

Is the equivalence for the response of static scalar sources in the Schwarzschild and Rindler spacetimes valid only in four dimensions?

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It was shown recently that in four dimensions scalar sources with fixed proper acceleration minimally coupled to a massless Klein-Gordon field lead to the same responses when they are (i) uniformly accelerated in Minkowski spacetime (in the inertial vacuum) and (ii) static in the Schwarzschild spacetime (in the Unruh vacuum). Here we show that this equivalence is broken if the spacetime dimension is more than four.

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Let us consider a pointlike scalar source with fixed proper acceleration, $a_0 = \text{const}$, minimally coupled to a massless Klein-Gordon field Φ through a small coupling constant q . It was shown recently that the source's response $R_S(r_0, M)$ to the Hawking radiation (associated with the Unruh vacuum) obtained when it lies at rest with (Schwarzschild) radial coordinate $r_0 = \text{const} > 2M$, outside a chargeless static black hole with mass M , is exactly the same as the response $R_M(a_0)$ of the source when it is uniformly accelerated (with the same proper acceleration as before) in the inertial vacuum of the Minkowski spacetime, or, equivalently, when it is static in the Fulling-Davies-Unruh thermal bath of the Rindler wedge [1].

The fact that this result is surprising can be seen as follows. First let us recall that in Schwarzschild spacetime we can express the source's radial coordinate r_0 in terms of its proper acceleration a_0 and the black hole mass M : $r_0 = r_0(a_0, M)$. Thus, it would be natural to expect that the response would depend on M as well as on a_0 , i.e., $R_S = R_S(a_0, M)$, rather than

$$R_S = R_S(a_0) = R_M(a_0) = q^2 a_0 / (4\pi^2). \quad (1)$$

We note that structureless static scalar sources can only interact with zero-energy field modes. Such modes probe the global geometry of spacetime and are accordingly quite different in Schwarzschild and Rindler spacetimes. Indeed the equivalence (1) is not valid, for instance, if one replaces the Unruh vacuum by the Hartle-Hawking one, in which case the source's response is [1]

$$R'_S(a_0, M) = q^2 a_0 / (4\pi^2) + q^2 / (16\pi^2 r_0^2 a_0),$$

nor when the massless Klein-Gordon field is replaced by electromagnetic [2] or massive Klein-Gordon [3] one. Moreover, the equivalence was shown to be broken also when the background spacetime is endowed with a cosmological constant [4] or when the black hole is given some electric charge [5].

It is hitherto unclear whether or not the equivalence found in Ref. [1] hides something deeper behind it. Even in the less interesting case where the equivalence turns out to be a “coincidence”, it will still be interesting to determine whether or not this is precisely restricted to the number of (macroscopic) dimensions of physical spacetimes, as we will do in this paper. Here we adopt natural units ($c = G = \hbar = k_B = 1$) and spacetime signature $(+ - \dots -)$.

The line element of a Schwarzschild spacetime with $N \equiv p + 2$ dimensions ($p \geq 2$) is [6]

$$ds^2 = f(r)dt^2 - f(r)^{-1}dr^2 - r^2 ds_p^2,$$

where $f(r) = 1 - (r_H/r)^{p-1}$ with $r_H = (2M)^{1/(p-1)}$ being the radius of the event horizon and ds_p^2 being the line element of a unit p -sphere S^p . This is assumed to be covered with angular coordinates $\{\theta_1, \dots, \theta_p\}$ and to be endowed with the standard metric $\tilde{\eta}_{ij}$ (and inverse metric $\tilde{\eta}^{ij}$) with signature $(+ \dots +)$. Here $i = 1, \dots, p$ and $j = 1, \dots, p$ are associated with the angular coordinates on S^p .

The Klein-Gordon equation $\square\Phi = 0$ can be written in this background as

$$f^{-1} \partial_t^2 \Phi - r^{-p} \partial_r [f r^p \partial_r \Phi] - r^{-2} \tilde{\nabla}^2 \Phi = 0, \quad (2)$$

where $\tilde{\nabla}^2 \equiv \tilde{\eta}^{ij} \tilde{\nabla}_i \tilde{\nabla}_j$ is the Laplacian and $\tilde{\nabla}_i$ is the associated covariant derivative on S^p . We look for positive frequency modes in the form

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$$u_{\omega lm}^{(n)}(t, r, \theta_i) = \psi_{\omega l}^{(n)}(r) Y_{lm}(\theta_i) e^{-i\omega t} \quad (3)$$

associated with the timelike Killing field $\xi = \partial/\partial t$, where $\omega \geq 0$, $n \Rightarrow$ and $\leftarrow$ label purely ingoing modes from the past white hole horizon $\mathcal{H}^-$ and from the past null infinity J^- , respectively, and $l = 0, 1, 2, \dots$, and m denotes a set $\{m_1, \dots, m_{p-1}\}$ of $p-1$ integers satisfying $l \geq m_{p-1} \geq \dots \geq m_2 \geq |m_1|$. The $u_{\omega lm}^{(n)}(x^\mu)$ modes are assumed to be orthonormalized with respect to the Klein-Gordon inner product [7]:

$$i \int_{\Sigma_t} d\Sigma^{p+1} n^\mu (u_{\omega lm}^{(n)*} \nabla_\mu u_{\omega' l' m'}^{(n')} - u_{\omega' l' m'}^{(n')} \nabla_\mu u_{\omega lm}^{(n)*}) = \delta_{nn'} \delta_{ll'} \delta_{mm'} \delta(\omega - \omega'),$$

$$i \int_{\Sigma_t} d\Sigma^{p+1} n^\mu (u_{\omega lm}^{(n)} \nabla_\mu u_{\omega' l' m'}^{(n')} - u_{\omega' l' m'}^{(n')} \nabla_\mu u_{\omega lm}^{(n)}) = 0, \quad (4)$$

where n^μ is the future-directed unit vector normal to the Cauchy surface Σ_t , e.g., $t = \text{const.}$ We note that modes $n \Rightarrow$ and $\leftarrow$ are orthogonal to each other. This fact can easily be seen by choosing $\Sigma_t = \mathcal{H}^- \cup J^-$ in Eq. (4) and recalling that $\psi_{\omega l}^{(\leftarrow)}(x)$ and $\psi_{\omega l}^{(\leftarrow)}(x)$ vanish on J^- and $\mathcal{H}^-$, respectively.

The modes $u_{\omega lm}^{(n)}$ and their respective complex conjugates form a complete orthonormal basis in the space of solutions of Eq. (2). As a result, we can expand the field operator as

$$\hat{\Phi}(x^\mu) = \sum_{n=\leftarrow}^{\rightarrow} \sum_{l=0}^{\infty} \sum_m \int_0^\infty d\omega [u_{\omega lm}^{(n)} \hat{a}_{\omega lm}^{(n)} + \text{H.c.}], \quad (5)$$

where $\hat{a}_{\omega lm}^{(n)}$ and $\hat{a}_{\omega lm}^{(n)\dagger}$ are annihilation and creation operators, respectively, and satisfy the usual commutation relations $[\hat{a}_{\omega lm}^{(n)}, \hat{a}_{\omega' l' m'}^{(n')\dagger}] = \delta_{nn'} \delta_{ll'} \delta_{mm'} \delta(\omega - \omega')$. The Boulware vacuum $|0\rangle$ is defined by $\hat{a}_{\omega lm}^{(n)} |0\rangle = 0$ for all n, ω, l and m . This is the state of “no particles” as defined by the static observers following integral curves of the vector field $\xi = \partial/\partial t$.

Next, by substituting Eq. (3) in the Klein-Gordon equation and using $\tilde{\nabla}^2 Y_{lm} = -l(l+p-1)Y_{lm}$ (for spherical harmonics on p -spheres see, e.g., Ref. [8]), we obtain

$$\frac{f}{r^p} \frac{d}{dr} \left(r^p f \frac{d\psi_{\omega l}^{(n)}}{dr} \right) + \left[\omega^2 - l(l+p-1) \frac{f}{r^2} \right] \psi_{\omega l}^{(n)} = 0. \quad (6)$$

Now we define $\varphi_{\omega l}^{(n)}(r) \equiv r^{p/2} \psi_{\omega l}^{(n)}(r)$ and $d/dx \equiv f(r) d/dr$ to cast Eq. (6) in the form

$$[d^2/dx^2 + \omega^2 - V_{\text{eff}}(x)] \varphi_{\omega l}^{(n)}(r) = 0, \quad (7)$$

where the scattering potential is

$$V_{\text{eff}}[x(r)] = f \left[\frac{p f'}{2r} + \frac{p}{2} \left(\frac{p}{2} - 1 \right) \frac{f}{r^2} + \frac{l(l+p-1)}{r^2} \right]$$

with $f' \equiv df/dr$ and

$$x(r) = \begin{cases} r + r_H \ln(r/r_H - 1) & \text{for } p = 2 \\ r F \left[\frac{1}{1-p}, 1; \frac{2-p}{1-p}; \left(\frac{r_H}{r} \right)^{p-1} \right] & \text{for } p \geq 3. \end{cases} \quad (8)$$

Close ($x < 0$, $|x| \gg r_H$) to and far away ($x \gg r_H$) from the horizon, we have $V_{\text{eff}}[x(r)] \approx 0$, and we write

$$\varphi_{\omega l}^{(\leftarrow)} \approx \begin{cases} A_{\omega l}^{(\leftarrow)} (e^{i\omega x} + \mathcal{R}_{\omega l}^{(\leftarrow)} e^{-i\omega x}) & (x < 0, |x| \gg r_H) \\ A_{\omega l}^{(\leftarrow)} \mathcal{T}_{\omega l}^{(\leftarrow)} e^{i\omega x} & (x \gg r_H) \end{cases} \quad (9)$$

and

$$\varphi_{\omega l}^{(\leftarrow)} \approx \begin{cases} A_{\omega l}^{(\leftarrow)} \mathcal{T}_{\omega l}^{(\leftarrow)} e^{-i\omega x} & (x < 0, |x| \gg r_H) \\ A_{\omega l}^{(\leftarrow)} (e^{-i\omega x} + \mathcal{R}_{\omega l}^{(\leftarrow)} e^{i\omega x}) & (x \gg r_H). \end{cases} \quad (10)$$

Here $|\mathcal{R}_{\omega l}^{(n)}|^2$ and $|\mathcal{T}_{\omega l}^{(n)}|^2$ are the reflection and transmission coefficients, respectively, satisfying the usual probability conservation equation $|\mathcal{R}_{\omega l}^{(n)}|^2 + |\mathcal{T}_{\omega l}^{(n)}|^2 = 1$. The normalization constants $A_{\omega l}^{(n)}$ can be obtained from the Klein-Gordon inner product (4), which implies

$$\int_{-\infty}^{+\infty} dx \varphi_{\omega l}^{(n)*}(r) \varphi_{\omega' l'}^{(n')}(r) = \delta_{nn'} (\omega + \omega')^{-1} \delta(\omega - \omega').$$

In order to transform the integral into a surface term (see [1] for more details in four dimensions), we use Eq. (7) in addition to $|\mathcal{R}_{\omega l}^{(n)}|^2 + |\mathcal{T}_{\omega l}^{(n)}|^2 = 1$, which leads (up to an arbitrary phase) to $A_{\omega l}^{(n)} = 1/(2\sqrt{\pi\omega})$.

Let us now describe our pointlike scalar source lying at (r_0, θ_{i0}) by

$$j(x^\mu) = (q/\sqrt{|h|}) \delta(r - r_0) \delta^p(\theta_i - \theta_{i0}), \quad (11)$$

where we recall that q is a small constant and $h = \det(g_{ij})$ is the determinant of the spatial metric on Σ_t . Note that $\int_{\Sigma_t} d\Sigma^{p+1} j(x^\mu) = q$ wherever the source lies. The absolute value of the source's four-acceleration $a_0 = |u^\mu \nabla_\mu u^\nu|$ is

$$a_0 = \frac{(p-1)r_H^{p-1}}{2r_0^p \sqrt{1 - (r_H/r_0)^{p-1}}}, \quad (12)$$

where we have used $u^\mu = f^{-1/2}(r_0) \delta_t^\mu$.

Now, let us couple our scalar source $j(x^\mu)$ to the Klein-Gordon field $\hat{\Phi}(x^\mu)$ as described by the interaction action

$$\hat{S}_I = \int dx^{p+2} \sqrt{|g|} j(x^\mu) \hat{\Phi}(x^\mu). \quad (13)$$

The total source response, i.e., total particle emission and absorption probabilities per proper time of the source, is given in a thermal bath by

$$R_S \equiv \sum_{n=\leftarrow}^{\rightarrow} \sum_{l=0}^{\infty} \sum_m \int_0^\infty d\omega R_{\omega lm}^{(n)}, \quad (14)$$

where

$$R_{\omega lm}^{(n)} \equiv \frac{1}{\tau} \{ |\mathcal{A}_{\omega lm}^{(n)\text{em}}|^2 [1 + n^{(n)}(\omega)] + |\mathcal{A}_{\omega lm}^{(n)\text{abs}}|^2 n^{(n)}(\omega) \} \quad (15)$$

and $\tau = 2\pi\sqrt{f(r_0)}\delta(0)$ is the source's total proper time [1]. Here $\mathcal{A}_{\omega lm}^{(n)\text{em}} \equiv \langle n\omega lm | \hat{S}_I | 0 \rangle$ and $\mathcal{A}_{\omega lm}^{(n)\text{abs}} \equiv \langle 0 | \hat{S}_I | n\omega lm \rangle$ are the emission and absorption amplitudes, respectively, of Boulware states $|n\omega lm\rangle$, at the tree level, and

$$n^{(n)}(\omega) \equiv \begin{cases} (e^{\omega\beta} - 1)^{-1} & \text{for } n \rightarrow \\ 0 & \text{for } n \leftarrow \end{cases}, \quad (16)$$

for the Unruh vacuum. We recall that the Unruh vacuum is characterized by a thermal flux leaving $\mathcal{H}^-$ with Hawking temperature β^{-1} at infinity and no thermal flux coming from J^- . Here $\beta^{-1} = \mathcal{K}/(2\pi)$ as is well known [9] with the surface gravity $\mathcal{K} = (p-1)/(2r_H)$. Since structureless static sources (11) can only interact with zero-energy modes, the total response of this source in the Boulware vacuum vanishes (for a more comprehensive discussion on zero-energy modes, see Ref. [10]). This is not so, however, in the presence of a background thermal bath since the absorption and (stimulated) emission rates render it non-zero. As a result, the only contribution in Eq. (15) comes from modes $n \rightarrow$ [see Eq. (16)]. Using the fact that $|\mathcal{A}_{\omega lm}^{(n)\text{abs}}| = |\mathcal{A}_{\omega lm}^{(n)\text{em}}|$, we write Eq. (14) as

$$R_S \equiv \frac{1}{\tau} \sum_{l=0}^{\infty} \sum_m \int_0^{\infty} d\omega |\mathcal{A}_{\omega lm}^{(\rightarrow)\text{em}}|^2 \coth(\omega\beta/2). \quad (17)$$

In order to deal with zero-energy modes, we need a “regulator” to avoid the appearance of intermediate indefinite results [10]. For this purpose we let the coupling constant q smoothly oscillate with frequency ω_0 , writing Eq. (11) in the form [see Ref. [11] for an alternative (but equivalent) regulator]

$$j_{\omega_0}(x^\mu) = (q_{\omega_0}/\sqrt{|h|})\delta(r-r_0)\delta^p(\theta_i - \theta_{i0}), \quad (18)$$

where $q_{\omega_0} \equiv \sqrt{2}q \cos(\omega_0 t)$ and take the limit $\omega_0 \rightarrow 0$ at the end. The factor $\sqrt{2}$ has been introduced to guarantee that the time average $\langle |q_{\omega_0}(t)|^2 \rangle_t$ equals q^2 . By using Eqs. (5), (18), and (13), we obtain

$$|\mathcal{A}_{\omega lm}^{(\rightarrow)\text{em}}|^2 = 2\pi^2 q^2 f(r_0) r_0^{-p} |\varphi_{\omega l}^{(\rightarrow)}|^2 |Y_{lm}|^2 [\delta(\omega - \omega_0)]^2. \quad (19)$$

Now we proceed to find the zero-energy modes with which our static source interacts. For this purpose we let $\omega = 0$ in Eq. (6) and make the change $r \mapsto z \equiv 2(r/r_H)^{p-1} - 1$, obtaining

$$(1-z^2) \frac{d^2 \psi_{0l}^{(n)}}{dz^2} - 2z \frac{d\psi_{0l}^{(n)}}{dz} + \frac{l(l+p-1)}{(p-1)^2} \psi_{0l}^{(n)} = 0, \quad (20)$$

where $1 < z < \infty$. Two linearly independent solutions of

Eq. (20) can be given as

$$P_\nu(z) = F[-\nu, \nu+1; 1; (1-z)/2], \quad (21)$$

and

$$Q_\nu(z) = \frac{\Gamma(\nu+1)\Gamma(1/2)}{(2z)^{\nu+1}\Gamma(\nu+\frac{3}{2})} F\left(\frac{\nu+2}{2}, \frac{\nu+1}{2}; \frac{2\nu+3}{2}; \frac{1}{z^2}\right) \quad (22)$$

with $\nu = l/(p-1)$. From the asymptotic behavior of the Legendre functions, $Q_\nu(z) \approx z^{-\nu-1}$ for $z \rightarrow \infty$ and $P_\nu(z) \approx 1$ for $z \approx 1$, we infer that, for $\omega \approx 0$,

$$\varphi_{\omega l}^{(\rightarrow)} \approx C_{\omega l}^{(\rightarrow)} r^{p/2} Q_{l/(p-1)}(z), \quad (23)$$

$$\varphi_{\omega l}^{(\leftarrow)} \approx C_{\omega l}^{(\leftarrow)} r^{p/2} P_{l/(p-1)}(z) \quad (24)$$

with $C_{\omega l}^{(n)}$ being normalization constants, generalizing a result with $p=2$ in Ref. [12]. Now, by using Eqs. (8.822.2) and (3.513.2) of Ref. [13], and $x \approx [r_H/(p-1)] \ln(r/r_H - 1)$ for $r \approx r_H$, we obtain for $x \rightarrow -\infty$ with $|\omega x| \ll 1$

$$\varphi_{\omega l}^{(\leftarrow)}(z) \approx -(C_{\omega l}^{(\leftarrow)} r^{p/2}/2) \ln(r/r_H - 1). \quad (25)$$

In order to find $C_{\omega l}^{(\leftarrow)}$, we first note from Eq. (9) that close to the horizon and for small enough frequencies ($x \rightarrow -\infty$, $|\omega x| \ll 1$)

$$\varphi_{\omega l}^{(\leftarrow)} \approx (4\pi\omega)^{-1/2} [(1 + \mathcal{R}_{\omega l}^{(\leftarrow)}) + i\omega x(1 - \mathcal{R}_{\omega l}^{(\leftarrow)})]. \quad (26)$$

Now, by comparing Eqs. (25) and (26), we note that $\mathcal{R}_{\omega l}^{(\leftarrow)} \rightarrow -1$ as $\omega \rightarrow 0$ and

$$C_{\omega l}^{(\leftarrow)} = -2i\sqrt{\omega/\pi} r_H^{1-p/2}/(p-1),$$

which allows us to write for $\omega \approx 0$ [see Eq. (23)]

$$\varphi_{\omega l}^{(\leftarrow)}(z) \approx -2i \frac{\sqrt{\omega/\pi}}{p-1} \frac{r^{p/2}}{r_H^{p/2-1}} Q_{l/(p-1)}(z). \quad (27)$$

Next, using Eq. (27) in Eq. (19) and letting $\omega_0 \rightarrow 0$, we can write the response (17) as

$$R_S = \frac{8q^2 f^{1/2}(r_0) r_H^{2-p}}{\beta(p-1)^2} \sum_{l=0}^{\infty} Q_{l/(p-1)}(z_0)^2 \sum_m |Y_{lm}|^2, \quad (28)$$

where $z_0 \equiv 2(r_0/r_H)^{p-1} - 1$. Using now

$$\sum_m |Y_{lm}|^2 = \frac{(2l+p-1)(p+l-2)! \Gamma[(p+1)/2]}{2\pi^{(p+1)/2} (p-1)! l!},$$

we eventually have

$$R_S = \frac{q^2 f^{1/2}(r_0) r_H^{1-p} \Gamma[(p+1)/2]}{(p-1)(p-1)! \pi^{(p+3)/2}} \sum_{l=0}^{\infty} \left[\frac{(2l+p-1)}{l!} \times (p+l-2)! (Q_{l/(p-1)}(z_0))^2 \right]. \quad (29)$$

The expression above will be compared with the total

response of the source when it is uniformly accelerated in $N(=p+2)$ dimensional Minkowski spacetime with the line element $ds^2 = dt^2 - dx^2 - d\mathbf{x}_\perp^2$ (i.e., static in the corresponding Rindler wedge), where $\mathbf{x}_\perp = (x_2, \dots, x_{p+1})$. We assume that our source $j(x^\mu)$ is uniformly accelerated along the x axis with proper acceleration a_0 and is coupled to the scalar field $\hat{\Phi}(x^\mu)$ through the interaction action (13). Here $\hat{\Phi}(x^\mu) = \int dk \int d\mathbf{k}_\perp [u_{k\mathbf{k}_\perp} \hat{a}_{k\mathbf{k}_\perp} + \text{H.c.}]$, where $u_{k\mathbf{k}_\perp} = [2\omega(2\pi)^{p+1}]^{-1/2} e^{-ik_\mu x^\mu}$, $k^\mu = (\omega, k, \mathbf{k}_\perp)$, $\omega = \sqrt{k^2 + \mathbf{k}_\perp^2}$ and $[\hat{a}_{k\mathbf{k}_\perp}, \hat{a}_{k'\mathbf{k}'_\perp}^\dagger] = \delta(k - k') \delta^p(\mathbf{k}_\perp - \mathbf{k}'_\perp)$. The total response in the Minkowski vacuum is $R_M \equiv \tau^{-1} \int dk \int d\mathbf{k}_\perp |\mathcal{A}_{k\mathbf{k}_\perp}^{\text{em}}|^2$, where τ is the source's total proper time and $\mathcal{A}_{k\mathbf{k}_\perp}^{\text{em}} \equiv \langle k\mathbf{k}_\perp | \hat{S}_I | 0 \rangle$. After integrating over the momentum k , we find for $N \geq 3$

$$R_M = \frac{2q^2}{(2\pi)^{p+1} a_0} \int d\mathbf{k}_\perp [K_0(k_\perp/a_0)]^2, \quad (30)$$

where $k_\perp = |\mathbf{k}_\perp|$. Now, using Eq. (6.576.4) of Ref. [13] and $\int d\mathbf{k}_\perp = \int_0^\infty dk_\perp k_\perp^{p-1} \int d\Omega_{p-1}$ for $p \geq 2$, where $d\Omega_{p-1}$ is the volume element of the unit $p-1$ sphere, we perform the integration in Eq. (30) (for $p=1$ the integration is trivial):

$$R_M = \frac{q^2 a_0^{p-1} [\Gamma(p/2)]^4 \Omega_{p-1}}{8\pi^{p+1} (p-1)!}, \quad (31)$$

where $\Omega_m = 2\pi^{(m+1)/2} / \Gamma[(m+1)/2]$ for $m \geq 1$, and $\Omega_m = 1$ for $m = 0$. (See Ref. [14] for related calculations.)

For $N=4$ the responses (29) and (31) can be shown analytically to be identical [and to satisfy Eq. (1)], by using the equation $\sum_{l=0}^\infty (2l+1)[Q_l(z)]^2 = 1/(z^2-1)$. For $N \geq 5$, we were only able to compare numerically the responses (29) and (31) (see Fig. 1). We first note that $R_S/R_M \approx 1$ for

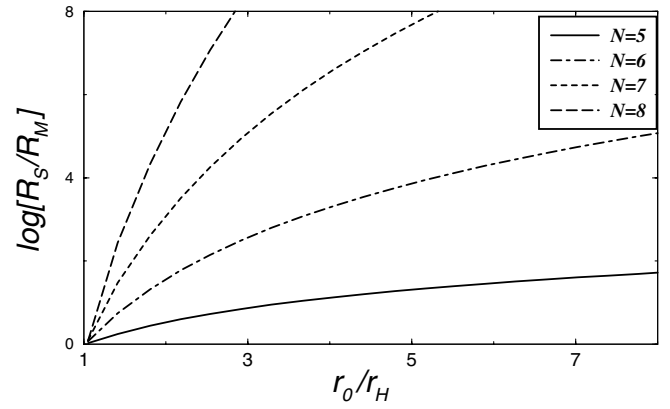


FIG. 1. Plot of the logarithm of the ratio R_S/R_M as a function of the source's radial coordinate r_0 (normalized by the Schwarzschild radius r_H) for $N=5, 6, 7, 8$. The source is assumed to have the same proper acceleration a_0 in Minkowski as in Schwarzschild spacetime.

$r_0 \approx r_H$ for every dimension $N \geq 4$. This is expected (see Ref. [1]) and can be seen as a consistency check for our results. It is also clear from the graph that the full equality $R_S = R_M$ found in [1] is not valid for $N \geq 5$. This is the main result of the paper. It may be that Eq. (1) turns out to be a coincidence rather than a result of a deep principle yet to be discovered. However, it is worthwhile to note that this remarkable relation appears precisely in spacetimes with the number of (macroscopic) dimensions of our physical world.

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