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On the higher order exterior and interior Whitehead products

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Abstract. We extend the notion of the exterior Whitehead product for maps $\alpha_i\colon \Sigma A_i \to X_i$ for $i = 1, \ldots, n$, where ΣA_i is the reduced suspension of A_i and then, for the interior product with $X_i = J_{m_i}(X)$, the m_ith-stage of the James construction $J(X)$ as well. The main result stated in Theorem 4.10 generalizes (Hardie in Q J Math Oxford Ser 12(2):196–204, 1961, Theorem 1.10) and concerns to the Hopf invariant of the generalized Hopf construction. We close the paper applying Gray's construction $\circ$ (called the Theriault product) to a sequence $X_1, \ldots, X_n$ of simply connected co-H-spaces to obtain a higher Gray–Whitehead product map

$$w_n\colon \Sigma^{n-2}(X_1 \circ \cdots \circ X_n) \to T_1(X_1, \ldots, X_n),$$

where $T_1(X_1, \ldots, X_n)$ is the fat wedge of $X_1, \ldots, X_n$.

1. Introduction

Whitehead products play an important role in algebraic topology and its applications. Higher Whitehead products are secondary, tertiary, etc. analogues of ordinary Whitehead products. They first appeared in the late 1960's as a part of a research theme studying higher products, or simply higher structures, in homotopy theory. They are vital for understanding the homotopy theory of certain basic objects, such as an iterated product of spaces, and their maps into other spaces. However, as indeterminacy is involved they are very tricky to work with. Recently, higher Whitehead products have re-emerged as key players in the homotopy theory of polyhedral products. These are important objects in toric topology and are being increasingly used in geometric group theory and graph theory.

Porter [20] has generalized Hardie's construction from [14] and introduced the notion of the nth order generalized Whitehead product of maps $\alpha_i\colon \Sigma A_i \to X$ for

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$i = 1, \ldots, n$, with $n \geq 2$ which is very useful in many mathematical constructions. For example, given a simplicial complex K on n vertices, Davis and Januszkiewicz [8] have associated two fundamental objects of toric topology: the moment–angle complex $\mathcal{Z}_K$ and the Davis–Januszkiewicz space DJ_K. The homotopy fibration sequence

$$\mathcal{Z}_K \stackrel{\tilde{\omega}}{\to} DJ_K \to \prod_{i=1}^{n} \mathbb{C}P^\infty$$

and its generalization have been studied in [11,15], respectively to show that $\tilde{\omega}\colon \mathcal{Z}_K \to DJ_K$ is a sum of higher and iterated Whitehead products for appropriate complexes K.

Next, let $\mathcal{F}(\mathbb{R}^{n+1}, m)$ be the Euclidean ordered configuration space. By Salvatore [26, Theorem 7], the homotopy type of $\mathcal{F}(\mathbb{R}^{n+1}, m)$ for $n \geq 2$ admits a minimal cellular model

$$* = X_0 \subseteq X_n \subseteq X_{2n} \subseteq \cdots \subseteq X_{mn}$$

whose cells are attached via higher order Whitehead products.

Hardie [14] has made use of the reduced product spaces $J_m(X)$, defined by James in [16], to study the interior Whitehead product of maps $\alpha_i\colon \mathbb{S}^{k_i} \to J_{m_i}(X)$ with $k_i, m_i \geq 1$ for $i = 1, \ldots, n$ as an element $\langle \alpha_1, \ldots, \alpha_n \rangle \in \pi_{k-1}(J_m(X))$, where $k = k_1 + \cdots + k_n$ and $m = m_1 + \cdots + m_n - \min\limits_{1 \leq i \leq n} \{m_i\}$. In addition, by means of the generalized Hopf construction from [13], for a given map $F\colon \mathbb{S}^{k_1} \times \cdots \times \mathbb{S}^{k_n} \to J(X)$ strongly of type $(\alpha_1, \ldots, \alpha_n)^{m-1}$, Hardie has defined an element $c(F) \in \pi_{k+1}(\Sigma X)$ and in particular, an element of order p in $\pi_{2p}(\mathbb{S}^3)$ analysed in [14].

The main result on the triple spherical Whitehead product from [12] has been generalized in [9] into suspensions. After necessary prerequisites exhibited in Sect. 2, we extend the notion of the exterior Whitehead product for maps $\alpha_i\colon \Sigma A_i \to X_i$ for $i = 1, \ldots, n$, where ΣA_i is the reduced suspension of A_i and then, for the interior product with $X_i = J_{m_i}(X)$ as well. Next, some properties of these products are presented in Sect. 3.

James has shown [17] that, with the exception of the toric constructions of Toda [28], the usual procedures for the construction of generators of homotopy groups of spheres give rise to no more elements that can be obtained by the Hopf construction together with the operation of composition (the Toda bracket). We follow Hardie [13] to adapt the necessary results on the generalized Hopf construction. First, we list in Sect. 4.1 some properties of the separation element $d(u, v)\colon \Sigma A \to Y$ of maps $u, v\colon C_f \to Y$ defined on the mapping cone C_f and studied in [30], for a given map $f\colon A \to X$. Then, the main result stated in Theorem 4.10 generalizes [13, Theorem 1.10] and concerns to the Hopf invariant of the generalized Hopf construction.

Recently, Gray has defined in [10] a functor $\circ$ (called the Theriault product) in the category $\mathcal{CO}$ of simply-connected co-H-spaces and co-H-maps, and also a natural transformation $X \circ Y \to X \vee Y$ generalizing the Whitehead product map.

In Sect. 5, we first make use of Gray's construction [10] to a sequence $X_1, \ldots, X_n$ of simply-connected co-H-spaces and a fiber sequence from [21] (as in the paper [11]) to obtain a higher Gray–Whitehead product map $w_n\colon \Sigma^{n-2}(X_1 \circ \cdots \circ X_n) \to T_1(X_1, \ldots, X_n)$, where $T_1(X_1, \ldots, X_n)$ is the fat wedge of $X_1, \ldots, X_n$. The map w_n is used to introduce higher order Whitehead product for maps defined on co-H-spaces. Its basic properties and applications extending those from [10] will be presented in a forthcoming paper.

Finally, applications of spherical interior Whitehead product on spheres and projective spaces are presented. We discuss some connections, via the interior Whitehead product, of the James construction with the symmetric product and then with projective spaces $\mathbb{F}P^n$, for $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ as well.

2. James construction and Hopf invariant

In this work all spaces are based, have the homotopy type of a CW-complex and we do not distinguish between a based map and its homotopy class. Given a well-pointed Hausdorff space $(X, *)$, the James construction $J(X)$ is the free associative monoid on X with $*$ as unit. More precisely, from [16], for each $n \geq 1$ let $J_n(X)$ be the quotient of $X \times \overset{\times n}{\cdots} \times X$, where $(x_1, \ldots, x_n) \sim (x'_1, \ldots, x'_n)$ provided they are equal after removing any occurrence of $*$. Then, $J(X) = \varinjlim J_n(X)$, where $X = J_1(X) \subseteq J_2(X) \subseteq \cdots \subseteq J_n(X) \subseteq J_{n+1}(X) \subseteq \cdots$ is the James filtration. Given $f\colon X \to Y$, there are maps $J_n(f)\colon J_n(X) \to J_n(Y)$ for $n \geq 1$ and $J(f)\colon J(X) \to J(Y)$. Further, there are natural multiplication maps

$$\mu_{m,n}(X)\colon J_m(X) \times J_n(X) \to J_{m+n}(X)$$

and

$$\mu(X)\colon J(X) \times J(X) \to J(X)$$

defined by the juxtaposition.

Let $\eta_X\colon X \to \Omega\Sigma X$ and $\varepsilon_X\colon \Sigma\Omega X \to X$ be the canonical maps determined by the pair of adjoint functors Σ and Ω.

By [32, Chapter VII], there is a canonical multiplicative extension

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \Omega\Sigma X \\ \big\downarrow & & \big\downarrow \simeq \\ J(X) & \overset{\bar{\phi}_X}{\dashrightarrow} & \Omega'\Sigma X, \end{array}$$

where $\Omega' X$ denotes the Moore loop space of X. Consequently, there is a homotopy equivalence $\phi_X\colon J(X) \xrightarrow{\simeq} \Omega\Sigma X$ with the adjoint map $\psi_X\colon \Sigma J(X) \to \Sigma X$. Writing $\pi(X, Y)$ for the set of homotopy classes of maps from X to Y, we have a commutative diagram

$$\begin{array}{ccc} \pi(A, J(X)) & \xrightarrow{\phi_{X*}} & \pi(A, \Omega\Sigma X) \\ \Sigma \big\downarrow & & \approx \big\downarrow \mathrm{Adj} \\ \pi(\Sigma A, \Sigma J(X)) & \xrightarrow{\psi_{X*}} & \pi(\Sigma A, \Sigma X) \end{array}$$

for any pointed space A. Also, by [16] there is a homotopy equivalence

$$\Sigma\Omega\Sigma X \xrightarrow{\simeq} \bigvee_{m\geq 1} \Sigma X^{\wedge m},$$

where $X^{\wedge m}$ denotes the iterated smash product $X \wedge \overset{\times m}{\cdots} \wedge X$. Then, for each $m \geq 1$ consider the composition

$$\Sigma\Omega\Sigma X \xrightarrow{\simeq} \bigvee_{m\geq 1} \Sigma X^{\wedge m} \xrightarrow{p_m} \Sigma X^{\wedge m}$$

where $p_m\colon \bigvee_{m\geq 1} \Sigma X^{\wedge m} \to \Sigma X^{\wedge m}$ is the projection map.

By the adjointness, we obtain the *mth Hopf–James invariant*

$$\mathcal{H}_m\colon \Omega\Sigma X \to \Omega\Sigma X^{\wedge m}$$

which induces (again by the adjointness) a map

$$h_m\colon \pi(\Sigma A, \Sigma X) \to \pi(\Sigma A, \Sigma X^{\wedge m}) \tag{1}$$

for any pointed space A.

Recall that:

1. $\varepsilon_{\Sigma X}\Sigma\eta_X = \mathrm{id}_{\Sigma X}$;
2. $\psi_X = \varepsilon_{\Sigma X}\Sigma\phi_X$;
3. $\phi_X = (\Omega\psi_X)\eta_{J(X)}$;
4. $\phi_X j_X = \eta_X$;
5. $\psi_X\Sigma j_X = \mathrm{id}_{\Sigma X}$

for the embedding map $j_X\colon X \hookrightarrow J(X)$.

Now, let $q\colon J(X) \to J(X^{\wedge m})$ be the combinatorial extension (see e.g., [7] for its construction) of the quotient map $q_m\colon (J_m(X), J_{m-1}(X)) \to (X^{\wedge m}, *)$ which collapses $J_{m-1}(X)$ to $*$ for $m \geq 1$. Given a map $f\colon \Sigma A \to \Sigma X$, we write $\tilde{f} = (\Omega f)\eta_A$ and get the following formula for the Hopf–James invariant:

$$h_m(f) = \varepsilon_{\Sigma X^{\wedge m}}\Sigma(\phi_{X^{\wedge m}} q\phi_X^{-1})\Sigma(\Omega f)\Sigma\eta_A = \psi_{X^{\wedge m}}\Sigma(q\phi_X^{-1})\Sigma\tilde{f}. \tag{2}$$

3. Generalized exterior and interior products

Given the spaces $X_1, \ldots, X_n$, write $\underline{X} = (X_1, \ldots, X_n)$. Let $T_i(\underline{X})$ be the subspace of $X_1 \times \cdots \times X_n$ consisting of those points with at least i coordinates at base points. In particular, $T_0(\underline{X}) = X_1 \times \cdots \times X_n$ and $T_{n-1}(\underline{X}) = X_1 \vee \cdots \vee X_n$. Denote by $\Lambda(\underline{X})$ the smash product $X_1 \wedge \cdots \wedge X_n$. Notice that for maps $\alpha_i\colon X_i \to Y_i$ with $i = 1, \ldots, n$, the induced map $\alpha_1 \times \cdots \times \alpha_n\colon T_0(\underline{X}) \to T_0(\underline{Y})$ restricts to maps

$$T_i(\underline{\alpha})\colon T_i(\underline{X}) \to T_i(\underline{Y})$$

for $i = 1, \ldots, n$, where $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n)$.

Given a space X, write CX for the cone of X and notice that there is a canonical embedding $\iota_X : X \hookrightarrow CX$. Further, for any map $f : A \to X$, we have a cofibration $X \hookrightarrow C_f \to \Sigma A$, where C_f is the mapping cone of f.

According to [20, Theorem (2.3)] there is a (up to homotopy) pushout

$$\begin{array}{ccc} \Sigma^{n-1}\Lambda(\underline{A}) & \xrightarrow{\omega_n} & T_1(\underline{\Sigma A}) \\ \big\downarrow & & \big\downarrow \\ C\Sigma^{n-1}\Lambda(\underline{A}) & \xrightarrow{\Omega_n} & T_0(\underline{\Sigma A}). \end{array} \tag{3}$$

Hence, we have the cofibre sequence

$$\Sigma^{n-1}\Lambda(\underline{A}) \xrightarrow{\omega_n} T_1(\underline{\Sigma A}) \hookrightarrow T_0(\underline{\Sigma A})$$

and write

$$\nu_{\omega_n} : C_{\omega_n} \xrightarrow{\simeq} T_0(\underline{\Sigma A})$$

for a homotopy equivalence. This yields the commutative (up to homotopy) square

$$\begin{array}{ccc} C\Sigma^{n-1}\Lambda(\underline{A}) & \xrightarrow{\Omega_n} & T_0(\underline{\Sigma A}) \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle s} \\ \Sigma^{n}\Lambda(\underline{A}) & \xrightarrow{\lambda} & \Lambda(\underline{\Sigma A}), \end{array} \tag{4}$$

where $\lambda : \Sigma^n\Lambda(\underline{A}) \xrightarrow{\simeq} \Lambda(\underline{\Sigma A})$ is a homotopy equivalence.

Definition 3.1. The *exterior Whitehead product* $\{\alpha_1, \ldots, \alpha_n\}$ of maps $\alpha_i : \Sigma A_i \to X_i$ for $i = 1, \ldots, n$ with $n \geq 2$ is the composition

$$T_1(\underline{\alpha})\omega_n : \Sigma^{n-1}\Lambda(\underline{A}) \to T_1(\underline{X}).$$

If $A_i = \mathbb{S}^{m_i}$ is the m_i-sphere with $m_i \geq 1$ for $i = 1, \ldots, n$ then $\{\alpha_1, \ldots, \alpha_n\}$ has been defined by Hardie in [14]. In the sequel, we refer to such a product as the *spherical* one.

From the above, we derive that $\{\alpha_1, \ldots, \alpha_n\} = 0$ if and only if there is a map $T' : T_0(\underline{\Sigma A}) \to T_1(\underline{X})$ such that the triangle

$$\begin{array}{ccc} T_1(\underline{\Sigma A}) & \xrightarrow{T_1(\underline{\alpha})} & T_1(\underline{X}) \\ \big\downarrow & \nearrow{\scriptstyle T'} & \\ T_0(\underline{\Sigma A}) & & \end{array}$$

commutes (up to homotopy).

We present below some further straightforward properties of the exterior Whitehead product and follow [14] to generalize the interior one. First, in view of

[20, Definition (2.10)], we say that two maps $f, g : T_1(\underline{\Sigma A}) \to X$ are *compatible off* the ith coordinate if they coincide on $T_0^{(i)}(\underline{\Sigma A})$, where $T_0^{(i)}(\underline{\Sigma A}) = \Sigma A_1 \times \cdots \times \Sigma A_{i-1} \times \Sigma A_{i+1} \times \cdots \times \Sigma A_n$ is canonically embedded into $T_1(\underline{\Sigma A})$ for $i = 1, \ldots, n$.

In addition, if A_{i_0} is a co-H-group with a comultiplication $\nu_{i_0} : A_{i_0} \to A_{i_0} \vee A_{i_0}$ then we follow [20, Definition (2.11)] to define:

$$
\begin{aligned}
&(f +^{(i_0)} g)((t_1, a_1), \ldots, (t_{i_0}, a_{i_0}), \ldots, (t_n, a_n)) \\
&\quad = \begin{cases} f((t_1, a_1), \ldots, (t_{i_0}, a'_{i_0}), \ldots, (t_n, a_n)), & \text{if } \nu_{i_0}(a_{i_0}) = (a'_{i_0}, *); \\ g((t_1, a_1), \ldots, (t_{i_0}, a''_{i_0}), \ldots, (t_n, a_n)), & \text{if } \nu_{i_0}(a_{i_0}) = (*, a''_{i_0}). \end{cases}
\end{aligned}
$$

Suppose that there are maps $T_1(\underline{\alpha}'), T_1(\underline{\alpha}'') : T_1(\underline{\Sigma A}) \to T_1(\underline{X})$ with $\underline{\alpha}' = (\alpha_1, \ldots, \alpha'_{i_0}, \ldots, \alpha_n)$ and $\underline{\alpha}'' = (\alpha_1, \ldots, \alpha''_{i_0}, \ldots, \alpha_n)$ which are clearly compatible off the i_0th coordinate. Then, $T_1(\underline{\alpha}') +^{(i_0)} T_1(\underline{\alpha}'')$ is defined and by means of an appropriate version of [20, Theorem (2.13)], we can state:

Proposition 3.2. *If $\alpha_i : \Sigma A_i \to X_i$ are maps for $i = 1, \ldots, n$ and A_{i_0} is a co-H-group for some $1 \le i_0 \le n$ then the exterior Whitehead product satisfies:*

(1) $\{\alpha_1, \ldots, \alpha'_{i_0}, \ldots, \alpha_n\} + \{\alpha_1, \ldots, \alpha''_{i_0}, \ldots, \alpha_n\} = \{\alpha_1, \ldots, \alpha'_{i_0} + \alpha''_{i_0}, \ldots, \alpha_n\}$*;*
(2) $\{\alpha_1, \ldots, -\alpha_{i_0}, \ldots, \alpha_n\} = -\{\alpha_1, \ldots, \alpha_n\}$.

Certainly, item (2) above is easily deduced from item (1).

We note that Proposition 3.2 has been shown in [14] for the spherical case using the star product $\star$ studied in [6].

Let $\theta_i : X_i \hookrightarrow T_2(\underline{X})$ and $\Psi_1^{(i)} : T_1^{(i)}(\underline{X}) \hookrightarrow T_2(\underline{X})$ for $i = 1, \ldots, n$ be the canonical embeddings. Given $\alpha_i : \Sigma A_i \to X_i$, we can consider the compositions $\theta_i \alpha_i$ and $\Psi_1^{(i)}\{\alpha_1, \ldots, \alpha_n\}^{(i)}$, explicitly defined by

$$
\Sigma A_i \xrightarrow{\alpha_i} X_i \xhookrightarrow{\theta_i} T_2(\underline{X})
$$

and

$$
\Sigma^{n-2}\Lambda^{(i)}(\underline{A}) \xrightarrow{\omega_n^{(i)}} T_1^{(i)}(\underline{\Sigma A}) \xrightarrow{T_1^{(i)}(\underline{\alpha})} T_1^{(i)}(\underline{X}) \xhookrightarrow{\Psi_1^{(i)}} T_2(\underline{X})
$$

for $i = 1, \ldots, n$, respectively.

Remark 3.3. In the sequel, given a permutation $\sigma \in S_n$ of the set $\{1, \ldots, n\}$ we write

$$
\hat{\sigma} : A_1 \wedge \cdots \wedge A_n \xrightarrow{\approx} A_{\sigma(1)} \wedge \cdots \wedge A_{\sigma(n)}
$$

for the associated homeomorphism.

Following the results from [14, Sect. 2] on the spherical exterior Whitehead product and the generalized [9, Lemma 4.1] boundary Nakaoka–Toda operation formula [19, Lemma (1.2)], we may state:

Proposition 3.4. (1) *If* $\sigma \in S_n$ *is a permutation then*

$$(\Sigma^{n-1}\hat{\sigma})^*\{\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(n)}\} = \bar{\sigma}\{\alpha_1, \ldots, \alpha_n\}$$

where $\bar{\sigma} : T_1(\underline{X}) \xrightarrow{\approx} T_1(\sigma\underline{X})$ *is the homeomorphism induced by* σ.

(2) *The exterior Whitehead product satisfies the Jacobi identity*

$$\sum_{i=1}^{n}(\Sigma^{n-2}\hat{\sigma}_i)^*[\theta_i\alpha_i, \Psi_1^{(i)}\{\alpha_1, \ldots, \alpha_n\}^{(i)}] = 0$$

in $\pi(\Sigma^{n-2}\Lambda(\underline{A}), T_2(\underline{X}))$ *where* $\hat{\sigma}_i : A_i \wedge \Lambda^{(i)}(\underline{A}) \xrightarrow{\approx} \Lambda(\underline{A})$ *is induced by an appropriate* $\sigma_i \in S_n$ *for* $i = 1, \ldots, n$.

(3) *If* $f_i : X_i \to Y_i$ *are maps for* $i = 1, \ldots, n$ *then*

$$\{f_1\alpha_1, \ldots, f_n\alpha_n\} = T_1(\underline{f})\{\alpha_1, \ldots, \alpha_n\},$$

where $T_1(\underline{f}) : T_1(\underline{X}) \to T_1(\underline{Y})$.

(4) *Let* $\beta_i : B_i \to A_i$ *and* $\alpha_i : \Sigma A_i \to X_i$ *be any maps for* $i = 1, \ldots, n$. *Then, the following Whitehead identity holds* (cf. [14, (2.4)] *and* [31, (3.59)])*:*

$$\{\alpha_1\Sigma\beta_1, \ldots, \alpha_n\Sigma\beta_n\} = \{\alpha_1, \ldots, \alpha_n\}\Sigma^{n-1}(\beta_1 \wedge \cdots \wedge \beta_n).$$

The Jacobi identity stated in Proposition 3.4(2) has been also considered in [15, Corollary 1.9] with a different approach.

Proposition 3.5. *Let* $\alpha_i : \Sigma A_i \to X_i$ *be maps for* $i = 1, \ldots, n$.

(1) *If* $\alpha_{i_0} = 0$ *for some* $1 \leq i_0 \leq n$ *then*

$$\{\alpha_1, \ldots, \alpha_{i_0-1}, 0, \alpha_{i_0+1}, \ldots, \alpha_n\} = 0;$$

(2) $\Sigma_*\{\alpha_1, \ldots, \alpha_n\} = 0$*, where* Σ_* *is the suspension homomorphism.*

Proof. (1): In virtue of Proposition 3.4(1), we can suppose that $i_0 = 1$. Define $T' : T_0(\Sigma\underline{A}) \to T_1(\underline{X})$ by $T'(a_1, \ldots, a_n) = T_1(\underline{\alpha})(*, a_2, \ldots, a_n)$ for any $(a_1, \ldots, a_n) \in T_0(\Sigma\underline{A})$. Then, T' is an extension (up to homotopy) of $T_1(\underline{\alpha})$ and the result follows.

(2): This is a direct consequence of [20, Corollary (4)]. □

Let $J_n(X)$ be the nth stage of the James construction $J(X)$ of a topological space X and write $j_n(X) : J_n(X) \hookrightarrow J(X)$ for the canonical inclusion map. Given $m_i \geq 1$ for $i = 1, \ldots, n$ with $n \geq 2$, write $\mathbf{m} = (m_1, \ldots, m_n)$ and let $\underline{J_{\mathbf{m}}(X)} = (J_{m_1}(X), \ldots, J_{m_n}(X))$. Set $m' = m_1 + \cdots + m_n$ and $m'' = m' - \min_{1\leq i\leq n}\{m_i\}$, and note that there is an inclusion $J_{m''}(X) \hookrightarrow J_{m'}(X)$. Next, consider the canonical multiplication $\mu_{\mathbf{m}}(X) : T_0(\underline{J_{\mathbf{m}}(X)}) \to J_{m'}(X)$ which restricts to a map

$$\mu_{\mathbf{m}}(X)_{|T_1(\underline{J_{\mathbf{m}}(X)})} : T_1(\underline{J_{\mathbf{m}}(X)}) \to J_{m''}(X)$$

commuting the diagram

$$\begin{array}{ccc} T_1(\underline{J_{\mathbf{m}}(X)}) & \xrightarrow{\mu_{\mathbf{m}}(X)_{|T_1(\underline{J_{\mathbf{m}}(X)})}} & J_{m''}(X) \\ \big\downarrow & & \big\downarrow \\ T_0(\underline{J_{\mathbf{m}}(X)}) & \xrightarrow{\mu_{\mathbf{m}}(X)} & J_{m'}(X). \end{array}$$

Definition 3.6. The *interior Whitehead product* $\langle \alpha_1, \ldots, \alpha_n \rangle$ of maps $\alpha_i : \Sigma A_i \to J_{m_i}(X)$ for $i = 1, \ldots, n$ is the composition

$$\Sigma^{n-1}\Lambda(\underline{A}) \xrightarrow{\{\alpha_1,\ldots,\alpha_n\}} T_1(\underline{J_{\mathbf{m}}(X)}) \xrightarrow{\mu_{\mathbf{m}}(X)_{|T_1(\underline{J_{\mathbf{m}}(X)})}} J_{m''}(X),$$

and thus $\langle \alpha_1, \ldots, \alpha_n \rangle \in \pi(\Sigma^{n-1}\Lambda(\underline{A}), J_{m''}(X))$.

Now, consider the composite maps $\alpha'_i : \Sigma A_i \xrightarrow{\alpha_i} J_{m_i}(X) \hookrightarrow J_{m''}(X)$ for $i = 1, \ldots, n$. Then, notice that $\langle \alpha_1, \ldots, \alpha_n \rangle$ represents an element of the higher order Whitehead product $[\alpha'_1, \ldots, \alpha'_n]$ considered in [20]. Thus, applying [20, Theorem (2.1)] for any map $f : J_{m''}(X) \to Y$ it follows that

$$f_*\langle \alpha_1, \ldots, \alpha_n \rangle \in f_*[\alpha'_1, \ldots, \alpha'_n] \subseteq [f\alpha'_1, \ldots, f\alpha'_n]. \quad (5)$$

The following properties of the interior Whitehead product are easily obtained from Proposition 3.4.

Corollary 3.7. (1) *Let* $\sigma \in S_n$ *be a permutation of the set* $\{1, \ldots, n\}$. *Then*

$$\langle \alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(n)} \rangle = (\Sigma^{n-1}\hat{\sigma})^*\langle \alpha_1, \ldots, \alpha_n \rangle,$$

where $\hat{\sigma} : \Lambda(\underline{A}) \xrightarrow{\approx} \Lambda(\underline{\sigma A})$ *is the associated homeomorphism.*

(2) *Denote by* $\delta_i : J_{m_i}(X) \hookrightarrow J_{m^*}(X)$ *the inclusions for* $i = 1, \ldots, n$, *where* $m^* = m' - \min\limits_{j<k}\{m_j + m_k\}$. *The interior Whitehead product satisfies the Jacobi identity*

$$\sum_{i=1}^{n} (\Sigma^{n-2}\hat{\sigma}_i)^*[\delta_i\alpha_i, \langle \alpha_1, \ldots, \alpha_n \rangle^{(i)}] = 0$$

as an element of $\pi(\Sigma^{n-2}\Lambda(\underline{A}), J_{m^*}(X))$.

(3) *If* $f : X \to Y$ *is any map then*

$$\langle J_{m_1}(f)\alpha_1, \ldots, J_{m_n}(f)\alpha_n \rangle = J_{m''}(f)\langle \alpha_1, \ldots, \alpha_n \rangle.$$

(4) *The following Whitehead identity holds:*

$$\langle \alpha_1\Sigma\beta_1, \ldots, \alpha_n\Sigma\beta_n \rangle = \langle \alpha_1, \ldots, \alpha_n \rangle \Sigma^{n-1}(\beta_1 \wedge \cdots \wedge \beta_n).$$

Remark 3.8. If $n = 2$, Corollary 3.7(2) is the result stated in [1, Theorem (5.4)] and [24, Theorem 1] (corrected in [25]).

4. Generalized Hopf construction

Let $\underline{A} = (A_1, \ldots, A_n)$ be a n-tuple of pointed spaces. From [23, Satz 19] there is a homotopy equivalence $\Sigma T_0(\underline{A}) \xrightarrow{\simeq} \bigvee_N \Sigma \bigwedge_{i\in N} A_i$, where N runs through all non-empty subsets of the set $\{1, \ldots, n\}$. On the other hand, Hardie has constructed in [13] a particular homotopy equivalence which possesses some useful properties. Hardie's construction uses a right lexicographic order between some subsets N. More precisely, for each $i = 1, \ldots, n$ let $c_i = \binom{n}{i}$ be the binomial coefficient. For $k = 1, \ldots, c_i$, denote by $N_{i,k}$ the kth subset of cardinality i in the ordered sequence

$$\{1, \ldots, i\} < \cdots < \{n-(i-1), \ldots, n-1, n\}.$$

Let $W(\underline{A}) = \bigvee_N \Sigma \bigwedge_{i\in N} A_i$. Following [13, (2.2)] and making use of the co-H-structure on $\Sigma T_0(\underline{A})$, we define

$$\theta = \sum_{i=1}^{n} \sum_{k=1}^{c_i} \theta_{i,k} : \Sigma T_0(\underline{A}) \to W(\underline{A}),$$

where

$$\theta_{i,k} : \Sigma T_0(\underline{A}) \to \Sigma \bigwedge_{j\in N_{i,k}} A_j \hookrightarrow W(\underline{A})$$

is determined by suspending the collapsing map $T_0(\underline{A}) \to \bigwedge_{j\in N_{i,k}} A_j$ and composing with the inclusion map $\Sigma \bigwedge_{j\in N_{i,k}} A_j \hookrightarrow W(\underline{A})$.

Theorem 4.1. ([13, Theorem 2.3] (cf. [23, Satz 19])) *The map $\theta : \Sigma T_0(\underline{A}) \to W(\underline{A})$ is a homotopy equivalence.*

Recall that given any based map $F : A_1 \times A_2 \to Z$, the Hopf construction on F leads to a map $H(F) : \Sigma(A_1 \wedge A_2) \to \Sigma Z$ which is given by the composition

$$\Sigma(A_1 \wedge A_2) \xrightarrow{\delta} \Sigma(A_1 \times A_2) \xrightarrow{\Sigma F} \Sigma Z,$$

where $\delta : \Sigma(A_1 \wedge A_2) \to \Sigma(A_1 \times A_2)$ is determined by the map $A_1 * A_2 \to \Sigma(A_1 \times A_2)$ for the join $A_1 * A_2$, with a natural homotopy equivalence $A_1 * A_2 \xrightarrow{\simeq} \Sigma(A_1 \wedge A_2)$. This yields a section of the cofibration

$$\Sigma(A_1 \vee A_2) \hookrightarrow \Sigma(A_1 \times A_2) \longrightarrow \Sigma(A_1 \wedge A_1). \quad (\delta: \Sigma(A_1 \wedge A_1) \dashrightarrow \Sigma(A_1 \times A_2))$$

Denoting by $\imath : \Sigma\Lambda(\underline{A}) \hookrightarrow W(\underline{A})$ the obvious inclusion, the composite map

$$\delta : \Sigma\Lambda(\underline{A}) \xhookrightarrow{\imath} W(\underline{A}) \xrightarrow{\theta^{-1}} \Sigma T_0(\underline{A})$$

is a generalization of the above and yields a section of the cofibration

$$\Sigma T_1(\underline{A}) \hookrightarrow \Sigma T_0(\underline{A}) \longrightarrow \Sigma\Lambda(\underline{A}) \quad (\delta: \Sigma\Lambda(\underline{A}) \dashrightarrow \Sigma T_0(\underline{A}))$$

which leads to a homotopy equivalence

$$\Sigma T_1(\underline{A}) \vee \Sigma\Lambda(\underline{A}) \xrightarrow{\simeq} \Sigma T_0(\underline{A}).$$

For the n-tuple $\underline{\Sigma A}$, another section $\delta' : \Sigma\Lambda(\underline{\Sigma A}) \to \Sigma T_0(\underline{\Sigma A})$ of the cofibration $\Sigma T_1(\underline{\Sigma A}) \hookrightarrow \Sigma T_0(\underline{\Sigma A}) \to \Sigma\Lambda(\underline{\Sigma A})$ might be also described as follows.

Write $i_X : X \hookrightarrow C_f$ and $i_{CA} : CA \to C_f$ for the canonical maps, $\tau_1 : C_{\Sigma f} \xrightarrow{\approx} \Sigma C_f$ and $\tau_2 : C\Sigma A \xrightarrow{\approx} \Sigma CA$ for the canonical homeomorphisms. Thus, $\tau_1 i_{C\Sigma A} = (\Sigma i_{CA})\tau_2 : C\Sigma A \to \Sigma C_f$.

Next, given a commutative square

$$\begin{array}{ccc} A' & \xrightarrow{f'} & X' \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ A & \xrightarrow{f} & X, \end{array} \tag{6}$$

the universal properties of the mapping cones $C_{f'}$ and C_f lead to a map $\gamma(\alpha, \beta) : C_{f'} \to C_f$ with $\gamma(\alpha, \beta)i_{X'} = i_X\beta$ and $\gamma(\alpha, \beta)i_{CA'} = i_{CA}C\alpha$. We notice that by [3, Theorem 6.2.8] the map $\gamma(\alpha, \beta)$ is a homotopy equivalence provided that the maps $\alpha : A' \to A$ and $\beta : X' \to X$ are homotopy equivalences as well. Further, the diagram

$$\begin{array}{ccccc} X' & \xhookrightarrow{i_{X'}} & C_{f'} & \xrightarrow{\pi_{X'}} & \Sigma A' \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma(\alpha,\beta)} & & \downarrow{\scriptstyle\Sigma\alpha} \\ X & \xhookrightarrow{i_X} & C_f & \xrightarrow{\pi_X} & \Sigma A \end{array} \tag{7}$$

commutes, where $\pi_{X'} : C_{f'} \to \Sigma A'$ and $\pi_X : C_f \to \Sigma A$ are the projection maps.

Notice that for the embedding map $\iota_A : A \hookrightarrow CA$ and for the constant map $\mathfrak{c}_A : A \to *$ there are canonical homeomorphisms $\nu_A : C_{\iota_A} \xrightarrow{\approx} \Sigma A$ and $\nu'_A : C_{\mathfrak{c}_A} \xrightarrow{\approx} \Sigma A$. Further, the commutative diagram

$$\begin{array}{ccc} A & \xhookrightarrow{\iota_A} & CA \\ {\scriptstyle\mathrm{id}_A}\| & & \downarrow{\scriptstyle\mathfrak{c}_{CA}} \\ A & \xrightarrow{\mathfrak{c}_A} & * \end{array}$$

determines a homotopy equivalence

$$\gamma_A = \gamma(\mathrm{id}_A, \mathfrak{c}_{CA}) : C_{\iota_A} \xrightarrow{\simeq} C_{\mathfrak{c}_A}$$

because the map $\mathfrak{c}_{CA} : CA \to *$ is a homotopy equivalence.

Now, because of a section $\Sigma T_0(\underline{\Sigma A}) \to \Sigma T_1(\underline{\Sigma A})$ for the inclusion map $\Sigma T_1(\underline{\Sigma A}) \hookrightarrow \Sigma T_0(\underline{\Sigma A})$, the map $\Sigma\omega_n : \Sigma^n\Lambda(\underline{A}) \to \Sigma T_1(\underline{\Sigma A})$ is homotopy trivial (cf. [20, Corollary (4)]). Consequently, $(\Sigma\omega_n)\lambda^{-1} : \Lambda(\underline{\Sigma A}) \to \Sigma T_1(\underline{\Sigma A})$

is also homotopy trivial and there is a map $\beta : C\Lambda(\underline{\Sigma A}) \to \Sigma T_1(\underline{\Sigma A})$ such that the diagram

$$\begin{array}{ccc} \Lambda(\underline{\Sigma A}) & \xhookrightarrow{\iota_{\Lambda(\underline{\Sigma A})}} & C\Lambda(\underline{\Sigma A}) \\ {\scriptstyle \lambda^{-1}}\downarrow & & \downarrow{\scriptstyle \beta} \\ \Sigma^n\Lambda(\underline{A}) & \xrightarrow{\Sigma\omega_n} & \Sigma T_1(\underline{\Sigma A}) \end{array} \tag{8}$$

commutes, where $\lambda : \Sigma^n\Lambda(\underline{A}) \xrightarrow{\simeq} \Lambda(\underline{\Sigma A})$ is given by diagram (4). It follows from diagrams (6) and (7) that there is a map

$$\gamma(\lambda^{-1}, \beta) : C_{\iota_{\Lambda(\underline{\Sigma A})}} \to C_{\Sigma\omega_n}.$$

Because the diagrams

$$\begin{array}{ccccccccc} \Sigma\Lambda(\underline{\Sigma A}) & \xrightarrow{\nu^{-1}_{\Lambda(\underline{\Sigma A})}} & C_{\iota_{\Lambda(\underline{\Sigma A})}} & \xrightarrow{\gamma(\lambda^{-1},\beta)} & C_{\Sigma\omega_n} & \xrightarrow{\tau_1} & \Sigma C_{\omega_n} & \xrightarrow{\Sigma\nu_{\omega_n}} & \Sigma T_0(\underline{\Sigma A}) \\ \| & & \downarrow & & \downarrow & & & & \downarrow{\scriptstyle \Sigma s} \\ \Sigma\Lambda(\underline{\Sigma A}) & = & \Sigma\Lambda(\underline{\Sigma A}) & \xrightarrow{\Sigma\lambda^{-1}} & \Sigma^{n+1}\Lambda(\underline{A}) & & \xrightarrow{\Sigma\lambda} & & \Sigma\Lambda(\underline{\Sigma A}), \end{array}$$

$$\begin{array}{ccc} \Sigma\Lambda(\underline{\Sigma A}) & \xrightarrow{\nu^{-1}_{\Lambda(\underline{\Sigma A})}} & C_{\iota_{\Lambda(\underline{\Sigma A})}} \\ \| & & \downarrow{\scriptstyle \gamma_{\Lambda(\underline{\Sigma A})}} \\ \Sigma\Lambda(\underline{\Sigma A}) & \xrightarrow{\nu'^{-1}_{\Lambda(\underline{\Sigma A})}} & C_{\mathfrak{c}_{\Lambda(\underline{\Sigma A})}} \end{array}$$

commute (up to homotopy) and $\gamma_{\Lambda(\underline{\Sigma A})} : C_{\iota_{\Lambda(\underline{\Sigma A})}} \to C_{\mathfrak{c}_{\Lambda(\underline{\Sigma A})}}$ is a homotopy equivalence, we may state the result needed in the sequel.

Proposition 4.2. *The maps*

$$\delta' : \Sigma\Lambda(\underline{\Sigma A}) \xrightarrow{\nu^{-1}_{\Lambda(\underline{\Sigma A})}} C_{\iota_{\Lambda(\underline{\Sigma A})}} \xrightarrow{\gamma(\lambda^{-1},\beta)} C_{\Sigma\omega_n} \xrightarrow{\tau_1} \Sigma C_{\omega_n} \xrightarrow{\Sigma\nu_{\omega_n}} \Sigma T_0(\underline{\Sigma A})$$

and

$$\delta'' : \Sigma\Lambda(\underline{\Sigma A}) \xrightarrow{\nu'^{-1}_{\Lambda(\underline{\Sigma A})}} C_{\mathfrak{c}_{\Lambda(\underline{\Sigma A})}} \xrightarrow{\gamma^{-1}_{\Lambda(\underline{\Sigma A})}} C_{\iota_{\Lambda(\underline{\Sigma A})}} \xrightarrow{\gamma(\lambda^{-1},\beta)} C_{\Sigma\omega_n} \xrightarrow{\tau_1} \Sigma C_{\omega_n} \xrightarrow{\Sigma\nu_{\omega_n}} \Sigma T_0(\underline{\Sigma A})$$

are a section and a section (*up to homotopy*) *of the cofibration*

$$\Sigma T_1(\underline{\Sigma A}) \hookrightarrow \Sigma T_0(\underline{\Sigma A}) \to \Sigma\Lambda(\underline{\Sigma A}),$$

respectively.

The *generalized Hopf construction* on a based map $F : T_0(\underline{A}) \to Z$ is the composition

$$H(F) : \Sigma\Lambda(\underline{A}) \xhookrightarrow{\iota} W(\underline{A}) \xrightarrow{\theta^{-1}} \Sigma T_0(\underline{A}) \xrightarrow{\Sigma F} \Sigma Z.$$

In particular, for $Z = J(X)$ Hardie has defined in [13] the element

$$c(F) = \psi_X H(F) \in \pi(\Sigma\Lambda(\underline{A}), \Sigma X).$$

By the adjointness, we obtain

$$\tilde{c}(F) \in \pi(\Lambda(\underline{A}), \Omega\Sigma X).$$

From (1), for each $m \geq 1$, we have

$$h_m(c(F)) \in \pi(\Sigma\Lambda(\underline{A}), \Sigma X^{\wedge m})$$

and by (2), it holds

$$h_m(c(F)) = \psi_{X^{\wedge m}} \Sigma(q\phi_X^{-1})\Sigma\tilde{c}(F). \tag{9}$$

Given based maps $\alpha_i : A_i \to J(X)$ for $i = 1, \ldots, n$, let $F' = \mu(X)T_0(\underline{\alpha}) : T_0(\underline{A}) \to J(X)$. From [13, Corollary 3.4], $c(F') : \Sigma\Lambda(\underline{A}) \to \Sigma X$ is trivial.

4.1. Separation element

Given a map $f : A \to X$, suppose that $u, v : C_f \to Y$ are maps such that $ui_X = vi_X$. We follow [30, Sect. 3] to define a map $w : \Sigma A \to Y$ given by

$$w(a,t) = \begin{cases} u(a, 2t), & \text{if } 0 \leq t \leq \frac{1}{2}, \\ v(a, 2-2t), & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

for $(a,t) \in \Sigma A$. The homotopy class of w is called the *separation element of* u, v and denoted by $d(u, v)$.

Remark 4.3. If $p_1, p_2 : CA \to \Sigma A$ are given by $p_1(a,t) = (a, \frac{t}{2})$ and $p_2(a,t) = (a, \frac{2-t}{2})$ for $a \in A$ and $0 \leq t \leq 1$, the separation element $w = d(u, v)$ satisfies $wp_1 = ui_{CA}$ and $wp_2 = vi_{CA}$. Further, the map $w : \Sigma A \to Y$ is uniquely determined by these two properties.

Lemma 4.4. *If* $u, v : \Sigma A \to Y$ *then* $d(u\nu'_A, v\nu'_A) = u - v$.

Proof. If $u, v : \Sigma A \to Y$ then $u\nu'_A, v\nu'_A : C_{\mathfrak{c}_A} \to Y$ satisfy $u\nu'_A i_* = v\nu'_A i_*$ and so $w = d(u\nu'_A, v\nu'_A)$ is defined. Thus, for any $(a,t) \in \Sigma A$ it holds

$$\begin{aligned} w(a,t) &= \begin{cases} wp_1(a, 2t) = u\nu'_A i_{CA}(a, 2t) = u(a, 2t), & \text{if } 0 \leq t \leq \frac{1}{2}, \\ wp_2(a, 2-2t) = v\nu'_A i_{CA}(a, 2-2t) = v(a, 2-2t), & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases} \\ &= (u-v)(a,t). \end{aligned}$$

Hence, $d(u\nu'_A, v\nu'_A) = u - v : \Sigma A \to Y$ and the result follows. □

Let $f' : A' \to X'$ and $f : A \to X$ be given by diagram (6). If $u, v : C_f \to Y$ satisfy $ui_X = vi_X$ then it is clear that $u\gamma(\alpha,\beta)i_{X'} = v\gamma(\alpha,\beta)i_{X'}$ and $w' = d(u\gamma(\alpha,\beta), v\gamma(\alpha,\beta)) : \Sigma A' \to Y$ is defined. To simplify notation, we write simply γ for $\gamma(\alpha,\beta)$ in the rest of this section.

We claim that:

Lemma 4.5. $d(u\gamma, v\gamma) = d(u,v)\Sigma\alpha$.

Proof. The separation elements $d(u,v)$ and $d(u\gamma, v\gamma)$ are defined by means of

$$w(a,t) = \begin{cases} u(a,2t), & \text{if } 0 \le t \le \frac{1}{2}, \\ v(a, 2-2t), & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

for $(a,t) \in \Sigma A$ and

$$w'(a',t) = \begin{cases} u\gamma(a',2t), & \text{if } 0 \le t \le \frac{1}{2}, \\ v\gamma(a', 2-2t), & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

for $(a',t) \in \Sigma A'$, respectively. Then,

$$(w\Sigma\alpha)(a',t) = \begin{cases} u(\alpha(a'),2t), & \text{if } 0 \le t \le \frac{1}{2}, \\ v(\alpha(a'), 2-2t), & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

for $(a',t) \in \Sigma A'$. Because $\gamma i_{CA'} = i_{CA} C\alpha$, we derive that $w' = w\Sigma\alpha$ and the proof follows. □

Denote by $\tilde{\sigma} : \Sigma^2 A \xrightarrow{\approx} \Sigma^2 A$ the homeomorphism defined by $(s,(s',a)) \mapsto (s',(s,a))$ for $(s,(s',a)) \in \Sigma^2 A = \Sigma(\Sigma A)$ and notice that $f\tilde{\sigma} = -f$ for any map $f : \Sigma^2 A \to X$. Since $u' = (\Sigma u)\tau_1$ and $v' = (\Sigma v)\tau_1 : C_{\Sigma f} \to \Sigma Y$ satisfy $u' i_{\Sigma X} = v' i_{\Sigma X}$ the map $w' = d(u',v') : \Sigma^2 A \to \Sigma Y$ is defined and satisfies $w' p_1' = u' i_{C\Sigma A}$ and $w' p_2' = v' i_{C\Sigma A}$, where $p_1', p_2' : C\Sigma A \to \Sigma^2 A$ are maps as in Remark 4.3.

Then, we can state (cf. [13, Lemma 4.1]):

Lemma 4.6. $d((\Sigma u)\tau_1, (\Sigma v)\tau_1) = \Sigma(d(u,v))\tilde{\sigma}$.

Proof. Just observe that $(\Sigma p_1)\tau_2 = \tilde{\sigma} p_1'$ and then the equalities $w' p_1' = (\Sigma u)\tau_1 i_{C\Sigma A} = (\Sigma u)\Sigma(i_{CA})\tau_2 = \Sigma(wp_1)\tau_2 = (\Sigma w)\tilde{\sigma} p_1'$ and similarly $w' p_2' = (\Sigma w)\tilde{\sigma} p_2'$ imply the result. □

Next, if $u, v : T_0(\underline{\Sigma A}) \to Y$ coincide on $T_1(\underline{\Sigma A})$ then there exist the separation elements $d(u\nu_{\omega_n}, v\nu_{\omega_n}) : \Sigma^n \Lambda(\underline{A}) \to Y$ and $d(\Sigma(u\nu_{\omega_n})\tau_1, \Sigma(v\nu_{\omega_n})\tau_1) : \Sigma^{n+1}\Lambda(\underline{A}) \to \Sigma Y$. Thus, diagram (8), Lemmas 4.5 and 4.6 lead to:

Corollary 4.7. *If* $\gamma = \gamma(\lambda^{-1}, \beta) : C_{\iota_{\Lambda(\underline{\Sigma A})}} \to C_{\Sigma\omega_n}$ *then*

$$d(\Sigma(u\nu_{\omega_n})\tau_1\gamma, \Sigma(v\nu_{\omega_n})\tau_1\gamma) = \Sigma(d(u,v))\tilde{\sigma}\Sigma\lambda^{-1}.$$

To state the main result of this section, we recall from [13] the notion of a map strongly of type.

Given $\alpha_i : A_i \to J_{m_i}(X)$ for $i = 1, \ldots, n$, let $F' = j_m(X)\mu_{\mathbf{m}}(X)T_0(\underline{\alpha}) : T_0(\underline{A}) \to J(X)$, where $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $\mathbf{m} = (m_1, \ldots, m_n)$ and $m = m_1 + \cdots + m_n$.

Definition 4.8. We say that $F : T_0(\underline{A}) \to J(X)$ is a map *strongly of type* $(\alpha_1, \ldots, \alpha_n)^k$ if its image $F(T_0(\underline{A}))$ is contained in $J_k(X)$ and the restrictions $F_{|T_1(\underline{A})}$, and $F'_{|T_1(\underline{A})}$ coincide.

Lemma 4.9. (cf. [13, Theorem 1.8]) *Let $F : T_0(\Sigma\underline{A}) \to J(X)$ be a map strongly of some type. Then $\phi_X d(F'\nu_{\omega_n}, F\nu_{\omega_n})\lambda^{-1} = \tilde{c}(F)$.*

Proof. Let $F' : T_0(\Sigma\underline{A}) \to J(X)$ be the map as in Definition 4.8 and let $\gamma = \gamma(\lambda^{-1}, \beta) : C_{\iota_{\Lambda(\Sigma\underline{A})}} \to C_{\Sigma\omega_n}$ as in Corollary 4.7. Because, in view of Proposition 4.2, the map $\delta'' = (\Sigma\nu_{\omega_n})\tau_1\gamma\gamma^{-1}_{\Lambda(\Sigma\underline{A})}\nu'^{-1}_{\Lambda(\Sigma\underline{A})}$ is a section (up to homotopy) of the cofibration

$$\Sigma T_1(\Sigma\underline{A}) \hookrightarrow \Sigma T_0(\Sigma\underline{A}) \to \Sigma\Lambda(\Sigma\underline{A}),$$

by Lemmas 4.4–4.6 we get:

$$\begin{aligned}
&\psi_X\Sigma(d(F'\nu_{\omega_n}, F\nu_{\omega_n}))\tilde{\sigma}\Sigma\lambda^{-1}\\
&\qquad = \psi_X d(\Sigma(F'\nu_{\omega_n})\tau_1, \Sigma(F\nu_{\omega_n})\tau_1)\Sigma\lambda^{-1}\\
&\qquad = \psi_X d(\Sigma(F'\nu_{\omega_n})\tau_1\gamma, \Sigma(F\nu_{\omega_n})\tau_1\gamma)\\
&\qquad = d(\psi_X\Sigma(F')\delta''\nu'_{\Lambda(\Sigma\underline{A})}\gamma_{\Lambda(\Sigma\underline{A})}, \psi_X\Sigma(F)\delta''\nu'_{\Lambda(\Sigma\underline{A})}\gamma_{\Lambda(\Sigma\underline{A})})\\
&\qquad = d(c(F')\nu'_{\Lambda(\Sigma\underline{A})}\gamma_{\Lambda(\Sigma\underline{A})}, c(F)\nu'_{\Lambda(\Sigma\underline{A})}\gamma_{\Lambda(\Sigma\underline{A})})\\
&\qquad = d(c(F')\nu'_{\Lambda(\Sigma\underline{A})}, c(F)\nu'_{\Lambda(\Sigma\underline{A})})\\
&\qquad = c(F') - c(F).
\end{aligned}$$

Since $\Sigma(d(F'\nu_{\omega_n}, F\nu_{\omega_n}))\tilde{\sigma} = -\Sigma d(F'\nu_{\omega_n}, F\nu_{\omega_n})$ and $c(F') = *$ ([13, Corollary 3.4]), we get $\psi_X\Sigma(d(F'\nu_{\omega_n}, F\nu_{\omega_n}))\Sigma\lambda^{-1} = c(F)$ which certainly implies

$$\phi_X d(F'\nu_{\omega_n}, F\nu_{\omega_n})\lambda^{-1} = \tilde{c}(F)$$

and this completes the proof. □

We finish this section with a generalization of [13, Theorem 1.10] as the main result.

Let $q_m : (J_m(X), J_{m-1}(X)) \to (X^{\wedge m}, *)$ be the quotient map and denote by $q : J(X) \to J(X^{\wedge m})$ its combinatorial extension. Given $\alpha_i : \Sigma A_i \to J_{m_i}(X)$ for $i = 1, \ldots, n$, we obtain the maps

$$\Sigma A_i \xrightarrow{\alpha_i} J_{m_i}(X) \xrightarrow{q_{m_i}} X^{\wedge m_i}$$

and the suspension of their smash products leads to

$$\Sigma(q_{m_1}\alpha_1 \wedge \cdots \wedge q_{m_n}\alpha_n) : \Sigma^{n+1}\Lambda(\underline{A}) \to \Sigma X^{\wedge m},$$

where $m = m_1 + \cdots + m_n$.

Theorem 4.10. *If* $\alpha_i \in \pi(\Sigma A_i, J_{m_i}(X))$ *for* $i = 1, \ldots, n$ *and* $F : T_0(\underline{\Sigma A}) \to J(X)$ *is strongly of type* $(\alpha_1, \ldots, \alpha_n)^{m-1}$ *then*

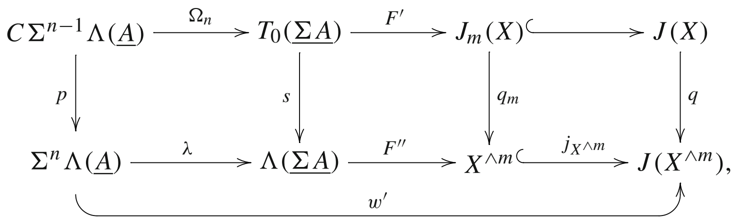
$$h_m(c(F)) = \Sigma(q_{m_1}\alpha_1 \wedge \cdots \wedge q_{m_n}\alpha_n).$$

Proof. In view of (9) and Lemma 4.9, we have

$$\begin{aligned} h_m(c(F)) &= \psi_{X^{\wedge m}}\Sigma(q\phi_X^{-1})\Sigma\tilde{c}(F) \\ &= \psi_{X^{\wedge m}}\Sigma(q\phi_X^{-1})\Sigma(\phi_X d(F'\nu_{\omega_n}, F\nu_{\omega_n})\lambda^{-1}) \\ &= \psi_{X^{\wedge m}}\Sigma(d(qF'\nu_{\omega_n}, qF\nu_{\omega_n}))\Sigma\lambda^{-1}. \end{aligned}$$

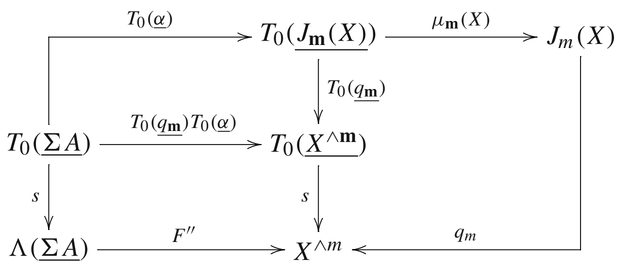
Since $F(T_0(\underline{\Sigma A})) \subseteq J_{m-1}(X)$ and q is the combinatorial extension of q_m then $qF\nu_{\omega_n} = *$. So $w = d(qF'\nu_{\omega_n}, qF\nu_{\omega_n}) : \Sigma^n\Lambda(\underline{A}) \to J(X^{\wedge m})$ satisfies $wp_1 = qF'\Omega_n$ and $wp_2 = *$. Consequently, w may be regarded as a map w' such that $w'p = qF'\Omega_n$, where $p : C\Sigma^{n-1}\Lambda(\underline{A}) \to \Sigma^n\Lambda(\underline{A})$ is the map from diagram (4).

Next, consider the following commutative diagram



where F'' is determined by $F''s = q_mF'$ and the other maps were already defined in the text. Hence, $h_m(c(F)) = \psi_{X^{\wedge m}}\Sigma(d(qF'\nu_{\omega_n}, qF\nu_{\omega_n}))\Sigma\lambda^{-1} = \psi_{X^{\wedge m}}\Sigma(j_{X^{\wedge m}}F''\lambda)\Sigma\lambda^{-1} = \psi_{X^{\wedge m}}\Sigma(j_{X^{\wedge m}}F'') = \Sigma F''$.

Since $F' = \mu_{\mathbf{m}}(X)T_0(\underline{\alpha})$, the commutativity of the diagram



where $\underline{q_{\mathbf{m}}} = (q_{m_1}, \ldots, q_{m_n})$ finishes the proof. □

Let $\alpha_i : \Sigma A_i \to X$ be maps for $i = 1, \ldots, n$. If $\langle \alpha_1, \ldots, \alpha_n \rangle = 0$ then there is a map $F : T_0(\underline{\Sigma A}) \to J(X)$ strongly of type $(\alpha_1, \ldots, \alpha_n)^{n-1}$. Thus, the generalized Hopf construction yields a map $c(F) : \Sigma\Lambda(\underline{\Sigma A}) \to \Sigma X$ and Theorem 4.10 supplies a criterion to check if the map $c(F)$ is non-trivial.

5. Applications

Fix $n \geq 2$ and suppose $A_i = A$ for $i = 1, \ldots, n$. Denote by $\mathrm{id}_{\Sigma A} : \Sigma A \to \Sigma A = J_1(\Sigma A)$ the identity map and by $\rho_n(\Sigma A) : T_0(\underline{\Sigma A}) \to J_n(\Sigma A)$ the quotient map. For the sequences $\mathbf{n} = (1, \overset{\times n}{\ldots}, 1)$ and $\underline{\mathrm{id}_{\Sigma A}} = (\mathrm{id}_{\Sigma A}, \overset{\times n}{\ldots}, \mathrm{id}_{\Sigma A})$ the map $\rho_n(\Sigma A)$ factors as

$$\rho_n(\Sigma A) : T_0(\underline{\Sigma A}) \xrightarrow{T_0(\underline{\mathrm{id}_{\Sigma A}})} T_0(\underline{J_{\mathbf{n}}(\Sigma A)}) \xrightarrow{\mu_{\mathbf{n}}(\Sigma A)} J_n(\Sigma A)$$

and restricts to $\rho_n(\Sigma A)_{|T_1(\underline{\Sigma A})} : T_1(\underline{\Sigma A}) \to J_{n-1}(\Sigma A)$. This leads to a (up to homotopy) pushout

$$\begin{array}{ccc} T_1(\underline{\Sigma A}) & \xrightarrow{\rho_n(\Sigma A)_{|T_1(\underline{\Sigma A})}} & J_{n-1}(\Sigma A) \\ \big\downarrow & & \big\downarrow \\ T_0(\underline{\Sigma A}) & \xrightarrow{\rho_n(\Sigma A)} & J_n(\Sigma A). \end{array}$$

Taking into account the pushout diagram (3), we get a (up to homotopy) pushout

$$\begin{array}{ccc} \Sigma^{n-1}\Lambda(\underline{A}) & \xrightarrow{\langle \mathrm{id}_{\Sigma A}, \overset{\times n}{\ldots}, \mathrm{id}_{\Sigma A}\rangle} & J_{n-1}(\Sigma A) \\ \big\downarrow & & \big\downarrow \\ C\Sigma^{n-1}\Lambda(\underline{A}) & \xrightarrow{\rho_n(\Sigma A)\Omega_n} & J_n(\Sigma A) \end{array} \qquad (10)$$

which yields the following result ([33, Proposition 1.1.1]):

Proposition 5.1. *Let A be a pointed space. Then there is a (functorial) cofibre sequence*

$$\Sigma^{n-1}\Lambda(\underline{A}) \xrightarrow{\rho_n(\Sigma A)_{|T_1(\underline{\Sigma A})}\omega_n} J_{n-1}(\Sigma A) \hookrightarrow J_n(\Sigma A).$$

Thus, the cofibre sequence

$$J_{n-1}(\Sigma A) \hookrightarrow J_n(\Sigma A) \to \Lambda(\underline{\Sigma A})$$

is principal.

5.1. Higher Gray–Whitehead interior product

Given simply-connected co-H-spaces X_1, X_2, Gray [10] has defined the Theriault product $X_1 \circ X_2$ as a retraction of $\Sigma(\Omega X_1 \wedge \Omega X_2)$. If

$$X_1 \circ X_2 \xrightarrow{\zeta} \Sigma(\Omega X_1 \wedge \Omega X_2) \xrightarrow{\kappa} X_1 \circ X_2$$

are maps with $\kappa\zeta = \mathrm{id}_{X_1 \circ X_2}$ then the homotopy fibration

$$\Sigma(\Omega X_1 \wedge \Omega X_2) \xrightarrow{w} X_1 \vee X_2 \hookrightarrow X_1 \times X_2$$

determines a natural transformation

$$w_2 : X_1 \circ X_2 \xrightarrow{\zeta} \Sigma(\Omega X_1 \wedge \Omega X_2) \xrightarrow{w} X_1 \vee X_2$$

generalizing the Whitehead product map.

We make use of the Theriault product to define the higher order Gray–Whitehead product for co-H-spaces $X_1, \ldots, X_n$. As in [11, Sect. 5], we start by recalling that Porter [21, Theorem 1] has shown that there is a homotopy fibration

$$\Sigma^{n-1}\Lambda(\underline{\Omega X}) \to T_1(\underline{X}) \hookrightarrow T_0(\underline{X}).$$

Because X_i are co-H-spaces, there are coretractions $\nu_i : X_i \to \Sigma\Omega X_i$ for $i = 1, \ldots, n$. Define the higher Gray–Whitehead product map

$$w_n : (X_1 \circ X_2) \wedge X_3 \wedge \cdots \wedge X_n \to T_1(\underline{X})$$

as the composite

$$(X_1 \circ X_2) \wedge X_3 \wedge \cdots \wedge X_n \xrightarrow{\zeta\wedge\nu_3\wedge\cdots\wedge\nu_n}$$
$$\Sigma(\Omega X_1 \wedge \Omega X_2) \wedge \Sigma\Omega X_3 \wedge \cdots \wedge \Sigma\Omega X_n = \Sigma^{n-1}\Lambda(\underline{\Omega X}) \to T_1(\underline{X}).$$

Notice that, by means of the above, basic results presented in previous sections might be generalized replacing suspended spaces by co-H-spaces.

Applying [10, Theorem 1], the inductive procedure shows that

$$(X_1 \circ X_2) \wedge X_3 \wedge \cdots \wedge X_n = \Sigma^{n-2}(X_1 \circ \cdots \circ X_n)$$

and consequently

$$w_n : \Sigma^{n-2}(X_1 \circ \cdots \circ X_n) \to T_1(\underline{X}).$$

5.2. Spherical interior Whitehead product on spheres

Let $\alpha_i : \mathbb{S}^{k_i} \to \mathbb{S}^m$ be maps with $m \geq 2$, $k_i \geq 2$ for $i = 1, \ldots, n$ and $n \geq 2$. Then, $\langle\alpha_1, \ldots, \alpha_n\rangle = 0$ in $\pi_{k_1+\cdots+k_n-1}(J_{n-1}(\mathbb{S}^m))$ yields a map $F : \mathbb{S}^{k_1} \times \cdots \times \mathbb{S}^{k_n} \to J(\mathbb{S}^m)$ and the generalized Hopf construction leads to possibly a non-trivial map $c(F) : \mathbb{S}^{k_1+\cdots+k_n+1} \to \mathbb{S}^{m+1}$.

Next, write $\iota_n = \mathrm{id}_{\mathbb{S}^n}$, $\iota_{m,n} : \mathbb{S}^n \hookrightarrow J_m(\mathbb{S}^n)$ for the canonical inclusion maps and $\eta_n \in \pi_{n+1}(\mathbb{S}^n)$ for generators with $n \geq 2$.

Proposition 5.2. (1) *The element $\langle\iota_n, \overset{\times m}{\ldots}, \iota_n\rangle$ is of infinite order provided n is odd and $m \neq 2$ or n is even;*

(2) $\pi_{mn-1}(J_{m-1}(\mathbb{S}^n)) \approx \mathbb{Z} \oplus \pi_{mn}(\mathbb{S}^{n+1})$ *and $\langle\iota_n, \overset{\times m}{\ldots}, \iota_n\rangle$ is a generator of the infinite cyclic group provided n is odd and $m \neq 2$ or n is even;*

(3) $[\iota_{m-2,n}, \langle\iota_n, \overset{\times(m-1)}{\ldots\ldots}, \iota_n\rangle] = 0$ *if and only if $n = 2$ and m is an odd prime; this element has order m otherwise;*

(4) $\langle\eta_2\eta_3\eta_4, \iota_2, \iota_2, \iota_2\rangle = 0$ *in* $\pi_{10}(J_3(\mathbb{S}^2))$;

(5) $\langle\eta_2\eta_3, \iota_2, \iota_2, \iota_2\rangle = 0$ *in* $\pi_9(J_3(\mathbb{S}^2))$;

(6) $\langle \eta_3, \eta_3, \iota_3 \rangle = 0$ *in* $\pi_{10}(J_2(\mathbb{S}^3))$.

Proof. (1): follows from [5,14,27].

(2): follows from [5,27].

(3): follows from [5,14].

(4): follows from [14].

(5)–(6): we make use of the isomorphisms $\pi_9(\mathbb{S}^2) \approx \mathbb{Z}_3$ and $\pi_{10}(\mathbb{S}^3) \approx \mathbb{Z}_{15}$ and then follow *mutatis mutandis* the proof of (4). □

Proposition 5.2(3) implies the existence of a map $F : \mathbb{S}^2 \times \mathbb{S}^{2m-3} \to J_{m-2}(\mathbb{S}^2)$ strongly of type $(\iota_{m-2,2}, \langle \iota_2, \overset{\times(m-1)}{\ldots\ldots}, \iota_2 \rangle)^{m-2}$ for an odd prime m which yields, in view of [14, Theorem 1.5], an element $c(F) : \mathbb{S}^{2m} \to \mathbb{S}^3$ of order m in $\pi_{2m}(\mathbb{S}^3)$.

Further, Proposition 5.2(4)–(6) implies also the existence of maps:

$$F_1 : \mathbb{S}^4 \times \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2 \to J(\mathbb{S}^2);$$
$$F_2 : \mathbb{S}^5 \times \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2 \to J(\mathbb{S}^2);$$
$$F_3 : \mathbb{S}^4 \times \mathbb{S}^4 \times \mathbb{S}^3 \to J(\mathbb{S}^3)$$

strongly of types $(\eta_2\eta_3, \iota_2, \iota_2, \iota_2)^3$, $(\eta_2\eta_3\eta_4, \iota_2, \iota_2, \iota_2)^3$ and $(\eta_3, \eta_3, \iota_3)^2$, respectively. Toda [29] has shown that $\pi_{11}(\mathbb{S}^3) \approx \mathbb{Z}/2\mathbb{Z}$, $\pi_{12}(\mathbb{S}^3) \approx \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and $\pi_{12}(\mathbb{S}^4) \approx \mathbb{Z}/2\mathbb{Z}$. As in [14, Corollary 1.4], in view of Theorem 4.10, we obtain:

Corollary 5.3. (1) $c(F_1)$ *is a non-zero element in* $\pi_{11}(\mathbb{S}^3)$*;*
(2) $c(F_2)$ *is a non-zero element in* $\pi_{12}(\mathbb{S}^3)$*;*
(3) $c(F_3)$ *is a non-zero element in* $\pi_{12}(\mathbb{S}^4)$.

Proof. (1): The map $F_1 : \mathbb{S}^4 \times \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2 \to J(\mathbb{S}^2)$ determines $c(F_1) : \mathbb{S}^{11} \to \mathbb{S}^3$. By Theorem 4.10 and [29, Proposition 5.3], it holds $h_4(c(F_1)) = \eta_9\eta_{10} = \eta_9^2 \neq 0$. This implies that $c(F_1)$ is a non-zero element in $\pi_{11}(\mathbb{S}^3)$.

(2): The map $F_2 : \mathbb{S}^5 \times \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2 \to J(\mathbb{S}^2)$ determines $c(F_2) : \mathbb{S}^{12} \to \mathbb{S}^3$. By Theorem 4.10 and [29, (5.5)], it holds $h_4(c(F_2)) = \eta_9\eta_{10}\eta_{11} = \eta_9^3 = 4\nu_9 \neq 0$. This implies that $c(F_2)$ is a non-zero element in $\pi_{12}(\mathbb{S}^3)$ (cf. [14, Corollary 1.4]).

(3): The map $F_3 : \mathbb{S}^4 \times \mathbb{S}^4 \times \mathbb{S}^3 \to J(\mathbb{S}^3)$ determines $c(F_3) : \mathbb{S}^{12} \to \mathbb{S}^4$. By Theorem 4.10 and [29, Propositions 3.1 and 5.3], it holds $h_3(c(F_3)) = \eta_7 \wedge \eta_3 = \eta_{10}\eta_{11} = \eta_{10}^2 \neq 0$. This implies that $c(F_3)$ is a non-zero element in $\pi_{12}(\mathbb{S}^4)$ and the proof is complete. □

5.3. Spherical interior Whitehead product on projective spaces

Write $SP_n(X)$ for the nth stage of the symmetric power of a space X for $n \geq 1$ and $SP(X) = \varinjlim SP_n(X)$. Because of the H-structure on $SP(X)$, the inclusion map $X \hookrightarrow SP(X)$ extends to a map

$$u(X) : J(X) \to SP(X)$$

which leads to a sequence of maps $u_n(X) : J_n(X) \to SP_n(X)$ for $n \geq 1$. Taking $u_{n-1}(\Sigma A)\langle \mathrm{id}_{\Sigma A}, \overset{\times n}{\dots}, \mathrm{id}_{\Sigma A}\rangle$ we get a (up to homotopy) pushout

$$\begin{array}{ccc} \Sigma^{n-1}\Lambda(\underline{A}) & \xrightarrow{u_{n-1}(\Sigma A)\langle \mathrm{id}_{\Sigma A}, \overset{\times n}{\dots}, \mathrm{id}_{\Sigma A}\rangle} & SP_{n-1}(\Sigma A) \\ \big\downarrow & & \big\downarrow \\ C\Sigma^{n-1}\Lambda(\underline{A}) & \xrightarrow{u_n(\Sigma A)\rho_n(\Sigma A)\Omega_n} & SP_n(\Sigma A). \end{array} \tag{11}$$

Notice that armed with the diagrams (10) and (11), the sequence of maps $u_n(\Sigma A) : J_n(\Sigma A) \to SP_n(\Sigma A)$ above for $n \geq 1$ might be derived by the inductive procedure as well.

In particular, this yields a sequence of maps $u_n(\mathbb{S}^1) : J_n(\mathbb{S}^1) \to SP_n(\mathbb{S}^1)$ for $n \geq 1$. On the other hand, the Segre map

$$\mathbb{R}P^m \times \mathbb{R}P^n \to \mathbb{R}P^{(m+1)(n+1)-1}$$

given by $([x_0 : \cdots : x_m], [y_0 : \cdots : y_n]) \mapsto [x_0y_0 : \cdots : x_iy_j : \cdots : x_my_n]$ leads to an abelian H-structure on the infinite real projective space $\mathbb{R}P^\infty$. Thus, the inclusion map $\mathbb{S}^1 = \mathbb{R}P^1 \hookrightarrow \mathbb{R}P^\infty$ extends to a map

$$v(\mathbb{S}^1) : J(\mathbb{S}^1) = \Omega\mathbb{S}^2 \to \mathbb{R}P^\infty$$

which leads to a sequence of maps $v_n(\mathbb{S}^1) : J_n(\mathbb{S}^1) \to \mathbb{R}P^n$ for $n \geq 1$. Further, the abelian H-structure on $\mathbb{R}P^\infty$ yields the factorization

$$\begin{array}{ccc} \mathbb{S}^1 & \hookrightarrow & \mathbb{R}P^\infty \\ \big\downarrow & \nearrow & \\ SP(\mathbb{S}^1) & & \end{array}$$

which in turn implies the sequence of maps $\varphi_n(\mathbb{S}^1) : SP_n(\mathbb{S}^1) \to \mathbb{R}P^n$ with commutative diagrams

$$\begin{array}{ccc} J_n(\mathbb{S}^1) & \xrightarrow{v_n(\mathbb{S}^1)} & \mathbb{R}P^n \\ {\scriptstyle u_n(\mathbb{S}^1)}\big\downarrow & \nearrow {\scriptstyle \varphi_n(\mathbb{S}^1)} & \\ SP_n(\mathbb{S}^1) & & \end{array}$$

for $n \geq 1$. Because, by means of [18], the space $SP_n(\mathbb{S}^1)$ has the homotopy type of the circle $\mathbb{S}^1$, the induced homomorphisms $\pi_k(v_n(\mathbb{S}^1))$ are trivial for $k > 1$ and $n \geq 1$.

Let now $\mathbb{C}P^n$ be the complex projective n-space. By the projective Viète's Theorem (see e.g., [4]), it holds $SP_n(\mathbb{S}^2) = \mathbb{C}P^n$ and we get a sequence of maps

$u_n(\mathbb{S}^2) : J_n(\mathbb{S}^2) \to \mathbb{C}P^n$ for $n \geq 1$. Notice that these maps are also determined by the H-structure (settled e.g., by the Segre map) on $\mathbb{C}P^\infty$ and the factorization

$$\begin{array}{ccc} \mathbb{S}^2 & \hookrightarrow & \mathbb{C}P^\infty \\ \downarrow & \nearrow & \\ J(\mathbb{S}^2). & & \end{array}$$

Write $\gamma_n : \mathbb{S}^{2n+1} \to \mathbb{C}P^n$ for the quotient map and $j_n(\mathbb{C}) : \mathbb{S}^2 = \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^n$ for the canonical inclusion. It is known from [2, Corollary 4.4] and [22, Corollary 2] that the set $[j_n(\mathbb{C}), \overset{\times(n+1)}{\ldots\ldots}, j_n(\mathbb{C})]$ of $(n+1)$st order Whitehead products contains a single element which is equal to $(n+1)!\,\gamma_n$. Consequently, by means of (5), for $n \geq 1$ the map $u_n(\mathbb{S}^2) : J_n(\mathbb{S}^2) \to \mathbb{C}P^n$ satisfies

$$u_n(\mathbb{S}^2)\langle \iota_2, \overset{\times(n+1)}{\ldots\ldots}, \iota_2\rangle = [j_n(\mathbb{C}), \overset{\times(n+1)}{\ldots\ldots}, j_n(\mathbb{C})] = (n+1)!\,\gamma_n.$$

Let now $\mathbb{H}$ be the quaternionic algebra and $j_n(\mathbb{H}) : \mathbb{S}^4 = \mathbb{H}P^1 \hookrightarrow \mathbb{H}P^n$ the canonical inclusion. Then, by [9, Remark 4.9(iii)], the higher order Whitehead product $[j_n(\mathbb{H}), \overset{\times(n+1)}{\ldots\ldots}, j_n(\mathbb{H})] = \emptyset$ for $n \geq 2$. Hence, the existence of a map $u_n(\mathbb{S}^4) : J_n(\mathbb{S}^4) \to \mathbb{H}P^n$ with properties as above leads to a contradiction $u_n(\mathbb{S}^4)\langle \iota_4, \overset{\times(n+1)}{\ldots\ldots}, \iota_4\rangle \in [j_n(\mathbb{H}), \overset{\times(n+1)}{\ldots\ldots}, j_n(\mathbb{H})] = \emptyset$.

Let $\alpha_i : \mathbb{S}^{k_i} \to \mathbb{S}^1$ be maps with $k_i \geq 1$ for $i = 1, \ldots, n$ and $n \geq 2$. If $k_{i_0} > 1$ for some $1 \leq i_0 \leq n$ then $\alpha_{i_0} = 0$ and certainly $\langle \alpha_1, \ldots, \alpha_n\rangle = 0$ in $\pi_{k_1+\cdots+k_n-1}(J_{n-1}(\mathbb{S}^1))$.

Next, consider maps $\alpha_i : \mathbb{S}^{k_i} \to X$ with $k_i \geq 2$ for $i = 1, \ldots, n$ and $n \geq 2$, where $X = \mathbb{S}^2$ or $\mathbb{R}P^2$. Then, $u_{n-1}(\mathbb{S}^2)\langle \alpha_1, \ldots, \alpha_n\rangle = 0$ provided $k_1 + \cdots + k_n < 2n$. Further, by [4, Theorem 2], it holds $SP_n(\mathbb{R}P^2) = \mathbb{R}P^{2n}$. Hence, $u_{n-1}(\mathbb{R}P^2)\langle \alpha_1, \ldots, \alpha_n\rangle = 0$ provided $k_1 + \cdots + k_n < 2n-1$.

We observe that [4, Theorem 2] also gives sequences of maps $A_n(\mathbb{S}^2)$ and $A_n(\mathbb{R}P^2)$ for $n \geq 1$, fitting together by the commutative diagrams

$$\begin{array}{ccc} J_n(\mathbb{S}^2) & \longrightarrow & J_n(\mathbb{R}P^2) \\ {\scriptstyle A_n(\mathbb{S}^2)}\downarrow & & \downarrow{\scriptstyle A_n(\mathbb{R}P^2)} \\ \mathbb{C}P^n & \longrightarrow & \mathbb{R}P^{2n}. \end{array}$$

In this sense, we close the paper with:

Conjecture 5.4. Let $\alpha_i : \mathbb{S}^{k_i} \to X$ be maps with $k_i \geq 2$ for $i = 1, \ldots, n$ and $n \geq 2$, where $X = \mathbb{S}^2$ or $\mathbb{R}P^2$. If $k_{i_0} > 2$ for some $1 \leq i_0 \leq n$ then $\langle \alpha_1, \ldots, \alpha_n\rangle = 0$ in $\pi_{k_1+\cdots+k_n-1}(J_{n-1}(X))$.

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