

Riemannian and Teleparallel Descriptions of the Scalar Field Gravitational Interaction

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Abstract

A comparative study between the metric and the teleparallel descriptions of gravitation is made for the case of a scalar field. In contrast to the current belief that only spin matter could detect the teleparallel geometry, scalar matter being able to feel the metric geometry only, we show that a scalar field is able not only to feel anyone of these geometries, but also to produce torsion. Furthermore, both descriptions are found to be completely equivalent, which means that in fact, besides coupling to curvature, a scalar field couples also to torsion.

I. INTRODUCTION

Since the early days of general relativity, the description of the gravitational interaction has been deeply connected to the geometry of spacetime. According to its postulates, the presence of gravitation produces a curvature in spacetime, the gravitational interaction being achieved by supposing a particle to freely follow its geodesics. Curvature, therefore, is considered to be an intrinsic attribute of spacetime.

On the other hand, theoretical developments have since long evoked the possibility of including torsion in the description of the gravitational interaction. In the usual approach to gravitation, torsion is set to vanish from the very beginning, and there seems to be no compelling experimental evidence not to set this condition. However, as we are going to see, in the context of the teleparallel equivalent of general relativity [1], one can not set the vanishing of torsion without vanishing the curvature as they are manifestations of the same gravitational field. In other words, the vanishing of torsion only, would spoil the alluded equivalence.

Curvature and torsion present completely different characteristics from the point of view of the gravitational interaction. Curvature, according to general relativity, is used to *geometrize* spacetime, and in this way successfully describe the gravitational interaction. On the other hand, teleparallelism attributes gravitation to torsion, but in this case torsion accounts for gravitation not by geometrizing the interaction, but by acting as a force [2].

This means that, in the teleparallel equivalent of general relativity, there are no geodesics, but force equations quite analogous to the Lorentz force equation of electrodynamics.

Differently from what is usually done in general relativity, in what follows we will benefit by separating the notions of space and connections. From a formal point of view, curvature and torsion are in fact properties of a connection [3], and a great many connections may be defined on the same space [4]. Strictly speaking, there is no such a thing as curvature or torsion of spacetime, but only curvature or torsion of connections. This becomes evident if we notice that different particles feel different connections, and consequently show distinct trajectories in spacetime. In the general relativity case, though, there is a point for taking the Levi-Civita connection of spacetime as part of its definition: universality of gravitation implies that all particles feel it the same, and this makes possible to interpret the curvature of the connection as the curvature of spacetime itself, leading thus to the general relativity scene. It seems far wiser, however, to take spacetime simply as a manifold, and connections (with their curvatures and torsions) as additional structures.

With the purpose of exploring the interaction of gravitation with a scalar field, as well as the role played by curvature and torsion in the description of this interaction, we assume the background of this work to be a spacetime manifold on which a nontrivial tetrad field is defined. The context may be, for example, that of a gauge theory for the translation group [2], which is the most common prototype of a tetrad theory. In this context, the gravitational field appears as the nontrivial part of the tetrad field. We will use the greek alphabet ($\mu, \nu, \rho, \dots = 1, 2, 3, 4$) to denote tensor indices, that is, indices related to spacetime. The latin alphabet ($a, b, c, \dots = 1, 2, 3, 4$) will be used to denote local Lorentz (or tangent space) indices. Of course, being of the same kind, tensor and local Lorentz indices can be changed into each other with the use of the tetrad, denoted by h^a_μ , and supposed to satisfy

$$h^a_\mu h_a^\nu = \delta_\mu^\nu \quad ; \quad h^a_\mu h_b^\mu = \delta^a_b . \quad (1)$$

II. THE SPACETIME GEOMETRY

As already discussed, curvature and torsion are properties of a connection, and many different connections may be defined on the same space. For example, denoting by η_{ab} the metric tensor of the tangent space, a nontrivial tetrad field can be used to define the riemannian metric

$$g_{\mu\nu} = \eta_{ab} h^a_\mu h^b_\nu , \quad (2)$$

in terms of which the Levi-Civita connection

$$\overset{\circ}{\Gamma}^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} [\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}] \quad (3)$$

can be introduced. Its curvature

$$\overset{\circ}{R}^\theta_{\rho\mu\nu} = \partial_\mu \overset{\circ}{\Gamma}^\theta_{\rho\nu} + \overset{\circ}{\Gamma}^\theta_{\sigma\mu} \overset{\circ}{\Gamma}^\sigma_{\rho\nu} - (\mu \leftrightarrow \nu) , \quad (4)$$

according to general relativity, accounts exactly for the gravitational interaction. Owing to the universality of gravitation, which means that all particles feel $\overset{\circ}{\Gamma}^\sigma_{\mu\nu}$ the same, it turns

out possible to describe the gravitational interaction by considering a Riemann spacetime with the curvature of the Levi–Civita connection, in which all particles will follow geodesics. This is the stage set of Einstein’s General Relativity, the gravitational interaction being mimicked by a geometrization of spacetime.

On the other hand, a nontrivial tetrad field can also be used to define the linear Cartan connection

$$\Gamma^\sigma_{\mu\nu} = h_a^\sigma \partial_\nu h^a_\mu , \quad (5)$$

with respect to which the tetrad is parallel:

$$\nabla_\nu h^a_\mu \equiv \partial_\nu h^a_\mu - \Gamma^\rho_{\mu\nu} h^a_\rho = 0 . \quad (6)$$

For this reason, tetrad theories have received the name of teleparallelism, or absolute parallelism. Plugging in Eqs.(2) and (3), we get

$$\Gamma^\sigma_{\mu\nu} = \overset{\circ}{\Gamma}^\sigma_{\mu\nu} + K^\sigma_{\mu\nu} , \quad (7)$$

where

$$K^\sigma_{\mu\nu} = \frac{1}{2} [T_\mu^\sigma{}_\nu + T_\nu^\sigma{}_\mu - T^\sigma{}_{\mu\nu}] \quad (8)$$

is the contorsion tensor, with

$$T^\sigma{}_{\mu\nu} = \Gamma^\sigma_{\nu\mu} - \Gamma^\sigma_{\mu\nu} \quad (9)$$

the torsion of the Cartan connection. If now, analogously to the way the Riemann spacetime was introduced, we try to introduce a spacetime with the same properties of the Cartan connection $\Gamma^\sigma_{\nu\mu}$, we end up with a Weitzenböck spacetime [5], a space presenting torsion, but no curvature. This spacetime is the stage set of the teleparallel description of gravitation. Considering that local Lorentz indices are raised and lowered with the Minkowski metric η^{ab} , tensor indices on it will be raised and lowered with the riemannian metric $g_{\mu\nu} = \eta_{ab} h^a_\mu h^b_\nu$ [1]. Universality of gravitation, in this case, means that all particles feel $\Gamma^\sigma_{\nu\mu}$ the same, which in turn means that they will also feel torsion the same.

From the above considerations, we can infer that the presence of a nontrivial tetrad field induces both, a riemannian and a teleparallel structures in spacetime. The first is related to the Levi–Civita connection, a connection presenting curvature, but no torsion. The second is related to the Cartan connection, a connection presenting torsion, but no curvature. It is important to remark that both connections are defined on the very same spacetime, a spacetime endowed with both a riemannian and a teleparallel structures.

As already remarked, the curvature of the Cartan connection vanishes identically:

$$R^\theta_{\rho\mu\nu} = \partial_\mu \Gamma^\theta_{\rho\nu} + \Gamma^\theta_{\sigma\mu} \Gamma^\sigma_{\rho\nu} - (\mu \leftrightarrow \nu) \equiv 0 . \quad (10)$$

Substituting $\Gamma^\theta_{\mu\nu}$ from Eq.(7), we get

$$R^\theta_{\rho\mu\nu} = \overset{\circ}{R}^\theta_{\rho\mu\nu} + Q^\theta_{\rho\mu\nu} \equiv 0 , \quad (11)$$

where $\overset{\circ}{R}{}^\theta_{\rho\mu\nu}$ is the curvature of the Levi-Civita connection, and

$$Q^\theta_{\rho\mu\nu} = D_\mu K^\theta_{\rho\nu} + K^\theta_{\sigma\mu} K^\sigma_{\rho\nu} - (\mu \leftrightarrow \nu) \quad (12)$$

with

$$D_\mu K^\theta_{\rho\nu} = \partial_\mu K^\theta_{\rho\nu} + (\Gamma^\theta_{\sigma\mu} - K^\theta_{\sigma\mu}) K^\sigma_{\rho\nu} - (\Gamma^\sigma_{\rho\mu} - K^\sigma_{\rho\mu}) K^\theta_{\sigma\nu} . \quad (13)$$

Equation (11) has an interesting interpretation: the contribution $\overset{\circ}{R}{}^\theta_{\rho\mu\nu}$ coming from the Levi-Civita connection, compensates exactly the contribution $Q^\theta_{\rho\mu\nu}$ coming from the contorsion tensor, yielding an identically zero Cartan curvature tensor $\overset{\circ}{R}{}^\theta_{\rho\mu\nu}$. This is a constraint satisfied by the Levi-Civita and Cartan connections, and is the fulcrum of the equivalence between the riemannian and the teleparallel descriptions of gravitation.

III. GENERAL RELATIVITY AND ITS TELEPARALLEL EQUIVALENT

According to general relativity, the description of the interaction between scalar matter and gravitation requires a spacetime endowed with a riemannian structure. The dynamics of the gravitational field, in this case, turns out to be described by a variational principle with the lagrangian

$$\mathcal{L}_G = \frac{\sqrt{-g} c^4}{16\pi G} \overset{\circ}{R} , \quad (14)$$

where $\overset{\circ}{R} = g^{\mu\nu} \overset{\circ}{R}{}^\rho_{\mu\rho\nu}$ is the scalar curvature of the Levi-Civita connection, and $g = \det(g_{\mu\nu})$. This lagrangian, which depends on the Levi-Civita connection only, can be rewritten in an alternative form depending on the Cartan connection only. In fact, substituting $\overset{\circ}{R}$ as obtained from Eq.(11), up to divergences we obtain

$$\mathcal{L}_G = \frac{hc^4}{16\pi G} \left[\frac{1}{4} T^\rho{}_{\mu\nu} T^\mu{}_{\rho}{}^{\nu} + \frac{1}{2} T^\rho{}_{\mu\nu} T^{\nu\mu}{}_{\rho} - T_{\rho\mu}{}^{\rho} T^{\nu\mu}{}_{\nu} \right] , \quad (15)$$

where $h = \det(h^a{}_\mu) = \sqrt{-g}$. This is exactly the lagrangian of a gauge theory for the translation group [2], which means that a translational gauge theory, with a lagrangian quadratic in the torsion field, is completely equivalent to general relativity, with its usual lagrangian linear in the scalar curvature [6]. As a consequence of this equivalence, therefore, gravitation might have two equivalent descriptions, one in terms of a metric geometry, and another one in which the underlying geometry is that provided by a teleparallel structure. It is important to remark that, in this approach, the lagrangian (15) has been obtained without requiring it to be local Lorentz invariant. The usual criticism [7] about the deduction of that lagrangian [8], therefore, does not apply here.

In the absence of gravitational field, the tetrad becomes trivial, $g_{\mu\nu}$ becomes the Minkowski metric, and both the curvature $\overset{\circ}{R}{}^\theta_{\rho\mu\nu}$ as well as the torsion $T^\theta_{\mu\nu}$ vanish simultaneously. In other words, it is not possible to set a vanishing torsion without having

a vanishing curvature, as they are manifestations of the same gravitational field. It is important to remember at this point that curvature and torsion are geometrical properties of different connections. There is no a connection presenting simultaneously non-vanishing curvature and torsion, which means that no Riemann–Cartan spacetime enters the description of the gravitational interaction. Furthermore, according to our approach, we can say that general relativity does not assume a vanishing torsion: despite always present, it simply does not make use of it. On the other hand, in consonance to what happens with the lagrangian of the gravitational field, there exists an alternative description of gravitation, the so called teleparallel description, which makes use of torsion only, not curvature. In this sense, the Cartan connection can be considered as a kind of *dual* to the Levi–Civita connection, the riemannian–teleparallel equivalence being a kind of *dual symmetry* presented by gravitation.

IV. SCALAR FIELDS: LAPLACE–BELTRAMI AND ITS TELEPARALLEL EQUIVALENT

Let us consider the lagrangian for a free scalar field ϕ in a Minkowski spacetime [9]:

$$\mathcal{L}_\phi = \frac{1}{2} \left[\eta^{ab} \partial_a \phi \partial_b \phi - m^2 \phi^2 \right] . \quad (16)$$

According to the usual minimal coupling prescription, which brings the free lagrangian to a lagrangian written in terms of the riemannian structure of spacetime, the gravitational interaction can be obtained through the replacements

$$\eta^{ab} \longrightarrow g^{\mu\nu} \quad (17)$$

$$\partial_a \longrightarrow \overset{\circ}{\nabla}_\mu , \quad (18)$$

with $g^{\mu\nu}$ a riemannian metric, and $\overset{\circ}{\nabla}_\mu$ the Levi–Civita covariant derivative which, for the specific case of a scalar field, is simply an ordinary derivative. Therefore, in terms of the riemannian structure, the lagrangian describing a scalar field in interaction with gravitation turns out to be

$$\mathcal{L}_\phi = \frac{\sqrt{-g}}{2} \left[g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - m^2 \phi^2 \right] . \quad (19)$$

By using the identity

$$\partial_\mu \sqrt{-g} = \frac{\sqrt{-g}}{2} g^{\rho\lambda} \partial_\mu g_{\rho\lambda} \equiv \sqrt{-g} \overset{\circ}{\Gamma}^\rho_{\mu\rho} ,$$

it is easy to show that the corresponding field equation is

$$\overset{\circ}{\square} \phi + m^2 \phi = 0 , \quad (20)$$

where

$$\overset{\circ}{\square} \phi = \overset{\circ}{\nabla}_\mu \partial^\mu \phi \equiv \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\rho} \partial_\rho \phi \right) \quad (21)$$

is the Laplace–Beltrami derivative of ϕ , with

$$\overset{\circ}{\nabla}_\mu = \partial_\mu + \overset{\circ}{\Gamma}^\rho_{\mu\rho} \quad (22)$$

the expression for the Levi–Civita covariant divergence of $\partial^\mu \phi$. We notice in passing that it is completely equivalent to apply the minimal coupling prescription in the lagrangian or in the field equations. In a locally inertial coordinate system, the Levi–Civita connection vanishes, and the Laplace–Beltrami becomes the free–field d’Alambertian operator. This is the usual version of the (strong) equivalence principle [10].

Let us consider now the total lagrangian $\mathcal{L} = \mathcal{L}_G + \mathcal{L}_\phi$, with \mathcal{L}_G given by Eq.(14) and \mathcal{L}_ϕ by Eq.(19). Variation in relation to the metric tensor $g^{\mu\nu}$ yields the gravitational field equation

$$\overset{\circ}{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \overset{\circ}{R} = \frac{8\pi G}{c^4} \mathcal{T}_{\mu\nu} , \quad (23)$$

where

$$\mathcal{T}_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_\phi}{\delta g^{\mu\nu}}$$

is the energy–momentum tensor of the scalar field. In the riemannian description of gravitation, therefore, the energy–momentum tensor of any matter field, as for example a scalar field, is able to produce curvature.

Now, we look for a minimal coupling prescription which brings the free lagrangian (16) to a lagrangian written in terms of the teleparallel structure of spacetime. This prescription is given by

$$\eta^{ab} \longrightarrow \eta^{ab} \quad (24)$$

$$\partial_a \longrightarrow D_a = h_a{}^\mu D_\mu , \quad (25)$$

where

$$D_\mu = \nabla_\mu - K_\mu \quad (26)$$

is the teleparallel version of the covariant derivative, with ∇_μ the Cartan covariant derivative, and K_μ the contorsion tensor. Therefore, in terms of the teleparallel structure, the scalar field lagrangian assumes the form

$$\mathcal{L}_\phi = \frac{h}{2} \left[\eta^{ab} D_a \phi D_b \phi - m^2 \phi^2 \right] , \quad (27)$$

where, for the specific case of a scalar field,

$$D_a = h_a{}^\mu \partial_\mu . \quad (28)$$

Using the identity

$$\partial_\mu h = h h_a{}^\rho \partial_\mu h^a{}_\rho \equiv h \Gamma^\rho_{\rho\mu} ,$$

it is easy to show that the corresponding field equation is

$$\square\phi + m^2\phi = 0 , \quad (29)$$

where

$$\square\phi = (\partial_\mu + \Gamma^\rho_{\rho\mu}) \partial^\mu\phi \equiv h^{-1} \partial_\rho (h \partial^\rho\phi) \quad (30)$$

is the teleparallel version of the Laplace–Beltrami operator. Because $\Gamma^\rho_{\rho\mu}$ is not symmetric in the last two indices, $(\partial_\mu + \Gamma^\rho_{\rho\mu})$ is not the expression for the Cartan covariant divergence of $\partial^\mu\phi$. By using Eq.(9), however, the expression for $\square\phi$ may be rewritten in the form

$$\square\phi = (\nabla_\mu + T^\rho_{\mu\rho}) \partial^\mu\phi , \quad (31)$$

where

$$\nabla_\mu = \partial_\mu + \Gamma^\rho_{\mu\rho}$$

is now the correct expression for the Cartan covariant divergence of $\partial^\mu\phi$. Making use of the identity

$$T^\rho_{\mu\rho} = -K^\rho_{\mu\rho} , \quad (32)$$

easily obtained from Eq.(8), the teleparallel version of the scalar field equation of motion is

$$D_\mu \partial^\mu\phi + m^2\phi = 0 , \quad (33)$$

with D_μ the teleparallel covariant derivative (26), here applied to the spacetime vector field $\partial^\mu\phi$. Besides justifying the form of the teleparallel minimal coupling prescription, therefore, we find that, also in this case, it is completely equivalent to apply the minimal coupling prescription in the lagrangian or in the field equation.

Now, Eq.(33) can be rewritten as

$$\nabla_\mu \partial^\mu\phi + m^2\phi = -T^\rho_{\mu\rho} \partial^\mu\phi \equiv K^\rho_{\mu\rho} \partial^\mu\phi . \quad (34)$$

In this form, it shows clearly that a scalar field, through its derivative $\partial^\mu\phi$, couples to, and therefore feels torsion. Moreover, it reveals that torsion plays a role similar to an external force [2], quite analogous to the role played by the electromagnetic field in the Lorentz force equation. On the other hand, from Eqs.(2) and (5) we have

$$\Gamma_{\rho\lambda\mu} = -\Gamma_{\lambda\rho\mu} + \partial_\mu g_{\rho\lambda} .$$

Thus, in a locally inertial coordinate system, the Cartan connection becomes skew symmetric in the first two indices, $\Gamma^\rho_{\rho\mu}$ consequently vanishes, and the teleparallel version of the Laplace–Beltrami operator becomes the free–field d’Alembertian operator. This is the teleparallel version of the (strong) equivalence principle.

Let us consider again the total lagrangian $\mathcal{L} = \mathcal{L}_G + \mathcal{L}_\phi$, but now with \mathcal{L}_G given by Eq.(15), and \mathcal{L}_ϕ by Eq.(27). Variation in relation to the tetrad field yields the teleparallel version of the gravitational field equation, which can be written in the form [2]

$$\partial_\rho S_\mu{}^{\nu\rho} - \frac{4\pi G}{c^4} t_\mu{}^\nu = \frac{4\pi G}{c^4} \mathcal{T}_\mu{}^\nu , \quad (35)$$

where $t_\mu{}^\nu$ is the energy–momentum (pseudo) tensor of the gravitational field,

$$\mathcal{T}_\mu{}^\nu = h^\alpha{}_\mu \left(-\frac{1}{h} \frac{\delta \mathcal{L}_\phi}{\delta h^\alpha{}_\nu} \right)$$

is the energy–momentum tensor of the scalar field, and

$$S_\mu{}^{\nu\rho} = \frac{1}{4} (T_\mu{}^{\nu\rho} + T^\nu{}_\mu{}^\rho - T^\rho{}_\mu{}^\nu) - \frac{1}{2} (\delta_\mu{}^\rho T_\theta{}^{\nu\theta} - \delta_\mu{}^\nu T_\theta{}^{\rho\theta}) .$$

In the teleparallel description of gravitation, therefore, energy and momentum are the source of the dynamical torsion, a point which is not in agreement with the usual belief that only a spin distribution could produce a torsion field [11]. Similar results have already been obtained in the literature [12], being in fact the correct source of torsion still an open problem.

We remark once more that the Levi–Civita and the Cartan connections are both defined on the very same spacetime, a manifold endowed with both a riemannian and a teleparallel structures. Moreover, it is possible to go from the riemannian to the teleparallel description through very simple manipulations. For example, take the teleparallel lagrangian (27), and substitute the covariant derivative (28). Then, by using Eq.(2), one can easily see that this lagrangian reduces to the riemannian lagrangian (19). Obviously, the same is true for the field equations: if we substitute relation (7) in the teleparallel Laplace–Beltrami (30), it is an easy task to verify that it reduces to the riemannian Laplace–Beltrami (21). We notice furthermore that, in both descriptions, it results completely equivalent to use the minimal coupling prescription in the lagrangians or in the equations of motion.

V. FINAL REMARKS

It has been known since long that only particles with spin could detect the teleparallel geometry, scalar matter being able to feel the metric geometry only [13]. However, as we have seen, the interaction of gravitation with scalar matter can be described alternatively in terms of magnitudes related to the riemannian or to the teleparallel structures defined in spacetime, which are structures related respectively to the Levi–Civita and the Cartan connections. Ultimately, this means that scalar matter is able to feel anyone of these geometries. In other words, scalar matter, through its derivative $\partial^\mu \phi$, is able to feel, and therefore couples to torsion. Furthermore, based on the equivalence of the corresponding lagrangians and field equations, we conclude that the description in terms of the teleparallel geometry is completely equivalent to the description in terms of the riemannian geometry. This means that, besides coupling to torsion, the scalar field, trough its energy–momentum tensor, can also be the source of torsion. It should be remarked, however, that in the teleparallel description, the gravitational interaction is not geometrized in the sense it is in general relativity, presenting characteristics quite analogous to those provided by gauge theories [14], with torsion playing the role of force. It is also important to remark that, according to this approach, no Riemann–Cartan geometry enters into the description of the gravitational interaction. As a matter of fact, no experimental evidence seems to indicate the necessity of including torsion, besides curvature, to correctly account for the gravitational interaction .

Finally, it is worth mentioning that the minimal coupling prescription (24)–(26), introduced here to describe a gravitationally coupled scalar field in the framework of the teleparallel equivalent of general relativity, can be consistently applied to other fields as well. In the case of the spin–one Maxwell field, for example, it is obtained as a consequence of its application that, besides being able to be minimally coupled to torsion, the electromagnetic field, through its energy–momentum tensor, can also produce torsion. Furthermore, this coupling of the electromagnetic field with torsion is found to preserve the local gauge invariance of Maxwell’s theory, yielding in this way a consistent description of such interaction [15].

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REFERENCES

- [1] Hayashi, K., and Shirafuji, T. (1979). *Phys. Rev.* **D19**, 3524.
- [2] de Andrade, V. C., and Pereira, J. G. (1997). *Phys. Rev.* **D**, in press (gr-qc/9703059).
- [3] Kobayashi, S., and Nomizu, K. (1963). *Foundations of Differential Geometry* (Interscience, New York).
- [4] Aldrovandi, R., and Pereira, J. G. (1995). *An Introduction to Geometrical Physics* (World Scientific, Singapore).
- [5] Weitzenböck, R. (1923). *Invariantentheorie* (Noordhoff, Gronningen).
- [6] Maluf, J. W. (1994). *J. Math. Phys.* **35**, 335.
- [7] Hayashi, K. (1977). *Phys. Lett* **B69**, 441.
- [8] Cho, Y. M. (1976). *Phys. Rev* **D14**, 2521.
- [9] See, for example: Birrel, N. D., and Davies, P. C. W. (1982). *Quantum Fields in Curved Spacetime* (Cambridge Univ. Press, Cambridge).
- [10] See, for example: Weinberg, S. (1972). *Gravitation and Cosmology* (Wiley, New York).
- [11] See, for example, Hammond, R. T. (1996). *Gen. Rel. Grav.* **28**, 749.
- [12] Saa, A. (1997). *Gen. Rel. Grav.* **29**, 205.
- [13] Hehl, F. W., von der Heyde, P., and Kerlick, G. D. (1976). *Rev. Mod. Phys.* **48**, 393.
- [14] Hehl, F. W., McCrea, J. D., Mielke, E. W., and Ne'eman, Y. (1995). *Phys. Rep.* **258**, 1.
- [15] de Andrade, V. C., and Pereira, J. G. (1997). *Torsion and the Electromagnetic Field*, Preprint IFT.P-97 (gr-qc/9708051).