q analogue realization of nucleon pairing

S. Shelly Sharma*
Instituto de Física Teórica, Universidade Estadual Paulista, Rua Pamplona-145, CEP 01405, São Paulo, São Paulo, Brazil
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A q-deformed analogue of zero-coupled nucleon pair states is constructed and the possibility of accounting for pairing correlations examined. For the single orbit case, the deformed pairs are found to be more strongly bound than the pairs with zero deformation, when a real-valued q parameter is used. It is found that an appropriately scaled deformation parameter reproduces the empirical few nucleon binding energies for nucleons in the $1f_{7/2}$ orbit and $1g_{9/2}$ orbit. The deformed pair Hamiltonian apparently accounts for many-body correlations, the strength of higher-order force terms being determined by the deformation parameter q. An extension to the multishell case, with deformed zero-coupled pairs distributed over several single particle orbits, has been realized. An analysis of calculated and experimental ground state energies and the energy spectra of three lowermost 0^+ states, for even-A Ca isotopes, reveals that the deformation simulates the effective residual interaction to a large extent.

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I. INTRODUCTION

The quantum group $SU_q(2)$, the q deformation of the Lie algebra of SU(2), has been studied extensively by Jimbo [1], Woronowicz [2], and Pasquier [3]. Macfarlane [4] and Biedenharn [5] have generalized to $SU_q(2)$, the Schwinger approach to the quantum theory of angular momentum, by constructing a q-deformed version of the quantum harmonic oscillator formalism. Such developments have motivated the application of q-deformed algebra to various physical situations in nuclear and molecular physics [6]. There has been a great interest in the application of quantum algebras to solvable models [7, 8]. Presently, we construct (i) the seniority scheme for deformed nucleon pairs in a single j orbit and (ii) zero seniority states for the case of deformed nucleon pairs distributed over m orbits, interacting through a pairing force. The object is to understand the physical significance of the deformation parameter q in the context of interacting multipair nucleon systems.

II. REVIEW OF SENIORITY SCHEME

In the description of even-even nuclei, where one kind of active nucleons in the valence orbit is considered, the pair scattering matrix elements, $G(j_1 j_1 j_2 j_2 0)$, are by far the largest due to the short-range nature of the interaction. The low-lying states in the energy spectrum of even nuclei are, as such, expected to be mainly composed of configurations with most of the particles appearing in pairs coupled to J=0. This concept is further supported by the empirical evidence that all ground states of even-even nuclei have spin and parity $J^\pi = 0^+$, and are considerably lower in energy than all the states with $J^\pi \neq 0^+$. Therefore it is meaningful to characterize shell model configurations by the number of valence particles occurring in $J^\pi = 0^+$ pairs. For this purpose the seniority scheme using the seniority quantum number v, denoting the number of unpaired particles, is used. It is equivalent to labelling a state by the number of zero-coupled pairs present.

We can write a zero-coupled pair for two nucleons [9] in a shell model orbit j as

$$Z_0 = -\frac{1}{\sqrt{2}}(A^j \times A^j)^0 \tag{1}$$

with

$$\overline{Z_0} = \frac{1}{\sqrt{2}}(B^j \times B^j)^0, \tag{2}$$

where

$$A_{jm} = a^\dagger_{jm}, \quad B_{jm} = (-1)^{m+j} a_{j,-m}. \tag{3}$$

The fermion creation and destruction operators $a^\dagger_{jm}$ and a_{jm} satisfy the usual commutation relations

$$[a^\dagger_{jm}, a_{jm}] = 1. \tag{4}$$

And the number operator is defined as

$$n_{op} = \sum_m a^\dagger_{jm} a_{jm}. \tag{5}$$

We can easily verify that

$$[Z_0, \overline{Z}_0] = \frac{n_{op}}{\Omega} - 1, \tag{6}$$

$$[n_{op}, Z_0] = 2\,Z_0, \quad [n_{op}, \overline{Z}_0] = -2\,\overline{Z}_0, \tag{7}$$

where $(2j+1) = N = 2\Omega$.

Next we may define the single orbit interaction operator

*On leave from Depto. de Fisica, Universidade Estadual de Londrina, Londrina, Brazil.

$$G_{op} = 2\,\Omega\, Z_0\, \overline{Z}_0. \tag{8}$$

We may note that

$$[G_{op}, Z_0] = 2\, Z_0\, (\Omega - n_{op}), \tag{9}$$

$$[G_{op}, \overline{Z}_0] = -2\,(\Omega - n_{op})\, \overline{Z}_0. \tag{10}$$

On the basis of these equations, we can catalog n-nucleon states of the configuration j^n. The nuclear state of n nucleons may be represented by $|n, v\rangle$, v being the number of unpaired nucleons in the state. In particular a state having no zero-coupled pairs, i.e., $n = v$, satisfies $G_{op}|v, v\rangle = 0$. The states with $n = v + 2, v + 4, \ldots$ may be constructed from this core state by successive applications of the operator Z_0. Again we can check that

$$G_{op}\,(Z_0)^p\,|v, v\rangle = 2\,p\,[\Omega - (v + p - 1)]\,(Z_0)^p\,|v, v\rangle. \tag{11}$$

Here $n = 2p + v$. The normalization factor for these states is easily obtained by successive application of the commutation relation between the operators Z_0 and $\overline{Z}_0$.

The pairing force Hamiltonian for identical particles in a single orbit is given by

$$H = \frac{A}{N}\, G_{op}, \tag{12}$$

where A is the strength parameter. It is easily verified that the pairing energy for the state $|n, v\rangle$ is

$$E(n, v) = \frac{A}{2N}\,(n - v)(N - n - v + 2). \tag{13}$$

III. THE q-DEFORMED PAIRS IN A SINGLE ORBIT

We may rewrite our pair operators in terms of the well-known quasispin operators by identifying

$$S_+ = \sqrt{\Omega}\, Z_0, \quad S_- = \sqrt{\Omega}\, \overline{Z}_0, \tag{14}$$

$$S_0 = \frac{(n_{op} - \Omega)}{2}. \tag{15}$$

S_+, S_-, and S_0 are the generators of Lie algebra of SU(2) and satisfy the same commutation relations as the angular momentum operators.

$$[S_+, S_-] = 2S_0, \tag{16}$$

$$[S_0, S_\pm] = \pm S_\pm. \tag{17}$$

Total quasispin operator is given by

$$S^2 = S_+S_- + S_0(S_0 - 1). \tag{18}$$

An equivalent description of the state $|n, v\rangle$ can be given in terms of the total quasispin quantum number s (related to the seniority quantum number v through $s = \frac{(\Omega - v)}{2}$) and the eigenvalue of operator S_0. The states $|s, s_0\rangle$ satisfy the following relations:

$$S^2\,|s, s_0\rangle = s(s + 1)\,|s, s_0\rangle, \quad S_0|s, s_0\rangle = s_0\,|s, s_0\rangle. \tag{19}$$

We may now define q-deformed pairs in terms of the generators of $SU_q(2)$ satisfying the commutation relations

$$[S_+(q), S_-(q)] = \{2S_0(q)\}_q, \tag{20}$$

$$[S_0(q), S_\pm(q)] = \pm S_\pm(q), \tag{21}$$

where

$$\{x\}_q = \frac{(q^x - q^{-x})}{(q - q^{-1})}. \tag{22}$$

Translated to Z operators the new commutation relations give

$$[Z_0, \overline{Z}_0] = \frac{\{n_{op} - \Omega\}_q}{\Omega}, \tag{23}$$

$$[n_{op}, Z_0] = 2\, Z_0, \quad [n_{op}, \overline{Z}_0] = -2\, \overline{Z}_0. \tag{24}$$

The commutator of the q-deformed interaction operator with the new deformed Z operators is found to be

$$[G_{op}, Z_0] = -2\, Z_0\, \{n_{op} - \Omega\}_q. \tag{25}$$

We define the vaccum state for q-deformed pairs through

$$G_{op}|v, v\rangle = 0. \tag{26}$$

Normalized n-nucleon states (normalization constant N_c), with $n = p + v$, are the eigenstates of the interaction operator G_{op}:

$$G_{op}\, N_c\, Z_0{}^p\,|v, v\rangle$$

$$= 2\sum_{s=0}^{p} \{\Omega - v - 2p + 2s + 2\}_q\, N_c\, Z_0{}^p\,|v, v\rangle. \tag{27}$$

For $q = e^\tau$, $\tau \neq 0$ being a real finite valued parameter, the expectation value of the pairing force Hamiltonian, $H = \frac{A}{N}\, G_{op}$, gives the energy of the n-nucleon state with p pairs and v unpaired nucleons as a function of deformation parameter τ:

$$E_q(n, v) = \langle n, v\,|\,H\,|\,n, v\rangle$$

$$= \frac{2\,A\,\sinh(p\tau)\,\sinh[(\Omega - v - p + 1)\tau]}{N\,\sinh^2(\tau)}. \tag{28}$$

As a first approximation, the ground state binding energies of $1f_{7/2}$ nucleons in even-A calcium isotopes may be obtained by treating the ground states as n-neutron zero senority states outside a ^{40}Ca core, with neutrons interacting via a pairing force. The calculated energies are, however, found to be less negative than the experimental binding energies. As explained by Lawson [10], the Talmi binding energy formula—including the effect of residual interaction due to two-nucleon coupling to $J^\pi = 2^+, 4^+, 6^+$ states—is seen to give a satisfactory agreement with the experiment. We have plotted in Fig. 1, the energies for the case of neutrons in $1f_{7/2}$ orbit, $n = 2, 4, 6, 8$, calculated by using Eq. (28), as a function of the parameter $\tau' = \frac{\tau}{p}$. The change of variable from τ to τ' has been effected to emphasize that the values of

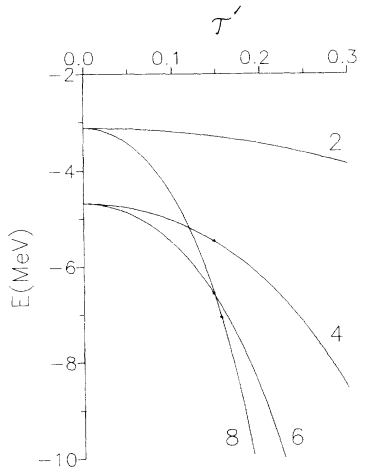

FIG. 1. The variation of calculated energies with deformation parameter, τ', for neutrons in $1f_{7/2}$ orbit ($n = 2,4,6,8$). $A = -3.12$ MeV. Experimental binding energies, indicated by markers, are reproduced for $\tau' \approx 0.147$.

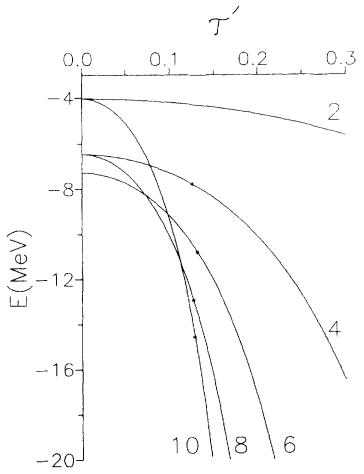

FIG. 3. The variation of calculated energies with deformation parameter, τ', for neutrons in $1g_{9/2}$ orbit ($n = 2,4,6,8,10$). $A = -4.05$ MeV. Experimental binding energies, indicated by markers, are reproduced for $\tau' \approx 0.127$.

deformation parameter τ, fitted to reproduce the experimental binding energies for nuclei with increasing number of active nucleons n, show a well-defined systematics. Single neutron energy from ^{41}Ca and zero-coupled pair energy from ^{42}Ca, have been used as input. We notice that for the deformation parameter value of $\tau' \approx 0.147$ the theoretical energies are in good agreement with the experimental results (shown in the figure by markers). Figure 2 shows a plot of proton binding energies versus τ' for protons in $1f_{7/2}$ orbit, $n = 2,4,6,8$, relative to ^{48}Ca core. In this case the experimental results are well reproduced for $\tau' \approx 0.176$. A similar calculation for neutrons in $1g_{9/2}$ orbit, $n = 2,4,6,8,10$, outside a ^{80}Zr core has been done using two nucleon energy from ^{82}Zr as the pair interaction energy and the results presented in Fig. 3. We have used a value of 9.83 MeV (average of neutron separation energies for ^{83}Zr,^{85}Zr, and ^{87}Zr) as the interaction energy of $1g_{9/2}$ neutron with the core. A good matching of experimental numbers with the theoretical results is obtained for $\tau' \approx 0.127$. It is interesting to note that in these cases the effect of residual interaction may be simulated through the deformed pairs. Besides that there is strong evidence of a simple relationship between the deformation parameter value τ that reproduces the experimental binding energy and the number of pairs present in the orbit.

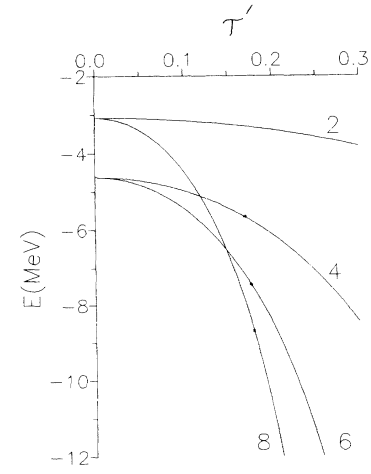

FIG. 2. The variation of calculated energies with deformation parameter, τ', for protons in $1f_{7/2}$ orbit ($n = 2,4,6,8$). $A = -3.08$ MeV. Experimental binding energies, indicated by markers, are reproduced for $\tau' \approx 0.176$.

IV. THE q-DEFORMED HAMILTONIAN AND PAIR CORRELATIONS

The quantum group $SU_q(2)$ and the group SU(2) being isomorphic there exists a homomorphism of the corresponding vector spaces V_q and V. In the limit $q \to 1$, as the homomorphism of vector spaces goes to automorphism, the q-deformed commutators reduce to normal Lie brackets. In order to understand the physical significance of deformation, in the context of interaction between the nucleons, we examine the deformation functionals which map the SU(2) generators into generators of $SU_q(2)$. Following Curtright and Zachos [11] for generic $q \neq 1$, the generators of $SU_q(2)$ as defined in Sec. III are related to the undeformed quasispin operators through

$$S_0(q) = S_0, \quad S_+(q) = \left[\frac{\{S_0 + S\}_q \{S_0 - 1 - S\}_q}{(S_0 + S)(S_0 - 1 - S)}\right]^{\frac{1}{2}} S_+, \tag{29}$$

$$S_- = [S_+(q)]^\dagger.$$

The functionals are invertible and when representations of SU(2) are substituted into Eq. (29), they yield the corresponding representations of $SU_q(2)$, of the same dimension. We can verify that in the space spanned by vectors $|s, s_0\rangle$, for (i) $\Omega = 1$, $s = \frac{1}{2}$, i.e. $s_{1/2}$ or $p_{1/2}$ orbit the representation of S_+ and S_0 maps to itself, i.e.,

$$S_+(q) = S_+, \quad S_0(q) = S_0. \tag{30}$$

(ii) $\Omega = 2$, $s = 1$, i.e., $p_{3/2}$ or $d_{3/2}$ orbit

$$S_+(q) = \left[\frac{\{2\}_q}{2}\right]^{\frac{1}{2}} S_+, \quad S_0(q) = S_0. \tag{31}$$

The pairing Hamiltonian

$$H = \frac{A}{N}\frac{\{2\}_q}{2} S_+ S_- \tag{32}$$

corresponds to a pairing Hamiltonian with q dependent pairing strength in this case. (iii) $\Omega = 3$, $s = \frac{3}{2}$, i.e., $d_{5/2}$ or $f_{5/2}$ orbit the representation of S_+ and S_0,

$$S_+ = \begin{pmatrix} 0 & \sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{\frac{3}{2}} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad S_0 = \begin{pmatrix} \frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{pmatrix} \tag{33}$$

maps to

$$S_+(q) = \begin{pmatrix} 0 & \sqrt{\frac{\{3\}_q}{2}} & 0 & 0 \\ 0 & 0 & \frac{\{2\}_q}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \sqrt{\frac{\{3\}_q}{2}} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad S_0(q) = S_0 \tag{34}$$

and so forth.

The pairing Hamiltonian for the general case may be written as

$$H = \frac{A}{N} S_+(q)\, S_-(q) = \frac{A}{N}\{S + S_0\}_q \{S - S_0 + 1\}_q. \tag{35}$$

For the special case, $q = e^\tau$, we examine the structure of H by making a Taylor expansion of the numerator. The terms containing the same powers of $(S + S_0)(S - S_0 + 1)$ are summed up to obtain the coefficients of each power. The expanded Hamiltonian is

$$H = \frac{A}{N}\frac{1}{\left(\sqrt{\frac{\pi}{2\tau}} I_{1/2}(\tau)\right)^2}\left[\sqrt{\frac{\pi}{2\sigma\tau}} I_{1/2}(\sigma\tau) S_+S_- - \frac{\tau}{\sigma}\sqrt{\frac{\pi}{2\sigma\tau}} I_{3/2}(\sigma\tau)\, S_+S_-S_+S_- + \frac{2\tau^2}{3\sigma^2}\sqrt{\frac{\pi}{2\sigma\tau}} I_{\frac{5}{2}}(\sigma\tau)\, S_+S_-S_+S_-S_+S_- \ldots.\right], \tag{36}$$

where $\sigma = (2S + 1)$ and $\sqrt{\frac{\pi}{2\tau}} I_{n+1/2}(\tau)$ are the modified spherical Bessel functions of the first kind [12]. We may note that the deformed pair Hamiltonian, besides having deformation dependent strength, contains operators for four-body, six-body, and other higher-order processes. Possibly such events amount to simulating the residual interaction partly. A more detailed examination of these questions is in progress.

V. THE q-DEFORMED PAIRS FOR MULTISHELL CASE

We consider here a special version of low-seniority shell model for even nuclei. The vacuum state for deformed pairs, $|0\rangle$, represents an inert core with magic proton and neutron numbers. The quantum numbers j then refer only to orbitals in the next major valence shell beyond this shell closure. The effect of mean field generated by core particles is included by using empirical single-particle energies, ϵ_j. One can write a seniority, v=0, multishell state for, $p = \frac{n}{2}$, deformed pairs distributed over m single-particle orbits as

$$|p_1, \ldots, p_m\rangle = \prod_{i=1}^{m}\left(\prod_{k=1}^{p_i}\{k\}_q\{\Omega - k + 1\}_q\right)^{-\frac{1}{2}} \times [S_+(q)]^{p_i}|0\rangle. \tag{37}$$

In the limit $q \to 0$, we recover the usual multishell state given by

$$|p_1, \ldots, p_m\rangle = \prod_{i=1}^{m}\left(\frac{(\Omega_i - p_i)!}{p_i!\Omega_i!}\right)^{\frac{1}{2}} (S_+)^{p_i}|0\rangle. \tag{38}$$

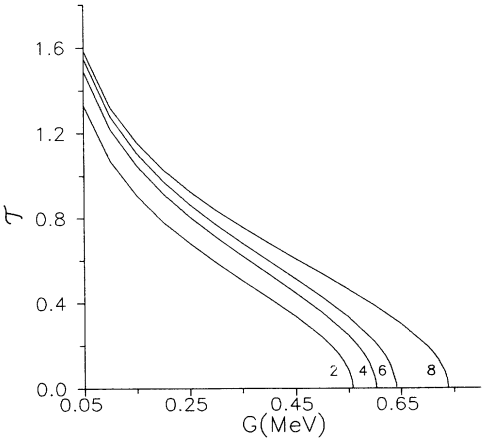

FIG. 4. The deformation parameter values that reproduce the correct ground state energies for ^{42}Ca, ^{44}Ca,^{46}Ca, and ^{48}Ca nuclei (n=2,4,6,8), vs pairing strength, G. Zero-coupled deformed pairs are distributed over four shell model orbits. Single-particle energies used are $\epsilon_{1f_{7/2}} = 0.0$ MeV, $\epsilon_{2p_{3/2}} = 2.1$ MeV,$\epsilon_{2p_{1/2}} = 4.4$ MeV, $\epsilon_{1f_{\frac{5}{2}}} = 8.2$ MeV.

The number of pairs in the ith valence shell satisfies

$$p_i \leq \Omega_i, \quad \sum_{i=1}^{m} p_i = p. \tag{39}$$

The dimension of the $v = 0$ space is equal to the number of different ways in which p pairs can be distributed over the valence subshells.

The Hamiltonian for the system may be written as

$$H = \sum_j \epsilon_j n_{op}^j + H_P, \tag{40}$$

where the pairing interaction is given by

$$H_P = -G \sum_{r,s} S_{r+}(q) S_{s-}(q). \tag{41}$$

G being the pairing interaction strength parameter. The matrix elements of the Hamiltonian between the basis states are obtained by using the quocommutation relations given in Eqs. (20) and (21). The diagonal matrix element is given by

$$\langle p_1, ..., p_m | H | p_1, ..., p_m \rangle = \sum_i^m 2\epsilon_i p_i - G \sum_r \{p_r\}_q \{\Omega - p_r + 1\}_q. \tag{42}$$

The off-diagonal matrix elements of the Hamiltonian are

$$\langle p_1, ..., p_m | H | p'_1, ..., p'_m \rangle = -G \sum_{r,s} \left(\{p_r\}_q \{\Omega - p_r + 1\}_q \{p_s + 1\}_q \{\Omega - p_s\}_q \right)^{\frac{1}{2}} \left(\prod_{i \neq r,s}^{m} \delta_{p_i, p'_i} \right) \left(\delta_{p_r, p_{r'}+1} \delta_{p_s+1, p'_s} \right). \tag{43}$$

The Hamiltonian matrix is set up and diagonalized to obtain the energy spectra of 0^+ states as a function of interaction parameter G and the parameter q.

For $q = e^{\tau}$, we have calculated the energy spectra of 0^+ states of even-A calcium isotopes. Active zero-coupled deformed nucleon pairs are allowed to occupy the complete (f-p) shell, outside the inert ^{40}Ca core. Single-particle energies used are $\epsilon_{1f_{7/2}} = 0.0$ MeV, $\epsilon_{2p_{3/2}} =$

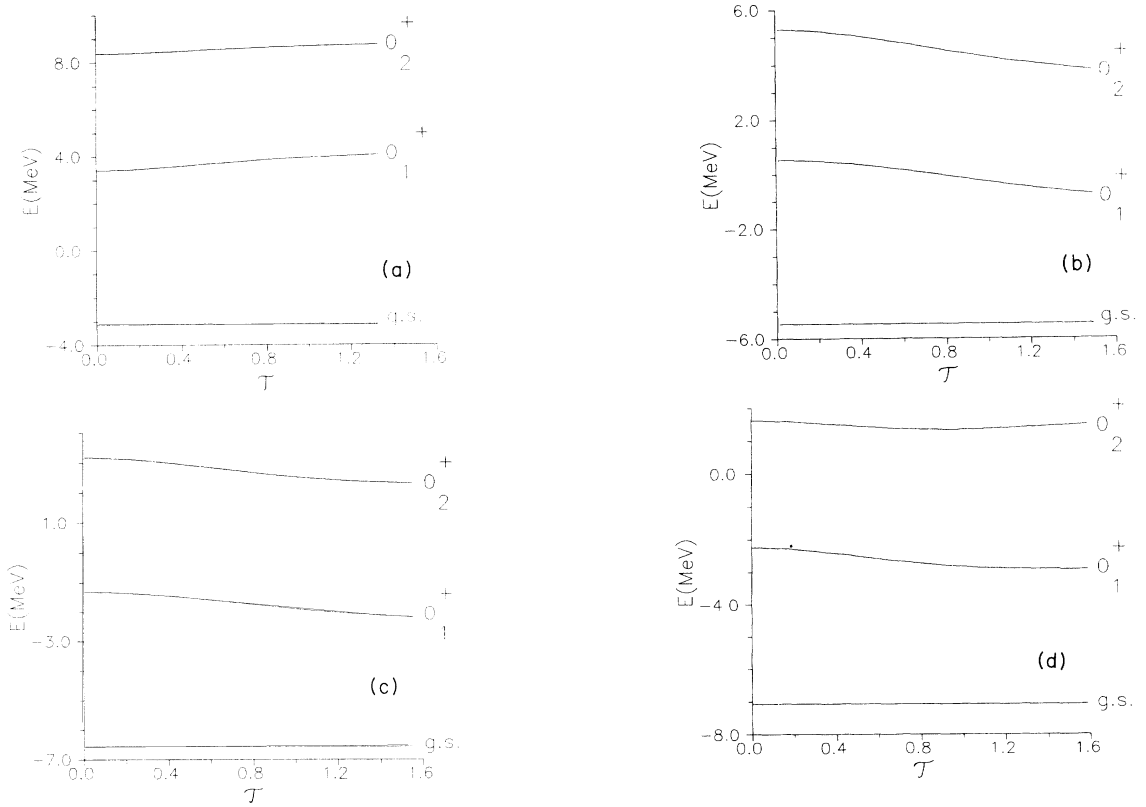

FIG. 5. (a) ^{42}Ca nucleus. Calculated energy spectra of three lowermost 0^+ states as a function of deformation parameter τ. At each point, the pairing interaction strength is choosen to correctly reproduce experimental ground state binding energy. Single-particle energies are the same as in Fig. 4. (b) ^{44}Ca nucleus. Same as in (a). (c) ^{46}Ca nucleus. Same as in (a). (d) ^{48}Ca nucleus. Same as in (a).

2.1 MeV, $\epsilon_{2p_{1/2}} = 4.4$ MeV, $\epsilon_{1f_{5/2}} = 8.2$ MeV. Figure 4 shows the deformation parameter values that reproduce the correct ground state energies for ^{42}Ca,^{44}Ca,^{46}Ca, and ^{48}Ca nuclei, (i.e., n=2,4,6,8) as a function of the pairing interaction strength parameter G. We may note that as the number of interacting nucleons increases, for a fixed value of G, succesively higher values of deformation τ are required to fully account for the internucleon interaction. In the range 0.2 MeV$\leq$G$\leq$0.5 MeV, τ is almost a linear function of G and can be written as

$$\tau = \tau_{0n} + G \tan(\theta_n), \tag{44}$$

where the constant parameters τ_{0n} and $\tan(\theta_n)$ characterize the curve for a given value of n. The scaling effect observed in the single orbit case is seen to be washed out in the multishell case as there is no simple relationship between the values of parameters describing the curves for different n values. In the extreme case, where very small values of G are used, large deformation τ simulates the interaction amongst the nucleons.

Figures 5(a)–5(c) represent a plot of the calculated energy spectra of three lowermost 0^+ states for the nuclei ^{42}Ca,^{44}Ca,^{46}Ca, and ^{48}Ca as a function of deformation parameter τ. At each point the pairing interaction strength is chosen to correctly reproduce the experimental ground state binding energy. We observe that in general weakly interacting heavily deformed pairs reproduce almost the same energy spectra as the strongly interacting weakly deformed pairs. In particular, as τ increases, the spectrum shows a tendency to expand in the nucleus ^{42}Ca, whereas an opposite trend is seen in the spectra of ^{44}Ca and ^{46}Ca. For the nucleus ^{48}Ca, the spectrum is first seen to shrink as τ varies from 0.0 to $\approx$0.8 and then it expands for larger values of τ. The maximum variation in the energy of a state is of the order of 1 MeV. As such it is possible to fix τ at a value so as to best reproduce the 0^+ spectra.

VI. CONCLUSIONS

By constructing a q-deformed analogue of zero-coupled nucleon pair states, we introduce an additional deformation parameter in the pairing interaction theory. For $q = e^{\tau}$, $\tau \neq 0$ being a real finite valued parameter, the deformed pairs are seen to be more strongly bound than the pairs with no deformation. Quantum algebra in general describes perturbation from some underlying symmetry structure—in the present case possibly corrections due to the residual interaction. An important conclusion is that the deformation simulates the effective residual interaction to a large extent. As such the possibility of constructing a deformation dependent analogue of BCS theory may be examined.

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