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Some Consequences of a Symmetry in Strong Distributions*

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Some polynomials and interpolatory quadrature rules associated with strong Stieltjes distributions are considered, especially when the distributions satisfy a certain symmetric property.

1. INTRODUCTION

Let the function $\psi(t)$, defined on (a, b), be real, bounded, and nondecreasing with infinitely many points of increase in (a, b), and such that the moments

$$\mu_m = \int_a^b t^m d\psi(t), \qquad m = 0, \pm 1, \pm 2, \ldots,$$

all exist. Then $d\psi(t)$ is called a strong distribution on (a, b).

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We refer to a strong distribution on (a, b) as a strong Stieltjes distribution on (a, b), or simply as a SS(a, b) distribution, when $(a, b) \subseteq (0, \infty)$. Given a SS$(a, b)$ distribution $d\psi(t)$ it is known, see for example [6], that

$$\int_a^b t^{-n+s} B_n(t)\, d\psi(t) = 0, \qquad 0 \le s \le n-1, n \ge 1, \tag{1.1}$$

defines a unique set of polynomials $B_n(z)$, taken to be monic, satisfying the three-term recurrence relation

$$B_n(z) = (z - \beta_n) B_{n-1}(z) - \alpha_n z B_{n-2}(z), \qquad n \ge 2, \tag{1.2}$$

with $B_0(z) = 1$, $B_1(z) = z - \beta_1$, and $\beta_n > 0$, $\alpha_{n+1} > 0$ for $n \ge 1$. For these results and others, but in terms of the equivalent orthogonal Laurent polynomials, see [4].

In [6], the polynomials $B_n(z)$ and the associated interpolatory quadrature rules have been investigated for a class of SS(a, b) distributions, denoted by ScS(a, b) distributions, which satisfy the symmetric (inverse) property

$$\frac{d\psi(t)}{\sqrt{t}} = -\frac{d\psi(c/t)}{\sqrt{c/t}}, \qquad t \in (a, b).$$

In this article we look at the real polynomials $B_n(\lambda, z)$ given by

$$B_n(\lambda, z) = B_n(z) - \lambda\, B_{n-1}(z) \qquad n \ge 1, \tag{1.3}$$

where $\lambda \in \Re$, and the associated interpolatory quadrature rules for a class of SS(a, b) distributions which possess the symmetric (inverse) property

$$d\psi(t) = -d\psi(c/t), \qquad t \in (a, b), \tag{1.4}$$

for $c > 0$. Just as in the ScS(a, b) distributions we must have $a = 0$ *iff* $b = \infty$ and, if $0 < a < b < \infty$ then $c = ab$. When $d\psi(t)$ can be given in the form $w(t)\, dt$ then $w(t)$ satisfies

$$t\, w(t) = (c/t)\, w(c/t), \qquad t \in (a, b).$$

For convenience we denote the class of distributions which satisfy (1.4) as S$\bar{c}$S(a, b) distributions.

LEMMA 1.1. *Let $d\psi(t)$ be a S$\bar{c}$S(a, b) distribution and let $f(t)$ be integrable with respect to $d\psi(t)$ on (a, b). Then*

$$\int_a^b f(t)\, d\psi(t) = \int_a^b f(c/t)\, d\psi(t).$$

The proof of this lemma follows from change of variable properties of Riemann–Stieltjes integrals and from the choice of c.

Some properties of the special $\mathrm{S\bar{c}S}(a,b)$ distributions, where $c = 1$, have been considered in [3]. The $\mathrm{S\bar{c}S}(a,b)$ distributions taking the form $w(t)\,dt$ have been studied in [8]. It was shown there that for any such distributions

$$\frac{z^n B_n(c/z)}{B_n(0)} = B_n(\alpha_{n+1}, z)$$
$$= B_n(z) - \alpha_{n+1}B_{n-1}(z)$$
$$= \{B_{n+1}(z) + \beta_{n+1}B_n(z)\}/z, \qquad n \geq 1, \tag{1.5}$$

and, further, if $\gamma_n = \beta_n + \alpha_{n+1}$, then

$$\frac{\gamma_{n+1}}{\gamma_n} = \frac{c}{\beta_n \beta_{n+1}}, \qquad n \geq 1, \tag{1.6}$$

with $\gamma_1 = c/\beta_1$. These results can easily be proved for all $\mathrm{S\bar{c}S}(a,b)$ distributions.

For a given $\mathrm{S\bar{c}S}(a,b)$ distribution $d\psi(t)$ it follows from Lemma 1.1 that $\mu_m = c^m \mu_{-m}$, $m \geq 0$. Consequently for the associated Hankel determinants

$$H_0^{(m)} = 1, \qquad H_n^{(m)} = \begin{bmatrix} \mu_m & \mu_{m+1} & \cdots & \mu_{m+n-1} \\ \mu_{m+1} & \mu_{m+2} & \cdots & \mu_{m+n} \\ \vdots & \vdots & & \vdots \\ \mu_{m+n-1} & \mu_{m+n} & \cdots & \mu_{m+2n-2} \end{bmatrix}, \qquad n \geq 1,$$

$m = 0, \pm 1, 2, \ldots$, the following holds:

$$H_n^{(m)} = c^{n(m+n-1)} H_n^{(-m-2n+2)}, \qquad m \geq 0, n \geq 0.$$

2. The Polynomials $B_n(\lambda, z)$

In this section $d\psi(t)$ is assumed to be any $\mathrm{SS}(a,b)$ distribution. In (1.3) the polynomials $B_n(\lambda, z)$ are defined by $B_n(z)$ in just the same way as the quasi-orthogonal polynomials (see, for example, [1]) are defined from orthogonal polynomials.

It follows from (1.1) and (1.3) that

$$\int_a^b t^{-n+s} B_n(\lambda, t)\, d\psi(t) = 0, \qquad 1 \leq s \leq n-1, n \geq 2. \tag{2.1}$$

A more general result is the following.

THEOREM 2.1. *Let $Q_n(z)$ be a real monic polynomial of degree $n \geq 2$ which satisfies the condition*

$$\int_a^b t^{-n+s} Q_n(t)\, d\psi(t) = 0, \qquad 1 \leq s \leq n-1.$$

Then there exists a $\lambda \in \Re$ such that $Q_n(z) = B_n(\lambda, z)$.

This result is easily proved by considering the linear combination of $Q_n(t)$ in terms of the polynomials $B_r(t)$, $r = 1, 2, \ldots, n$.

It can be established from (2.1) that the zeros of $B_n(\lambda, z)$ are all real and distinct and that at least $n-1$ of them lie inside (a, b). Let these zeros, in increasing order, be denoted by $z_{n,1}^{(\lambda)}, z_{n,2}^{(\lambda)}, \ldots, z_{n,n}^{(\lambda)}$. From (1.2) and (1.3), $B_n(\lambda, 0) = (-1)^n \beta_1 \ldots \beta_{n-1}(\beta_n + \lambda)$ and, hence, for example, we can say that if $\lambda > -\beta_n$ then $z_{n,1}^{(\lambda)} > 0$ while if $\lambda < -\beta_n$ then $z_{n,1}^{(\lambda)} < 0$.

THEOREM 2.2. *For $n \geq 1$, if $0 \leq \lambda \leq \alpha_{n+1}$ then all the zeros of $B_n(\lambda, z)$ lie inside the interval (a, b).*

Proof. It is known that the zeros of $B_n(0, z) = B_n(z)$ and $B_n(\alpha_{n+1}, z)$ lie inside (a, b). Thus, we obtain

$$(-1)^n B_n(0, a) > 0, \qquad (-1)^n B_n(\alpha_{n+1}, a) > 0, \qquad B_n(0, b) > 0, \qquad B_n(\alpha_{n+1}, b) > 0$$

for all $n \geq 1$. From this and from

$$B_n(\lambda, z) = \left(1 - \frac{\lambda}{\alpha_{n+1}}\right) B_n(0, z) + \frac{\lambda}{\alpha_{n+1}} B_n(\alpha_{n+1}, z),$$

we then have for $0 \leq \lambda \leq \alpha_{n+1}$

$$(-1)^n B_n(\lambda, a) > 0, \qquad B_n(\lambda, b) > 0, \qquad n \geq 1. \tag{2.2}$$

It is true that $z_{n,n-1}^{(\lambda)} < b$. Now, if $z_{n,n}^{(\lambda)} > b$ then we must have $B_n(\lambda, b) < 0$, which is a contradiction to (2.2). Hence, $z_{n,n}^{(\lambda)} < b$.

Similarly, we can see that $z_{n,1}^{(\lambda)} < a$ also leads to a contradiction of (2.2). Hence the theorem is proved.

From (1.2) and (1.3) it can be seen that when $n \geq 2$, the zeros of $B_n(\lambda, z)$ for any λ, are also the eigenvalues of the Hessenberg matrix

$$\mathbf{H}_n(\lambda) = \begin{bmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_{n-1} & \beta_n + \lambda \\ \alpha_2 & \gamma_2 & \cdots & \gamma_{n-1} & \beta_n + \lambda \\ 0 & \alpha_3 & \cdots & \gamma_{n-1} & \beta_n + \lambda \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \gamma_{n-1} & \beta_n + \lambda \\ 0 & 0 & \cdots & \alpha_n & \beta_n + \lambda \end{bmatrix}.$$

The proof of this for $\lambda = 0$ can be found in [7]. Now let

$$A_n(\lambda, z) = \int_a^b \frac{B_n(\lambda, z) - B_n(\lambda, t)}{z - t}\, d\psi(t), \qquad n \geq 1.$$

Then it follows from (2.1) that

$$A_n(\lambda, z) = \int_a^b t^{-p} \frac{t^p B_n(\lambda, z) - z^p B_n(\lambda, t)}{z - t}\, d\psi(t), \qquad n \geq 1, \tag{2.3}$$

where p is any integer such that $0 \leq p \leq n - 1$. Thus, in the relation

$$\frac{A_n(\lambda, z)}{B_n(\lambda, z)} = \sum_{r=1}^{n} \frac{w_{n,r}^{(\lambda)}}{z - z_{n,r}^{(\lambda)}}, \qquad n \geq 1,$$

we have

$$w_{n,r}^{(\lambda)} = \frac{A_n(\lambda, z_{n,r}^{(\lambda)})}{B_n'(\lambda, z_{n,r}^{(\lambda)})} = \frac{\{z_{n,r}^{(\lambda)}\}^p}{B_n'(\lambda, z_{n,r}^{(\lambda)})} \int_a^b \frac{t^{-p} B_n(\lambda, t)}{t - z_{n,r}^{(\lambda)}}\, d\psi(t), \qquad r = 1, 2, \ldots, n. \tag{2.4}$$

Now, by considering the interpolation of $t^{n-1} f(t)$ on the zeros of $B_n(\lambda, z)$ we obtain the quadrature rule

$$\int_a^b f(t)\, d\psi(t) = \sum_{r=1}^{n} w_{n,r}^{(\lambda)} f(z_{n,r}^{(\lambda)}) + \mathbf{E}_n(\lambda, f), \tag{2.5}$$

where $\mathbf{E}_n(\lambda, f) = 0$ for $t^{n-1} f(t) \in \mathbf{P}_{2n-2}$.

As $\mathbf{E}_n(\lambda, f) = 0$ whenever $f(t) \in \mathbf{P}_{n-1}$, the above is also the usual interpolatory quadrature rule. Taking $f(t) = t^{-n+1}\{B_n(\lambda, t)/(t - z_{n,r}^{(\lambda)})\}^2$, we find that $w_{n,r}^{(\lambda)}$ is positive. Moreover, taking $f(t) = 1$ we get $\sum_{r=1}^{n} w_{n,r}^{(\lambda)} = \mu_0$.

3. The $S\bar{c}S(a, b)$ Distributions and $B_n(\lambda, z)$

We assume throughout the rest of this article that $d\psi(t)$ is a $S\bar{c}S(a, b)$ distribution and that $\lambda \in \Re$ is such that $B_n(\lambda, 0) \neq 0$ (i.e., $\lambda \neq -\beta_n$).

In (2.1), by applying Lemma 1.1 we obtain

$$\int_a^b t^{-n+s} t^n B_n(\lambda, c/t)\, d\psi(t) = 0, \qquad 1 \leq s \leq n-1, n \geq 2.$$

Thus from Theorem 2.1, for each $\lambda \in \Re$ there exists an $\eta \in \Re$ such that

$$\frac{z^n B_n(\lambda, c/z)}{B_n(\lambda, 0)} = B_n(\eta, z), \qquad n \geq 1. \tag{3.1}$$

Here we have also included $n = 1$, which is easily seen to hold. Since $B_n(z) = B_n(0, z)$, we have from (1.3)

$$z^n B_n(\lambda, c/z) = z^n B_n(0, c/z) - \lambda z^n B_{n-1}(0, c/z), \qquad n \geq 1.$$

Thus from (1.5)

$$\begin{aligned} z^n B_n(\lambda, c/z) &= B_n(0,0)\{B_n(0, z) - \alpha_{n+1} B_{n-1}(0, z)\} \\ &-\lambda B_{n-1}(0,0)\{B_n(0, z) + \beta_n B_{n-1}(0, z)\}, \qquad n \geq 1. \end{aligned}$$

Regrouping again leads to the relation (3.1), where we now find

$$\eta = \frac{\alpha_{n+1} B_n(0,0) + \lambda\, \beta_n B_{n-1}(0,0)}{B_n(0,0) - \lambda\, B_{n-1}(0,0)}, \qquad n \geq 1.$$

Since $B_n(0, 0) = -\beta_n B_{n-1}(0, 0)$, the following result is established.

Theorem 3.1. *For any $n \geq 1$, if*

$$\eta = \frac{\beta_n(\alpha_{n+1} - \lambda)}{\beta_n + \lambda}, \tag{3.2}$$

then the polynomials $B_n(\lambda, z)$ and $B_n(\eta, z)$ are related to each other by the relation (3.1). *Furthermore, when $\lambda, \eta > -\beta_n$,*

$$z_{n,r}^{(\lambda)} = c/z_{n,n+1-r}^{(\eta)}, \qquad r = 1, 2, \ldots, n,$$

and when λ, $\eta < -\beta_n$,

$$z_{n,1}^{(\lambda)} = c/z_{n,1}^{(\eta)}, \qquad z_{n,r}^{(\lambda)} = c/z_{n,n+2-r}^{(\eta)}, \qquad r = 2, 3, \ldots, n.$$

The following is a special case of the above theorem.

THEOREM 3.2. *For any* $n \geq 1$, *if* λ *is equal to*

$$\hat{\lambda}_n = \sqrt{\beta_n}(\sqrt{\gamma_n} - \sqrt{\beta_n}) \qquad or \qquad \tilde{\lambda}_n = \sqrt{\beta_n}(-\sqrt{\gamma_n} - \sqrt{\beta_n})$$

then

$$\frac{z^n B_n(\lambda, c/z)}{B_n(\lambda, 0)} = B_n(\lambda, z). \tag{3.3}$$

In particular, $z_{n,r}^{(\hat{\lambda}_n)} = c/z_{n,n+1-r}^{(\hat{\lambda}_n)}$, $r = 1, 2, \ldots, n$, *and* $z_{n,1}^{(\tilde{\lambda}_n)} = -\sqrt{c}$, $z_{n,r}^{(\tilde{\lambda}_n)} = c/z_{n,n+2-r}^{(\tilde{\lambda}_n)}$, $r = 2, 3, \ldots, n$.

Proof. In Theorem 3.1 letting $\eta = \lambda$ we get $\lambda^2 + 2\beta_n\lambda - \beta_n\alpha_{n+1} = 0$. We take $\hat{\lambda}_n$ as the positive and $\tilde{\lambda}_n$ as the other solution of this quadratic equation. Since $\tilde{\lambda}_n \leq -\beta_n$, the polynomial $B_n(\tilde{\lambda}_n, z)$ has that one negative zero and (3.3) implies that this zero must be equal to $-\sqrt{c}$. This completes the proof.

EXAMPLE. We consider the shifted log-normal distribution

$$d\psi(t) = \frac{1}{2\kappa\sqrt{\pi}} t^{-1} e^{-(\ln(t)/2\kappa)^2}\, dt, \qquad t \in (0, \infty).$$

This distribution, which has also been considered in [3], is a $S\bar{c}S(0, \infty)$ distribution with $c = 1$. For this the coefficients of the recurrence relation (1.2) are

$$\beta_n = q^{1/2}, \qquad \alpha_{n+1} = q^{1/2}(q^{-n} - 1), \qquad n \geq 1,$$

where $q = e^{-2\kappa^2}$. It is interesting to note that this distribution is also a $ScS(0, \infty)$ distribution with $c = q$.

It is easily verified that these coefficients satisfy (1.6). By following reasonings similar to those of [2, 5], we obtain

$$B_n(z) = \sum_{r=0}^{n} (-1)^r q^{-r(n-r)} \begin{Bmatrix} n \\ r \end{Bmatrix}_q q^{r/2} z^{n-r}, \qquad n \geq 1,$$

where $\begin{Bmatrix} n \\ r \end{Bmatrix}_q$ are the q-binomial coefficients given by

$$\begin{Bmatrix} n \\ 0 \end{Bmatrix}_q = \begin{Bmatrix} n \\ n \end{Bmatrix}_q = 1, \qquad \begin{Bmatrix} n \\ r \end{Bmatrix}_q = \frac{\prod_{k=1}^{n}(1-q^k)}{\prod_{k=1}^{r}(1-q^k)\prod_{k=1}^{n-r}(1-q^k)}, \qquad 1 \le r \le n-1,$$

for $n \ge 0$. We can write

$$B_n(\lambda, z) = z^n + \sum_{r=1}^{n}(-1)^r q^{-r(n-r)}\begin{Bmatrix} n \\ r \end{Bmatrix}_q \left[1 + \lambda q^{n-r}\frac{(1-q^r)}{(1-q^n)}\right] q^{r/2} z^{n-r} \quad n \ge 1.$$

Hence, for $\lambda = \hat{\lambda}_n = q^{1/2}(q^{-n/2} - 1)$, we get

$$B_n(\hat{\lambda}_n, z) = \sum_{r=0}^{n}(-1)^r q^{-r(n-r)}\begin{Bmatrix} n \\ r \end{Bmatrix}_q \left[\frac{q^{r/2} + q^{(n-r)/2}}{1 + q^{n/2}}\right] z^{n-r}, \qquad n \ge 1$$

and for $\lambda = \tilde{\lambda}_n = q^{1/2}(-q^{-n/2} - 1)$, we get

$$B_n(\tilde{\lambda}_n, z) = \sum_{r=0}^{n}(-1)^r q^{-r(n-r)}\begin{Bmatrix} n \\ r \end{Bmatrix}_q \left[\frac{q^{r/2} - q^{(n-r)/2}}{1 - q^{n/2}}\right] z^{n-r}, \qquad n \ge 1.$$

From the symmetry of the coefficients of $B_n(\hat{\lambda}_n, z)$ and $B_n(\tilde{\lambda}_n, z)$ we see that these polynomials satisfy the relation (3.3).

4. The S$\bar{\text{c}}$S(a, b) Distributions and $w_{n,r}^{(\lambda)}$

From (3.1) we have

$$B_n'(\lambda, c/z) = \frac{nz}{c}B_n(\lambda, c/z) - \frac{z^{-n+2}}{c}B_n(\lambda, 0)\, B_n'(\eta, z), \qquad n \ge 1. \quad (4.1)$$

Now in (2.3) with $p = 0$, substituting c/u for t and then replacing z by c/z, we obtain

$$A_n(\lambda, c/z) = \frac{z}{c}\int_a^b u\frac{B_n(\lambda, c/z) - B_n(\lambda, c/u)}{u - z}\, d\psi(u).$$

Hence, from (3.1)

$$A_n(\lambda, c/z) = \frac{z^{-n+1}}{c} B_n(\lambda, 0) \int_a^b u^{-n+1} \frac{u^n B_n(\eta, z) - z^n B_n(\eta, u)}{u - z} d\psi(u).$$

Writing $u^n B_n(\eta, z) - z^n B_n(\eta, u)$ as $zu^{n-1} B_n(\eta, z) - z^n B_n(\eta, u) + u^{n-1}(u - z) B_n(\eta, z)$, immediately gives

$$A_n(\lambda, c/z) = \frac{z^{-n+1}}{c} B_n(\lambda, 0)[\mu_0 B_n(\eta, z) - z A_n(\eta, z)], \qquad n \geq 1. \quad (4.2)$$

THEOREM 4.1. *For any* $\mathrm{S\bar{c}S}(a, b)$ *distribution* $d\psi(t)$, *if* $\lambda, \eta > -\beta_n$ *and they satisfy* (3.2), *then the weights of the quadrature rule* (2.5) *satisfy*

$$w_{n,r}^{(\lambda)} = w_{n,n+1-r}^{(\eta)}, \qquad r = 1, 2, \ldots, n, n \geq 1.$$

In particular,

$$w_{n,r}^{(\hat{\lambda}_n)} = w_{n,n+1-r}^{(\hat{\lambda}_n)}, \qquad r = 1, 2, \ldots, n, n \geq 1.$$

Proof. In (4.1) and (4.2) substituting $z = z_{n,n+1-r}^{(\eta)}$ (i.e., by Theorem 3.1, $c/z = z_{n,r}^{(\lambda)}$), and then using (2.4) immediately gives the required result.

Similarly, we can also obtain the following.

THEOREM 4.2. *For any* $\mathrm{S\bar{c}S}(a, b)$ *distribution* $d\psi(t)$, *if* $\lambda, \eta < -\beta_n$ *and they satisfy* (3.2), *then the weights of the quadrature rule* (2.5) *satisfy*

$$w_{n,1}^{(\lambda)} = w_{n,1}^{(\eta)}, \qquad w_{n,r}^{(\lambda)} = w_{n,n+2-r}^{(\eta)}, \qquad r = 2, 3, \ldots, n, n \geq 2.$$

In particular,

$$w_{n,r}^{(\check{\lambda}_n)} = w_{n,n+2-r}^{(\check{\lambda}_n)}, \qquad r = 2, 3, \ldots, n, n \geq 1.$$

We now define the step functions $\psi_n(t)$, $n \geq 1$, on the interval (a, b) by

$$\psi_n(t) = \begin{cases} 0 & a < t \leq z_{n,1}^{(\hat{\lambda})}, \\ \sum_{s=1}^{r} w_{n,s}^{(\hat{\lambda})}, & z_{n,r}^{(\hat{\lambda})} < t \leq z_{n,r+1}^{(\hat{\lambda})}, r = 1, 2, \ldots, n-1. \\ \mu_0, & z_{n,n}^{(\hat{\lambda})} < t < b, \end{cases}$$

Then Theorems 3.2 and 4.1 tell us that the distributions given by $d\psi_n(t)$ are all S$\bar{c}$S(a, b) distributions.

EXAMPLE. We consider the distribution

$$d\bar{\psi}(t) = \frac{1 + \sqrt{ab/t}}{\sqrt{b-t}\sqrt{t-a}}\, dt, \qquad t \in (a, b),$$

where $0 < a < b < \infty$. This distribution, which was the principle object of study in [8], is a S$\bar{c}$S(a, b) distribution with $c = ab$.

THEOREM 4.3. *For the distribution $d\bar{\psi}(t)$ we have for $n \geq 1$*

$$z_{n,n+1-r}^{(\hat{\lambda}_n)} = (\beta + \alpha v_{n,r}) + \sqrt{(\beta + \alpha v_{n,r})^2 - \beta^2}, \qquad z_{n,r}^{(\hat{\lambda}_n)} = c/z_{n,n+1-r}^{(\hat{\lambda}_n)},$$

for $r = 1, 2, \ldots, [(n+1)/2]$ and

$$w_{n,r}^{(\hat{\lambda}_n)} = \frac{2\pi}{n}, \qquad \text{for } r = 1, 2, \ldots, n.$$

Here,

$$v_{n,r} = 1 + \cos((2r-1)\pi/n), \beta = \sqrt{ab}, \text{ and } \alpha = (\sqrt{b} - \sqrt{a})^2/4.$$

Proof. Clearly these results satisfy the required conditions given by Theorems 3.2 and 4.1. First we consider the polynomials $\hat{B}_n(z)$, $n \geq 1$, defined by

$$\int_a^b t^{-n+2}\, \hat{B}_n(t) \frac{1}{\sqrt{b-t}\sqrt{t-a}}\, dt = 0, \qquad 0 \leq s \leq n-1, n \geq 1. \quad (4.3)$$

The distribution in (4.3) is a ScS(a, b) distribution. The polynomials $\hat{B}_n(z)$ have been studied in some detail in [6], where it was shown that

$$\frac{z^n \hat{B}_n(ab/z)}{\hat{B}_n(0)} = \hat{B}_n(z), \qquad n \geq 1. \quad (4.4)$$

Now in (4.3), if we substitute t by ab/t and then use the above relation, we obtain

$$\int_a^b t^{-n+s}\, \hat{B}_n(t) \frac{\sqrt{ab/t}}{\sqrt{b-t}\sqrt{t-a}}\, dt = 0, \qquad 1 \leq s \leq n, n \geq 1.$$

Adding this with (4.3) gives

$$\int_a^b t^{-n+2}\,\hat{B}_n(t)\,d\bar{\psi}(t) = 0, \qquad 1 \leq s \leq n-1,\ n \geq 1,$$

which together with (4.4) yields

$$\hat{B}_n(z) = B_n(\hat{\lambda}_n, z), \qquad n \geq 1.$$

Here, $B_n(\hat{\lambda}_n, z)$ are the polynomials given by Theorem 3.2 in relation to the $\mathrm{S\bar{c}S}(a, b)$ distribution $d\bar{\psi}(t)$.

The proof of the theorem is completed by using a result given in [9].

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