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Piecewise smooth vector fields: Closing lemma, Topological entropy and Shifts

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Tese apresentada como parte dos requisitos para obtenção do título de Doutor em Matemática, junto ao Programa de Pós-Graduação em Matemática, do Instituto de Biociências, Letras e Ciências Exatas da Universidade Estadual Paulista "Júlio de Mesquita Filho", Câmpus de São José do Rio Preto.

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A minha família,
dedico

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"A preocupação com o próprio homem e seu destino deve ser sempre o principal objetivo de todos os empreendimentos técnológicos(...) para que criações de nossa mente sejam uma benção, e não uma maldição, para a humanidade. Nunca se esqueçam disso quando estiverem às voltas com os seus diagramas e equações."

(Einstein, 1931, citado em [14], tradução nossa)

RESUMO

Recentemente, a teoria sobre campos vetoriais suaves por partes (CVSPs) tem passado por importantes desenvolvimentos. Uma linha de investigação desses campos vetoriais é procurar estabelecer resultados análogos aos já clássicos sobre campos vetoriais suaves, como os teoremas de Poincaré-Bendixson e Índice de Poincaré. Nessa linha de trabalho, nós abordamos o clássico problema do Closing Lemma para CVSPs e apresentamos uma resposta positiva para o caso C^0.

Outra possível linha de investigação é estudar as diferenças entre CVSPs e suaves. A maior parte delas surge do fato de que não há unicidade de trajetória por um ponto para CVSPs. Isto implica resultados como a existência de um CVSP planar caótico. Nesta linha de trabalho, propomos um novo modo de abordar CVSPs, através da construção de um espaço métrico de todas as possíveis trajetórias, usamos isso para definir entropia topológica de um CVSP; provamos a existências de CVSPs com entropia positiva (finita e infinita) e damos condições suficientes para que um CVSP tenha entropia infinita. Além disso, a partir desse espaço métrico, propomos um modo de conjugar a dinâmica de um CVSP com a aplicação shift em espaços de sequências.

Palavras-chave: Sistemas dinâmicos, Campos de vetores suaves por partes, Closing Lemma, Entropia topológica, Shifts.

ABSTRACT

Recently, the theory concerning piecewise smooth vector fields (PSVFs for short) has been undergoing important improvements. One line of investigation of these vector fields is to seek to establish results analogous to those already well known for the smooth case, such as Poincaré Bendixson and Poincaré Index Theorems. On this line of work, we tackle the classical problem of Closing Lemma in the setting of PSVFs and provide a positive answer for the case C^0*.*
Another possible line of investigation is studying the differences between PSVFs and smooth vector fields. Most of them arises from the fact that there is no uniqueness of trajectory passing through a point for a PSVF. It implies results like the existence of a planar PSVF that is chaotic. On this line, we propose a new way of looking at PSVFs, by the construction of a metric space of all possible trajectories, we use it to define topological entropy of a PSVF; prove the existence of PSVFs of positive (finite and infinite) entropy and give a sufficient condition for a PSVF to have infinite topological entropy. Moreover, from this metric space, we propose a way of conjugating the dynamics of PSVFs and shift spaces.

Keywords: Dynamical Systems, Piecewise smooth vector fields, Closing Lemma, Topological entropy, Shifts.

Contents

CHAPTER 1

Introduction

Ordinary Differential Equations (ODEs) has been extensively studied since mid 17th century from pure and applied points of view and had contributions from many important mathematicians like Newton, Euler, the Bernoulli family, Laplace, Cauchy and others. But it was Poincaré, at the end of 19th century ([30]), that started what today is called Qualitative Theory of Ordinary Differential Equations. Since then, many tools were developed to study the behaviour of solutions of an ODE without explicitly solving it, with aid from all three major areas: Topology, Analysis and Algebra.

From the beginning, it was clear the vast applications of these equations in modelling natural phenomena, like Newton's second law of motion and Maxwell's equations of electromagnetism. More recently, ODEs and their associated vector fields were used to model phenomena that are subject to sudden change (e.g. caused by a on-off switch), so they have two or more distinct ODEs that rule this phenomena. These systems with two or more laws of interaction were called Piecewise Smooth Vector Fields (PSVFs for short). Some examples of these are: a system that model the dynamics of a cancer patient being submitted to a severe drug treatment that is abruptly interrupted in order to provide the immune system recovery (see [37, 38]). The same holds for recent protocols of HIV treatment (see [9]). There is also examples in ecology, modelling predator preys populations when predator suddenly change its food preference (see [11]; and electric engineering: [12, 43].

The general mathematical theory about PSVFs is being constructed in recent years. The textbook [13] is a good reference for general theory and some applications, and also [19] and [24] are very comprehensive on theory of PSVFs. There are some similarities with the smooth vector fields, and some versions of classical theorems have been published (e.g. the *Poincaré-Bendixson Theorem*, *Poincaré Index Theorem*, *Poincaré recurrence Theorem*, and *Peixoto Theorem* see [5, 7, 15, 40]), but there are many behaviours which are typical to PSVFs that are not possible in the smooth case. Describing this has been a main source of research for the mathematicians dedicated to this area. For example: it is possible to construct a planar PSVF that is chaotic ([6]), whereas this feature is impossible for the smooth case. The main aspect that leads to these differences is the non uniqueness of trajectories passing through a point of the domain of the PSVF. In Chapter 2, we provide the main ideas and general definitions concerning PSVFs, topological entropy and shift spaces that we use along this text.

Chapter 3 is dedicated to a version of the historical problem of Closing Lemma for PSVFs. The contents of this chapter is compiled at preprint [3]. The question here is the following:

"Given a nonperiodic point x_0 such that the trajectory through x_0 return to an arbitrarily small neighborhood of x_0 infinitely many times, does there exist an arbitrarily small C^r perturbation of this system which has a closed trajectory through x_0?"

Probably who first stated this problem was Poincaré ([31]) and the fact that this problem is the *10th Problem* of the 21st century list compiled by Smale ([39]) certifies the relevance of this question.

Nowadays, the term Closing Lemma actually refers to a family of different results, that involves a point p with some recurrence property and some additional hypothesis on the dimension of space, on the vector field (or diffeomorphism) and others in order to achieve the existence of the closed orbit through p.

Because of this, many versions of Closing Lemmas were published through the years. In some of them a closed trajectory is achieved (see [28, 29, 33, 34, 35]) but in other cases it is not (see [20, 22, 36]). The surveys [1, 32] give a good vision of this problem's history and advances achieved in this subject.

As far as the author know, the paper [8] is the only one studying a version of the Closing Lemma for PSVFs. In fact, it proves that the Closing Lemma is false for a specific example of PSVF.

In this work, we use the same recurrence property of [8] that is the following: "p is *non-trivially recurrent* of Z if it belongs to α or $\omega-$ limit sets of itself and it is not periodic" (see Definition 2.1.6), and require an additional hypothesis on the transversality between the trajectory through p and the switching manifold Σ. With this we provide a *positive answer* for the Closing Lemma C^0 for PSVFs.

Let γ_p denote a trajectory through $p \in \Sigma$, and p_{-1}, p_1 denote the first hits of this trajectory in the discontinuity Σ for negative and positive time, respectively. The main result of Chapter 3 is the following:

Theorem 3.1.1 . *Let $Z = (X, Y)$ be a PSVF defined in $\mathbb{R}^n$, with switching manifold given by $\Sigma = f^{-1}(0)$ and $p \in \Sigma$ a nontrivially recurrent point for Z. Suppose one of the following holds:*

(a) $p \in \omega(p)$ and γ_p is transversal to Σ at p and p_{-1}.

(b) $p \in \alpha(p)$ and γ_p is transversal to Σ at p and p_1.

Then for every C^0-neighbourhood $\mathcal{V}$ of Z, there exists $\widetilde{Z} \in \mathcal{V}$ and a closed trajectory $\gamma_p^{\widetilde{Z}}$ of $\widetilde{Z}$ passing through p.

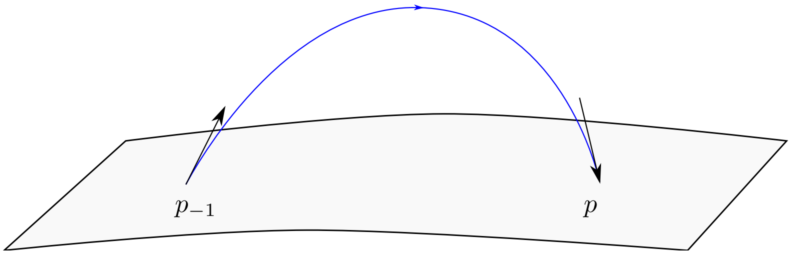

Figure 1.1: Case (a) of Theorem 3.1.1. Source: Prepared by the author.

Chapters 4 and 5 are devoted to present a new way of looking at PSVFs and how that made it possible to study *topological entropy* of these systems and conjugate its dynamics with *symbolic dynamics*. The content of these chapters are compiled at preprints [2, 4].

Given that one of the most relevant characteristic about PSVFs is the non-uniqueness of trajectory through a point it is hard to define a time-one map and define the topological entropy from that, as it is done in the smooth case. So, in Chapter 4, a space Ω of all possible trajectories of a given PSVF Z is constructed with an appropriate metric, and a flow $\{F_t\}_{t\in\mathbb{R}}$ that summarizes all the dynamics of the PSVF. With this flow, topological entropy of a PSVF Z ($h(Z)$) is defined as $h(Z) := h(F_1)$. Details are given in Sections 2.2 and 4.1. The main results of Chapter 4 are:

Theorem 4.2.1 . *Given $\alpha \in \mathbb{Z}, \alpha \geqslant 2$ there exists a PSVF Z such that $h(Z) = \log \alpha$*

Theorem 4.3.1 . *The PSVF Z defined by:*

$$Z(x,y) = \begin{cases} X(x,y) = (1, -2x), & \textit{for} \quad y \geq 0 \\ Y(x,y) = (-2, -4x^3 + 2x), & \textit{for} \quad y \leq 0 \end{cases}. \tag{1.1}$$

has infinite topological entropy.

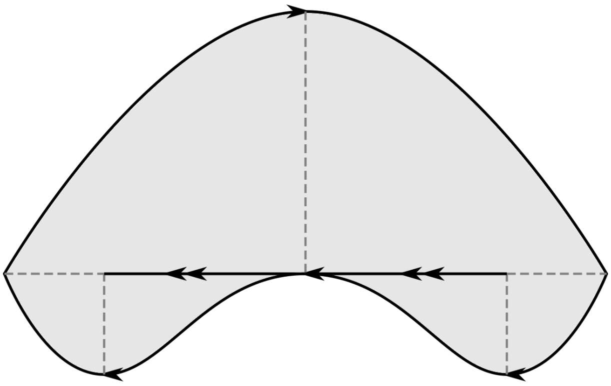

Figure 1.2: Vector field from Theorem 4.3.1. Source: [4].

Based on Theorem 4.3.1, sufficient conditions for a PSVF Z to have infinite topological entropy are given. Moreover, some examples of planar PSVFs with null, positive and infinite entropy are presented. Achieving this in $\mathbb{R}^2$ is one more example of how complex the dynamics of PSVFs can be. To the best of author's knowledge, there is no work related to the topological entropy of PSVFs in the literature.

Chapter 5 also features a metric space Ω of all possible trajectories of a PSVF Z, but in this one, the metric is defined in a slightly different manner, but there is still a flow $\{T_t\}$ on Ω inherited from the PSVF and conjugations between T_1 and *shift spaces* are constructed.

The first results concern the full shift of two symbols, and subshifts of finite type with $2k$ symbols, for any $k \in \mathbb{N}$. In the other ones we use shift spaces with infinite symbols: a subshift of $\mathbb{Z}^{\mathbb{Z}}$ and $(0,1]^{\mathbb{Z}}$.

Moreover, we define a geometric structure (*homoclinic loops*) that enables us to construct these conjugations on a larger class of PSVFs:

Definition 5.2.1 . *A* k-homoclinic loop *of a planar PSVF $Z = (X, Y)$ is a global trajectory presenting k distinct visible-visible two-fold singularities $p_1, \dots, p_k$ in such a way that, after it passes through p_i, the trajectory reaches Σ either in p_{i-1} or p_{i+1} when $i = 2, \dots, k-1$. When $i = 1$ (resp., $i = k$), after passes through p_i, the trajectory reaches Σ either in a crossing point or p_{i+1} (resp., p_{i-1}) (see Figure 1.3).*

Moreover, an ∞-homoclinic loop *of a planar PSVF is a global trajectory of Z presenting ∞ visible-visible two-fold singularities $p_1, p_2, \dots$ in such a way that, after it passes through p_i, the trajectory reaches Σ either in p_{i-1} or p_{i+1}.*

The main results of Chapter 5 are the following:

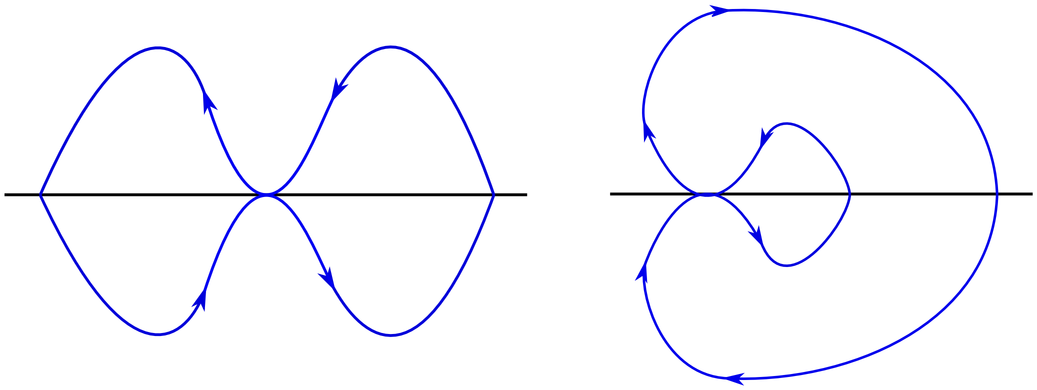

Figure 1.3: 2-homoclinic loops. Source: [2].

Theorem 5.2.1 . *(i) There exists a conjugacy between the time-one map of the PSVF*

$$Z_2(x,y)=\begin{cases} X_2(x,y)=\left(1,\frac{x}{2}-4x^3\right), & \textit{for} \quad y\geq 0\\ Y_2(x,y)=\left(-1,\frac{x}{2}-4x^3\right), & \textit{for} \quad y\leq 0 \end{cases}, \tag{1.2}$$

restricted to an invariant compact set $\Lambda_2\subset\mathbb{R}^2$ and the full shift of two symbols $\sigma:\{0,1\}^{\mathbb{Z}}\to\{0,1\}^{\mathbb{Z}}$

(ii) For each $k\geqslant 3$, there exists a planar PSVF

$$Z_k(x,y)=\begin{cases} X_k(x,y)=(1,P_k'(x)), & \textit{for} \quad y\geq 0\\ Y_k(x,y)=(-1,P_k'(x)), & \textit{for} \quad y\leq 0 \end{cases}, \tag{1.3}$$

such that, restricted to an invariant compact set Λ_k, its time-one map is conjugate to a subshift of $2(k-1)$ symbols.

(iii) There exists a conjugacy between the time-one map of the PSVF

$$Z_\infty(x,y)=\begin{cases} X_\infty(x,y)=(1,2\,\mathrm{sen}(2\pi x)), & \textit{for} \quad y\geq 0\\ Y_\infty(x,y)=(-1,2\,\mathrm{sen}(2\pi x)), & \textit{for} \quad y\leq 0 \end{cases}, \tag{1.4}$$

restricted to an invariant set $\Lambda_\infty\subset\mathbb{R}^2$ and a subshift over a countable alphabet $\sigma:\Theta_\infty\to\Theta_\infty$, where $\Theta_\infty\subset\mathbb{Z}^{\mathbb{Z}}$.

(iv) The return map of the PSVF

$$Z(x,y)=\begin{cases} X(x,y)=(1,-2x), & \textit{for} \quad y\geq 0\\ Y(x,y)=(-2,-4x^3+2x), & \textit{for} \quad y\leq 0 \end{cases}, \tag{1.5}$$

restricted to an invariant compact set Λ, is conjugated to $\sigma:(0,1]^{\mathbb{Z}}\to(0,1]^{\mathbb{Z}}$.

In the next theorem, we use the $\Sigma-$equivalence between two PSVFs, that, in vague words, is stated as the existence of a homeomorphism that takes the switching manifold and the orbits of the first PSVF to the switching manifold and orbits of the other one, respectively (see details in Definition 5.2.2).

Theorem 5.2.2 . *(i) The PSVF Z_2 of Theorem A, restrict to Λ_2, is Σ-equivalent to any PSVF presenting a 1-homoclinic loop.*

(ii) *The PSVF* Z_k *of Theorem A, restrict to* Λ_k*, is* Σ*-equivalent to any PSVF presenting a* $(k-1)$*-homoclinic loop.*

(iii) *The PSVF* Z_∞ *of Theorem A, restrict to* Λ_∞*, is* Σ*-equivalent to any PSVF presenting a* ∞*-homoclinic loop.*

(iv) *The PSVF* Z *of Theorem A, restrict to* Λ*, is* Σ*-equivalent to any PSVF* $\widetilde{Z}$ *presenting a compact region* $\widetilde{\Lambda}$ *bounded by a trajectory of* $\widetilde{Z}$ *passing through a invisible-visible two-fold* $\widetilde{p}$*. Moreover, except for* $\widetilde{p}$*, the PSVF* $\widetilde{Z}$ *has just more two invisible tangential singularities.*

In the final chapter, some concluding remarks and future perspectives of this work are presented.

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