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Lotka-Voterra competition model with nonlocal coefficient diffusion

Marcos Antonio Viana Costa [a], Cristian Morales-Rodrigo [b], Antonio Suárez [b,*]

[a] *Dpto. de Matemática e Computação, Universidade Estadual Paulista (UNESP), Faculdade de Ciências e Tecnologia, Presidente Prudente, Brazil*

[b] *Dpto. de Ecuaciones Diferenciales y Análisis Numérico and IMUS, Universidad de Sevilla, Facultad de Matemáticas, Sevilla, Spain*

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Abstract

We consider the classical Lotka-Volterra competition system with non-local diffusion, specifically, the diffusion coefficients depend on the total population in a nonlinear way. This kind of diffusion models that the species tends to leave crowded areas or is attracted to regions with higher population density, depending on whether the nonlinear function increases or decreases, respectively. The inclusion of these non-local terms in the diffusion coefficients entails significant technical difficulties. We show results of the existence and non-existence of coexistence states of the models depending on the coefficients of the model.

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1. Introduction

In this paper, we deal with the existence of coexistence states of the following nonlocal elliptic system:

* Corresponding author.
E-mail address: suarez@us.es (A. Suárez).

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$$\begin{cases} -m\left(\displaystyle\int_\Omega u\right)\Delta u = \lambda u - u^2 - cuv & \text{in } \Omega, \\ -n\left(\displaystyle\int_\Omega v\right)\Delta v = \mu v - v^2 - duv & \text{in } \Omega, \\ \qquad u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where Ω is a bounded regular domain in $\mathbb{R}^N$, $N \geq 1$, $\lambda, \mu \in \mathbb{R}$, $c, d > 0$ and $m, n : \mathbb{R} \to (0, +\infty)$ are continuous functions.

From the perspective of Population Dynamics, (P) models the behavior of two species with population densities $u(x)$ and $v(x)$, inhabiting a habitat Ω. Since we consider homogeneous Dirichlet boundary conditions, the habitat is entirely surrounded by an inhospitable region.

The right-hand side of (P) is called reaction terms, which describe the dynamics of growth, decay, and interaction between populations. In our model, we consider the classical Lotka-Volterra type, and since we are assuming $c, d > 0$ the interaction between the populations will be competitive. Here, c and d represent the interaction rates between species, while λ and μ represent the growth rates of species. Specifically, when positive, the population grows, and when negative, it decays.

The left-hand side of the equations of u and v are called diffusion terms, which describe the spatial movement of the species. In this case, we find that the velocity of the diffusion of each species depends on the population of the species itself. Let us explain the meaning of this term on the diffusion.

In population dynamics, the nonlocal term was included in [16]. The authors analyzed several biological models involving a single species, in which they included and described the importance of nonlocal interactions, such as competition for shared resources. The non-local term, addressed in the text, plays a central role in extending the traditional modeling based on reaction-diffusion equations, which typically consider only local interactions. Its introduction makes possible to represent situations where interactions between populations are not restricted to immediate proximity, such as in resource competition. This approach adds realism to the models, allowing for the analysis of more complex spatial patterns.

Subsequently, M. Chipot and collaborators, see, for instance, [3] and [4], studied a class of elliptic problems with non-local terms, addressing both theoretical and applied aspects. These problems are characterized by the global dependence of the solution, which distinguishes them from traditional local problems. Moreover, the authors included and analyzed the following nonlocal term in the diffusion

$$-a\left(\int_\Omega u\right)\Delta u$$

to model the dependence of migration velocity on the total population within the domain. Here, a is an increasing function if the species tends to leave crowded areas. In contrast, if the species is attracted to regions with higher population density in Ω, it is assumed that a is decreasing.

Hence, in (P) two nonlocal diffusion coefficients have been included in each species, see for instance [7], [15] and [18] for previous works. Assume, for example, that m is increasing. Then, if the total population of the species u is large, the species u tends to leave the crowded areas. However, if the total population of species u is small, species u is attracted to regions with higher population density despite competition with v. The opposite effect occurs when m is decreasing.

Note that, when one species is absent, the other one follows the following logistic equation with a nonlocal diffusivity coefficient:

$$\begin{cases} -g\left(\int_\Omega w\right)\Delta w = \gamma w - w^2 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\gamma \in \mathbb{R}$ and $g : \mathbb{R} \longrightarrow (0, +\infty)$ is a continuous function. This problem has been studied in [14] in a more general situation. In this paper, we complete the results of [14], and we prove:

1. If $\gamma \le g_L := \inf_{s\in\mathbb{R}} g(s)$, (1) does not possess positive solutions.
2. If $g(0) > 0$, there exists at least a positive solution of (1) if

$$\gamma > g(0)\lambda_1,$$

   where λ_1 is the principal eigenvalue of the Laplacian under homogeneous Dirichlet boundary conditions. Moreover, if g is increasing, there exists a unique positive solution of (1) in and only if $\gamma > g(0)\lambda_1$.
3. Let w_γ a positive solution of (1). Then,

$$\int_\Omega w_\gamma dx \to \infty \quad \text{and} \quad w_\gamma(x) \to \infty \quad \text{as } \gamma \to \infty.$$

These results represent a drastic change compared to the local diffusion case, that is, compared to the problem:

$$\begin{cases} -d\Delta w = \gamma w - w^2 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where d is a positive constant. In this case, there exists a unique positive solution if, and only if,

$$\gamma > d\lambda_1.$$

Hence, in the linear case, there exists at most a unique positive solution, while in the nonlocal case, the structure of positive solutions is more complex. Moreover, in the linear case, the behavior for large growth rate depends on the quotient γ/d, while in the nonlocal case the solutions diverge as $\gamma \to \infty$ independently of the behavior of g.

Now, we consider the system (P). In the case of local and linear diffusion, that is, $m \equiv n \equiv 1$, (P) has been extensively studied. Without wishing to be exhaustive in the references, we mention, for instance, the general references [2,19], and [1], [9], [10], [12], [22], [17,22] and references therein, where different techniques have been used (sub-supersolution, bifurcation, index point fixed, etc.) to prove existence, non-existence, uniqueness/stability and non-uniqueness of positive solutions of (P).

In this paper, the main goal is to provide sufficient conditions to assure the existence of coexistence states, that is, solutions of (P) with both components being positive. In the following, we outline our main results.

1. The competitive exclusion principle holds, that is, if the growth rate of a species is fixed, the species do not coexist if the other growth rate is large. Hence, both species can not coexist when the growth rate of one species is much higher than that the other one.
2. If λ and μ verify

$$\lambda > m_M\lambda_1 + c\mu \quad \text{and} \quad \mu > n_M\lambda_1 + d\lambda, \tag{3}$$

   where $m_M := \sup_{s\in\mathbb{R}} m(s)$ and $n_M := \sup_{s\in\mathbb{R}} n(s)$, then (P) possesses at least a coexistence state.
3. Assume that m is increasing. Fixed $\lambda > m(0)\lambda_1$ there exists a unique semitrivial solution $(\theta_{[\lambda,m(\cdot)]}, 0)$ of (P). Moreover, there exists a value $F(\lambda)$ such that at $\mu = F(\lambda)$ emanates from the semitrivial solution $(\theta_{[\lambda,m(\cdot)]}, 0)$ a continuum $\mathcal{C}$ of positive solutions of (P). Furthermore, there exists a sequence $(\mu_n, u_n, v_n) \in \mathcal{C}$ such that $(\mu_n, u_n, v_n) \to (\mu^*, 0, v_{\mu^*})$ where $(0, v_{\mu^*})$ is a semitrivial solution. In this case, there exists a coexistence state of (P) if

$$\mu \in (\min\{F(\lambda), \mu^*(\lambda)\}, \max\{F(\lambda), \mu^*(\lambda)\}). \tag{4}$$

   Furthermore, if n is increasing, then there exists a unique semitrivial solution $(0, \theta_{[\mu,n(\cdot)]})$ and μ^* is uniquely determined.
4. If (λ, μ) verifies (3), then it also verifies (4).

Some remarks are in order:

1. It is quite surprising that the competitive exclusion principle is satisfied independently of the behavior of the functions m and n. Indeed, it is well-known that the principle holds in the linear diffusion case $m \equiv n \equiv 1$, however it is also satisfied whatever the functions m and n are.
2. Condition (3) is obtained by the Sub-Supersolution Method and does not require any monotonicity condition in either n or m, however, it is necessary that $cd < 1$.
3. Unlike the linear diffusion case, the bifurcation method cannot be applied in general for two reasons. On the one hand, there is no unique semitrivial solution, and on the other hand, it is not, in general, nondegenerate. Under one condition, see (42) below, it is verified that bifurcates a continuum of positive solutions. In particular, (42) is verified if m is increasing.
4. In the linear diffusion case, the continuum of positive solutions bifurcates from a semitrivial solution and goes to the other one. In the nonlocal case, the behavior of the continuum is not predetermined, and various behaviors could occur.
5. When the bifurcation method can be applied, the results obtained are more robust than those achieved through the method of sub- and supersolutions, expanding the region of coexistence.
6. The inclusion of the non-local terms implies a more complex behavior of the system, giving, for example, positive solution multiplicity under similar conditions where the system with linear diffusion possesses a unique coexistence state.

The structure of the paper is as follows: Section 2 presents basic concepts and results used throughout the paper. Section 3 studies the logistic equation with a nonlocal diffusivity coefficient, arising as a particular case of (P) when one species is absent. Section 4 establishes a priori bounds for the coexistence state and derives non-existence results. Section 5 proves the existence of a coexistence state for (P) using the Sub-Supersolution Method. Section 6 applies the Crandall-Rabinowitz Theorem to ensure a local curve of non-trivial solutions for (P). Finally, Section 7 analyzes the global behavior of this curve, proving the existence of a continuum of positive solutions.

2. Preliminaries and notation

In this section we collect some basic concepts and results that we will use throughout this paper.

First, for $f \in C(\overline{\Omega})$ we denote by

$$f_M := \max_{x\in\overline{\Omega}} f(x), \qquad f_L := \max_{x\in\overline{\Omega}} f(x).$$

Given $d > 0$, $b \in L^\infty(\Omega)$ and Ω_0 a regular subdomain of Ω, we denote by $\sigma_1^{\Omega_0}[-d\Delta + b]$ the principal eigenvalue of the problem:

$$\begin{cases} -d\Delta u + b(x)u = \lambda u & \text{in } \Omega_0, \\ u = 0 & \text{on } \partial\Omega_0. \end{cases} \tag{5}$$

It is well known that the map

$$\mathbb{R} \times L^\infty(\Omega) \ni (d, b) \longmapsto \sigma_1^{\Omega_0}[-d\Delta + b] \in \mathbb{R} \tag{6}$$

is continuous and increasing.

When $\Omega_0 = \Omega$, we remove the superscript. Moreover, when $d = 1$ and $b \equiv 0$ in Ω, we simply denote by λ_1 the principal eigenvalue of (5), that is

$$\lambda_1 := \sigma_1^{\Omega}[-\Delta],$$

and by φ_1 the principal positive eigenvalue associated with $\|\varphi_1\|_\infty = 1$.

Moreover, we will use the solutions of the classical logistic equation:

$$\begin{cases} -d\Delta w = \gamma w - w^2 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{7}$$

where $d > 0$ and $\gamma \in \mathbb{R}$. In the following result, we present several well-known results concerning to (7), see for instance [11].

Theorem 2.1. *Problem (7) admits a unique classical positive solution if, and only if,*

$$\gamma > d\lambda_1.$$

In this case, we denote this solution by $\theta_{[\gamma,d]}$. *Moreover, the following properties hold:*

(a) *The application* $\gamma \in [d\lambda_1, \infty) \mapsto \theta_{[\gamma,d]} \in C_0^2(\overline{\Omega})$ *is derivable and increasing.*

(b)

$$(\gamma - d\lambda_1)\varphi_1 \le \theta_{[\gamma,d]} < \gamma \quad \textit{in } \Omega.$$

(c)

$$\theta_{[\gamma,d]} = \frac{\gamma - d\lambda_1}{\displaystyle\int_\Omega \varphi_1^3}\varphi_1 + G_{\gamma,d}(x), \qquad \textit{as } \gamma \approx d\lambda_1,$$

where

$$\lim_{\gamma \to d\lambda_1} \frac{G_{\gamma,d}(x)}{\gamma - d\lambda_1} = 0 \quad \textit{in } C^2(\overline{\Omega}).$$

(d) $\sigma_1\left[-d\Delta + 2\theta_{[\gamma,d]} - \gamma\right] > 0$.

Remark 2.1. Observe that dividing the equation (7) by d^2, we get

$$-\Delta\left(\frac{w}{d}\right) = \frac{\gamma}{d}\left(\frac{w}{d}\right) - \left(\frac{w}{d}\right)^2.$$

Then, $\theta_{[\gamma,d]}/d = \theta_{[\gamma/d,1]}$, that is,

$$\theta_{[\gamma,d]} = d\theta_{[\gamma/d,1]}. \tag{8}$$

In the following result, we study a nonlocal problem that will be used in an upcoming section.
Consider, for $d > 0$, $\alpha \in L^\infty(\Omega)$ such that $\sigma_1[-d\Delta + \alpha] > 0$ and $\beta \in L^\infty(\Omega)$, the following nonlocal problem:

$$\begin{cases} -d\Delta w + \alpha(x)w + \beta(x)\displaystyle\int_\Omega w = f(x) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{9}$$

Since $\sigma_1[-d\Delta + \alpha] > 0$, from [20] there exists a unique positive solution, denoted by $e \in C_0^2(\overline{\Omega})$, of the equation

$$\begin{cases} -d\Delta e + \alpha(x)e = 1 & \text{in } \Omega, \\ e = 0 & \text{on } \partial\Omega. \end{cases} \tag{10}$$

The next result guarantees that (9) admits a unique solution under certain conditions on the coefficients of the problem.

Theorem 2.2. *Assume that* $\sigma_1[-d\Delta + \alpha] > 0$ *and*

$$1 + \int_\Omega \beta(x)e(x) \neq 0, \tag{11}$$

where e *is the unique positive solution of (10).*

For each $f \in L^2(\Omega)$, (9) *admits a unique solution* $w \in W^{2,2}(\Omega)$. *Moreover,* $w \in W^{2,p}(\Omega)$ *when* $f \in L^p(\Omega)$, $p > 1$. *If* $\alpha, \beta, f \in C^\rho(\overline{\Omega})$, *then* $w \in C^{2,\rho}(\overline{\Omega})$ *for any* $\rho \in (0, 1)$.

Proof. We will use some ideas of Lemma 3.1 of [5]. Since $\sigma_1[-d\Delta + \alpha] > 0$, there exists $w_0 \in W^{2,2}(\Omega)$, unique solution of the following problem

$$\begin{cases} -d\Delta w_0 + \alpha(x)w_0 = f(x) - \dfrac{\displaystyle\int_\Omega f(x)e}{1 + \displaystyle\int_\Omega \beta(x)e}\beta(x) & \text{in } \Omega, \\ w_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{12}$$

We will show that w_0 is the unique solution of (9). Note that proving this statement is equivalent to show that

$$\int_\Omega w_0 = \frac{\displaystyle\int_\Omega f(x)e}{1 + \displaystyle\int_\Omega \beta(x)e}.$$

Multiplying the equation (12) by e and integrating over Ω, it follows that

$$\int_\Omega w_0 = \int_\Omega e(x)f - \frac{\displaystyle\int_\Omega f(x)e}{1 + \displaystyle\int_\Omega \beta(x)e}\int_\Omega \beta(x)e = \frac{\displaystyle\int_\Omega f(x)e}{1 + \displaystyle\int_\Omega \beta(x)e}.$$

Now, we show the uniqueness of solution of (9). Let w be a solution of (9). Then, multiplying (9) by e and integrating in Ω, we get

$$\int_\Omega w\left[1 + \int_\Omega \beta(x)e\right] = \int_\Omega f(x)e,$$

and then

$$\int_\Omega w = \frac{\displaystyle\int_\Omega f(x)e}{1+\displaystyle\int_\Omega \beta(x)e}.$$

Hence, $w = w_0$ and the uniqueness follows. The results concerning the regularity of the solution follow by classical elliptic regularity. This completes the proof. □

3. Logistic equation with nonlocal diffusivity coefficient

In this section, we will study the logistic equation with a nonlocal diffusivity coefficient, which arises from (P) when one species is absent.

Observe that (P) admits three types of non-negative solutions, namely:

(S_1) The trivial solution $(0,0)$.

(S_2) The semi-trivial solutions $(u,0)$ and $(0,v)$, where $u \not\equiv 0$ and $v \not\equiv 0$ are positive solutions of

$$\begin{cases} -m\left(\displaystyle\int_\Omega u\right)\Delta u = \lambda u - u^2 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{13}$$

and

$$\begin{cases} -n\left(\displaystyle\int_\Omega v\right)\Delta v = \mu v - v^2 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{14}$$

respectively.

(S_3) The coexistence states (u,v), where both components are non-negative and non-trivial. In fact, thanks to the strong maximum principle any (u,v) non-negative and non-trivial solution of (P) is, in fact, strongly positive.

Now, we will study the semi-trivial solutions, that is equations (13) and (14). For that, consider the following general problem:

$$\begin{cases} -g\left(\displaystyle\int_\Omega w\right)\Delta w = \gamma w - w^2 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{15}$$

where g is a continuous positive function and $\gamma \in \mathbb{R}$.

The first result of this section ensures the existence and uniqueness of a positive solution for (15), as well as some properties of this solution, see [14].

Theorem 3.1.

1. *If* $\gamma \le g_L \lambda_1$, (15) *does not possess a positive solution, where*

$$g_L := \inf_{\mathbb{R}} g.$$

2. *Any positive solution* u_γ *of* (15) *satisfies*

$$\left(\gamma - g\left(\int_\Omega u_\gamma\right)\lambda_1\right)\varphi_1 \le u_\gamma < \gamma \qquad \text{in } \Omega.$$

3. *Assume* $g(0) > 0$. *Then, from the trivial solution* $w \equiv 0$ *emanates at the value* $\gamma = g(0)\lambda_1$ *an unbounded continuum* $\mathcal{C}_\gamma \subset \mathbb{R} \times C_0^1(\overline{\Omega})$ *of positive solutions of* (15) *such that*

$$(g(0)\lambda_1, +\infty) \subset Proj_{\mathbb{R}}(\mathcal{C}_\gamma),$$

where for $(\gamma, u) \in \mathcal{C}_\gamma$, $Proj_{\mathbb{R}}(\gamma, u) = \gamma$.
As a consequence, (15) *admits a positive solution when* $\gamma > g(0)\lambda_1$.
4. *If* g *is increasing, there exists a unique positive solution if and only if* $\gamma > g(0)\lambda_1$.

Remark 3.1. When g is regular, in [14] and using local bifurcation techniques, it was proved that if $g'(0)$ is negative and large, then there exist at least two positive solutions of (15) for $\gamma \in (\gamma^*, g(0)\lambda_1)$ for some $\gamma^* > 0$.

As a consequence of this result, problems (13) and (14) admit positive solutions when $\lambda > m(0)\lambda_1$ and $\mu > n(0)\lambda_1$, respectively.

In the next result, we determine the behavior of any positive solution of (15) as $\gamma \to \infty$.

Proposition 3.1. *Let* $\theta_{[\gamma, g(\cdot)]}$ *be a positive solution of* (15). *Then,*

$$\lim_{\gamma \to +\infty} \int_\Omega \theta_{[\gamma, g(\cdot)]} = +\infty. \tag{16}$$

Moreover, for any $x \in \Omega$

$$\lim_{\gamma \to +\infty} \theta_{[\gamma, g(\cdot)]}(x) = +\infty. \tag{17}$$

Proof. Using Theorem 3.1 (2), we obtain

$$\left(\gamma - g\left(\int_\Omega \theta_{[\gamma, g(\cdot)]}\right)\lambda_1\right)\varphi_1 \le \theta_{[\gamma, g(\cdot)]}, \tag{18}$$

and integrating

$$\int_\Omega \theta_{[\gamma,g(\cdot)]} + g\left(\int_\Omega \theta_{[\gamma,g(\cdot)]}\right)\|\varphi_1\|_1\lambda_1 \geq \gamma\|\varphi_1\|_1,$$

whence we deduce (16).

Now, we prove (17). We consider two cases:

(1) Assume that $g(\infty) := \lim_{s\to\infty} g(s) < \infty$. In this case, we have

$$g(s) \leq g_M := \sup_{\mathbb{R}} g < \infty \quad \forall s \in \mathbb{R}.$$

Then, using (18) we obtain

$$\theta_{[\gamma,g(\cdot)]} \geq \left(\gamma - g\left(\int_\Omega \theta_{[\gamma,g(\cdot)]}\right)\lambda_1\right)\varphi_1 \geq (\gamma - g_M\lambda_1)\varphi_1,$$

whence we deduce (17).

(2) Assume that $g(\infty) = \infty$. In this case, by (16) we have that

$$\lim_{\gamma\to+\infty} g\left(\int_\Omega \theta_{[\gamma,g(\cdot)]}\right) = +\infty. \tag{19}$$

Denote by

$$d = g\left(\int_\Omega \theta_{[\gamma,g(\cdot)]}\right). \tag{20}$$

Then, dividing the equation (15) by d^2, we get, see (8), that

$$\theta_{[\gamma,g(\cdot)]} = d\theta_{[\gamma/d,1]}. \tag{21}$$

We divide the proof into two cases:

(a) If $\gamma/d \geq b > \lambda_1$ for some b, by Theorem 2.1 (a) we get

$$\theta_{[\gamma/d,1]} \geq \theta_{[b,1]}$$

and consequently

$$\theta_{[\gamma,g(\cdot)]} = d\theta_{[\gamma/d,1]} \geq d\theta_{[b,1]}$$

and we deduce the result from (19) and (20).

(b) Assume that there exists a sequence such that

$$\lim_{n\to+\infty} \gamma_n/d_n = \lambda_1.$$

By Theorem 2.1 (c), we deduce

$$\theta_{[\gamma_n/d_n,1]} = \left(\frac{\gamma_n}{d_n} - \lambda_1\right)\frac{\varphi_1}{\int_\Omega \varphi_1^3} + G_n(x) \tag{22}$$

where

$$\lim_{\gamma_n/d_n\to\lambda_1} \frac{G_n(x)}{\left(\frac{\gamma_n}{d_n} - \lambda_1\right)} = 0 \quad \text{in } C^2(\overline{\Omega}). \tag{23}$$

Then, by (21)

$$\begin{aligned}\theta_{[\gamma_n,g(\cdot)]} &= d_n\left[\left(\frac{\gamma_n}{d_n} - \lambda_1\right)\frac{\varphi_1}{\int_\Omega \varphi_1^3} + G_n(x)\right] = (\gamma_n - d_n\lambda_1)\frac{\varphi_1}{\int_\Omega \varphi_1^3} + d_n\cdot G_n(x)\\ &= (\gamma_n - d_n\lambda_1)\left[\frac{\varphi_1}{\int_\Omega \varphi_1^3} + \frac{G_n(x)}{\left(\frac{\gamma_n}{d_n} - \lambda_1\right)}\right].\end{aligned} \tag{24}$$

Integrating the previous expression over Ω, it is apparent that

$$\int_\Omega \theta_{[\gamma_n,g(\cdot)]} = (\gamma_n - d_n\lambda_1)\left[\frac{1}{\int_\Omega \varphi_1^3}\int_\Omega \varphi_1 + \frac{\int_\Omega G_n(x)}{\left(\frac{\gamma_n}{d_n} - \lambda_1\right)}\right].$$

Using (23) and (16), we get that

$$\lim_{n\to+\infty} (\gamma_n - d_n\lambda_1) = +\infty. \tag{25}$$

Hence, coming back to (24), it is evident that

$$\lim_{n\to+\infty} \theta_{[\gamma,g(\cdot)]}(x) = +\infty \quad \text{for any } x\in\Omega. \quad \square$$

Remark 3.2. Observe that the previous result reveals a remarkable feature: any solution $\theta_{[\gamma,g(\cdot)]}$ of (15) diverges to infinity as $\gamma\to+\infty$, regardless of the specific behavior of the function g. This particularity indicates that the influence of g on the problem is limited regarding the behavior of $\theta_{[\gamma,g(\cdot)]}$ as $\gamma\to+\infty$. In other words, even if g possesses specific properties, such as growth or decay, it cannot prevent $\theta_{[\gamma,g(\cdot)]}$ from diverging to infinity as γ grows indefinitely.

4. A priori bounds and non-existence of coexistence states

In this section, we study some a priori bounds for the coexistence states and, through them, prove some non-existence results.

Proposition 4.1. *If* (u, v) *is a coexistence state of* (P) *then*

$$u \le \lambda \tag{26}$$

and

$$v \le \mu. \tag{27}$$

Proof. Consider (u, v) a coexistence state of (P). Denote by $x_M, y_M \in \Omega$ such that

$$u(x_M) = \max_{x \in \overline{\Omega}} u(x) =: u_M \qquad \text{and} \qquad v(y_M) = \max_{x \in \overline{\Omega}} v(x) =: v_M.$$

Note that

$$-m\left(\int_\Omega u\right)\Delta u(x_M) = \lambda u_M - u_M^2 - c u_M v(x_M) \ge 0$$

and, consequently,

$$u(x) \le u_M \le \lambda.$$

Analogously, it is possible to prove (27). $\square$

As a consequence of the previous result, we can ensure the following non-existence result for the coexistence state.

Corollary 4.1. *For the competition case,* (P) *does not possess coexistence state when* $\lambda \le 0$ *or* $\mu \le 0$.

Now, we prove the main result of this section, which confirms the principle of competitive exclusion. That is, fixed $\mu > 0$ (resp. $\lambda > 0$) (P) does not possess coexistence states for sufficiently large parameter λ (resp. $\mu > 0$.)

Proposition 4.2. *Fix* $\mu > 0$. *Then, there is no positive solution to* (P) *for* λ *large enough. Analogously, for* $\lambda > 0$ *fixed, there is no positive solution to* (P) *for* μ *large enough.*

Proof. We fix $\mu > 0$ and suppose, by contradiction, that there exists at least one coexistence state (u, v) for (P) for λ large enough. Since (u, v) is a coexistence state, we have that

$$\mu = \sigma_1 \left[-n \left(\int_\Omega v \right) \Delta + v + du \right].$$

Using the properties of the principal eigenvalue, see (6), and Proposition 4.1, we obtain

$$\mu \geq \sigma_1 \left[-n_\mu \Delta + du \right], \tag{28}$$

where $n_\mu := \min_{0 \leq v \leq \mu} n \left(\int_\Omega v \right)$. On the other hand, once again by Proposition 4.1,

$$-m \left(\int_\Omega u \right) \Delta u \geq u(\lambda - c\mu - u).$$

Hence, u is supersolution of (7) with

$$d = m \left(\int_\Omega u \right), \qquad \gamma = \lambda - c\mu.$$

Then,

$$u \geq \theta_{[\gamma,d]},$$

and consequently, by Theorem 2.1 (b) we get

$$u \geq (\gamma - d\lambda_1)\varphi_1 = \left[\lambda - c\mu - \lambda_1 m \left(\int_\Omega u \right) \right] \varphi_1. \tag{29}$$

Integrating the previous expression over Ω and rearranging the terms, we obtain

$$\int_\Omega u + \lambda_1 m \left(\int_\Omega u \right) \|\varphi_1\|_1 \geq (\lambda - c\mu)\|\varphi_1\|_1,$$

and, consequently,

$$\lim_{\lambda \to +\infty} \int_\Omega u = +\infty. \tag{30}$$

We distinguish two cases:

(1) Assume that $m_M < \infty$. Then, from (29)

$$u(x) \geq \left[\lambda - c\mu - \lambda_1 m\left(\int_\Omega u\right)\right]\varphi_1(x) \geq [\lambda - c\mu - m_M]\varphi_1(x).$$

Now, using the monotony of the principal eigenvalue on (28), see (6), we obtain

$$\mu \geq \sigma_1\left[-n_\mu \Delta + d(\lambda - c\mu - m_M)\varphi_1\right]$$

and by [20] we get

$$\lim_{\lambda\to+\infty} \sigma_1\left[-n_\mu \Delta + d(\lambda - c\mu - m_M)\varphi_1\right] = \infty,$$

a contradiction.

(2) Assume that $m_M = \infty$. Then, by (30) we get

$$\lim_{\lambda\to+\infty} m\left(\int_\Omega u\right) = +\infty \tag{31}$$

On the other hand, note that, dividing the first equation of (P) by $\left[m\left(\int_\Omega u\right)\right]^2$ and applying Proposition 4.1,

$$\begin{aligned} -\Delta\left(\frac{u}{m\left(\int_\Omega u\right)}\right) &= \frac{u}{m\left(\int_\Omega u\right)}\left[\frac{\lambda - cv}{m\left(\int_\Omega u\right)} - \frac{u}{m\left(\int_\Omega u\right)}\right] \\ &\geq \frac{u}{m\left(\int_\Omega u\right)}\left[\frac{\lambda - c\mu}{m\left(\int_\Omega u\right)} - \frac{u}{m\left(\int_\Omega u\right)}\right] \end{aligned}$$

then, $u/m\left(\int_\Omega u\right)$ is supersolution of (7) with $d = 1$ and

$$\gamma = \frac{\lambda - c\mu}{m\left(\int_\Omega u\right)}.$$

Consequently,

$$u \geq m\left(\int_\Omega u\right)\theta_{[\gamma,1]}. \tag{32}$$

We distinguish two cases:

1. If $\gamma \geq b > \lambda_1$ for some b, we obtain by (32) and Theorem 2.1 (1) that

$$u \geq m\left(\int_\Omega u\right)\theta_{[\gamma,1]} \geq m\left(\int_\Omega u\right)\theta_{[b,1]}$$

and we conclude by combining (28), (31) and again by [20] that

$$\mu \geq \sigma_1\left[-n_\mu\Delta + dm\left(\int_\Omega u\right)\theta_{[b,1]}\right] \to \infty \quad \text{as } \lambda \to \infty,$$

again a contradiction.

2. Assume that there exists a sequence $\lambda_n \to \infty$ such that

$$\gamma_n = \frac{\lambda_n - c\mu}{m\left(\int_\Omega u_n\right)} \to \lambda_1. \tag{33}$$

Then, by (32) and Theorem 2.1, we get

$$u_n \geq m\left(\int_\Omega u_n\right)\left[(\gamma_n - \lambda_1)\frac{\varphi_1}{\int_\Omega \varphi_1^3} + G_n(x)\right],$$

with $G_n(x) = o(\gamma_n - \lambda_1)$ in $C^2(\overline{\Omega})$.
Note that

$$\begin{aligned}-\Delta\left(\frac{u_n}{m\left(\int_\Omega u_n\right)}\right) &= \frac{u_n}{m\left(\int_\Omega u_n\right)}\left[\frac{\lambda_n - cv_n}{m\left(\int_\Omega u_n\right)} - \frac{u_n}{m\left(\int_\Omega u_n\right)}\right] \\ &\leq \frac{u_n}{m\left(\int_\Omega u_n\right)}\left[\frac{\lambda_n}{m\left(\int_\Omega u_n\right)} - \frac{u_n}{m\left(\int_\Omega u_n\right)}\right]\end{aligned}$$

and, consequently,

$$u_n \leq m\left(\int_\Omega u_n\right)\theta_{[\tilde{\gamma}_n,1]}$$

where

$$\tilde{\gamma}_n = \frac{\lambda_n}{m\left(\int_\Omega u_n\right)}.$$

It is straightforward to see that

$$\tilde{\gamma}_n = \gamma_n + \frac{c\mu}{m\left(\int_\Omega u_n\right)}$$

and then by (33) we have

$$\tilde{\gamma}_n \to \lambda_1.$$

Thus

$$u_n \leq m\left(\int_\Omega u_n\right)\left[(\tilde{\gamma}_n - \lambda_1)\frac{\varphi_1}{\int_\Omega \varphi_1^3} + \tilde{a}_n(x)\right],$$

where $\tilde{a}_n(x) = o(\tilde{\gamma}_n - \lambda_1)$. Integrating the previous expression over Ω, we obtain

$$\int_\Omega u_n \leq m\left(\int_\Omega u_n\right)(\tilde{\gamma}_n - \lambda_1)\frac{\int_\Omega \varphi_1}{\int_\Omega \varphi_1^3} + m\left(\int_\Omega u_n\right)\int_\Omega \tilde{a}_n(x). \tag{34}$$

We claim that,

$$\lim_{\lambda_n \to +\infty} m\left(\int_\Omega u_n\right)(\tilde{\gamma}_n - \lambda_1) = +\infty. \tag{35}$$

Assume by contradiction that

$$m\left(\int_\Omega u_n\right)(\tilde{\gamma}_n - \lambda_1) \leq C \quad \text{for some } C > 0.$$

Thus

$$\lim_{\lambda_n \to +\infty} m\left(\int_\Omega u_n\right)\int_\Omega \tilde{a}_n(x) = \lim_{\lambda_n \to +\infty} m\left(\int_\Omega u_n\right)(\tilde{\gamma}_n - \lambda_1)\frac{\int_\Omega \tilde{a}_n(x)}{\tilde{\gamma}_n - \lambda_1} = 0,$$

and, consequently,

$$\int_\Omega u_n \leq C,$$

a contradiction with (30). This proves (35). Then,

$$\lim_{\lambda_n \to +\infty} m\left(\int_\Omega u_n\right)(\gamma_n - \lambda_1) = +\infty.$$

Using the expression of γ_n,

$$\lim_{\lambda_n \to +\infty} \left(\lambda_n - c\mu - m \left(\int_\Omega u_n \right) \lambda_1 \right) = +\infty.$$

We get

$$\mu \geq \lim_{\lambda_n \to +\infty} \sigma_1 \left[-n_\mu \Delta + d \left[\lambda_n - c\mu - \lambda_1 m \left(\int_\Omega u_n \right) \right] \varphi_1 \right] = \infty.$$

Then, we get a contradiction and we complete the proof. □

5. Sub-supersolution analysis

In this section, we prove the existence of coexistence states for (P) using the Sub-Supersolution Method. To do this, we will first present some definitions to apply this method to the system. For more details on the method applied to systems, we recommend the reference [13]

Definition 5.1. We say that the pair $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$, with $\underline{u}, \overline{u}, \underline{v}, \overline{v} \in H^2(\Omega) \cap L^\infty(\Omega)$, is a pair of sub-supersolution of (P) when:

(1) $\underline{u} \leq \overline{u}$, $\underline{v} \leq \overline{v}$ in Ω and $\underline{u} \leq 0 \leq \overline{u}$, $\underline{v} \leq 0 \leq \overline{v}$ on $\partial\Omega$.
(2) The pair $(\underline{u}, \overline{u})$ satisfies, for all $u \in [\underline{u}, \overline{u}]$,

$$-m\left(\int_\Omega u\right)\Delta\underline{u} - \lambda\underline{u} + \underline{u}^2 + c\underline{u}\overline{v} \leq 0 \leq -m\left(\int_\Omega u\right)\Delta\overline{u} - \lambda\overline{u} + \overline{u}^2 + c\overline{u}\underline{v}.$$

(3) The pair $(\underline{v}, \overline{v})$ satisfies, for all $v \in [\underline{v}, \overline{v}]$,

$$-n\left(\int_\Omega v\right)\Delta\underline{v} - \mu\underline{v} + \underline{v}^2 + d\overline{u}\underline{v} \leq 0 \leq -n\left(\int_\Omega v\right)\Delta\overline{v} - \mu\overline{v} + \overline{v}^2 + d\underline{u}\overline{v}.$$

The following result guarantees the existence of coexistence states for problem (P) through the existence of a sub-supersolution.

Theorem 5.1. *Assume that $cd < 1$. If*

$$\lambda > m_M\lambda_1 + c\mu \quad \text{and} \quad \mu > n_M\lambda_1 + d\lambda, \tag{36}$$

then (P) possesses at least a coexistence state.

Proof. We claim that the pairs $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$, given by

$$(\underline{u}, \underline{v}) = (\varepsilon_1\varphi_1, \varepsilon_2\varphi_1) \quad \text{and} \quad (\overline{u}, \overline{v}) = (\lambda, \mu),$$

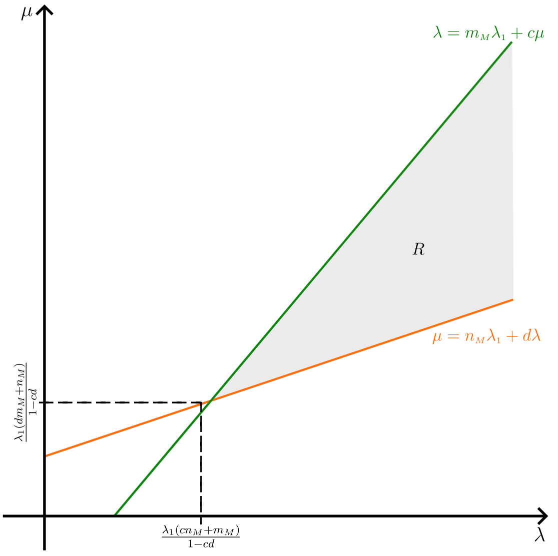


Fig. 1. Region defined by (36). If $(\lambda, \mu) \in R$, then (P) possesses at least a coexistence state.

are sub-supersolution of (P) provided that $\varepsilon_1, \varepsilon_2 > 0$ are small enough. Indeed, we must verify items (1)-(3) of Definition 5.1. Note that, (1) is clear, so we will only check (2) and (3). Observe that they are sub-supersolution of (P) provided that

$$m\left(\int_\Omega u\right)\lambda_1 \leq \lambda - \varepsilon_1\varphi_1 - c\mu \quad \text{for all } u \in [\underline{u}, \overline{u}],$$

$$n\left(\int_\Omega v\right)\lambda_1 \leq \mu - \varepsilon_2\varphi_1 - d\lambda \quad \text{for all } v \in [\underline{v}, \overline{v}].$$

If (λ, μ) satisfies (36), then we can take $\varepsilon_1, \varepsilon_2$ small enough such that the above inequalities hold.

Thus, $(\varepsilon_1\varphi_1, \varepsilon_2\varphi_1)$ and (λ, μ) are sub-supersolutions of (P). Therefore, there exists at least one coexistence state (u^*, v^*). $\square$

In Fig. 1 we present the condition (36). Here, the gray area represents the coexistence region

$$R := \{(\lambda, \mu) \in \mathbb{R}^2 : \ (\lambda, \mu) \text{ verifies } (36)\}.$$

Observe that $R \neq \emptyset$ if $cd < 1$.

6. Local bifurcation analysis

In this section, we assume that $m, n \in C^2(\mathbb{R})$ and we apply the Crandall-Rabinowitz Theorem to ensure the existence of a local curve of non-trivial solutions for Problem (P) close to the bifurcation point.

Unlike the case of linear diffusion, (P) generally does not admit a unique semitrivial solution. Therefore, in order to prove bifurcation from any semi-trivial solution we need to impose some conditions.

We fix $\lambda \in \mathbb{R}$ such that there exists a positive solution $\theta_{[\lambda,m(\cdot)]}$ of (13), we study the bifurcation from the point $\left(\theta_{[\lambda,m(\cdot)]}, 0\right)$ with μ as a bifurcation parameter.

Consider throughout this section $E := C_0^2(\overline{\Omega})$ and $F := C(\overline{\Omega})$ and we define the operator $\mathcal{F} : \mathbb{R} \times E^2 \to F^2$ as follows

$$\mathcal{F}(\mu, u, v) = \begin{bmatrix} -m\left(\displaystyle\int_\Omega u\right)\Delta u - \lambda u + u^2 + cuv \\ -n\left(\displaystyle\int_\Omega v\right)\Delta v - \mu v + v^2 + duv \end{bmatrix}. \tag{37}$$

It is clear that $\mathcal{F} \in C^2(\mathbb{R} \times E; F)$ and $(u, v) \in E^2$ is a non-negative strong solution of (P) if, and only if,

$$\mathcal{F}(\mu, u, v) = 0_{F\times F} \quad \text{for all } \mu \in \mathbb{R}. \tag{38}$$

In particular

$$\mathcal{F}(\mu, \theta_{[\lambda,m(\cdot)]}, 0) = 0_{F\times F} \tag{39}$$

for all $\mu \in \mathbb{R}$, where $\theta_{[\lambda,m(\cdot)]}$ is a positive solution of (13).

Denote by

$$F(\lambda) := \sigma_1\left[-n(0)\Delta + d\theta_{[\lambda,m(\cdot)]}\right].$$

Observe that by Theorem 2.1 (d), we get

$$\sigma_1\left[-m\left(\int_\Omega \theta_{[\lambda,m(\cdot)]}\right)\Delta + 2\theta_{[\lambda,m(\cdot)]} - \lambda\right] > 0, \tag{40}$$

and then there exists a unique positive solution e_λ of

$$\begin{cases} -m\left(\displaystyle\int_\Omega \theta_{[\lambda,m(\cdot)]}\right)\Delta e_\lambda + (2\theta_{[\lambda,m(\cdot)]} - \lambda)e_\lambda = 1 & \text{in } \Omega, \\ e_\lambda = 0 & \text{on } \partial\Omega. \end{cases} \tag{41}$$

The next result guarantees the existence and uniqueness of a curve of non-trivial solutions emanating from point $(\theta_{[\lambda,m(\cdot)]}, 0)$ for a value of μ.

Theorem 6.1. *Assume that $\lambda > 0$. Suppose there exists a positive solution $\theta_{[\lambda,m(\cdot)]}$ of (13) such that*

$$1+\frac{m'\left(\int_\Omega \theta_{[\lambda,m(\cdot)]}\right)}{m\left(\int_\Omega \theta_{[\lambda,m(\cdot)]}\right)}\int_\Omega (\lambda\theta_{[\lambda,m(\cdot)]}-\theta^2_{[\lambda,m(\cdot)]})e_\lambda \neq 0. \tag{42}$$

Then,

$$(\mu,u,v)=\left(F(\lambda),\theta_{[\lambda,m(\cdot)]},0\right) \tag{43}$$

is a bifurcation point from the semitrivial solution $(\mu,\theta_{[\lambda,m(\cdot)]},0)$. Moreover, there exist $\varepsilon > 0$ and applications of class C^1

$$\mu:(-\varepsilon,\varepsilon)\longrightarrow \mathbb{R} \qquad \psi:(-\varepsilon,\varepsilon)\longrightarrow Z \qquad \varphi:(-\varepsilon,\varepsilon)\longrightarrow Z,$$

where Z is a topological complementary of $Ker\left[\mathcal{F}_{(u,v)}(F(\lambda),\theta_{[\lambda,m(\cdot)]},0)\right]$ in E^2, such that

$$\begin{cases} \mu(s)=F(\lambda)+s\mu_1+\rho(s)\\ u(s)=\theta_{[\lambda,m(\cdot)]}+s(\varphi_1+\varphi(s))\\ v(s)=s(\psi_1+\psi(s)) \end{cases} \tag{44}$$

where φ_1 and ψ_1 are functions, which will be detailed below, $\psi(0)=\varphi(0)=0$ and $\rho(0)=0$. Moreover, the set of coexistence states of (P) in a neighborhood of $(F(\lambda),\theta_{[\lambda,m(\cdot)]},0)$ consists of the curves $(\mu(s),u(s),v(s))$, for each $s\in(0,\varepsilon)$. Furthermore,

$$\mu_1=\frac{n'(0)\displaystyle\int_\Omega \psi_1\int_\Omega|\nabla\psi_1|^2+\int_\Omega\psi_1^3+d\int_\Omega\psi_1^2\varphi_1}{\displaystyle\int_\Omega\psi_1^2}. \tag{45}$$

Finally, for $\lambda\approx m(0)\lambda_1$ we have

$$sign(\mu_1)=sign\left(n'(0)\lambda_1\int_\Omega\Phi_1\int_\Omega\Phi_1^2+(1-cd)\int_\Omega\Phi_1^3\right), \tag{46}$$

where Φ_1 is the principal eigenfunction associated to λ_1 with $\|\Phi_1\|_\infty=1$.

Proof. We denote the Fréchet differential

$$\mathcal{L}(\mu):=\mathcal{F}_{(u,v)}(\mu,\theta_{[\lambda,m(\cdot)]},0).$$

We have that:

$$\mathcal{L}(\mu)(\xi,\eta)=\begin{bmatrix} A(\xi,\eta) \\ -n(0)\Delta\eta + d\theta_{[\lambda,m(\cdot)]}\eta - \mu\eta \end{bmatrix},$$

where

$$\begin{aligned} A(\xi,\eta) =& -\left[m'\left(\int_\Omega \theta_{[\lambda,m(\cdot)]}\right)\int_\Omega \xi \right]\Delta\theta_{[\lambda,m(\cdot)]} - m\left(\int_\Omega \theta_{[\lambda,m(\cdot)]}\right)\Delta\xi \\ & -\lambda\xi + 2\theta_{[\lambda,m(\cdot)]}\xi + c\theta_{[\lambda,m(\cdot)]}\eta \\ =& -m\left(\int_\Omega \theta_{[\lambda,m(\cdot)]}\right)\Delta\xi + (2\theta_{[\lambda,m(\cdot)]}-\lambda)\xi + \beta(x)\int_\Omega \xi + c\theta_{[\lambda,m(\cdot)]}\eta \end{aligned}$$

and

$$\beta(x) := \frac{m'\left(\int_\Omega \theta_{[\lambda,m(\cdot)]}\right)}{m\left(\int_\Omega \theta_{[\lambda,m(\cdot)]}\right)}\left(\lambda\theta_{[\lambda,m(\cdot)]} - \theta^2_{[\lambda,m(\cdot)]}\right). \tag{47}$$

To demonstrate this result, we will apply Theorem 1.7 of [8]. For that, let us consider μ as the main bifurcation parameter and we need to show that for

$$\mu^* = F(\lambda)$$

the map $\mathcal{F}$ verifies the following conditions:

(CR_1) $\mathrm{Ker}\left[\mathcal{L}(\mu^*)\right] = \mathrm{Span}\{(\varphi^*,\psi^*)\}$;
(CR_2) $\mathrm{codim}\left[\mathrm{Rg}\,(\mathcal{L}(\mu^*))\right] = 1$; and
(CR_3) $\mathcal{F}_1(\mu^*) := \mathcal{F}_{\mu(u,v)}(\mu^*,\theta_{[\lambda,m(\cdot)]},0)(\varphi^*,\psi^*) \notin \mathrm{Rg}\left[\mathcal{L}(\mu^*)\right]$,

for some φ^* and ψ^* functions, which will be detailed below.
(CR_1): We show that

$$\dim\left[\mathrm{Ker}(\mathcal{L}(\mu^*))\right] = 1.$$

Note that, $(\xi,\eta) \in \mathrm{Ker}\left[\mathcal{L}(\mu^*)\right]$ if, and only if, (ξ,η) is solution of

$$\begin{cases} A(\xi,\eta) = 0 & \text{in } \Omega, \\ -n(0)\Delta\eta + d\theta_{[\lambda,m(\cdot)]}\eta = \mu^*\eta & \text{in } \Omega, \\ \xi = \eta = 0 & \text{on } \partial\Omega. \end{cases} \tag{48}$$

Since $\mu^* = F(\lambda)$, the solutions of the second equation are given by $C\psi_1$, where $C \in \mathbb{R}$ and ψ_1 is the principal eigenfunction associated to $F(\lambda)$ with $||\psi_1||_\infty = 1$, that is,

$$-n(0)\Delta\psi_1 + d\psi_1\theta_{[\lambda,m(\cdot)]} = F(\lambda)\psi_1 \quad \text{in } \Omega, \quad \psi_1 = 0 \quad \text{on } \partial\Omega. \tag{49}$$

Hence, $\eta \in \mathrm{Span}\{\psi_1\}$. For the first equation, we can write $A(\xi,\eta)=0$ as follows

$$\begin{cases} -m\left(\int\limits_\Omega \theta_{[\lambda,m(\cdot)]}\right)\Delta\xi + (2\theta_{[\lambda,m(\cdot)]}-\lambda)\xi + \beta(x)\int\limits_\Omega \xi = -c\theta_{[\lambda,m(\cdot)]}\eta \text{ in } \Omega, \\ \xi = 0 \text{ on } \partial\Omega. \end{cases} \tag{50}$$

Moreover, for $d \equiv m\left(\int\limits_\Omega \theta_{[\lambda,m(\cdot)]}\right)$ and $\alpha \equiv 2\theta_{[\lambda,m(\cdot)]}-\lambda$, by Theorem 2.1 (d) and (40), we obtain

$$\sigma_1\left[-d\Delta + \alpha(x)\right] > 0.$$

Consequently, using condition (42), it follows by Theorem 2.2, that (50) admits a unique positive solution in $W^{2,p}(\Omega)$, $p>1$, which we will denote as φ_1. Thus, the solutions of (48) are given by $(C\varphi_1, C\psi_1)$, with $C\in\mathbb{R}$ and, consequently,

$$\mathrm{Ker}\left[\mathcal{L}(\mu^*)\right] = \mathrm{Span}\{(\varphi_1,\psi_1)\},$$

where ψ_1 and φ_1 are solutions of (49) and (50) with $\eta=\psi_1$, respectively.
(CR_2): Note that

$$\mathrm{Rg}\left[\mathcal{L}(\mu^*)\right] = \left\{(u,v)\in E^2;\ \int\limits_\Omega v\psi_1 = 0\right\}. \tag{51}$$

Indeed, given $(\varphi,\psi)\in \mathrm{Rg}\left[\mathcal{L}(\mu^*)\right]$, there exists $(u,v)\in E^2$ such that

$$\begin{cases} A(u,v) = \varphi & \text{in } \Omega, \\ -n(0)\Delta v + d\theta_{[\lambda,m(\cdot)]}v - F(\lambda)v = \psi & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{52}$$

Consequently, $\psi \in \mathrm{Rg}(-n(0)\Delta + d\theta_{[\lambda,m(\cdot)]} - F(\lambda))$. Hence, according to (49), ψ is such that

$$\int\limits_\Omega \psi\psi_1 = 0.$$

Arguing as in the previous condition, it follows that for every $\varphi\in E$, there exists $u\in E$ solution of the first equation of (52). Thus, the claim holds and, consequently,

$$\mathrm{codim}\left[\mathrm{Rg}\left(\mathcal{L}(F(\lambda))\right)\right] = 1.$$

(CR_3): Note that

$$\mathcal{L}_1(F(\lambda)) := D_\mu \mathcal{L}(F(\lambda)) = \begin{bmatrix} 0 & 0 \\ 0 & -\mathrm{Id} \end{bmatrix}.$$

Arguing by contradiction, if $\mathcal{L}_1(F(\lambda))\,(\varphi_1, \psi_1) \in \mathrm{Rg}\,[\mathcal{L}(F(\lambda))]$, there exists $(\xi, \eta) \in E^2$ such that

$$\mathcal{L}(F(\lambda))(\xi, \eta) = \mathcal{L}_1\,(F(\lambda))\,(\varphi_1, \psi_1)\,.$$

Consequently, η satisfies

$$\begin{cases} -n(0)\Delta\eta + d\theta_{[\lambda,m(\cdot)]}\eta - F(\lambda)\eta = -\psi_1 & \text{in } \Omega, \\ \eta = 0 & \text{on } \partial\Omega. \end{cases}$$

Multiplying the previous equation by ψ_1, integrating over Ω and taking into account (49), we obtain

$$\int_\Omega \psi_1^2 = 0,$$

a contradiction. Thus

$$\mathcal{L}_1\left(F(\lambda), \theta_{[\lambda,m(\cdot)]}, 0\right)(\varphi_1, \psi_1) \notin \mathrm{Rg}\,[\mathcal{L}(F(\lambda))]\,.$$

Therefore, by Theorem 1.7 of [8], the solutions of (P) in a neighborhood of $\left(F(\lambda), \theta_{[\lambda,m(\cdot)]}, 0\right)$ are given by $(\mu(s), u(s), v(s))$ defined in (44), for all $s \in (-\varepsilon, \varepsilon)$.

Now, we will study the direction bifurcation from the semi-trivial solution. For that, we determine the value of μ_1 in (44). Note that, using the expressions of (44) in the second equation of (P), we get

$$\begin{aligned} -n\left(\int_\Omega s(\psi_1 + \psi(s))\right) s\Delta(\psi_1 + \psi(s)) =& s(F(\lambda) + s\mu_1 + o(s))(\psi_1 + \psi(s)) \\ & - s^2(\psi_1 + \psi(s))^2 - d\theta_{[\lambda,m(\cdot)]}s(\psi_1 + \psi(s)) \\ & - ds^2(\varphi_1 + \varphi(s))(\psi_1 + \psi(s)) \end{aligned}$$

Dividing the previous expression by s, taking the Taylor Expansion for the function $n\left(\int_\Omega v(s)\right)$ and substituting it, we obtain

$$\begin{aligned} -\left(n(0) + n'(0)s\int_\Omega \psi_1 + o(s)\right)\Delta(\psi_1 + \psi(s)) =& (F(\lambda) + s\mu_1 + o(s))(\psi_1 + \psi(s)) \\ & - s(\psi_1 + \psi(s))^2 - d\theta_{[\lambda,m(\cdot)]}(\psi_1 + \psi(s)) \end{aligned}$$

$$- ds(\varphi_1 + \varphi(s))(\psi_1 + \psi(s)).$$

Taking the terms independent of s, we get (49). Moreover, taking the terms of first order in s, it can be easily verified that

$$\begin{aligned}-n(0)\Delta\psi(s) - n'(0)\Delta(\psi_1 + \psi(s))\int_\Omega \psi_1 =& F(\lambda)\psi(s) + \mu_1(\psi_1 + \psi(s)) \\ & - (\psi_1 + \psi(s))^2 - d\theta_{[\lambda,m(\cdot)]}\psi(s) \\ & - d(\varphi_1 + \varphi(s))(\psi_1 + \psi(s)),\end{aligned}$$

and, consequently,

$$\begin{aligned}-n(0)\Delta\psi(s) - F(\lambda)\psi(s) + d\theta_{[\lambda,m(\cdot)]}\psi(s) =& n'(0)\Delta(\psi_1 + \psi(s))\int_\Omega \psi_1 \\ & + \mu_1(\psi_1 + \psi(s)) - (\psi_1 + \psi(s))^2 \\ & - d(\varphi_1 + \varphi(s))(\psi_1 + \psi(s)).\end{aligned}$$

Multiplying by ψ_1 and integrating, we get

$$\begin{aligned}n'(0)\int_\Omega \psi_1 \int_\Omega \Delta(\psi_1 + \psi(s))\psi_1 =& \mu_1 \int_\Omega (\psi_1 + \psi(s))\psi_1 - \int_\Omega (\psi_1 + \psi(s))^2\psi_1 \\ & - d\int_\Omega (\varphi_1 + \varphi(s))(\psi_1 + \psi(s))\psi_1.\end{aligned}$$

Taking $s \to 0$, we obtain

$$n'(0)\int_\Omega \psi_1 \int_\Omega \Delta\psi_1\psi_1 + \mu_1 \int_\Omega \psi_1^2 - \int_\Omega \psi_1^3 - d\int_\Omega \varphi_1\psi_1^2 = 0,$$

whence we deduce (45).

To complete the proof it remains to show (46). Using (49) we obtain that

$$\mu_1 \int_\Omega \psi_1^2 = \frac{n'(0)}{n(0)} \int_\Omega \psi_1 \int_\Omega \left(F(\lambda) - d\theta_{[\lambda,m(\cdot)]}\right)\psi_1^2 + \int_\Omega \psi_1^3 + d\int_\Omega \psi_1^2\varphi_1.$$

Observe that

$$\varphi_1 = -c\eta_1$$

where η_1 is the unique solution of (50) with $c = 1$ and $\eta = \psi_1$. Consequently,

$$\mu_1 \int_\Omega \psi_1^2 = \frac{n'(0)}{n(0)} \int_\Omega \psi_1 \int_\Omega \left(F(\lambda) - d\theta_{[\lambda,m(\cdot)]}\right) \psi_1^2 + \int_\Omega \psi_1^3 - cd \int_\Omega \psi_1^2 \eta_1. \tag{53}$$

Now, we consider $\lambda \approx m(0)\lambda_1$. For that, we follow the ideas of Theorem 5.2 in [12]. Observe that in this case, see [14], the map $\lambda \mapsto \theta_{[\lambda,m(\cdot)]}$ is regular, and then, by Theorem 2.1 (c),

$$\theta_{[\lambda,m(\cdot)]} \cong (\lambda - m(0)\lambda_1) \frac{\Phi_1}{\int_\Omega \Phi_1^3}, \quad \text{for } \lambda \approx m(0)\lambda_1, \tag{54}$$

where Φ_1 is the principal eigenfunction associated to λ_1 with $\|\Phi_1\|_\infty = 1$. Consequently, also the map $\lambda \mapsto F(\lambda)$ is also regular, and then (see Lemma 4.3 in [11])

$$F(\lambda) = \sigma_1 \left[-n(0)\Delta + d\theta_{[\lambda,m(\cdot)]}\right] \cong n(0)\lambda_1 + d(\lambda - m(0)\lambda_1) \frac{\Phi_1}{\int_\Omega \Phi_1^3}, \quad \text{for } \lambda \approx m(0)\lambda_1. \tag{55}$$

Since ψ_1 is the principal eigenfunction of $F(\lambda)$ it follows that $\psi_1 = \psi_1(\lambda)$ admits a expansion of the form

$$\psi_1(\lambda) \cong \Phi_1 + (\lambda - m(0)\lambda_1) \frac{\eta}{\int_\Omega \psi_1^3} + o(\lambda - m(0)\lambda_1), \quad \text{for } \lambda \approx m(0)\lambda_1. \tag{56}$$

Finally, from (50) taking $\lambda \to m(0)\lambda_1$, we obtain

$$\eta_1 \cong \Phi_1 + o(\lambda - m(0)\lambda_1), \quad \text{for } \lambda \approx m(0)\lambda_1. \tag{57}$$

Using (54)-(55) in (53) and taking $\lambda \to m(0)\lambda_1$, it becomes apparent that

$$\mu_1 \int_\Omega \Phi_1^2 = n'(0)\lambda_1 \int_\Omega \Phi_1 \int_\Omega \Phi_1^2 + (1 - cd) \int_\Omega \Phi_1^3.$$

This ends the proof. $\square$

Remark 6.1. From the expression of μ_1, when $\lambda \approx m(0)\lambda_1$ the direction of the bifurcation depends on the signs of $1 - cd$ and $n'(0)$. When $cd < 1$, it is supercritical when $n'(0) > 0$ and subcritical when $n'(0) \le e < 0$ for e large enough.

Remark 6.2. Condition (42) deserves a comment. Observe that $\theta_{[\lambda,m(\cdot)]} \le \lambda$ in Ω, and then

$$\lambda\theta_{[\lambda,m(\cdot)]} - \theta^2_{[\lambda,m(\cdot)]} \ge 0 \quad \text{in } \Omega.$$

On the other hand, by the maximum principle, the solution e_λ of (41) is positive. Hence,

$$\int_\Omega (\lambda\theta_{[\lambda,m(\cdot)]} - \theta^2_{[\lambda,m(\cdot)]}) e_\lambda \ge 0.$$

Then, if $m' \ge 0$ it is obvious that (42) holds.

7. Global bifurcation analysis

In this section, we will analyze the global behavior of the curve of non-trivial solutions of (P). To do this, we use the results from the previous section and apply Theorem 6.4.3 of [23], proving the existence of a continuum of positive solutions of (P).

Consider $Y := C_0^1(\overline{\Omega})$ and the positive cone

$$P_Y = \{u \in Y;\ u(x) \geq 0 \text{ for all } x \in \Omega\}.$$

In the next result, we prove the existence of a continuum of positive solutions of (P), that is, a maximal connected and closed set in the set of positive solutions.

Theorem 7.1. *Assume that $\lambda > 0$. Suppose that there exists a positive solution $(\theta_{[\lambda,m(\cdot)]}, 0)$ verifying (42). Thus, from the point*

$$(\mu, u, v) = \big(F(\lambda), \theta_{[\lambda,m(\cdot)]}, 0\big) \tag{58}$$

bifurcates a continuum $\mathfrak{C}_{(\mu,\theta,0)} \subset \mathbb{R} \times Int(P_Y) \times Int(P_Y)$ of coexistence states of (P). Moreover,

$$Proj_{\mathbb{R}}(\mathfrak{C}_{(\mu,\theta,0)}) \quad \text{is bounded in } \mathbb{R}, \tag{59}$$

where $Proj_{\mathbb{R}}(\mu, u, v) := \mu$ for $(\mu, u, v) \in \mathfrak{C}_{(\mu,\theta,0)}$. In fact, one of the following options occurs:

1. *There exists a sequence $(\mu_n, u_n, v_n) \in \mathfrak{C}_{(\mu,\theta,0)}$ such that $(\mu_n, u_n, v_n) \to (\mu_*, u_*, 0)$, when $n \to +\infty$, where u_* is a positive solution of (13), $u_* \neq \theta_{[\lambda,m(\cdot)]}$ and*

$$\mu_* = \sigma_1[-n(0)\Delta + du_*].$$

2. *There exists a sequence $(\mu_n, u_n, v_n) \in \mathfrak{C}_{(\mu,\theta,0)}$ such that $(\mu_n, u_n, v_n) \to (\mu^*, 0, v_\mu^*)$, when $n \to +\infty$, such that v_μ^* is a positive solution of (14) and*

$$\lambda = G(\mu^*) := \sigma_1\left[-m(0)\Delta + cv_\mu^*\right]. \tag{60}$$

3. *There exists a sequence $(\mu_n, u_n, v_n) \in \mathfrak{C}_{(\mu,\theta,0)}$ such that $(\mu_n, u_n, v_n) \to (n(0)\lambda_1, 0, 0)$, when $n \to +\infty$, and $\lambda = m(0)\lambda_1$.*

Proof. Consider the operator $\mathcal{G} : \mathbb{R} \times Y^2 \to Y^2$ defined by

$$\mathcal{G}(\mu, u, v) = \begin{pmatrix} u - L(Mu + \dfrac{1}{m(\int_\Omega u)}(\lambda u - u^2 - cuv)) \\ v - L(Mv + \dfrac{1}{n(\int_\Omega v)}(\mu v - v^2 - duv)) \end{pmatrix}, \tag{61}$$

where $M > 0$ is a constant and $L := (-\Delta + M)^{-1}$ under homogeneous Dirichlet conditions. It is clear that (u, v) is a non-negative solution of (P) if, and only if, $\mathcal{G}(\mu, u, v) = 0$. Moreover, $\mathcal{G}(\mu, \theta_{[\lambda,m(\cdot)]}, 0) = 0$ for all $\mu \in \mathbb{R}$.

Theorem 6.1 ensures that from the semitrivial solution $(\theta_{[\lambda,m(\cdot)]}, 0)$ bifurcates a branch of positive solution at $\mu = F(\lambda)$.

Now, we can apply Theorem 6.4.3 of [23], and following exactly the argument of Theorem 1.1 of [6] (see also Theorem 4.1 of [21]) and conclude that there exists a continuum $\mathfrak{C}_{(\mu,\theta,0)}$ of coexistence states for (P) that emanates from $(F(\lambda), \theta_{[\lambda,m(\cdot)]}, 0)$ and satisfies at least one of the following alternatives:

(G$_1$) $\mathfrak{C}_{(\mu,\theta,0)}$ is unbounded in $\mathbb{R} \times Y^2$.
(G$_2$) There exists $(\mu_n, u_n, v_n) \in \mathfrak{C}_{(\mu,\theta,0)}$ such that $(\mu_n, u_n, v_n) \to (\overline{\mu}, 0, 0)$.
(G$_3$) There exists $(\mu_n, u_n, v_n) \in \mathfrak{C}_{(\mu,\theta,0)}$ such that $\mu_n \to \overline{\mu}$ and $(u_n, v_n) \to (u_*, 0)$, with $u_* > 0$ solution of (13) and $u_* \neq \theta_{[\lambda,m(\cdot)]}$.
(G$_4$) There exists $(\mu_n, u_n, v_n) \in \mathfrak{C}_{(\mu,\theta,0)}$ such that $\mu_n \to \overline{\mu}$ and $(u_n, v_n) \to \left(0, v_\mu^*\right)$, with $v_\mu^* > 0$ solution of (14).

First, we show that (G_1) is not possible. Indeed, by Proposition 4.1 the positive solutions of (P) are bounded in $L^\infty(\Omega)$, and then, by elliptic regularity in Y.

On the other hand, by Corollary 4.1 and Proposition 4.2, we obtain that $Proj_{\mathbb{R}}(\mathfrak{C}_{(\mu,\theta,0)})$ is bounded in $\mathbb{R}$.

Assume that (G$_2$) holds. Suppose there exists $(\mu_n, u_n, v_n) \in \mathfrak{C}_{(\mu,\theta,0)}$ such that $\mu_n \to \overline{\mu}$ and $(u_n, v_n) \to (0, 0)$. Note that

$$\mu_n = \sigma_1 \left[-n \left(\int_\Omega v_n \right) \Delta + v_n + d u_n \right], \tag{62}$$

and

$$\lambda = \sigma_1 \left[-m \left(\int_\Omega u_n \right) \Delta + u_n + c v_n \right]. \tag{63}$$

Since $(u_n, v_n) \longrightarrow (0, 0)$ in Y^2, we conclude that

$$\lim_{n\to+\infty} \mu_n = n(0)\lambda_1 \quad \text{and} \quad \lambda = m(0)\lambda_1.$$

In this case, (3) occurs.

Now, assume that (G$_3$) occurs. Suppose there exists $(\mu_n, u_n, v_n) \in \mathfrak{C}_{(\mu,\theta,0)}$ such that $\mu_n \to \overline{\mu}$ and $(u_n, v_n) \to (u_*, 0)$, when $n \to +\infty$, with $u_* > 0$ solution of (13). Since from $(\theta_{[\lambda,m(\cdot)]}, 0)$ the unique bifurcation point is $\mu = F(\lambda)$ we get that $u_* \neq \theta_{[\lambda,m(\cdot)]}$.

Since $(u_n, v_n) \to (u_*, 0)$ in Y^2, we take limit as $n \to +\infty$ in (62), to conclude that

$$\lim_{n\to+\infty} \mu_n = \sigma_1 \left[-n(0)\Delta + d u_* \right].$$

In this case, (1) is true.

Finally, assume that (G_4) occurs. In this case, since $\mu_n \to \overline{\mu}$ and $(u_n, v_n) \to \left(0, v_\mu^*\right)$ as $n \to +\infty$, taking limit in (62) we get that

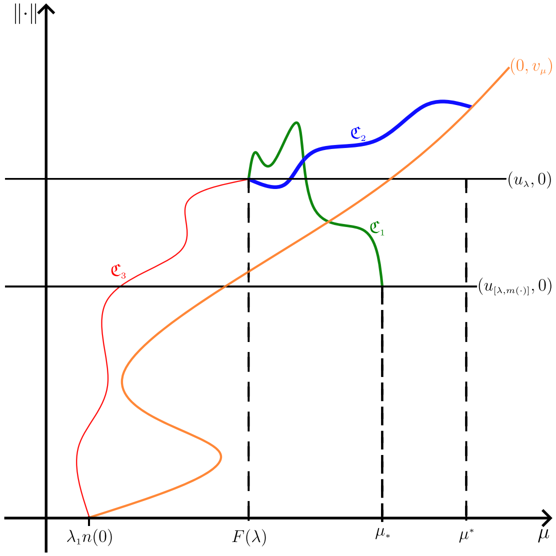


Fig. 2. Possible bifurcation diagrams in the case of two semitrivial solutions $(u, 0)$ and one semtrivial solution $(0, v)$.

$$\overline{\mu} = \sigma_1 \left[-n \left(\int_\Omega v_\mu^* \right) \Delta + \overline{v} \right].$$

Analogously, from (63) we obtain

$$\lambda = \sigma_1[-m(0)\Delta + dv_\mu^*],$$

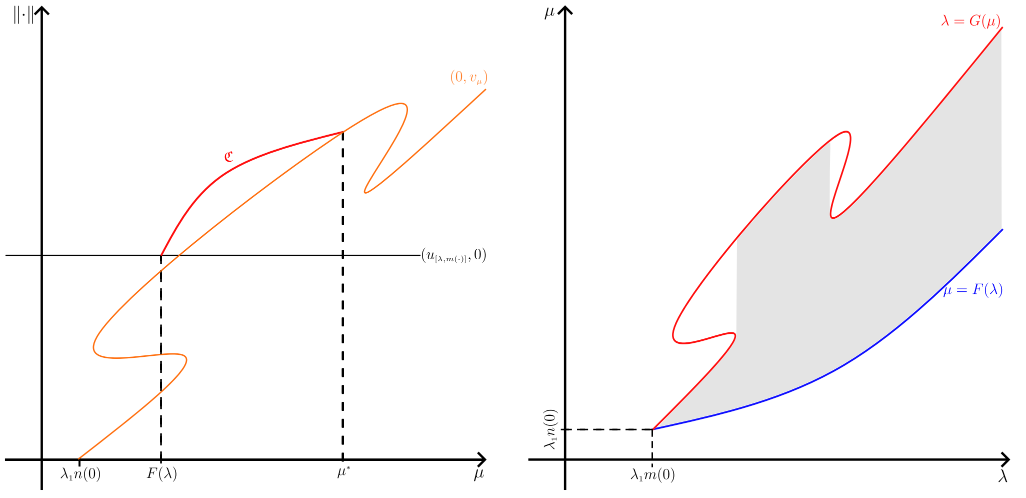
in this case, we have (2).

This concludes the proof. □

In Fig. 2 we have sketched the different behavioral options of the continuum $\mathfrak{C}_{(\mu,\theta,0)}$ of positive solutions of (P). In this figure we have represented a situation with two semitrivial solutions of the form $(u, 0)$, which we have denoted $(u_\lambda, 0)$ and $(u_{[\lambda,m(\cdot)]}, 0)$. On the other hand, a continuum of semitrivial solution $(0, v_\mu)$ exists for $\mu > \lambda_1 n(0)$. The continuum $\mathfrak{C}_{(\mu,\theta,0)}$ bifurcates from the semitrivial solution $(u_\lambda, 0)$ at $\mu = F(\lambda)$. When (1) occurs, $\mathfrak{C}_{(\mu,\theta,0)}$ behaves as $\mathcal{C}_1$, that is, the continuum emanates from the semitrivial $(u_\lambda, 0)$ and goes to the another semitrivial $(u_{[\lambda,m(\cdot)]}, 0)$ at $\mu = \mu_*$. In this case, there exists at least a coexistence state for $\mu \in (F(\lambda), \mu_*)$. $\mathfrak{C}_{(\mu,\theta,0)}$ behaves as $\mathcal{C}_2$ and $\mathcal{C}_3$ when (2) and (3) occurs, respectively. In (2) (resp. (3)) the continuum goes to the semitrivial solution $(0, v_\mu)$ (resp. the trivial solution $(0, 0)$) at $\mu = \mu^*$ (resp. $\mu = \lambda_1 n(0)$) and then there exists at least a coexistence state for $\mu \in (F(\lambda), \mu^*)$ (resp. $\mu \in (\lambda_1 n(0), F(\lambda))$). In general, it is difficult to find out which of the alternatives of Theorem 7.1 is true. When m is increasing, we can assume that one of them always occurs.

Corollary 7.1. *Assume that m is increasing and $\lambda > m(0)\lambda_1$. Then, from the point*

$$(\mu, u, v) = \left(F(\lambda), \theta_{[\lambda,m(\cdot)]}, 0\right) \tag{64}$$



(a) Behaviour of the global continuum $\mathfrak{C}_{(\mu,\theta,0)}$. (b) A possible coexistence region.

Fig. 3. Case m increasing.

bifurcates a continuum $\mathfrak{C}_{(\mu,\theta,0)} \subset \mathbb{R} \times Int(P_Y) \times Int(P_Y)$ of coexistence states of (P)*. Moreover, there exists a sequence $(\mu_n, u_n, v_n) \in \mathfrak{C}_{(\mu,\theta,0)}$ such that $(\mu_n, u_n, v_n) \to (\mu^*, 0, v^*_\mu)$, where v^* is a positive solution of* (14) *and*

$$\lambda = G(\mu^*).$$

As consequence, (P) *possesses at least a coexistence state when*

$$\mu \in (\min\{F(\lambda), \mu^*(\lambda)\}, \max\{F(\lambda), \mu^*(\lambda)\}).$$

Proof. Assume that m is increasing. Then, by Remark 6.1, condition (42) holds. In this case, there exists a unique solution $\theta_{[\lambda,m(\cdot)]}$ of (13). Then, (1) and (3) do not occur. This ends the proof. $\square$

In Fig. 3 we have represented the case m increasing, and then there is a unique semitrivial solution $(u, 0)$. However, since we are not assuming that n is increasing, given $\mu > 0$ there could be several solutions of the form $(0, v)$. In Fig. 3 (a) we have represented a continuum of semitrivial solution $(0, v_\mu)$ and describe the bifurcation diagram. In this case, the global continuum $\mathfrak{C}_{(\mu,\theta,0)}$ goes to the semitrivial $(0, v_\mu)$. Observe that in this case $G(\mu)$ defined in (60) is not a curve, not even a function. Hence, we cannot guarantee the existence of a coexistence state for any (λ, μ) within the region delimited by the map $\mu = F(\lambda)$ and the graph $\lambda = G(\mu)$. In Fig. 3 (b) we have represented a possible coexistence region, the grey subset. We can not ensure the existence of coexistence states in the white area.

In Fig. 4, we represent the case where m is increasing, $\lambda \approx m(0)\lambda_1$, and $n'(0) < 0$, so that the bifurcation direction is subcritical. Due to the subcritical nature of the bifurcation, there exist at least two coexistence states for $\mu \in (\mu_*, F(\lambda))$, for some $\mu_* < F(\lambda)$.

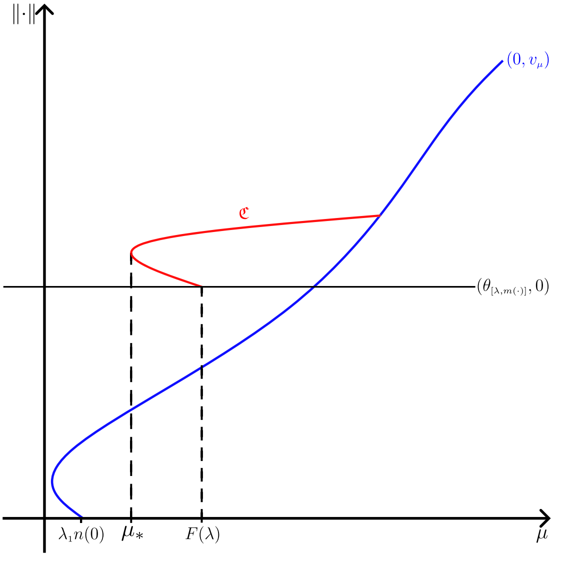


Fig. 4. Bifurcation diagram for $\lambda \approx m(0)\lambda_1$, $cd < 1$ and $n'(0) < 0$. In this case, the bifurcation is subcritical.

Finally, we consider the case where m and n both increase. In this case, there exist unique semi-trivial solutions $\theta_{[\lambda,m(\cdot)]}$ and $\theta_{[\mu,n(\cdot)]}$ for $\lambda > m(0)\lambda_1$ and $\mu > n(0)\lambda_1$, respectively. Hence, the following maps

$$F(\lambda) := \sigma_1\left[-n(0)\Delta + d\theta_{[\lambda,m(\cdot)]}\right], \quad G(\mu) := \sigma_1\left[-m(0)\Delta + c\theta_{[\mu,n]}\right],$$

are well-defined, continuous, increasing and

$$\lim_{\lambda\to+\infty} F(\lambda) = +\infty, \qquad \lim_{\mu\to+\infty} G(\mu) = +\infty.$$

Theorem 7.2. *Assume that m and n are increasing.*

1. *If $\lambda \le m(0)\lambda_1$ or $\mu \le n(0)\lambda_1$, (P) does not possess coexistence states.*
2. *If*

$$(\mu - F(\lambda)) \cdot (\lambda - G(\mu)) > 0 \tag{65}$$

then (P) possesses at least a coexistence state.

Proof. (1) Consider $(u, v) \in Y^2$ a coexistence state of (P). Then, since m is increasing,

$$\lambda = \sigma_1\left[-m\left(\int_\Omega u\right)\Delta + u + cv\right] > \sigma_1[-m(0)\Delta] = m(0)\lambda_1.$$

Analogously, we can show that $\mu > n(0)\lambda_1$.

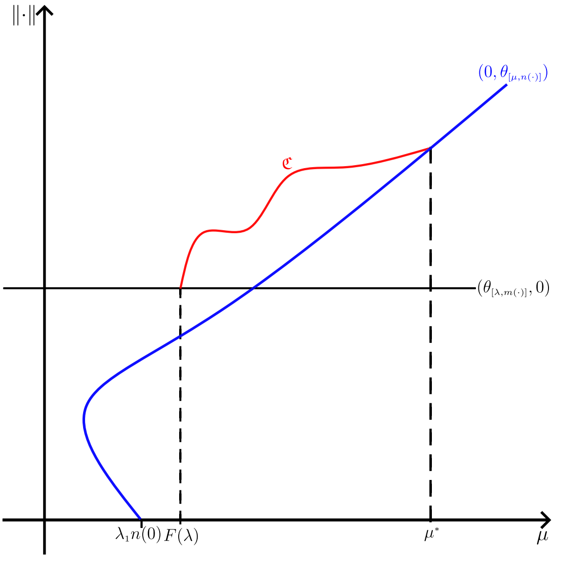


Fig. 5. Bifurcation diagram when m and n both increase. In this case, there exists a unique semitrivial solution $(u_\lambda, 0)$ and $(0, v_\mu)$.

(2) Assume $\lambda > m(0)\lambda_1$ and $\mu > n(0)\lambda_1$. Since m and n are increasing, there exists a unique semi-trivial solution $(\theta_{[\lambda,m(\cdot)]}, 0)$ and $(0, \theta_{[\mu,n(\cdot)]})$. By Corollary 7.1, there exists at least a coexistence state if

$$\mu \in (\min\{F(\lambda), \mu^*(\lambda)\}, \max\{F(\lambda), \mu^*(\lambda)\}). \tag{66}$$

Assume (65), for example $\mu > F(\lambda)$ and $\lambda > G(\mu)$. Since $G(\mu^*(\lambda)) = \lambda > G(\mu)$ and G is increasing, it follows that $\mu < \mu^*(\lambda)$. Hence,

$$F(\lambda) < \mu < \mu^*(\lambda)$$

and then μ satisfies (66), and as consequence (P) possesses at least a coexistence state.

We can argue in the same way if $\mu < F(\lambda)$ and $\lambda < G(\mu)$. □

In Fig. 5 we have represented the bifurcation diagram when m and n increase. In the following result, we will show that by assuming m and n are increasing, we can improve the result in Theorem 5.1, which did not impose any conditions on the functions m and n. In Fig. 6, we compare the coexistence region in both cases.

Lemma 7.1. *If (λ, μ) verifies (36), then $\mu > F(\lambda)$ and $\lambda > G(\mu)$.*

Proof. Assume that (λ, μ) verifies (36). Then

$$\mu > \lambda_1 n_M + d\lambda = \sigma_1[-n_M\Delta + d\lambda] > \sigma_1\left[-n(0)\Delta + d\theta_{[\lambda,m(\cdot)]}\right] = F(\lambda).$$

Analogously, $\lambda > \lambda_1 m_M + c\mu > G(\mu)$. □

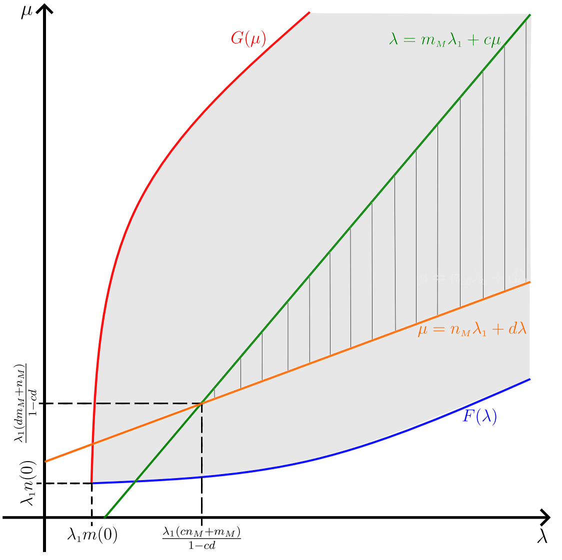


Fig. 6. In this figure we compare the coexistence regions defined by (36) and (65). It is clear that region (65) defines a region larger than (36).

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Data availability

No data was used for the research described in the article.

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