS SANTOS, 2017; CARDOSO DOS SANTOS et al., 2017), thus extending the lifetime of orbits by decreasing the disturbing effects over the eccentricity (see the region around $g_0 = 270^{\circ}$ in figure 24(b)).

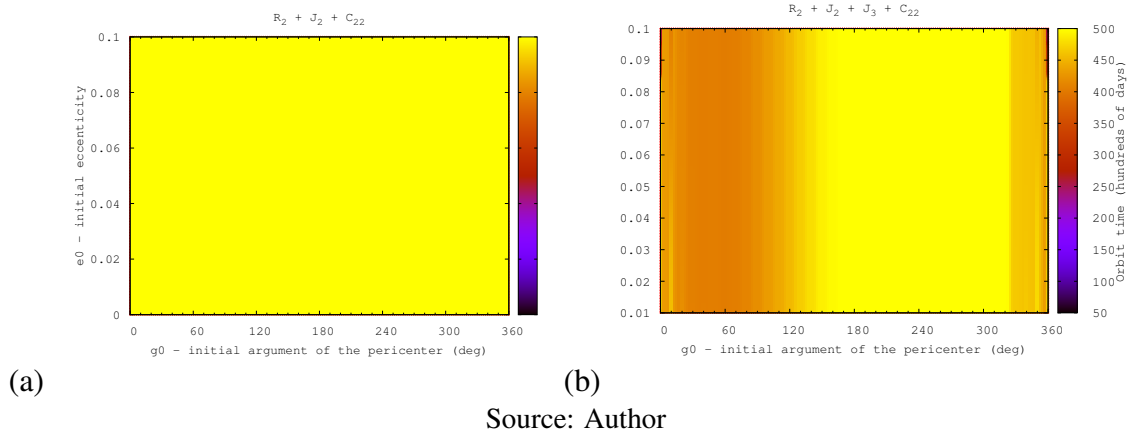
The resulting effect of this behavior appears in figure 25, where despite the inclusion of the non-symmetric disturbance due to J_3 , the same region around $g_0 = 270^{\circ}$ maintains orbits with longer surviving time. In fact, this characteristics helps to map initial conditions for the different parameters of the system, so that a search for frozen orbits can be

Figure 24: Final eccentricity map e versus g . Initial Conditions: $a = 2740.0$ km, $i = 90^\circ$, $h = 90^\circ$, $\beta = 0$. Perturbations: (a) $J_2 + C_{22} + R_{23b}$ and (b) $J_2 + J_3 + C_{22} + R_{23b}$



performed.

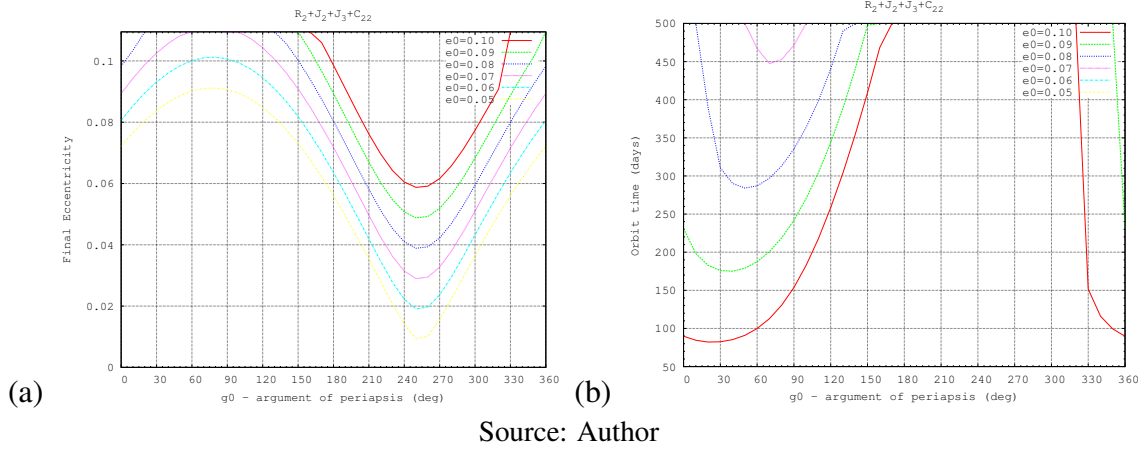
Figure 25: Orbit time map e versus g . Initial Conditions: $a = 2740.0$ km, $i = 90^\circ$, $h = 90^\circ$, $\beta = 0$. Perturbations: (a) $J_2 + C_{22} + R_{23b}$ and (b) $J_2 + J_3 + C_{22} + R_{23b}$



Once we know that the region around $g_0 = 270^\circ$ produces longer lifetimes, in averaged dynamics it might mean that, by adopting this value of g_0 , the dynamical system will present smaller oscillations in eccentricity. Thus, simulations for different values of e_0 are made to evaluate the impact of different values of g_0 over the eccentricity. Also, simulations using $g_0 = 270^\circ$ can be performed to evaluate if this value is helpful on finding other orbital parameters which produces small oscillations in the orbital elements, thus contributing to find regions with potential to host frozen orbits.

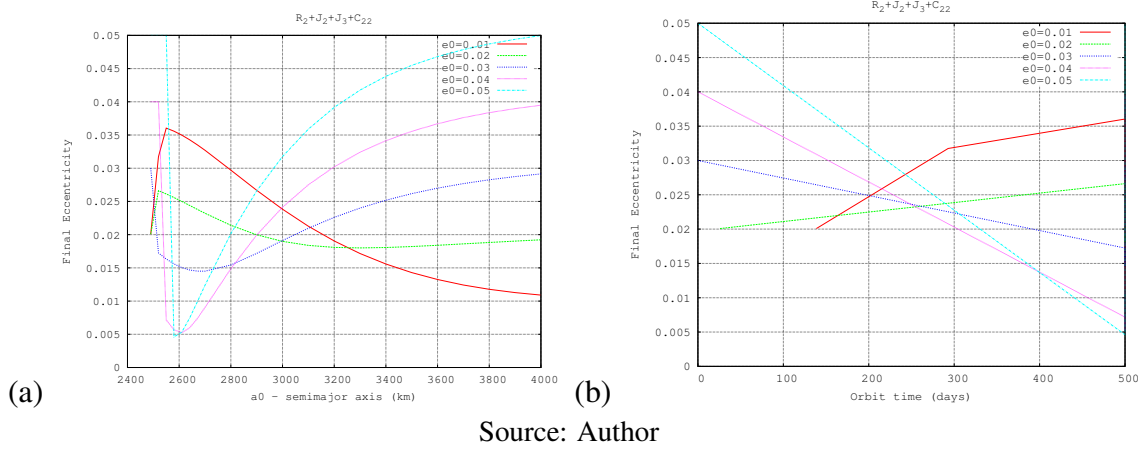
Figures 26 and 27 present the outcome of these numerical analysis, where it can be highlighted that: (A) by adopting a value for g_0 in the region around $g_0 = 270^\circ$, the orbital eccentricity of the spacecraft will really be less disturbed (figure 26(a)); (B) As a consequence of the fact (A), lesser disturbed eccentricities will usually provide lesser approaches of Mercury's surface, decreasing the risk of impacts and meaning longer orbit

Figure 26: Representation of final eccentricity vs g_0 and orbit time versus g_0 considering different values for e_0 . Initial Conditions: $a = 2740.0$ km, $i = 90^\circ$, $h = 90^\circ$, $\beta = 0$. Perturbations: $J_2 + J_3 + C_{22} + R_{23b}$



Source: Author

Figure 27: Representation of final eccentricity vs a_0 and final eccentricity versus orbit time considering different values for e_0 . Initial Conditions: $i = 90^\circ$, $g = 270^\circ$, $h = 90^\circ$, $\beta = 0$. Figure (b) also adopts $a = 2740.0$ km. Perturbations: $J_2 + J_3 + C_{22} + R_{23b}$

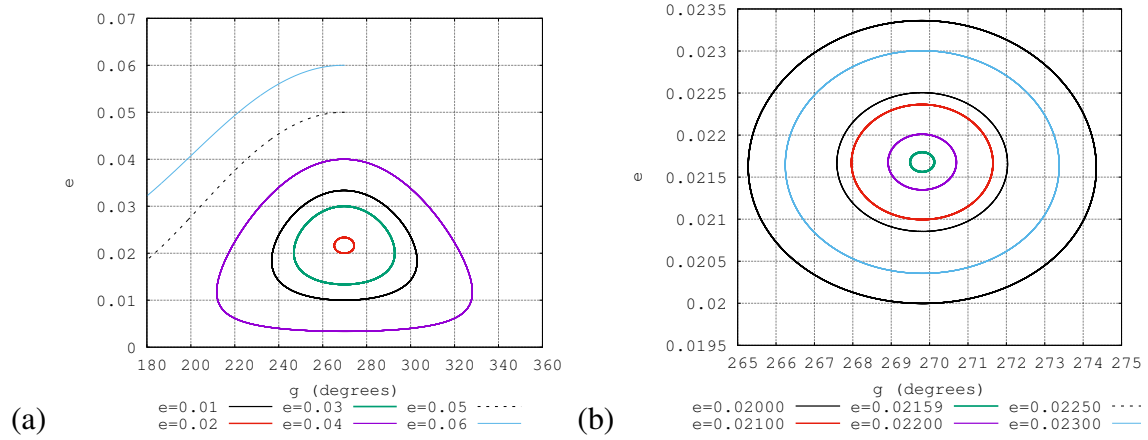


Source: Author

lifetimes (figure 26(b)); (C) the value of initial eccentricity $e_0 = 0.02$ produces smaller oscillations in final eccentricity along different values of semi-major axes (figure 27(a)) and orbit times (figure 27(b)), indicating possibly that a good candidate for a frozen orbit would be close to this value. Another comment on these figures is that the figure 27(a) seems to indicate a region ($2600 \text{ km} \leq a_0 \leq 2800 \text{ km}$) where eccentricities tend to get smaller values.

All these previous indicators are in accordance with the result obtained by Carvalho et al. (2018) and reproduced in figure 28, where the term J_3 is able to protect against the increase of the eccentricity, meanwhile proper values can be selected after using the previous figures, to find conditions for a frozen orbit. It is worth of mention that, in figure 28, some orbits librate around an equilibrium point and others are destroyed, *i.e.*, the orbits with eccentricity 0.05 and 0.06 circulates. In the latter case there is a precession in the

Figure 28: Integration of e versus g . Initial Conditions: $a = 2700.0$ km, $i = 90^\circ$, $g = 270^\circ$, $h = 90^\circ$, $\beta = 0$. Perturbations: (a) $J_2 + J_3 + C_{22} + R_{23b}$ and (b) $J_2 + J_3 + C_{22} + R_{23b}$ (zoom).



Source: Carvalho et al. (2018)

argument of the periapsis (CARVALHO et al., 2018).

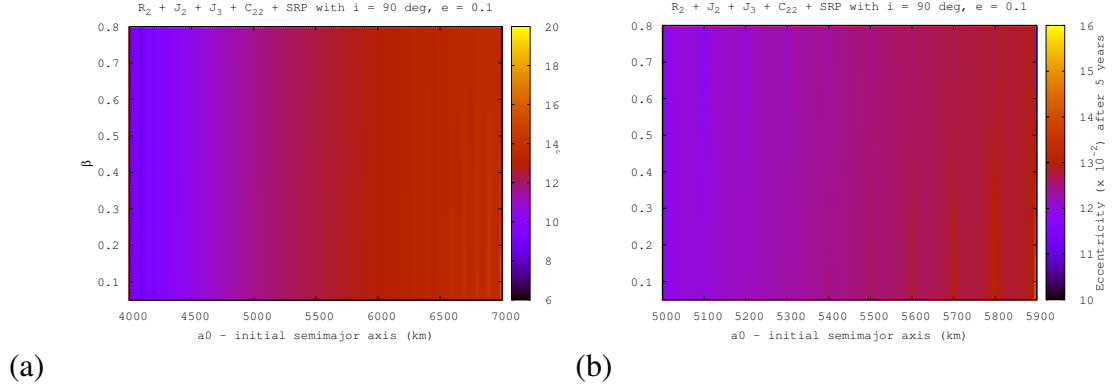
The diagram shown in figure 28(b) provides a zoom in the region of eccentricities between 0.0195 and 0.0235 where it is possible to note a number of frozen orbits. The conditions to find the frozen orbit with smallest amplitude are: $a = 2700.0$ km, $i = 90^\circ$, $g = 270^\circ$, $h = 90^\circ$ and $e = 0.02159$, which corresponds to an altitude of the periapsis of 201.707 km. Finally, when a similar analysis is made adopting $g = 180^\circ$, some orbits may collide rapidly or exhibit large amplitudes, showing that these orbits are not frozen (CARVALHO et al., 2018), as indicated by figure 24(b). All these values are close to the values mentioned used the analysis developed in the present section, point out it as a useful method to search and find orbits with smaller perturbations and, sometimes, also frozen ones.

3.2.2 High-altitude orbits with solar radiation pressure

The high-altitude orbits are strongly influenced by the gravitational perturbation of the Sun and SRP, especially when the spacecraft has a solar sail. It is evident when one analyzes figure 29(a), which shows a mapping for the value of the eccentricity as a function of initial conditions for β and the semi-major axis a . It is possible to see that, after five years of integration, the value of the eccentricity did change significantly when $a > 4600$ km (around 2140 km of altitude from the surface), *i.e.*, when the eccentricity departs significantly from 0.01 (blue colors). This is the region where SRP starts to play significant impact on the dynamics.

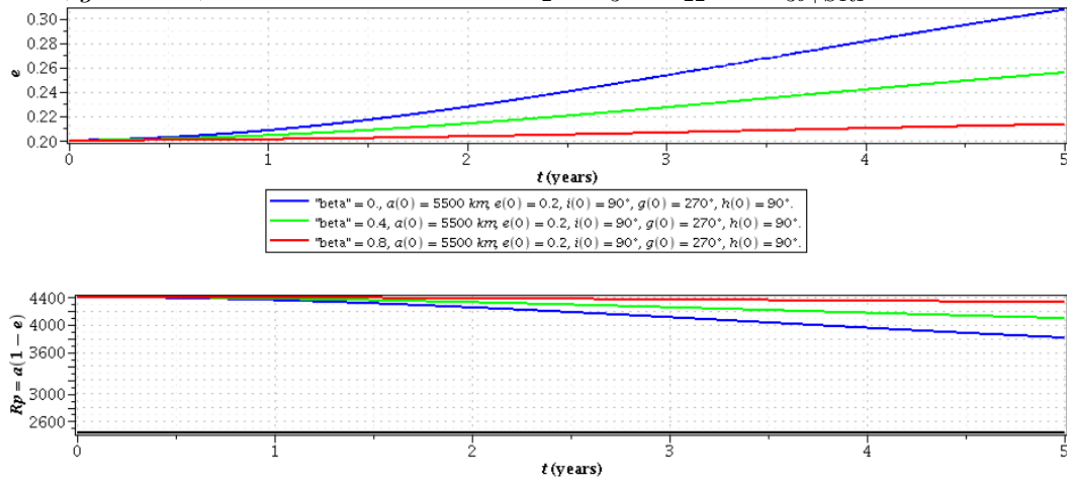
In order to highlight the protection mechanism of the solar radiation pressure against the perturbations due to other forces over the eccentricity, figure 29(b) presents a zoom in the region presented in figure 29(a). Thus, the higher is the value of β , the less the eccentricity will grow. This effect appears in figure 29(b) rather than in figure 29(a) because

Figure 29: Final eccentricity map varying the parameter β and the semi-major axis a . Initial Condition: $e = 0.1, i = 90^\circ, g = 270^\circ, h = 90^\circ$. Perturbations: $J_2 + J_3 + C_{22} + R2_{3b+SRP}$. (a) $4000 \text{ km} < a < 7000 \text{ km}$, (b) $5000 \text{ km} < a < 5900 \text{ km}$ (zoom)



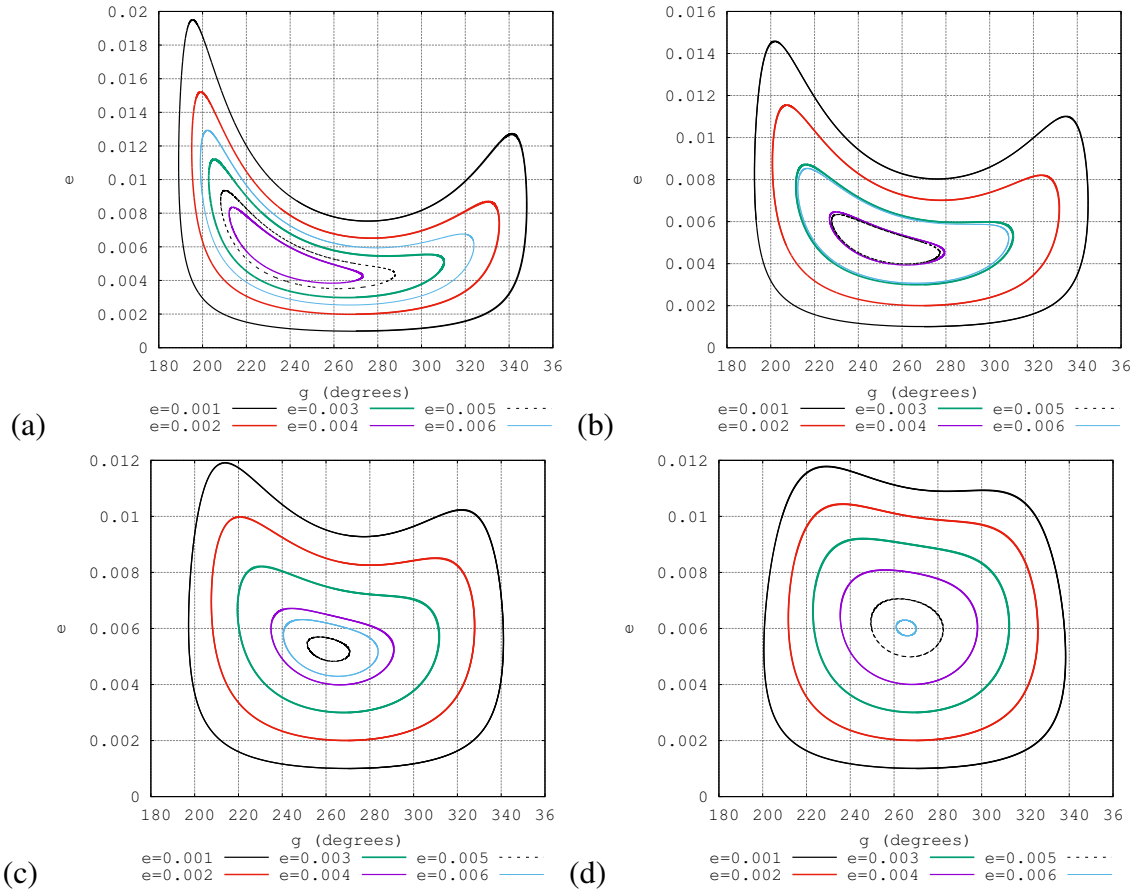
SRP effects are not significant in the low-altitudes when compared with other forces. This result is confirmed by results appearing in figure 30, where simulations were carried out for high-altitude eccentric orbits assuming different values for β , namely 0.0, 0.4 and 0.8. Thus, the effects due to SRP need to be accounted for, meanwhile the dynamics is controlled by β , a parameter which depends on the mass/area ratio. More details can be seen in McInnes (1999) and Tresaco; Elipse; Carvalho (2016). Therefore, the effects due to SRP are included when simulations for high-altitude orbits are carried out, as well as including the perturbations studied in the previous subsection.

Figure 30: Sample integration of e and $R_p = a(1 - e)$ for high-altitude eccentric orbits assuming different values for $\beta = \{0, 0.4, 0.8\}$. Initial Conditions: $a = 5500 \text{ km}, e = 0.2, i = 90^\circ, g = 270^\circ, h = 90^\circ$. Perturbations: $J_2 + J_3 + C_{22} + R2_{3b+SRP}$.



A similar analysis can be performed as done in section 3.2.2 to find frozen orbits or at least orbits with smaller amplitudes on desired orbital elements as possible. Just as a matter of illustration the results, after several simulations carried out, Carvalho et al.

Figure 31: Integration of e versus g . Initial Conditions: $a = 6000$ km, $i = 90^\circ$, $g = 270^\circ$, $h = 90^\circ$. Perturbations: $J_2 + J_3 + C_{22} + R2_{3b+SRP}$. (a) $\beta = 0$, (b) $\beta = 0.1$, (c) $\beta = 0.3$, (d) $\beta = 0.5$.



Source: Carvalho et al. (2018)

(2018) observed that the frozen orbits for high altitudes only appear for eccentricities of the order of 10^{-3} , where these results appear in figure 31. These results, along with the ones presented in figure 30 point out to the fact that the choice of area and mass of the solar sail is very important when high-altitude orbits are to be used. Thus, it is possible to play with the parameter β (ratio of area and mass) in order to find a desired motion for the solar sail around Mercury. By increasing the value of this parameter, the orbits will librate around the equilibrium point with smaller amplitudes, thus producing specific frozen orbits for each type of solar sail. The results also indicate the importance on the choice of the initial eccentricity is important (CARVALHO et al., 2018).

These results presented in this subsection show some characteristics of orbits taking into consideration different values for the parameter β . All the technique used during this investigation can support the proper choice of physical characteristics for a solar sail around Mercury, such as area and mass. Thus, the idea is the use of the parameter β and the appropriate initial conditions for the orbital motion of the solar sail, so that less disturbed orbit can be identified and used in space mission planning.

3.2.3 Low-altitude orbits, another approach without solar radiation pressure

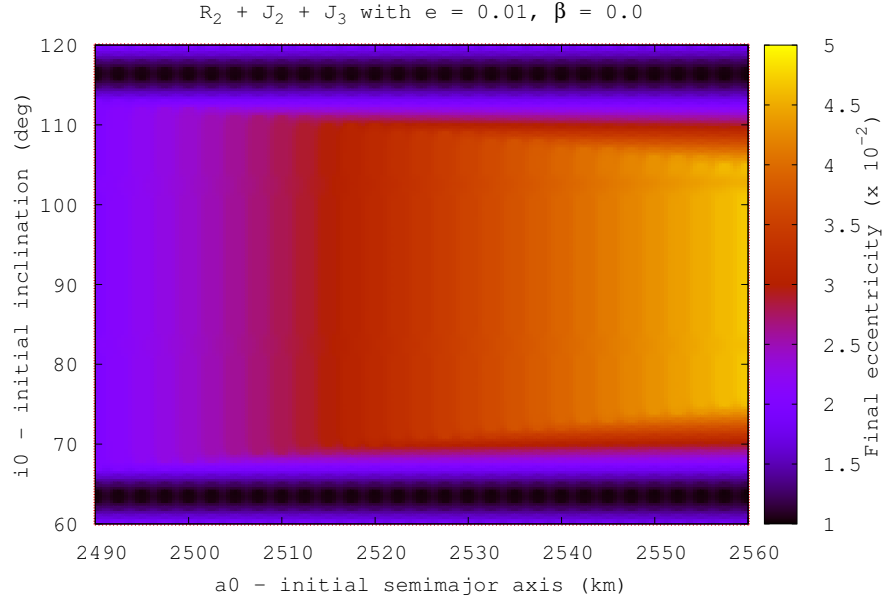
Using the present doubly-averaged models and performing an analysis of the low-altitude and high inclined region, we found a region ($2490 \text{ km} < a < 2550 \text{ km}$) where the perturbations over the orbit of the spacecraft increase in accordance with the Kozai-Lidov mechanism (KOZAI, 1962; LIDOV, 1962). It is in accordance with the results presented in the previous chapter where a Callisto orbiter was studied. It means that the more inclined are the orbits, the larger is the disturbance, with special importance on the growth of the eccentricity, impacting their lifetimes. The reason why this effect appears in this low altitude can be connected to the fact that the disturbing potential due to the irregular shape of Mercury is not able to affect significantly the orbiting mass in high altitudes and low inclinations, decreasing the effects due to the third body. In this region, the effects due to SRP are not significant also, so that the third-body perturbation is dominant, providing the conditions to the Kozai-Lidov mechanism to appear here.

In order to observe the effects due to each perturbation, maps i versus a are presented in figures 32 to 35. Two couples of maps are presented: one for the final eccentricity and one for the lifetime, both for an integration of 500 days. The maps considering perturbations due to R_2 , J_2 and C_{22} are omitted, once they did not present significant perturbations; In these cases, the orbits survived during all the 500 days of integration. However, when the perturbation due to J_3 is included, the Kozai-Lidov disturbing effect appears strong in the region analyzed (see figures 32 and 33). At these low altitudes, SRP does not contribute significantly. Despite J_3 having one order of magnitude smaller than J_2 and C_{22} , this term is responsible for describing the pear-shape of the body, which will impact significantly in high-altitude orbits and does not present the force compensations over one orbital period, as the other forces considered here. It is also important to note the maps presented in figures 34 and 35, where the effects due to C_{22} is added to the dynamics and decreases the disturbing effects in the region. In fact, this natural "protection mechanism" due to C_{22} is expected, when it is coupled with other terms of the disturbing potential (CARVALHO et al., 2012A; CARDOSO DOS SANTOS et al., 2017).

The distance of the periapsis given by the expression $R_p = a(1 - e)$ is an important parameter that impacts the surviving time of the orbits. As the present work deals with doubly-averaged models (where the semi-major axis a does not change in time), an analysis of the disturbing effects over the orbital eccentricity is important to observe the impact of each force in the lifetimes of the orbits. Figure 36 presents the results of the contribution of different forces for a polar orbit ($i = 90^\circ$) with the increase of the eccentricity up to the its critical value, where the collision with Mercury's surface occurs. The results show the higher impact due to J_3 and the decreasing effect resulting from the coupling with C_{22} , which is in agreement with the previous perturbation maps (compare figure 32 with figure 34 and figure 33 with figure 35). Finally, a diagram showing the contributions of different inclinations is shown in figure 37. It is possible to observe the existence of symmetry between prograde and retrograde orbits. The expected higher perturbation (and

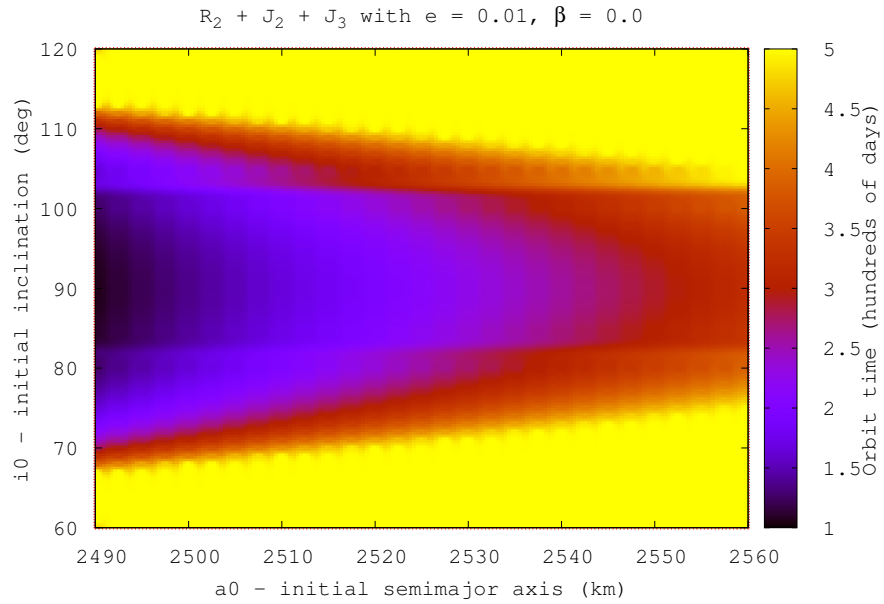
shorter lifetimes) due to $i = 90^\circ$ consistently appears in this diagram.

Figure 32: Lifetime map for a low-altitude orbit considering the final eccentricity of the orbit after 500 days. Initial Condition: $e = 0.01$, $g = 270^\circ$, $h = 90^\circ$, $\beta = 0$. Perturbations: $J_2 + J_3 + R_{23b}$



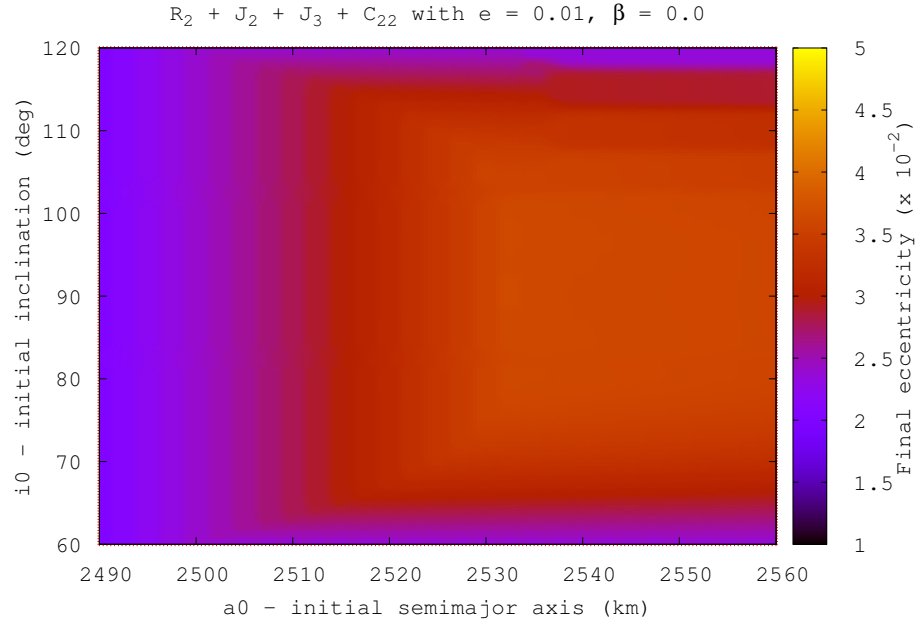
Source: Adapted from Carvalho et al. (2018)

Figure 33: Lifetime map for a low-altitude orbit considering the lifetime of the orbit for an integration of 500 days. Initial Condition: $e = 0.01$, $g = 270^\circ$, $h = 90^\circ$, $\beta = 0$. Perturbations: $J_2 + J_3 + R_{23b}$



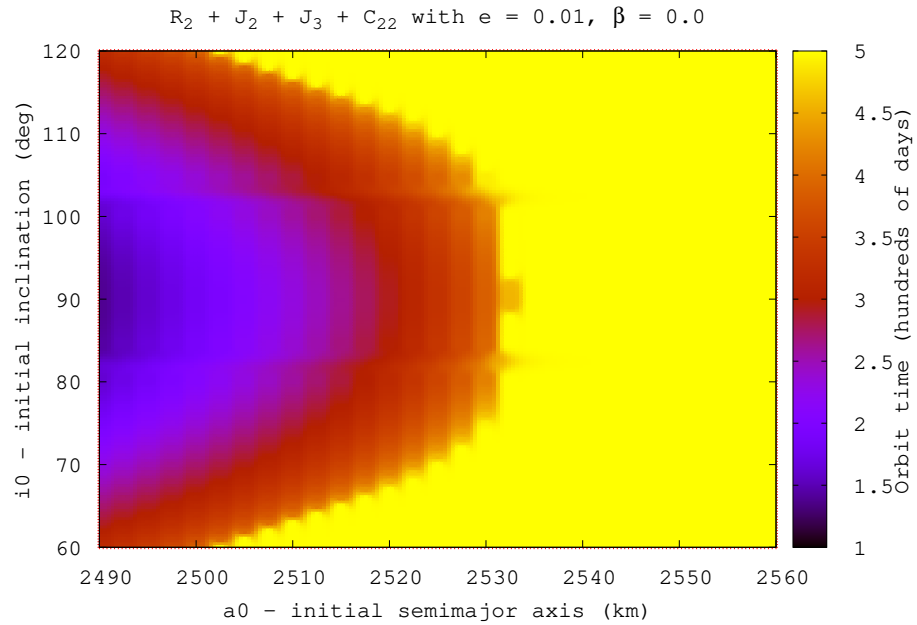
Source: Adapted from Carvalho et al. (2018)

Figure 34: Lifetime map for a low-altitude orbit considering the final eccentricity of the orbit after 500 days. Initial Conditions: $e = 0.01$, $g = 270^\circ$, $h = 90^\circ$, $\beta = 0$. Perturbations: $J_2 + J_3 + C_{22} + R_{23b}$



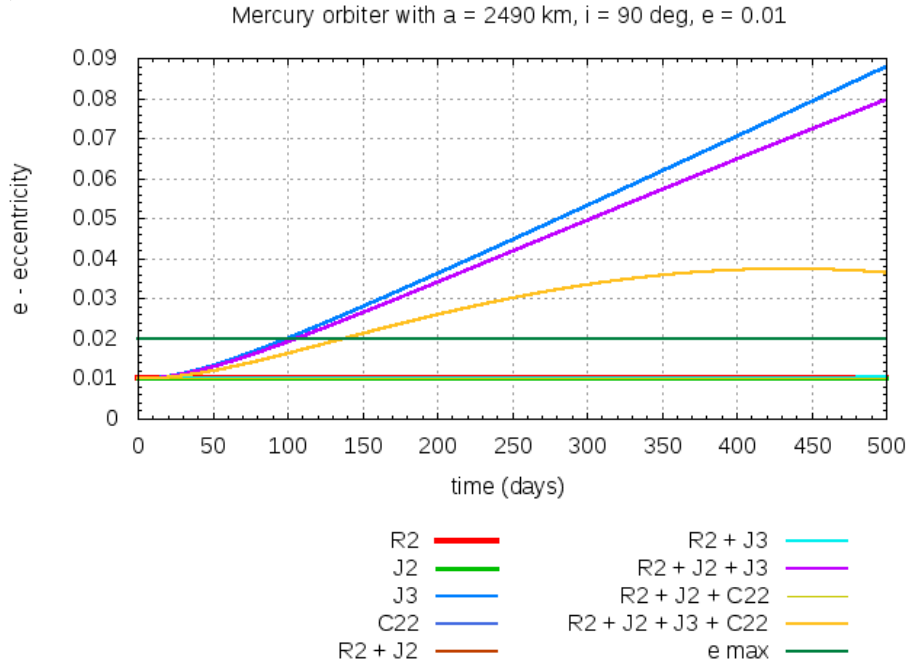
Source: Adapted from Carvalho et al. (2018)

Figure 35: Lifetime map for a low-altitude orbit considering the lifetime of the orbit for an integration of 500 days. Initial Condition: $e = 0.01$, $g = 270^\circ$, $h = 90^\circ$, $\beta = 0$. Perturbations: $J_2 + J_3 + C_{22} + R_{23b}$



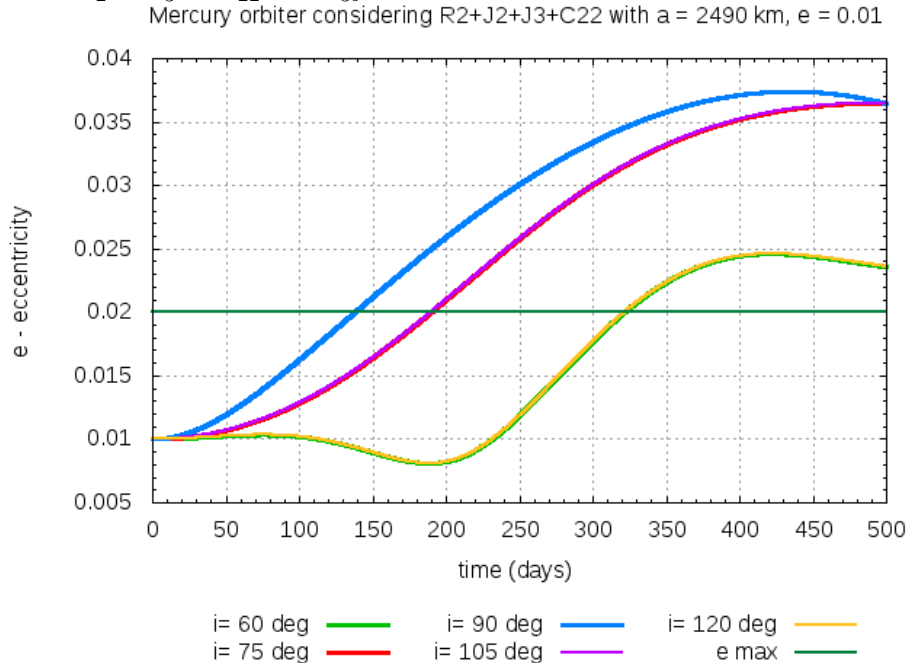
Source: Adapted from Carvalho et al. (2018)

Figure 36: Diagram e versus t (days) for different perturbations acting in an orbit. Time of integration of 500 days. Initial Condition: $a = 2490$ km, $e = 0.01$, $i = 90^\circ$, $g = 270^\circ$, $h = 90^\circ$, $\beta = 0$



Source: Carvalho et al. (2018)

Figure 37: Diagram e versus t (days) for different values of inclination. Time of integration of 500 days. Initial Conditions: $a = 2490$ km, $e = 0.01$, $g = 270^\circ$, $h = 90^\circ$, $\beta = 0$. Perturbations: $J_2 + J_3 + C_{22} + R_{23b}$



Source: Carvalho et al. (2018)

3.3 DISCUSSION

In this chapter, it is presented a study about the dynamics of a spacecraft using a solar sail in orbit around Mercury and under the influence of external perturbations. The per-

turbations considered in the dynamical system are the ones due to the irregular shape of Mercury (central body), the gravitational attraction of the Sun (third body) and the solar radiation pressure (SRP). The Sun is assumed to be a disturbing body in an elliptical and inclined orbit. Using the approach and the methodology adopted in this chapter, it was possible to identify regions to place a solar sail around Mercury with smaller amplitude in the variations of some orbital elements, as eccentricity, inclination and argument of the periapsis. These regions are, therefore, seem as potential candidates to present frozen orbits, once the small variations on these mentioned parameters are necessary to define this special class of orbits. In the case of low-altitude orbits, after several numerical simulations a frozen orbit was obtained with the following initial conditions: $a = 2700.0$ km, $i = 90^\circ$, $g = 270^\circ$, $h = 90^\circ$ and $e = 0.02159$, corresponding to an altitude of the periapsis of 201.70 km, and that has potential to contribute on the mission planning and design of a space mission such as BepiColombo. We also have shown that the pear-shape distribution of mass (J_3) of Mercury and its equatorial ellipticity (C_{22}), affected significantly the dynamics, the first generally improving the perturbations and second strongly reducing the growth of orbital eccentricity, keeping the orbit frozen. Perturbation maps to study the low-altitude and high-altitude regions around Mercury were presented, so that this impact of the J_3 term and the decreasing effect resulted from the coupling with C_{22} were evident. It means the these parameters can also contribute for the decrease or for the extension of the long-period lifetime of the orbits, respectively. For some orbital parameters tested during the study, for instance the region around $g = 270^\circ$, the J_3 term was able to help on freezing the orbit. Therefore, the C_{22} term contributes to decrease the disturbing effects in this region of low-altitude orbits around Mercury, as well as the J_3 term does for some regions identified.

The case of high-altitude orbits presents motion strongly perturbed by the gravitational attraction of the Sun and smaller effects due to its direct radiation (SRP), especially when the spacecraft has a solar sail. Due to the sensibility of the region, after performing several simulations, the results point out that the frozen orbits in high altitudes only appear for small eccentricities of the order of 10^{-3} . However, the methodology used in the present investigation is not only important to identify regions with small perturbations and frozen orbits, but also helpful on providing support to select the suitable physical parameters that a solar sail must have to accomplished the desired dynamics for each type of mission.

Finally, the ideas developed in the present investigation can also be applied to study the lifetime of small debris around celestial objects under the influence of SRP effect, as in the case of systems of rings.

Some of the main results of the study presented in this chapter were published in Carvalho et al. (2018). Some other results are a complement of this work and will be presented and discussed in a conference held in Germany (CARDOSO DOS SANTOS et al., 2018).

4 SEARCH FOR LESS DISTURBED REGIONS FOR SPACECRAFTS IN PLANETARY SYSTEMS

Potential celestial bodies to receive deeper in situ studies in a near future are the giant planets and some of their planetary moons. These bodies and their respective systems of moons are an open laboratory for scientists to obtain a better understanding about many phenomena, like the atmospheric chemistry and dynamics of all the planets, including the Earth. These studies are performed under more critical conditions than in the Earth, open a deeper range of possible analysis for the understanding about many phenomena that affect our daily lives. Among such topics are: atmospheric studies, gravitational and magnetic fields, volcanism, tidal, meteoric impacts and so forth. These topics bring several questions that can be better answered after in situ investigations using different equipments coupled to spacecrafts designed to visit these planetary systems. The use of perturbation maps (SANCHEZ; HOWELL; PRADO, 2016) to quantitatively describe the effects of different forces and for different regions around these planetary systems has become an important topic in astrodynamics and its investigation is of fundamental importance for saving costs in space mission operations. The findings of less disturbed regions is, for example, very useful for spacecraft observations and estimation of station-keeping costs. Generally, such orbits are also easier to be controlled and they also have lower changes in the orbital energy, which help the design of specific science orbits having these characteristics. Concerning the applications for the astrodynamics of these missions, several studies have tried to explore the orbital dynamics of such systems (GIACAGLIA; MURPHY; FELSENTERGER, 1970; SCHEERES; GUMAN; VILLAC, 2001; PRADO, 2003; LARA; SAN JUAN, 2005; DE SAEDELEER, 2006; LARA; RUSSELL, 2007; PASKOWITZ; SCHEERES, 2006; RUSSELL; BRINCKERHOFF, 2009; CARVALHO; VILHENA DE MORAES; PRADO, 2010; CARVALHO et al., 2012A; CARVALHO; DE MORAES; PRADO, 2013; CARDOSO DOS SANTOS et al., 2015; CARVALHO et al., 2018; CARDOSO DOS SANTOS et al., 2017). In this context, the present investigation proposes the use of some techniques, called *PerturbationIntegrals* or *IntegralIndexes* (PRADO, 2013, 2014; LARA, 2016), to search for orbits that have smaller perturbations, to help to get the less disturbed regions for a scientific orbit, extracted from the perturbation maps. Thus, it will be an useful tool to provide physical information about the system, supporting not only the design and planning of a space mission, but also making contributions on the astronomical side. It is confirmed by the recent discovery of a ring system around the dwarf planet Haumea (ORTIZ et al., 2017):

the region where the rings were discovered fits the region with smaller net perturbations in the system and which was predicted by (SANCHEZ; HOWELL; PRADO, 2016) using the *Integral Indexes*. In this context, the present chapter proposes to use techniques similar to the ones introduced by Prado et al. (2016) and Lara (2016). These techniques based in integrals, which are useful to make perturbation maps of regions around any celestial body, are then applied in order to map different regions around giant planets with massive moons, including regions between some of their main moons.

An analytical model for each perturbation is used considering the restricted three-body problem and the effects of the irregular gravity field of the central body of the system. Here, the central body is assumed to be a giant planet being orbited by a spacecraft, where the flattening of the planet is described by the J_2 harmonic term. The disturbances due to a third-body in the system are assumed to be due to the planetary moons. The present investigation intends to complement studies found in the literature.

The basic structure of this chapter is shown next. Section 4.1 introduces the concept of *Integral Indexes*, used to quantify different disturbing regions present in the dynamical system, as well the methods and equations of motion adopted; in section 4.2, numerical results for the different types of perturbation integrals are presented for the systems of Jupiter, Saturn and Neptune; finally, the section 4.3 presents a discussion about the results, as well some conclusions.

4.1 THEORETICAL FUNDAMENTS

The concept of using "*Integral Indexes*" as an useful tool to analyze the dynamics of a spacecraft's orbit was introduced recently in literature (PRADO, 2013, 2014; LARA, 2016). Since these very first works, this technique has been used for several studies in literature, having applications in estimating orbital maneuvers, fuel consumption, maneuvering time, and so forth (CARVALHO et al., 2013; OLIVEIRA; PRADO, 2014; SANCHEZ; PRADO; YOKOYAMA, 2014; CARVALHO; VILHENA DE MORAES; PRADO, 2014; CARDOSO DOS SANTOS et al., 2015). In these applications, the main effect of the forces involved in the system were to change the velocity of the spacecraft, according to the physical law of impulse (CARDOSO DOS SANTOS et al., 2015).

The present study uses three different *Integral Indexes* that were defined and studied in the literature (PRADO et al., 2016; PRADO, 2013; LARA, 2016) to analyze the orbital dynamics around different celestial bodies, specially in the context of the design of science missions. Each *Integral Index* will be denoted by PI_1 , PI_2 and PI_3 .

4.1.1 The Integral Approaches and concepts

The first PI considered in the present investigation is called PI_1 and it is defined as (PRADO et al., 2016):

$$PI_1 = \frac{1}{T} \int_0^T |\mathbf{a}| dt \quad (58)$$

where $\mathbf{a}$ is the disturbing acceleration vector, t is the time and T is the period of the orbit.

This perturbation integral is useful to map orbits with respect to oscillations during the orbital period, when compared to a Keplerian reference orbit. Once this method measures the norm of the perturbations over the time, it is useful to observe the orbits which are more or less disturbed during one orbital period of the spacecraft.

The second model for the *Integral Indexes* considered here is called PI_2 and it is defined as (PRADO et al., 2016):

$$PI_2 = \frac{1}{T} \int_0^T \mathbf{a} \cdot \mathbf{v} dt \quad (59)$$

where $\mathbf{a}$ is the disturbing acceleration vector experienced by the spacecraft, $\mathbf{v}$ is the velocity vector of the spacecraft, t is the time and T is the period of the orbit.

This approach is applied in order to measure the energy transferred to the spacecraft after one revolution, so it can detect effects of close approaches (PRADO; BROUCKE, 1995; GOMES; PRADO, 2010).

The third integral considered in this study is the one defined in Lara (2016):

$$PI_3 = \frac{1}{T} \left[\left(\int_0^T a_x dt \right)^2 + \left(\int_0^T a_y dt \right)^2 + \left(\int_0^T a_z dt \right)^2 \right]^{\frac{1}{2}} \quad (60)$$

where $\mathbf{a} = (a_x, a_y, a_z)$ is the disturbing acceleration which the spacecraft is subject to, and once again, t is the time and T is the period of the orbit.

This integral is useful to map orbits with respect to the net result due to the perturbations after one revolution, not considering the oscillations during the orbital period. Thus, it is the best choice to understand the long-term evolution.

4.1.2 Methods

A recent study (PRADO et al., 2016) used these three different *Integral Indexes* in order to investigate orbital motion of a spacecraft orbiting the Galilean moons of Jupiter. Perturbations maps were obtained to quantify the effects due to different forces that a particle experiences when traveling around these planetary moons. Similarly, the present study considers spacecrafts orbiting different regions around the giant planets Jupiter, Saturn and Neptune, assuming their flattening and some of the more massive moons as disturbing objects. Perturbation maps are presented for different regions by quantifying the values of the *Integral Indexes* in terms of different orbital elements. All the results are obtained by performing numerical integrations using proper codes for the software Mathematica and are useful for applications in astrodynamics, aerospace engineering, and celestial mechanics.

In the present gravitational model for the orbits, the influence of the flattening at the poles of the giant planet is considered in the disturbing potential. This point is particu-

larly important for orbits close to the planet. In the present analysis, the gravity coefficient considered is J_2 . The corrections due to the flattening at the poles of the central body help to obtain a model taking into account some anomalies that occur when the true gravitational potential is compared with the potential of a perfect spherical body. Similarly to this process, more gravity coefficients can be included in the analysis. For instance, in the study for orbits around the Galilean moons (PRADO et al., 2016), the effects due to the ellipticity of the central body equator was also included (C_{22} term). A study for the Haumea system included the effects of the irregular shape of this dwarf planet up to fourth order, *i.e.*, C_{40} , C_{42} and C_{44} terms (SANCHEZ; HOWELL; PRADO, 2016).

The potential related to J_2 can be written as (BROUWER; CLEMENCE, 1961; PASKOWITZ; SCHEERES, 2009):

$$R = -\frac{\mu J_2}{2} \left(\frac{3z^2}{d_{ps}^5} - \frac{1}{d_{ps}^3} \right) \quad (61)$$

where μ is the gravitational parameter of each central body (the resulting product of the universal gravitational constant times the summation of the masses of the planet and the spacecraft), J_2 is the term due to flattening at the poles of the planet, d_{ps} is the norm of the radius vector (distance) planet-spacecraft $\mathbf{d}_{ps}$ and z is the norm of the third component of $\mathbf{d}_{ps}$.

4.1.3 Equations of motion

The geometry of the problem considered in the present study takes into account the classical three body problem and the J_2 term of the gravity field of the central body (the planet). A geometrical representation of the system can be visualized in figure 2 and reproduced again in figure 38, where the reference plane is assumed as the equatorial plane of the central body/planet.

The equation (62) shows the model for the accelerations which the particle will experience during the flight through the system

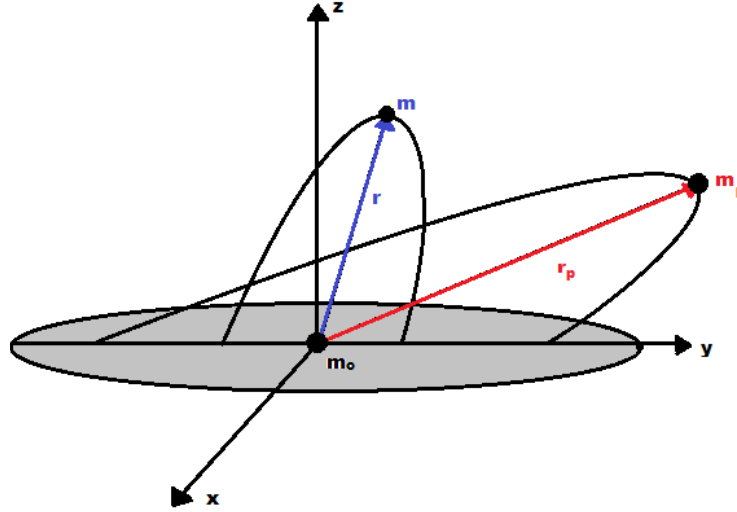
$$\mathbf{a}_t = \mathbf{a}_{planet} + \mathbf{a}_{moons} + \mathbf{a}_{J2} \quad (62)$$

where $\mathbf{a}_{planet}$ is the Newtonian acceleration due to the planet's central field, $\mathbf{a}_{moons}$ is the disturbing acceleration due to the moons considered in the system, $\mathbf{a}_{J2}$ is the disturbing acceleration due to the planet's flattening at the poles (computed using the expression (61)) and $\mathbf{a}_t = (a_x, a_y, a_z)$ is the resulting acceleration due to all these forces assumed in the system.

In the process of determining the motion of the spacecraft, it is possible to relate the Cartesian coordinates to the orbital elements of each orbit through classical equations from celestial mechanics¹. The orbital elements are here defined as: mean anomaly M that describe the motion of the spacecraft in a specific orbit; semi-major axis a , eccentricity e ,

¹see Brouwer; Clemence (1961), Murray; Dermott (2000) and Vallado (2007)

Figure 38: Geometrical scheme of the dynamical system



Source: Cardoso dos Santos et al. (2016a)

inclination i , argument of the pericenter ω and the longitude of the ascending node Ω . We refer the reader to the works of Brouwer; Clemence (1961), Murray; Dermott (2000) and Vallado (2007), as well to previous chapter for more details about these variables and the orbital motion.

4.2 NUMERICAL SIMULATIONS

The numerical simulations considering spacecrafts traveling in the systems of Jupiter, Saturn and Neptune are presented in tables 5 and 6.

The perturbation maps considering a spacecraft traveling in the system of moons of Jupiter include the Galilean moons Io, Europa, Ganymede and Callisto. The results for this system are presented in figures 39, 41 and 43. The tests considering a spacecraft traveling in the Saturnian system of moons include the moons Enceladus, Thetys, Dione, Rhea and Titan. As Titan is much more massive than these other small moons, the results show its much stronger contribution than these ones. These results are presented in figures 40, 42 and 44. The simulations for the Neptunian system considers Triton, which is much more massive and closer to the planet than the other moons. The results for this system are presented in figures 45 - 47.

The analysis of some results for Jupiter and Saturn is very similar. For example, the figure 39 and 40, respectively for Jupiter and Saturn, show the results considering the semi-major axis and the inclination as variables, with the other orbital elements fixed as shown in the legend. It is clear the symmetry with respect to the inclination, so direct and retrograde orbits have the same level of perturbation using all the three indexes. The same figures show that the perturbation decrease with semi-major axis.

Figures 41 and 42 show the results considering the eccentricity and the inclination as variables, with the other orbital elements fixed, as shown in the legend. Indexes 1 (PI_1)

Table 5: Gravity coefficients for Jupiter, Saturn and Neptune

Planet	Jupiter	Saturn	Neptune
$J_2 \times 10^6$	14695.62	16290.71	3408.43
$\mu (km^3/sec^2)$	126712765	37940585.0	6836527.0

Source: Author. Data collected from Nasa/SSD (2018)

Table 6: Physical and orbital data for objects considered in simulations

Moon	$\mu (km^3/sec^2)$	a_m (km)	e_m	ω_m (deg)	Ω_m (deg)	i_m (deg)	M_m (deg)
Io	5959.916	421800	0.0041	84.129	43.977	0.036	342.021
Europa	3202.739	671100	0.0094	88.970	219.106	0.466	171.016
Ganymede	9887.834	1070400	0.0013	192.417	63.552	0.177	317.540
Callisto	7179.289	1882700	0.0074	52.643	298.848	0.192	181.408
Enceladus	7.2027	238042	0.0000	0.076	342.507	0.003	199.686
Thetys	41.2067	294672	0.0001	45.202	259.842	1.091	243.367
Dione	73.1146	377415	0.0022	284.315	290.415	0.028	322.232
Rhea	153.9426	527068	0.0002	241.619	351.042	0.333	179.781
Titan	8978.1382	1221865	0.0288	180.532	28.060	0.306	163.310
Triton	1427.6	354759	0.0000	66.142	335.570	156.865	3522.257

Source: Author. Data collected from Nasa/SSD (2018)

and 3 (PI_3) keep the symmetry with respect to the inclination. The same figures show that the perturbation increases with the eccentricity. These results are confirmed by figures 43 and 44 which show the results considering the eccentricity and semi-major axis as variables, with the other orbital elements fixed as shown in the legend.

When we consider the Neptune-Triton system, the figures 45 to 47 show that the same behavior occurs. Thus, in order to illustrate the presence of regions where we have a more predominant effects of some of the disturbing forces, the figure 48 also present results for a spacecraft orbiting Neptune considering the effects due to Neptune's flattening at the pole and gravitational attraction of Triton. These results are obtained for a spacecraft which is co-orbital to Triton, a configuration where disturbing effects due to Triton are expected to be more intense. This effect appears, for instance, when one looks to the line/border that separates orange and brown regions in these figures, i.e., regions separated by the surface line with $PI_2 = 0$. Lighters colors have positive values for PI_2 . Darker colors present negative values (closer to Triton). So, we have regions where energy is being transfered to the spacecraft, and other regions where the spacecraft is loosing energy. The border where $PI_2 = 0$ is a region where we have a balance between the disturbing forces, so that the spacecraft would keep its energy constant. In practice, regions near to these borders can be useful for keeping spacecrafts, if we intend to minimize changes in the orbital elements.

A similar analysis than the one developed in the previous paragraph can be extended

to the results with minimal transfers of energy presented in plots (labels (b)) for PI_2 of the previous figures for the systems of Jupiter and Saturn, namely figures 39 to 41 and 40 and 42. On find these regions, we can find insights about the values for orbital parameters which minimize the orbital perturbations over the spacecraft motion. Indeed, the opposite idea can also be helpful, once to find regions with high values of perturbation is able to support the mission control to save costs during an orbital maneuver. In this case, the perturbations in the identified highly perturbed regions can be potentially used to modify the trajectory of the spacecraft instead of using the spacecraft's thrusters, providing savings regarding fuel-consumption and, consequently, related operational savings.

Figure 49(a) shows PI_1 due to the irregular shape of Saturn as a function of the semi-major axis of the orbit for a circular and planar orbit. Note the expected decay with the distance. Figure 49(b) shows the same quantity due to the perturbations coming from the moons. Note the peaks near the locations of the orbits of the moons. Figure 49(c) shows the same quantity due to the perturbations coming from the moons and the irregular shape of Saturn. Since the magnitude of the perturbation coming from Saturn is much larger, the peaks coming from the moons do not appear, in general. The exception happens for the last peak, where the effects of Saturn are smaller at this distance and Titan's non-negligible mass attracting the particle produces significant effects. Similar results will naturally happen when one analyzes the integral index PI_2 for a system with multiple massive moons, as the case of Jupiter and its Galilean moons. These results appear in figure 50, where we easily observe that closer to these moons, there are positive peaks in the energy exchanged with the particle, meaning that a close encounter with the moon has happened. Gravitational captures are also possible.

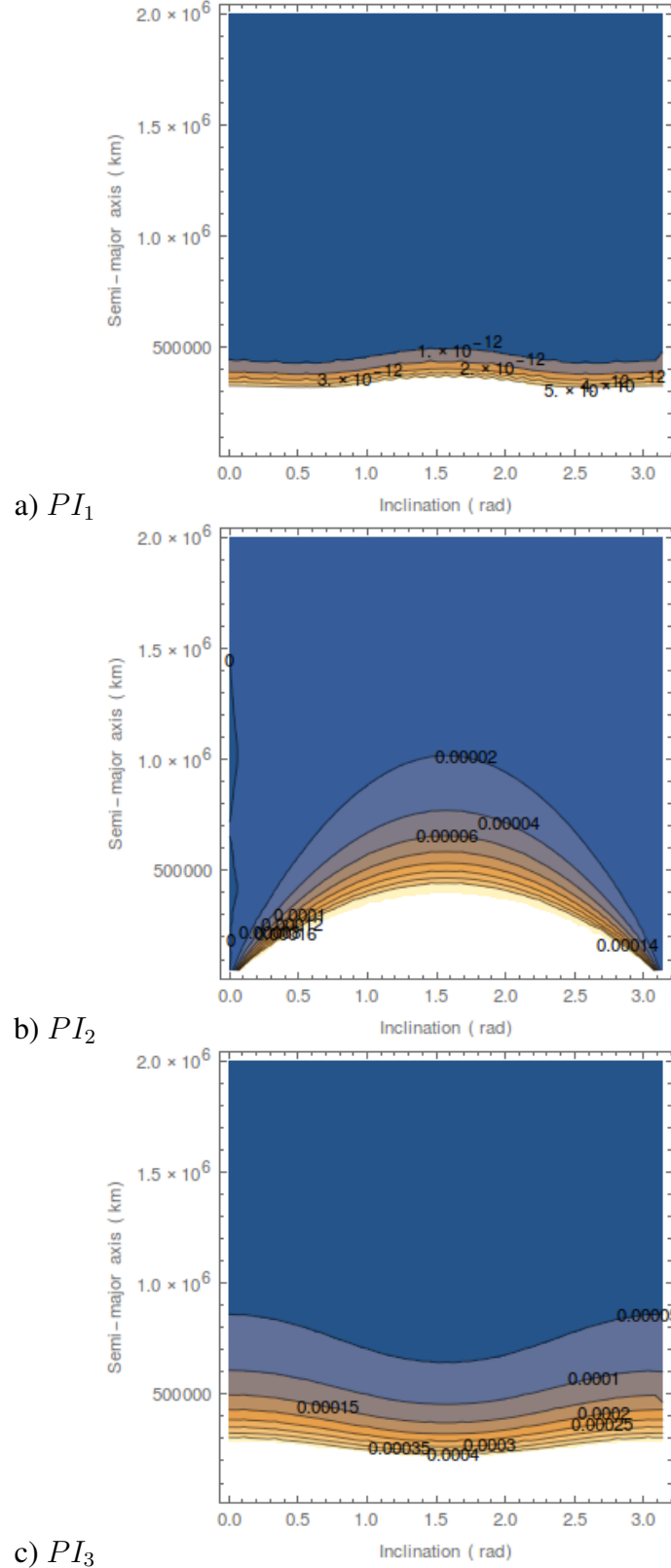
Figure 51(a) shows the evolution of PI_2 as a function of the inclination considering the total perturbation (moons+ J_2) for a circular orbit with semi-major axis 500,000 km. PI_2 is zero for the perturbations coming from the irregular shape of Saturn, which means that a plot considering only the perturbations coming from the five moons is similar to Figure 49(a). Note that there is always gains of energy, but the magnitude oscillates.

Figure 51(b) shows the evolution of PI_2 as a function of the semi-major axis considering the total perturbation (moons+ J_2) for a circular planar orbit. PI_2 is zero for the perturbations coming from the irregular form of Saturn, which means that a plot considering only the perturbations coming from the five moons is similar to Figure 49(a). Note that there are variations of energy, indicating that some orbits gain energy and some others loses energy from the perturbations due to the moons.

The results have shown that J_2 perturbation does not change the energy of the spacecraft, as well as one single third-body; those results are according to the expected, but when the four moons of Jupiter and the five moons of Saturn are included in the model, the combinations of the four and five "third-body" perturbations, respectively, changed the energy of the orbit. There are regions of gains and losses of energy, as showed in Figure 51. Of course, there are equilibrium points, where the energy transfered during

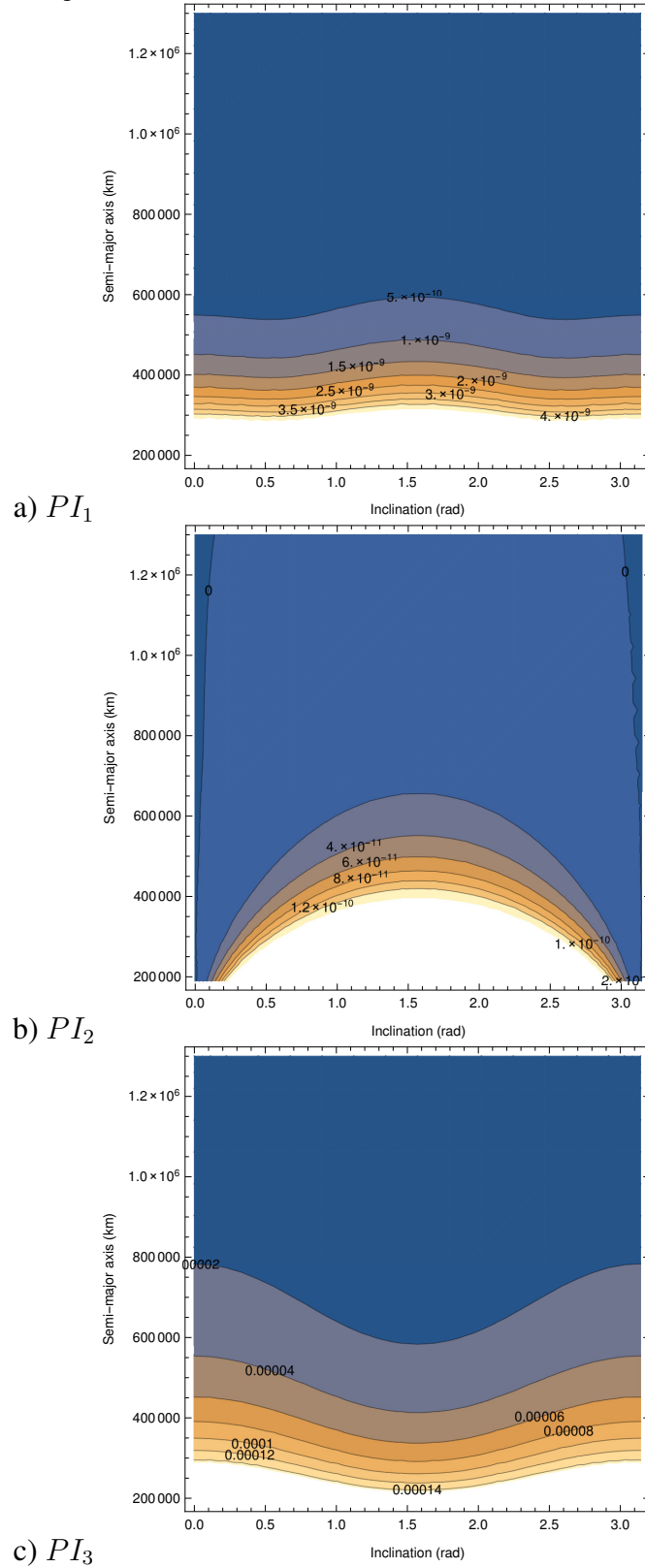
one orbital period is null, *i.e.*, the orbital energy is constant ($PI_2 = 0$).

Figure 39: Perturbation Integrals as functions of a (km) and i (radians) of the spacecraft and considering the effects of the J_2 of Jupiter + 4 Galilean moons. Initial conditions: $e = 0.1$, $\Omega = \frac{\pi}{4}$ rad, $\omega = \frac{\pi}{4}$ rad



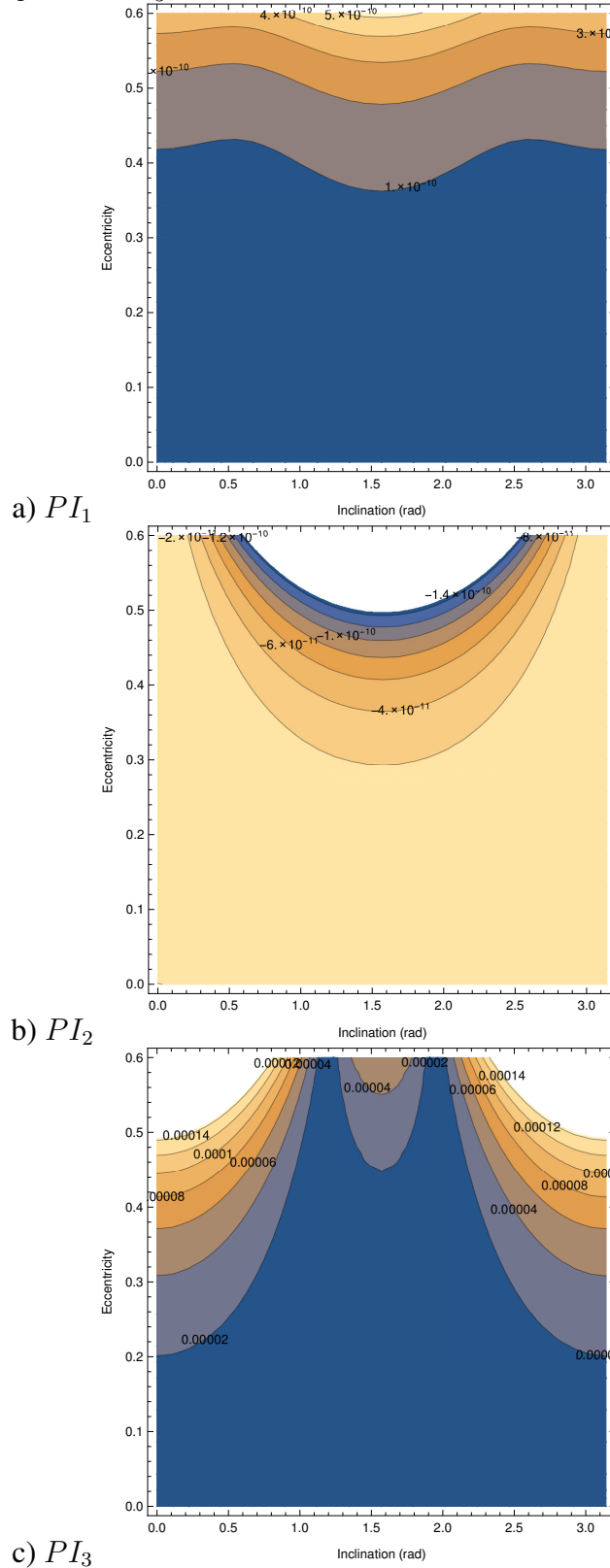
Source: Cardoso dos Santos et al. (2016b)

Figure 40: Perturbation Integrals as functions of a (km) and i (radians) of the spacecraft and considering the effects of the J_2 of Saturn + 5 Saturnian moons. Initial conditions: $e = 0.1$, $\Omega = \frac{\pi}{4}$ rad, $\omega = \frac{\pi}{4}$ rad



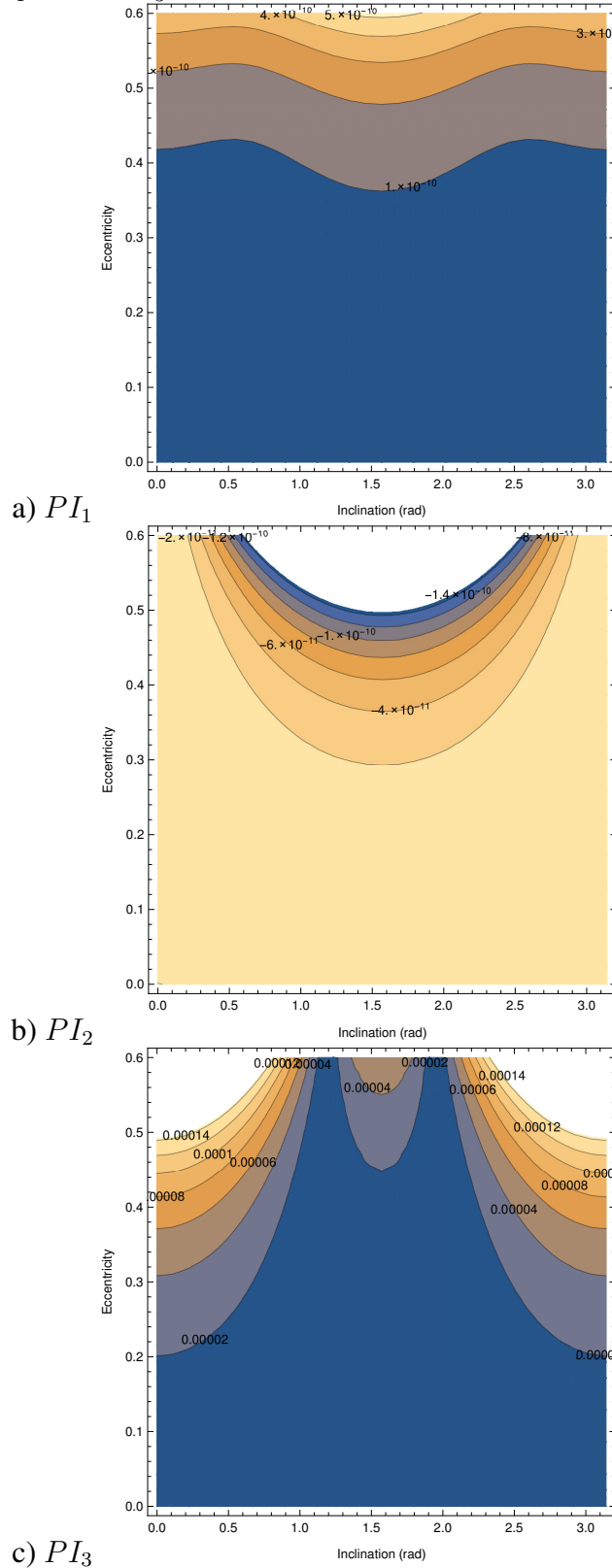
Source: Cardoso dos Santos et al. (2016a)

Figure 41: Perturbation Integrals as functions of e and i (radians) of the spacecraft and considering the effects of the J_2 of Jupiter + 4 Galilean moons. Initial conditions: $a = 1.200.000$ km, $\Omega = \frac{\pi}{4}$ rad, $\omega = \frac{2\pi}{3}$ rad



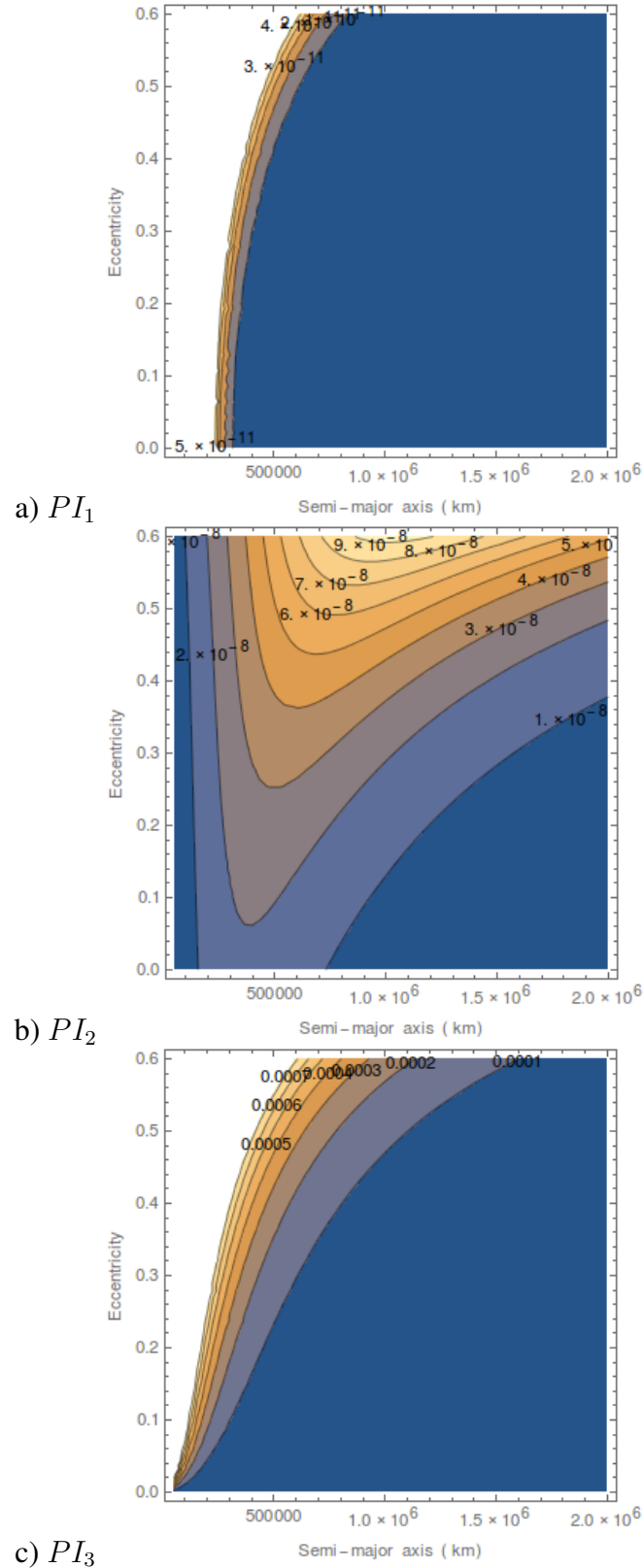
Source: Cardoso dos Santos et al. (2016b)

Figure 42: Perturbation Integrals as functions of e and i (radians) of the spacecraft and considering the effects of the J_2 of Saturn + 5 Saturnian moons. Initial conditions: $a = 1.200.000$ km, $\Omega = \frac{\pi}{4}$ rad, $\omega = \frac{2\pi}{3}$ rad



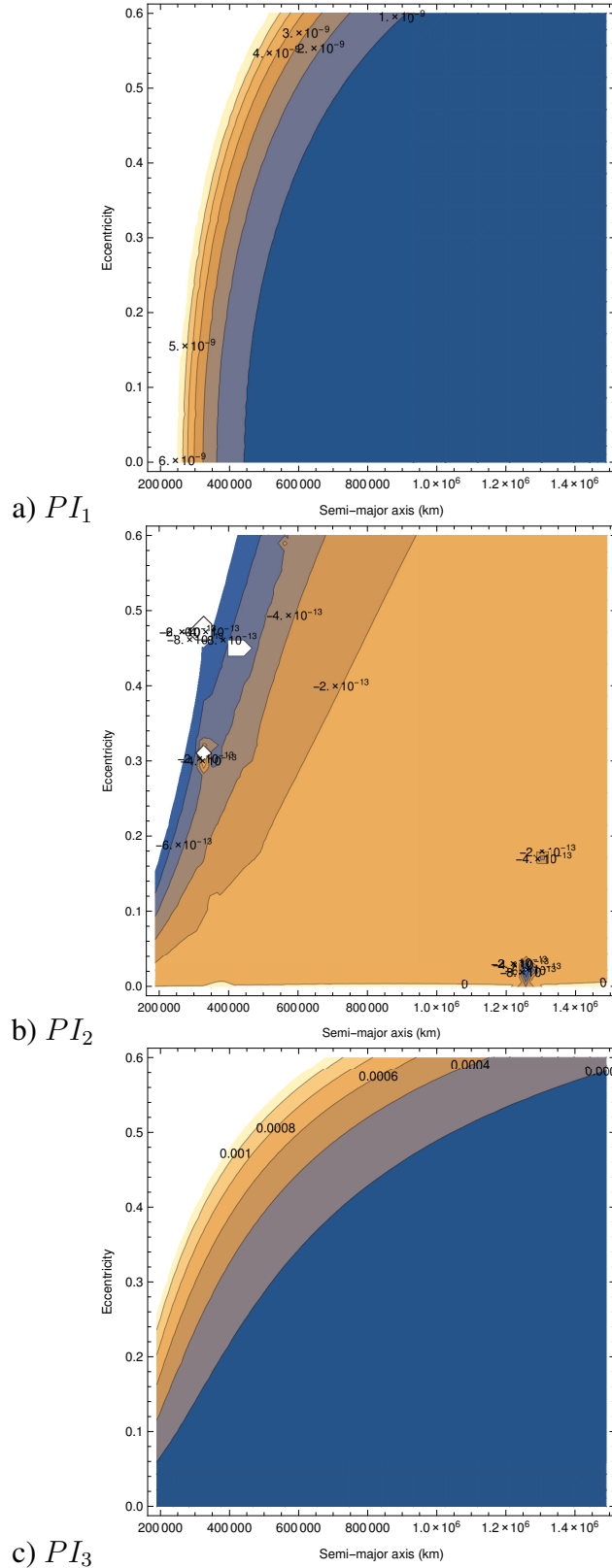
Source: Cardoso dos Santos et al. (2016a)

Figure 43: Perturbation Integrals as functions of e and a (km) of the spacecraft and considering the effects of the J_2 of Jupiter + 4 Galilean moons. Initial conditions: $i = 0$ rad, $\Omega = \frac{\pi}{4}$ rad, $\omega = \frac{\pi}{2}$ rad



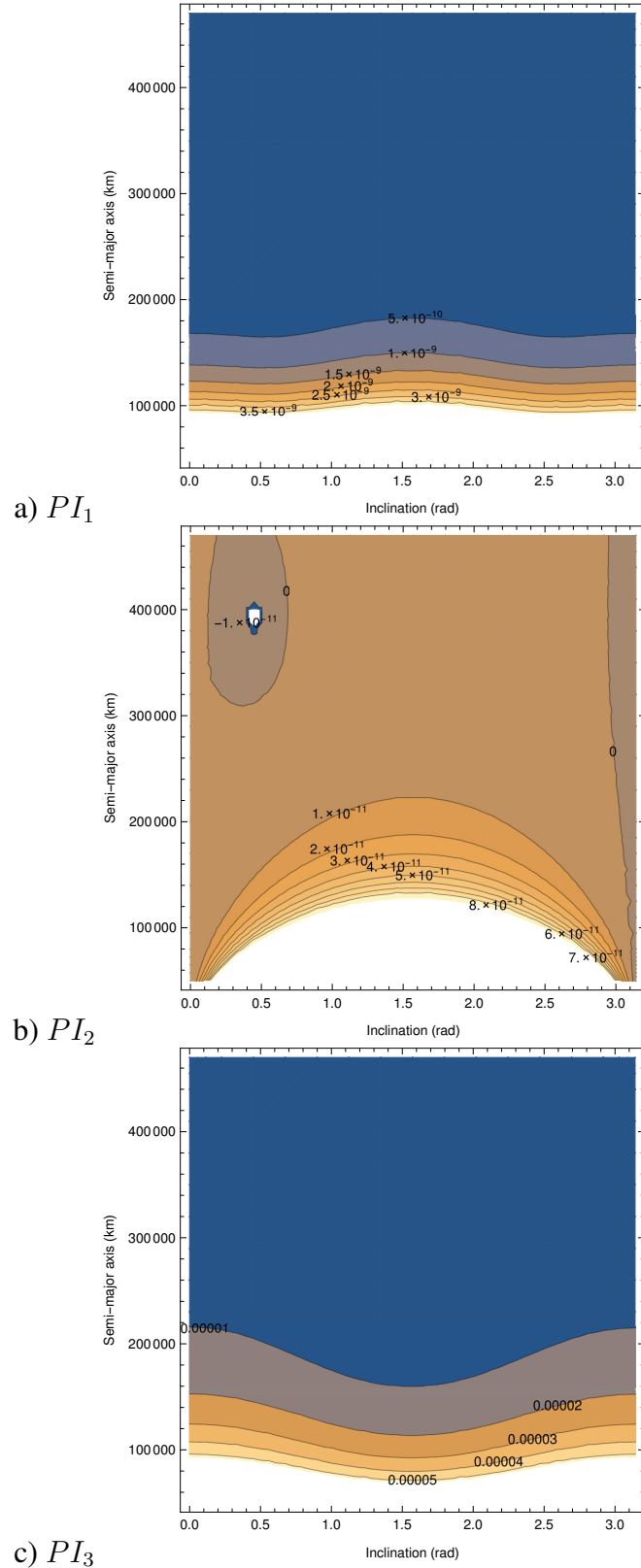
Source: Cardoso dos Santos et al. (2016b)

Figure 44: Perturbation Integrals as functions of e and a (km) of the spacecraft and considering the effects of the J_2 of Saturn + 5 Saturnian moons. Initial conditions: $i = 0$ rad, $\Omega = \frac{\pi}{4}$ rad, $\omega = \frac{2\pi}{3}$ rad



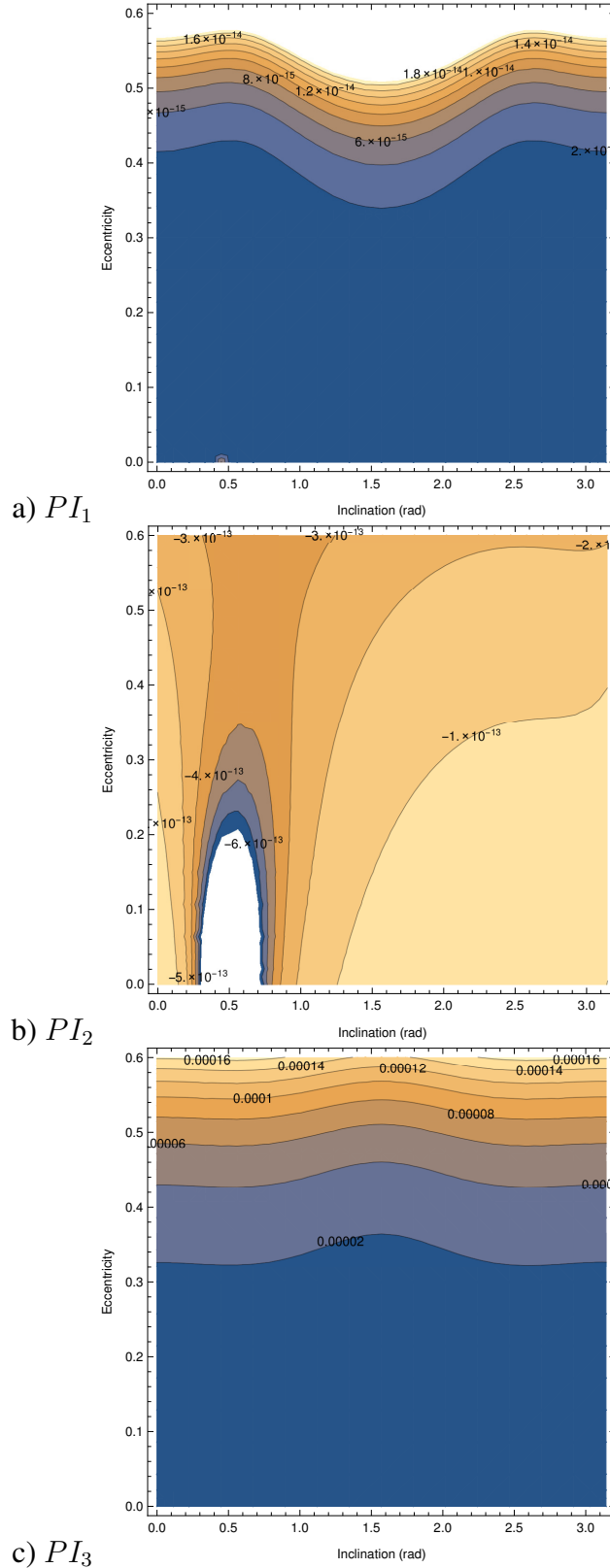
Source: Cardoso dos Santos et al. (2016a)

Figure 45: Perturbation Integrals as functions of a (km) and i (radians) of the spacecraft and considering the effects of the J_2 of Neptune + Triton. Initial conditions: $e = 0.1$, $\Omega = \frac{\pi}{4}$ rad, $\omega = \frac{\pi}{4}$ rad



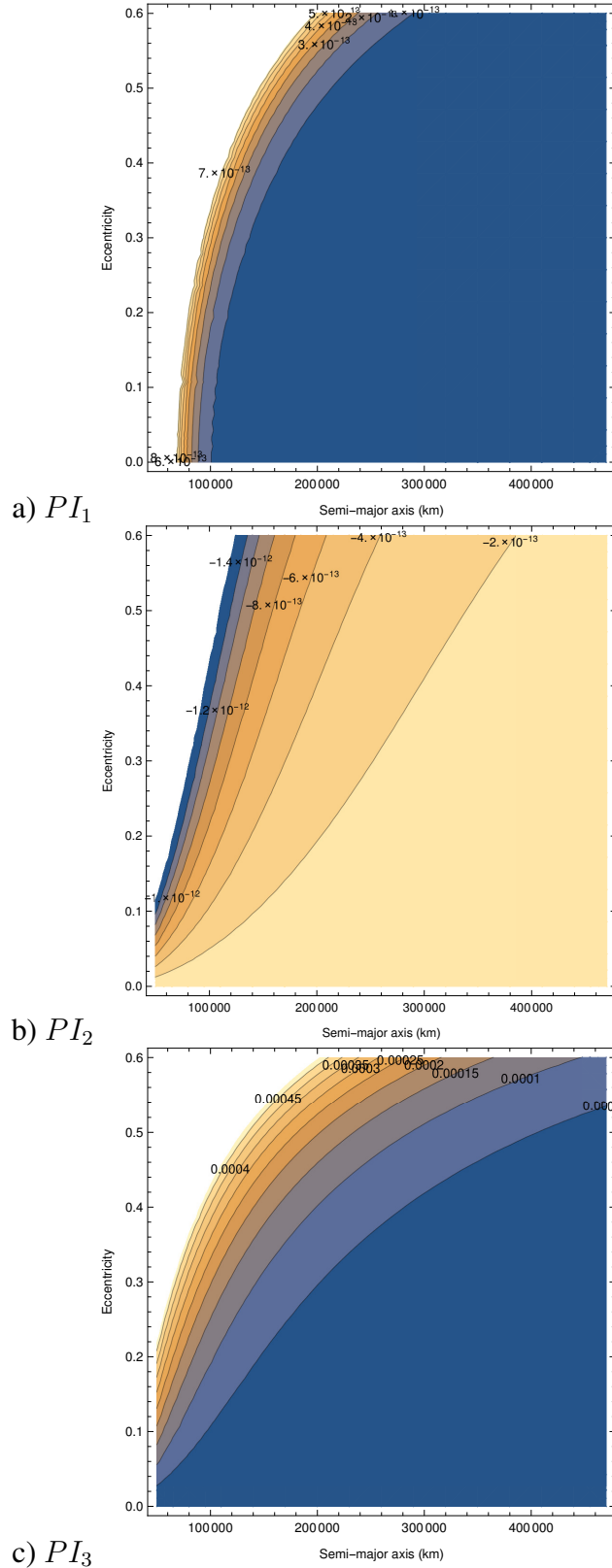
Source: Cardoso dos Santos et al. (2016a)

Figure 46: Perturbation Integrals as functions of e and i (radians) of the spacecraft and considering the effects of the J_2 of Neptune + Triton. Initial conditions: $a = 350.000$ km, $\Omega = \frac{\pi}{4}$ rad, $\omega = \frac{\pi}{2}$ rad



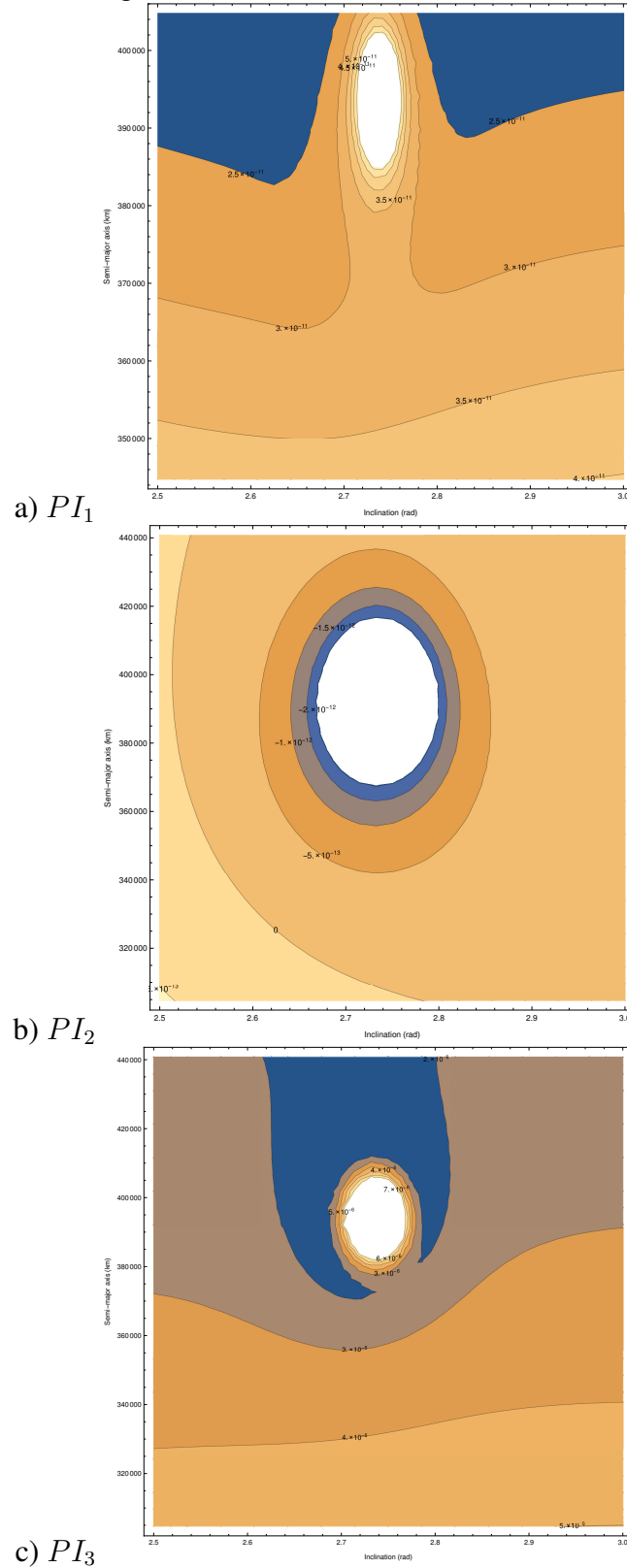
Source: Cardoso dos Santos et al. (2016a)

Figure 47: Perturbation Integrals as functions of e and a (km) of the spacecraft and considering the effects of the J_2 of Neptune + Triton. Initial conditions: $i = 0$ rad, $\Omega = \frac{\pi}{4}$ rad, $\omega = \frac{\pi}{2}$ rad



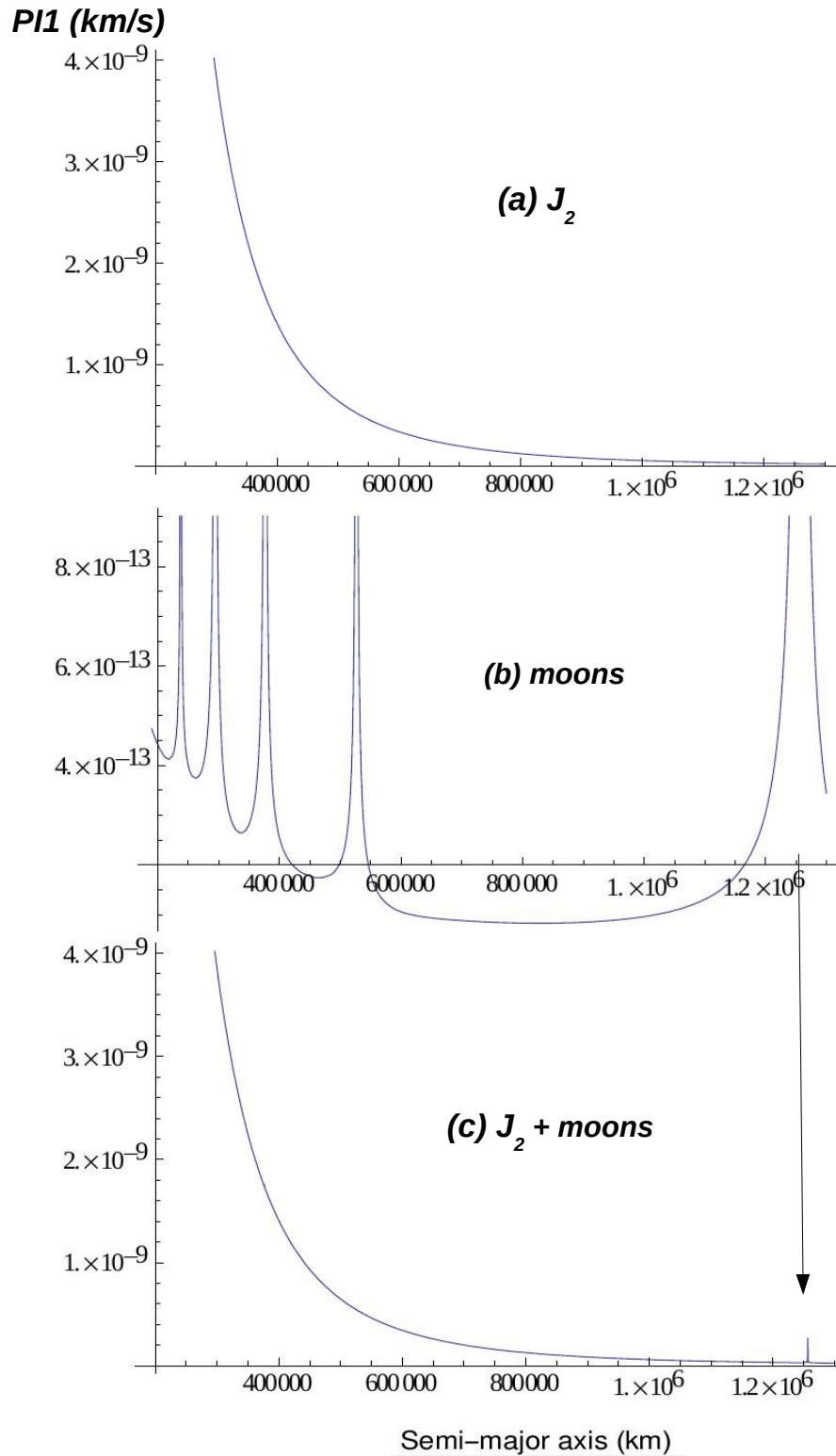
Source: Cardoso dos Santos et al. (2016a)

Figure 48: Perturbation Integrals as functions of a (km) and i (radians) of the spacecraft and considering the effects of the J_2 of Neptune + Triton. Initial conditions: $e = 0.1$, $\omega = 335.570$ deg, $\Omega = 66.142$ deg



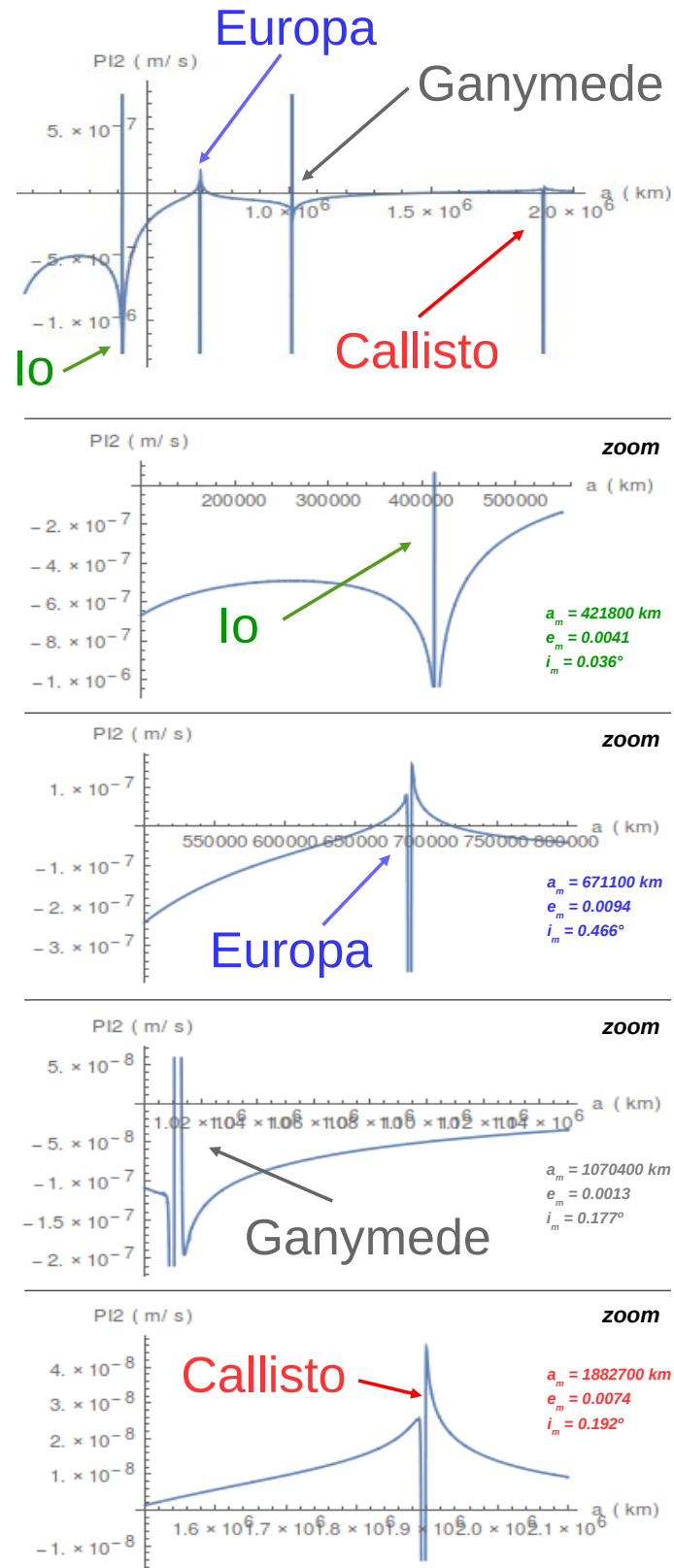
Source: Cardoso dos Santos et al. (2016a)

Figure 49: PI_1 as a function of a (km) of the spacecraft considering different disturbing effects in a system with Saturn and 5 moons (Enceladus, Tethys, Rhea, Dione and Titan): (a) J_2 of Saturn; (b) moons; and (c) J_2 of Saturn + 5 Saturnian moons.



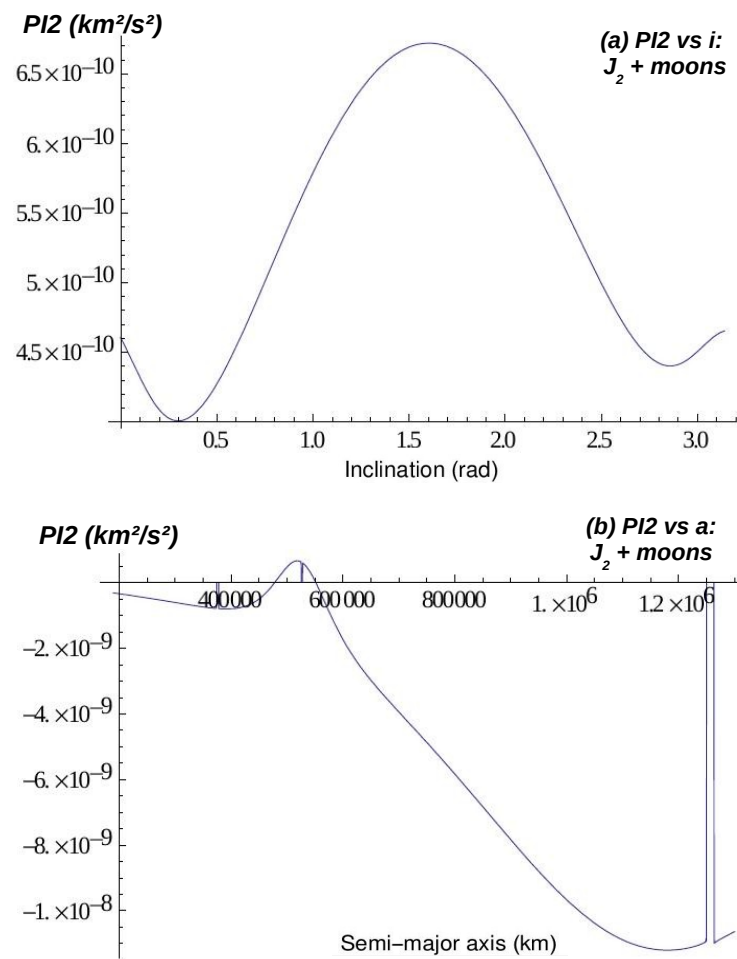
Source: Cardoso dos Santos et al. (2016a)

Figure 50: PI_2 as a function of the (a) inclination i (radians) and (b) semi-major axis a (km) of the spacecraft considering the disturbing effects due to the J_2 of Jupiter + 4 Galilean moons.



Source: Cardoso dos Santos et al. (2016b)

Figure 51: PI_2 as a function of the (a) inclination i (radians) and (b) semi-major axis a (km) of the spacecraft considering the disturbing effects due to the J_2 of Saturn + 5 Saturnian moons.



Source: Cardoso dos Santos et al. (2016a)

4.3 DISCUSSION

In the continuation of the present work, it is possible to consider more effects in the disturbing acceleration, like those due to the irregular shape of the planets. A study seeking orbits close to the moons is also currently under development in order to analyze important information like the possibilities of collision. The propagation of the orbits can be performed using numerical integrations in order to observe the dynamical behavior of the orbits characterized using the perturbation maps.

The present research shows the first steps of using "*Integral Indexes*" to measure the perturbation level of orbits in a system composed by a central planet and several moons. The gravity fields of all the bodies involved and the J_2 term of the gravity field of the central body were considered in the dynamical model. The results quantified the increase of the effects with the eccentricity of the orbits, as well as a decrease in the effects with the semi-major axis. Oscillations showing maximum and minimum points appeared in the study with respect to the inclination. There are some equilibrium points, where losses and gain of energy compensates each other and the net result is a constant energy. Those results require further studies and comparison with numerical simulations of the orbits to validate the perturbation maps developed here. The magnitude of the perturbations coming from the central body (irregular shape) are usually much larger than the ones coming from the moons, but they give a zero net result concerning variations of energy. So, from this point of view, it is important to study those perturbations individually.

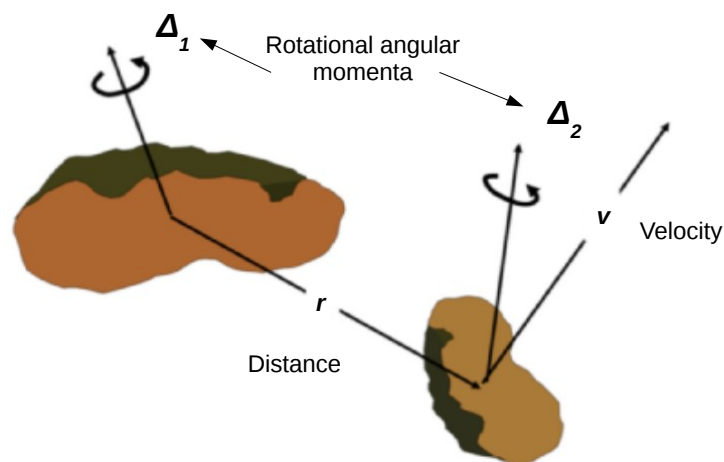
A final comment refers to the fact that this work is an extension of work of Prado et al. (2016), with the test of the "*Integral Indexes*" for planet-moon systems.

The study presented in this chapter concerning the Saturn and Neptune systems was published in a paper for a conference held in USA (CARDOSO DOS SANTOS et al., 2016A) and the other results for the Jupiter system were presented and discussed in a conference held in Mexico (CARDOSO DOS SANTOS et al., 2016B).

5 ROTO-ORBITAL DYNAMICS USING INTERMEDIARIES

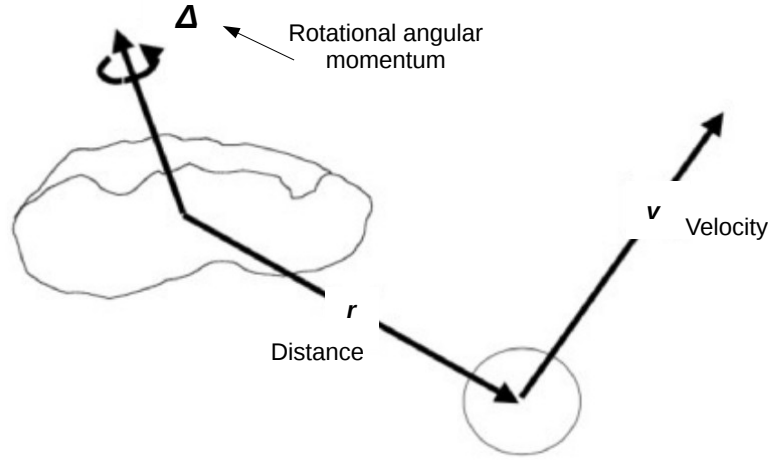
Analytical methods perform a key role on the study of a wide range of problems in celestial mechanics and astrodynamics. These techniques especially have the potential to provide a fast identification of useful motions for general bodies that respect similar mathematical structures for the equations of motion. Inside of the scope of applications of such methods is the roto-orbital dynamics, which requires the understanding of the full two-body problem and deals with the generic time evolution of the orbit and attitude/rotation of two bodies having specific masses and shapes (see figure 52). This problem is analytically not integrable, once it presents many degrees of freedom. However, restricted models for this problem can be assumed in order to find approximated solutions for its dynamics. One assumption usually adopted is to consider one of the two bodies, assumed to be the primary, as an spherical body and the other one, denoted by secondary and the one in which we are focused in to understand its motion, as a body with a irregular shape (see figure 53). This model works especially well for large distances or when the effects due to the shape of the primary spherical body are negligible. Thus, the adoption of adequate sets of orbital and rotational variables is necessary for a deeper understand the mathematical and physical nature of this problem.

Figure 52: Geometry of the Full Two-Body Problem.



Source: Author

Figure 53: Restricted Model for the Two-Body Problem.



Source: Author

The Cartesian variables (which represent the state vector of the rigid body's orbital motion), the orbital elements and the Euler angles are among the main and well known sets of variables used in such studies. But, as these sets of variables are not canonical, it restricts their application in important analytical studies. In order to take advantage of some useful canonical techniques to study the roto-orbital problem, a canonical set of variables ($r, \phi, \psi, \theta, \delta, \nu, R, \Phi, \Psi, \Theta, \Delta, N$) is considered. This one is built through a process which joints the Hill-Whittaker variables ($r, \theta, h, R, \Theta, H$) to describe the orbital motion of the center of mass of a rigid body (DEPRIT, 1981) and the Andoyer (also known as Andoyer-Deprit) variables ($\nu, \mu, \lambda, N, M, \Lambda$) to describe the rotational dynamics (ANDOYER, 1923; DEPRIT, 1967). Both sets of variables are canonical as well. The process to build these variables is presented in Ferrandiz; Sansaturio (1989) and the use of these ones brings the benefit of reducing two degrees of freedom of the problems, *i.e.*, the nodes are eliminated.

The elimination of the parallax was devised on the context of building a compact higher order analytical theory in closed form for the main problem of the artificial satellite. In that case it results in a dramatic reduction in the number and complexity of the perturbation terms, which proved very efficient for carrying out a subsequent Delaunay normalization in closed form. But the advantages provided by this preparatory transformation are neither limited to the main problem nor to problems with axial symmetry, and this transformation was soon applied to different models in artificial satellite theory (see for instance Deprit (1981) and references therein). This canonical procedure is a Lie transformation that is not limited exclusively to Hamiltonian normalization Deprit; Ferrer (1989). Recently, Lara; San-Juan; López-Ochoa (2014) followed Deprit's work (developed on the context of the main problem of artificial satellite theory) and applied this technique using

Delaunay variables. Similar procedure was used by Ferrer; Lara (2012) to obtain a simplified Hamiltonian for the roto-orbital problem, which is split in two parts: The first one is called the “*Intermediary*”; the second one is the “*Remainder*”. An intermediary Hamiltonian model which produces a system integrable through elliptic functions is built for the resulting axisymmetric Hamiltonian obtained after some considerations. However, the technique of eliminating the parallax is used in order to build a second intermediary Hamiltonian which is integrable in terms of elementary quadratures. The procedure used in the present investigation for this second intermediary construction follows similar methodology used by Ferrer; Lara (2012) and the resulting equations of motion for both intermediaries (without and with the application of the elimination of the parallax) are presented. The introduction of parameters of mass (q) and shape (q_s) helps on the investigation of the dynamics of the problem.

Numerical propagations are considered for different values of shape, orbital eccentricity and attitude of a slow-rotating rigid body, with basic parameters based in the system presented in Ferrer; Lara (2012). These changes are performed in order to compare the results provided by the intermediaries with the analytical models without the simplifications made using the concepts of intermediary and elimination of the parallax. The results present significant precision for both intermediaries and final discussions are made, including a proposal of continuation of the investigation in order to test the intermediaries on representing more realistic models, with potential of application in different classes of problem in celestial mechanics and astrodynamics.

Finally, the present chapter is divided in the following sections: A general overview of the problem appears in the present introductory section; Section 5.2 deals with the roto-orbital dynamics with the introduction of the roto-orbital canonical variables and the application of the elimination of the nodes; Section 5.4 presents the concept of elimination of the parallax; The concept of intermediary appears in section 5.3; Section 5.5 shows the development of two roto-orbital intermediaries (with one model obtained after the elimination of the parallax); Numerical experiments are presented and discussed in section 5.6; Finally, section 5.7 ends the study with the presentation of preliminary conclusions and possibilities of investigation for the continuation of the project.

5.1 UNPERTURBED ORBITAL AND ROTATIONAL MODELS

After considering the short historical review presented in the previous chapter, it is necessary the adoption of adequate sets variables to describe the orbital and rotational dynamics. In fact, before we enter in the study of the roto-orbital dynamics, it is important to review some important aspects of both dynamics when unperturbed: Keplerian motion and free-rigid body spin motion. This information is used as reference to understand the differences that appear in the dynamics with the inclusion of the coupling effects.

5.1.1 Keplerian motion using Hill-Whittaker variables

The Kepler problem is a classical and well known problem in celestial mechanics. It also has a major importance in the foundations of the astrodynamics. In fact, the study of this problem plays a central role in the history of science. Due to these, extensive literature regarding this problem, its different formulations and applications in these areas is available (BROUWER; CLEMENCE, 1961; DANBY, 1992; MURRAY; DERMOTT, 2000; VALLADO, 2007). The Keplerian motion is the solution of the two-body problem and can be described by different sets of variables. These sets of variables are usually referred to the frame fixed in space (usually defined as a plane parallel to the equatorial plane). The most known and used sets are:

- Cartesian variables: composed by the components of position $\mathbf{x} = (x, y, z)$ and velocity $\mathbf{X} = (v_x, v_y, v_z)$ vectors of the orbiting body;

- Orbital elements: semi-major axis a , eccentricity e , orbital inclination I_o , argument of the pericenter ω , longitude of the ascending node Ω , true anomaly f^1 . This set of variables shapes (a, e) and places (I_o, ω, Ω) the orbit in space; the angle f helps to define the position of the body at the orbit. In many applications for the classical two-body problem, the orbital elements are usually preferred in comparison with the Cartesian variables due to its characteristic of having all parameters constant $(a, e, I_o, \omega, \Omega)$ with exception for τ , while all the parameters in Cartesian system present time variation.

Both coordinate systems are fundamentally important in several applications in celestial mechanics and astrodynamics. However, these sets of variables are not the more convenient ones to provide a solid and more compact mathematical foundation in investigations about some topics such as stability, resonances and periodic orbits for applications in astrodynamics. The Cartesian variables and the orbital elements present restricted applications when it is necessary to take advantage of analytical methods that require specific classes of canonical variables for simplifying the problem. In fact, the Hamiltonian formulation is usually the more convenient for this purpose and the one that is usually chosen by mathematicians. These formulations of the mechanics take advantage of the sets of canonical variables, which correspond to sets of variables that under processes called “canonical transformations”, that are special coordinate transformations², which keep the “transformed” Hamilton’s equations in the exactly same structure and form as the original Hamilton’s equations using a previous set of variables. In other words, one

¹Sometimes it can be replaced by the eccentric anomaly f or by the mean anomaly $l = nt - \tau$, where $n = \sqrt{k/a^3}$ is the mean motion, $k = G_{CG}(m_1 + m_2)$ is the gravitational parameter, G_{CG} is the gravitational constant, m_1 and m_2 are the masses of the bodies of the system and τ is referred to an specific date when the orbiting body is at the pericenter of its orbit.

²The reader is directed to Goldstein; Poole; Safko (2002) and Lemos (2007) to more details about the fundamentals and applications in mechanics of these variables and transformations.

do not need to modify the differential equations (equations of motion) in order to perform a canonical transformation of coordinates. Instead, it is only necessary to transform the Hamiltonian of the problem and the transformed equations of motion will be obtained from the computation of the Hamilton's equations of the transformed Hamiltonian.

In this context, the Hill-Whittaker variables $(r, \theta, h, R, \Theta, H)$ are a set of canonical variables that can be used to describe the orbital motion of the center of mass of a rigid body (see figure 54), so we replace the Cartesian coordinates by the Hill-Whittaker variables through a canonical transformation presented in Whittaker (1989).

This new set of variables is obtained from Cartesian state variables by mean of a canonical transformation $(\mathbf{x}, \mathbf{X}) = (x, y, z, v_x, v_y, v_z) \rightarrow (r, \theta, h, R, \Theta, H)$ that is carried out to obtain Hill-Whittaker variables on the symplectic manifold $(\mathfrak{R}^6, \Upsilon^2)$, once we considered the Cartesian vector space $\mathfrak{R}^6$ with coordinates (x, y, z, v_x, v_y, v_z) and the differential 2-form (DEPRIT, 1981)

$$\Upsilon^2 = x \wedge v_x + y \wedge v_y + z \wedge v_z \quad (63)$$

The Hamiltonian of the orbital motion under the influence a Newtonian attraction is given by

$$\mathcal{H}_O = \frac{1}{2m} (\mathbf{X} \cdot \mathbf{X}) - \frac{km}{\|\mathbf{x}\|} \quad (64)$$

on the symplectic manifold $(\mathfrak{R}^6, \Upsilon^2)$ the Hamiltonian (64) is called a Keplerian flow.

For any function $\mathcal{F} \equiv \mathcal{F}(x, y, z, v_x, v_y, v_z)$, the Lie derivative of $\mathcal{F}$ in the Keplerian flow is the function (DEPRIT, 1981)

$$\mathcal{L}_0(\mathcal{F}) = \{\mathcal{F}; \mathcal{H}_0\} \quad (65)$$

which is defined as the Poisson bracket of the function $\mathcal{F}$ and the Hamiltonian $\mathcal{H}_0$ in the symplectic manifold $(\mathfrak{R}^6, \Upsilon^2)$.

Now, it is necessary to perform a change of variables from the Cartesian state coordinates to the by polar nodal variables on the symplectic manifold by mean of a canonical transformation to simplify the evaluation of the Lie derivatives, which were used in several studies available in literature (HORI, 1966; DEPRIT, 1969; FERRAZ-MELLO, 2007; FERRER; LARA, 2010; CECCARONI; BISCANI; BIGGS, 2014).

Let it be recalled that the rotation of coordinates (DEPRIT, 1981):

$$x = x' \cos(h) - y' \sin(h) \cos(I_o) \quad (66)$$

$$y = x' \sin(h) + y' \cos(h) \cos(I_o) \quad (67)$$

$$z = y' \sin(I_o) \quad (68)$$

which is extended to momenta (velocities)

$$v_x = v'_x \cos(h) - v'_y \sin(h) \cos(I_o) \quad (69)$$

$$v_y = v'_x \sin(h) + v'_y \cos(h) \cos(I_o) \quad (70)$$

$$v_z = v'_y \sin(I_o) \quad (71)$$

These equations keep invariant the differential identity

$$v_x dx + v_y dy + v_z dz = v'_x dx' + v'_y dy' + H dh \quad (72)$$

when the orbital inclination I_o is determined by the relations

$$G_O = x' v'_y - y' v'_x \quad (73)$$

$$\cos(I_o) = H/G_O \quad (74)$$

where G_O is the norm of the orbital angular momentum vector $\mathbf{G}_O$ and H is its third component.

By respecting these conditions, the substitution $(x, y, z, v_x, v_y, v_z) \rightarrow (x', y', h, v'_x, v'_y, H)$ defines a canonical transformation that is homogeneous and that is composed with the extension from the Cartesian to polar coordinates $(x', y', v'_x, v'_y) \rightarrow (r, \theta, R, \Theta)$ which completes the canonical transformation from Cartesian coordinates to Hill-Whittaker coordinates $(x, y, z, v_x, v_y, v_z) \rightarrow (r, \theta, h, R, \Theta, H)$, that is also known as Whittaker transformation. The remaining relations to complete this transformation are given by

$$x' = r \cos(\theta) \quad (75)$$

$$y' = r \sin(\theta) \quad (76)$$

$$v'_x = R \cos(\theta) - \frac{\Theta}{r} \sin(\theta) \quad (77)$$

$$v'_y = R \sin(\theta) + \frac{\Theta}{r} \cos(\theta) \quad (78)$$

As we have that $\mathbf{X} \cdot \mathbf{X} = v_x^2 + v_y^2 + v_z^2 = v'^2_x + v'^2_y$ and $r = \|\mathbf{x}\|$, the Hamiltonian of the Keplerian motion (64) can be rewritten in terms of the Hill-Whittaker variables as:

$$\mathcal{H}_{Kepler} = \mathcal{H}_O = \frac{1}{2m} \left(R^2 + \frac{\Theta^2}{r^2} \right) - \frac{km}{r} \quad (79)$$

where $\mathbf{X} \cdot \mathbf{X} = [R \cos(\theta) - \frac{\Theta}{r} \sin(\theta)]^2 + [R \sin(\theta) + \frac{\Theta}{r} \cos(\theta)]^2 = R^2 + \frac{\Theta^2}{r^2}$.

The use of Hill-Whittaker variables can provide simplifications to obtain solutions for some problems. In the present they are used for applying the elimination of the nodes to simplify and reduce the degrees of freedom of system. Once the process is completed and results are obtained, it is possible to perform an inverse transformation to obtain the Cartesian variables from the Hill-Whittaker variables by mean of the equations that relate both systems of variables (expressions (66) to (71)). After obtaining the resulting information in terms of Cartesian variables, it is possible to evaluate the orbital elements using classical relations of the celestial mechanics presented in literature (BROUWER; CLEMENCE, 1961; DANBY, 1992; MURRAY; DERMOTT, 2000; VALLADO, 2007).

5.1.2 Free-rigid body motion

The kinetic energy for the rotational motion of a rigid body about its center of mass is given by

$$T_R = \frac{1}{2} (A\omega_1^2 + B\omega_2^2 + C\omega_3^2) \quad (80)$$

where (A, B, C) are principal moments of inertia of the rigid body defined, respectively along axes x, y, z and $\omega_1, \omega_2, \omega_3$ are the components of its instantaneous rotational (spin) velocity vector in the frame principal axis of the body.

An alternative definition for the kinetic energy may be expressed as

$$T_R = \frac{1}{2} \left(\frac{g_1^2}{A} + \frac{g_2^2}{B} + \frac{g_3^2}{C} \right) \quad (81)$$

where g_1, g_2, g_3 are the components x, y, z , respectively, of the angular momentum vector in the body frame.

The Hamiltonian of the torque-free rotation is obtained from the kinetic energy by expressing either the angular velocity or the angular momentum in a convenient set of canonical variables like, for instance, Andoyer angles and their canonical extension. For describing the rotational motion of a rigid body free from external torques, it is possible to consider the very known Eulerian angles ϕ_1, ϕ_2, ϕ_3 and their conjugate momenta $p_{\phi_1}, p_{\phi_2}, p_{\phi_3}$. By mean of a canonical transformation $(\phi_1, \phi_2, \phi_3, p_{\phi_1}, p_{\phi_2}, p_{\phi_3}) \rightarrow (\nu, \mu, \lambda, N, M, \Lambda)$ it is possible to move from this set of variables to the canonical set of variables that is the Andoyer chart³ which is defined as: ν, μ, λ are angles and N, M, Λ are their respective canonical momenta. In fact, the transformation must to satisfy

$$N d\nu + M d\mu + \Lambda d\lambda = p_{\phi_1} d\phi_1 + p_{\phi_2} d\phi_2 + p_{\phi_3} d\phi_3 \quad (82)$$

Specifically, the Hamiltonian is reduced to one degree of freedom when using Andoyer

³See further details in Deprit (1967)

chart $(\nu, \mu, \lambda, N, M, \Lambda)$ that links the body and inertial frames through an intermediate frame with the z -axis defined by the direction of the angular momentum vector, the x -axis defined by the intersection of the plane perpendicular to the angular momentum vector and the inertial $x - y$ plane, and the y axis defining a direct frame (ANDOYER, 1923). Then the position of the intermediate plane with respect to the inertial frame is defined by its argument of the node λ and inclination $I_{ram} = \arccos(\frac{\Lambda}{M})$, where Λ is the projection of the angular momentum vector (given by $M = G_R$) on the z -axis of the inertial frame. The position of the equatorial plane of the rigid body with respect to the intermediate plane is defined by its argument of the node μ , and inclination $\sigma = \arccos(N/M)$, where N is the projection of the angular momentum vector over the z -axis of the body frame. Finally, the x -axis of the body frame is located on the equatorial plane of the body through the angle ν (see for instance the figure 55).

In order to perform this transformation, it is important to define first the quantities.

$$\Lambda = M \cos(I_{ram}) \quad (83)$$

$$N = M \cos(\sigma) \quad (84)$$

where I_{ram} and σ are, respectively, the attitude inclinations of the rigid body in the inertial frame and body-fixed frame with respect to the invariant plane perpendicular to the rotational angular momentum $\mathbf{M}$. In fact, I_{ram} is the inclination of the invariable plane, i.e., the plan perpendicular to the rotational angular momentum.

Now, it is necessary to define the transformation between the momenta in Euler angles and momenta in Andoyer variables. We define the following relations

$$p_{\phi_1} = \Lambda \quad (85)$$

$$p_{\phi_2} = M \sin(\sigma) \sin(\nu - \phi_3) \quad (86)$$

$$p_{\phi_3} = N \quad (87)$$

Through spherical trigonometry, it is possible to relate $\phi_1 - \lambda$, ϕ_2 , $\phi_3 - \nu$ to determine angles I_{ram} , σ and I_r , where I_r is the inclination of the rotational plane regarding the inertial frame.

$$\cos(I_{ram}) = \cos(\sigma) \cos(I_r) + \sin(\sigma) \sin(I_r) \cos(\phi_3 - \nu) \quad (88)$$

$$\cos(I_r) = \cos(\sigma) \cos(I_{ram}) - \sin(\sigma) \sin(I_{ram}) \cos(\phi_1 - \lambda) \quad (89)$$

$$\cos(\sigma) = \cos(I_r) \cos(I_{ram}) + \sin(I_r) \sin(I_{ram}) \cos(\mu) \quad (90)$$

where ν measures the rotation of the rigid body, μ measures the precession of the body-fixed frame (in this case the frame whose axis are along the principal axis of inertia) and λ measures the precession of the invariable plane (perpendicular to the rotational angular momentum) regarding the inertia frame (frame fixed in space).

The Euler equations are given by

$$\omega_1 = \dot{\phi}_1 \sin(\phi_2) \sin(\phi_3) + \dot{\phi}_2 \cos(\phi_3) \quad (91)$$

$$\omega_2 = \dot{\phi}_1 \sin(\phi_2) \cos(\phi_3) + \dot{\phi}_2 \sin(\phi_3) \quad (92)$$

$$\omega_3 = \dot{\phi}_1 \cos(\phi_2) + \dot{\phi}_3 \quad (93)$$

whose Lagrangian describing the free rotation of a rigid body is given by expression (80), i.e.

$$\mathcal{L}_R = \mathcal{L}_R(\omega_1, \omega_2, \omega_3) = \frac{1}{2} (A\omega_1^2 + B\omega_2^2 + C\omega_3^2) \quad (94)$$

By inverting the relations, we get

$$p_{\phi_1} = \frac{\partial \mathcal{L}_R}{\partial \dot{\phi}_1} = A\omega_1 \frac{\partial \omega_1}{\partial \dot{\phi}_1} + B\omega_2 \frac{\partial \omega_2}{\partial \dot{\phi}_1} + C\omega_3 \frac{\partial \omega_3}{\partial \dot{\phi}_1} \quad (95)$$

$$p_{\phi_2} = \frac{\partial \mathcal{L}_R}{\partial \dot{\phi}_2} = A\omega_1 \frac{\partial \omega_1}{\partial \dot{\phi}_2} + B\omega_2 \frac{\partial \omega_2}{\partial \dot{\phi}_2} + C\omega_3 \frac{\partial \omega_3}{\partial \dot{\phi}_2} \quad (96)$$

$$p_{\phi_3} = \frac{\partial \mathcal{L}_R}{\partial \dot{\phi}_3} = A\omega_1 \frac{\partial \omega_1}{\partial \dot{\phi}_3} + B\omega_2 \frac{\partial \omega_2}{\partial \dot{\phi}_3} + C\omega_3 \frac{\partial \omega_3}{\partial \dot{\phi}_3} \quad (97)$$

where

$$\begin{pmatrix} \frac{\partial \omega_1}{\partial \dot{\phi}_1} & \frac{\partial \omega_2}{\partial \dot{\phi}_1} & \frac{\partial \omega_3}{\partial \dot{\phi}_1} \\ \frac{\partial \omega_1}{\partial \dot{\phi}_2} & \frac{\partial \omega_2}{\partial \dot{\phi}_2} & \frac{\partial \omega_3}{\partial \dot{\phi}_2} \\ \frac{\partial \omega_1}{\partial \dot{\phi}_3} & \frac{\partial \omega_2}{\partial \dot{\phi}_3} & \frac{\partial \omega_3}{\partial \dot{\phi}_3} \end{pmatrix} = \begin{pmatrix} \sin(\omega_2) \sin(\omega_3) & \sin(\omega_2) \cos(\omega_2) & \cos(\omega_2) \\ \cos(\omega_3) & -\sin(\omega_3) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (98)$$

After inverting and rewriting the expressions (95) to (97) using the values for ω_n and $\frac{\partial \omega_n}{\partial \dot{\phi}_m}$ (with $n, m = \{1, 2, 3\}$) presented in expressions (91) to 93 and 98 we get

$$A\omega_1 = \left[\frac{p_{\phi_1} - p_{\phi_3} \cos(\phi_2)}{\sin(\phi_2)} \right] \sin(\phi_3) + p_{\phi_2} \cos(\phi_3) \quad (99)$$

$$B\omega_2 = \left[\frac{p_{\phi_1} - p_{\phi_3} \cos(\phi_2)}{\sin(\phi_2)} \right] \cos(\phi_3) - p_{\phi_2} \sin(\phi_3) \quad (100)$$

$$C\omega_3 = p_{\phi_3} \quad (101)$$

which are the components of the rotational angular momentum.

Once we have the relation

$$\begin{aligned} A\omega_1^2 + B\omega_2^2 &= \frac{(A\omega_1)^2}{A} + \frac{(B\omega_2)^2}{B} = \left[\frac{p_{\phi_1} - p_{\phi_3} \cos(\phi_2)}{\sin(\phi_2)} \right]^2 \left[\frac{\sin^2(\phi_3)}{A} + \frac{\cos^2(\phi_3)}{B} \right] \\ &\quad + \left[\frac{p_{\phi_1} - p_{\phi_3} \cos(\phi_2)}{\sin(\phi_2)} \right] \left[\frac{1}{A} - \frac{1}{B} \right] \sin(2\phi_3) \end{aligned} \quad (102)$$

$$C\omega_3^2 = \frac{(C\omega_3)^2}{C} = [p_{\phi_3}]^2 \quad (103)$$

it is possible to rewrite the Lagrangian of the torque-free motion $\mathcal{L}_R = \mathcal{L}_R(\omega_1, \omega_2, \omega_3)$ (expression (94)) in terms of $\phi_1, \phi_2, \phi_3, p_{\phi_1}, p_{\phi_2}$ and p_{ϕ_3} . From this Lagrangian, it is obtained the Hamiltonian $\mathcal{H}_R = \mathcal{H}_R(\phi_1, \phi_2, \phi_3, p_{\phi_1}, p_{\phi_2}, p_{\phi_3}) = \mathcal{H}_R(-, \phi_2, \phi_3, p_{\phi_1}, p_{\phi_2}, p_{\phi_3})$ where the angles ϕ_1 is replaced by dash (-) to represent its character as an ignorable (and cyclic) variable, i.e., a constant of motion of the free-torque motion. By performing the canonical transformation from Euler chart to Andoyer chart, and that takes advantage of expressions like 83 to 87, it is possible to obtain (DEPRIT, 1967)

$$A\omega_1 = M \sin(\sigma) \sin(I_{ram}) = \sqrt{M - N} \sin(\nu) \quad (104)$$

$$\longrightarrow A\omega_1^2 = \frac{M^2}{A} \sin^2(\sigma) \sin^2(I_{ram}) = \frac{M - N}{A} \sin^2(\nu) \quad (105)$$

$$B\omega_2 = M \sin(\sigma) \cos(I_{ram}) = \sqrt{M - N} \cos(\nu) \quad (106)$$

$$\longrightarrow B\omega_2^2 = \frac{M^2}{B} \sin^2(\sigma) \cos^2(I_{ram}) = \frac{M - N}{B} \cos^2(\nu) \quad (107)$$

$$C\omega_3 = N \longrightarrow C\omega_3^2 = \frac{N^2}{C} \quad (108)$$

where $M \sin(\sigma) = \sqrt{M - N}$.

By mean of the expressions (105) to (108) it is possible to write $\mathcal{H}_R = \mathcal{H}_R(\phi_1, \phi_2, \phi_3,$

$p_{\phi_1}, p_{\phi_2}, p_{\phi_3}$) as $\mathcal{H}_{\mathcal{R}} = \mathcal{H}_R(\nu, \mu, \lambda, N, M, \Lambda)$.

$$A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = \left(\frac{\sin^2(\nu)}{A} + \frac{\cos^2(\nu)}{A} \right) (M^2 - N^2) + \frac{N^2}{C} \quad (109)$$

Therefore, the transformation from Euler to Andoyer variables is canonical (DEPRIT, 1967), and gives the Hamiltonian $\mathcal{H}_{\mathcal{R}} = \mathcal{H}_R(\nu, \mu, \lambda, N, M, \Lambda)$ of the torque-free motion

$$\mathcal{H}_{Euler} = \mathcal{H}_R = \mathcal{H}_R(\nu, -, -, N, M, -) = \frac{1}{2} \left[(M^2 - N^2) \left(\frac{\sin^2(\nu)}{A} + \frac{\cos^2(\nu)}{B} \right) + \frac{N^2}{C} \right] \quad (110)$$

where μ, λ and Λ are cyclic and therefore $\mu = \mu_0, \lambda = \lambda_0$ and $\Lambda = \Lambda_0$ are constant. This last result means that the third component of the rotational angular momentum is conserved in the inertia frame. The integration of the Hamilton equations for ν, μ and N can be solved by quadrature, and accepts a closed form solution in elliptic integrals. For the case where the rigid body is an axisymmetric body ($A = B$), the Hamiltonian (110) is reduced to

$$\mathcal{H}_{RAB} = \mathcal{H}_{RAB}(-, -, -, N, M, -) = \frac{1}{2} \left[\left(\frac{M^2 - N^2}{A} \right) + \frac{N^2}{C} \right] \quad (111)$$

whose non-null equations of motion are

$$\dot{\nu} = \frac{\partial \mathcal{H}_{RAB}}{\partial N} = \left(\frac{1}{C} - \frac{1}{A} \right) N \quad (112)$$

$$\dot{\mu} = \frac{\partial \mathcal{H}_{RAB}}{\partial M} = \left(\frac{1}{A} \right) M \quad (113)$$

Once one assumed the symmetric shape for the rigid body ($A = B$), the expressions (112) and (113) can be solved by quadrature and present linear evolution with fix rate. Because of this assumption, it was also possible to eliminate the variable ν from the Hamiltonian, which implies in an extra cyclic variable: $\dot{N} = 0 \rightarrow N = N_0$. The meaning of this result is that the third component of the rotational angular momentum is conserved in the body frame, i.e., the frame whose axis are along the principal axis of inertia of the rigid body. Looking into the inertial frame, it is possible to see that for the free-torque motion, the angular momentum vector preserves its modulus and direction⁴.

⁴Constant inclination of the rotational plane (or Andoyer plane) regarding the invariant plane defined as the plane orthogonal to the vector $\mathbf{G}_R$, so that, $G_R = \sqrt{g_1^2 + g_2^2 + g_3^2} = \sqrt{(A\omega_1)^2 + (B\omega_2)^2 + N^2}$ and I_{ram} are constant.

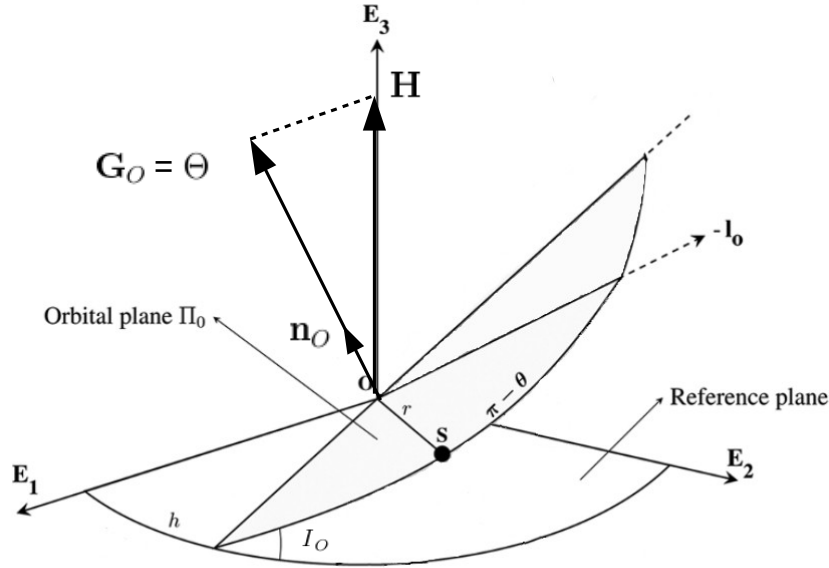
5.2 ROTO-ORBITAL DYNAMICS

Despite of the advances in celestial mechanics and astrodynamics, the roto-orbital dynamics of the full two-body problem still is a challenge in these areas. This problem is usually studied by mean of proposed models that rely on simplifying assumptions, that sometimes can lead to integrable systems with limited, but valuable information. Among the topics where the roto-orbital dynamics presents important application are the case of natural celestial bodies of the solar system, where it is commonly assumed a system having a body behaving as a central gravitational field, and the studies for artificial satellites and small moons, where the small ratio of mass of these bodies allows the use of restricted models that open possibilities of applications of several tools of analysis of the dynamical system. However, even these simplifications are very useful for many cases, they do not solve the full roto-orbital problem, which is not integrable, and because of that, it is convenient the use of techniques from perturbation theories to approach the full problem.

5.2.1 Roto-orbital canonical variables

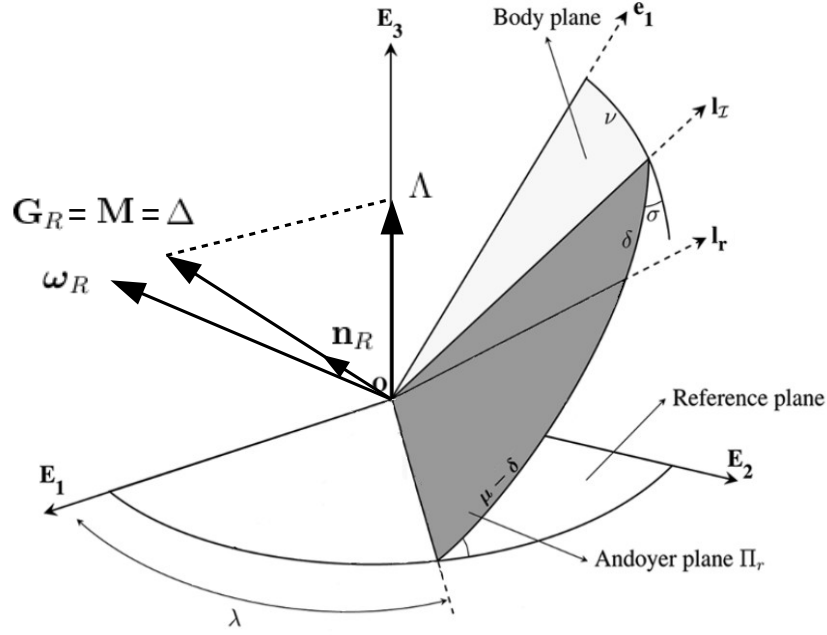
Before we present the model for the roto-orbital problem, it is necessary to define planes of reference. Some of these planes of reference are presented in figures 54 and 55. These figures present two sets of canonical variables, where the figure 54 presents the Hill-Whittaker variables for the orbital motion and the figure 55 presents the Deprit-Andoyer variables for the rotational motion.

Figure 54: Orbital reference system containing Hill-Whittaker variables and polar nodal variables associated with orbital motion.



Source: Author

Figure 55: Rotational reference system containing Andoyer variables and polar nodal variables associated with rotational motion.



Source: Author

It is also necessary to define an appropriate set of variables to work with, which are built in terms of these orbital and rotational canonical sets. It is made following a procedure presented in Ferrandiz; Sansaturio (1989). When compared to the classical approach referred to an inertial frame, the choice for this new set of variables reduces the number of terms in the expansions drastically Ferrandiz; Sansaturio; Caballero (1993). The planes referred to both dynamics are presented in figure 56, where: $\mathbf{S}$ stands for the position of the 'satellite' rigid body in the inertial frame, ν is the precession angle on the plane orthogonal to $\mathbf{G}_R$, h is the longitude of the ascending node of the orbiting body, λ is the node of the Andoyer plane regarding the space frame reference direction $\mathbf{E}_1$. Further details about the variables and planes appearing in this figure can be seen in Ferrandiz; Sansaturio (1989) (see also Cardoso dos Santos; Ferrer; Scheeres (2019)).

Following the development presented by Ferrandiz and Sansaturio (see also Deprit; Eliepe (1993)), it is possible to obtain the Cartan's 1-form equation as (FERRANDIZ; SANSATURIO, 1989)

$$\mathcal{B} = Rdr + (\mathbf{G} \cdot \mathbf{E}_3)d\phi + (\mathbf{G} \cdot \mathbf{n})d\psi + G_Od\theta + G_Rd\delta + (\mathbf{G}_R \cdot \mathbf{e}_3)d\nu \quad (114)$$

This expression connects both sets of variables by creating a connection between the angular momentum planes to each individual dynamics with the total angular momentum plane and under the light of the of the expression (114), it only remains the definition of

the new conjugate momenta of the variables $(r, \phi, \psi, \theta, \delta, \nu)$ through

$$\begin{aligned} R &= R, & \Phi &= \mathbf{G} \cdot \mathbf{E}_3, & \Psi &= \mathbf{G} \cdot \mathbf{n} = G\mathbf{n} \cdot \mathbf{n} = G, \\ \Theta &= G_O, & \Delta &= G_R, & N &= \mathbf{G}_R \cdot \mathbf{e}_3 \end{aligned} \quad (115)$$

By looking to the figure 56 it is possible to get the geometrical meaning of these new set of variables and their canonical momenta. Therefore, we have

$$(r, \phi, \psi, \theta, \delta, \nu, R, \Phi, \Psi, \Theta, \Delta, N) \quad (116)$$

which is a set of canonical variables. The momenta are identified as:

- R : Radial velocity of the center of mass.
- Φ : Third component of the total angular momentum vector in space frame.
- Ψ : Magnitude of the total angular momentum vector.
- Θ : Magnitude of the orbital angular momentum of the system.
- Δ : Magnitude of the rotational angular momentum of the rigid body.
- N : Third component of the rotational angular momentum of the rigid body in the body frame (principal axes of inertia).

This set of variables presented in expression (116) is canonical and we direct the reader to Deprit (1981) and Ferrandiz; Sansaturio (1989) for further details and proof of the symplectic character of this transformation.

The mathematical expressions that present the relative position (inclination) of each reference plane as dependent on the momenta are as follows Ferrandiz; Sansaturio (1989); Ferrer; Lara (2012)

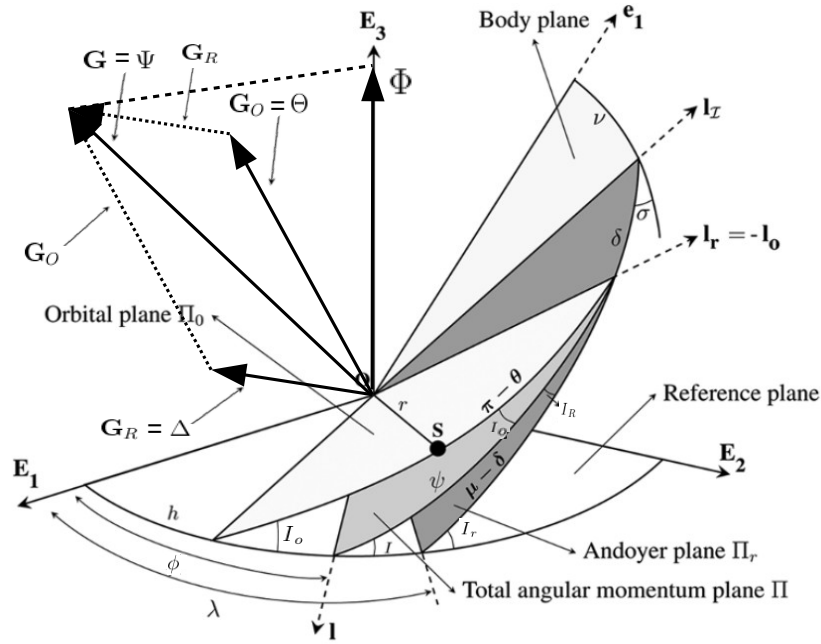
$$\begin{aligned} \cos(I_O) &= \frac{\Psi^2 + \Theta^2 - \Delta^2}{2\Psi\Theta}; & \cos(I_R) &= \frac{\Psi^2 + \Delta^2 - \Theta^2}{2\Psi\Delta} \\ \cos(I_O + I_R) &= \frac{\Psi^2 - \Delta^2 - \Theta^2}{2\Theta\Delta}; & \cos(\sigma) &= \frac{N}{\Delta}; & \cos(I) &= \frac{\Phi}{\Psi} \end{aligned} \quad (117)$$

As we already have defined an appropriate set of variables to deal with the roto-orbital dynamics, we need to move on stating this problem in terms of this set of new variables along with the concept of Intermediary and the parallax elimination that are the in the core of the present investigation.

5.2.2 Roto-orbital Hamiltonian

In the work of Ferrandiz; Sansaturio (1989), a Hamiltonian for the roto-orbital motion is obtained from the Keplerian flow $\mathcal{H}_O = \mathcal{H}_{Kepler}$ using Hill-Whittaker variables and from the Hamiltonian for the free-rotation of a rigid-body $\mathcal{H}_R = \mathcal{H}_{Euler}$ using Andoyer variables. To study the coupling of these two dynamics, it is necessary to consider an extra term in the potential expanded up to the first order of the perturbation, so that the

Figure 56: Roto-orbital canonical variables and reference planes. Adapted from Ferrer; Lara (2012)



Source: Cardoso dos Santos; Ferrer; Scheeres (2019)

Hamiltonian of the roto-orbital problem can be written as follows Ferrer; Lara (2012)

$$\mathcal{H} = \mathcal{H}_O + \mathcal{H}_R + \mathcal{H}_{Pert} \quad (118)$$

where

$$\mathcal{H}_O = \mathcal{H}_{Kepler} = \frac{1}{2m} \left(R^2 + \frac{\Theta^2}{r^2} \right) - \frac{km}{r} \quad (119)$$

$$\mathcal{H}_R = \mathcal{H}_{Euler} = \frac{m}{2} \left[(\Delta^2 - N^2) \left(\frac{\sin^2(\nu)}{A} + \frac{\cos^2(\nu)}{B} \right) + \frac{N^2}{C} \right] \quad (120)$$

$$\mathcal{H}_{Pert} = -\frac{km}{2m_a r^3} \left[(A - B) (1 - 3 \cos^2(\alpha)) + (C - A) (1 - 3 \cos^2(\gamma)) \right] \quad (121)$$

where m_a and m_s are the masses of the rigid triaxial body and the spherical body, G is the gravitational constant, $k = G(m_s + m_a)$ and $m = \frac{m_s m_a}{m_s + m_a}$; A, B, C are the principal moments of inertia of the body m_a . The Hamiltonian $\mathcal{H}_{Pert}$ is the part that offers the coupling between the orbital and the rotational motions and it is obtained as a truncation of the MacCullagh expansion (FERRANDIZ; SANSATURIO, 1989; ELIPE; ABAD; ARRIBAS, 1992; SCHAUB; JUNKINS, 2003; FERRER; LARA, 2012) up to r^{-3} and this approach works in almost all the cases of general interest. However, rather expansions of the mutual

gravitay potential are necessary in specific cases where the dimensions of the objects and their mutual distance start to become comparable, as the cases of binary asteroids in close approaches (see the appendix A for a brief discussion).

The cosines of α , β and γ are the direction cosines of the body m_s with respect to system of principal axes of the rigid body m_a . Their expression can be obtained in the new set of variables by mean of the equality:

$$\begin{pmatrix} \cos(\alpha) \\ \cos(\beta) \\ \cos(\gamma) \end{pmatrix} = \mathbf{R}_3^t(\nu) \mathcal{R}_1^t(\sigma) \mathcal{R}_3^t(\delta) \mathcal{R}_1^t(I_O + I_R) \mathcal{R}_3^t(\pi - \theta) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad (122)$$

where $\mathcal{R}_i^t(\zeta)$ is the transposed matrix of the rotation matrix $\mathcal{R}_i(\zeta)$.

By defining $\iota = I_O + I_R$, we get from the above relations what follows:

$$\cos(\alpha) = [(s_\nu s_\sigma) s_\iota - (c_\nu s_\delta + s_\nu c_\delta c_\sigma) c_\iota] \sin(\theta) + [s_\nu s_\delta c_\sigma - c_\nu c_\delta] \cos(\theta) \quad (123)$$

$$\cos(\beta) = [(c_\nu s_\sigma) s_\iota + (s_\nu s_\delta - c_\nu c_\delta c_\sigma) c_\iota] \sin(\theta) + [c_\nu s_\delta c_\sigma + s_\nu c_\delta] \cos(\theta) \quad (124)$$

$$\cos(\gamma) = [c_\sigma s_\iota + c_\delta s_\sigma c_\iota] \sin(\theta) - [s_\delta s_\sigma] \cos(\theta) \quad (125)$$

where $c_x \equiv \cos(x)$ and $s_x \equiv \sin(x)$.

The use of internal units is usually useful for the precision of the numerical integrations. Having this in mind, it is possible to choose convenient units to scale each problem. Here, we will follow the same scaling performed by Ferrer; Lara (2012), where it is used the following assumptions: (1) scaling of the momenta ($R, \Phi, \Psi, \Theta, \Delta, N$) per unit of mass m ($m = \frac{m_s m_a}{m_s + m_a}$); (2) the scaling of the moments of inertia (A, B and C) per unit of mass m_a of the triaxial body.

Using these new scaled values, the Hamiltonian (118) becomes:

$$\begin{aligned} \mathcal{H} = & \frac{1}{2m} \left(R^2 + \frac{\Theta^2}{r^2} \right) m^2 + \frac{m}{2} \left[(\Delta^2 - N^2) \left(\frac{\sin^2(\nu)}{A} + \frac{\cos^2(\nu)}{B} \right) + \frac{N^2}{C} \right] \frac{m^2}{m_a} \\ & - \frac{km}{r} - \frac{km}{2m_a r^3} [(A - B) (1 - 3 \cos^2(\alpha)) + (C - A) (1 - 3 \cos^2(\gamma))] m_a \end{aligned} \quad (126)$$

By choosing $q = \frac{m}{m_a}$ and the unit of mass $m = 1$, we get the following Hamiltonian:

$$\begin{aligned} \mathcal{H} = & \frac{1}{2} \left(R^2 + \frac{\Theta^2}{r^2} \right) - \frac{k}{r} + \frac{q}{2} \left[(\Delta^2 - N^2) \left(\frac{\sin^2(\nu)}{A} + \frac{\cos^2(\nu)}{B} \right) + \frac{N^2}{C} \right] \\ & - \frac{k}{2r^3} [(A - B) (1 - 3 \cos^2(\alpha)) + (C - A) (1 - 3 \cos^2(\gamma))] \end{aligned} \quad (127)$$

The Hamiltonian (127) has the following form

$$\mathcal{H} = \mathcal{H}(r, \phi, \psi, \theta, \delta, \nu, R, \Phi, \Psi, \Theta, \Delta, N) = \mathcal{H}(r, -, -, \theta, \delta, \nu, R, -, \Psi, \Theta, \Delta, N) \quad (128)$$

where ϕ , ψ , Φ are cyclic variables, which reduces the problem from a system with six-degree of freedom to a system with four-degree of freedom.

For the case when we have symmetry ($A = B$), we have:

$$\mathcal{H} = \frac{1}{2} \left(R^2 + \frac{\Theta^2}{r^2} \right) - \frac{k}{r} + \frac{q}{2} \left[\left(\frac{\Delta^2 - N^2}{A} \right) + \frac{N^2}{C} \right] - \frac{k}{2r^3} [(C - A) (1 - 3 \cos^2(\gamma))] \quad (129)$$

where the analysis of the expression (129) shows that this Hamiltonian, which we denominate as *original Hamiltonian*, produces a non-analytically integrable system of equations of motion. It happens because $\cos(\gamma)$ depends explicitly on θ and δ , which gives two extra degrees of freedom for the system. Thus, the usual way of procedure is to solve this system through numerical techniques of integrations. However, it is desirable to have an analytical solution for this problem, allowing different kinds of investigations such as stability analysis. Out of fact, for decades people have proposed analytical models which are approximations of full non-integrable problems. These models make possible to build analytically integrable systems very similar to the original ones. The interest on having such approached systems is that on investigating their analytical solutions, it is possible to better understand intrinsic dynamical structures and characteristics of the original problem through analytical techniques.

In the present study, we use *perturbation theory* to approach the roto-orbital Hamiltonian system described by expression (129), where the concept of *Intermediary* is used to build two roto-orbital Hamiltonians whose equations of motion present analytical solutions. The main difference to obtain these two models, namely radial roto-orbital intermediaries, lies on the use of the elimination of the parallax as a preliminary canonical transformation before the process of building one of the intermediaries. Both models are built departing from the expression (129) and were introduced recently in literature: Crespo et al. (2018) presented a roto-orbital intermediary which produces equations of motion integrable in terms of elliptic functions (due to the factor r^{-3} in the coupling term); Ferrer; Lara (2012) develop a study where the roto-orbital intermediary is built after the *elimination of the parallax* is carried out to turn the coupling term, which depends on r^{-3} , in a function of the factor r^{-2} . Due to this transformation, this model presents equations of motion integrable in terms of elementary quadratures. The present work intends to complement the available analysis for these models through numerical experiments, while comparisons are made between them.

5.3 INTERMEDIARIES

In the present study, the key concept of intermediary requires to be properly clarified, so that the reader can be sure which are the ones adopted along the text:

Common Intermediaries: The classical concept of intermediary Garfinkel; Aksnes

(1970) is used in Crespo et al. (2018) to build a roto-orbital intermediary with coupling term proportional to r^{-3} . The main ideas behind the procedure lies on the search of an intermediary which contains at least part of the first-order terms of Hamiltonian and makes possible the integration of the problem in terms of elliptic functions. This classical concept is also named in Deprit (1981) as *common intermediaries*, once in this work the author defines a different type of intermediary denoted as *natural intermediaries*.

Natural Intermediaries: Using this concept of intermediary, the system becomes integrable in terms of trigonometric functions, where the process of elimination of the parallax is used to reduce the order of the disturbing term to r^{-2} . This different concept was introduced by Deprit (1981) and is the one used in Ferrer; Lara (2012) to build a roto-orbital intermediary with coupling term proportional r^{-2} . In fact, when Deprit was working in the development of the technique of elimination of the parallax, he made very clear that “the elimination of the parallax is an instance of a general technique which, in retrospect, may be related to an approach taken by Sterne, Garfinkel and others”. Following his development we can define what is called “natural intermediary”. If we consider the Hamiltonians $\mathcal{J} \equiv \mathcal{J}(p, P; \epsilon)$ and

$$\mathcal{M} \equiv \mathcal{M}(q, Q; \epsilon) = \mathcal{M}_0(q, Q) + \epsilon \mathcal{M}_1(q, Q) + \mathcal{O}(\epsilon^2) \quad (130)$$

the Hamiltonian $\mathcal{J}$ will be called the “natural intermediary of $\mathcal{M}$ ” in the case where we have two conditions being satisfied:

- (1) $\mathcal{J}$ is integrable in the map (p, P) about $\epsilon = 0$;
- (2) There exists a canonical transformation $\mathcal{D} : (q, Q; \epsilon) \rightarrow (p, P)$ such that

$$\mathcal{M}[q(p, P; \epsilon), Q(p, P; \epsilon); \epsilon] = \mathcal{J}(p, P; \epsilon) + \mathcal{O}(\epsilon^n) \quad (n \geq 2) \quad (131)$$

The meaning of this definition in practical terms is that the dynamical system $\mathcal{J}$ may be solved in the phase space (p, P) and that its solution constitutes an approximation to order 1 at least of the solution for the dynamical system $\mathcal{M}$ in the map (p, P) . Replacing the Hamiltonian $\mathcal{M}$ by its *natural intermediary* $\mathcal{J}$ amounts to pushing down the perturbation from the first order to a higher order. Whenever it is feasible, the downgrading is a very effective way of advancing the solution of the original problem.

Deprit’s work presents a detailed discussion about intermediaries where after discussing different models for intermediaries (radial, zonal, polar, etc), it is presented that the *common intermediaries* can all be recovered as *natural intermediaries* for the main problem in the theory of artificial satellites. By the ‘criteria of merit’ proposed by Garfinkel; Aksnes (1970), the radial intermediaries are rated higher than the zonal ones since the separation of variables in their Hamilton-Jacobi equation does not involve elliptic integral for the argument of latitude. We refer the reader to references Deprit (1981) for further details.

What is essential to emphasize in the present context is: the same principles remain for other problems, which includes the roto-orbital dynamics. Regarding this, there exist recent works discussing different models for the intermediaries in such dynamics Molero (2013); Crespo (2014). Thus, a possible choice for $\mathcal{M}_0$ is $\mathcal{M}_0 = \mathcal{H}_{0,0} = \mathcal{H}_{Unperturbed} = \mathcal{H}_{Kepler} + \mathcal{H}_{Euler}$, once $\mathcal{H}_{Unperturbed}$ is an integrable and separable dynamical system whose solution constitutes a free-rotating rigid body in a Kepler-type orbit. Additionally, the perturbation of this system will depends on the small parameter ϵ which here appears in the term describing the coupling between the two dynamics.

5.4 ELIMINATION OF THE PARALLAX

According to Poincaré (1893), most of the problems related to orbital and rotational motion in Celestial Mechanics can be formulated as a perturbed integrated system. Once we assume the perturbation Hamiltonian $\mathcal{M} = \mathcal{M}_0 + \mathcal{P}$, where $\mathcal{M}_0$ is integrable, the disturbing term $\mathcal{P} = \sum_{m \geq 1} \left(\frac{\epsilon^m}{m!} \right) \mathcal{M}_{m,0}$ will be given by its expansion in powers of some small parameter ϵ . The perturbation theory presented in Deprit (1969) allows to compute a simplified new Hamiltonian $\mathcal{M}' = \sum_{m \geq 1} \left(\frac{\epsilon^m}{m!} \right) \mathcal{M}_{0,m}$ in a new set of variables, as well as the explicit transformation from old to new variables and vice versa. This new set of variables can be the resulting ones after the elimination of the parallax. In the present study, the expansion in power series is truncated to first order and we get $\mathcal{P} = \epsilon \mathcal{M}_1 = \mathcal{H}_1$. Thus, it is possible to write the Hamiltonian (129) as $\mathcal{M} = \mathcal{M}_0 + \epsilon \mathcal{M}_1$ which means that it can be split in two parts⁵: $\mathcal{M}_0 = \mathcal{H}_{Unperturbed} = \mathcal{H}_{Kepler} + \mathcal{H}_{Euler}$ defines an integrable system and $\mathcal{P} = \epsilon \mathcal{M}_1 = \mathcal{H}_{1,0}$ is called the perturbation of the Hamiltonian system defined by $\mathcal{M}_0$. Thus, we have

$$\mathcal{H}_{0,0} = \mathcal{H}_{Unperturbed} = \frac{1}{2} \left(R^2 + \frac{\Theta^2}{r^2} \right) - \frac{k}{r} + \frac{q}{2} \left[\left(\frac{\Delta^2 - N^2}{A} \right) + \frac{N^2}{C} \right] \quad (132)$$

and

$$\mathcal{H}_{1,0} = -\frac{k}{2r^3} [(C - A) (1 - 3 \cos^2(\gamma))] \quad (133)$$

where the first term $\mathcal{H}_{0,1}$ is the new first order Hamiltonian (explicitly independent on θ) given by

$$\mathcal{H}_{0,1} = -\left(\frac{1}{r^2} \right) \left(\frac{k}{2p} \right) (C - A) \left[1 - \frac{3}{2} (b_1^2 + b_2^2) \right] \quad (134)$$

where p is the semilatus rectum, $b_1 = c_\sigma s_\iota + s_\sigma c_\iota \cos(\delta)$ and $b_2 = s_\sigma \sin(\delta)$. The Hamiltonian $\mathcal{H}_{0,1}$ is not the averaged Hamiltonian $\langle \mathcal{H}_{0,1} \rangle$ because it still depends on $r = r(\theta)$.

The application of standard perturbation methods allows us to eliminate periodic terms

⁵In the main problem of the artificial satellite theory, Deprit (1981) defines the integrable system as $\mathcal{M}_0 = \mathcal{H}_0 = \mathcal{H}_{0,0} = \mathcal{H}_{Kepler}$.

and the elimination of the parallax is a technique introduced to perturbation theories in non-linear and celestial mechanics. On the context of the main problem of artificial satellite theory, it is used to convert a Keplerian system in a central force field r^{-m} ($m \geq 3$) into a “quasi-Keplerian” system with a variable angular momentum Deprit (1981) and it also may be used to define a chain of natural intermediaries (see section 5.5 about roto-orbital intermediaries). In Deprit’s work the small parameter ϵ is given by $\epsilon = -C_{2,0}$, while in the present context it is connect to the coupling between the orbital and rotational dynamics. It is important to state that, excluding the sub-manifold $\Theta = 0$, the transformation is applied everywhere in the phase space.

Presently, we are considering a study with perturbation up to first order ($m = 1$). Therefore, after the procedure for elimination of the parallax Deprit (1981); Ferrer; Lara (2012), the canonical transformation $\mathcal{W} = \mathcal{W}_1$ is written as

$$\begin{aligned} \mathcal{W}_1 = \frac{C-A}{16} & \left\{ 8 \left(1 - 3s_{\sigma'}^2 \sin^2(\delta') \right) \left(\frac{S'_p}{p'} \cos(\theta') - \frac{C'_p}{p'} \sin(\theta') \right) \right. \\ & + [2s_{\sigma'} (c_{\sigma'} s_{\iota'} \sin(\delta') + s_{\sigma'} c_{\iota'} \sin(2\delta'))] \\ & \times \left[3 \frac{\Theta'}{p'^2} \cos(2\theta') + \frac{S'_p}{p'} (\sin(3\theta') - 3 \sin(\theta')) + \frac{C'_p}{p'} (\cos(3\theta') + 3 \sin(\theta')) \right] \\ & - [(2 - 3s_{\sigma'}^2) s_{\iota'}^2 + 4c_{\sigma'} c_{\iota'} s_{\sigma'} s_{\iota'} \cos(\delta') + (1 + c_{\iota'}^2) s_{\sigma'}^2 \cos(2\delta')] \\ & \times \left[3 \frac{\Theta'}{p'^2} \sin(2\theta') - \frac{S'_p}{p'} (\cos(3\theta') - 9 \cos(\theta')) + \frac{C'_p}{p'} (\sin(3\theta') - 3 \sin(\theta')) \right] \Big\} \end{aligned} \quad (135)$$

where the primed functions mean functions of the primed variables and with transformation equations being given by

$$x_i = x_i' + \{x_i'; \mathcal{W}_1\}; \quad x_i' \in (r, \theta, R, \Theta, \delta, \Delta, \nu, \phi) \quad (136)$$

$$y_i = y_i'; \quad y_i' \in (\psi, \Phi, \Psi, N) \quad (137)$$

Indeed, we have performed the transformation of variables of the form

$$(r, \phi, \psi, \theta, \delta, \nu, R, \Phi, \Psi, \Theta, \Delta, N) \longrightarrow (r', \phi', \psi', \theta', \delta', \nu', R', \Phi', \Psi', \Theta', \Delta', N') \quad (138)$$

whose generating function is given by equation (135). It justifies the more proper use of primed variables to represent the new first order perturbation Hamiltonian $\mathcal{H}_{0,1}$ (expression (134)) as

$$\begin{aligned} \mathcal{H}'_{0,1} = - \left(\frac{k}{p'} \right) \frac{(C-A)}{8r'^2} & [(1 - 3s_{\iota'}^2) (1 - 3s_{\sigma'}^2) \\ & - 12c_{\sigma'} c_{\iota'} s_{\sigma'} s_{\iota'} \cos(\delta') + 3s_{\sigma'}^2 s_{\iota'}^2 \cos(2\delta')] \end{aligned} \quad (139)$$

Therefore, the new Hamiltonian $\mathcal{H}'_p = \mathcal{H}'_{0,0} + \mathcal{H}'_{0,1}$, denominated “parallax Hamiltonian” is

$$\mathcal{H}'_p = \frac{1}{2} \left(R'^2 + \frac{\Theta'^2}{r'^2} \right) - \frac{k}{r'} + \frac{q}{2} \left[\left(\frac{\Delta'^2 - N'^2}{A} \right) + \frac{N'^2}{C} \right] \quad (140)$$

$$- \left(\frac{k}{p'} \right) \frac{(C - A)}{8r'^2} \left[(1 - 3s_{\iota'}^2) (1 - 3s_{\sigma'}^2) - 12c_{\sigma'}c_{\iota'}s_{\sigma'}s_{\iota'} \cos(\delta') + 3s_{\sigma'}^2s_{\iota'}^2 \cos(2\delta') \right]$$

The elimination of the parallax reduced the factor r^{-3} in expression (129) to r^{-2} as seen in expression (140), which turns the process of finding an analytical solution for the system of equations of motion simpler, avoiding the use of elliptic integrals by the use of elementary integrals. In the other hand, this technique requires to understand how significant is the magnitude of the orbital eccentricity for each case, which can require higher order terms on the process of building the model through expansion. In section 5.6, tests for systems with different eccentricities are made to help on this understanding.

5.5 ROTO-ORBITAL INTERMEDIARIES AND EQUATIONS OF MOTION

When elimination of the parallax was applied to our roto-orbital model, the Hamiltonian $\mathcal{H}_{1,0}$ was replaced by $\mathcal{H}'_{0,1}$. In order to build an intermediary for the roto-orbital problem, we need to split neither $\mathcal{H}_{1,0}$ or $\mathcal{H}'_{0,1}$ (once the elimination of the parallax is previously applied) in two parts. We call the model before the elimination of the parallax as “original Hamiltonian” (expression (129)) and the model after the use of this canonical technique as “parallax Hamiltonian” (expression (140)). The intermediaries that built from these two models are named as Intermediary 1 ($\mathcal{I}_o$) and Intermediary 2 ($\mathcal{I}_p$), respectively.

Regarding the original Hamiltonian (expression (129)), we can split $\mathcal{H}_{1,0}$ as $\mathcal{I}_p^{1,0} + \mathcal{R}_o$ and we get

$$\mathcal{H}_{original} = \mathcal{H}_{0,0} + \mathcal{H}_{1,0} = \mathcal{I}_o + \mathcal{R}_o \quad (141)$$

where $\mathcal{I}_o = \mathcal{H}_{0,0} + \mathcal{I}_o^{1,0}$ is the Intermediary and $\mathcal{R}_o$ is the Remainder. The term $\mathcal{I}_o^{1,0}$ is the part of $\mathcal{H}_{1,0}$ that is independent on fast angles (θ for translation and δ for rotation).

For the parallax Hamiltonian (expression (140)), we can split $\mathcal{H}'_{0,1}$ as $\mathcal{I}_p^{0,1} + \mathcal{R}_p$ and we will have

$$\mathcal{H}_{parallax} = \mathcal{H}'_{0,0} + \mathcal{H}'_{0,1} = \mathcal{I}_p + \mathcal{R}_p \quad (142)$$

where $\mathcal{I}_p = \mathcal{H}'_{0,0} + \mathcal{I}_p^{0,1}$ will be the Intermediary and $\mathcal{R}_p$ will be the Remainder. The term $\mathcal{I}_p^{0,1}$ is the part of $\mathcal{H}'_{0,1}$ that is independent on fast angles (in this case: δ').

An intermediary Hamiltonian that intends to approach the system needs to be very

close to the original one, which means to consider especially the long-period terms related to the angles that are dependent on the canonical momenta (σ , I_O and I_R), namely the inclinations between the “fundamental” planes given in expressions (117). At the same time, the fast-angles (θ and δ) will carry the information about short-period behavior.

It is important to make clear the intermediary as not being an “averaged model” due to the presence of implicit internal fast angles, e.g., $r = r(\theta)$. To build the remainder by splitting the terms with explicit dependence on the fast angles, it is assumed their effects as not having significant role in the dynamical system. Additionally, the effects are assumed to be recovered using the expressions (136) and (137). For the intermediary 1 ($\mathcal{I}_o$) we have:

$$\mathcal{I}_o = \mathcal{H}_{0,0} + \mathcal{I}_o^{1,0} = \mathcal{H}_{Unperturbed} + \frac{(A-C)k}{8r^3} [(1-3c_\iota^2)(1-3c_\sigma^2)] \quad (143)$$

$$\mathcal{R}_o = \mathcal{H}_{1,0} - \mathcal{I}_o^{1,0} = \frac{3(A-C)k}{16r^3} \mathcal{R}_{\theta\delta} \quad (144)$$

$$\mathcal{R}_{\theta\delta} = 3s_\sigma^2 [(1-c_\iota)^2 C_{2,2,0} + (1+c_\iota)^2 C_{-2,2,0}] \quad (145)$$

$$+ 6s_\iota^2 [s_\sigma^2 C_{0,2,0} - (1-3c_\sigma^2) C_{2,0,0}] \quad (146)$$

$$- 12c_\sigma s_\iota s_\sigma [(1-c_\iota) C_{2,1,0} + 2c_\iota C_{0,1,0} - (1+c_\iota) C_{-2,1,0}] \quad (147)$$

where the notation $C_{i,j,k} \equiv \cos(i\theta + j\delta + k\nu)$ is adopted. The angles σ and $\iota = I_O + I_R$ are related to the new canonical variables (see expressions (117)), which means that the disturbing potential of the intermediary depends on the canonical momenta through these angles between the “fundamental” planes.

Similarly, for the intermediary 2 ($\mathcal{I}_p$) we have:

$$\mathcal{I}_p = \mathcal{H}'_{0,0} + \mathcal{I}_o^{0,1} = \mathcal{H}'_{Unperturbed} + \frac{(A-C)k^2}{8r'^2\Theta'^2} [(1-3c_{\iota'}^2)(1-3c_{\sigma'}^2)] \quad (148)$$

$$\mathcal{R}_p = \mathcal{H}'_{0,1} - \mathcal{I}_p^{0,1} = -\frac{3(A-C)k^2}{8r'^2\Theta'^2} s_{\iota'} s_{\sigma'} [4c_{\iota'} c_{\sigma'} \cos(\delta') - s_{\iota'} s_{\sigma'} \cos(2\delta')] \quad (149)$$

For the radial intermediary problem without the elimination of the parallax (model $\mathcal{I}_o$) we have the following system

$$\dot{\psi} = -3\epsilon \frac{p\Theta}{r^3} \left[c_\iota (1-3c_\sigma^2) \frac{\Psi}{\Delta} \right] \quad (150)$$

$$\dot{\delta} = \frac{q}{A} \Delta + 3\epsilon \frac{p\Theta}{r^3} \left[c_\iota (1-3c_\sigma^2) + (c_\iota^2 + c_\sigma^2 - 6c_\iota^2 c_\sigma^2) \frac{\Theta}{\Delta} \right] \quad (151)$$

$$\dot{\nu} = q \left(\frac{1}{C} - \frac{1}{A} \right) N - 3\epsilon \frac{p\Theta}{r^3} \left[c_\sigma^2 (1 - 3c_\iota^2) \frac{\Theta}{N} \right] \quad (152)$$

$$\dot{\theta} = \frac{\Theta}{r^2} \left[1 + 3\epsilon \frac{p}{r} c_\iota (1 - 3c_\sigma^2) \left(c_\iota + \frac{\Theta}{\Delta} \right) \right] \quad (153)$$

$$\dot{R} = -\frac{k}{r^2} + \frac{\Theta^2}{r^3} \left[1 + \frac{3p}{2r} \epsilon (1 - 3c_\iota^2) (1 - 3c_\sigma^2) \right] \quad (154)$$

$$\dot{r} = R \quad (155)$$

where the parameter ϵ is given by $\epsilon = \frac{A}{(2p)^2} (1 - q_s)$, with $p = \Theta^2/k$ and $q_s = \frac{C}{A}$.

The parameter q_s is introduced to represent the shape of the rigid body and have the properties:

(1) $q_s = 1 \rightarrow \epsilon = 0$ (spherical shape)

(2) $q_s > 1 \rightarrow \epsilon < 0$ (oblate shape)

(3) $q_s < 1 \rightarrow \epsilon > 0$ (prolate shape)

For the radial intermediary problem after the elimination of the parallax (model $\mathcal{I}_p$) we have the following system⁶

$$\dot{\psi}' = -3\epsilon' \frac{\Theta'}{r'^2} \left[c_{\iota'} \left(1 - 3c_{\sigma'}^2 \right) \frac{\Psi'}{\Delta'} \right] \quad (156)$$

$$\dot{\delta}' = \frac{q}{A} \Delta' + 3\epsilon' \frac{\Theta'}{r'^2} \left[c_{\iota'} \left(1 - 3c_{\sigma'}^2 \right) + \left(c_{\iota'}^2 + c_{\sigma'}^2 - 6c_{\iota'}^2 c_{\sigma'}^2 \right) \frac{\Theta'}{\Delta'} \right] \quad (157)$$

$$\dot{\nu}' = q \left(\frac{1}{C} - \frac{1}{A} \right) N' - 3\epsilon' \frac{\Theta'}{r'^2} \left[c_{\sigma'}^2 \left(1 - 3c_{\iota'}^2 \right) \frac{\Theta'}{N'} \right] \quad (158)$$

$$\dot{\theta}' = \frac{\Theta'}{r'^2} \left[1 - \epsilon' \left(1 - 3c_{\sigma'}^2 \right) \left(1 - 3\frac{\Theta'}{\Delta'} c_{\iota'} - 6c_{\iota'}^2 \right) \right] \quad (159)$$

$$\dot{R}' = -\frac{k}{r'^2} + \frac{\Theta'}{r'^3} \left[1 - \epsilon' \left(1 - 3c_{\iota'}^2 \right) \left(1 - 3c_{\sigma'}^2 \right) \right] \quad (160)$$

⁶Here we emphasize the primes to avoid confusion with the original variables

$$\dot{r}' = R' \quad (161)$$

with parameter $\epsilon' = \frac{A}{(2p')^2} (1 - q_s)$ and $p' = \Theta'^2/k$.

The remaining equations of motion for both systems are

$$\dot{\phi} = \dot{\Phi} = \dot{\Psi} = \dot{\Theta} = \dot{\Delta} = \dot{N} = 0 \quad (162)$$

$$\dot{\phi}' = \dot{\Phi}' = \dot{\Psi}' = \dot{\Theta}' = \dot{\Delta}' = \dot{N}' = 0 \quad (163)$$

which give us $\phi, \Phi, \Psi, \Theta, \Delta, N, \phi', \Phi', \Psi', \Theta', \Delta', N'$ as constants of motion.

The radial part composed by equations (161) and (160) ($\dot{r}'$ and $\dot{R}'$ for the model after the elimination of parallax) can be separated and solved independently. The solution of r for this Kepler-type orbit can be replaced in the equations to $\dot{\psi}', \dot{\delta}', \dot{\nu}'$ and $\dot{\theta}'$ in order to obtain solutions by mean of quadratures. This system is precisely the one that was obtained by Ferrer; Lara (2012), where a discussion about some equilibrium conditions is presented.

Before moving to numerical simulations, it is important to note ϵ (and ϵ' , as well) as the parameter that represents the coupling/perturbation in the dynamics. It especially depends on the shape on the rigid body ($q_s = q_s(A, C)$) and its orbital configuration (the kind of orbit is connected with $p(\Theta) = \frac{\Theta^2}{k}$). This special dependence on p also brings the necessity of investigating different radial distances $r = r(p)$. In fact, it is easy to verify loss of precision with the increase of the perturbation by analyzing the magnitude of ϵ (and q_s) for different values of A and C . The greater is the difference between these moments of inertia and the values for ϵ will increase. It departs from the assumption used for perturbation theory regarding a small ϵ , *i.e.*, a small perturbation assumed is in the system. The more spherical and distant from the spherical primary body is the rigid body, weaker will be the coupling between the orbital and the rotational dynamics, and consequently less disturbed will be the motion. Once the perturbation techniques will give better approaches when we have less disturbing effects in the system, it is essential to investigate the ranges where key parameters of the system will produce results to increase the precision and time for what the intermediary models are valid. It is clear the preference of using models where this precision and time satisfy the requirements of a specific investigation, for instance, a model that is able to determine the attitude dynamics of an artificial satellite for a long-term mission. In fact, the maximization of the precision and extension of time of where the models are valid support applications in a wider range of different classes of problem in celestial mechanics and astrodynamics. In connection with this purpose, the parameters introduced as q_s and q are very important in our study, once they present, respectively, the information about the shape of the rigid body and the relative size between the two bodies.

5.6 NUMERICAL SIMULATIONS AND ANALYSIS

The numerical validation of the two intermediaries models is of fundamental importance in order to provide support for their application in different classes of problems, with special interest in the model $\mathcal{I}_p$, which has an analytical solution for the equations of motion easier to be obtained in comparison with the model $\mathcal{I}_o$, whose solution requires the use of elliptic integrals. However, both systems are of interest once as integrable systems, they bring the benefits of allowing studies about stability, relative equilibria, attitude prediction, etc (see for instance Crespo; Ferrer (2018)). At this moment, the use of primes in the variables will be omitted for simplicity. In simulations of the present study, the results presented in Ferrer; Lara (2012) were reproduced successfully and used as reference to validate the numerical integrator developed using the software Maple 16. After this validation process, which also includes the verification of the conservation of the total energy of the system, several numerical propagations were performed for different dynamical models, considering two sets of initial conditions with different attitude angles for the rigid body. As the perturbation between the two dynamical motions depends on the parameter ϵ , which depends on the shape of the rigid body, all the tests were performed considering different shapes, from a very oblate body to an almost spherical one.

5.6.1 Setting models and test parameters

As a first analysis of the intermediaries presented previously, we propagate the dynamics of six different models: (1) Unperturbed (yellow lines), (2) triaxial (green lines; expression (127)), (3) original (aquamarine lines; expression (129)); (4) intermediary 1 ($\mathcal{I}_o$) (red lines; expression (143)), (5) intermediary 2 ($\mathcal{I}_p$) with primed variables (q', p') (blue lines; expression (148)); and (6) intermediary 2 ($\mathcal{I}_p(q, p)$) with short-period corrections (black lines; expressions (148), (136) and (137)). As the plots for many variables are almost perfectly superposed many numerical propagations, the timespans for each model are presented truncated to highlight each model's result.

We fix the values $A = 6.9 \times 10^{-4}$ and $C = 7.0 \times 10^{-4}$ for the moments of inertia (in internal units), considering the following restrictions (BROUWER; CLEMENCE, 1961): if $C > B > A \rightarrow A + B > C$. We also set $B = A$ for the comparisons with the triaxial model. The values considered for all simulations are given by (in internal units and degrees) are:

$$\begin{aligned} k &= (m_s + m_s)G_{new} = 1.0005, G_{new} = 0.16008, q = 0.8 \\ r_0 &= 1.3, \theta_0 = 0.0, \delta_0 = 0.1, \phi_0 = 0.5^\circ, \psi_0 = 0.4^\circ, \nu_0 = 0.2^\circ, I_0 = 9.5^\circ \\ R_0 &= 0.0, \Theta_0 = 1, \Delta_0 = 0.03, \Phi_0 = \Psi_0 \cos(I_0), \Psi_0 = \Theta_0 + \frac{\Delta_0}{2}, N_0 = \\ &\Delta_0 \cos(\sigma_0) \end{aligned}$$

where the parameter $q = 0.8$ corresponds to masses of $m_s = 5$ for the spherical primary body and $m_a = 5/4$ for the rigid body. It also comes from these variables that the orbital eccentricity, semi-major axis, true anomaly and argument of the periapsis are given by $e = 0.23115$, $a = 1.05592$, $f = 180.1^\circ$ and $\omega = -180.1^\circ$, respectively. It means an orbit propagation departing from the apoapsis.

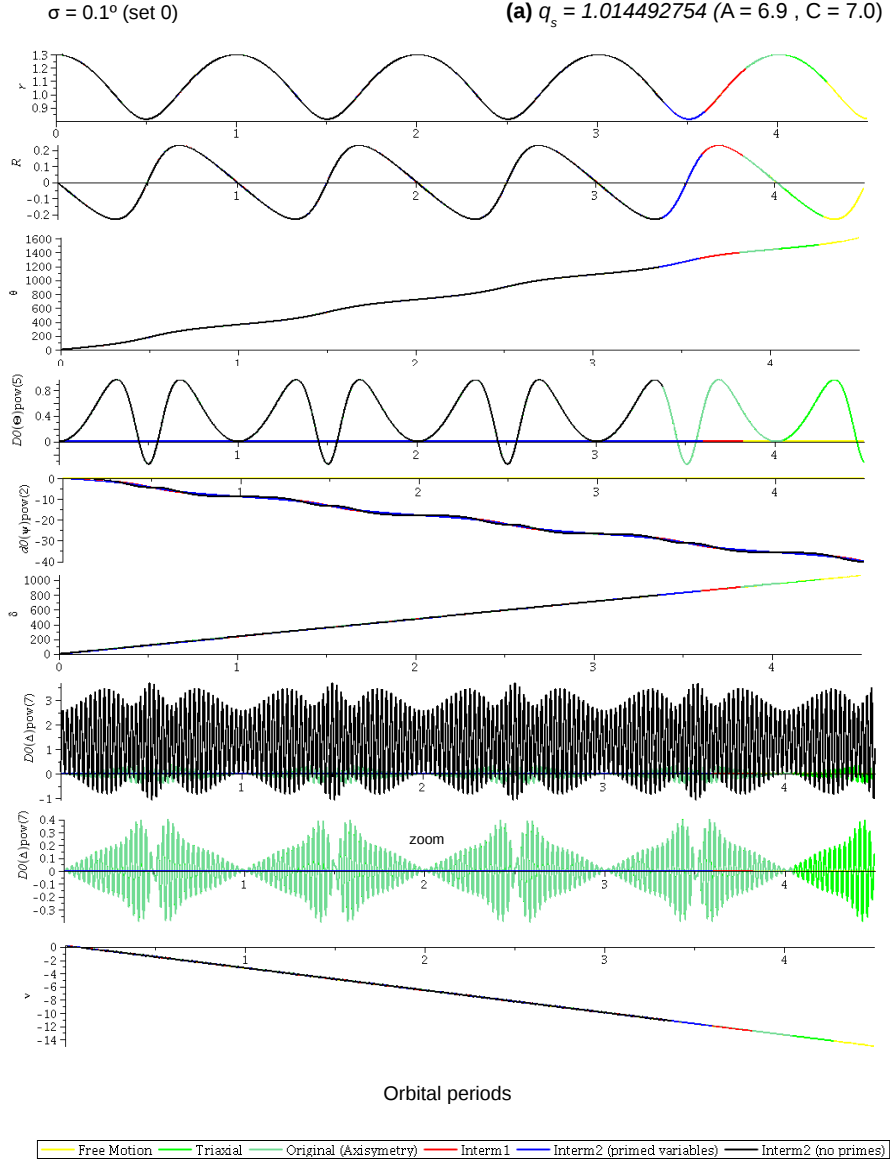
In fact, by setting physical units, it is possible to have the information about the system in terms of the International System (IS) of units (or any other system of preference). An example of this is that by assuming $uds = 10 \times 10^3$ km (unity of distance), $ums = 6.557058953 \times 10^{18}$ kg (unity of mass) and $uts = 604800$ seconds (1 week, unity of time), the parameters chosen for the present numerical experiments are given by (in non-scaled units):

$$\begin{aligned} k &= G_{IS}(m_s + m_a) = 1.0005, G_{IS} = 6.67428 \times 10^{-20} km^3 kg^{-1} s^{-22}, q = 0.8, \\ m \text{ (IS)} &= 1, m_s(IS) = 3.278529476 \times 10^{19} kg, m_a(IS) = 8.196323691 \times 10^{18} kg, \\ r_0 &= 13000 km, R_0 = mR_0 = 0.0 km/s, \\ \Theta_0 &= m\Theta_0 = 1.084169800 \times 10^{21} kg.km^2/s, \\ \Delta_0 &= m\Delta_0 = 3.252509401 \times 10^{19} kg.km^2/s, \\ \Psi_0 &= m\Psi_0 = m\Theta_0 + \frac{m\Delta_0}{2} = 1.100432348 \times 10^{21} kg.km^2/s, \\ \Phi_0 &= m\Phi_0 = m\Psi_0 \cos(I_0) = 1.100432348 \times 10^{21} \cos(I_0) kg.km^2/s, \\ N_0 &= mN_0 = m\Delta_0 \cos(\sigma_0) = 3.252509401 \times 10^{19} \cos(\sigma_0) kg.km^2/s \end{aligned}$$

Using the value of attitude angle $\sigma_0 = 0.1^\circ$ (set 0 of initial conditions), we get the initial conditions presented in Ferrer; Lara (2012). A sample propagation of this set of initial conditions considering the original model is made and the results are in accordance with the ones presented by Ferrer and Lara work. These results are presented in figure 57.

A short discussion about the meaning of these values is valuable: Taking the mass of Charon ($\approx 1.586 \times 10^{21}$ kg) as reference, the masses $m_s \sim 3.279 \times 10^{19}$ kg for the spherical body and $m_a \sim 8.196 \times 10^{18}$ kg for the rigid body will correspond to $\sim 2.07\%$ and 0.517% of the mass of Charon, respectively. Regarding the averaged dimensions for these bodies, an estimate of hypothetical averaged radius for these bodies is possible by assuming hypothetical densities taken other bodies as reference. Taking again Charon, we can use its radius and mass to perform a simplified computation assuming uniform density $\rho_{averaged}$. We get the value $\rho_{averaged} \sim 7.127 \text{ g/cm}^3$, which is very different of the estimated value $\rho_{estimated} \sim 1.707 \text{ g/cm}^3$. Assuming the first value for the density, we get the averaged values of ~ 166.315 and ~ 104.772 km for the radius of the bodies “s” and “a”, respectively. By using the relation “ $radius_{estimated} \approx radius_{averaged} \times (\rho_{averaged}/\rho_{estimated})^{1/3}$ ” and assuming the second (real) value for the Charonian density, we obtain the values of ~ 267.803 and ~ 168.705 km, respectively. In order to find the

Figure 57: Sample propagation considering the set 0 of initial conditions ($\sigma = 0.1^\circ$). Notation: $dO(x) \equiv x - x_0$, $DO(x) \equiv 1 - x/x_0$ and $pow(x) \equiv 10^x$. Angles in radians and remaining values in internal units.



Source: Author

values of the non-specific moments of inertia in terms of the specific ones A , B and C with respect to these units, it is necessary to multiply the value of each of last ones by the value of the mass m_a of the rigid body. Additionally, the values considered for the normalized moments of inertia $A = 6.9 \times 10^{-4}$ and $C = 7.0 \times 10^{-4}$ denote an oblate rigid body presenting the expected retrograde rotation around itself (see the time evolution of the rotation angle ν in figure 57). The parameter q_s for this case is equals to $C/A = 1.014492754$. Finally, the semi-major axis is $a = 10559.2$ km. In fact, this discussion about the physical parameters is not the main focus of the present work and it is presented to help the reader's overview about the hypothetical system considered using Ferrer and Lara like parameters (FERRER; LARA, 2012) and which were adopted in the present

study to facilitate the analysis of the results.

5.6.2 Comparisons of different dynamical models for different physical parameters

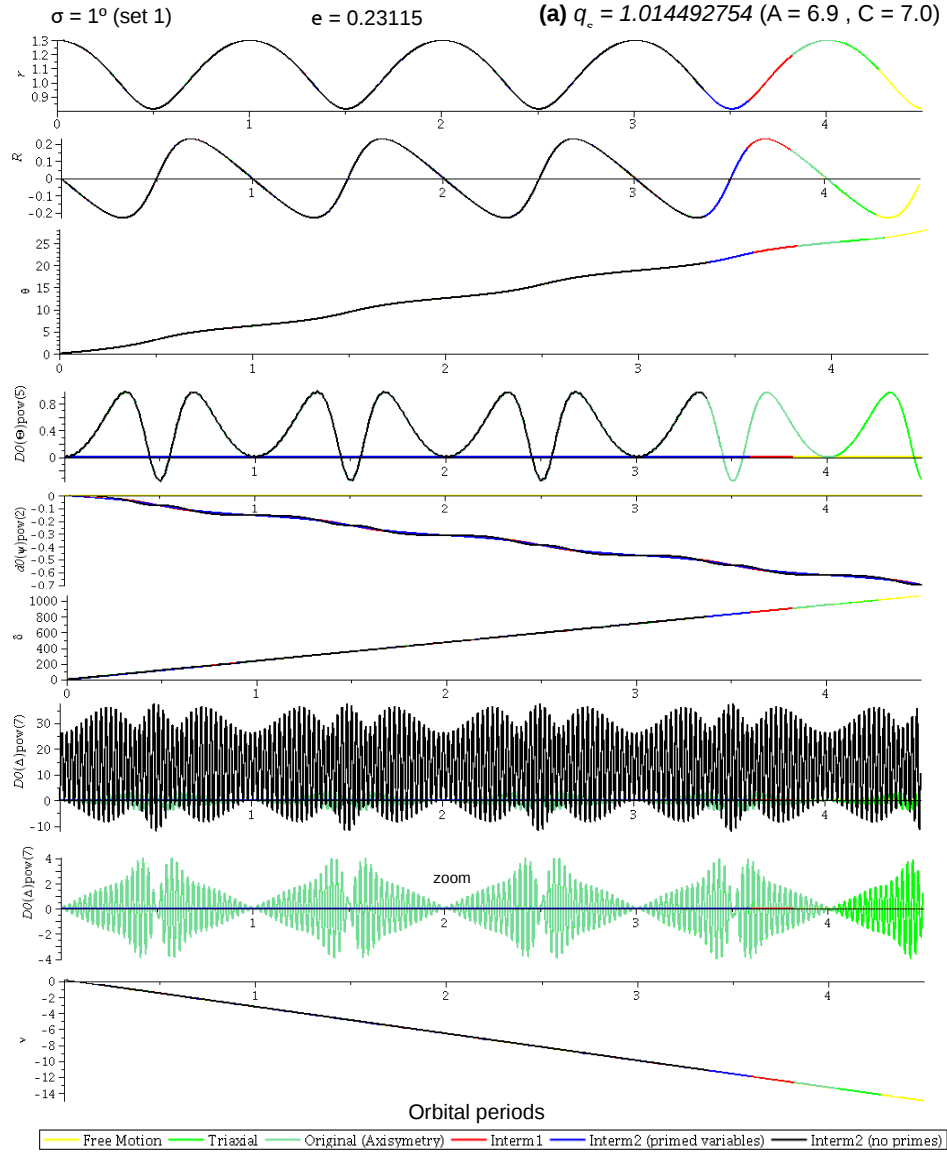
Numerical propagations of different models⁷ are presented in figures 58 to 64. First of all, numerical experiments are made for three similar sets of initial conditions with differences on the values related to the obliquity σ_0 of the rigid body (σ is the attitude angle/inclination between the inertia/body fixed frame and the rotational frame). Figures 58 to 60 present the results for the sets 1, 2 and 3 of initial conditions, where the values of $\sigma_0 = 1.0^\circ$, $\sigma_0 = 45.1^\circ$ and $\sigma_0 = 90.1^\circ$ were, respectively, considered. Worth of mention is that the value $\sigma_0 = 1.0^\circ$ is assumed to avoid problems with numerical integrations close to the singularity in $\sigma_0 = 0.0^\circ$. Once $q_s > 1$ (oblate body), in all these cases the rotation angle ν present a retrograde motion, as previously mentioned in section 5.6.1. For the cases where $\sigma_0 = 45.1^\circ$ and 90.1° , the body is naturally evolving slower once the gravity gradient effect is minimized by its attitude (see variable ν in figures 59 and 60). It means that these cases present the component of the force along the direction of this motion with a smaller magnitude than in the case where $\sigma_0 = 1.0^\circ$ or any smaller angle.

In these simulations, the initial conditions are not the ones which produce some equilibrium points in the system defined by the intermediary $\mathcal{I}_p$ (see section 5.5). Figures 58 to 60 show this model and the intermediary $\mathcal{I}_o$ producing very close approaches of the dynamics, as it follows both, the orbital and the attitude, evolutions (see plots for r , R , θ , δ and ν) when compared with the original Hamiltonian (aquamarine lines) and triaxial Hamiltonian (green lines). The expected perturbed time evolution for the angles θ and ν (linear component with slightly periodic oscillations) in both intermediaries is very close to the one due to the original model given by expression (129).

In general, the simulations present precise approaches, especially regarding the orbital motion. The main differences appear on the magnitude of angular momenta: Θ for orbital motion and Δ for rotational motion. These differences appear when one moves from set 1 in figure 58 ($\sigma_0 = 1.0^\circ$) to set 3 in figure 60 ($\sigma_0 = 90.1^\circ$). The further the attitude angle σ of the rigid body departs from the equilibrium position ($\sigma = 0^\circ$), the more significant will be the differences presented in the magnitude of these momenta. This result comes from the physical fact that the closer is the angle σ of the value 0° (a rotation around the axis of greater moment of inertia C aligned with the z -axis of the rotation inertial frame), the symmetry in shape (x and y axes) will decrease the gravity gradient that disturbs the attitude. In the case of a triaxial body, this equilibrium position refers to the known

⁷The presence of six plots for almost all figures is due to the fact that the results for the Hamiltonian $\mathcal{I}_p$ are presented in primed variables (blue lines - after the elimination of the parallax) and with no primes (black lines - after the return to the original variables with short-period corrections). Because of the higher amplitude in the oscillation of the magnitude of some variables for this last model, the plots denoted by zoom are the ones where the results for this model do are not able to appear when plotted together with all the other models (black lines missing). In fact, the results in black lines for the intermediary $\mathcal{I}_p$ were obtained by using the generating function $\mathcal{W}_1$ (expression (135) and the transformation equations given in expressions (136) and (137) to obtain short-period corrections in original (non-primed) variables.

Figure 58: Sample propagation considering the set 1 of initial conditions ($\sigma = 1.0^\circ$). Notation: $dO(x) \equiv x - x_0$, $DO(x) \equiv 1 - x/x_0$ and $pow(x) \equiv 10^x$. Angles in radians and remaining values in internal units.

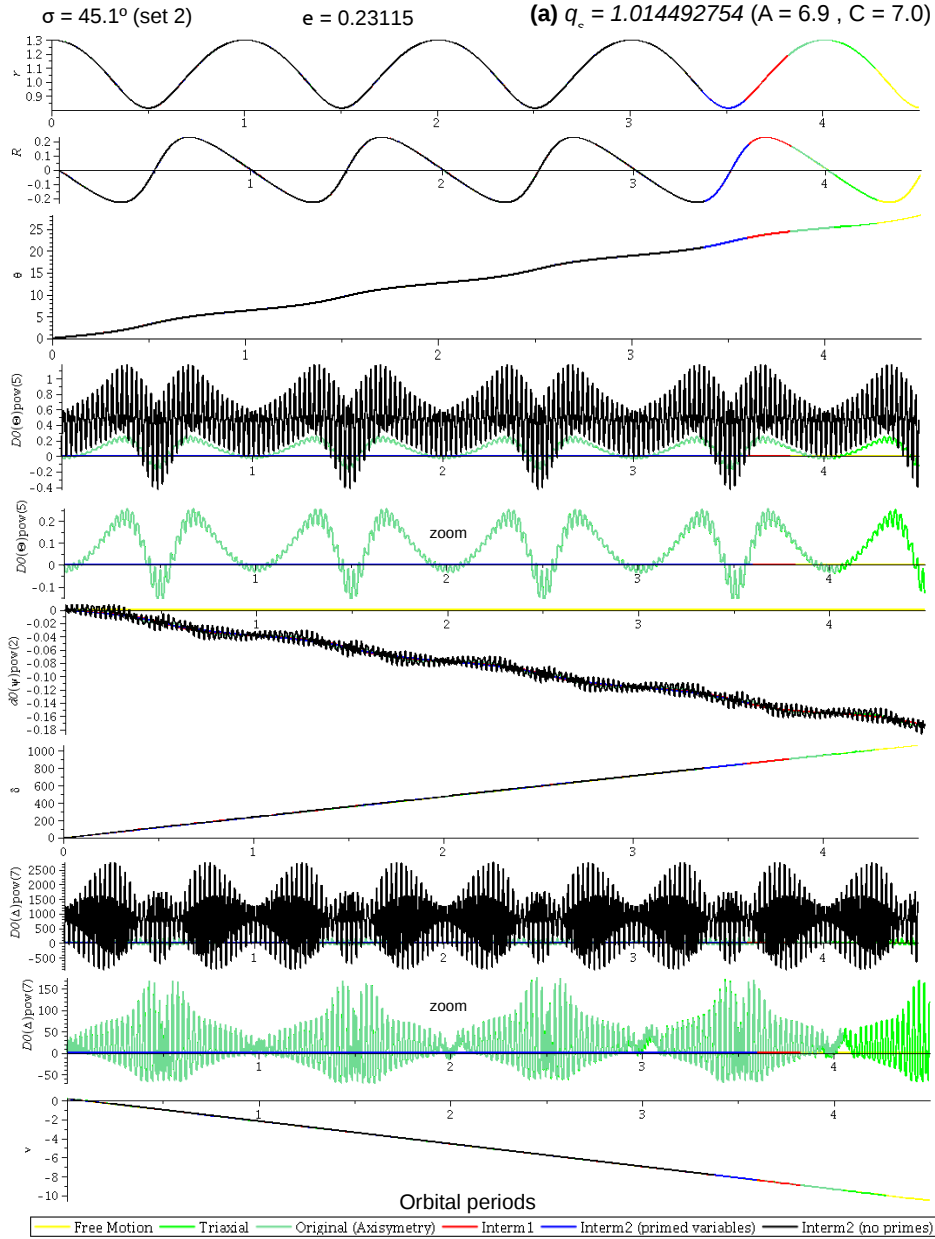


Source: Cardoso dos Santos; Ferrer; Scheeres (2019)

stability around the axis with maximum moment of inertia (C). Even in the triaxial case, as one increases the attitude σ , the curves will reach the separatrices that split the rotational motion in different regimes of motion, as the case of libration and circulation. More detailed investigation can be done in this topic, which is however out of the scope of the present study that deals with an axisymmetric body.

It is clear the presence of a slight precession of the node ψ due to perturbations coming from the term which couples both dynamics (the unperturbed case presents a constant ψ). Regarding the time propagation of this angle, both intermediaries ($\mathcal{I}_o$ and $\mathcal{I}_p$) are able to follow the actual precession due to the original models. For the unperturbed case, the rotation angles ν and δ are free and evolve linearly with time. However, the presence

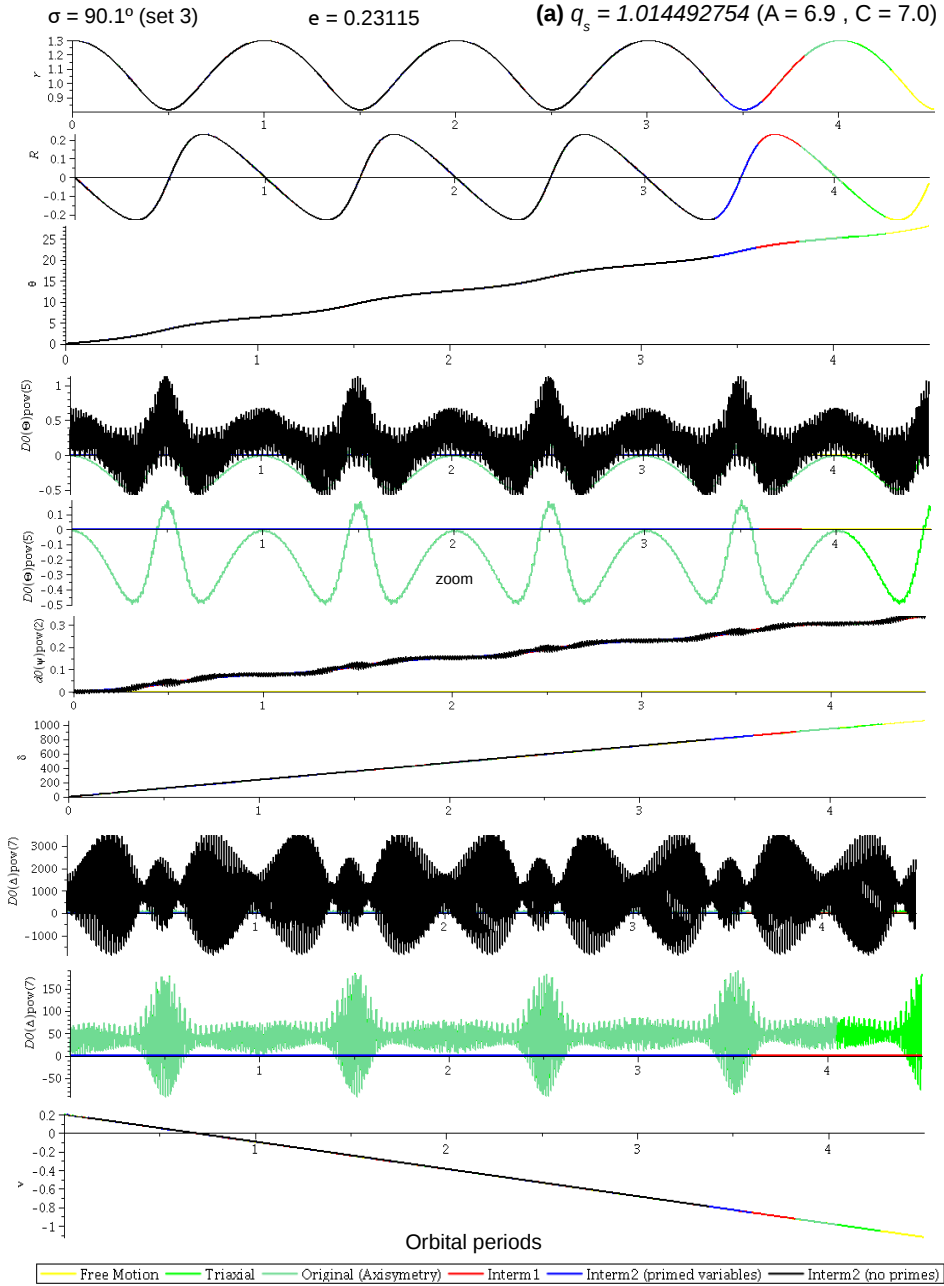
Figure 59: Sample propagation considering the set 2 of initial conditions ($\sigma = 45.1^\circ$). Notation: $dO(x) \equiv x - x_0$, $DO(x) \equiv 1 - x/x_0$ and $pow(x) \equiv 10^x$. Angles in radians and remaining values in internal units.



Source: Cardoso dos Santos; Ferrer; Scheeres (2019)

of the coupling with the orbital motion (see, the second term in the right-hand side of the expressions (151), (152), (157) and (158)) did not create significant changes in these angles. For the angular momenta Θ and Δ , the intermediaries also produce constant values for these physical quantities (red and blue lines), once the process to build these models removes the terms explicitly dependent on fast angles θ and δ . Therefore, for each intermediary, the derivatives of these momenta are null, *i.e.*, Θ and Δ are constant. The presence of these angles in the original Hamiltonian produces time variations on these momenta. However, the intermediary $\mathcal{I}_p$ is able to recover some short-period effects after

Figure 60: Sample propagation considering the set 3 of initial conditions ($\sigma = 90.1^\circ$). Notation: $dO(x) \equiv x - x_0$, $DO(x) \equiv 1 - x/x_0$ and $pow(x) \equiv 10^x$. Angles in radians and remaining values in internal units.

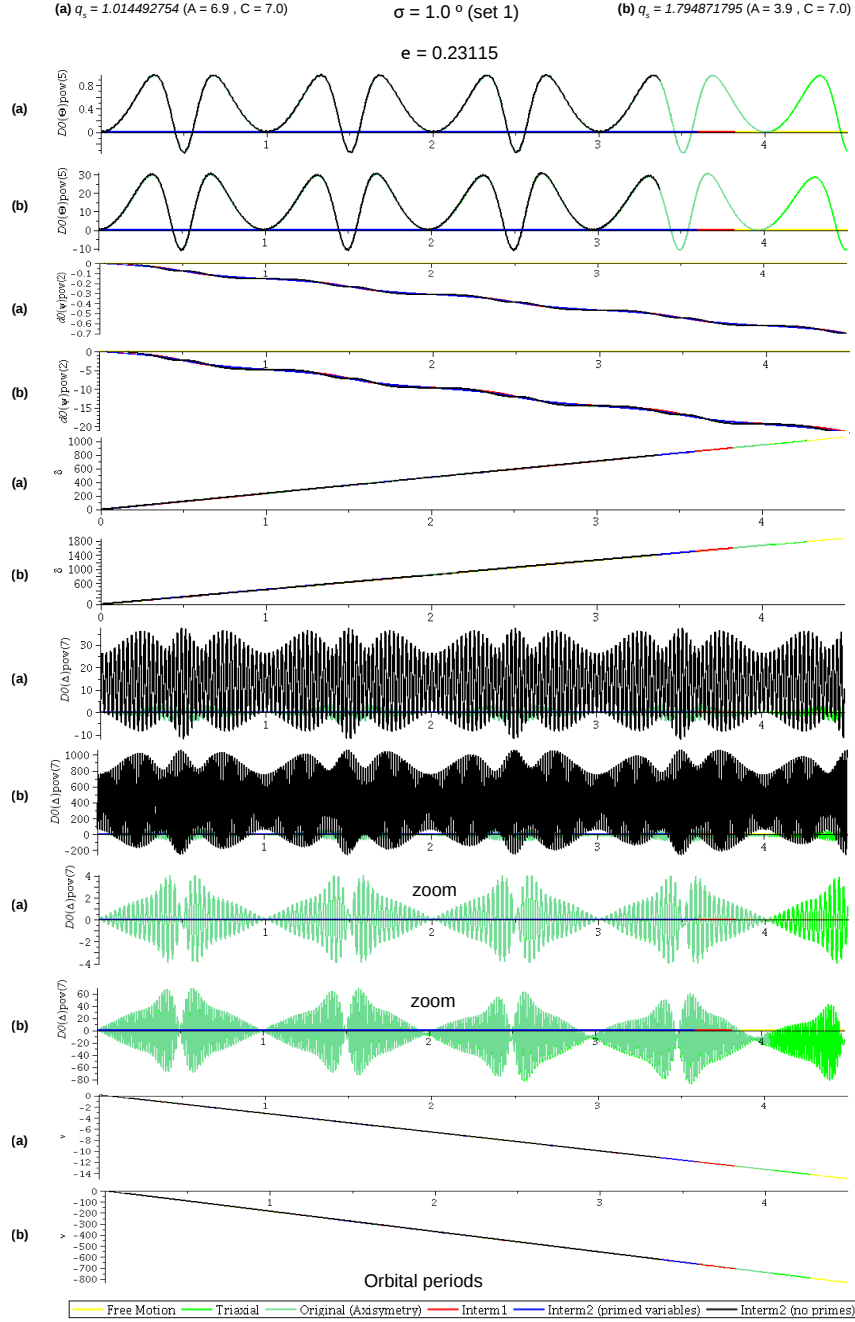


Source: Cardoso dos Santos; Ferrer; Scheeres (2019)

the use of the transformation equations (136) and (137) to return from primed to original variables (see black lines in figures 58 to 62), especially when one chooses small values for σ_0 in scenarios with different shapes and orbital eccentricities. It appears when one observes, for instance, the results for $\sigma_0 = 1.0^\circ$ in figures 61 and 62.

In general the amplitudes of the short-period oscillations recovered for Δ and Θ are bigger for high values of σ_0 than in the original Hamiltonians. Further analysis can be done about these higher values of amplitude in a future investigation. However, when

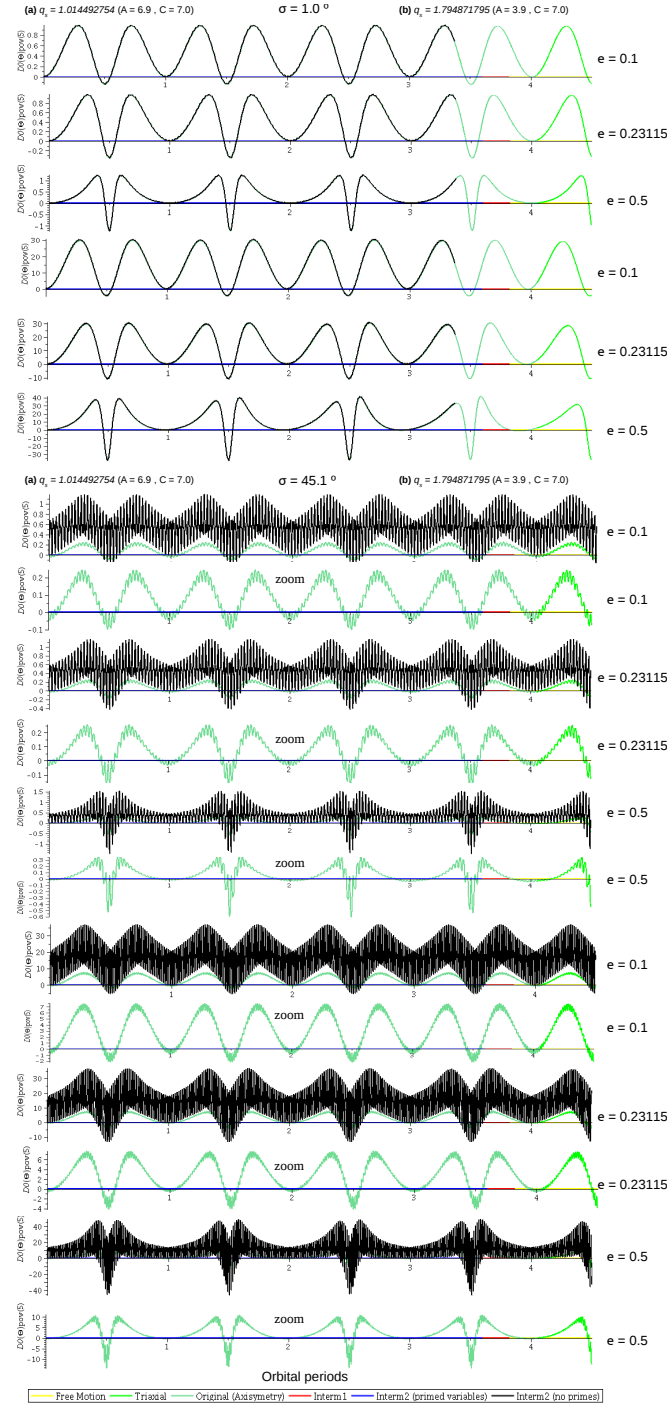
Figure 61: Sample propagation considering the set 1 of initial conditions ($\sigma = 1.0^\circ$). Different shapes are assumed for an oblate rigid body with fixed $C = 7.0 \times 10^{-4}$ and $q_s =$ (a) 1.014492754 and (b) 1.794871795. Notation: $dO(x) \equiv x - x_0$, $DO(x) \equiv 1 - x/x_0$ and $pow(x) \equiv 10^x$. Angles in radians and remaining values in internal units.



Source: Author

one chooses $\sigma_0 = 1.0^\circ$ for scenarios with different shapes and orbital eccentricities, the short-period corrections are quite precise, especially for Θ (compare the results presented in figure 62). Once this model carries information about the angles θ and δ , it might be useful for further analysis such as the investigation of resonances in solar system. These scenarios with different parameters are more properly discussed on the following paragraphs.

Figure 62: Sample propagation for Θ and considering three different orbital eccentricities of the set 2 ($\sigma = 45.1^\circ$). Different shapes are assumed for an oblate rigid body with fixed $C = 7.0 \times 10^{-4}$ and $q_s =$ (a) 1.014492754 and (b) 1.794871795. Notation: $dO(x) \equiv x - x_0$, $DO(x) \equiv 1 - x/x_0$ and $pow(x) \equiv 10^x$. Angles in radians and remaining values in internal units.



Source: Cardoso dos Santos; Ferrer; Scheeres (2019)

As the parameter ϵ depends on the parameter q_s , which measures the ratio between the two moments of inertia A and C , it is also important to investigate the effects for small and big differences between these physical parameters. It means to investigate the decrease

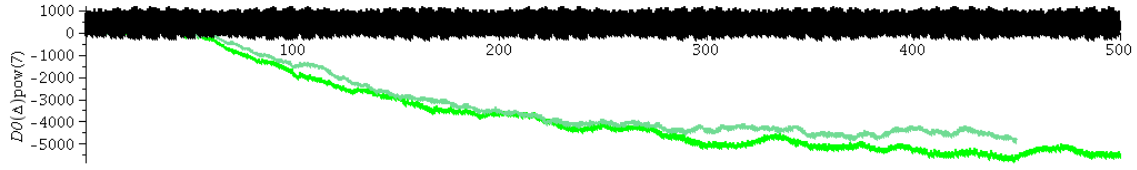
and increase of the perturbation due to the coupling, respectively. It is done by choosing different values for $A = \{(a) 6.9; (b) 3.9\} \times 10^{-4}$ while the value $C = 7.0 \times 10^{-4}$ is fixed, producing values of $q_s = \{(a) 1.014492754; (b) 1.794871795\}$, respectively. These values for the moments of inertia are given in internal units and the results are presented in figures 61, 62, 64 and 65.

The results for the intermediaries show a decrease in their precision with the increase of the difference between A and C , which impacts the assumption used to consider perturbation theory to build these models. In fact, the roto-orbital Hamiltonian was assumed having a coupling producing a small perturbation (ϵ assumed small) over the system described by $\mathcal{H}_{Unperturbed} = \mathcal{H}_{Kepler} + \mathcal{H}_{Euler}$. In the case of a significant difference between A and C , the parameter ϵ might become not small, impacting the precision of the intermediaries on approaching the more general models in the vicinity of the points under analysis in the canonical map where the roto-orbital dynamics is defined. Thus, the value ($\sigma = 1.0^\circ$) which produces less disturbed results in the previous analysis is select to better contrast the impacts caused by changing the values q_s , *i.e.*, the shape of the rigid body. As expected, the more oblate in shape is the rigid body, the more disturbed will be its orbital and rotational dynamics. Significant changes are noted in the attitude angles δ and ν , with significant increase in the precession rate as the rigid body is more oblate. The same behavior is observed for the node ψ . However, when one varies the shape of the rigid body (q_s) and its orbital eccentricity (e), both intermediaries still models present quite precise results on following the time propagation of these angles when compared to the results provided by the original and triaxial models (these other results are omitted here). Indeed, the more significant differences between the intermediaries and the original models appear in the magnitudes of the angular momenta Θ and Δ due to similar the reasons explained above.

In order to verify how significant is the accumulated error after several orbital periods, the models were propagated for 500 orbital periods, where different shapes for an oblate rigid body were also considered. As previously seen in figures 58 to 60, the variable which presented the more significant differences in magnitude was the rotational angular momentum Δ . These results are omitted here and the essential is that both intermediaries are still quite precise for the magnitude of this variable, despite the propagation for significantly long time spans. The most disturbed result for Δ helps to illustrate these differences: considering the values for shape $q_s = 1.794871795$ ($A = 3.9 \times 10^{-4}$, $C = 7.0 \times 10^{-4}$) and attitude $\sigma_0 = 1.0^\circ$ after 500 orbital revolutions, the averaged difference between the intermediaries with respect to the original models was about -5000×10^{-7} ($1 - \Delta/\Delta_0 \approx -0.05\%$ (see figure 63). Indeed, the propagation up to 50 revolutions presented in average differences of the order of $10^{-5}\%$. Smaller values in σ_0 maximized the long-term action of the gravity gradient torque, which increases the disturbances over the attitude dynamical parameters after several revolutions.

At the present moment, the capacity of recovering short-period effects in some vari-

Figure 63: Sample propagation of Δ for 500 orbits and with initial condition $\sigma = 45.1^\circ$. Notation: $DO(x) \equiv 1 - x/x_0$ and $pow(x) \equiv 10^x$. Angles in radians and remaining values in internal units.



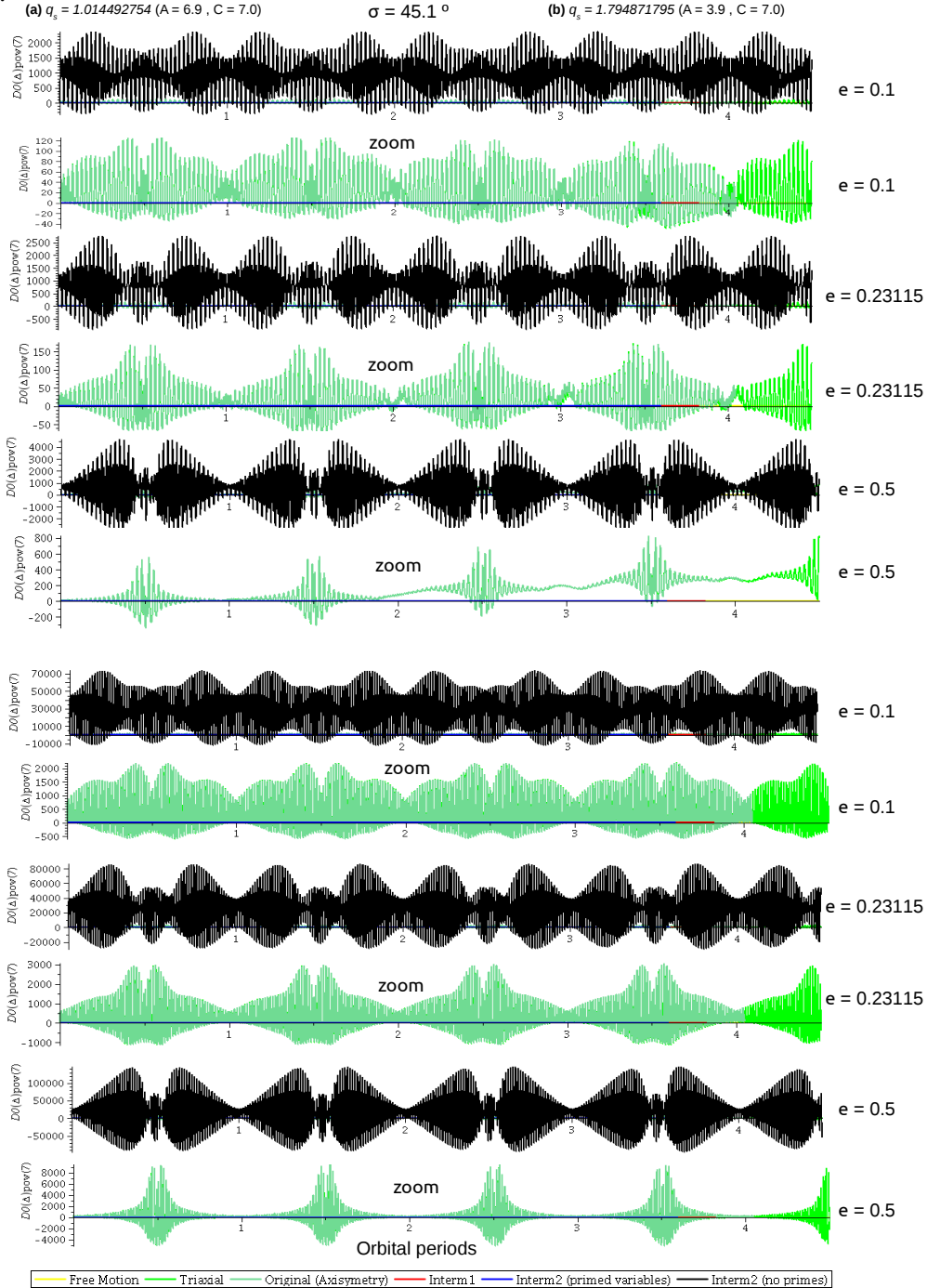
Source: Author

ables and its simplicity to provide an analytical solution in terms of trigonometric quadratures seem to indicate the adoption of the intermediary $\mathcal{I}_p$ in favor of the use of $\mathcal{I}_o$. However, important remarks require to be highlighted: (1) On choosing the $\mathcal{I}_o$ instead of $\mathcal{I}_p$, one avoids the use of canonical transformation to simplify the system; (2) Additional to (1), the intermediary $\mathcal{I}_p$ might require higher order expansion (and terms in the generating function). This intermediary is especially affected for the assumption about the shape of rigid body because the more spherical is this one, less disturbed will be its eccentricity through the orbit-attitude coupling, which avoids the need of expanding the potential up to terms of higher order to better approach its motion. This dependence also appears with respect to the ratio of mass and relative distances between the bodies. Depending on the number of disturbances and order of the terms involved in a specific Hamiltonian problem under study, it is possible the necessity of expansion of the transformation up to same order. This possibility of higher order terms for the transformation $\mathcal{W}$ also need to be considered for rather eccentric systems.

Thus, numerical experiments are made adopting: $e = 0.1$, $e = 0.23115$ and $e = 0.5$. The figures 62 and 64 present some results for variables Θ and Δ when $\sigma = 45.1^\circ$. The results for the cases with $\sigma = 1.0^\circ$ present less divergence and are omitted in figure 64. When this figure is analyzed, the intermediary $\mathcal{I}_o$ seems to lose precision, especially when short-period effects increase together with the increase of q_s , σ and e . In the other hand, the intermediary $\mathcal{I}_p$ is in general pretty efficient, notably on recovering the short-period effects of Θ , especially when σ is close to $\sigma = 0^\circ$, which represents a rotation around the z -axis of the rotation inertial frame and can be observed in figure 62. Departing from the equilibrium position, the differences due to short-period effects become bigger as higher values of σ_0 are adopted. This result shows the disturbing effect of the attitude over the orbital motion.

These previous results point out that if one considers a rigid body with significant high values for flattening, orbital eccentricity, attitude inclination and close approaches of the central body, the results provided by the intermediaries will be less precise in some roto-orbital parameters, notably, the magnitude of angular momenta Θ and Δ . The figures 62, 64 and 65 appears to highlight this conclusion by adopting high values for parameters of shape, eccentricity and attitude ($q_s = 1.794871795$, $e = 0.5$ and $\sigma = 45.1^\circ$). In general, the short-period effects (black lines) for $\mathcal{I}_p$ present bigger magnitudes than the original

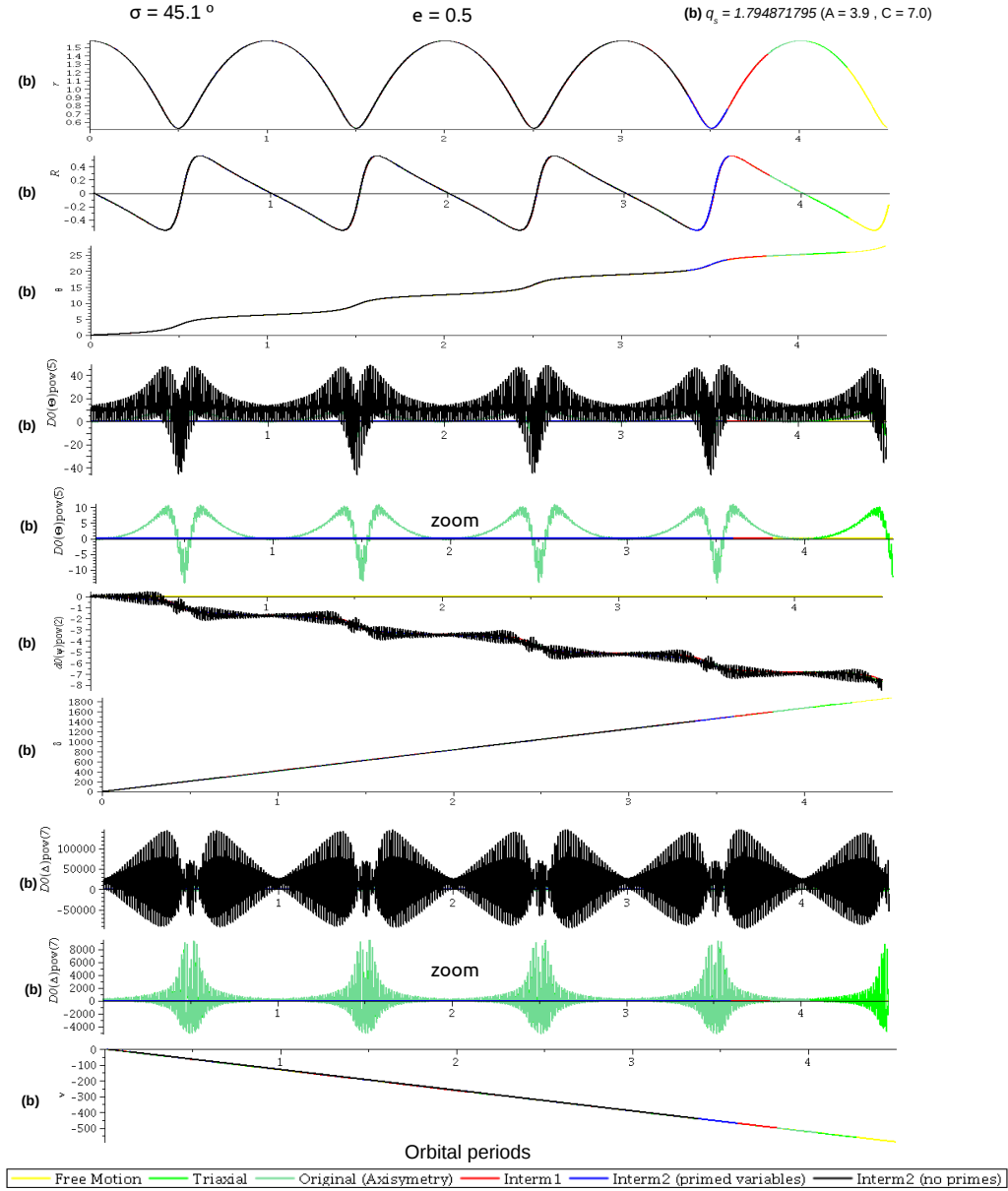
Figure 64: Sample propagation for Δ and considering three different orbital eccentricities of the set 2 ($\sigma = 45.1^\circ$). Different shapes are assumed for an oblate rigid body with fixed $C = 7.0 \times 10^{-4}$ and $q_s =$ (a) 1.014492754 and (b) 1.794871795. Notation: $dO(x) \equiv x - x_0$, $DO(x) \equiv 1 - x/x_0$ and $pow(x) \equiv 10^x$. Angles in radians and remaining values in internal units.



Source: Cardoso dos Santos; Ferrer; Scheeres (2019)

Hamiltonians and they could be used as a safety-range in some problems. Here, the bigger differences presented for $\mathcal{I}_p$ are between -90000×10^{-7} ($1 - \Delta/\Delta_0 \approx -0.9\%$) and $+1.4\%$ (see figures 64 and 65), which may bring difficulties in some applications which require high precision.

Figure 65: Sample propagation for orbital eccentricity $e = 0.5$, shape parameter $q_s = 1.794871795$ and $\sigma = 45.1^\circ$. Notation: $dO(x) \equiv x - x_0$, $DO(x) \equiv 1 - x/x_0$ and $pow(x) \equiv 10^x$. Angles in radians and remaining values in internal units.



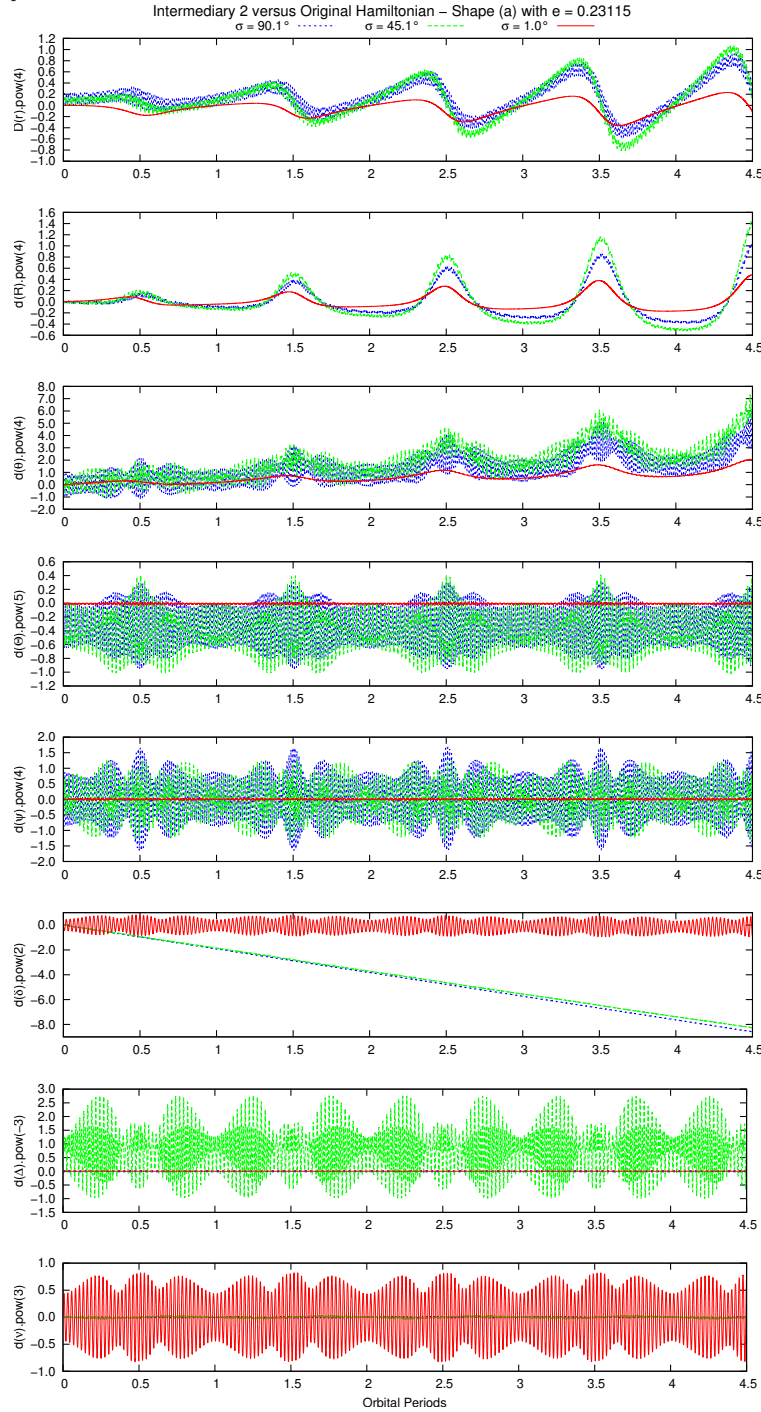
Source: Author

5.6.3 Computation of errors

In order to explore the previous conclusions, the results of a study of errors between two different models is presented in figures 66 and 67. In these figures, the differences between numerical results for the intermediary $\mathcal{I}_p$ and the original Hamiltonian are presented. Note, specially, the figure 67, which present plots where the system is assumed to have a very flat ($q_s = 1.794871795$) rigid body in a very eccentric orbit ($e = 0.5$). The analysis made in the previous section is supported by these results. In these plots, differences between numerical results for the intermediary $\mathcal{I}_p$ and the original Hamiltonian are

presented. The conclusions are the same obtained previously.

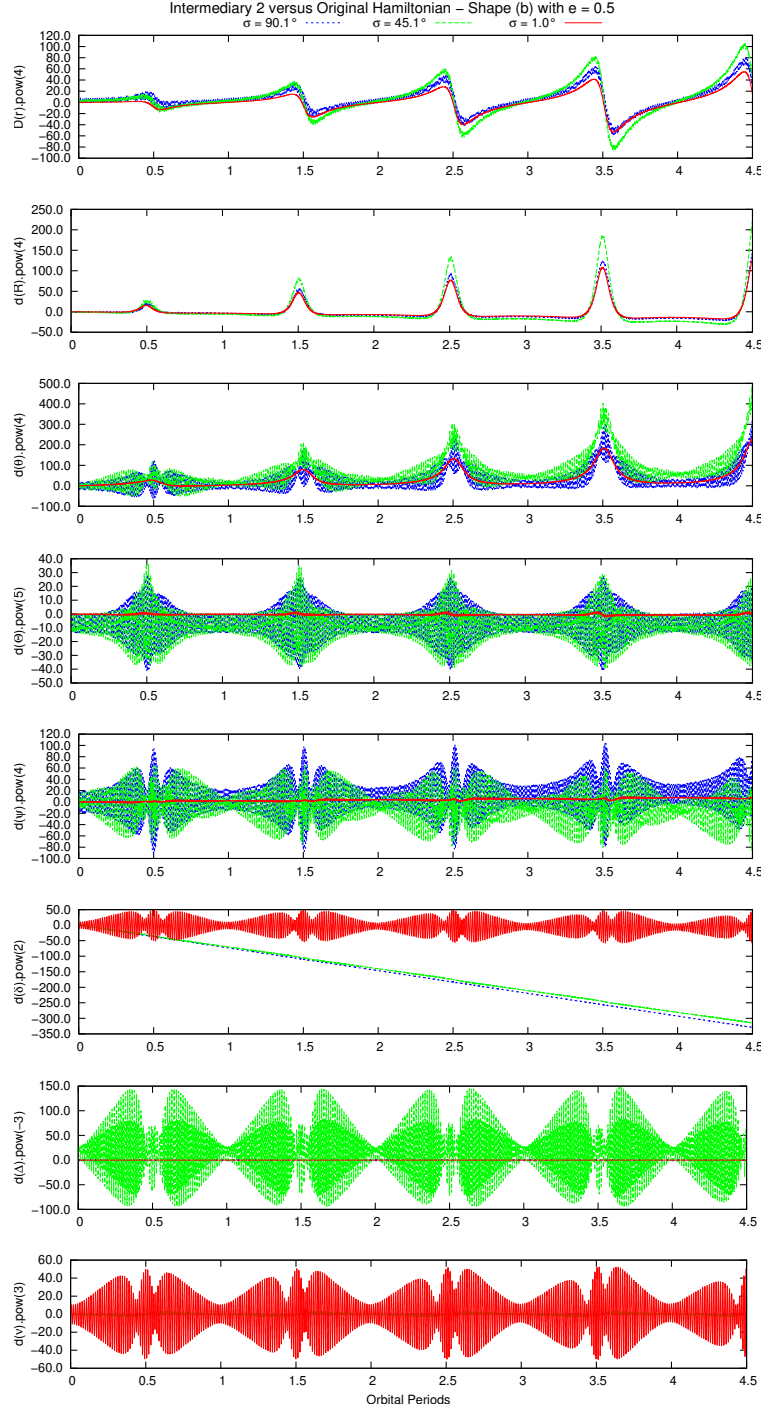
Figure 66: Sample propagation showing deviations between the short-period correction in $\mathcal{I}_p$ and the original Hamiltonian. Shape parameters fixed as $A = 6.9 \times 10^{-4}$ and $C = 7.0 \times 10^{-4}$. Attitude angles $\sigma = 1.0^\circ$, 45.1° and 90.1° . Notation: $d(x) \equiv x_{int} - x_{or}$, $D(x) \equiv 1 - x_{or}/x_{int}$ and $pow(x) \equiv 10^x$. Values x_{or} and x_{int} refer to Original Hamiltonian and Intermediary $\mathcal{I}_p$, respectively. Angles in radians and remaining values in internal units.



Source: Cardoso dos Santos; Ferrer; Scheeres (2019)

A final study to verify the validity of the previous conclusions in a more general scenario is necessary, so that the conclusions can be extended for a wide range of conditions

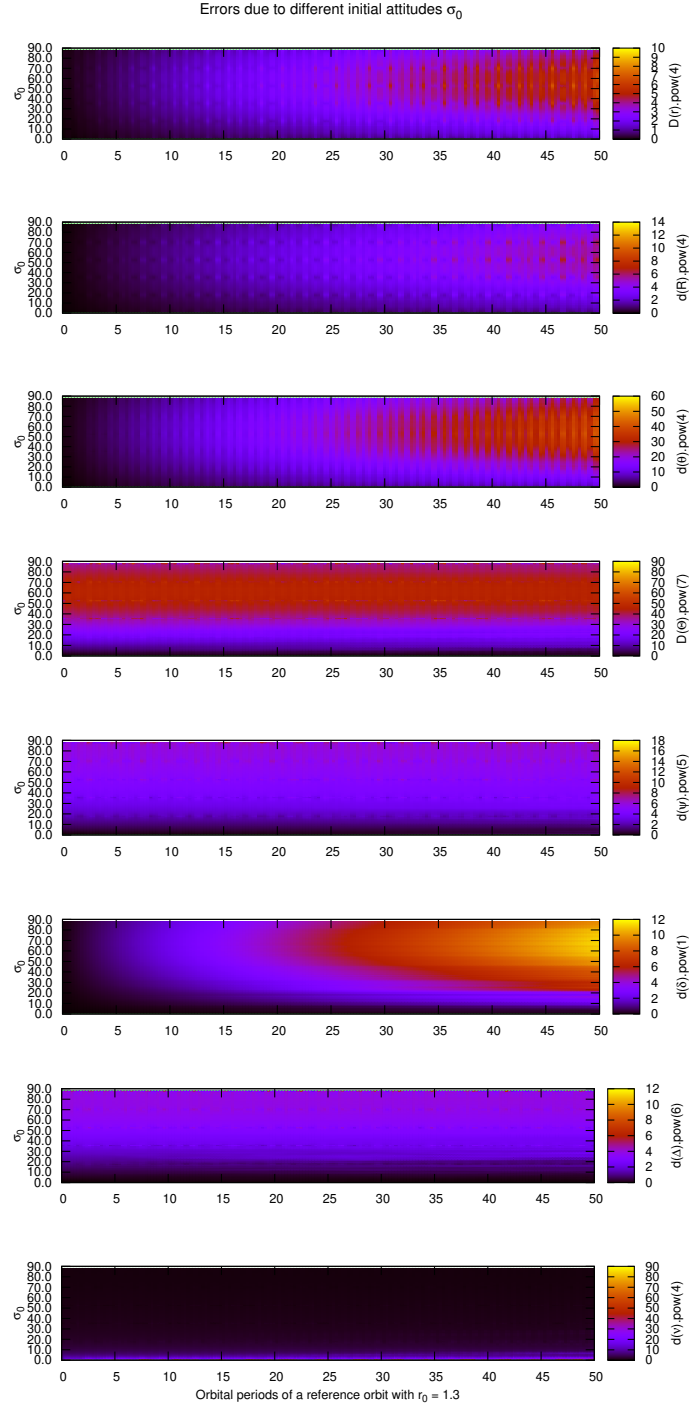
Figure 67: Sample propagation of 4.5 orbits showing deviations between the short-period correction in $\mathcal{I}_p$ and the original Hamiltonian. Shape parameters fixed as $A = 3.9 \times 10^{-4}$ and $C = 7.0 \times 10^{-4}$. Attitude angles $\sigma = 1.0^\circ$, 45.1° and 90.1° . Notation: $d(x) \equiv x_{int} - x_{or}$, $D(x) \equiv 1 - x_{or}/x_{int}$ and $pow(x) \equiv 10^x$. Values x_{or} and x_{int} refer to Original Hamiltonian and Intermediary $\mathcal{I}_p$, respectively. Angles in radians and remaining values in internal units.



Source: Cardoso dos Santos; Ferrer; Scheeres (2019)

adopted for attitude inclination, shape and eccentricity/distance. The figures 68, 69 and 70 present the mapping of the errors between the intermediary $\mathcal{I}_p$ with short-period corrections and the original Hamiltonian. The errors in different variables of the problem due

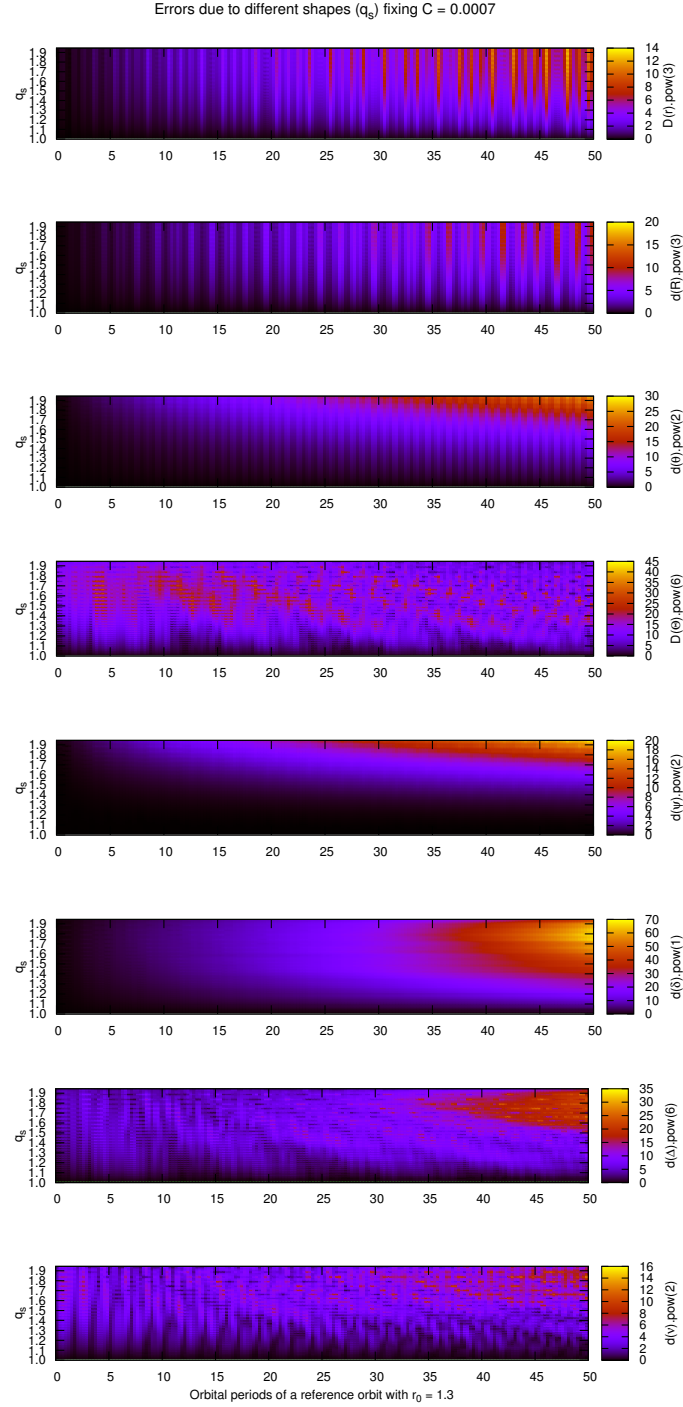
Figure 68: Error maps for the short-period correction in $\mathcal{I}_p$ with respect to the original Hamiltonian. Sample propagation of 50 orbits with initial radial distance at the orbit apoapsis $r_0 = 1.3$. Notation: $d(x) \equiv x_{int} - x_{or}$, $D(x) \equiv 1 - x_{or}/x_{int}$ and $pow(x) \equiv 10^x$. Values x_{or} and x_{int} refer to Original Hamiltonian and Intermediary $\mathcal{I}_p$, respectively. Other values of eccentricities presented similar results. Angles in radians and remaining values in internal units.



Source: Cardoso dos Santos; Ferrer; Scheeres (2019)

to different initial values in attitude (σ), shape q_s and distance (r) are highlighted. All the maps confirm the validity of the previous analysis, again with significant loss of precision

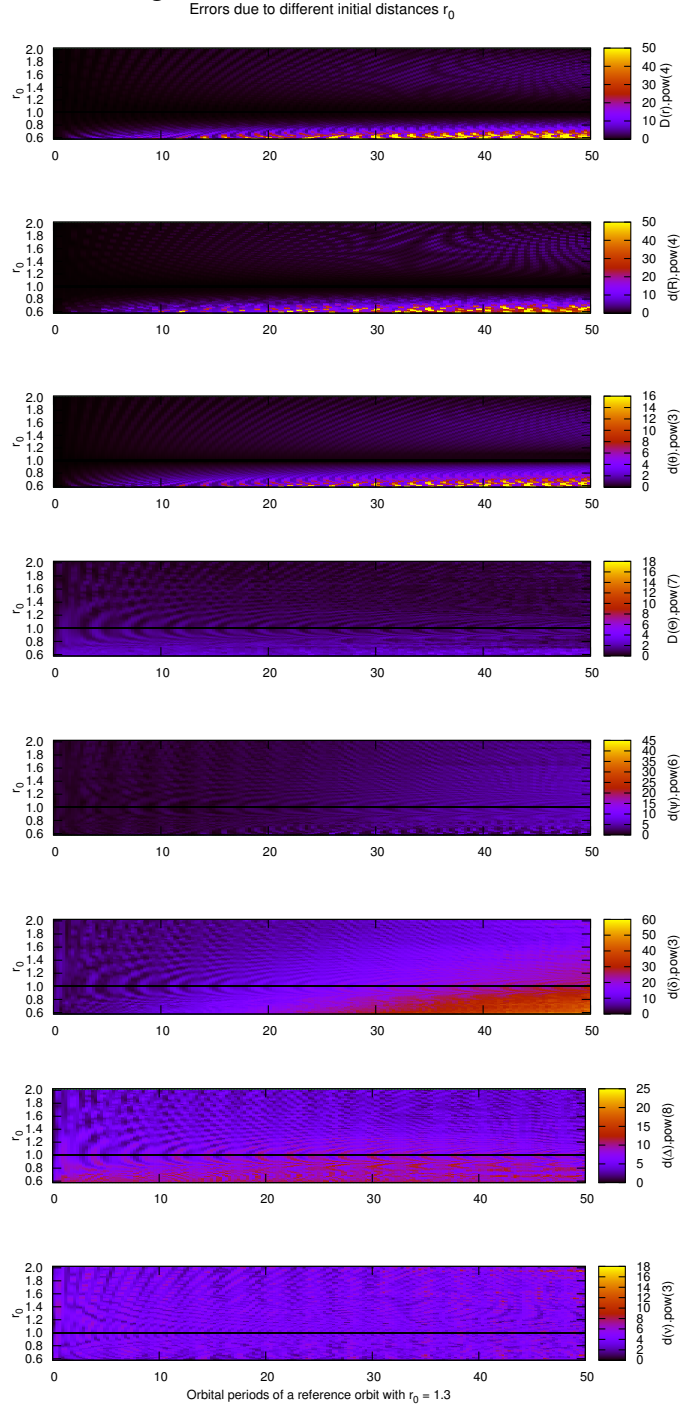
Figure 69: Error maps for the short-period correction in $\mathcal{I}_p$ with respect to the original Hamiltonian. Sample propagation of 50 orbits with initial radial distance at the orbit apoapsis $r_0 = 1.3$. Notation: $d(x) \equiv x_{int} - x_{or}$, $D(x) \equiv 1 - x_{or}/x_{int}$ and $pow(x) \equiv 10^x$. Values x_{or} and x_{int} refer to Original Hamiltonian and Intermediary $\mathcal{I}_p$, respectively. Other values of eccentricities presented similar results. Angles in radians and remaining values in internal units.



Source: Cardoso dos Santos; Ferrer; Scheeres (2019)

for high values of attitude σ (figure 68) and non-sphericity q_s (figure 69). Figures 68 and 69 point out for higher values of errors in orbital variables appearing when the values σ

Figure 70: Error maps for the short-period correction in $\mathcal{I}_p$ with respect to the original Hamiltonian. Sample propagation of 50 orbits. Notation: $d(x) \equiv x_{int} - x_{or}$, $D(x) \equiv 1 - x_{or}/x_{int}$ and $pow(x) \equiv 10^x$. Values x_{or} and x_{int} refer to Original Hamiltonian and Intermediary $\mathcal{I}_p$, respectively. Other values of eccentricities presented similar results. Angles in radians and remaining values in internal units.



Source: Cardoso dos Santos; Ferrer; Scheeres (2019)

$\approx 55^\circ$ and $q_s \approx 1.944$ are adopted (see plots for r , R and θ). These figures also indicate the values $\sigma \approx 70^\circ$ and $q_s \approx 1.8$ as the ones producing less precision for rotational variables (see plots for δ and Δ). Results in figure 70 show the loss of precision when using

such model to study bodies too close, so that it is an important issue to be checked before adopting them in any investigation. It is worth of mention that the ranges for r_0 presented in figure 70 are $[0.57769 : 0.99950]$ (below the black line) and $[0.99950 : 2.02230]$ (above the black line) and refer, respectively, to the ranges $[0 : 0.73015]$ and $[0 : 0.50576]$ for the initial orbital eccentricity e_0 . The first interval (below) corresponds to orbits starting in the orbit periapsis (fixing $f = \omega = 0^\circ$). The second interval (above) denotes orbits starting close to the orbit apoapsis (fixing $f = -\omega = 180.1^\circ$). It happen because, for simplicity, the initial values $\Theta_0 = 1$ and $\theta_0 = 0^\circ$ were assumed in all simulations.

On using the physical parameter for the present hypothetical systems, it was not require the expansion of the perturbation technique up to higher order terms to efficiently approach the original model. The intermediaries appear precise enough for a significant range of parameters considered, especially for almost spherical bodies rotating with small obliquity and with no very close approaches with the central body. Similar analysis for classes of systems having different ratios of mass can be done to expand the present analysis, providing extra understanding of the some limits of these two approached models. However, the present results are enough to conclude that an overall analysis shows both intermediaries as efficient models to approach the original roto-orbital Hamiltonian of an axisymmetric rigid body. This analysis also indicates an advantage for the model with elimination of the parallax $\mathcal{I}_p$, once it requires only elementary quadratures to be solved and it is also able to recover some short-period effects. This last characteristic is also relevant once it allows the expansion of the applications of this model to a wide range of problems such as: systems where fast angles have key role in the dynamics, as and maybe resonant systems as well (see for instance Vilhena de Moraes; da Silva (1990)).

This section is finalized suggesting the reader to verify the results presented in the Appendix B. There, an analysis of the phase space is made to compare the values of the Hamiltonians (129), $\mathcal{I}_o$ (expression (143) and $\mathcal{I}_p$ (expression (148) for different parameters. The analysis presented in this appendix is in good accordance with the analysis presented in this chapter. Thus, it provides a good complementary material to the present investigation.

5.7 DISCUSSION

A Hamiltonian theory is studied, where the roto-orbital problem is shaped as a perturbed form of the Keplerian motion and torque-free rotation. A procedure to obtain two intermediary Hamiltonians ($\mathcal{I}_o$ and $\mathcal{I}_p$), where the first one was obtained without the step of elimination of the parallax) was used in order to obtain an analytically integrable dynamical system. The first intermediary produces equations of motion integrable using elliptic integrals. The second intermediary, as a result of the elimination of the parallax, generates a system of equations that is integrable using elementary quadratures, simplifying the integration of the system in comparison with the first case. The tests for this study were performed for a system containing a central and spherical body with five units of

mass and an orbiting rigid body with $5/4$ of units of mass ($q = 0.8$). Numerical propagations of both intermediaries are presented and discussed for different initial attitudes and shapes of the secondary rigid body. Simulations also considered different values for its orbital eccentricity. The results point out that both intermediary models present significant precision when compared with an axisymmetric and a triaxial model, with more advantages for the model $\mathcal{I}_p$ obtained after the elimination of the parallax (blue and black lines) for the simplicity of its solution and for dealing with short-term dynamics. In general, the intermediaries seem to be efficient approaches, including when one considers rather eccentric systems, with high values of attitude inclination and very flat orbiting bodies.

Also worth of mention is the importance of the parameters q (parameter of mass), q_s (parameter of shape) and semilatus rectum p (function of r , orbital eccentricity, etc) which have the fundamental information that impacts significantly the precision provided by the intermediaries. Therefore, there exist a wide number of possible investigations related to the present problem which give significant importance on its continuation for the benefit of the scientific community.

Thus, detailed analytical investigations, such as the stability analysis of the systems, are possible to be carried out. A study focusing in the analytical aspect of $\mathcal{I}_o$ is presented in Crespo et al. (2018). However, it is also necessary to verify the range where the parameters of the roto-orbital problem will produce a close approach regarding a complete, and therefore, more realistic model. The discussions presented here can help on this issue. As the radial distance r is the only parameter that is function of time in the equations of motion (156) to (161), it is interesting to note that the more is the distance between the centers of mass, less disturbed will be the node ψ of total angular momentum plane. Thus, the amplitude of the oscillations around the linear precession decrease. The opposite is also true: The closer are the bodies, more disturbed will be this angle. Results in figure 70 also show the loss of precision when using these models to study bodies too close, an issues that need to be checked before adopting them in any investigation. More investigations varying the mass ratio q is required to strength the analysis of these models, allowing its application in a wider range of cases that will allow applications in different topics in astronomy and astrodynamics.

Finally, further investigations are under present analysis: (1) A recent study, on the context of the main problem of artificial satellites, was developed by Lara; San-Juan; López-Ochoa (2014), where the authors have worked with the topic of the parallax elimination in terms of Delaunay variables to present characteristics of this transformation using this specific chart, an attempt of extension of Deprit's formulation in this chart. According to Deprit (1981), a consequence of choosing the Delaunay variables for the orbital dynamics is the reduction of the total number of terms by about 10%. Therefore, the elimination of the parallax for the roto-orbital problem in terms Delaunay variables is a topic presently under investigation; (2) The extension of the present analysis of the roto-orbital dynamics to the case of an axisymmetric rigid body under the gravitational

field of an oblate central body is also under preparation.

The main results of the study presented in this chapter were submitted to a scientific journal (CARDOSO DOS SANTOS; FERRER; SCHEERES, 2019) and are in final process of analysis. Some other results are a complement of this work.

6 VARIATIONAL EQUATIONS FOR FORMATION FLIGHT WITH PHOBOS

The Hill problem is a limit case of the three-body problem where two of the masses of the system are assumed to be very small when compared to the third dominant mass (HENON; PETIT, 1986; LARA, 2018), which we denote by "primary". Case one of the assumed small masses is negligible when compared to the other small mass, which we denote by secondary body of the system, and this "massless" object moves close enough to the secondary, the Hill problem can be considered as a particular case of the restricted three-body problem (SZEBEHELY, 1967). One of the most important merits of Hill's formulation is its capability to be conveniently modeled in a nondimensionalized form, making it independent of physical parameters. This feature brings the possibility to apply this problem to a wide range of problems in celestial mechanics and astrodynamics. We can cite a few of these problems, namely the dynamics of planetary rings (PETIT; HENON, 1986) and planetary satellite orbiters (SCHEERES, 2012), Sun–Earth–Moon orbital interaction, Hill's original application (HILL, 1879), and the relative dynamics of two artificial satellites in formation flight (CLOHESSY; WILTSHIRE, 1960; SCHAUB; JUNKINS, 2003).

When one thinks about the analytical integrability of the equations of motion, the system can be solved when the small masses are far enough away from each other, when their mutual attraction can be neglect so that the system is reduced to the dynamics of two Keplerian orbits around the bigger dominant body of the system (LARA, 2018). However, the system turns into non-integrable when the gravitational interaction between the two small masses becomes. In this scenario, particular cases and simplifications need to be assumed to provide further analytical analysis of system. Out of these scenarios, it is necessary to deal with numerical approaches for the Hill dynamics (see more details in (LARA, 2018) and references therein).

Recently, the case of quasi-satellite orbits (QSOs) has taken the attention of the scientific community when dealing with the Hill problem (WIESEL, 1993; GIL; SCHWARTZ, 2010; CABRAL, 2011), once these orbits can be efficiently applied to study problems of great interest as the case of the recent discoveries of near-Earth objects co-orbiting the Sun with our planet (see de la Fuente Marcos; de la Fuente Marcos (2016) and de la Fuente Marcos; de la Fuente Marcos (2014) and references therein) and the design of science missions to small bodies of the solar system, namely the JAXA's Martian Moons Exploration (MMX), the NASA's Asteroid Redirect Mission and others to be proposed. A study about the orbit operational strategy that MMX is planning to use in the vicinity

of Phobos, which includes the use of QSOs, has been recently published (IKEDA et al., 2017).

Therefore, in this chapter, the use of Gauss variational equations for designing of QSOs in the context of the Elliptical Hill problem, a particular case of the elliptic restricted three-body problem, is made in order to provide a preliminary space mission design perspective to the Martian Moons Exploration mission (MMX)¹. The study presented here is a joint investigation lead by Nicola Baresi (ISAS/JAXA and MMX mission flight dynamics team), which is leading the numerical investigation meanwhile co-leading the analytical study in partnership with the author of this thesis, and also with the collaboration of Lamberto Dell’Elce (Université Côte d’Azur and Inria) and Yasuhiro Kawakatsu (ISAS/JAXA and MMX mission leader).

The basic structure of this chapter is described as follows. Section 6.1 presents the challenges of orbiting Phobos and the proper strategy adopted to deal with the orbit dynamics about this body; 6.2 presents a solution for the unperturbed motion of a particle under a Hill approach to three-body problem; in section 6.3, variational equations are developed in order to find a solution for the disturbed motion around Phobos, meanwhile also introducing as different set of orbital elements; following, the section 6.4 presents a discussion about the alternative orbital elements introduced in section 6.3; the 6.5 deals with the new linearized equations of motion including the gravitational pull due to Phobos; in section 6.6 moves the investigation forward on providing new averaged equations of motion and their analysis taking into account Phobos’ gravity; and finally, in section 6.7, a discussion about the results is presented.

6.1 THE CHALLENGE OF ORBITING PHOBOS

The desired of orbiting and studying the Martian is not a new subject of study. Several works in literature have dealt with this very challenging problem before (WIESEL, 1993; GIL; SCHWARTZ, 2010; CABRAL, 2011; ZAMARO; BIGGS, 2015; SCHEERES et al., 2017; CANALIAS; LORDA; LAURENT-VARIN, 2018). Some characteristics about the physical data of the Phobos-Mars system are presented in tables 7 and 8, where a , e and i are the semi-major axis, eccentricity and orbital inclination of Phobos’ orbit around Mars.

In order to model the motion of a spacecraft/particle in the vicinity of Phobos, it is necessary to have information about the region around this body. The figures 71 and 72 present how critical is to maintain a spacecraft orbiting this body using the two-body problem. The figure 71 presents a view of the xy -plane where the Lagrangian equilibrium points L_1 and L_2 around Phobos appear very close to its surface. When one looks especially to figure 72, it is possible to observe the equipotential lines within the xy -plane that have the same value for the Jacobi constant (and therefore the same energy level) as these equilibrium points. For the Roche lobe (closed curve for L_1 point), all the stationary

¹See MMX mission official website: <http://mmx.isas.jaxa.jp/en/index.html>

Table 7: Phobos' data around Mars

a (km)	e	i (deg)	Obliquity (deg)	Mean sidereal period (days)
9377.2	0.0151	1.082	Tidally locked	0.318910

Source: Author. Data collected from Gil; Schwartz (2010)

Table 8: Phobos' physical data (Ellipsoidal shape)

mass (kg)	μ (km/s^2)	Major axis (km)	Interm. axis (km)	Minor axis (km)	J_2
1.08×10^{16}	7.1×10^{-4}	13.1	11.1	9.3	0.10053

Source: Author. Data collected from Gil; Schwartz (2010)

particles which lie within the region defined by this curve are energetically bound to remain in the vicinity of Phobos, meaning that, energetically, any of the stationary particles located outside of the Roche lobe could escape from Phobos' vicinity through the neck region around L_1 , departing further away from this body. A return to Phobos' vicinity is also possible, but not guaranteed within finite time, with a Mars capture as a more likely result. As the Roche lobe constructed using the L_2 point is a curve which lies very close to the Roche lobe due to L_1 point, most

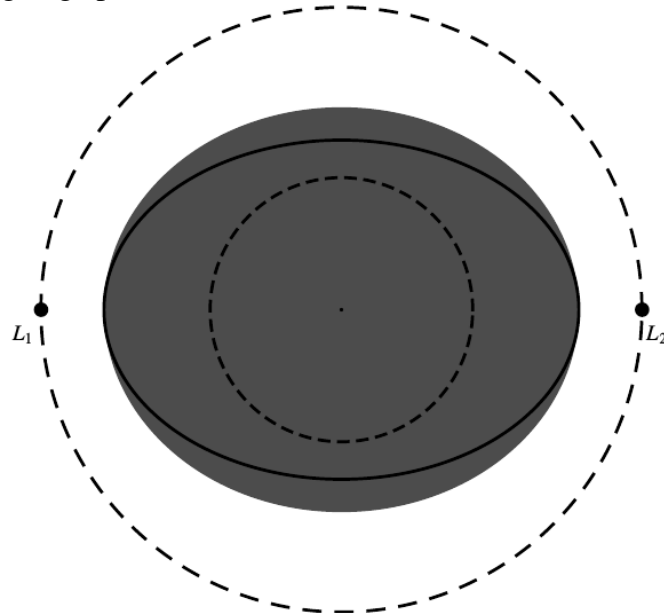
particles on the leading/trailing edge therefore also have the energy to escape through L_1 . However, this does not mean those particles have a physical path along which to escape. Indeed, a stationary particle on Phobos' leading/trailing edge would have to burrow through the surface in order to reach either of the neck regions (SCHEERES et al., 2017), p. 3.

Indeed, Keplerian orbits about Phobos are not possible. As shown in figures 71 and 72, the Hill sphere, which determines the region where a body is the main attracting force, is close to the surface of Phobos, not allowing the use of two-body dynamics to place a spacecraft around it. In figure 71, the thinnest axis (≈ 9.3 km, see table 8) is represented by the solid line, the Hill's sphere just above Phobos (outer dashed line) has a radius $R_{Hill} \approx 16.6$ km and the region of influence below the surface (inner dashed line) presents a radius $R_{Inf} \approx 7.2$ km. These values can be estimated using the expressions (166) (GIL, 2011):

$$R_{Hill} = a \left(\frac{\mu}{3\mu_M} \right)^{\frac{1}{3}} \quad ; \quad R_{Inf} = a \left(\frac{\mu}{\mu_M} \right)^{\frac{2}{5}} \quad (166)$$

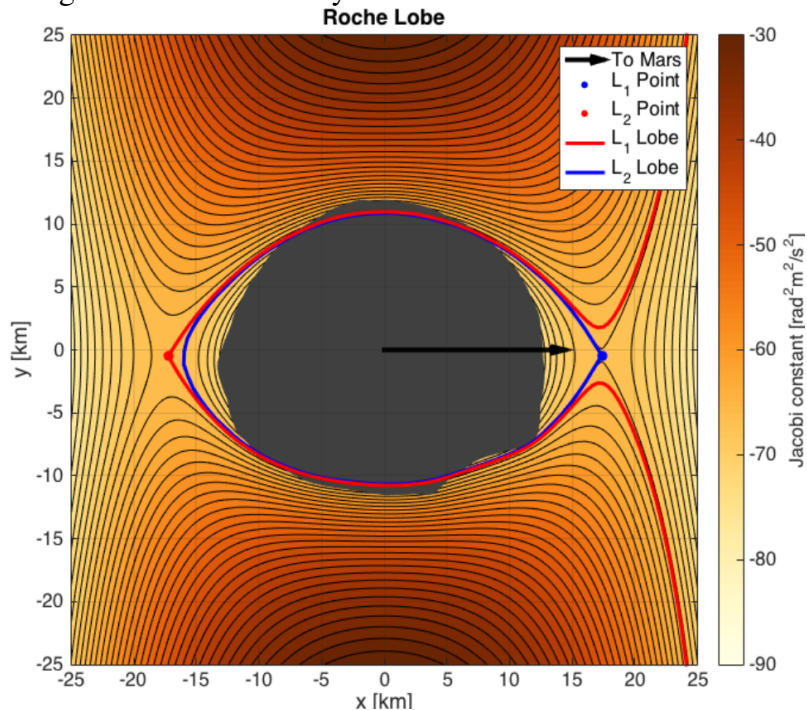
where μ and $\mu_M = \mu_{Mars}$ are the mass parameters of Phobos and Mars, respectively. The parameter a is the semi-major axis of Phobos around Mars.

Figure 71: Sketch of the ellipsoid representing Phobos as viewed from the North Pole looking down on the equator. The minor axis is presented as a solid line. The interior and exterior dashed lines are, respectively, the region of influence and the Hill sphere containing the Lagrange points L_1 and L_2



Source: Gil; Schwartz (2010), p. 2

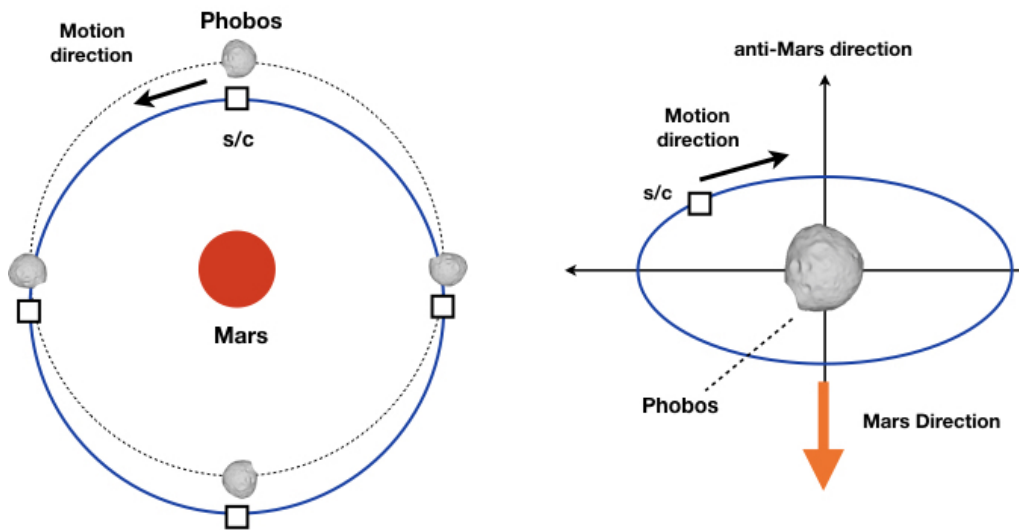
Figure 72: Zero-velocity curves and Roche lobe of Phobos



Source: Scheeres et al. (2017), p. 3

After this analysis, it is clear that the influence of Mars influence needs to be accounted for when modeling such dynamics and that usual orbits are impossible. Fortunately, it is possible to find sufficiently stable orbits around Phobos in the scenario of the restricted

Figure 73: Scheme of a QSO in Mars-Phobos system



Source: JAXA/MMX (2018)

three-body problem, when one of the primaries has a mass much larger than the other. Several works in literature have pointed out that a special type of orbit assumes particular interest, namely the ones known as quasi-satellite orbits (QSOs) (KOGAN, 1990; GIL; SCHWARTZ, 2010; CABRAL, 2011; SIDORENKO et al., 2014; POUSSE; ROBUTEL; VIENNE, 2017; BEZROUK; PARKER, 2017; LARA, 2018). These orbits are a special because despite not being closed periodic trajectories, they tend to stay in the same region in space. They can be used, for example, in space maneuvers called *rendezvous*, potentially helping also on the landing phase of a sample return mission, as the case of MMX mission (JAXA). The study of QSOs dates back at least to Jackson (1913), and since then, these orbits became also known as quasi-synchronous orbits or distant satellite orbits (DSOs), happening that the class of orbits known as distant retrograde orbits (DROs) are a particular family of QSOs (KOGAN, 1989; LIDOV; VASHKOV'YAK, 1994; GIL; SCHWARTZ, 2010). As properly defined by Gil and Schwartz,

QSOs exist beyond the Lagrange points, as seen from the rotating synodic reference frame usually used in three-body problems. In this frame, a coplanar QSO relative to the orbital plane of Phobos can be described as an elliptical reference trajectory drifting mainly back and forth in the direction of the orbital velocity of Phobos. In the inertial reference, the spacecraft (S/C) is still orbiting Mars with its orbit slightly perturbed by Phobos. In some cases, this motion can resemble the epicycloidal motion in the old Ptolemaic system, with the important difference that in this case it is a dynamic effect originated by the existence of Phobos (GIL; SCHWARTZ, 2010), p. 1.

A graphical scheme for a QSO around Phobos can be seen in figure 73.

6.2 GENERAL SOLUTION OF THE KEPLERIAN RELATIVE MOTION PROBLEM

Now, it is necessary to model the main forces to describe the motion. The net acceleration experience by a particle with position $\mathbf{r}$ and velocity $\dot{\mathbf{r}}$ in the Phobos-fixed frame can be written as (SCHEERES et al., 2017)

$$\ddot{\mathbf{r}} = \nabla U(\mathbf{r}) - \boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{r} - 2\boldsymbol{\omega} \times \dot{\mathbf{r}} + \mu_M \left(\frac{\mathbf{r}_M - \mathbf{r}}{|\mathbf{r}_M - \mathbf{r}|^3} - \frac{\mathbf{r}_M}{|\mathbf{r}_M|^3} \right) \quad (167)$$

where the potential $U(\mathbf{r})$ could be written as $U(\mathbf{r}, t)$, *i.e.*, as a potential which is time dependent, allowing these equations to represent dynamics near a small body with a generic shape and with a rotation state that is independent of its orbit with the larger primary (BROSCHART, 2006). However, in the study presented in this chapter, we do not take into account the shape of the small body, thus adopting the notation $U(\mathbf{r})$.

The dynamics is defined with $\omega = \dot{\nu}$ as the rate of change of the true anomaly. The vector $\boldsymbol{\omega} = \omega \hat{\mathbf{z}}$ is the angular velocity vector of the Martian moon around the planet with a constant direction taken to be the $\hat{\mathbf{z}}$ and varying magnitude in accordance with the following relations (SCHEERES; MARZARI, 2002)

$$p = a(1 - e^2) \quad (168)$$

$$r_M = \frac{p}{1 + e \cos \nu} \quad (169)$$

$$\omega = \frac{\sqrt{p\mu_M}}{(r_M)^2} \quad (170)$$

$$\dot{\omega} = -2\omega \sqrt{\frac{\mu_M}{p}} \frac{e \sin \nu}{r_M} \quad (171)$$

The expression (167) has close affinity with the classical equations of motion for the elliptic restricted three-body problem, with the addition of the Hill's approximation will be assumed.

The Hill approximation applies well to the motion of a spacecraft near to a small bodies orbiting the Sun or near to a moon orbiting its mother planet. However, it generally is not applied to study motion near one of the objects which compose a binary asteroid system, once the mass parameter μ for the orbited body is too large in comparison with the mass parameter of the other object². Indeed, the Hill's approach of the small body gravity potential uses a series expansion assuming $\xi = (\mu_{moon}/\mu_{planet})^{1/3}$ a small parameter (see more details below). In this formulation, if necessary, the effects of the solar radiation pressure (SRP) can be included (SCHEERES; MARZARI, 2002). The frame of these equations is again the rotating three-body frame, centered at the center of mass of the small body. The larger primary is modeled as a point mass on the negative x -axis.

The model presented in equation (167) is composed by four acceleration terms: the gravitational attraction from Phobos $\nabla U(\mathbf{r})$, the centrifugal acceleration $-\omega \times \omega \times \mathbf{r} = -\omega \hat{\mathbf{z}} \times \omega \hat{\mathbf{z}} \times \mathbf{r} = \omega^2 [-\mathbf{r} + (\hat{\mathbf{z}} \cdot \mathbf{r}) \hat{\mathbf{z}}]$, the Coriolis acceleration $-2\omega \times \dot{\mathbf{r}} = -2\omega \hat{\mathbf{z}} \times \dot{\mathbf{r}}$, and the gravitational acceleration from Mars as the term containing the parameter μ_M . In this expression, $\mu_M = 4.2828372854 \times 10^{13} m^3/s^2$ is the gravitational parameter of Mars, and $\mathbf{r}_M = [r_M, 0, 0]^T$ is the position vector of Mars in Phobos frame.

Using the more standard treatment of the Elliptical Hill Problem (EHP), we introduce the vector $\mathbf{R}$ as the spacecraft position in this rotating frame, finding the new equations of motion in the form (SCHEERES; MARZARI, 2002)

$$\ddot{\mathbf{R}} + \omega \times \omega \times \mathbf{R} + 2\omega \times \dot{\mathbf{R}} + \dot{\omega} \times \mathbf{R} = \nabla U(\mathbf{R}) \quad (172)$$

where the left hand side of the equations of motion (172) correspond to the acceleration of a vector in an arbitrarily rotating frame. Here, we introduce the so-called pulsating frame $\mathcal{S}'$, where the spacecraft position is scaled by the Phobos-Mars distance r_M and time is replaced with Phobos' true anomaly ν . In this formulation, as the time dependence is replaced by a dependence in terms of the true anomaly, the "dots" related to the notation of time derivatives are suppressed by the use of "primes", these ones denoting derivatives in terms of the true anomaly ν . Once the orbital eccentricity of Phobos around Mars is small ($e_{Phobos} = 0.0151$), the correction to return to time dependence by performing a multiplication by the factor $d\nu/dt = \dot{\nu} \approx 1$ can be avoided.

²See for instance, the ratio of mass parameter q , which is used in the study presented in chapter 5.

Despite the complexity introduced using these concepts, the introduction of this frame is essential to derive the equations of motion that can be stated in a simple form, allowing its analysis. The following relations are applied in order to find the desired equations of motion (SCHEERES; MARZARI, 2002):

$$r'_M = \frac{(r_M)^2}{p} e \sin \nu \quad (173)$$

$$r''_M = \frac{e \cos \nu}{p} r_M^2 + 2 \frac{(r'_M)^2}{r_M} \quad (174)$$

$$\omega = \frac{\sqrt{p\mu_M}}{(r_M)^2} \quad (175)$$

$$\omega' = -2 \frac{\omega r'_M}{r_M} = -2\omega \sqrt{\frac{\mu_M}{p}} \frac{e \sin \nu}{r_M} \quad (176)$$

$$\boldsymbol{\omega}' = \omega' \boldsymbol{\omega} \quad (177)$$

$$\mathbf{R} = \xi r_M \mathbf{r} \quad (178)$$

$$\dot{\mathbf{R}} = \xi \omega (r'_M \mathbf{r} + r_M \mathbf{r}') \quad (179)$$

$$\ddot{\mathbf{R}} = \xi \omega (\omega' r'_M + \omega r''_M) \mathbf{r} + \xi \omega (\omega' r_M + 2\omega r'_M) \mathbf{r}' + \xi \omega^2 r_M \mathbf{r}'' \quad (180)$$

where $\mathbf{r}$ is the new position vector in pulsating frame $\mathcal{S}'$ and $\mathbf{r}'$ denotes the derivative of this vector with respect to Phobos' true anomaly.

Thus, the equations of motion (172) become

$$\mathbf{r}'' + 2\omega \hat{\mathbf{z}} \times \mathbf{r}' + \omega^2 (\hat{\mathbf{z}} \cdot \mathbf{r}) \hat{\mathbf{z}} = \nabla U(\mathbf{r}) \quad (181)$$

Now, we need to determine the potential $\nabla U(\mathbf{r})$. For our case, it depends on the small parameter $\xi = (\mu/\mu_M)^{1/3}$, as in the standard derivation of Hill's equations

$$\nabla U(\mathbf{r}) = \frac{\partial U_\xi}{\partial \mathbf{r}} \quad (182)$$

$$U_\xi = \frac{1}{1 + e \cos \nu} \left[\frac{1}{|\mathbf{r}|} + \frac{1 - \zeta/\mu_M}{\xi^2} \frac{1}{|\hat{\mathbf{d}} + \xi \mathbf{r}|} + \frac{1}{\xi} \hat{\mathbf{d}} \cdot \mathbf{r} + \frac{1}{2} \mathbf{r} \cdot \mathbf{r} \right] \quad (183)$$

where the parameter ζ is selected to zero, once it refers to SRP, whose effects can be neglected for practical purposes of the present study. Here, $\hat{\mathbf{d}}$ is the unit vector pointing from the planet to the planetary moon.

Assuming ξ as a small parameter, we have the Taylor's expansion

$$\frac{1}{|\hat{\mathbf{d}} + \xi \mathbf{r}|} \approx 1 - \xi \hat{\mathbf{d}} \cdot \mathbf{r} - \frac{\xi^2}{2} \left\{ \mathbf{r} \cdot \mathbf{r} - 3 \left(\hat{\mathbf{d}} \cdot \mathbf{r} \right)^2 \right\} + \mathcal{O}(\xi^3) \quad (184)$$

so that

$$U_\xi = \frac{1}{1 + e \cos \nu} \left[\frac{1}{\xi^2} + \frac{1}{|\mathbf{r}|} + \frac{3}{2} \left(\hat{\mathbf{d}} \cdot \mathbf{r} \right)^2 + \mathcal{O}(\xi) \right] \quad (185)$$

To complete the approximation, we discard the constant term ($\frac{1}{\xi^2}$) in the potential and drop the higher order terms. The final form of our equations of motion is

$$\mathbf{r}'' + 2\omega \hat{\mathbf{z}} \times \mathbf{r}' + \omega^2 (\hat{\mathbf{z}} \cdot \mathbf{r}) \hat{\mathbf{z}} = \frac{\partial U_\xi}{\partial \mathbf{r}} \quad (186)$$

and

$$U_\xi = \frac{1}{1 + e \cos \nu} \left[\frac{1}{|\mathbf{r}|} + \frac{3}{2} \left(\hat{\mathbf{d}} \cdot \mathbf{r} \right)^2 \right] \quad (187)$$

At this moment, the Elliptical Hill three-body problem, which is a valid approximation of the elliptical restricted three-body problem³ for spacecraft motion near the smaller primary (the planetary moon in our case) when μ_{moon} is small, is now defined. This approximation is valid when spacecraft motion occurs inside the small body's Hill sphere, defined in expression (166) as $R_{Hill} = a (\mu/(3\mu_M))^{\frac{1}{3}}$. Here, again we remember that a is the semi-major axis of Phobos around Mars and the units of time and length where normalized such that: (1) One Phobos' orbital revolution around Mars is 2π , i.e. $\omega = 1$; (2) The value for R_{Hill} is equal to 1.

Assuming in this study that $\hat{\mathbf{d}}$ is defined in the same direction than the x -axis unit vector, i.e., $\hat{\mathbf{d}} = \hat{\mathbf{x}}$, and rewriting the equations in scalar form, we obtain

$$x'' - 2y' = \frac{1}{1 + e \cos \nu} \left[-\frac{x}{r^3} + 3x \right] \quad (188)$$

$$y'' + 2x' = \frac{1}{1 + e \cos \nu} \left[-\frac{y}{r^3} \right] \quad (189)$$

³ Similarly to the case where the Hill three-body problem which is a valid approximation of the circular restricted three-body problem when μ_{moon} is small.

$$z'' + z = \frac{1}{1 + e \cos \nu} \left[-\frac{z}{r^3} \right] \quad (190)$$

where ω was normalized to 1.

As discussed above, the equations of motion of a mass particle subject to the gravitational pull of Phobos and Mars can be reduced in the neighborhood of the Martian moon to the equations of the EHP as

$$x' = v_x \quad (191)$$

$$y' = v_y \quad (192)$$

$$z' = v_z \quad (193)$$

$$v'_x = \frac{1}{\gamma} (g_x + 3x) + 2v_y \quad (194)$$

$$v'_y = \frac{1}{\gamma} (g_y) - 2v_x \quad (195)$$

$$v'_z = \frac{1}{\gamma} (g_z) - z \quad (196)$$

where $\gamma = (1 + e \cos \nu)$, $\mathbf{g} = [g_x, g_y, g_z]^T$ is the gravitational attraction of a co-orbiter body in normalized units, meanwhile $\mathbf{r} = [x, y, z]^T$ and $\mathbf{r}' = \mathbf{v} = [v_x, v_y, v_z]^T$ are the position and velocity vectors of a mass particle as seen from the barycenter of the Martian moon. It is important to mention that all coordinates are expressed in the pulsating synodic reference frame $\mathcal{S}'$ properly discussed in Scheeres; Marzari (2002).

In order to use perturbation theory, we intend to use variational equations which will describe perturbations over the unperturbed solution of the system. Thus, we neglect the gravitational attraction of the moon and focus on firstly work on finding a solution for a simpler set of equations of motion given by

$$x' = v_x \quad (197)$$

$$y' = v_y \quad (198)$$

$$z' = v_z \quad (199)$$

$$v'_x = \frac{1}{\gamma}(3x) + 2v_y \quad (200)$$

$$v'_y = -2v_x \quad (201)$$

$$v'_z = -z \quad (202)$$

The system of equations (197) - (202) is equivalent to the equations of the relative motion of two neighboring spacecrafts in eccentric Keplerian orbits and has an analytical solution provided by Yamanaka; Arkersen (2002), where the authors obtained a simple and well-arranged transition matrix applicable to arbitrary elliptic orbits, and that provides an expression for the relative position and velocity vectors in terms of six integrals of motion $\mathbf{K} = [K_1, K_2, K_3, K_4, K_5, K_6]^T$, plus the integral term $J = \int_{\nu_0}^{\nu} \gamma(\tau)^{-2} d\tau$. This solution is as follows

$$x(\nu) = K_2\gamma \sin \nu + K_3\gamma \cos \nu + K_4(2 - 3eJ\gamma \sin \nu) \quad (203)$$

$$y(\nu) = K_1 + K_2(1 + \gamma) \cos \nu - K_3(1 + \gamma) \sin \nu + -3K_4J\gamma^2 \quad (204)$$

$$z(\nu) = K_5 \sin \nu + K_6 \cos \nu \quad (205)$$

$$v_x(\nu) = K_2s' + K_3c' - 3eK_4 \left(\frac{\sin \nu}{\gamma} + Js' \right) \quad (206)$$

$$v_y(\nu) = -2K_2\gamma \sin \nu - K_3(2\gamma \cos \nu - e) - 3K_4(1 - 2eJ\gamma \sin \nu) \quad (207)$$

$$v_z(\nu) = K_5 \cos \nu - K_6 \sin \nu \quad (208)$$

where $s' = \cos \nu + e \cos 2\nu$, $c' = -(\sin \nu + e \sin 2\nu)$, and can be easily rewritten in matrix form as

$$\mathbf{X}(\nu) = \Phi(\mathbf{r}, \mathbf{v}, \nu)\mathbf{K} \quad (209)$$

where the determinant of the state transition matrix $\Phi(\mathbf{r}, \mathbf{v}, \nu)$ which correlates both sets of variables is $(e^2 - 1)$. It means that this matrix is non-singular for all the admissible values of eccentricity $e \in [0, 1)$. This characteristic is important, once it allows to invert the matrix to find the elements K_i in terms of the coordinates of position and velocity in

all the possible eccentricities of our interest. This inverse transformation can be written as

$$\mathbf{K} = \Phi^{-1}(\mathbf{r}, \mathbf{v}, \nu) \mathbf{X}(\nu) = \begin{pmatrix} C_{1,1} & C_{1,2} & 0 & C_{1,4} & C_{1,5} & 0 \\ C_{2,1} & 0 & 0 & C_{2,4} & C_{2,5} & 0 \\ C_{3,1} & 0 & 0 & C_{3,4} & C_{3,5} & 0 \\ C_{4,1} & 0 & 0 & C_{4,4} & C_{4,5} & 0 \\ 0 & 0 & C_{5,3} & 0 & 0 & C_{5,6} \\ 0 & 0 & C_{6,3} & 0 & 0 & C_{6,6} \end{pmatrix} \mathbf{X}(\nu) \quad (210)$$

with

$$C_{1,1} = -3 \frac{e(1+\gamma) \sin \nu - J\gamma(e^2 + 3e \cos \nu + 2)}{\gamma(1-e^2)} \quad (211)$$

$$C_{1,2} = 1 \quad (212)$$

$$C_{1,4} = \frac{e\gamma \cos \nu + 3eJ\gamma \sin \nu - 2}{(1-e^2)} \quad (213)$$

$$C_{1,5} = -\frac{e(1+\gamma) \sin \nu - 3J\gamma^2}{\gamma(1-e^2)} \quad (214)$$

$$C_{2,1} = -3 \frac{(\gamma + e^2) \sin \nu - eJ\gamma(e^2\gamma + e(2+3\gamma) \cos \nu + 2)}{\gamma(1-e^2)} \quad (215)$$

$$C_{2,4} = \frac{\gamma \cos \nu + 3e^2J\gamma \sin \nu - 2e}{(1-e^2)} \quad (216)$$

$$C_{2,5} = -\frac{(1+\gamma) \sin \nu - 3eJ\gamma^2}{(1-e^2)} \quad (217)$$

$$C_{3,1} = -3 \frac{(e + \cos \nu)}{(1-e^2)} \quad (218)$$

$$C_{3,4} = -\frac{\gamma \sin \nu}{(1-e^2)} \quad (219)$$

$$C_{3,5} = -\frac{e + (1 + \gamma) \cos \nu}{(1 - e^2)} \quad (220)$$

$$C_{4,1} = \frac{3\gamma + e^2 - 1}{(1 - e^2)} \quad (221)$$

$$C_{4,4} = \frac{e\gamma \sin \nu}{(1 - e^2)} \quad (222)$$

$$C_{4,5} = \frac{\gamma^2}{(1 - e^2)} \quad (223)$$

$$C_{5,3} = \sin \nu \quad (224)$$

$$C_{5,6} = \cos \nu \quad (225)$$

$$C_{6,3} = \cos \nu \quad (226)$$

$$C_{6,6} = -\sin \nu \quad (227)$$

6.3 GAUSS VARIATIONAL EQUATIONS

It is possible to verify in equation (210) that the values of the $\mathbf{K}$ integrals can be specified in terms of the coordinates of position and velocity through the inverse matrix $\Phi(\mathbf{r}, \mathbf{v}, \nu)$ such that $\mathbf{K} = \mathbf{K}(\mathbf{r}, \mathbf{v}, \nu)$. Thus, we have the total differential of $\mathbf{K}$ defined as

$$\mathbf{K}' = \frac{\partial \mathbf{K}}{\partial \mathbf{r}} \mathbf{r}' + \frac{\partial \mathbf{K}}{\partial \mathbf{v}} \mathbf{v}' + \frac{\partial \mathbf{K}}{\partial \nu} \quad (228)$$

where the value of the components of $\mathbf{K}$ have to be equal to zero for the unperturbed case

$$\mathbf{v}' = \mathbf{a} = \left[3\frac{x}{\gamma} + 2v_y, -2v_x, -z \right]^T.$$

For the perturbed system

$$\mathbf{r}' = \mathbf{v} \quad (229)$$

$$\mathbf{v}' = \mathbf{a} + \mathbf{u} \quad (230)$$

where $\mathbf{u} = [u_x, u_y, u_z]^T$ is a vector of disturbing accelerations in the pulsating synodic reference frame $\mathcal{S}'$. By replacing the equations (229) and (230) into equation Kdifferential, we get the expressions of *Gauss Variational Equations* (GVE) for the $\mathbf{K}$ integrals of motion:

$$\mathbf{K}' = \frac{\partial \mathbf{K}}{\partial \mathbf{v}} \mathbf{u} \quad (231)$$

with

$$K'_1 = \left(\frac{e\gamma \cos \nu + 3eJ\gamma \sin \nu - 2}{1 - e^2} \right) u_x - \left(\frac{e(1 + \gamma) \sin \nu - 3J\gamma^2}{1 - e^2} \right) u_y \quad (232)$$

$$K'_2 = \left(\frac{\gamma \cos \nu + 3e^2J\gamma \sin \nu - 2e}{1 - e^2} \right) u_x - \left(\frac{(1 + \gamma) \sin \nu - 3eJ\gamma^2}{1 - e^2} \right) u_y \quad (233)$$

$$K'_3 = - \left(\frac{\gamma \sin \nu}{1 - e^2} \right) u_x - \left(\frac{e + (1 + \gamma) \cos \nu}{1 - e^2} \right) u_y \quad (234)$$

$$K'_4 = \left(\frac{e\gamma \sin \nu}{1 - e^2} \right) u_x + \left(\frac{\gamma^2}{1 - e^2} \right) u_y \quad (235)$$

$$K'_5 = (\cos \nu) u_z \quad (236)$$

$$K'_6 = -(\sin \nu) u_z \quad (237)$$

6.4 GEOMETRICAL INTERPRETATION AND NEW DYNAMICAL EQUATIONS

The equations of motion defined by the equations (232) - (237) is a set of nonsingular equations of motion for the osculating orbit element $\mathbf{K}$. However, as they do not provide a clear geometrical/physical of the problem, it may be better to introduce auxiliary osculating variables to provide a more direct and useful geometry of the relative trajectory⁴

$$A_x = K_3 \quad (238)$$

⁴See for instance the studies presented by Canalias et al. (2017) and Cabral (2011). The first one uses a similar formulation to perform a numerical analysis of the problem using the multiple shooting and numerical continuation methods, meanwhile the second one present a preliminary analytical analysis with some discrepancies that will be mentioned when the averaged linearized equations are introduced in section 6.6.

$$A_y = K_2 - 3eJK_4 \quad (239)$$

$$\delta_x = 2K_4 \quad (240)$$

$$\delta_y = K_1 - 3JK_4 \quad (241)$$

$$A = \sqrt{A_x^2 + A_y^2} \quad (242)$$

$$\alpha = \arctan\left(\frac{-A_y}{A_x}\right) \quad (243)$$

$$B = \sqrt{K_5^2 + K_6^2} \quad (244)$$

$$\beta = \arctan\left(\frac{K_6}{K_5}\right) \quad (245)$$

such that

$$x(\nu) = \gamma A \cos(\nu + \alpha) + \delta_x \quad (246)$$

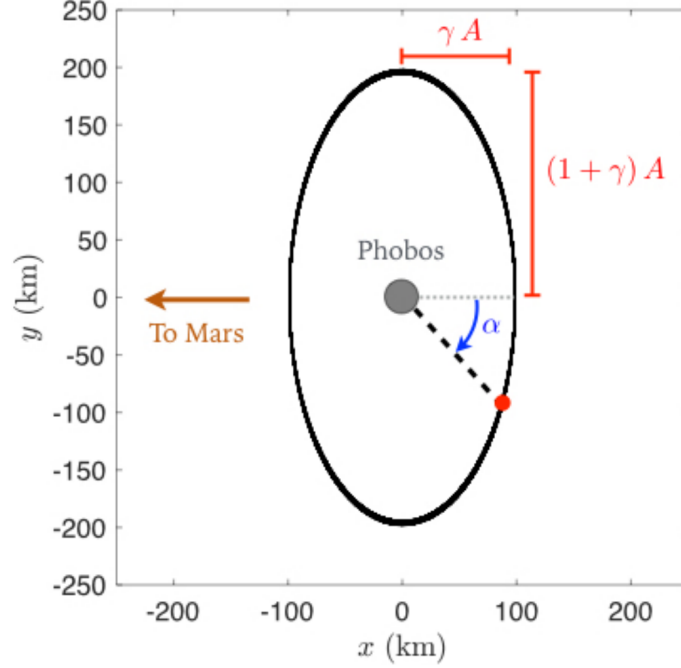
$$y(\nu) = -(1 + \gamma)A \sin(\nu + \alpha) + \delta_y \quad (247)$$

$$z(\nu) = B \sin(\nu + \beta) \quad (248)$$

where the equations (246) - (248) show that a mass particle in orbit around a massless Phobos would remain in the vicinity of this planetary moon for as long as the variables A , δ_x and δ_y do not grow secularly along the time. As A_y and δ_y depend on the integral term $J = \int_{\nu_0}^{\nu} d\tau / \gamma(\tau)^2$, which in first order approach grows linearly, the condition to avoid secular growth on these variables, regardless the integral term, is satisfied when $K_4 = 0$. In this scenario, the trajectory described by the spacecraft in the equatorial plane becomes a pulsating ellipse with frequency $\omega_{xy} = 1$ and semi-major and semi-minor axes given by $(1 + \gamma)A$ and γA , respectively. The angle α represents only a phase angle, the longitude of the spacecraft at the orbit periapsis (also denominated as perimartem for the case of Mars), *i.e.*, $\nu = 0$. The parameters δ_x and δ_y denote displacements of the QSO along the x and y axes, respectively. A representation of the in-plane motion is provided in figure 74. Regarding the out-of-plane motion, it is decoupled from the planar one and follows

an harmonic oscillation with amplitude B and frequency $\omega_z = 1$. Similarly to the case of α , the angle β represents a phase angle describing the latitude of the spacecraft position when at the orbit's pericenter ($\nu = 0$), thus being interpreted as the latitude at perimartem.

Figure 74: Initial position of a spacecraft in a planar QSO at perimartem with $\delta_x = \delta_y = 0$



Source: Baresi et al. (2020)

As these new orbital variables provide better insights on the understanding and work with the dynamics of QSOs, it is important to obtain similar equations for the time variation of A , α , δ_x , δ_y , B , and β than those for the integrals K_i presented in equations (232) - (237). For example, for the parameter A we can write

$$A' = \frac{A_x}{A} A_x' + \frac{A_y}{A} A_y' = \cos(\alpha) A_x' - \sin(\alpha) A_y'$$

$$A' = \cos(\alpha) K_3' - \sin(\alpha) \left(K_2' - 3eJ K_4' - 3e \frac{K_4}{\gamma^2} \right) \quad (249)$$

which, after replacing the expressions of K_i and K_i' , provides the rate of change in variable A in terms of the new parameters. Accordingly, it is possible to find the new set of equations of motion for A , α , δ_x , δ_y , B , and β

$$A' = - \left[\frac{\gamma \sin(\nu + \alpha) - 2e \sin \alpha}{1 - e^2} \right] u_x - \left[\frac{e \cos \alpha + (1 + \gamma) \cos(\nu + \alpha)}{1 - e^2} \right] u_y + \frac{3e \sin \alpha}{2\gamma^2} \delta_x \quad (250)$$

$$\alpha' = - \left[\frac{\gamma \cos(\nu + \alpha) - 2e \cos \alpha}{A(1 - e^2)} \right] u_x + \left[\frac{e \sin \alpha + (1 + \gamma) \sin(\nu + \alpha)}{A(1 - e^2)} \right] u_y + \frac{3e \cos \alpha}{2A\gamma^2} \delta_y \quad (251)$$

$$\delta'_x = \left[\frac{2e\gamma \sin \nu}{1 - e^2} \right] u_x + \left[\frac{2\gamma^2}{1 - e^2} \right] u_y \quad (252)$$

$$\delta'_y = \left[\frac{e\gamma \cos \nu - 2}{1 - e^2} \right] u_x - \left[\frac{e(1 + \gamma^2) \sin \nu}{1 - e^2} \right] u_y - \frac{3}{2\gamma^2} \delta_x \quad (253)$$

$$B' = [\cos(\nu + \beta)] u_z \quad (254)$$

$$\beta' = - \left[\frac{\sin(\nu + \beta)}{B} \right] u_z \quad (255)$$

where the equation (237) shows that the rate of change of the latitude at perimartem is inversely proportional to the magnitude of the out-of-plane motion B . Thus, it is preferable to work with equations (236) and (237) instead of (254) and (255), making possible to avoid singularities when studying the evolution of the orbital elements of a planar QSO orbit such as the case presented in figure 74. Therefore, the final set of osculating elements we choose to work with is defined by $\mathbf{A} = [A, \alpha, \delta_x, \delta_y, B, \beta]^T$ and whose correspondent equations of motion are

$$\mathbf{A}' = G(\mathbf{A}, \nu) \mathbf{u} + \mathbf{F}(\mathbf{A}, \nu) \quad (256)$$

with

$$\mathbf{F}(\mathbf{A}, \nu) = \begin{pmatrix} \frac{3e \sin \alpha}{2\gamma^2} \delta_x \\ \frac{3e \cos \alpha}{2A\gamma^2} \delta_y \\ 0 \\ -\frac{3}{2\gamma^2} \delta_x \\ 0 \\ 0 \end{pmatrix} \quad (257)$$

and

$$G(\mathbf{A}, \nu) = \begin{pmatrix} G_{1,1} & G_{1,2} & 0 \\ G_{2,1} & G_{2,2} & 0 \\ C_{3,1} & G_{3,2} & 0 \\ C_{4,1} & G_{4,2} & 0 \\ 0 & 0 & C_{5,3} \\ 0 & 0 & C_{6,3} \end{pmatrix} \quad (258)$$

where the elements of the six-by-three matrix $G(\mathbf{A}, \nu)$ are

$$G_{1,1} = -\frac{\gamma \sin(\nu + \alpha) - 2e \sin \alpha}{1 - e^2} \quad (259)$$

$$G_{1,2} = -\frac{e \cos \alpha + (1 + \gamma) \cos(\nu + \alpha)}{1 - e^2} \quad (260)$$

$$G_{2,1} = -\frac{\gamma \cos(\nu + \alpha) - 2e \cos \alpha}{A(1 - e^2)} \quad (261)$$

$$G_{2,2} = \frac{e \sin \alpha + (1 + \gamma) \sin(\nu + \alpha)}{A(1 - e^2)} \quad (262)$$

$$G_{3,1} = \frac{2e\gamma \sin \nu}{1 - e^2} \quad (263)$$

$$G_{3,2} = \frac{2\gamma^2}{1 - e^2} \quad (264)$$

$$G_{4,1} = \frac{e\gamma \cos \nu - 2}{1 - e^2} \quad (265)$$

$$G_{4,2} = -\frac{e(1 + \gamma^2) \sin \nu}{1 - e^2} \quad (266)$$

$$G_{5,3} = \cos \nu \quad (267)$$

$$G_{6,3} = -\sin \nu \quad (268)$$

After organizing the new set of equations presented in equation (256), they can be now integrated to obtain the time histories of the osculating orbit elements directly, rather than by conversion from the Cartesian coordinates obtained from equations (229) and (230). Additionally to this, analytical expressions for the effects of Phobos' gravity on the osculating orbit elements $\mathbf{A}$ can be derived and analyzed as in the following sections.

6.5 THE EFFECTS OF PHOBOS'S GRAVITY AND LINEARIZED MODEL

In this section, it is desired to include the effects gravitational attraction on the analysis of the orbital motion. Thus, the osculating orbit elements $\mathbf{A}$ will be disturbed by the arbitrary accelerations u_x , u_y and u_z appearing in equation (256), which are due to the gravitational attraction of the Martian moon. These accelerations are represented in normalized units and in the pulsating synodic reference frame $\mathcal{S}'$:

$$\mathbf{u} = -\frac{1}{\gamma r^3} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad (269)$$

where $r = \sqrt{x^2 + y^2 + z^2}$.

When the values for $x(\nu)$, $y(\nu)$ and $z(\nu)$ given in expressions (246) to (248) are introduced in the components of the vector presented in expression (269). Thus, we are able to obtain a fairly complicated expression for the distance which separates the spacecraft and the barycenter of the Martian moon. It is worth of mention, however, that a spacecraft on a QSO orbit will most likely remain in the equatorial plane of Phobos with δ_x and δ_y close to zero. In addition to that, the orbital eccentricity e of the Martian moon is relatively small, allowing the use of simplifications in the expressions of the GVE derived in section 6.4, such as the development of a linearized model in terms of this parameter. It means that we expand the equations of motion up to the first order term in e by using auxiliary variables ($\delta = \frac{\delta_x}{A}$, $\epsilon = \frac{\delta_y}{A}$, $\eta_5 = \frac{K_5}{A}$, $\eta_6 = \frac{K_6}{A}$). It is neglected the cross-coupling and second order terms by assuming that δ , ϵ , η_5 , $\eta_6 \approx \mathcal{O}(e) \ll 1$. These assumptions are proven to be reliable by numerical experiments for high-altitude QSOs above 80 km. In this orbital regime, the equations for x , y , and z which appear in expressions (246) to (248) will imply

$$r(\mathbf{e}) \approx A \left\{ d^2 + 2e \cos \nu [\cos^2 \theta + 2 \sin^2 \theta] + 2\delta \cos \theta - 4\epsilon \sin \theta \right\} \quad (270)$$

where $\mathbf{e} = [e, \delta, \epsilon]^T$, $d = r(\mathbf{0})/A = \sqrt{\cos^2 \theta + 4 \sin^2 \theta}$ and $\theta = \nu + \alpha$.

Now, we move forward on obtaining the linearized GVEs by the u_x , u_y and u_z terms given in expression (269) and using in them the expression (270). We can start, for instance, with the expression for δ_x' presented in equation (252):

$$\begin{aligned} \delta_x' &\approx \frac{2A}{[r(\mathbf{e})]^3} \left\{ 2 \sin \theta + e [3 \cos \nu \sin \theta - \sin \nu \cos \theta] - \epsilon \right\} \\ &\approx \frac{2}{A^2} \left\{ \left[\frac{2 \sin \theta + e [3 \cos \nu \sin \theta - \sin \nu \cos \theta] - \epsilon}{d^3} \right] \right. \\ &\quad \left. - 6 \sin \theta \left[\frac{e \cos \nu [\cos^2 \theta + 2 \sin^2 \theta] + \delta \cos \theta - 2\epsilon \sin \theta}{d^5} \right] \right\} \end{aligned} \quad (271)$$

Following the same procedure, it is possible to obtain linearized equations for the other orbit elements

$$A' \approx \frac{1}{A^2 d^3} \left\{ -\frac{3}{2} \sin(2\theta) - 2e \left[\sin(\nu + 2\alpha) - \frac{1}{4} \cos \nu \sin(2\theta) \right] + \delta \sin \theta + 2\epsilon \cos \theta \right. \\ \left. + \frac{9}{2} \sin(2\theta) \left[\frac{e \cos \nu (\cos^2 \theta + 2 \sin^2 \theta) + \delta \cos \theta - 2\epsilon \sin \theta}{d^2} \right] \right\} \quad (272)$$

$$\alpha' \approx \frac{1}{A^3} \left\{ \frac{1}{d} - e \frac{\cos \nu}{d} - e \left[\frac{2 \cos(\nu + 2\alpha) + \cos \nu (\cos^2 \theta + 2 \sin^2 \theta)}{d^3} \right] \right. \\ \left. + \frac{2\delta \cos \theta - 4\epsilon \sin \theta}{d^3} \right\} \quad (273)$$

$$\delta'_y \approx \frac{1}{A^2 d^3} \left\{ 2 \cos \theta - e [\cos \nu \cos \theta + 4 \sin \nu \sin \theta] + \delta \left(\frac{4 - 3A^3 d^3}{2} \right) \right. \\ \left. - 6 \cos \theta \left[\frac{e \cos \nu (\cos^2 \theta + 2 \sin^2 \theta) + \delta \cos \theta - 2\epsilon \sin \theta}{d^2} \right] \right\} \quad (274)$$

$$K'_5 \approx -\frac{\cos \nu}{A^2} \left(\frac{\eta_5 \sin \nu + \eta_6 \cos \nu}{d^3} \right) \quad (275)$$

$$K'_6 \approx \frac{\sin \nu}{A^2} \left(\frac{\eta_5 \sin \nu + \eta_6 \cos \nu}{d^3} \right) \quad (276)$$

The set of expressions (271) to (276) provides a set of linearized equations of motion (LM) which accurately predicts the dynamics of mass particles in high-altitude QSOs for several orbital periods around Phobos. It can also be represented in the vectorial form

$$\mathbf{A}' = \mathbf{g}(\mathbf{A}, \nu) \mathbf{u} \quad (277)$$

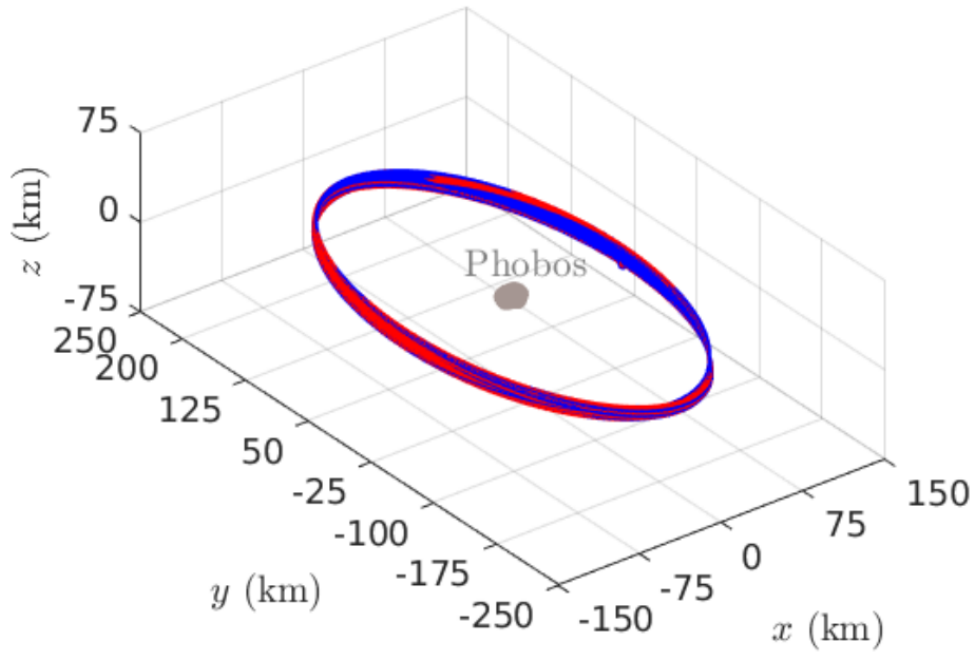
The GVE and LM equations are confirmed to be precise approaches of the dynamics by the sequence of plots presented in figures 75 to 79, which present numerical simulations to compare the different systems of equations presented in this chapter: 1) System EHP (Elliptical Hill Problem): equations (229) and (230) or equations (191) to (196); 2) System GVE (Gauss Variational Equations): equations (250) to (253), (236) and (237) or equations (256); 3) System LM (Linearized Model/Equations): equations (271) to (276) or equations (277). These numerical experiments were performed considering the fol-

lowing initial conditions in Cartesian coordinates ($\mathbf{X}(\nu_0)$) and osculating orbit elements ($\mathbf{A}(\nu_0)$)

$$\mathbf{X}(\nu_0) = \begin{pmatrix} 4.223784177246 \\ -0.0814069532286406 \\ -0.317146285024353 \\ 0.0342016222056316 \\ -8.42418511932641 \\ 0.139224827215046 \end{pmatrix}, \mathbf{A}(\nu_0) = \begin{pmatrix} 4.22922122381657 \\ 0.61341331263205 \\ -0.0576706532250935 \\ -0.0778356681681636 \\ 0.296336249720383 \\ -0.179304617116979 \end{pmatrix} \quad (278)$$

where the initial true anomaly is set as $\nu_0 = 324.8780$ deg.

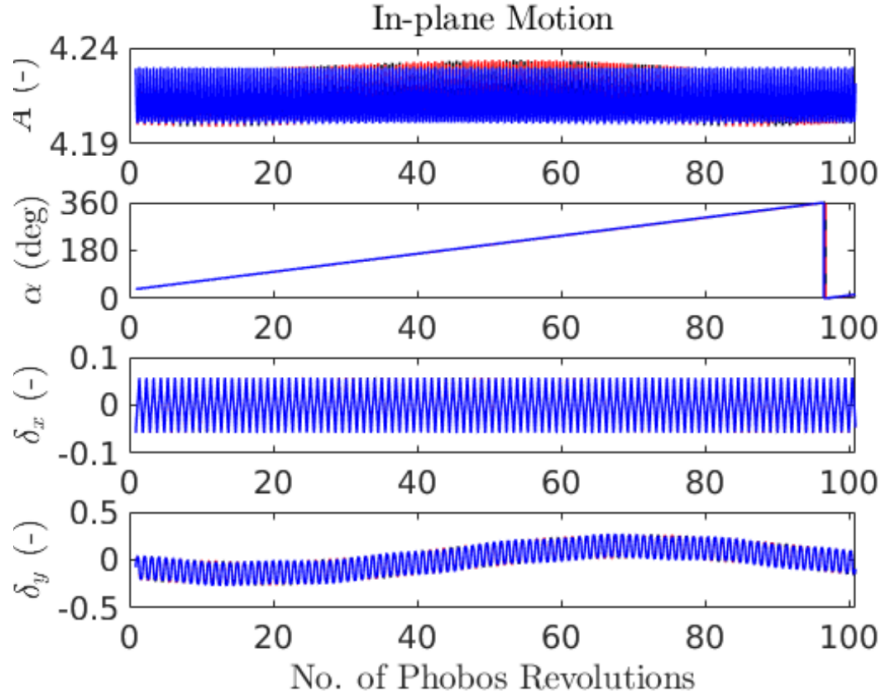
Figure 75: Quasi-QSO orbit integrated over 100 Phobos revolution via the LM (blue), GVEs (red), and EHP (black)



Source: Baresi et al. (2020)

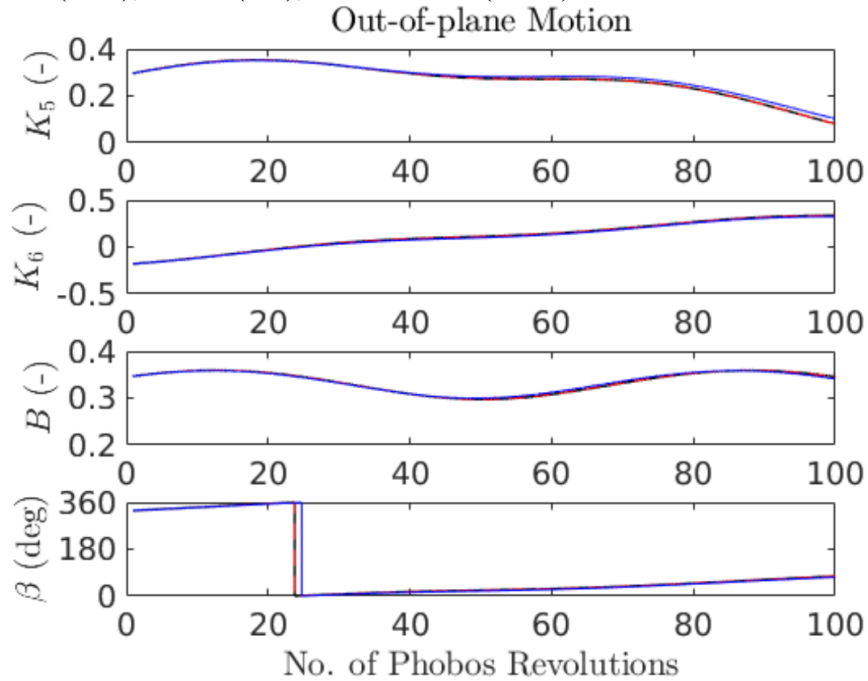
As expected, the figure 78 points out that considering a mass particle or a satellite in mid- and high-altitude QSO orbits, the error produced between equations (256) and the equations (229) and (230) is purely numerical, being perfectly negligible even at the end of the numerical simulations assuming an orbit propagation time of 100 revolutions of the Martian moon, which corresponds to ≈ 32 days for the case of Phobos. This period is selected because it is close to the estimated time required for MMX mission operations close to this moon. Differently from the result for GVE, but still inside a precise margin of error, the differences with the linearized equations (277) grow more rapidly (see figure 79), however never exceeding 5 kilometers over the same period of time. Such an error is deemed reasonable for the purpose of this study, as navigation and mismodeled dynamics-related errors will most likely overtake the inaccuracies of LM over the period of a few

Figure 76: Time histories of the in-plane osculating orbit elements A , α , δ_x , δ_y obtained with the LM model (blue), GVEs (red), and the EHP (black)



Source: Baresi et al. (2020)

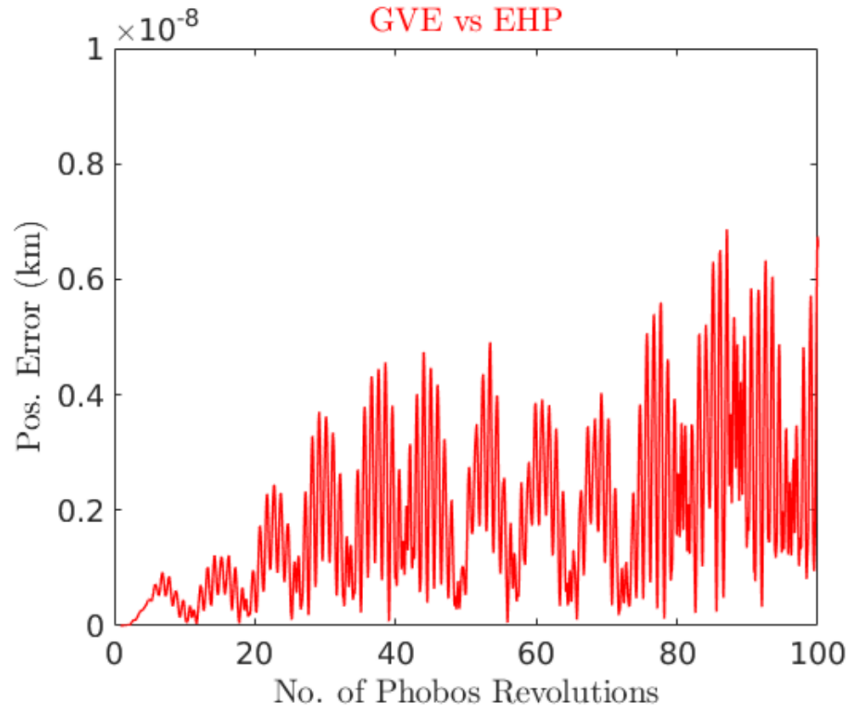
Figure 77: Time histories of the osculating orbit elements K_5 , K_6 , B , β obtained with the LM model (blue), GVEs (red), and the EHP (black)



Source: Baresi et al. (2020)

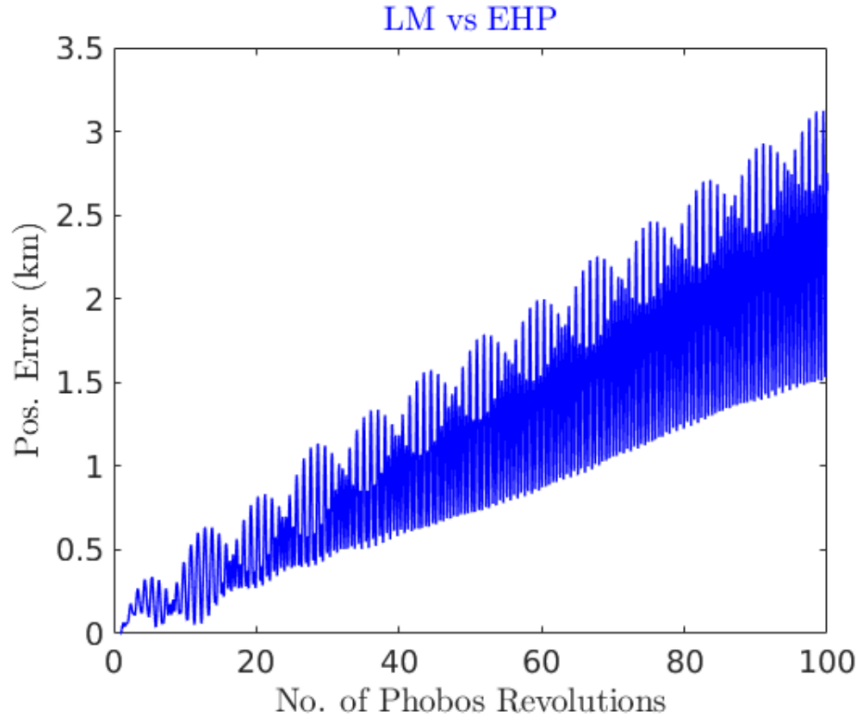
days. Considering this time span, the equations for LM are considered good candidates to describe the long-term motion of satellites in QSO orbits and can be used for the analysis of its secular dynamical evolution as it will be made in section 6.6.

Figure 78: Position error between GVEs and EHP



Source: Baresi et al. (2020)

Figure 79: Position error between LM and EHP



Source: Baresi et al. (2020)

6.6 AVERAGED EQUATIONS

After the analysis of the linearized system of equations (271) to (276), it seems a good idea to seek the understanding of its long-term evolution by applying the averaging

technique in terms of the fast angle θ . Thus, it is possible to grasp the secular evolution of a QSO. Thus, by using the definition of averaging $\langle \mathbf{A} \rangle = \frac{1}{2\pi} \int_0^{2\pi} (\mathbf{A}) d\theta$, we find

$$\bar{A}' \approx 0 \quad (279)$$

$$\bar{\alpha}' \approx \frac{\mathcal{K}}{\pi \bar{A}^3} \quad (280)$$

$$\bar{\delta}'_x \approx -2 \frac{e}{\bar{A}^2} \left(\frac{\mathcal{K} - 7\mathcal{E}}{9\pi} \right) \sin \bar{\alpha} + 2 \frac{\bar{\delta}_y}{\bar{A}^3} \left(\frac{\mathcal{K} - \mathcal{E}}{3\pi} \right) \quad (281)$$

$$\bar{\delta}'_y \approx -\frac{3}{2} \bar{\delta}_x + \frac{e}{\bar{A}^2} \left(\frac{\mathcal{K} - 52\mathcal{E}}{9\pi} \right) \cos \bar{\alpha} + \frac{\bar{\delta}_x}{\bar{A}^3} \left(\frac{\mathcal{K} - 4\mathcal{E}}{3\pi} \right) \quad (282)$$

$$\bar{K}'_5 \approx \left(\frac{\bar{K}_6 \cos(2\bar{\alpha}) - \bar{K}_5 \sin(2\bar{\alpha})}{\bar{A}^3} \right) \left(\frac{2\mathcal{K} - 5\mathcal{E}}{6\pi} \right) - \frac{\bar{K}_6}{\bar{A}^3} \frac{\mathcal{E}}{2\pi} \quad (283)$$

$$\bar{K}'_6 \approx \left(\frac{\bar{K}_5 \cos(2\bar{\alpha}) + \bar{K}_6 \sin(2\bar{\alpha})}{\bar{A}^3} \right) \left(\frac{2\mathcal{K} - 5\mathcal{E}}{6\pi} \right) + \frac{\bar{K}_5}{\bar{A}^3} \frac{\mathcal{E}}{2\pi} \quad (284)$$

where $\mathcal{K} \approx 2.156516$ and $\mathcal{E} \approx 1.211056$ are the complete elliptic integrals of the first and second kind, respectively, evaluated with parameter $m = \sqrt{3}/2$.

We note that the averaged equations (279) to (284) are different from ones presented in Cabral (2011) and similar to the ones presented in Kogan (1989), presenting cross-coupling terms between some of the orbit elements, as well as long-term oscillations due to the secular growth⁵ of the spacecraft's longitude at the perimartem $\bar{\alpha}$. This dynamical feature changes along the time the geometrical configuration between the spacecraft, Mars and Phobos, and especially at the perimartem, when the gravitational attraction due to Mars presents its strongest magnitude.

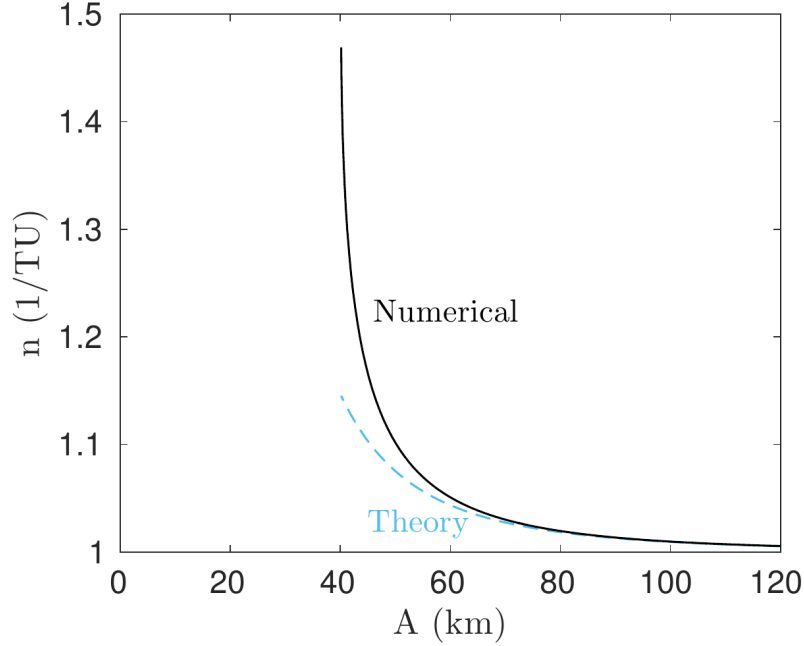
As the longitude $\bar{\alpha}$ grows linearly with time and with a frequency that depends on the amplitude of the in-plane motion:

$$\bar{\alpha}(\nu) = \omega_\alpha (\nu - \nu_0) + \bar{\alpha}(0) \quad (285)$$

where $\omega_\alpha = (\mathcal{K}/\pi) \bar{A}^{-3} \approx 0.686440 \bar{A}^{-3}$ and $\bar{\alpha}(0) = \bar{\alpha}(\nu_0)$ is Phobos' true anomaly at epoch. It is worth to mention that, once $P = 2\pi/n_{QSO}$ and $n_{QSO} = \dot{\theta} = \dot{\nu} + \dot{\alpha} \approx 1 + \omega_\alpha$, the frequency ω_α will also controls the period of the QSO around Phobos. This result is demonstrated with the plot presented in figure 80, which illustrates the comparison

⁵The secular growth in $\bar{\alpha}$ is due to the term $2\mathcal{K}/\pi \bar{A}^3$ in equation (280)

Figure 80: Comparison between the numerical value of n_{QSO} and its theoretical prediction $1 + \frac{\dot{\bar{\alpha}}}{\bar{\alpha}}$



Source: Baresi et al. (2020)

between the analytical prediction for the "*pseudo-mean motion*" n_{QSO} and its numerical value computed with the GMOS algorithm ⁶. Once again, we find that the proposed linearization and averaging approach is only valid when $A > 80$ km. In the validity regime of the averaged equations (279) to (284), the in-plane and out-of-plane motion can be effectively decoupled and solved for by means of the uncoupled linear systems

$$\begin{pmatrix} \bar{\delta}'_x \\ \bar{\delta}'_y \end{pmatrix} = \begin{pmatrix} 0 & D_x \\ D_y & 0 \end{pmatrix} \begin{pmatrix} \bar{\delta}_x \\ \bar{\delta}_y \end{pmatrix} - e \begin{pmatrix} d_x \sin \bar{\alpha} \\ d_y \cos \bar{\alpha} \end{pmatrix} \quad (286)$$

$$\begin{pmatrix} \bar{K}'_5 \\ \bar{K}'_6 \end{pmatrix} = \begin{pmatrix} -\zeta \sin(2\bar{\alpha}) & -E + \zeta \cos(2\bar{\alpha}) \\ E + \zeta \cos(2\bar{\alpha}) & \zeta \sin(2\bar{\alpha}) \end{pmatrix} \begin{pmatrix} \bar{K}_5 \\ \bar{K}_6 \end{pmatrix} \quad (287)$$

where

$$D_x = 2 \left(\frac{\mathcal{K} - \mathcal{E}}{3\pi \bar{A}^3} \right) > 0, \quad D_y = -\frac{3}{2} + 2 \left(\frac{\mathcal{K} - 4\mathcal{E}}{3\pi \bar{A}^3} \right) < 0 \quad (288)$$

$$d_x = 2 \left(\frac{\mathcal{K} - 7\mathcal{E}}{9\pi \bar{A}^3} \right) < 0, \quad d_y = \left(\frac{52\mathcal{K} - \mathcal{E}}{9\pi \bar{A}^3} \right) > 0 \quad (289)$$

$$\zeta = \left(\frac{2\mathcal{K} - 5\mathcal{E}}{6\pi \bar{A}^3} \right), \quad E = \left(\frac{\mathcal{E}}{2\pi \bar{A}^3} \right) \quad (290)$$

⁶The main idea of this numerical algorithm (named after the work of Gerard Gomez, Josep Mondelo, Zubin Olikara and Daniel Scheeres) is that quasi-periodic invariant tori can be calculated via invariant curves of a stroboscopic mapping. An advantage of the GMOS algorithm is that additional frequencies can be taken into account in order to search for higher-dimensional tori. This feature is hereby adopted for calculating quasi-periodic QSOs that extend beyond the equatorial plane of Phobos. We refer to this type of orbits as "3D QSOs". For further details, we refer the reader to Baresi et al. (2020) and references therein.

The systems of equations (286) and (287) can be both solved analytically⁷ providing the following solutions:

$$\begin{aligned}\bar{\delta}_x(\nu) = & \bar{\delta}_{x,0} \cos(\omega_d \bar{\nu}) + \bar{\delta}_{y,0} \left(\frac{D_x}{\omega_d} \right) \sin(\omega_d \bar{\nu}) + e \left(\frac{D_x d_y + \omega_\alpha d_x}{\omega_\alpha^2 - \omega_d^2} \right) \cos(\bar{\alpha}) \\ & + \frac{e}{\omega_\alpha^2 - \omega_d^2} \left[\omega_d \sin(\bar{\alpha}_0) \sin(\omega_d \bar{\nu}) \left(d_x - \omega_\alpha \frac{d_y}{D_y} \right) \right. \\ & \left. - \omega_\alpha \cos(\bar{\alpha}_0) \cos(\omega_d \bar{\nu}) \left(d_x + \frac{1}{\omega_\alpha} D_x d_y \right) \right]\end{aligned}\quad (291)$$

$$\begin{aligned}\bar{\delta}_y(\nu) = & \bar{\delta}_{x,0} \left(\frac{D_y}{\omega_d} \right) \sin(\omega_d \bar{\nu}) + \bar{\delta}_{y,0} \cos(\omega_d \bar{\nu}) + e \left(\frac{D_y d_x + \omega_\alpha d_y}{\omega_\alpha^2 - \omega_d^2} \right) \sin(\bar{\alpha}) \\ & + \frac{e}{\omega_\alpha^2 - \omega_d^2} \left[\omega_d \cos(\bar{\alpha}_0) \sin(\omega_d \bar{\nu}) \left(d_y - \omega_\alpha \frac{d_x}{D_x} \right) \right. \\ & \left. + \omega_\alpha \sin(\bar{\alpha}_0) \cos(\omega_d \bar{\nu}) \left(d_y - \frac{1}{\omega_\alpha} D_y d_x \right) \right]\end{aligned}\quad (292)$$

which can be rewritten as (BARESI et al., 2020)

$$\begin{aligned}\bar{\delta}_x(\nu) = & + \left[\bar{\delta}_{y,0} \left(\frac{D_x}{\omega_d} \right) + e \left(\frac{\omega_d}{D_y} \right) d_2 \sin(\bar{\alpha}_0) \right] \sin(\omega_d \bar{\nu}) \\ & + [\bar{\delta}_{x,0} - e d_1 \cos(\bar{\alpha}_0)] \cos(\omega_d \bar{\nu}) \dots + e d_1 \cos(\bar{\alpha})\end{aligned}\quad (293)$$

$$\begin{aligned}\bar{\delta}_y(\nu) = & \left[\bar{\delta}_{x,0} \left(\frac{D_y}{\omega_d} \right) + e \left(\frac{\omega_d}{D_x} \right) d_1 \cos(\bar{\alpha}_0) \right] \sin(\omega_d \bar{\nu}) \dots \\ & + [\bar{\delta}_{y,0} - e d_2 \sin(\bar{\alpha}_0)] \cos(\omega_d \bar{\nu}) + e d_2 \sin(\bar{\alpha})\end{aligned}\quad (294)$$

where $d_1 = \left(\frac{D_x d_y + \omega_\alpha d_x}{\omega_\alpha^2 - \omega_d^2} \right)$, $d_2 = \left(\frac{D_y d_x - \omega_\alpha d_y}{\omega_\alpha^2 - \omega_d^2} \right)$, $\bar{\nu} = \nu - \nu_0$, $\bar{\delta}_{x,0} = \bar{\delta}_x(\nu_0)$ and $\bar{\delta}_{y,0} = \bar{\delta}_y(\nu_0)$.

The in-plane motion is characterized by two frequencies ω_α and ω_d . The ratio between these two parameters is $\frac{\omega_\alpha}{\omega_d} \approx 1.251283 \bar{A}^{-\frac{3}{2}}$, once $\omega_d^2 = -D_x D_y \approx 0.30095 \bar{A}^{-3} - 0.11443 \bar{A}^{-6}$.

The out-plane motion can also be solved using auxiliary variables, providing the following solution (BARESI et al., 2020):

$$\bar{K}_5(\nu) = \bar{B}(\nu) \cos \bar{\beta}(\nu) = \mathcal{B} [1 - B_\phi \cos(2\bar{\phi})]^{-\frac{1}{2}} \cos(\bar{\alpha} - \bar{\phi}) \quad (295)$$

$$\bar{K}_6(\nu) = \bar{B}(\nu) \sin \bar{\beta}(\nu) = \mathcal{B} [1 - B_\phi \cos(2\bar{\phi})]^{-\frac{1}{2}} \sin(\bar{\alpha} - \bar{\phi}) \quad (296)$$

where the following expression were used $\phi = \alpha - \beta$ and

$$\bar{B}(\nu) = \mathcal{B} \sqrt{1 - B_\phi \cos(2\bar{\phi})} \quad (297)$$

⁷See Baresi et al. (2020) for details about the steps of integrations of these equations.

$$\mathcal{B} = \bar{B}(\nu_0) \sqrt{1 - B_\phi \cos(2\bar{\phi}_0)} \quad (298)$$

$$B_\phi = \frac{2\mathcal{K} - 5\mathcal{E}}{6\mathcal{K} - 3\mathcal{E}} \quad (299)$$

The computation of the expressions (297) and (298) requires the use of the following expressions

$$\bar{\phi}(\nu) = \arctan \left[\frac{1}{2} \sqrt{\frac{2\mathcal{K} + \mathcal{E}}{\mathcal{K} - \mathcal{E}}} \tan[\omega_\phi(\nu - \nu_0) + \psi] \right] \quad (300)$$

$$\omega_\phi = \frac{\sqrt{2\mathcal{K}^2 - \mathcal{K}\mathcal{E} - \mathcal{E}^2}}{3\pi\bar{A}^3} \approx 0.484965\bar{A}^{-3} \quad (301)$$

and finally

$$\psi = \arctan \left[2 \sqrt{\frac{\mathcal{K} - \mathcal{E}}{2\mathcal{K} - \mathcal{E}}} \tan \bar{\phi}_0 \right] \quad (302)$$

with $\bar{\phi}_0 = \bar{\phi}(\nu_0)$.

By inspection of the expression (297), it is possible to see that the amplitude of the out-of-plane motion oscillates with a frequency of $2\omega_\phi$. We also note that these solutions for the in-plane (expressions (293) and (294)) and for the out-plane (expressions (295) and (296)) depend on sines and cosines multiplied by time-periodic amplitudes at worst. It confirms that 3D quasi-QSO orbits are stable for as long as the assumptions of the LM model are valid. For more detailed steps to obtain these solutions and for further discussions, the reader can check Baresi et al. (2020).

6.7 DISCUSSION

This chapter presents an investigation about the use of variational equations to study quasi-satellite orbits around Phobos. The equations of motion are modeled in the context of the Elliptic Hill Problem (EHP), which is a valid approximation of the restricted three-body problem in the physical model studied. The formulation uses a coordinate frame that is scaled by the Phobos-Mars distance with Phobos true anomaly as an independent parameter. After verifying the solution for the unperturbed motion, new orbit parameters are obtained and whose geometrical interpretation is presented, meanwhile the equations of motions in terms of these parameters are provided (GVE). The analysis of these equations is made where these new parameters are verified more useful for orbit design than state vectors for position and velocity, once they provide a straight geometrical visualization of the orbit. During the analytical study of the equations of motion GVE, they are linearized in terms of the first order in eccentricity producing the system of equations de-

nominated LM. These linearized equations are validated by numerical simulations which prove them very precise above 80 km of altitude. Averaged equations are obtained from these linearized equations LM. These averaged equations present cross-coupling terms, differently from what was previously found in literature (CABRAL, 2011). Based on these equations, analytical solutions for both, the in-plane and out-of-plane, components of the motion are proposed and used to verify the stability of 3D QSO orbits disturbed by the gravitational attraction of a spherical Phobos.

Concerning the use of variational equations to describe the dynamics of a spacecraft, the problem of quasi-satellite orbits can also be treated, by mean of the inclusion of other forces contributing with components along the different coordinate axes, with the possibility of using again the averaging technique. Thus, it will possibly enable the analysis of the long-period motion, now considering other forces in the system. However, as it is of great interest, the development of a mean-to-osculating relative orbit element mapping is desired, once missions in the category of MMX require deeper details about the dynamics. It means to be necessary to obtain corrections at least in a few orders of the short-period effects removed during the averaging process (SCHAUB; JUNKINS, 2003; LARA, 2015; HAUTESERRES; LARA, 2017). Among the techniques which can be considered for to reach this goal, it is possible to cite Brouwer's gravity theory (BROUWER, 1959; LYD-DANE, 1963) and the theory of Hamiltonians called "Intermediaries" (DEPRIT, 1981; FERRER; LARA, 2012; GURFIL; LARA, 2014; HAUTESERRES; LARA, 2017; CRESPO et al., 2018; CARDOSO DOS SANTOS; FERRER; SCHEERES, 2019), which were used in chapter 5. An alternative proposal of analytical solution to recover short-period effects in low orders of perturbation has been recently implemented by Lara (2018) where the author uses the Lindstedt–Poincaré method, an approach in terms of series expansions called Lindstedt series (FERRAZ-MELLO, 2007). As in the previous proposed methods, the Hamiltonian form of the system is obtained. This investigation provides valuable insights about the implementation of the previously mentioned methods, once it also required the use of a Hamiltonian formalism to describe the problem. In both cases, the techniques use Lie series (LIE, 1888), which impose that a proper set of canonical variables is required to be defined on each of these analytical treatments (FERRAZ-MELLO, 2007).

The main results of the study presented in this chapter were submitted to a scientific journal (BARESI et al., 2020), where further studies are performed, as the case of a numerical mean-to-osculating orbital element mapping that provides reliable initial conditions for bounded relative motion. The reader is also referred to other outcomes of this research presented in (BARESI et al., 2018), namely the presentation of a nonlinear control law which provides useful numbers for orbit maintenance strategies. The author specially thanks Nicola Baresi for providing the plots presented in this chapter.

7 FINAL COMMENTS

The objectives of the present thesis were to provide the analysis of different problems of astrodynamics and celestial mechanics in the context of using analytical and semi-analytical methods, meanwhile highlighting their importance and merit for current scientific investigations. Interesting ideas and methods to be applied to a wide range of problems in astrodynamics and celestial mechanics were presented and studied. The results available in literature and in this thesis show the present investigations with a good potential to be continued and expanded. Thus, in each chapter these ideas and methods were also pointed out together with other ones for the development of extensions of the presented studies. In order to provided a focused analysis of each topic discussed in the present thesis, throughout this document each investigation is presented in a specific chapter finalized with a section containing a proper discussion of the results and a particular conclusion concerning that specific study. Finally, the author also hopes that the present document will serve as a complementary material for students and researchers working in the topics of Celestial Mechanics and Astrodynamics discussed here.

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APPENDIX A: DISCUSSION OF ATTITUDE EFFECTS OVER THE ORBITAL MOTION IN BINARY SYSTEMS

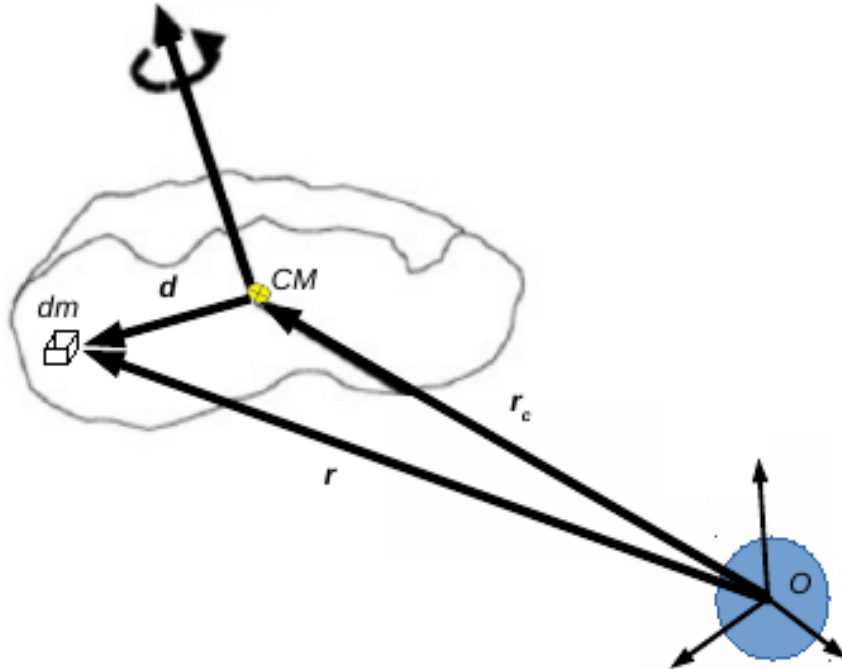
This appendix presents a short discuss of the importance of including more terms in the disturbing potential given by the MacCullagh expansion (subsection 5.2.2). In order to simplify the discussion, a vectorial treatment is considered.

Lets assume the system represented in figure 81. For case of a mass element dm being attracted by a central body with mass M_e , the element of gravity force $d\mathbf{F}_G$ is given by:

$$d\mathbf{F}_G = -G \frac{M_e dm}{|\mathbf{r}|^2} \frac{\mathbf{r}}{|\mathbf{r}|} \quad (303)$$

where G is the gravitational universal constant and $\mathbf{r} = \mathbf{r}_C + \mathbf{d}$. The vectors $\mathbf{r}$, $\mathbf{r}_C$ and $\mathbf{d}$ denote, respectively, the position of the mass element dm with respect to the primary' center of mass, the position of the secondary's center of mass with respect to the same origin than $\mathbf{r}$, and $\mathbf{d}$ is the position of the mass element dm with respect to the secondary's center of mass.

Figure 81: Geometry of the gravitational force over a mass element dm .



Source: Author

The total force is computed by integrating over the entire body and it is given by

$$\mathbf{F}_G = \int_B d\mathbf{F}_G = -\mu \int_B \frac{\mathbf{r}}{|\mathbf{r}|^3} dm \quad (304)$$

where $\mu = GM_e$ is a mass parameter.

We have that for the mutual potential of two objects (WERNER; SCHEERES, 2005)

$$\frac{1}{|\mathbf{r}|} = (r_C^2 + d^2 + 2\mathbf{r}_C \cdot \mathbf{d})^{-\frac{1}{2}} = \frac{1}{r_C} \left[1 + \left(\frac{d}{r_C} \right)^2 - 2 \left(\frac{d}{r_C} \right) \left(\frac{-\mathbf{r}_C \cdot \mathbf{d}}{r_C d} \right) \right]^{-\frac{1}{2}} \quad (305)$$

thus given the expansion

$$\frac{1}{|\mathbf{r}|} = \frac{1}{r_C} \sum_{n=0}^{\infty} \left(\frac{d}{r_C} \right)^n P_n \left(\frac{-\mathbf{r}_C \cdot \mathbf{d}}{r_C d} \right) \quad (306)$$

where P_n are Legendre polynomials.

In order to provide an example of the coupling between orbit and attitude, we take the simplifying assumption (KANE; LIKINS; LEVINSON, 1983; SCHAUB, 2017)

$$\frac{1}{|\mathbf{r}|^3} \approx \frac{1}{r_C^3} \left[1 - \frac{3}{2} \left(2 \frac{\mathbf{r}_C \cdot \mathbf{d}}{r_C^2} + \frac{\mathbf{d} \cdot \mathbf{d}}{r_C^2} \right) + \frac{15}{2} \frac{(\mathbf{r}_C \cdot \mathbf{d})^2}{r_C^4} \right] \quad (307)$$

Thus, the gravity force for two mutual irregular bodies can be described by expanding the terms in expression (307) and using them in expression (304). However, only as a practical discussion to show the importance of the other terms in the expression of the coupling of two bodies with comparable sizes in close distances, we look for the gravity force of a primary body as a spherical object written as

$$\mathbf{F}_G \approx -\frac{\mu}{r_C^3} \left[\int_B \mathbf{r} dm - \frac{3}{2} \int_B \left(2 \frac{\mathbf{r}_C \cdot \mathbf{d}}{r_C^2} + \frac{\mathbf{d} \cdot \mathbf{d}}{r_C^2} \right) \mathbf{r} dm + \frac{15}{2} \int_B \frac{(\mathbf{r}_C \cdot \mathbf{d})^2}{r_C^4} \mathbf{r} dm \right] \quad (308)$$

After using $\mathbf{r} = \mathbf{r}_C + \mathbf{d}$, we obtain

$$\begin{aligned} \mathbf{F}_G \approx -\frac{\mu}{r_C^3} & \left[\int_B \mathbf{r}_C dm - \frac{3}{2} \int_B \left(2 \frac{\mathbf{r}_C \cdot \mathbf{d}}{r_C^2} + \frac{\mathbf{d} \cdot \mathbf{d}}{r_C^2} \right) \mathbf{r}_C dm + \frac{15}{2} \int_B \frac{(\mathbf{r}_C \cdot \mathbf{d})^2}{r_C^4} \mathbf{r}_C dm \right. \\ & \left. + \int_B \mathbf{d} dm - \frac{3}{2} \int_B \left(2 \frac{\mathbf{r}_C \cdot \mathbf{d}}{r_C^2} + \frac{\mathbf{d} \cdot \mathbf{d}}{r_C^2} \right) \mathbf{d} dm + \frac{15}{2} \int_B \frac{(\mathbf{r}_C \cdot \mathbf{d})^2}{r_C^4} \mathbf{d} dm \right] \end{aligned} \quad (309)$$

The expression (309) can be simplified using

$$\begin{aligned} \int_B \mathbf{d} dm &= \mathbf{0}, \\ \int_B \alpha \mathbf{r}_C dm &= \mathbf{r}_C \int_B \alpha dm \end{aligned} \quad (310)$$

where α is a scalar function and $\mathbf{r}_C$ is a fixed vector during the integration. Thus, equation (309) is

$$\mathbf{F}_G \approx -\frac{\mu}{r_C^3} \left[m\mathbf{r}_C - \frac{3}{r_C^2} \int_{\mathcal{B}} (\mathbf{r}_C \cdot \mathbf{d}) \mathbf{d} dm - \frac{3\mathbf{r}_C}{2r_C^2} \int_{\mathcal{B}} (\mathbf{d} \cdot \mathbf{d}) dm + \frac{15\mathbf{r}_C}{2r_C^2} \int_{\mathcal{B}} \frac{(\mathbf{r}_C \cdot \mathbf{d})^2}{r_C^2} dm \right] \quad (311)$$

Now, the following expressions are useful to simplify and rearrange the expression for the force¹ (LEMOS, 2007; SCHEERES, 2012; SCHAUB, 2017)

$$\mathbf{d} \cdot \mathbf{d} = d^2 \quad (312)$$

$$\int_{\mathcal{B}} d^2 dm = \frac{1}{2} \text{tr}([I]) \quad (313)$$

$$(\mathbf{r}_C \cdot \mathbf{d}) \mathbf{d} = \mathbf{d} \times (\mathbf{d} \times \mathbf{r}_C) + (\mathbf{d} \cdot \mathbf{d}) \mathbf{r}_C = \mathbf{d} \times (\mathbf{d} \times \mathbf{r}_C) + d^2 \mathbf{r}_C \quad (314)$$

$$\mathbf{i}_C = \frac{\mathbf{r}_C}{r_C} \quad (315)$$

$$\frac{(\mathbf{r}_C \cdot \mathbf{d})^2}{r_C^2} = \frac{(\mathbf{r}_C \cdot \mathbf{d}) \mathbf{d} \cdot \mathbf{r}_C}{r_C^2} = \mathbf{d} \times (\mathbf{d} \times \mathbf{i}_C) \cdot \mathbf{i}_C + d^2 \mathbf{i}_C \cdot \mathbf{i}_C \quad (316)$$

$$\overset{\leftrightarrow}{\mathbf{I}} = \int_{\mathcal{B}} \left(d^2 \overset{\leftrightarrow}{\mathbf{I}} - \mathbf{d} \cdot \mathbf{d} \right) dm \quad (317)$$

$$I_{ij} = \hat{\mathbf{e}}_i^T \cdot \overset{\leftrightarrow}{\mathbf{I}} \cdot \hat{\mathbf{e}}_j \quad (318)$$

$$[I_C] = - \int_{\mathcal{B}} \left(\tilde{\mathbf{d}} \cdot \tilde{\mathbf{d}} \right) dm \quad (319)$$

where $\overset{\leftrightarrow}{\mathbf{I}}$ is the unity dyadic, $\hat{\mathbf{e}}_i$ and $\hat{\mathbf{e}}_j$ are unit vectors and $\tilde{\mathbf{d}}$ is denoting the cross-product dyad (WERNER; SCHEERES, 2005; SCHEERES, 2006). The second term in expression (317) is also a dyadic (see Scheeres (2012)).

We can also assume that the inertia matrix is taken with respect to a principal coordinate system, which means that it is a diagonal matrix of the form

$$[I] = \begin{pmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix} \quad (320)$$

where I_{11} , I_{22} and I_{33} can be denoted as A , B and C , respectively.

Thus, we finally obtain

$$\mathbf{F}_G \approx -\frac{\mu m}{r_C^3} \left[1 + \frac{3}{mr_C^2} \left([I] + \frac{1}{2} \left(\text{tr}([I]) - 5 (\mathbf{i}_C^T [I] \mathbf{i}_C) \right) [I_{3 \times 3}] \right) \right] \mathbf{r}_C \quad (321)$$

where $[I_{3 \times 3}]$ is an Identity matrix.

¹Including expressions for the inertia matrix

In order to illustrate the importance of considering the effects of the attitude of an object over its orbital motion in a few specific cases, the MacCullagh expansion which is used in general problems (as the case of the one used in subsection 5.2.2) accounts for the expansion neglecting, for instance, the last term of the expression (321), namely

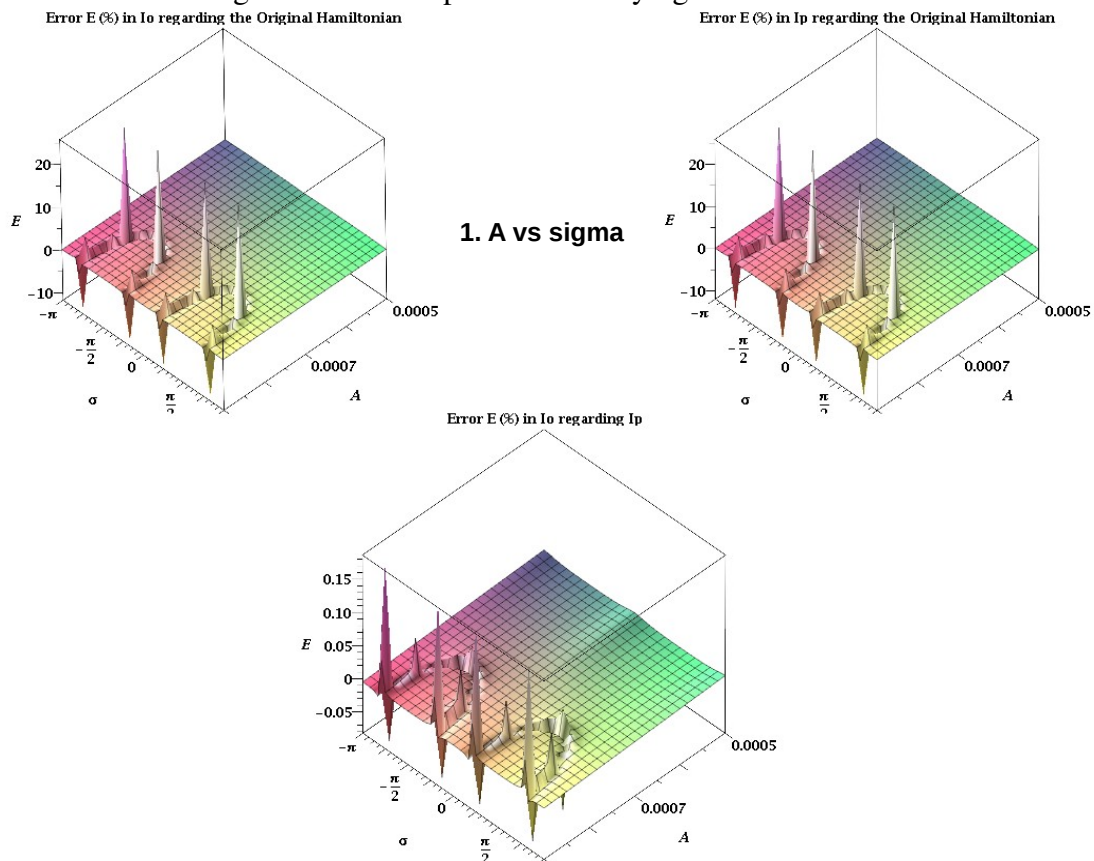
$$\frac{3}{mr_C^2} \left(-\frac{5}{2} \left(\mathbf{i}_C^T [I] \mathbf{i}_C \right) [I_{3 \times 3}] \right) = -\frac{15}{2mr_C^2} \left(\mathbf{i}_C^T [I] \mathbf{i}_C \right) [I_{3 \times 3}] \quad (322)$$

After the analysis of the expression (322), we observe that for most of the cases known in celestial mechanics and astrodynamics, this term can be neglect. It happens because, once the moments of inertia present in the inertia matrix $[I]$ are dependent of the square of the averaged semidiameters of the body, they are usually many orders of magnitude smaller than the square of the distance (r_C^2). However, if we assume binary asteroids or a spacecraft flying around a very small object, scenarios where the dimensions of the orbiting body start to become comparable to its distance to the center of attraction, the differences between these two quantities become non-negligible and they must be accounted for in the dynamics (especially due to cumulative effects which appear after several revolutions). The author refers the reader to discussions presented in Kane; Likins; Levinson (1983), Bucci (2015) and Schaub (2017).

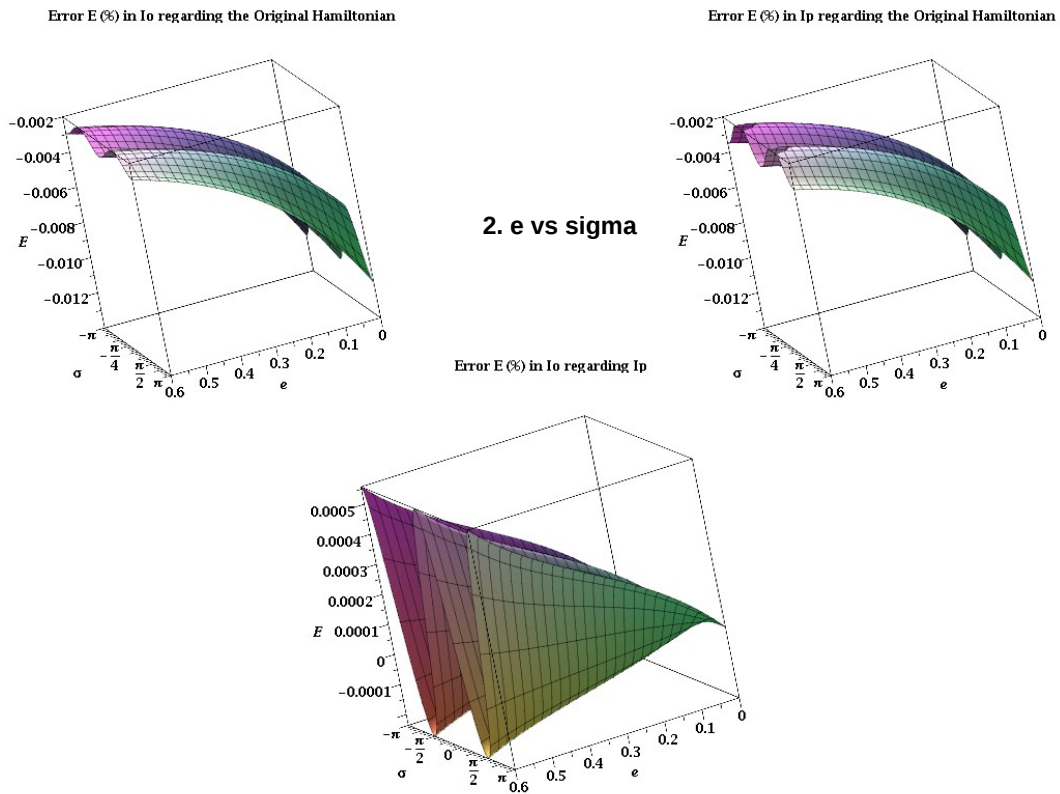
APPENDIX B: PHASE SPACE ANALYSIS OF ROTO-ORBITAL INTERMEDIARIES

In this appendix, it is presented a list of plots which provides comparisons in the phase space of three different Hamiltonians presented in chapter 5: The original Hamiltonian for an axisymmetric body (expression (129)), the intermediary $\mathcal{I}_o$ (expression (143)) and the intermediary $\mathcal{I}_p$ (expression (148)). The last two Hamiltonians were built with the purpose to approach the first one, meanwhile providing an analytical approximate solution for the dynamical system described by the first one, and which is not analytically integrable. Thus, different plots of errors between these Hamiltonians are presented here where, with exception of the variables varying in each axis, all the parameters are fixed as the same ones which generated the results presented in figure 57. The results shown in figures 82 to 90 present good accordance with the study developed in chapter 5.

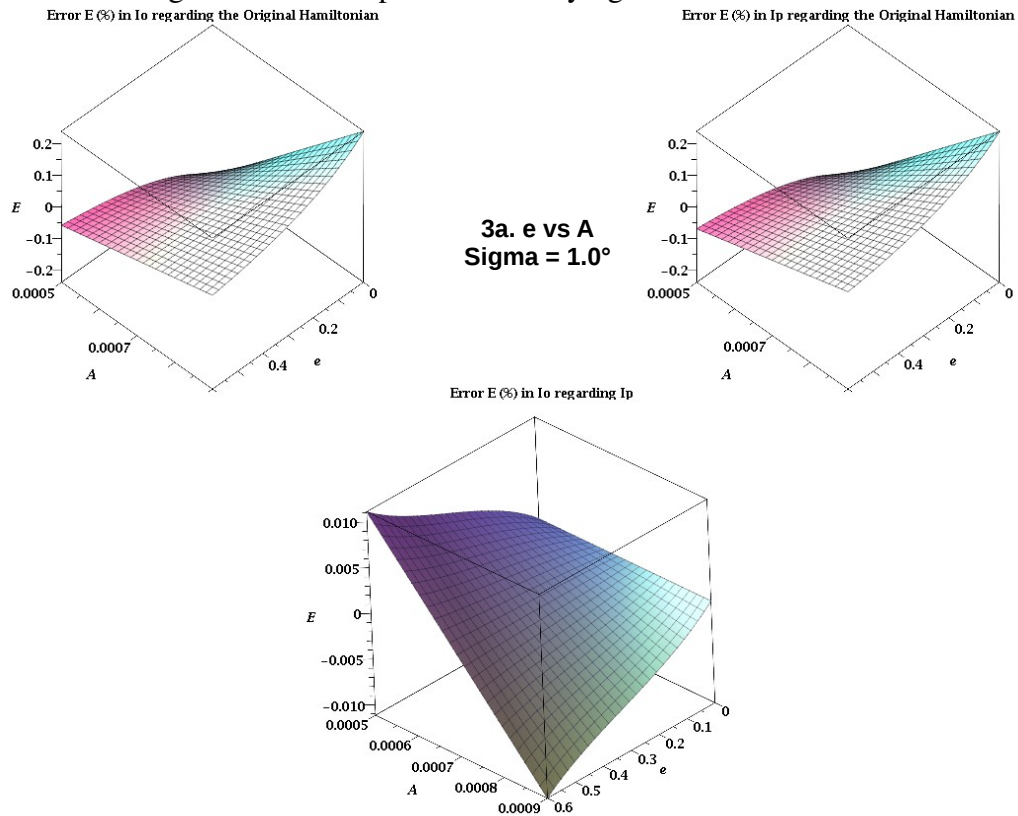
Figure 82: Phase space errors varying A and σ .



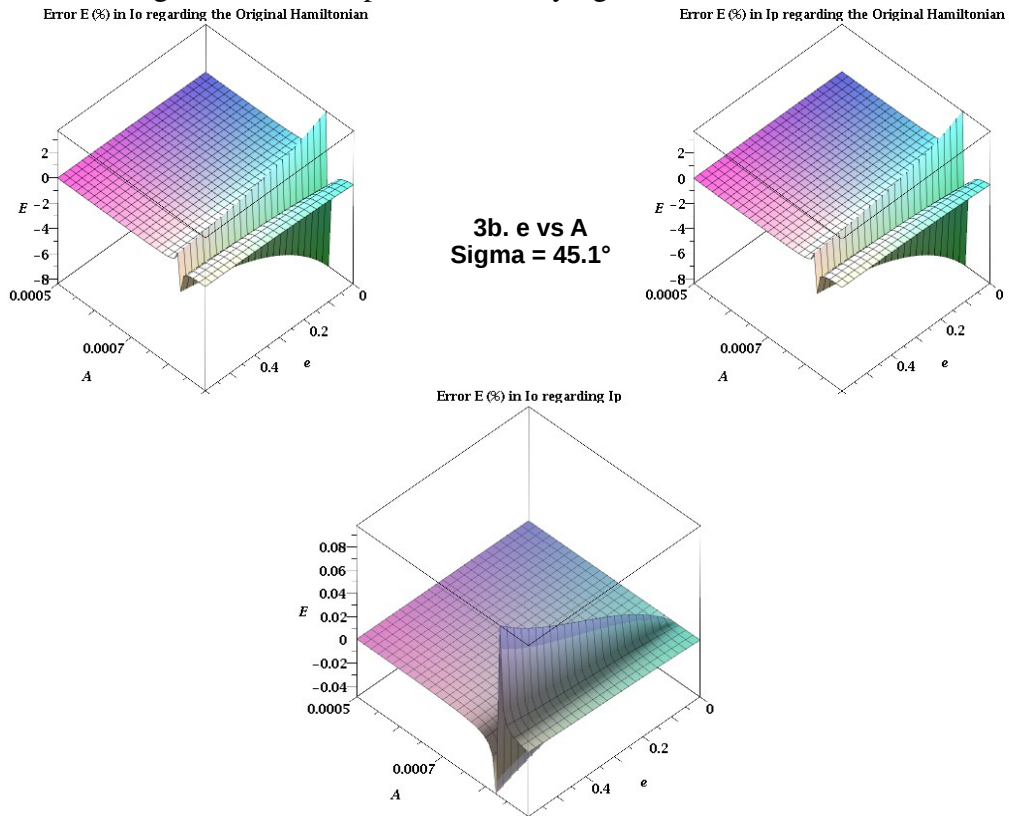
Source: Author

Figure 83: Phase space errors varying e and σ 

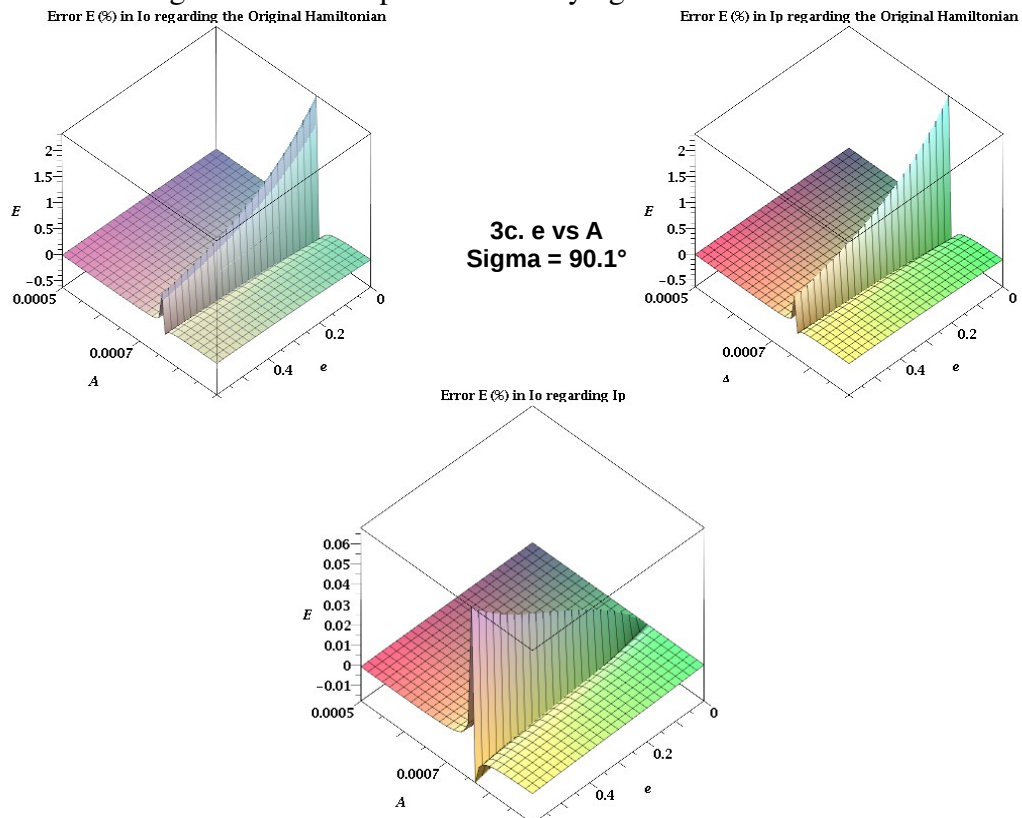
Source: Author

Figure 84: Phase space errors varying e and A for $\sigma = 1.0^\circ$ 

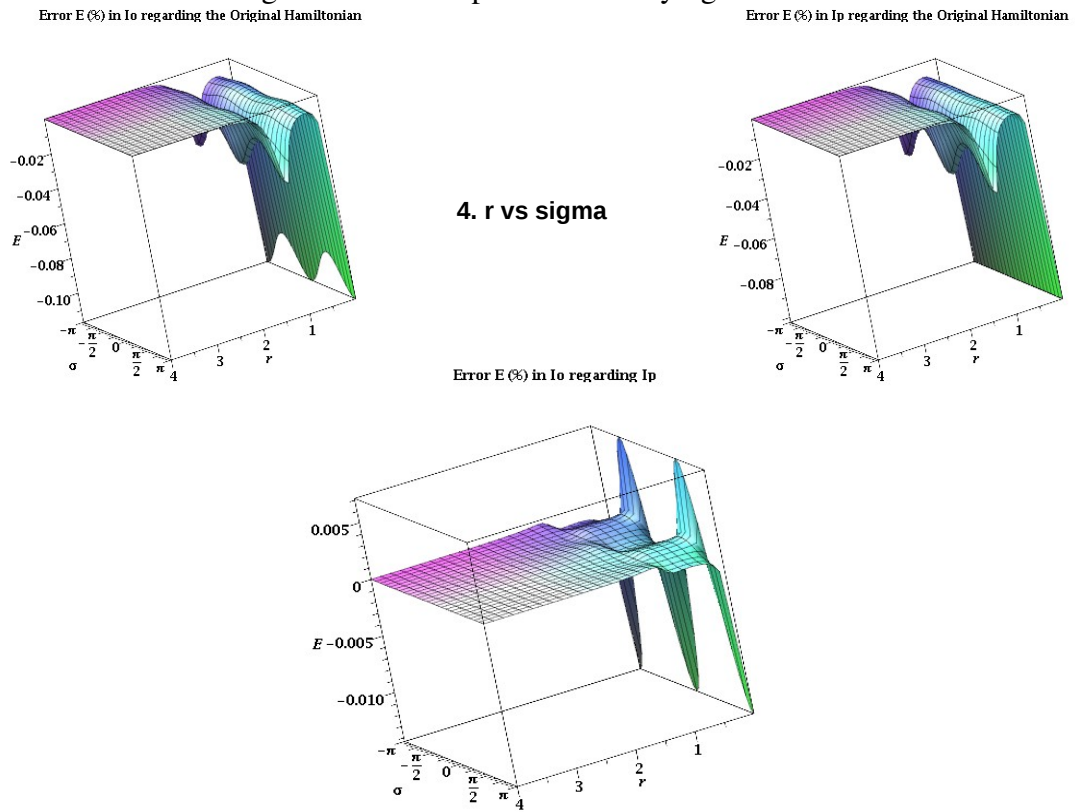
Source: Author

Figure 85: Phase space errors varying e and A for $\sigma = 45.1^\circ$ 

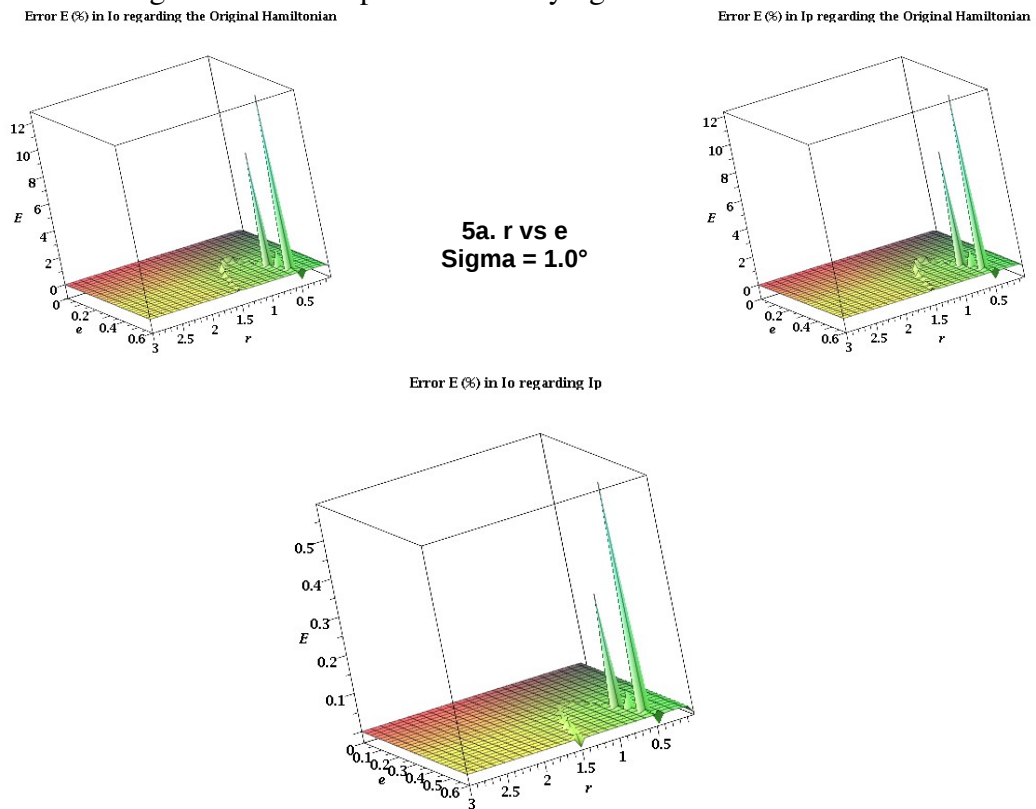
Source: Author

Figure 86: Phase space errors varying e and A for $\sigma = 90.1^\circ$ 

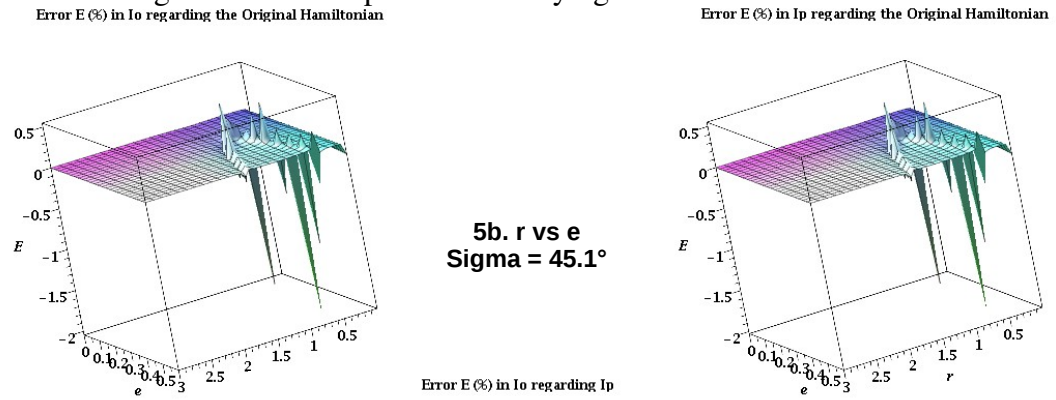
Source: Author

Figure 87: Phase space errors varying r and σ 

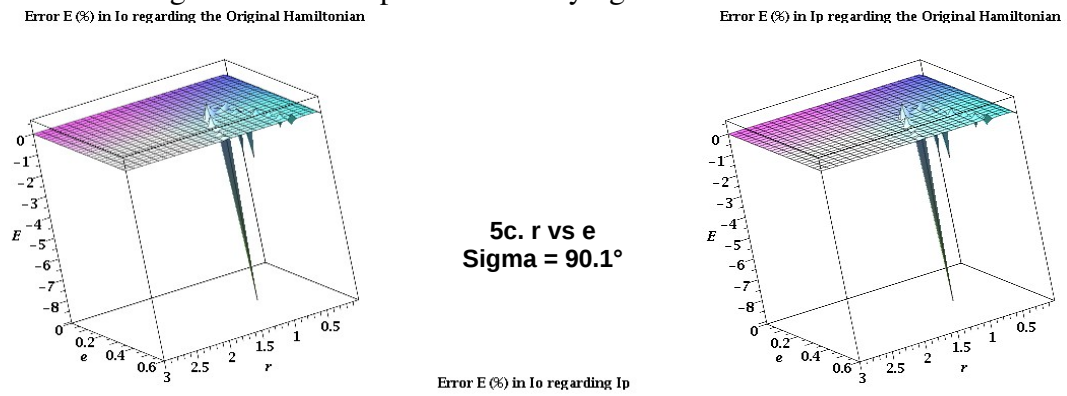
Source: Author

Figure 88: Phase space errors varying r and e for $\sigma = 1.0^\circ$ 

Source: Author

Figure 89: Phase space errors varying r and e for $\sigma = 45.1^\circ$ 

Source: Author

Figure 90: Phase space errors varying r and e for $\sigma = 90.1^\circ$ 

Source: Author