Exact general solutions for the mode shapes of longitudinally vibrating non-uniform rods via Lie symmetries

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Abstract

A Lie symmetry method-based approach is proposed for systematically computing general solutions in closed-form for the mode shape equation of non-uniform and unconventional vibrating rods. The mode shape equation is modeled by the elementary rod theory, addressing polynomial, exponential, trigonometric, and hyperbolic cross-section variations. The method provides algorithmic order-reduction steps for solving the investigated mode shape equation, producing a first-order Riccati equation whose integration reveals the aimed solutions for the problem. Illustrative examples are presented, including original solutions in closed-form as well as solutions previously obtained in the literature by other approaches. Mode shapes from general solutions with appropriate rod boundary conditions are also considered for different examples.

Keywords: General solutions, Non-uniform rods, Mode shapes, Elementary rod theory, Lie symmetries

1. Introduction

The study of structural elements undergoing longitudinal vibration is crucial for several scientific and industrial applications' design, safety, and efficiency

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[1, 2, 3]. Rod models comprise the longitudinal motion of thin elastic structures under many loads, geometry configurations, and frequency ranges, whose response predictions usually derive from their modal properties, i.e., mode shapes, modal frequencies, and modal damping [4, 5, 6, 7]. Few exact solutions are available in the literature for the mode shapes of non-uniform rods once the complexity of the associated differential equations (DEs) makes it difficult for their computation. When existing, these solutions occur from apparently unconnected ad hoc methods for solving DEs. This paper presents the computation of general solutions via the Lie symmetry method for the mode shape equation of longitudinally vibrating non-uniform rods, modeled according to the elementary rod theory. From that, a systematic and elegant way of deriving general solutions arises for non-uniform configurations, recalling previous literature cases and addressing new expressions, from where it assesses mode shapes together with rod boundary conditions.

Exact mode shape solutions for some non-uniform rods have been documented in the literature. In what follows, all displayed variables but x, the position along the rod, refer to constant cross-section area parameters. Eisenberger computed solutions for non-uniform rods with the polynomial cross-section area variation $\sum_{i=0}^{n} A_i\, x^i$ via infinite power series [8]. Raman found families of rod shapes for which the one-dimensional Schroedinger and Sturm-Liouville equations provide solutions for conical, catenoid, exponential, and trigonometric cross-sections of non-uniform rods [9]. Kumar and Sujith generalized Abrate's study for quadratic cross-sections [10] by considering polynomial area $(ax+b)^n$ and sinusoidal $A_0 \sin^2(ax+b)$ variations [11]. Li investigated similar polynomials and exponential functions $a\, \mathrm{e}^{bx}$ [12], while Raj expanded solutions for the exponential shapes $k\, x^n\, \mathrm{e}^{bx^i}$ $(i = 1, 2)$ and $k\, \mathrm{e}^{bx}\, \mathrm{e}^{n\mathrm{e}^{mx}}$ [13]. Calio's and Elishakoff's study comprises the cross-section shapes $a+b\sin(cx)$ and $a+b\cos(cx)$ [14]. Yardimoglu and Aydin showed exact mode shapes for $A_0 \sin^n(ax+b)$ and $A_0 \cos^n(ax+b)$ cross-sections [15, 16], then Guo and Yang extended solutions for the cases $A_0 \cosh^2(cx+s)/\cosh^2(s)$ and $A_0 \cos^2(cx+s)/\cos^2(s)$ [17]. Additionally, spectral and general wave approaches may be convenient for rod configurations of higher

complexity, e.g., phononic systems and metamaterial applications [18, 19].

Computational advances brought helpful methods for investigating the behavior of elastic structures, addressing the computation of either exact or numerical solutions. Against this backdrop, the latter is almost a routine task without considering the former [20, 21]. Despite the practicality of numerical methods, remarkable benefits arise from exact solution methods, such as avoiding convergence studies and concerns about cumulative errors, reducing computational time, and providing insights into the nature of the problem through relations between the well-known component functions of the solution, either expressed in closed form or via infinite series [22, 23, 24]. However, problem-solving aspects of exact solutions should not overlay numerical ones and vice versa. One intrinsic limitation of such methods is the usual requirement of cumbersome symbolic operations. Symmetry methods are intrinsically algorithmic and likely handled by symbolic manipulation, and computer algebra software [25, 26, 27]. Within the computational advance framework, the interest in symmetry methods has been growing over the last decades.

The Lie symmetry method unifies and extends many integration techniques for DEs through a systematic algorithm based on the existence of Lie symmetry groups [28, 29]. A Lie symmetry group for given DEs is a local group of local transformations that maps their solutions into other solutions. The knowledge of a Lie symmetry group leads to changes in the equation's coordinates, retaining solutions invariant. This invariance may produce analytical order reductions, linearizations, and domain compressions for directly constructing exact solutions. For the case of ordinary differential equations (ODEs), a one-parameter Lie symmetry group provides a single reduction of order, leading to a quadrature integration in the particular case of first-order ODEs [30, 31, 32]. From the literature, Lie's approach provided exact solutions for the axially accelerating string problem, reducing the associated DE to a more convenient form [33]. At the same time, group theory analysis assisted eigenvalue computations for vibrating symmetric and polygonal structures efficiently [34, 35, 36].

This paper is organized as follows. Section 2 derives an equivalent equation

to the mode shape equation for rods with only varying cross-section areas, encompassing material and geometrical non-uniformities. Section 3 introduces the Lie symmetry method for ODEs. Section 4 provides the equivalent mode shape equation with an order reduction via the Lie symmetry method and establishes a systematic procedure for computing its general solutions in closed form. This section also considers two examples for deriving mode shapes from the general solutions, illustrating the interchangeability of solutions between completely non-uniform and simply varying cross-section area rods. General exact solutions for many different cross-section variations are also presented. Finally, Section 5 brings the concluding remarks.

2. Mode shape equation of non-uniform rods

The elastodynamic equation from the elementary model for rods with non-uniform cross-sections undergoing free longitudinal vibrating follows as [37]:

$$\frac{\partial}{\partial x}\Big(E\,S\,\frac{\partial}{\partial x}\tilde{u}(x,t)\Big) - \rho\,S\,\frac{\partial^2}{\partial t^2}\tilde{u}(x,t) = 0, \quad x \in \,]\,0, l\,[\,, \;\; t > 0, \tag{1}$$

where $\tilde{u}(x,t)$ is the rod's time-dependent longitudinal displacement, x represents the position along the structure, and t denotes time, whereas $E = E(x)$ is the Young's modulus, $S = S(x)$ is the cross-section area function and $\rho = \rho(x)$ is the density of the rod of length l.

By introducing the new variable [17, 38]:

$$y = \int_0^x \left(\frac{\rho(\tau)/\rho_0}{E(\tau)/E_0}\right)^{\frac{1}{2}} d\tau, \tag{2}$$

the combined use of the chain rule and the inverse function theorem leads to:

$$\frac{\partial}{\partial y}\Big(E_0\,\overline{S}\,\frac{\partial}{\partial y}\tilde{u}(y,t)\Big) - \rho_0\,\overline{S}\,\frac{\partial^2}{\partial t^2}\tilde{u}(y,t) = 0, \tag{3}$$

for E_0 and ρ_0 being the Young's modulus and the density of the rod at its cross-section labeled as the origin, respectively, and $\overline{S} = [E\,\rho/(E_0\,\rho_0)]^{\frac{1}{2}} S$ when $\overline{S} = \overline{S}(y)$.

Let $\tilde{u}(y,t) = u(y)\,\mathrm{e}^{\mathrm{j}\omega t}$ be a non-trivial harmonic solution, where $u = u(y)$ and ω represent the general solution for the mode shapes and the natural angular

frequencies of the rod, respectively, and j be the imaginary unit. From this, Eq. (3) can be written equivalently to the mode shape equation as follows:

$$\frac{1}{\overline{S}}\frac{\mathrm{d}}{\mathrm{d}y}\left(\overline{S}\,\frac{\mathrm{d}u}{\mathrm{d}y}\right)+\lambda_0^2\,u=0, \tag{4}$$

where $\lambda_0=(\rho_0/E_0)^{\frac{1}{2}}\,\omega$ is the *quasi*-longitudinal wavenumber at the origin [39].

3. Lie symmetries

This section introduces the Lie symmetry method for ODEs. The reader is referred to [28, 30, 40, 41, 42, 43] for a comprehensive description of symmetry methods.

3.1. Lie symmetries of a given ODE

Consider a Nth-order ODE written in the form:

$$\mathcal{F}\left(y,u,u_1,...,u_N\right)=0, \tag{5}$$

where u and y denote the dependent and independent variables, respectively, and $u_k=\dfrac{d^k u}{dy^k}$ from the jet notation, for $k=1,\ldots,N$. Let $\mathcal{X}$ represent a smooth vector field defined on an open set of the space with coordinates (y,u) as the infinitesimal generator [30]:

$$\mathcal{X}=\xi\frac{\partial}{\partial y}+\eta\frac{\partial}{\partial u}, \tag{6}$$

where the functions $\xi=\xi(y,u)$ and $\eta=\eta(y,u)$ are the so-called infinitesimals of $\mathcal{X}$. Such vector field can be prolonged to the space with the coordinates $(y,u,u_1,\ldots,u_N)$ in which Eq. (5) is defined. Their Nth-order prolongation is given by [31, 40]:

$$\mathcal{X}^{(N)}=\mathcal{X}+\eta^{(1)}\frac{\partial}{\partial u_1}+\cdots+\eta^{(N)}\frac{\partial}{\partial u_N}, \tag{7}$$

where the prolonged infinitesimals are, for $\eta^{(0)}=\eta$, equal to:

$$\eta^{(i)}=\mathcal{D}(\eta^{(i-1)})-u_i\mathcal{D}(\xi),\quad i=1,\cdots,N, \tag{8}$$

and $\mathcal{D}$ is the total derivative operator:

$$\mathcal{D} = \frac{\partial}{\partial y} + u_1 \frac{\partial}{\partial u} + u_2 \frac{\partial}{\partial u_1} + \cdots \tag{9}$$

In general terms, Lie point symmetries of Eq. (5) are vector fields of the form of Eq. (6) whose corresponding flows map solutions of the equation into other solutions. This condition can be given in terms of the following infinitesimal criterion of invariance [30]:

$$\mathcal{X}^{(N)} \mathcal{F}\big(y, u, u_1, ..., u_N\big) = 0 \quad \text{when} \quad \mathcal{F}\big(y, u, u_1, ..., u_N\big) = 0, \tag{10}$$

which is split into a system of linear partial differential equations for the infinitesimals functions ξ and η by equating the derivative's coefficients to zero with $u_N{=}f(y, u, u_1, \ldots, u_{N-1})$ from Eq. (5). This system is known as the system of determining equations. Symbolic manipulation packages from computer algebra can assist the solution of such equations and, hence, the Lie symmetries computation of the analyzed ODE [26, 44, 45, 46].

3.2. Order reduction by one Lie point symmetry

Each Lie symmetry of a given ODE provides new functionally independent coordinates, which can be used to reduce the order of the equation by one. These new coordinates can be computed as follows:

$$\mathcal{X}\, r = 0, \qquad \mathcal{X}^{(1)}\, v = 0. \tag{11}$$

for $r = r(y, u)$ and $v = v(y, u, u_1)$. Solutions of Eq. (11) arise as the constants from the integration of the characteristic equation:

$$\frac{\mathrm{d}y}{\mathrm{d}\xi} = \frac{\mathrm{d}u}{\mathrm{d}\eta} = \frac{\mathrm{d}u_1}{\mathrm{d}\eta^{(1)}}. \tag{12}$$

Computing by derivation:

$$v_k = \frac{\mathcal{D}\, v_{k-1}}{\mathcal{D}\, r}, \quad k = 1, \cdots, N-1, \tag{13}$$

thus, $\{r, v, v_1, \ldots, v_{N-1}\}$ is a complete set of differential invariants of the vector field $\mathcal{X}^{(N)}$ [47, 48]. The infinitesimal criterion in Eq. (10) assures that Eq. (5)

can be written in terms of these new coordinates, providing a reduced equation of the form:

$$\mathcal{G}\left(r, v, v_1, ..., v_{N-1}\right) = 0. \tag{14}$$

Assuming that $v = G(r; \mathcal{C}_1, \ldots, \mathcal{C}_{N-1})$ is the general solution of Eq. (14), then, the general solution to Eq. (5) can be reconstructed through the auxiliary first-order equation:

$$v(y, u, u_1) = G(r(y, u); \mathcal{C}_1, \ldots, \mathcal{C}_{N-1}), \tag{15}$$

Whose integration recovers u through a systematic procedure since it inherits $\mathcal{X}$ from the Lie symmetry transformation.

4. General solutions for the equivalent mode shape equation of non-uniform rods via Lie symmetries

Let Eq. (7) be the prolonged Lie point symmetry of Eq. (4), written according to the jet notation. The infinitesimal criterion of invariance from Eq. (10) follows as:

$$\mathcal{X}^{(2)}\Big(u_2 + \frac{1}{\overline{S}}\frac{\mathrm{d}\overline{S}}{\mathrm{d}y}u_1 + \lambda_0^2 u\Big) = 0, \quad \text{when} \quad u_2 + \frac{1}{\overline{S}}\frac{\mathrm{d}\overline{S}}{\mathrm{d}y}u_1 + \lambda_0^2 u = 0, \tag{16}$$

whose evaluation yields the system of determining equations:

$$\begin{gathered}
\frac{\partial^2 \xi}{\partial u^2} = 0, \\
\frac{\partial^2 \eta}{\partial u^2} - 2\,\frac{\partial^2 \xi}{\partial y \partial u} + 2\,\frac{1}{\overline{S}}\frac{\mathrm{d}\overline{S}}{\mathrm{d}y}\frac{\partial \xi}{\partial u} = 0, \\
\frac{\partial^2 \xi}{\partial y^2} - 2\,\frac{\partial^2 \eta}{\partial y \partial u} - \frac{1}{\overline{S}}\frac{\mathrm{d}\overline{S}}{\mathrm{d}y}\frac{\partial \xi}{\partial y} - 3\,{\lambda_0}^2 u\,\frac{\partial \xi}{\partial u} - \left[\frac{\mathrm{d}^2\overline{S}}{\mathrm{d}y^2} - \frac{1}{\overline{S}}\left(\frac{\mathrm{d}\overline{S}}{\mathrm{d}y}\right)^2\right]\frac{\xi}{\overline{S}} = 0, \\
\frac{\partial^2 \eta}{\partial y^2} + {\lambda_0}^2 u\left(2\,\frac{\partial \xi}{\partial y} - \frac{\partial \eta}{\partial u}\right) + \frac{1}{\overline{S}}\frac{\mathrm{d}\overline{S}}{\mathrm{d}y}\frac{\partial \eta}{\partial y} + {\lambda_0}^2 \eta = 0.
\end{gathered} \tag{17}$$

The pair $\xi = 0$ and $\eta = u$ consists in an admitted solution for the system of Eq. (17), associated with the vector field $\mathcal{X}_1 = u\,\partial/\partial u$. Within such Lie point

transformation, the invariance relations from Eq. (11) produce the differential invariants:

$$r = y, \quad v = \frac{u_1}{u}, \tag{18}$$

then, the first-order differential invariant simply follows from Eq. (13) as:

$$v_1 = \frac{u_2}{u} - \Big(\frac{u_1}{u}\Big)^2. \tag{19}$$

The new space of the coordinates (r, v, v_1) provides Eq. (4) with a change of variables in such a way that this ODE is transformed into the Riccati equation:

$$\frac{1}{\overline{S}}\frac{\mathrm{d}}{\mathrm{d}r}\big(\overline{S}\,v\big) + v^2 + \lambda_0^2 = 0, \tag{20}$$

for $v = v(r)$ and $\overline{S} = \overline{S}(r)$.

The solution $v = v(r; \mathcal{C}_1)$ of Eq. (20) for specific cross-section variations allows to recover general solutions for Eq. (4) in closed form through the auxiliary first-order Eq. (15) resolution:

$$u = \mathcal{C}_2\, \mathrm{e}^{\int v(y;\mathcal{C}_1)\,dy}, \tag{21}$$

where $\mathcal{C}_1$ and $\mathcal{C}_2$ are integration constants.

4.1. Example 1: mode shapes of a clamped-clamped rod with the cross-section area $S = a_1\, a_2^{a_3\, x}$

Consider a rod with the homogeneous material distribution $E = E_0$ and $\rho = \rho_0$ and the cross-section area variation $S = a_1\, a_2^{a_3\, x}$ for $a_1, a_2, a_3 \in \mathbb{R}$ and $a_2 > 0$. In this case, Eq. (20) assumes the form:

$$v_1 + v^2 + \ln(a_2)\, a_3\, v + \lambda_0^2 = 0, \tag{22}$$

for $y = x$ and $\overline{S} = S$, whose resolution reduces to the computation of the primitive

$$-\int \frac{\mathrm{d}v}{v^2 + \ln(a_2)\, a_3\, v + \lambda_0^2},$$

since it is a separable equation [49]. The following three cases are distinguished. If $4\,\lambda_0^2 > \ln^2(a_2)\,a_3^2$:

$$v(r) = -\frac{\sqrt{4\,\lambda_0^2-\ln^2(a_2)\,a_3^2}}{2}\,\tan\Big(\frac{\sqrt{4\,\lambda_0^2-\ln^2(a_2)\,a_3^2}}{2}\,(\mathcal{C}_1+r)\Big) - \frac{\ln(a_2)\,a_3}{2}, \tag{23}$$

if $4\,\lambda_0^2 < \ln^2(a_2)\,a_3^2$, then:

$$v(r) = \frac{\sqrt{\ln^2(a_2)\,a_3^2-4\,\lambda_0^2}}{2}\,\tanh\Big(\frac{\sqrt{\ln^2(a_2)\,a_3^2-4\,\lambda_0^2}}{2}\,(\mathcal{C}_1+r)\Big) - \frac{\ln(a_2)\,a_3}{2}, \tag{24}$$

and for $4\,\lambda_0^2 = \ln^2(a_2)\,a_3^2$:

$$v(r) = \frac{1}{(\mathcal{C}_1+r)} - \frac{\ln(a_2)\,a_3}{2}. \tag{25}$$

According to Eq. (21), the general solution for each case follows as:

$$u(x) = \mathcal{C}_2\,a_2^{-\frac{a_3}{2}(\mathcal{C}_1+x)}\cos\Big(\frac{\sqrt{4\,\lambda_0^2-\ln^2(a_2)\,a_3^2}}{2}\,(\mathcal{C}_1+x)\Big), \quad 4\,\lambda_0^2 > \ln^2(a_2)\,a_3^2, \tag{26}$$

$$u(x) = \mathcal{C}_2\,a_2^{-\frac{a_3}{2}(\mathcal{C}_1+x)}\cosh\Big(\frac{\sqrt{\ln^2(a_2)\,a_3^2-4\,\lambda_0^2}}{2}\,(\mathcal{C}_1+x)\Big), \quad 4\,\lambda_0^2 < \ln^2(a_2)\,a_3^2, \tag{27}$$

and

$$u(x) = \mathcal{C}_2\,a_2^{-\frac{a_3}{2}\,x}\,(\mathcal{C}_1+x), \quad 4\,\lambda_0^2 = \ln^2(a_2)\,a_3^2. \tag{28}$$

Determining mode shapes from the presented general solutions involves knowing the boundary condition of the rod. For example, take a clamped-clamped rod whose boundary conditions are $u(0) = u(l) = 0$ and the cross-section area variation $S = a_1\,\mathrm{e}^{a_3\,x}$ as in Fig. 1.

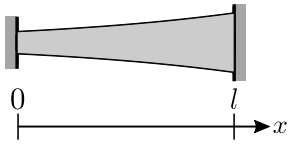


Figure 1: Clamped-clamped rod for $S = a_1\,\mathrm{e}^{a_3\,x}$

This problem was considered in [17], where the authors found the following general solution for the particular case of $a_2 = \mathrm{e}$:

$$u(x) = \mathrm{e}^{-\frac{a_3}{2}\,x}\left[\mathcal{B}_1\sin\Big(\sqrt{\lambda_0^2-b}\,x\Big) + \mathcal{B}_2\cos\Big(\sqrt{\lambda_0^2-b}\,x\Big)\right], \tag{29}$$

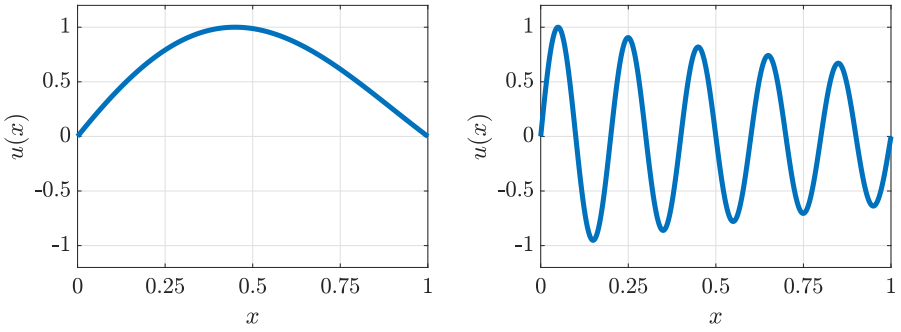
for $b = a_3^2/4$ and where the integration constants for the boundary conditions are: $\mathcal{B}_1$ arbitrary, and $\mathcal{B}_2 = 0$.

For this example, the assumption of $4\,\lambda_0^2 > a_3^2$ leads to the general solution of Eq. (26) with $a_2 = \mathrm{e}$. The integration constant from the left boundary condition of the rod yields $\mathcal{C}_1 = \pi/\sqrt{4\,\lambda_0^2 - a_3^2}$. The remaining boundary condition provides $\mathcal{C}_2$ arbitrary and different from zero for not having a trivial solution, while the remaining terms from such condition must be equal to zero. Then, the characteristic frequency equation arises:

$$\sin\left(\frac{\sqrt{4\,\lambda_0^2 - a_3^2}}{2}\,l\right) = 0. \tag{30}$$

If we consider the geometric parameters $a_1 = 0.05$ m, $a_3 = 1$ m^{-1}, and $l = 1$ m for a slender rod, then equation (30) has the roots $\lambda_{0(1)} = 3.18$ m^{-1} and $\lambda_{0(10)} = 31.42$ m^{-1} for the first and tenth wavenumbers, respectively.

Fig. 2 displays the corresponding normalized mode shapes, whose asymmetry occurs from the decaying transmissibility due to the exponential cross-section area enlargement along the horizontal axis. After symbolic manipulations, It can be found that Eqs. (26) and (29) are equivalent.



(a) 1st mode shape

(b) 10th mode shape

Figure 2: Mode shapes of the clamped-clamped rod of example 1

4.2. Example 2: mode shapes of a clamped-free rod with the cross-section variation $\overline{S} = a_1 \cosh(a_2 + a_3\, y)$

For a more general example, let $\overline{S} = a_1 \cosh(a_2 + a_3\, y)$ be the cross-section variation shape for the clamped-free rod of Fig. 3 with the boundary conditions $u(0) = u_1(l) = 0$ for $l = 1$ m. Let us consider $\rho = 2500\,[1 + x/(4l)]^2$ kg/m^3, $E = 70$ GPa, and $S = 0.05\,[1+x/(4l)]^{-1} \cosh\big[-1.5+3\,x\,[1+x/(8l)]\big]$ m^2, which lead to the values $a_1 = 0.05$ m, $a_2 = -1.5$ m, and $a_3 = 3$.

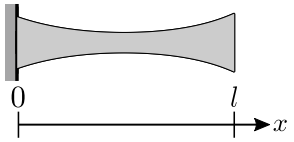


Figure 3: Clamped-free rod for $S = 0.05\,[1 + x/(4l)]^{-1} \cosh\big[-1.5 + 3\,x\,[1 + x/(8l)]\big]$

From this, Eq. (14) follows as:

$$v_1 + v^2 + a_3 \tanh(a_2 + a_3\, r)\, v + \lambda_0^2 = 0, \tag{31}$$

whose solution in closed form, according to the auxiliary first-order equation of Eq. (15) and $y = x\,(1 + x/8)$ from Eq. (2), turns into the general solution:

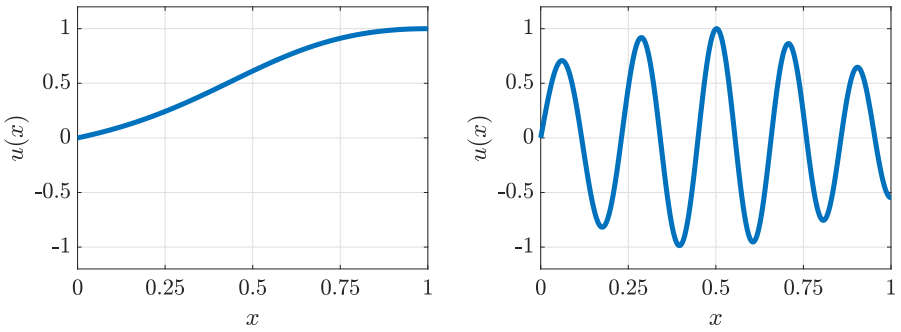
$$u(x) = \mathcal{C}_1 \sinh\big[a_2 + a_3\, x\big(1 + \tfrac{x}{8}\big)\big]\, {}_2\mathrm{F}_1\Big[\tfrac{3}{4} - \tfrac{\mu_5}{2},\, \tfrac{3}{4} + \tfrac{\mu_5}{2},\, \tfrac{3}{2},\, -\sinh^2\big[a_2 + a_3\, x\big(1 + \tfrac{x}{8}\big)\big]\Big] + \mathcal{C}_2\, {}_2\mathrm{F}_1\Big[\tfrac{1}{4} + \tfrac{\mu_5}{2},\, \tfrac{1}{4} - \tfrac{\mu_5}{2},\, \tfrac{1}{2},\, -\sinh^2\big[a_2 + a_3\, x\big(1 + \tfrac{x}{8}\big)\big]\Big], \tag{32}$$

where $\mu_5 = \frac{\sqrt{n^2 a_3^2 - 4\,\lambda_0^2}}{2\,a_3}$ for $n = 1$ and ${}_2\mathrm{F}_1$ is the ordinary hypergeometric function [50].

Fig. 4 shows the first and tenth normalized mode shapes associated with the wavenumbers $\lambda_{0(1)} = 1.20$ m^{-1} and $\lambda_{0(10)} = 26.55$ m^{-1}, whose computation, similarly to the first example, follows from the solution of the characteristic frequency equation when taking the appropriate boundary conditions.

4.3. General solutions for the mode shape equation of non-uniform rods

Table 1 presents general solutions for the mode shape equation of non-uniform rods computed in accordance with the prior systematical procedure



(a) 1st mode shape

(b) 10th mode shape

Figure 4: Mode shapes of the clamped-free rod of example 2

via Lie symmetries. Several cross-section variations are addressed, indicating referred cases from the literature and including new general solutions. The notation of the special functions follows as: J_σ and Y_σ are the first and second kind Bessel functions of order σ, respectively, HeunC and HeunG are the Heun confluent and general functions and $'$ represents their first derivatives, ${}_2\mathrm{F}_1$ is the ordinary hypergeometric function, P_σ^γ and Q_σ^γ are the associated first and second kind Legendre functions of degree σ and order γ, where $\gamma = 0$ for non-associated Legendre functions [50, 51]. In addition, the constants ν_{i_1} for $i_1 = 1, 2, \ldots, 6$ and μ_{i_2} for $i_2 = 1, 2, \ldots, 5$ and the functions $H_{i_3}(y)$ for $i_3 = 1, 2, \ldots, 12$ are defined below the table.

Table 1: General solutions for rods with non-uniform cross-sections

Cross-section variation $\bar{S}(y)$	General solution $u(y)$	Literature solution
$(a_0+a_1 y)^n$	$(a_0+a_1 y)^{-\nu_1}\left\{\mathcal{C}_1 \mathrm{J}_{\nu_1}\left[\lambda_0\left(\frac{a_0}{a_1}+y\right)\right]+\mathcal{C}_2 \mathrm{Y}_{\nu_1}\left[\lambda_0\left(\frac{a_0}{a_1}+y\right)\right]\right\}$, for $\nu_1 \in \mathbb{Z}$ $(a_0+a_1 y)^{-\nu_1}\left\{\mathcal{C}_1 \mathrm{J}_{\nu_1}\left[\lambda_0\left(\frac{a_0}{a_1}+y\right)\right]+\mathcal{C}_2 \mathrm{J}_{-\nu_1}\left[\lambda_0\left(\frac{a_0}{a_1}+y\right)\right]\right\}$, for $\nu_1 \notin \mathbb{Z}$	[11, 12]
$(a_0+a_1 y+a_2 y^2)^n$	$[\mathcal{C}_1 H_1(y)+\mathcal{C}_2]\, \mathrm{e}^{j\lambda_0 y}\mathrm{HeunC}\left(4\mu_1\nu_6, 0, 0, 0, 0, \frac{a_1+2a_2 y}{2\mu_1}+\frac{1}{2}\right)$ and Eq. (33), for $n=1$ $\mathcal{C}_2\, \mathrm{e}^{-\frac{H_2(y)}{2a_2}}$ and Eq. (34)	—
$a_1 y+a_3 y^3$	$\mathcal{C}_2\, \mathrm{e}^{H_3(y)}$ and Eq. (35)	—
$a_2 y^2+a_3 y^3$	$\mathcal{C}_2\, \mathrm{e}^{H_4(y)}$ and Eq. (36)	—
$a_2 y^2+a_4 y^4$	$\frac{\mathcal{C}_1}{y}\mathrm{HeunC}\left(0, -\frac{1}{2}, 0, -\lambda_0^2\frac{a_2}{4a_4}, \lambda_0^2\frac{a_2}{4a_4}-\frac{1}{4}, -\frac{a_4}{a_2}y^2\right)+\mathcal{C}_2\mathrm{HeunC}\left(0, \frac{1}{2}, 0, -\lambda_0^2\frac{a_2}{4a_4}, \lambda_0^2\frac{a_2}{4a_4}-\frac{1}{4}, -\frac{a_4}{a_2}y^2\right)$	—
$a_3 y^3+a_4 y^4$	$\mathcal{C}_2\, \mathrm{e}^{H_5(y)}$ and Eq. (37)	—
$a_0+a_1\, a_2^{a_3 y}$	$\mathcal{C}_2\, a_2^{-\frac{a_3}{2}(\mathcal{C}_1+y)}\cos\left[\frac{\sqrt{4\lambda_0^2-\ln^2(a_2)a_3^2}}{2}(\mathcal{C}_1+y)\right]$, for $a_0=0$ and $4\lambda_0^2>\ln^2(a_2)a_3^2$ [17] $\mathcal{C}_2\, a_2^{-\frac{a_3}{2}(\mathcal{C}_1+y)}\cosh\left[\frac{\sqrt{\ln^2(a_2)a_3^2-4\lambda_0^2}}{2}(\mathcal{C}_1+y)\right]$, for $a_0=0$ and $4\lambda_0^2<\ln^2(a_2)a_3^2$ $\mathcal{C}_2\, a_2^{-\frac{a_3}{2}y}(\mathcal{C}_1+y)$, for $a_0=0$ and $4\lambda_0^2=\ln^2(a_2)a_3^2$ —
$a_0+a_1 \sin^n(a_2+a_3 y)$	$\mathcal{C}_2\, \mathrm{e}^{\lambda_0 H_6(y)}$ and Eq. (38) $\mathcal{C}_2\, \mathrm{e}^{\frac{a_3}{2}H_7(y)}$ and Eq. (39), for $n=1$ [14] $\sin^{-\nu_1}(a_2+a_3 y)\left\{\mathcal{C}_1 \mathrm{Q}^{\nu_1}_{\nu_3-\frac{1}{2}}[\cos(a_2+a_3 y)]+\mathcal{C}_2\mathrm{P}^{\nu_1}_{\nu_3-\frac{1}{2}}[\cos(a_2+a_3 y)]\right\}$, for $a_0=0$ [15]
$a_1 \tan^n(a_2 y)$	$\mathcal{C}_1 \sin(2a_2 y)\tan^{-n}(a_2 y)\, {}_2\mathrm{F}_1\left[1-\frac{\lambda_0}{2a_2}, 1+\frac{\lambda_0}{2a_2}, 1-\nu_1, \sin^2(a_2 y)\right]+\mathcal{C}_2\cos^{n+1}(a_2 y)\, {}_2\mathrm{F}_1\left[1-\frac{\lambda_0}{2a_2}+\nu_1, 1+\frac{\lambda_0}{2a_2}+\nu_1, 1+\nu_1, \sin^2(a_2 y)\right]$	—
$a_0+a_1 \sinh^n(a_2+a_3 y)$	$\mathcal{C}_1\mathrm{Q}_{\mu_4-\frac{1}{2}}[\cosh(a_2+a_3 y)]+\mathcal{C}_2\mathrm{P}_{\mu_4-\frac{1}{2}}[\cosh(a_2+a_3 y)]$, for $a_0=0$ and $n=1$ $\mathcal{C}_1\frac{\cosh(\sqrt{a_3^2-\lambda_0^2}\, y)}{\sinh(a_2+a_3 y)}+\mathcal{C}_2\frac{\sinh(\sqrt{a_3^2-\lambda_0^2}\, y)}{\sinh(a_2+a_3 y)}$, for $a_0=0$, $n=2$ and $\lambda_0^2\leq a_3^2$ $\mathcal{C}_1\frac{\cos(\sqrt{\lambda_0^2-a_3^2}\, y)}{\sinh(a_2+a_3 y)}+\mathcal{C}_2\frac{\sin(\sqrt{\lambda_0^2-a_3^2}\, y)}{\sinh(a_2+a_3 y)}$, for $a_0=0$, $n=2$ and $\lambda_0^2>a_3^2$ — $\mathcal{C}_2\, \mathrm{e}^{-\frac{H_8(y)}{4a_3}}$, and Eq. (40), for $a_0=0$ $\mathcal{C}_2\, \mathrm{e}^{\frac{a_3}{4}H_9(y)}$ and Eq. (41), for $n=1$

Table 1: General solutions for rods with non-uniform cross-sections (continued)

Cross-section variation $\bar{S}(y)$	General solution $u(y)$	Literature solution
$a_0+a_1\cosh^n(a_2+a_3y)$	$\mathcal{C}_1\sinh(a_2+a_3y)\,{}_2\mathrm{F}_1\left[\frac{3}{4}-\frac{\mu_5}{2},\frac{3}{4}+\frac{\mu_5}{2},\frac{3}{2},-\sinh^2(a_2+a_3y)\right]+\mathcal{C}_2\,{}_2\mathrm{F}_1\left[\frac{1}{4}+\frac{\mu_5}{2},\frac{1}{4}-\frac{\mu_5}{2},\frac{1}{2},-\sinh^2(a_2+a_3y)\right]$, for $a_0{=}0$ and $n{=}1$	—
	$\mathcal{C}_1\frac{\cosh(\sqrt{a_3^2-\lambda_0^2}\,y)}{\cosh(a_2+a_3y)}+\mathcal{C}_2\frac{\sinh(\sqrt{a_3^2-\lambda_0^2}\,y)}{\cosh(a_2+a_3y)}$, for $a_0{=}0$, $n{=}2$ and $\lambda_0^2{\leq}a_3^2$	[17]
	$\mathcal{C}_1\frac{\cos(\sqrt{\lambda_0^2-a_3^2}\,y)}{\cosh(a_2+a_3y)}+\mathcal{C}_2\frac{\sin(\sqrt{\lambda_0^2-a_3^2}\,y)}{\cosh(a_2+a_3y)}$, for $a_0{=}0$, $n{=}2$ and $\lambda_0^2{>}a_3^2$	
	$\mathcal{C}_2\,\mathrm{e}^{\frac{H_{10}(y)}{4a_3}}$ and Eq. (42), for $a_0{=}0$	—
	$\mathcal{C}_2\,\mathrm{e}^{a_3H_{11}(y)}$ and Eq. (43), for $n{=}1$	—
	$\mathcal{C}_2\,\mathrm{e}^{2a_2H_{12}(y)}$ and Eq. (44), for $a_0{=}0$	
$a_0+a_1\tanh^n(a_2y)$	$\mathcal{C}_1\cosh^{2\nu_6}(a_2y)\,\mathrm{HeunG}\left[-\nu_3,(2\,\nu_3-1)\,\nu_6+\lambda_0^2\frac{\nu_3}{a_2^2},-2\,\nu_6,2-2\,\nu_6,1-2\,\nu_6,1-2\,\nu_6,\frac{1}{2}+\frac{\tanh(a_2y)}{2}\right]+$	—
	$2j\sqrt{2}\,\mathcal{C}_2(-2\mathrm{e}^{2a_2y})^{\nu_6}\mathrm{HeunG}\left[-\nu_3,\nu_6,0,2,1{+}2\,\nu_6,1-2\,\nu_6,\frac{1}{2}+\frac{\tanh(a_2y)}{2}\right]$, for $n{=}1$	

Source: Prepared by the authors.

$$\nu_1{=}\frac{n-1}{2},\quad \nu_2{=}j\,\frac{\lambda_0}{\ln(a_2)a_3},\quad \nu_3{=}\frac{a_0}{2\,a_1}-\frac{1}{2},\quad \nu_4{=}j\,\frac{a_0}{2\,a_1}+\frac{1}{2},\quad \nu_5{=}j\,\frac{a_0^2+a_1^2}{2\,a_1(a_0+j\,a_1)},\quad \nu_6{=}j\,\frac{\lambda_0}{2\,a_2},$$
$$\mu_1{=}\sqrt{a_1^2-4a_0a_2},\quad \mu_2{=}\frac{\sqrt{\ln^2(a_2)a_3^2-4\lambda_0^2}}{2\ln(a_2)a_3},\quad \mu_3{=}\frac{\sqrt{a_3^2n^2+4\lambda_0^2}}{2\,a_3},\quad \mu_4{=}\frac{\sqrt{a_3^2-4\lambda_0^2}}{2\,a_3},\quad \mu_5{=}\frac{\sqrt{a_3^2n^2-4\lambda_0^2}}{2\,a_3}.$$

$$H_1'(y) \;=\; \frac{\left[\mathrm{e}^{j\lambda_0y}\mathrm{HeunC}\left(4\mu_1\nu_6,0,0,0,0,\frac{a_1+2a_2y}{2\mu_1}+\frac{1}{2}\right)\right]^{-2}}{a_0+a_1y+a_2y^2}. \tag{33}$$

$$\begin{aligned}H_2'(y) \;=\;& \left\{\mathcal{C}_1h_2(y)\left[\mu_1[\nu_6(\mu_1+a_1+2a_2y)-2\nu_1]\,\mathrm{HeunC}\left(4\mu_1\nu_6,-2\nu_1,2\nu_1,0,\tfrac{1}{2}-\tfrac{n^2}{2},\tfrac{a_1+2a_2y}{2\mu_1}+\tfrac{1}{2}\right)+\frac{1}{2a_2}\mathrm{HeunC}'\left(4\mu_1\nu_6,-2\nu_1,2\nu_1,0,\tfrac{1}{2}-\tfrac{n^2}{2},\tfrac{a_1+2a_2y}{2\mu_1}+\tfrac{1}{2}\right)\right]+\right.\\ &\left.\mu_1\nu_6\,\mathrm{HeunC}\left(4\mu_1\nu_6,2\nu_1,2\nu_1,0,\tfrac{1}{2}-\tfrac{n^2}{2},\tfrac{a_1+2a_2y}{2\mu_1}+\tfrac{1}{2}\right)+\frac{1}{2}\mathrm{HeunC}'\left(4\mu_1\nu_6,2\nu_1,2\nu_1,0,\tfrac{1}{2}-\tfrac{n^2}{2},\tfrac{a_1+2a_2y}{2\mu_1}+\tfrac{1}{2}\right)\right\}\\ &\left\{\mathcal{C}_1h_2(y)(\mu_1+a_1+2a_2y)\,\mathrm{HeunC}\left(4\mu_1\nu_6,-2\nu_1,2\nu_1,0,\tfrac{1}{2}-\tfrac{n^2}{2},\tfrac{a_1+2a_2y}{2\mu_1}+\tfrac{1}{2}\right)+\mathrm{HeunC}\left(4\mu_1\nu_6,2\nu_1,2\nu_1,0,\tfrac{1}{2}-\tfrac{n^2}{2},\tfrac{a_1+2a_2y}{2\mu_1}+\tfrac{1}{2}\right)\right\}^{-1},\end{aligned} \tag{34}$$

and

$$h_2(y) \;=\; (a_0+a_1y+a_2y^2)^{-\frac{n}{2}}\left(\frac{\mu_1-a_1-2a_2y}{\mu_1+a_1+2a_2y}\right)^{\frac{n}{2}}.$$

$$H_3'(y) \;=\; h_3'(y)\left\{\frac{\mathcal{C}_1}{\mathcal{C}_1h_3(y)+1}-\frac{2a_3}{a_1}\,y\,(a_1y+a_3y^3)\,\mathrm{HeunC}\left(0,0,0,-\lambda_0^2\frac{a_1}{4a_3},\lambda_0^2\frac{a_1}{4a_3},-\frac{a_3}{a_1}y^2\right)\mathrm{HeunC}'\left(0,0,0,-\lambda_0^2\frac{a_1}{4a_3},\lambda_0^2\frac{a_1}{4a_3},-\frac{a_3}{a_1}y^2\right)\right\}, \tag{35}$$

and

$$h_3'(y) \;=\; \frac{\mathrm{HeunC}\left(0,0,0,-\lambda_0^2\frac{a_1}{4a_3},\lambda_0^2\frac{a_1}{4a_3},-\frac{a_3}{a_1}y^2\right)^{-2}}{a_1y+a_3y^3}.$$

$$H_4'(y) \;=\; \frac{\mathcal{C}_1 h_4'(y)}{\mathcal{C}_1 h_4(y)+1} - j\lambda_0 - \frac{a_3}{a_2}\,\frac{\mathrm{HeunC}'\left(2j\lambda_0\frac{a_2}{a_3},1,0,0,-\frac{1}{2},-\frac{a_3}{a_2}y\right)}{\mathrm{HeunC}\left(2j\lambda_0\frac{a_2}{a_3},1,0,0,-\frac{1}{2},-\frac{a_3}{a_2}y\right)}, \tag{36}$$

and

$$h_4'(y) \;=\; \frac{\left[\mathrm{e}^{j\lambda_0 y}\mathrm{HeunC}\left(2j\lambda_0\frac{a_2}{a_3},1,0,0,-\frac{1}{2},-\frac{a_3}{a_2}y\right)\right]^{-2}}{a_2y^2+a_3y^3}.$$

$$H_5'(y) \;=\; \frac{\mathcal{C}_1 h_5'(y)}{\mathcal{C}_1 h_5(y)+1} - j\lambda_0 - \frac{a_4}{a_3}\,\frac{\mathrm{HeunC}'\left(2j\lambda_0\frac{a_3}{a_4},2,0,0,-1,-\frac{a_4}{a_3}y\right)}{\mathrm{HeunC}\left(2j\lambda_0\frac{a_3}{a_4},2,0,0,-1,-\frac{a_4}{a_3}y\right)}, \tag{37}$$

and

$$h_5'(y) \;=\; \frac{\left[\mathrm{e}^{-j\lambda_0 y}\mathrm{HeunC}\left(2j\lambda_0\frac{a_3}{a_4},2,0,0,-1,-\frac{a_4}{a_3}y\right)\right]^{-2}}{a_3y^3+a_4y^4}.$$

$$\begin{aligned} H_6'(y) \;=\;& \Big\{ j\,h_6(y)^{2\nu_2}\,{}_2\mathrm{F}_1\left(\nu_2-\mu_2+\tfrac{1}{2},\mu_2+\nu_2+\tfrac{1}{2},1+2\nu_2,-h_6(y)\right) - j\,\mathcal{C}_1\,{}_2\mathrm{F}_1\left(-\mu_2-\nu_2+\tfrac{1}{2},\mu_2-\nu_2+\tfrac{1}{2},1-2\nu_2,-h_6(y)\right) - \\ & h_6(y)\,\frac{\ln(a_2)a_3}{\ln^2(a_2)a_3^2+4\lambda_0^2}\left[\mathcal{C}_1(j\ln(a_2)a_3-2\lambda_0)\,{}_2\mathrm{F}_1\left(\mu_2-\nu_2+\tfrac{3}{2},-\mu_2-\nu_2+\tfrac{3}{2},2-2\nu_2,-h_6(y)\right)+h_6(y)^{2\nu_2}(j\ln(a_2)a_3+2\lambda_0)\,{}_2\mathrm{F}_1\left(\nu_2-\mu_2+\tfrac{3}{2},\mu_2+\nu_2+\tfrac{3}{2},2+2\nu_2,-h_6(y)\right)\right]\Big\}\cdot \\ & \Big\{\mathcal{C}_1\,{}_2\mathrm{F}_1\left(-\mu_2-\nu_2+\tfrac{1}{2},\mu_2-\nu_2+\tfrac{1}{2},1-2\nu_2,-h_6(y)\right)+h_6(y)^{2\nu_2}\,{}_2\mathrm{F}_1\left(\nu_2-\mu_2+\tfrac{1}{2},\mu_2+\nu_2+\tfrac{1}{2},1+2\nu_2,-h_6(y)\right)\Big\}^{-1}, \end{aligned} \tag{38}$$

and

$$h_6(y) \;=\; \frac{a_1\,a_2^{a_3y}}{a_0}.$$

$$\begin{aligned} H_7'(y) \;=\;& \Big\{2\,\mathcal{C}_1[\sin(a_2+a_3y)-1]\left[\mathrm{HeunG}\left(-\nu_3,\tfrac{1}{2}-\nu_3\,\mu_4^2,1+\mu_3,1-\mu_3,\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2}+\frac{\sin(a_2+a_3y)}{2}\right)+[\sin(a_2+a_3y)+1]\,\mathrm{HeunG}'\left(-\nu_3,\tfrac{1}{2}-\nu_3\,\mu_4^2,1+\mu_3,1-\mu_3,\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2}+\frac{\sin(a_2+a_3y)}{2}\right)\right]+ \\ & \cos(a_2+a_3y)\sqrt{2-2\sin(a_2+a_3y)}\,\mathrm{HeunG}'\left(-\nu_3,\lambda_0^2\frac{\nu_3}{a_3^2},\tfrac{1}{2}+\mu_3,\tfrac{1}{2}-\mu_3,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}+\frac{\sin(a_2+a_3y)}{2}\right)\Big\}\cdot \\ & \left[\sqrt{2-2\sin(a_2+a_3y)}\,\mathrm{HeunG}\left(-\nu_3,\lambda_0^2\frac{\nu_3}{a_3^2},\tfrac{1}{2}+\mu_3,\tfrac{1}{2}-\mu_3,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}+\frac{\sin(a_2+a_3y)}{2}\right)-2\,\mathcal{C}_1\cos(a_2+a_3y)\,\mathrm{HeunG}\left(-\nu_3,\tfrac{1}{2}-\nu_3\,\mu_4^2,1+\mu_3,1-\mu_3,\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2}+\frac{\sin(a_2+a_3y)}{2}\right)\right]^{-1}. \end{aligned} \tag{39}$$

$$\begin{aligned} H_8'(y) \;=\;& \Big\{\left(\nu_1+\mu_5+\tfrac{1}{2}\right)[2\sinh(2a_2+2a_3y)-\sinh(4a_2+4a_3y)]\left[\mathcal{C}_1\mathrm{Q}^{\nu_1}_{\mu_5-\frac{1}{2}}[\cosh(a_2+a_3y)]+\mathrm{P}^{\nu_1}_{\mu_5-\frac{1}{2}}[\cosh(a_2+a_3y)]\right]+ \\ & 2\left(\nu_1-\mu_5-\tfrac{1}{2}\right)[3\sinh(a_2+a_3y)-\sinh(3a_2+3a_3y)]\left[\mathcal{C}_1\mathrm{Q}^{\nu_1}_{\mu_5+\frac{1}{2}}[\cosh(a_2+a_3y)]+\mathrm{P}^{\nu_1}_{\mu_5+\frac{1}{2}}[\cosh(a_2+a_3y)]\right]\Big\}\cdot \\ & \Big\{[4\cosh(2a_2+2a_3y)-\cosh(4a_2+4a_3y)-3]\left[\mathcal{C}_1\mathrm{Q}^{\nu_1}_{\mu_5-\frac{1}{2}}[\cosh(a_2+a_3y)]+\mathrm{P}^{\nu_1}_{\mu_5-\frac{1}{2}}[\cosh(a_2+a_3y)]\right]\Big\}^{-1}. \end{aligned} \tag{40}$$

$$\begin{aligned} H_9'(y) \;=\;& \Big\{\mathcal{C}_1\,\mathrm{HeunG}\left[\nu_4,\tfrac{1}{2}+\nu_5\,\mu_3^2,1+\mu_4,1-\mu_4,\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2}-j\frac{\sinh(a_2+a_3y)}{2}\right]-j\sqrt{\sinh(a_2+a_3y)+j}\,\mathrm{HeunG}'\left[\nu_4,\lambda_0^2\frac{\nu_5}{a_3^2},\tfrac{1}{2}+\mu_4,\tfrac{1}{2}-\mu_4,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}-j\frac{\sinh(a_2+a_3y)}{2}\right]- \\ & \mathcal{C}_1[j\sinh(a_2+a_3y)-1]\,\mathrm{HeunG}'\left[\nu_4,\tfrac{1}{2}+\nu_5\,\mu_3^2,1+\mu_4,1-\mu_4,\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2}-j\frac{\sinh(a_2+a_3y)}{2}\right]\Big\}\cdot \\ & \left\{\frac{\sqrt{\sinh(a_2+a_3y)+j}}{2\cosh(a_2+a_3y)}\left[\mathcal{C}_1\sqrt{\sinh(a_2+a_3y)+j}\,\mathrm{HeunG}\left[\nu_4,\tfrac{1}{2}+\nu_5\,\mu_3^2,1+\mu_4,1-\mu_4,\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2}-j\frac{\sinh(a_2+a_3y)}{2}\right]+\mathrm{HeunG}\left[\nu_4,\lambda_0^2\frac{\nu_5}{a_3^2},\tfrac{1}{2}+\mu_4,\tfrac{1}{2}-\mu_4,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}-j\frac{\sinh(a_2+a_3y)}{2}\right]\right]\right\}^{-1}. \end{aligned} \tag{41}$$

$$\begin{aligned} H_{10}'(y) \;=\;& \Big\{-\left(\nu_1+\mu_5+\tfrac{1}{2}\right)[2\sinh(2a_2+2a_3y)+\sinh(4a_2+4a_3y)]\left[\mathcal{C}_1\mathrm{Q}^{\nu_1}_{\mu_5-\frac{1}{2}}[j\sinh(a_2+a_3y)]+\mathrm{P}^{\nu_1}_{\mu_5-\frac{1}{2}}[j\sinh(a_2+a_3y)]\right]+ \\ & 2j\left(\nu_1-\mu_5-\tfrac{1}{2}\right)[3\cosh(a_2+a_3y)+\cosh(3a_2+3a_3y)]\left[\mathcal{C}_1\mathrm{Q}^{\nu_1}_{\mu_5+\frac{1}{2}}[j\sinh(a_2+a_3y)]+\mathrm{P}^{\nu_1}_{\mu_5+\frac{1}{2}}[j\sinh(a_2+a_3y)]\right]\Big\}\cdot \\ & \Big\{[4\cosh(2a_2+2a_3y)+\cosh(4a_2+4a_3y)+3]\left[\mathcal{C}_1\mathrm{Q}^{\nu_1}_{\mu_5-\frac{1}{2}}[j\sinh(a_2+a_3y)]+\mathrm{P}^{\nu_1}_{\mu_5-\frac{1}{2}}[j\sinh(a_2+a_3y)]\right]\Big\}^{-1}. \end{aligned} \tag{42}$$

$$
\begin{aligned}
H'_{11}(y) \;=\;& [\cosh(a_2{+}a_3y){-}1]^{\frac{1}{2}}\Big\{\mathcal{C}_1\,\mathrm{HeunG}\Big[-\nu_3,\tfrac{1}{2}{-}\nu_3\,\mu_3^2,1{+}\mu_5,1{-}\mu_5,\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2}{+}\frac{\cosh(a_2{+}a_3y)}{2}\Big]+\\
&\mathcal{C}_1[\cosh(a_2{+}a_3y){+}1]\,\mathrm{HeunG}'\Big[-\nu_3,\tfrac{1}{2}{-}\nu_3\,\mu_3^2,1{+}\mu_5,1{-}\mu_5,\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2}{+}\frac{\cosh(a_2{+}a_3y)}{2}\Big]{+}\cosh\Big(\frac{a_2{+}a_3y}{2}\Big)\mathrm{HeunG}'\Big[-\nu_3,-\lambda_0^2\frac{\nu_3}{a_3^2},\tfrac{1}{2}{+}\mu_5,\tfrac{1}{2}{-}\mu_5,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}{+}\frac{\cosh(a_2{+}a_3y)}{2}\Big]\Big\}.\\
&\Big\{4\,\mathcal{C}_1\cosh\Big(\frac{a_2{+}a_3y}{2}\Big)\mathrm{HeunG}\Big[-\nu_3,\tfrac{1}{2}{-}\nu_3\,\mu_3^2,1{+}\mu_5,1{-}\mu_5,\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2}{+}\frac{\cosh(a_2{+}a_3y)}{2}\Big]{+}\mathrm{HeunG}'\Big[-\nu_3,-\lambda_0^2\frac{\nu_3}{a_3^2},\tfrac{1}{2}{+}\mu_5,\tfrac{1}{2}{-}\mu_5,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}{+}\frac{\cosh(a_2{+}a_3y)}{2}\Big]\Big\}^{-1}.
\end{aligned}
\tag{43}
$$

$$
\begin{aligned}
H'_{12}(y) \;=\;& \Big\{-\nu_6\Big(1{-}\frac{\nu_6}{\nu_1{+}1}\Big)\tanh^{n+1}(a_2y)\,{}_2\mathrm{F}_1[1{-}\nu_6,2{+}\nu_1{-}\nu_6,2{+}\nu_1,\tanh^2(a_2y)]{+}\mathcal{C}_1(\nu_6\sinh^2(a_2y){-}\nu_1)\,{}_2\mathrm{F}_1[1{-}\nu_6,-\nu_1{-}\nu_6,1{-}\nu_1,\tanh^2(a_2y)]-\\
&\mathcal{C}_1\Big[\nu_6\Big(1{+}\frac{\nu_6}{\nu_1{-}1}\Big){-}\frac{\nu_1}{\nu_1{-}1}\Big]\tanh^2(a_2y){}_2\mathrm{F}_1[2{-}\nu_6,1{-}\nu_1{-}\nu_6,2{-}\nu_1,\tanh^2(a_2y)]{+}\nu_6\tanh^{n+1}(a_2y)\cosh^2(a_2y)\,{}_2\mathrm{F}_1[-\nu_6,1{+}\nu_1{-}\nu_6,1{+}\nu_1,\tanh^2(a_2y)]\Big\}.\\
&\Big\{\cosh^2(a_2y)\Big[\mathcal{C}_1\tanh(a_2y)\,{}_2\mathrm{F}_1[1{-}\nu_6,-\nu_1{-}\nu_6,1{-}\nu_1,\tanh^2(a_2y)]{+}\tanh^n(a_2y)\,{}_2\mathrm{F}_1[-\nu_6,1{+}\nu_1{-}\nu_6,1{+}\nu_1,\tanh^2(a_2y)]\Big]\Big\}^{-1}.
\end{aligned}
\tag{44}
$$

5. Conclusion

A Lie symmetry method-based approach has been proposed to compute general solutions for the mode shape equation of non-uniform rods modelled according to the elementary rod theory. An admitted one-parameter Lie symmetry group provides an order reduction to such mode shape equation, transforming it into a first-order Riccati equation. Finding closed-form solutions for the reduced equation and recovering them in the space of the mode shape variables provides a systematic procedure for determining general solutions for non-uniform rods, whose mode shapes arise by the appropriate boundary conditions.

Such Lie symmetry method retrieves numerous literature solutions based on ad hoc methods with the advantage of dealing with a reduced equation that belongs to a well-known class of equations. This reduction makes available the great database of solutions for Riccati equations towards the exact resolution of the mode shape equation. However, the proposed method does not guarantee solutions to the problem once there may not be solutions to the Riccati equation. On the other hand, the applied coordinate transformation implies getting a nonlinear equation for the problem, whose accomplishment requires preliminary calculations inherent to the method and an additional step to recover the general solutions into the space of variables of the mode shape equation. Concerning these calculations, there is considerable dependence on symbolic manipulation software for complex analytical derivations from the algorithmic steps of the Lie symmetry method, which may preclude its application.

Multiple techniques from the literature for computing solutions of non-uniform rods rely on the idea of applying coordinate transformations to solve mode shape equations [11, 12, 15, 17]. Generally, the considered equation is transformed into another second-order equation whose general solution is known. Besides, this is done only for a few types of cross-section area functions. In contrast, the Lie symmetry method systematically performs an order reduction which is valid for any cross-section area function. In this way, general solutions from the literature can be retrieved by using the presented approach.

Extensions of the method to different elastic vibrating structures, e.g., non-uniform beams and plates, or even alternative models for non-uniform rods, could lead to systematically determining general solutions for the corresponding mode shape equations, investigated by the authors for future works. The nature and the number of the admitted symmetry transformations play a significant role in integrating such equations, and the Lie symmetry method may be insufficient for integrating the equation. However, there are some generalizations of the classical notion of Lie symmetry, such that λ-symmetries, Lie-Bäcklund, nonlocal, and generalized symmetries [52, 53, 54, 55], which may help to overcome these difficulties.

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Conflict of interest

The authors declare that they do not have any conflict of interest.

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