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Why the Light Front Time x^+ is the Best Time Variable

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Abstract Investigating the coordinate transformations from the usual Minkowski's space-time to the light-front coordinates, we obtain two possible forms to represent the time coordinate in the light-front, x^+ and/or x^-, which usually is said to be equivalent and symmetric. We observe that this equivalence and symmetry is only apparent, since in the ultra-relativistic limit $v \to c$ the former x^+ is well defined while the latter x^- becomes singular and not well-defined. From this setting and choice we analyze the concept of time in quantum mechanics tunneling effect using the light front tools.

1 Introduction

In 1932 MacColl [1] studied the transmission of a wave packet through a square potential barrier and concluded "that there is no appreciable delay" comparing the times of the packet that enters and leaves the barrier. Thus, the question we pose is what do we understand by tunneling time? From Newtonian classical mechanics, time can be viewed as the distance traveled by the wave packet divided by the speed of the packet. However, this cannot be applied here. First because classically, there is no tunneling effect in such a setting; secondly once the quantum system can cross the region of the barrier potential that is forbidden classically, and MacColl's argue that there is no apparent delay, we are interested in knowing how long actually the wave packet will take to cross through the barrier using the variable x^+ as the time in the light front.

In 1949 Dirac [2] constructed three different forms of relativistic dynamics, depending on the hypersurface chosen to describe the temporal evolution of the system: The instant form, in which the initial conditions (position and velocity) are given at $t = 0$; the point form, in which the initial conditions are set on a hyperboloid defined by $x^2 = a^2$, with a constant and $t > 0$ and the front form, in which initial conditions are set on the surface tangent to the light cone, $t + z/c = 0$. The latter form is also commonly referred to as the light front. In the light front the x^+ coordinate is usually taken as the parameter for the temporal evolution for the system. Many applications in light front has shown interesting results that deserve their special attention [3]. The change to the light-front frame is not a Lorentz transformation, nor the limit of a Lorentz reference frame

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moving in the z direction with the speed of light [4], which is commonly known as the infinite momentum frame. Since x^+ and x^- are on the very light-cone and anything on this line moves with the speed of light, is it possible and meaningful to define Lorentz transformations on these light cone variables? Our work shows it is feasible and with surprising effects and results.

2 Light Front Coordinates and Time Variable

The light-cone coordinates are given by $x^{\pm} = \frac{1}{\sqrt{2}}(ct \pm z)$ and $\mathbf{x}_{\perp} = x\hat{\imath} + y\hat{\jmath} = x^1\hat{\imath} + x^2\hat{\jmath}$, where as usual, $\hat{\imath}$ and $\hat{\jmath}$ are respectively the unit vectors along coordinates x and y, such that for a Lorentz transformation along the $x^3 = z$ axis this implies $\mathbf{x}'_{\perp} = \mathbf{x}_{\perp}$. Our light-front metric $(+, -, 1, 2)$ is therefore $g^{+-} = -g^{11} = -g^{22} = 1$. Scalar product of two vectors is then $a \cdot b = a^+b^- + a^-b^+ - \mathbf{a}_{\perp} \cdot \mathbf{b}_{\perp}$.

From the definition of the light-cone coordinates we may define the following intervals:

$$\begin{aligned} \Delta x^{\pm} &= \frac{1}{\sqrt{2}}\left(\Delta x^0 \pm \Delta x^3\right) \\ &= \frac{1}{\sqrt{2}}\Delta t\,(c \pm v) \end{aligned} \tag{1}$$

so that we can express the usual time interval in terms of light-cone variables

$$\Delta t = \sqrt{2}\left(\frac{\Delta x^+}{c+v}\right) \tag{2}$$

$$= \sqrt{2}\left(\frac{\Delta x^-}{c-v}\right) \tag{3}$$

where v is referential velocity along the $z = x^3$ direction.

The two Eqs. (2) and (3) are proportional to the variations in the coordinates $x^+$and x^- respectively. We note that the definition of time in terms of Δx^+, Eq. (2) is well defined in the limit $v \to c$, whereas diverges in Eq. (3) for there is a singularity in the same limit. Therefore, the coordinate x^+ in light front dynamics is the best choice to represent the "time" variable.

3 The Tunneling Time

According to the laws of Newtonian classical mechanics, if a particle has a total energy that is smaller than the height of a potential barrier, it cannot cross the barrier. The reason for this is that the total energy of the particle is the sum of its kinetic energy and its potential energy. For conservative systems, the total mechanical energy is a constant of motion and therefore as the particle passes from a region of less intense potential to another of more intense potential, its kinetic energy will decrease until it eventually vanishes. At this point it defines the point of return, i.e., the particle reverses its direction of motion and is bounced back to where it came from.

In quantum mechanics, the scenario is quite different. Here there exists a non zero probability of the particle crossing the potential barrier with finite height. This phenomenon is known as tunneling effect. MacColl [1] studied cases in which the incident packet is a superposition of states with energies smaller than the height of the potential and concluded that the transmitted packet appears at the end extremum of the barrier approximately at the instant in which the incident packet hits the initial extremum of the barrier, that is, there is apparently no delay! Thus begins the study of tunneling time (and/or delay time).

Let us consider solutions to the one-dimensional Schrödinger equation, but restricting ourselves to a constant potential barrier, i.e., $V(z) = V = constant$ for $0 < z < L$ and $V(z) = 0$ outside this region, for $z < 0$ and $z > L$. Let us also consider an incident wave packet made up of a superposition of plane waves in the form

$$\psi_{\text{plane}}(z, t) = e^{i(kz - Et)}, \qquad z < 0, \tag{4}$$

where for simplicity we have taken $\hbar = 1$. Then

$$\psi_{\text{inc}}(z,t) = \int_0^\infty dE A(E) e^{i[k(E)z-Et]} \equiv \int_0^\infty dE |A(E)| e^{i\varphi(E)}$$
$$= \int_0^\infty dE |A(E)| \cos\varphi(E) + i \int_0^\infty dE |A(E)| \sin\varphi(E), \tag{5}$$

where the coefficient $A(E)$ is the energy distribution for the incident wave and $\varphi(E) = kz - Et + \phi(E)$.

The behavior of the sinusoidal functions in Eq. (5) with respect to the energy E is such that if the coefficient $|A(E)|$ remains approximately constant around a given point E_0 along with large variations of the phase $\phi(E)$, the integration is approximately zero because the integrand behaves as a sinusoidal function modulated by an almost constant function. A nonvanishing result for the integration results for the opposite situation in which $|A(E)|$ is a peak function around E_0 and the phase $\phi(E)$ remains stationary in its vicinity. This characterizes our incident wave packet. The stationary phase condition is therefore obtained by the following condition

$$\left.\frac{d\varphi(E)}{dE}\right|_{E_0} = 0. \tag{6}$$

For a wave packet, with a defined peak centered on E_0, the stationary phase condition is therefore the best first approximation where we have the largest contribution to the integration. In this manner, we may write the incident (I), reflected (R) and transmitted (T) wave packets at the barrier respectively as

$$\psi_{\text{I}}(t,z,k) = \int dE\, |A_1(E)|\, e^{-i[kz-Et+\phi_1(E)]}, \tag{7}$$

$$\psi_{\text{R}}(t,z,k) = \int dE\, |A_2(E)|\, e^{i[kz-Et+\phi_2(E)]}, \tag{8}$$

$$\psi_{\text{T}}(t,z,k) = \int dE\, |A_3(E)|\, e^{-i[kz-Et+\phi_3(E)]}, \tag{9}$$

where to be consistent with our front notation afterwards, we have taken the potential barrier along the $z = x^3$ direction.

As the incident wave "hits" the barrier, both amplitude and phase become affected; so we may express the reflection and transmission phases in terms of the incident phase as follows:

$$\phi_2(E) = \phi_1(E) + \phi_{\text{R}}(E), \tag{10}$$
$$\phi_3(E) = \phi_1(E) + \phi_{\text{T}}(E), \tag{11}$$

where $\phi_{\text{R}}(E)$ and $\phi_{\text{T}}(E)$ are the phases for the reflection and transmission coefficients respectively. By the stationary phase method, Eq. (6), we have

$$-\frac{dk}{dE}z + t - \frac{d\phi_1(E)}{dE} = 0.$$

Thus, the time of incidence of the wave packet is given by

$$t_{\text{I}} = \frac{dk}{dE}z + \frac{d\phi_1(E)}{dE}. \tag{12}$$

In a similar manner we may write for the time of reflection of the wave packet as

$$t_{\text{R}} = -\frac{dk}{dE}z + \frac{d\phi_1(E)}{dE} + \frac{d\phi_{\text{R}}(E)}{dE}, \tag{13}$$

and for the time of transmission of the wave packet,

$$t_{\text{T}} = \frac{dk}{dE}z + \frac{d\phi_1(E)}{dE} + \frac{d\phi_{\text{T}}(E)}{dE}. \tag{14}$$

The instant in which the incident wave packet enters the barrier (we may make a mental visualization of this as the instant in which the peak of the wave packet is at $z = 0$) is defined as

$$t_{\text{en}} \equiv t_{\text{I}}|_{z=0} = \frac{d\phi_1(E)}{dE}, \tag{15}$$

where $t_{\text{I}}(0)$ corresponds to that time at which the mentalized peak of the packet is at $z = 0$ (, i.e., the definition of the instant in which the wave packet enters into the barrier). In the same vein, we may consider the mental construct of the instant in which the peak of the packet exits the barrier as being given by

$$t_{\text{ex}} \equiv t_{\text{T}}|_{z=L} = \frac{dk}{dE}L + \frac{d\phi_1(E)}{dE} + \frac{d\phi_{\text{T}}(E)}{dE}, \tag{16}$$

where we have used Eq. (14) at position $z = L$. The difference between these two instants of time is defined as the time of tunneling,

$$T_{\text{tun}} = t_{\text{ex}} - t_{\text{en}} = t_{\text{vacuum}} + \frac{d\phi_{\text{T}}(E)}{dE}, \tag{17}$$

where $t_{\text{vacuum}} = \dfrac{dk}{dE}L$ is called time in vacuum, that is, in the absence of a potential barrier, the wave packet travels the distance L corresponding to the width of the barrier in such a time, and the other term, namely, $\dfrac{d\phi_{\text{T}}(E)}{dE}$ is known as the delay time.

In a similar manner, we may take the instant in which the reflected peak exits back from the barrier as

$$t_{\text{ex,R}} \equiv t_{\text{R}}|_{z=0} = \frac{d\phi_1(E)}{dE} + \frac{d\phi_{\text{R}}(E)}{dE}. \tag{18}$$

Thus, the time interval from entrance to reflection is the difference between the two as follows

$$T_{\text{R}} = t_{\text{ex,R}} - t_{\text{en}} = \frac{d\phi_{\text{R}}(E)}{dE}. \tag{19}$$

From this last equation we can note that the wave packet is not, in general, reflected instantaneously by the potential, but has also what we may think of as a delaying time for the reflection to occur.

4 Time of Tunneling in the Light Front

The fundamental reason for the existence of a focused study on the time of tunneling is found in the fact that time is not an observable in quantum mechanics [5]. By this very reason, there are several definitions for the time of tunneling which in a majority of cases are equivalent among themselves.

Our aim here is to present a version using the delay time of the signal for a wave in the light front, taking as the front time the variable x^+ as introduced in our previous Sect. 2. Since the front form may be understood as a change in coordinate system from the instant form, we can here also consider the stationary solutions for a barrier with potential energy V between $0 < x^- < L$ and zero outside this region. Though this barrier resembles the one we considered before, this is quite a distinct potential, defined in a different coordinate system. Since we are dealing with a one-dimensional problem, we set to zero from the start all the transverse components, $\mathbf{x}_\perp = \mathbf{0}$ and $\mathbf{k}_\perp = \mathbf{0}$.

For a particle with total energy E which in the light front is defined by $E = k^-$, we have the incident, reflected and transmitted waves in the light front as

$$\psi_{\text{I}}(x^+, x^-, k^+) = \int dk^- \left|A_1(k^-)\right| e^{-i[k^- x^+ + k^+ x^- + \phi_1(k^-)]}, \tag{20}$$

$$\psi_{\text{R}}(x^+, x^-, k^+) = \int dk^- \left|A_2(k^-)\right| e^{i[k^- x^+ + k^+ x^- + \phi_2(k^-)]}, \tag{21}$$

$$\psi_{\text{T}}(x^+, x^-, k^+) = \int dk^- \left|A_3(k^-)\right| e^{-i[k^- x^+ + k^+ x^- + \phi_3(k^-)]}. \tag{22}$$

As in the instant form case, we may write the complex amplitude of the wave packet as $A_{\mathrm{j}}(k^-) = \left|A_{\mathrm{j}}(k^-)\right| e^{i\phi_{\mathrm{j}}(k^-)}$, for $j = 1, 2, 3$; the difference now is that the energy dependence is given by k^-. As before, the action of the potential on each of the components of the reflected and transmitted waves results in modulation in their amplitudes and a change in their phases, changes that in general depend on the energy of each component. Therefore we may write

$$\phi_2(k^-) = \phi_1(k^-) + \phi_{\mathrm{R}}(k^-), \tag{23}$$

$$\phi_3(k^-) = \phi_1(k^-) + \phi_{\mathrm{T}}(k^-), \tag{24}$$

where $\phi_{\mathrm{R}}(k^-)$ and $\phi_{\mathrm{T}}(k^-)$ are the phases of the reflection and transmission coefficients respectively. We may follow the same line of reasoning as before and using the light front equivalent for the method of stationary phase, Eq. (6), for a one-dimensional wave propagation along x^-, i.e., $\mathbf{k}_\perp = 0$ and $\mathbf{x}_\perp = 0$ in Eq. (20), we get

$$-x^+ - \frac{d}{dk^-}(k^+x^-) - \frac{d\phi_1(k^-)}{dk^-} = 0. \tag{25}$$

Thus, in an analogous way as we did in the instant form, the light front time of incidence of the wave packet may be defined as

$$x_{\mathrm{I}}^+ = -\frac{d}{dk^-}(k^+x^-) - \frac{d\phi_1(k^-)}{dk^-}. \tag{26}$$

Similarly, from the other two equations, Eqs. (21) and (22), we have the reflected and transmitted light front times respectively as

$$x_{\mathrm{R}}^+ = -\frac{d}{dk^-}(k^+x^-) - \frac{d\phi_1(k^-)}{dk^-} - \frac{d\phi_{\mathrm{R}}(k^-)}{dk^-}, \tag{27}$$

and

$$x_{\mathrm{T}}^+ = -\frac{d}{dk^-}(k^+x^-) - \frac{d\phi_1(k^-)}{dk^-} - \frac{d\phi_{\mathrm{T}}(k^-)}{dk^-}. \tag{28}$$

The light front time x_{R}^+ is valid for $x^- < 0$ whereas x_{T}^+ is valid for $x^- > L$. The light front times Eqs. (26), (27) and (28), as in the instant form, may be thought of as associated to the peaks of the incident, reflected and transmitted packets respectively, though not necessarily so experimentally or analytically speaking, but only as a mental construct to help visualize the relevant instants of time considered.

We may now calculate some important instants. For example, the instant of entrance of the incident packet into the barrier,

$$x_{\mathrm{en}}^+ = x_{\mathrm{I}}^+\big|_{x^-=0} = -\left.\frac{d}{dk^-}(k^+x^-)\right|_{x^-=0} - \frac{d\phi_1(k^-)}{dk^-}. \tag{29}$$

Another instant of time with particular importance is the time of exit from the barrier of the transmitted packet:

$$x_{\mathrm{ex}}^+ = x_{\mathrm{T}}^+\big|_{x^-=L} = -\left.\frac{d}{dk^-}(k^+x^-)\right|_{x^-=L} - \frac{d\phi_1(k^-)}{dk^-} - \frac{d\phi_{\mathrm{T}}(k^-)}{dk^-}. \tag{30}$$

The difference between these two instants of time may be defined as the tunneling time in the light front

$$\begin{aligned} X_{\mathrm{tun}}^+ = x_{\mathrm{ex}}^+ - x_{\mathrm{en}}^+ &= -\left.\frac{d}{dk^-}(k^+x^-)\right|_{x^-=L} + \left.\frac{d}{dk^-}(k^+x^-)\right|_{x^-=0} - \frac{d\phi_{\mathrm{T}}(k^-)}{dk^-}, \\ &= -L\frac{dk^+}{dk^-} - \frac{d\phi_{\mathrm{T}}(k^-)}{dk^-}. \end{aligned} \tag{31}$$

This result, Eq. (31), when compared to the result in the work of Winful [6] reproduced in Eq. (17), differs by an overall sign. This sign change can be traced back from the fact that in the argument of the exponential for the wave, we have in the instant form a term that is proportional to the product $-Et$, whereas in the front form, this is written correspondingly as k^-x^+. The light front "vacuum" time is given correspondingly by

$$x_{\mathrm{vacuum}}^+ = -L\frac{dk^+}{dk^-}, \tag{32}$$

where, following Winful, we introduce the light front "vacuum" time as the time the wave packet would travel the distance L in the case the barrier were absent.

Thus, taking this into account we may write Eq. (31) as

$$X^{+}_{\rm tun} = x^{+}_{\rm vacuum} + \Delta x^{+},$$

where Δx^{+} is the light front delay time and is related to the delay experienced by the packet due to the presence of the potential, expressed as,

$$\Delta x^{+} = -\frac{d\phi_{\rm T}(k^{-})}{dk^{-}}. \tag{33}$$

We may also consider the instant of exiting from the region of the potential by the reflected wave packet, which is

$$x^{+}_{\rm ex,R} = x^{+}_{\rm R}\big|_{x^{-}=0} = -\frac{d}{dk^{-}}(k^{+}x^{-})\bigg|_{x^{-}=0} - \frac{d\phi_{1}(k^{-})}{dk^{-}} - \frac{d\phi_{\rm R}(k^{-})}{dk^{-}}. \tag{34}$$

The difference between this exit time $x^{+}_{\rm ex,R}$ and the time of entrance of the incident wave packet Eq. (29) is

$$X^{+}_{\rm R} = x^{+}_{\rm ex,R} - x^{+}_{\rm en} = -\frac{d\phi_{\rm R}(k^{-})}{dk^{-}}. \tag{35}$$

Thus, we see that in general, the incident wave packet in the light front is not reflected instantaneously by the potential. This result also agrees with the one in [6], modulo the overall sign the origin of which may be retraced back to the difference in signs in the definition of the scalar product between k and x in the front form when compared to the instant form.

5 Conclusion

We have shown that the choice for the time coordinate in the light front possesses peculiarities that often go unrecognized or even neglected. We have shown that the choice of x^{+} as the temporal parameter is more advantageous because this choice does not have the problem of singularity in the limit $v \to c$.

As we considered the time of tunneling of a wave packet across a rectangular potential barrier in the light front time x^{+}, we concluded that our result agrees with the ones obtained in the instant form time, modulo an overall sign. We obtain a similar expression for the “vacuum” time, as well as tunneling time and/or delay time for the wave packets hitting a rectangular barrier.

Our results are very preliminary, necessitating more thorough investigation to deepen our understanding of the concepts of time, vacuum and so forth in the light front.

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