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de Sitter Special Relativity: Implications for the Mass of Clusters of Galaxies

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Resumo

No cálculo da velocidade Doppler de galáxias, assume-se implicitamente que a galáxia está se movendo em um espaço-tempo de Minkowski. Usando essas velocidades, a massa virial de aglomerados de galáxias é duas ordens de magnitude maior que a massa obtida através da luminosidade de cada uma das galáxias do aglomerado. Por outro lado, de acordo com a relatividade especial invariante de de Sitter, qualquer sistema físico induz um termo cosmológico local na região que ocupa. Isso significa que a galáxia está, na verdade, movendo-se em um espaço-tempo de de Sitter. Como consequência, o tempo próprio muda, produzindo mudanças concomitantes na velocidade Doppler da galáxia. Usando valores fisicamente aceitáveis para os parâmetros galácticos, a velocidade modificada de de Sitter para as galáxias será uma ordem de magnitude menor que os valores atualmente aceitos. Como a massa virial de aglomerados depende do quadrado da velocidade média das galáxias pertencentes ao aglomerado, a massa virial modificada de de Sitter encontrada será duas ordens de grandeza menor que os valores atuais. Considerando que a discrepância usual entre as massas virial e luminosa é também de duas ordens de magnitude, a massa virial modificada de de Sitter de aglomerados de galáxias concorda com a massa luminosa e, consequentemente, nenhuma matéria escura é necessária para explicar a dinâmica dos aglomerados de galáxias.

Palavras-chave:*Relatividade especial de de Sitter; Massa de aglomerados de galáxias*

Áreas do conhecimento: *Gravitação e cosmologia*

Abstract

In the computation of the Doppler velocity of galaxies, one implicitly assumes that the galaxy moves in a Minkowski background spacetime. In this case, the ensuing virial mass of galaxies clusters is usually found to be two orders of magnitude larger than the mass obtained from the galaxy luminosity. On the other hand, according to the de Sitter-invariant special relativity, any physical system induces a local cosmological term in the region of space it occupies. This means that the galaxy is actually moving in a local de Sitter spacetime. Consequently, the proper time changes, producing concomitant changes in the Doppler velocity of the galaxy. Using physically acceptable values for the galactic parameters, the de Sitter-modified velocity of galaxies is one order of magnitude smaller than the currently accepted values. Since the virial mass of clusters depends on the mean square velocity of the galaxies belonging to the cluster, the de Sitter-modified virial mass is two orders of magnitude smaller than the current values. Considering that the usual discrepancy between the virial and the luminous masses is also two orders of magnitude, the de Sitter-modified virial mass of galaxies clusters agrees with the luminosity mass. Consequently, no dark matter is necessary to explain the dynamics of clusters of galaxies.

Keywords: *de Sitter special relativity; Mass of cluster of galaxies*

Knowledge Areas: *Gravitation and cosmology*

Contents

1 Introduction

In the study of the mass of the Coma cluster in the early nineteen-thirties, Fritz Zwicky found that the virial mass-to-light ratio M/L of the cluster was much larger than the mass-to-light ratio of the visible region of individual galaxies. More precisely, he found that [1]

$$\frac{M}{L} \simeq 500 \, \frac{M_\odot}{L_\odot} \, , \tag{1.1}$$

where $M_\odot$ and $L_\odot$ are, respectively, the mass and the absolute luminosity of the Sun. This discrepancy between M/L and $M_\odot/L_\odot$ became initially known as the *missing mass problem*. Since then, this result has been corroborated by many other studies of the Coma cluster [2]. In addition, similar results have been obtained for other rich clusters of galaxies. Estimates of M/L have generally given results of order 200 to $350 \, h \, M_\odot/L_\odot$ [3], where h is the Hubble constant in units of $100 \, Km \, s^{-1} \, Mpc^{-1}$. For example, a study of 16 clusters of galaxies with redshifts between 0.17 and 0.55 gave $M/L \simeq 213 \, h \, M_\odot/L_\odot$ [4]. Another application of the virial theorem to 459 clusters of galaxies has found $M/L \simeq 348 \, h \, M_\odot/L_\odot$ [5].

The visible light from clusters comes almost entirely from their galaxies. If most of the mass of these galaxies were in the galaxy's luminous central regions, then the rotation speeds of stars outside this region would follow the Kepler law $v \propto r^{-1/2}$. However, as first observed in the nineteen sixties and seventies, the velocity v of stars outside the central region is roughly constant, even beyond the visible disk of the galaxy [6, 7]. More than 60 spiral galaxies were studied, and the same results were borne out in all cases. Further studies have shown that this is what one would expect if a non-luminous spherical halo with a mass density decreasing as $1/r^2$ were present in every galaxy. These findings are usually considered to be a piece of evidence for the existence of halos of dark matter around galaxies. The existence of such halos could eventually solve—or at least attenuate—the missing mass problem of clusters of galaxies.

The above arguments favoring the existence of dark matter presupposes that the local spacetime kinematics is correctly described by ordinary special relativity at all energy scales and that the gravitational interaction, as described by general relativity—or by its Newtonian limit—remains valid at galactic and extra-galactic scales. However, this is a speculative assumption in the sense that there is no evidence that this could be true. The inability to explain the observed rotation curves of galaxies could quite reasonably be interpreted to indicate that both special and general relativities fail at the galactic scale. The same interpretation could be used to explain the inability of these theories to determine the mass of clusters of galaxies correctly.

Considering that the search for fundamental particles that could play the role of dark matter has, up to now, returned no results, we are going to explore in this work whether the missing mass problem in clusters of galaxies could find a solution based on a modification of ordinary special relativity. More specifically, we will adopt the approach based on replacing the Poincaré-invariant Einstein special relativity with a de Sitter-invariant special relativity.

The work will develop according to the following scheme. In Section 2, for the sake of completeness, we present the fundamentals of the de Sitter space, group, and algebra. In Section 3, we discuss the rationale behind the replacement of Poincaré by de Sitter as the group governing the local kinematics of spacetime. Then we introduce the dispersion relation of the de Sitter-invariant special relativity. Now, if special relativity changes, general relativity must undergo concomitant changes. The de Sitter-modified general relativity is discussed in Section 4, where it is also discussed how a physical system creates a local cosmological term Λ in the region of space it occupies. Due to this property, instead of traveling in a Minkowski spacetime, all galaxies are actually traveling in a local de Sitter spacetime, which, as we discuss in Section 5, produces a change in the galaxy proper time. This, in turn, will produce a change in the Doppler velocity of the galaxy. In Section 6, we discuss how such a change gives rise to a further change in the galaxy's virial mass and consequently in the virial mass of clusters of galaxies. Finally, in Section 7, we explore the implications of this result for the physics of galaxy clusters, and in particular of the Coma cluster.

2 de Sitter Space, Group and Algebra

2.1 The de Sitter spacetime

The maximally symmetric de Sitter spacetime, denoted dS, can be seen as a hypersurface in a host pseudo-Euclidean 5-space with metric $\eta_{AB} = (+1, -1, -1, -1, -1)$ $(A, B, \ldots = 0, \ldots, 4)$, whose points in Cartesian coordinates χ^A satisfy the relation [8]

$$\eta_{AB}\, \chi^A \chi^B = -\, l^2, \tag{2.1}$$

or equivalently, in four-dimensional coordinates,

$$\eta_{\mu\nu}\, \chi^\mu \chi^\nu - \left(\chi^4\right)^2 = -\, l^2. \tag{2.2}$$

It has the de Sitter group $SO(4,1)$ as group of motions, and is homogeneous under the Lorentz group $\mathcal{L} = SO(3,1)$, that is,

$$dS = SO(4,1)/\mathcal{L}\,. \tag{2.3}$$

In Cartesian coordinates χ^A, the generators of the infinitesimal de Sitter transformations are written in the form

$$L_{AB} = \eta_{AC}\, \chi^C \frac{\partial}{\partial \chi^B} - \eta_{BC}\, \chi^C \frac{\partial}{\partial \chi^A}\,. \tag{2.4}$$

They satisfy the commutation relations

$$[L_{AB}, L_{CD}] = \eta_{BC} L_{AD} + \eta_{AD} L_{BC} - \eta_{BD} L_{AC} - \eta_{AC} L_{BD}\,. \tag{2.5}$$

Among all coordinate systems used to describe the de Sitter metric, the four-dimensional stereographic coordinates $\{x^\mu\}$ emerge as a special one, in the sense that, for a vanishing cosmological term, they reduce to the Cartesian coordinates of Minkowski spacetime. They are obtained through a stereographic projection from the de Sitter hypersurface into a target Minkowski spacetime. However, in order to obtain the stereographic coordinates, it is necessary to use two different parameterizations: one appropriate for large values of the de Sitter parameter l, and another appropriate for small values of l. Considering that the cosmological term Λ depends on l according to $\Lambda \sim l^{-2}$, a large l means small Λ whereas a small l means a large Λ. Here we are going to consider only the small Λ parameterization.

The reference value for defining small and large l is the Planck length l_P. Accordingly, a large l is represented by the condition

$$l \gg l_P\,,$$

which is equivalent to

$$\Lambda\, l_P^2 \ll 1\,.$$

On the other hand, assuming that the Planck length is the smallest possible length in nature, a small l is represented by

$$l \gtrsim l_P,$$

which is equivalent to

$$\Lambda\, l_P^2 \lesssim 1\,.$$

This last expression can be rephrased as

$$\Lambda \lesssim \Lambda_P$$

with $\Lambda_P \sim l_P^{-2}$ the Planck cosmological constant.

2.2 Stereographic coordinates: the small Λ case

In the parameterization appropriate to deal with small values of Λ, the stereographic coordinates are defined by [9]

$$\chi^\mu = \Omega\, x^\mu \tag{2.6}$$

and

$$\chi^4 = -\,l\,\Omega\left(1 + \sigma^2/4l^2\right)\,, \tag{2.7}$$

where $\Omega \equiv \Omega(x)$ is given by

$$\Omega = (1 - \sigma^2/4l^2)^{-1} \tag{2.8}$$

with σ^2 the Lorentz invariant quadratic form $\sigma^2 = \eta_{\mu\nu}\, x^\mu x^\nu$. In these coordinates, the infinitesimal de Sitter quadratic interval

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta \tag{2.9}$$

is found to be the conformally flat metric

$$g_{\alpha\beta} = \Omega^2\, \eta_{\alpha\beta}. \tag{2.10}$$

The Christoffel connection is

$$\Gamma^\lambda{}_{\mu\nu} = \frac{\Omega}{2l^2}\left(\delta^\lambda_\mu\, \eta_{\nu\alpha}\, x^\alpha + \delta^\lambda_\nu\, \eta_{\mu\alpha}\, x^\alpha - \eta_{\mu\nu}\, x^\lambda\right). \tag{2.11}$$

The corresponding Riemann tensor, which is defined by

$$R^\mu{}_{\nu\rho\sigma} = \partial_\sigma \Gamma^\mu{}_{\nu\rho} - \partial_\rho \Gamma^\mu{}_{\nu\sigma} + \Gamma^\mu{}_{\alpha\sigma}\Gamma^\alpha{}_{\nu\rho} - \Gamma^\mu{}_{\alpha\rho}\Gamma^\alpha{}_{\nu\sigma}, \tag{2.12}$$

has the form

$$R^\mu{}_{\nu\rho\sigma} = \frac{1}{l^2}\left(\delta^\mu_\rho\, g_{\nu\sigma} - \delta^\mu_\sigma\, g_{\nu\rho}\right). \tag{2.13}$$

The Ricci tensor and the scalar curvature are, consequently,

$$R_{\nu\sigma} = \frac{3\Omega^2}{l^2}\,\eta_{\nu\sigma} \qquad \text{and} \qquad R = \frac{12}{l^2}\,. \tag{2.14}$$

In terms of the stereographic coordinates $\{x^\mu\}$, the de Sitter generators (2.4) are written in the form

$$L_{\mu\nu} = \eta_{\mu\rho}\, x^\rho\, P_\nu - \eta_{\nu\rho}\, x^\rho\, P_\mu \tag{2.15}$$

and

$$L_{4\mu} = l\, P_\mu - \frac{1}{4l}\, K_\mu\,, \tag{2.16}$$

where

$$P_\mu = \partial_\mu \quad \text{and} \quad K_\mu = \left(2\eta_{\mu\nu} x^\nu x^\rho - \sigma^2 \delta_\mu^\rho\right)\partial_\rho \tag{2.17}$$

are, respectively, the generators of translations and proper conformal transformations [10]. They can be rewritten in the form

$$P_\mu = \delta_\mu^\alpha \partial_\alpha \quad \text{and} \quad K_\mu = \bar{\delta}_\mu^\alpha \partial_\alpha \tag{2.18}$$

where δ_μ^α and $\bar{\delta}_\mu^\alpha = 2\eta_{\mu\nu} x^\nu x^\alpha - \sigma^2 \delta_\mu^\alpha$ are, respectively, the Killing vectors of translations and proper conformal transformations. Generators $L_{\mu\nu}$ refer to the Lorentz subgroup, whereas the elements $L_{4\mu}$ define the transitivity on the homogeneous space. From Eq. (2.16) it follows that the de Sitter spacetime is transitive under a combination of translations and proper conformal transformations—usually called de Sitter "translations". The relative importance of these two transformations is clearly determined by the value of the pseudo-radius l.

In order to study the limit of large values of l, it is necessary to parameterise the generators (2.16) according to

$$\Pi_\mu \equiv \frac{L_{4\mu}}{l} = P_\mu - \frac{1}{4l^2}\, K_\mu\,. \tag{2.19}$$

In terms of these generators, the de Sitter algebra (2.5) assumes the form

$$\left[L_{\mu\nu}, L_{\rho\sigma}\right] = \eta_{\nu\rho}\, L_{\mu\sigma} + \eta_{\mu\sigma}\, L_{\nu\rho} - \eta_{\nu\sigma}\, L_{\mu\rho} - \eta_{\mu\rho}\, L_{\nu\sigma}, \tag{2.20}$$

$$\left[\Pi_\mu, L_{\rho\sigma}\right] = \eta_{\mu\rho}\Pi_\sigma - \eta_{\mu\sigma}\Pi_\rho, \tag{2.21}$$

$$\left[\Pi_\mu, \Pi_\rho\right] = l^{-2} L_{\mu\rho}. \tag{2.22}$$

The last commutator shows that the de Sitter "translation" generators are not really translations, but rotations — hence the quotation marks.

2.3 The contraction limit $\Lambda \to 0$

In the limit $\Lambda \to 0$, which corresponds to $l \to \infty$, we see from Eq. (2.19) that the de Sitter generators Π_μ reduce to generators of ordinary translations

$$\Pi_\mu \to P_\mu. \tag{2.23}$$

Concomitantly, the de Sitter algebra (2.20-2.22) contracts to

$$\left[L_{\mu\nu}, L_{\rho\sigma}\right] = \eta_{\nu\rho}\, L_{\mu\sigma} + \eta_{\mu\sigma}\, L_{\nu\rho} - \eta_{\nu\sigma}\, L_{\mu\rho} - \eta_{\mu\rho}\, L_{\nu\sigma} \tag{2.24}$$

$$\left[P_\mu, L_{\rho\sigma}\right] = \eta_{\mu\rho} P_\sigma - \eta_{\mu\sigma} P_\rho \tag{2.25}$$

$$\left[P_\mu, P_\rho\right] = 0 \tag{2.26}$$

which is the Lie algebra of the Poincaré group $\mathcal{P} = \mathcal{L} \oslash \mathcal{T}$, the semi-direct product of the Lorentz $\mathcal{L}$ and the translation $\mathcal{T}$ groups. As a result of this algebra and group deformations, the de Sitter spacetime dS contracts to the flat Minkowski space M: $dS \rightarrow M$. In fact, as a simple inspection shows, the de Sitter metric (2.10) reduces to the Minkowski metric

$$g_{\mu\nu} \rightarrow \eta_{\mu\nu}, \tag{2.27}$$

and the Riemann, Ricci and scalar curvatures vanish identically:

$$R^{\mu}{}_{\nu\rho\sigma} \rightarrow 0, \quad R_{\nu\sigma} \rightarrow 0, \quad R \rightarrow 0. \tag{2.28}$$

Furthermore, we see from Eq. (2.23) that Minkowski spacetime is transitive under ordinary translations.

3 The de Sitter-Invariant Special Relativity

3.1 Why a new special relativity

Due to the existence of an invariant length at the Planck scale, given by the Planck length, ordinary special relativity fails at that scale. In fact, the Lorentz group is well-known not to allow the existence of an invariant length. Since Lorentz is a subgroup of Poincaré—which is the group that rules the kinematics in ordinary special relativity—the kinematics at Planck scale cannot be described by ordinary special relativity. For this reason, whenever looking for a Planck scale (quantum) kinematics, the first idea that comes to mind is that, to allow the existence of an invariant length, Lorentz symmetry should somehow be broken down. However, as we are going to discuss next, this is not necessarily true.

To begin with, let us recall that Lorentz transformations can be performed only in *homogeneous spacetimes*. In addition to Minkowski, therefore, they can be performed in de Sitter and anti-de Sitter spaces, which are the unique homogeneous spacetimes in $(1+3)$-dimensions [11]. From now on, our interest will be restricted to the de Sitter spacetime. As a homogeneous space, the de Sitter spacetime has constant sectional curvature. Of course, the Ricci scalar is also constant and has the form

$$R = 12\, l^{-2}\,, \tag{3.1}$$

where l is the de Sitter length-parameter, aka pseudo-radius. Now, by definition, Lorentz transformations do not change the curvature of the homogeneous spacetime in which they are performed. As a consequence, Lorentz transformations are found not to change the length parameter l either [12]. Although not immediately visible in Minkowski spacetime, because what is left invariant in this case is an infinite length, in de Sitter spacetime, whose pseudo-radius is finite, this property becomes manifest. Contrary to the usual belief, therefore, Lorentz transformations do leave invariant a very particular length parameter: that defining the scalar curvature of the homogeneous spacetime in which they are performed. If the Planck length l_P is to be invariant under Lorentz transformations, it must represent the pseudo-radius of spacetime at the Planck scale, which will be a de Sitter space with the Planck cosmological term

$$\Lambda_P = \frac{3}{l_P^2} \simeq 1.2 \times 10^{70}\,\mathrm{m}^{-2}\,. \tag{3.2}$$

In this case, the existence of an invariant length-parameter at the Planck scale does not clash with Lorentz invariance, which remains a symmetry at all energy scales.

The de Sitter spacetime is usually interpreted as the simplest *dynamical* solution of the sourceless Einstein equation in the presence of a cosmological constant, standing on an equal footing with all other gravitational solutions, like for example Schwarzschild and Kerr. However, as a non-gravitational spacetime, the de Sitter solution should instead be interpreted as a fundamental background for the construction of physical theories, standing on an equal footing with the Minkowski solution. General relativity, for instance, can be constructed on any one of them. In either case, gravitation will have the same dynamics, only their local kinematics will be different. If the underlying spacetime is Minkowski, which implies a vanishing Λ, the local kinematics will be ruled by the Poincaré group of special relativity. If the underlying spacetime is de Sitter, the local kinematics will be governed by the de Sitter group, which amounts then to replace ordinary special relativity by a de Sitter-invariant special relativity [13, 14].*

3.2 Dispersion relation of the de Sitter special relativity

We proceed now to obtain the dispersion relation of a de Sitter-invariant special relativity. For the sake of completeness, we obtain first the usual dispersion relation of ordinary special relativity.

Ordinary special relativity

The dispersion relation of ordinary special relativity can be obtained from the first Casimir operator of the Poincaré group, which is the kinematic group of Minkowski spacetime. As is well known, it is given by

$$\hat{C}_P = \eta^{\mu\nu} P_\mu P_\nu, \tag{3.3}$$

with $P_\mu = \partial_\mu$ the translation generators. Its eigenvalue c_P, on the other hand, is

$$c_P = -\frac{m^2c^2}{\hbar^2}. \tag{3.4}$$

From the identity $\hat{C}_P = c_P$, we obtain

$$\eta^{\mu\nu} P_\mu P_\nu = -\frac{m^2c^2}{\hbar^2}. \tag{3.5}$$

Applied to a scalar field ϕ, it yields the Klein-Gordon equation

$$\Box\phi + \frac{m^2c^2}{\hbar^2}\phi = 0, \tag{3.6}$$

with $\Box$ the Minkowski d'Alembertian operator.

The dispersion relation of ordinary special relativity can be obtained from (3.5) by replacing

$$P_\mu \to -\frac{i}{\hbar}\, p_\mu \tag{3.7}$$

*It is interesting to observe that, in the same way Einstein special relativity may be interpreted as a generalization of Galilei relativity for velocities near the velocity of light, the de Sitter-invariant special relativity may be interpreted as a generalization of Einstein special relativity for energies near the Planck energy. It holds, for this reason, at any energy scale.

with $p_\mu = mcu_\mu$ the four-momentum of a particle of mass m and four-velocity u_μ. As can be easily verified, the result is

$$\eta_{\mu\nu} p^\mu p^\nu = m^2 c^2, \tag{3.8}$$

which is the dispersion relation of ordinary special relativity.

de Sitter special relativity

For large values of the de Sitter length parameter l, the first Casimir invariant operator of the de Sitter group is written in the form [9]

$$\hat{C}_{dS} = -\frac{1}{2l^2}\,\eta^{AC}\,\eta^{BD}\,J_{AB}\,J_{CD}, \tag{3.9}$$

where

$$J_{AB} = L_{AB} + S_{AB}, \tag{3.10}$$

with L_{AB} the orbital generators (2.4), and S_{AB} the matrix spin generators. For the sake of simplicity, we assume the case of spinless particles, in which case the Casimir operator reduces to

$$\hat{C}_{dS} = -\frac{1}{2l^2}\,\eta^{AC}\,\eta^{BD}\,L_{AB}L_{CD}. \tag{3.11}$$

In terms of stereographic coordinates, it assumes the form

$$\hat{C}_{dS} = \eta^{\alpha\beta}\,\Pi_\alpha\Pi_\beta - \frac{1}{2l^2}\,\eta^{\alpha\beta}\,\eta^{\gamma\delta}\,L_{\alpha\gamma}L_{\beta\delta}. \tag{3.12}$$

with Π_α the generators (2.19). Through a lengthy, but otherwise straightforward computation, one can verify that

$$\hat{C}_{dS} = g^{\mu\nu}\nabla_\mu\nabla_\nu \equiv \Box \tag{3.13}$$

with $\Box$ the Laplace-Beltrami operator in the de Sitter metric (2.10).

On the other hand, for representations of the *principal series*, the eigenvalues of the Casimir operator are given by [15]

$$c_{dS} = -\frac{m^2c^2}{\hbar^2} - \frac{1}{l^2}\,[\mathsf{s}(\mathsf{s}+1)-2]\,, \tag{3.14}$$

with s the spin of the particle under consideration. For the case of spinless particles ($\mathsf{s} = 0$), they assume the form

$$c_{dS} = -\frac{m^2c^2}{\hbar^2} + \frac{2}{l^2}. \tag{3.15}$$

From the identity $\hat{C}_{dS} = c_{dS}$, we obtain

$$g^{\mu\nu}\nabla_\mu\nabla_\nu = -\frac{m^2c^2}{\hbar^2} + \frac{2}{l^2}. \tag{3.16}$$

Applied to a scalar field ϕ, it yields the Klein-Gordon equation in a de Sitter spacetime

$$\Box\phi + \frac{m^2c^2}{\hbar^2}\phi - \frac{1}{6}R\,\phi = 0 \tag{3.17}$$

where $R = 12/l^2$ is the scalar curvature of the de Sitter spacetime [see Eq. (2.14)]. For $m = 0$, it is the conformal invariant equation for a scalar field.

Similarly to ordinary special relativity, the dispersion relation of the de Sitter special relativity can be obtained from (3.16) by replacing

$$\nabla_\mu \to -\frac{i}{\hbar}\,\pi_\mu \tag{3.18}$$

where

$$\pi_\mu = p_\mu - (1/4l^2)\,k_\mu \tag{3.19}$$

is the de Sitter four-momentum of the particle, with p_μ the ordinary four-momentum and k_μ the proper conformal four-momentum [16]. The dispersion relation of de Sitter special relativity is then found to be [17]

$$g_{\mu\nu}\,\pi^\mu\,\pi^\nu = m^2c^2 - \frac{2\hbar^2}{l^2}. \tag{3.20}$$

In the limit $l \to \infty$, the de Sitter spacetime contracts to Minkowski, and it reduces to the dispersion relation (3.8) of ordinary special relativity.

4 The de Sitter-Modified General Relativity

On account of the strong equivalence principle, if special relativity changes, general relativity must change accordingly, giving rise to what can be called *de Sitter-modified general relativity*. In what follows, we give a description of the main properties of this theory.

4.1 The de Sitter-modified Einstein equation

A crucial property of the de Sitter-invariant special relativity is that any physical system induces a local cosmological term Λ in the region of spacetime it occupies, which is necessary to comply with the spacetime kinematics, now governed by the de Sitter group. To clarify the mechanism behind this process, let us note first that, if special relativity changes, general relativity must change accordingly, giving rise to what can be called *de Sitter-modified general relativity*. In this theory, ordinary Einstein equation changes to the de Sitter-modified Einstein equation [18]

$$\mathcal{R}^{\rho\mu} - \frac{1}{2} g^{\rho\mu} \mathcal{R} = -\frac{8\pi G}{c^4} \, \Pi^{\rho\mu} \, . \tag{4.1}$$

The curvature tensor $\mathcal{R}^{\rho\mu}$ in this equation represents both the kinematic curvature of the background de Sitter and the dynamic curvature of general relativity. In ordinary general relativity, any solution to Einstein equation is a spacetime that reduces locally to Minkowski, in agreement to the Poincaré-invariant special relativity. On the other hand, any solution to the de Sitter-modified Einstein equation (4.1) will be a spacetime that reduces locally to de Sitter [19], in agreement to the de Sitter-invariant special relativity.

A crucial point of the field equation (4.1) is that the dynamic curvature of gravitation and the kinematic curvature of the local de Sitter spacetime are both included in the same Riemann tensor $\mathcal{R}^{\rho}{}_{\mu\alpha\beta}$. Differently from the usual Einstein equation, the cosmological term Λ does not appear explicitly in the de Sitter-modified Einstein equation. As a consequence, the (contracted form of the second) Bianchi identity does not restrict it to be constant, which means that Λ is allowed to change in space and time.

In stereographic coordinates, the covariantly conserved current $\Pi^{\rho\mu}$ splits in the form

$$\Pi^{\rho\mu} = T^{\rho\mu} - \frac{1}{4l^2} K^{\rho\mu} \, , \tag{4.2}$$

where $T^{\rho\mu} \equiv \delta^{\rho}_{\alpha} T^{\alpha\mu}$ is the symmetric energy-momentum current, and

$$K^{\rho\mu} \equiv \bar{\delta}^{\rho}_{\alpha} T^{\alpha\mu} = \left(2\eta_{\alpha\beta} x^{\beta} x^{\rho} - \sigma^2 \delta^{\rho}_{\alpha} \right) T^{\alpha\mu} \tag{4.3}$$

is the proper conformal current [20], with, δ^{ρ}_{α} and $\bar{\delta}^{\rho}_{\alpha}$, respectively, the Killing vectors of translations and proper conformal transformations. In these coordinates, therefore, equation (4.1) assumes the form

$$\mathcal{R}^{\rho\mu} - \frac{1}{2} g^{\rho\mu} \mathcal{R} = -\frac{8\pi G}{c^4} \left[T^{\rho\mu} - (1/4l^2) K^{\rho\mu} \right] . \tag{4.4}$$

As a consequence of the source decomposition, which occurs only in stereographic coordinates, the Einstein tensor can also be decomposed in the form

$$\mathcal{R}^{\rho\mu} - \frac{1}{2} g^{\rho\mu} \mathcal{R} = \left(R^{\rho\mu} - \tfrac{1}{2} g^{\rho\mu} R \right) - \left(\hat{R}^{\rho\mu} - \tfrac{1}{2} g^{\rho\mu} \hat{R} \right). \tag{4.5}$$

In this expression, $R^{\rho\mu}$ represents the dynamical curvature of gravitation whereas $\hat{R}^{\rho\mu}$ is the kinematic curvature of the local de Sitter spacetime. The sign *minus* of the decomposition was chosen to be consistent with the sign *minus* in the decomposition of the source current. Substituting in Eq. (4.4), we get

$$\left(R^{\rho\mu} - \tfrac{1}{2} g^{\rho\mu} R \right) - \left(\hat{R}^{\rho\mu} - \tfrac{1}{2} g^{\rho\mu} \hat{R} \right) = -\frac{8\pi G}{c^4} \Big[T^{\rho\mu} - (1/4l^2) K^{\rho\mu} \Big]. \tag{4.6}$$

When written in a spacetime with a general metric $g^{\mu\nu}$, the Ricci and scalar curvatures of the de Sitter background are given by

$$\hat{R}^{\rho\mu} = \Lambda g^{\rho\mu} \qquad \text{and} \qquad \hat{R} = 4\Lambda. \tag{4.7}$$

Since the metric $g^{\rho\mu}$ reduces locally to the de Sitter metric $\hat{g}^{\rho\mu}$, these tensors reduce locally to the Ricci and scalar curvatures of the background de Sitter spacetime. Using these relations, Eq. (4.6) assumes the form

$$R^{\rho\mu} - \tfrac{1}{2} g^{\rho\mu} R + g^{\rho\mu} \Lambda = -\frac{8\pi G}{c^4} \Big[T^{\rho\mu} - (1/4l^2) K^{\rho\mu} \Big] . \tag{4.8}$$

4.2 The local value of Λ

In the de Sitter-modified Einstein's equation (4.8), whereas the energy-momentum current keeps its role of dynamic source of gravitation, the proper conformal current appears as the kinematic source of the background de Sitter spacetime. In other words, the proper conformal current of ordinary matter appears as source of the local cosmological term Λ. Let us explore this property in some more details.

In locally Minkowski spacetimes, the only curvature present is the dynamic curvature of general relativity. In locally de Sitter spacetimes, however, the curvature is composed of both the kinematic curvature of the background de Sitter and the dynamic curvature of general relativity. To isolate the scalar curvature related to the local de Sitter kinematics—which is proportional to the local cosmological term Λ—we multiply both sides of the field equation (4.8) by $g_{\rho\mu}$. The result is

$$R - 4\Lambda = \frac{8\pi G}{c^4} \left[T^{\mu}{}_{\mu} - (1/4l^2) K^{\mu}{}_{\mu} \right] . \tag{4.9}$$

Without loss of generality, we consider now the case in which the source is the electromagnetic field. In this case, the trace $T^\mu{}_\mu$ of the energy-momentum current vanishes, and consequently the corresponding dynamic scalar curvature vanishes as well. Equation (4.9) reduces then to an algebraic equation for the cosmological term Λ:

$$\Lambda = \frac{2\pi G}{c^4} \frac{K^\mu{}_\mu}{4l^2} \,. \tag{4.10}$$

We see from this equation that the source of the local cosmological term Λ is the trace $K^\mu{}_\mu$ of the proper conformal current (4.3) of ordinary matter.

It should be remarked that equation (4.10) is valid, not only for the electromagnetic field, but for any kind of source field. The difference is that, since the gravitational field produced by the electromagnetic field has vanishing scalar curvature, the only contribution to the cosmological term Λ comes from the trace of the proper conformal current of the electromagnetic field. In the cases where the trace of the energy-momentum tensor does not vanish, the scalar curvature will include also a contribution coming from the dynamic curvature of spacetime, as given by ordinary Einstein equation. In either case, we can identify

$$\frac{K^\mu{}_\mu}{4l^2} = \varepsilon_\Lambda \,, \tag{4.11}$$

with ε_Λ the energy density associated to the cosmological term Λ—that is to say, the dark energy density. In this case, relation (4.10) assumes the form

$$\Lambda = \frac{2\pi G}{c^4}\, \varepsilon_\Lambda \,. \tag{4.12}$$

Considering that the dark energy density ε_Λ and the matter energy density ε_m differ by the conformal factor $\bar{\delta}^\rho_\alpha$, defined in Eq. (2.18), for small energies they can be assumed to be of the same order. In the present day universe, for example, they are roughly related by $\varepsilon_\Lambda \simeq 2\varepsilon_m$. Adopting the same relation, equation (4.12) becomes [21]

$$\Lambda = \frac{4\pi G}{c^4}\, \varepsilon_m \,. \tag{4.13}$$

As an application, let us consider the present day universe, for which the space section is known to be nearly flat: $k \simeq 0$. As a consequence, the mean energy density of the universe is of the order of the critical energy density

$$\varepsilon_c = \frac{3H_0^2 c^2}{8\pi G} \simeq 10^{-9}\,\mathrm{Kg\,m^{-1}\,s^{-2}} \,, \tag{4.14}$$

with $H_0 \simeq 75\, km\, s^{-1}\, Mpc^{-1}$ the Hubble parameter. Using this energy density in the relation (4.13), the effective cosmological term is found to be

$$\Lambda \simeq 10^{-52}\, m^{-2} \,, \tag{4.15}$$

which coincides with the order of magnitude obtained from observations [22, 23, 24].

It should be noted that the local cosmological term (4.13) is different from the usual notion in the sense that it is no longer required to be constant [25]. Outside the region occupied by the physical system, where the matter energy density ε_m vanishes, the cosmological term Λ vanishes as well. It is also important to remark that, since Λ is now connected to the local spacetime kinematics, equation (4.13) is a kinematic (algebraic) equation.

4.3 New physics from de Sitter-invariant relativity

The *de Sitter-modified general relativity* gives rise to new gravitational physics only for the case of physical systems that can be described by a continuous medium, or fluid. Since the fluid has an energy density $\varepsilon_m(r)$, it will induce a cosmological term $\Lambda(r)$ in the region of spacetime it occupies, as given by equation (4.13). The presence of this Λ in the whole physical system gives rise to a de Sitter-modified Einstein equation, which may give rise to new gravitational physics in relation to ordinary general relativity. Of course, in the weak field approximations, the ensuing de Sitter-modified Newtonian limit can be used. An example of such fluid system is the universe itself. For this case, the de Sitter-modified FLRW solution in the Newtonian limit has already been shown to yield a reasonable account of the present day universe [26]. Another example is a galaxy, in which the stars can be considered as a fluid with a mass density $\rho(r)$ that decays with the distance to the galactic center. The de Sitter-modified Newtonian limit has already been shown to provide a possible explanation for the flat rotation curve of galaxies, without the necessity of supposing the existence of dark matter [18].

For systems that cannot be described by a continuous medium, on the other hand, the de Sitter-modified *general* relativity gives the same results of ordinary general relativity. This is the case of the solar system: since the mass density in the inter-planetary space vanishes, the cosmological term Λ vanishes in that region as well. As a consequence, the Newtonian limit of the de Sitter-modified general relativity coincides with the usual Newtonian limit of general relativity. Another example is clusters of galaxies, in which the inter-galactic mass density vanishes. As a consequence, the cosmological term Λ vanishes as well, and the Newtonian limit of the de Sitter-modified general relativity coincides with the usual Newtonian limit of general relativity. However, on account of the different local kinematics, the *de Sitter-invariant special relativity* can give rise to mensurable differences in relation to ordinary special relativity.

For example, let us consider the propagation of light. According to the de Sitter-invariant special relativity, the electromagnetic radiation produces a local cosmological term Λ in the region it is located. This means that the light is actually travelling in a local de Sitter spacetime. Let us turn now to case of clusters of galaxies. When measuring the Doppler velocity of a galaxy, one usually assumes that the galaxy is moving in a Minkowski background. However, similarly to the light, a galaxy induces a local cosmological term in the region it is located. This means that the galaxy is also moving in a local de Sitter spacetime. As a consequence, the proper time will change, producing concomitant changes in the computation of the velocity of the galaxy. Since the virial mass of clusters depends on the mean (mass weighted) square velocity of the galaxies relative to the center of mass of the cluster, the virial mass will change accordingly. In what follows we are going to study this kinematic induced change.

5 Relativistic Doppler Velocity of Galaxies

5.1 Proper time and the cosmological term Λ

In this section, we study how the presence of a cosmological term Λ changes the very notion of proper time. For the sake of completeness, we review first the usual notion of proper time in Minkowski spacetime.

5.1.1 Proper time in Minkowski spacetime

Suppose that in a certain inertial frame we observe clocks which are moving relative to us in an arbitrary manner. At each different moment of time this motion can be considered as uniform. Thus, at each moment of time we can introduce a coordinate system rigidly linked to the moving clock, which with the clocks constitute an inertial frame.

In the course of an infinitesimal time interval dt (as read by a clock in the rest frame) the moving clocks travel a distance

$$\sqrt{dx^2 + dy^2 + dz^2} \tag{5.1}$$

in a Minkowski spacetime with Cartesian coordinates $\{x^\mu\}$, in which case the metric has the form

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \,. \tag{5.2}$$

Denoting the proper time by t_0, we ask now what time interval dt_0 is indicated for this period by the moving clocks. In a frame linked to the moving clocks, the latter are at rest: $dx_0 = dy_0 = dz_0 = 0$. Owing to the invariance of intervals, we can write

$$c^2 dt^2 - \left(dx^2 + dy^2 + dz^2\right) = c^2 dt_0^2 \,, \tag{5.3}$$

which is equivalent to

$$dt_0 = \frac{dt}{\gamma} \tag{5.4}$$

where

$$\gamma = \left(1 - \beta^2\right)^{-1/2} \tag{5.5}$$

is the relativistic factor, with $\beta = v/c$ the velocity of the moving clock in units of the speed of light. Integrating (5.4) we obtain

$$t_0 = \frac{t}{\gamma} \,, \tag{5.6}$$

where t_0 is the proper time and t is the observer time.

5.1.2 Proper time in de Sitter spacetime

Let us take the de Sitter metric in inflationary coordinates,

$$ds^2 = c^2 dt^2 - n^2 (\Lambda)\ \delta_{ij}\, dx^i dx^j\ , \tag{5.7}$$

where

$$n(\Lambda) = \exp\left[(\Lambda/3)^{\frac{1}{2}}\ \tau\right] \tag{5.8}$$

is the conformal factor, with τ a length parameter usually assumed to represent the characteristic dimension of the physical system. Using the relation $\Lambda = 3/l^2$, it can be rewritten as

$$n(\Lambda) = \exp\left[\tau/l\right]\ . \tag{5.9}$$

In a frame attached to the moving clock, t_0 will be the proper time, and $dx_0 = dy_0 = dz_0 = 0$. Owing to the invariance of intervals, we can write

$$c^2 dt^2 - n^2(\Lambda)\ \left[dx^2 + dy^2 + dz^2\right] = c^2 dt_0^2\ . \tag{5.10}$$

This expression is equivalent to

$$dt_0 = \frac{dt}{\gamma\ (n)}\ , \tag{5.11}$$

where

$$\gamma(n) = \left[1 - n^2(\Lambda)\, \beta^2\right]^{-1/2} \tag{5.12}$$

is the de Sitter-modified relativistic factor. Integrating (5.11) for a constant Λ , we obtain

$$t_0 = \frac{t}{\gamma(n)}\ . \tag{5.13}$$

5.2 Doppler velocity of galaxies

We review first the usual Doppler effect in Minkowski spacetime, and then we discuss the changes that occur in a de Sitter spacetime.

5.2.1 Doppler velocity in Minkowski spacetime

Instead of a clock, let us consider a galaxy moving in Minkowski spacetime with velocity v in relation to an observer. If the galaxy emits an electromagnetic radiation of wavelength λ , and since this radiation moves with velocity c , we can write

$$\lambda - vt = ct\ . \tag{5.14}$$

Using the identity $\lambda = c/\nu$, with ν the wave frequency, we obtain

$$t = \frac{1}{(1+\beta)\,\nu}\ . \tag{5.15}$$

Now, due to the relativistic time dilation, the observer will measure this time as $t_0 = t/\gamma$. The corresponding observed frequency is

$$\nu_0 = \frac{1}{t_0} = \frac{\gamma}{t} = \left(\frac{1+\beta}{1-\beta}\right)^{1/2} \nu \tag{5.16}$$

where we have used equations (5.6) and (5.15). The ratio

$$\frac{\nu}{\nu_0} = \left(\frac{1-\beta}{1+\beta}\right)^{1/2} \tag{5.17}$$

is called the Doppler factor of the source relative to the observer. Solving this expression for β, we get

$$\beta = \frac{\nu_0^2 - \nu^2}{\nu_0^2 + \nu^2}\,. \tag{5.18}$$

The corresponding galaxy velocities are consequently given by

$$v = \frac{\nu_0^2 - \nu^2}{\nu_0^2 + \nu^2}\, c\,. \tag{5.19}$$

5.2.2 Doppler velocity in de Sitter spacetime

According to the de Sitter-invariant special relativity, a galaxy induces, in the region it is located, a cosmological term Λ proportional to the its mass density, as given by equation (4.13). This means that the galaxy is actually moving in a local de Sitter spacetime. As a consequence, the notion of proper time changes, producing concomitant changes in the galaxy velocity seen from a distant observer. Instead of v, the galaxy velocity turns out to be $n(\Lambda)v$, with $n(\Lambda)$ given by (5.9). In this case, relation (5.14) assumes the form

$$\lambda - n(\Lambda) v t = c t\,. \tag{5.20}$$

Using the identity $\lambda = c/\nu$, with ν the wave frequency, we obtain

$$t = \frac{1}{[1 + n(\Lambda)\beta]\,\nu}\,. \tag{5.21}$$

Now, due to the relativistic time dilation, the observer will measure this time as $t_0 = t/\gamma$. The corresponding observed frequency is

$$\nu_0 = \frac{1}{t_0} = \frac{\gamma(n)}{t} = \left[\frac{1 + n(\Lambda)\beta}{1 - n(\Lambda)\beta}\right]^{1/2} \nu \tag{5.22}$$

where we have used equations (5.13) and (5.21). The Doppler factor has now the form

$$\frac{\nu}{\nu_0} = \left[\frac{1 - n(\Lambda)\beta}{1 + n(\Lambda)\beta}\right]^{1/2}. \tag{5.23}$$

Solving this expression for $n(\Lambda)\beta$, we get

$$n(\Lambda)\beta = \frac{\nu_0^2 - \nu^2}{\nu_0^2 + \nu^2}\,. \tag{5.24}$$

The corresponding galaxy velocities are consequently given by

$$v = \frac{\nu_0^2 - \nu^2}{\nu_0^2 + \nu^2}\,\frac{c}{n(\Lambda)}\,. \tag{5.25}$$

Denoting the Minkowski and the de Sitter Doppler velocities respectively by v_m and v_{ds}, a comparison of (5.25) with (5.19) yields the relation

$$v_{ds} = \frac{v_m}{n(\Lambda)} \, . \tag{5.26}$$

Since $n(\Lambda) \geqslant 1$, the velocity v_{ds} of a galaxy moving in a de Sitter spacetime will be smaller than the velocity v_m of a galaxy moving in a Minkowski spacetime.

6 Mass of Cluster of Galaxies

6.1 Virial theorem

The virial theorem establishes a relationship between the time average kinetic energy and the time average potential energy of a classical system in dynamic equilibrium. To derive this relationship we will consider a gravitationally bound system of n point masses, the equation of motion of the n^{th} mass is given by

$$m_n \ddot{x}_n^i = -\frac{\partial V}{\partial x_n^i} \,. \tag{6.1}$$

Where m_n is the mass of the n^{th} particle, $\vec{x}_n$ is its position vector with relation to the center of mass of the system and V the potential energy of the system. Remembering that the potential energy of the system is equal to the sum

$$V = -\frac{1}{2}\sum_{l=0}^{n}\sum_{p=0}^{n}\frac{Gm_p m_l}{\left|\vec{x}_p - \vec{x}_l\right|} \,, \tag{6.2}$$

to $p \neq l$, where G is the gravitational constant. And the kinetic energy T of the system, disregarding the movement of the center of mass, is given by

$$T = \frac{1}{2}\sum_{l=0}^{n} m_l \left(\dot{\vec{x}}_l \cdot \dot{\vec{x}}_l\right) \,. \tag{6.3}$$

Now, multiplying the equation (6.1) by x_l^i and summing all the terms, we can write

$$-\sum_{l=0}^{n} x_l^i \frac{\partial V}{\partial x_l^i} = \frac{1}{2}\frac{d^2}{dt^2}\sum_{l=0}^{n} m_l \left(\vec{x}_l \cdot \vec{x}_l\right) - \sum_{l=0}^{n} m_l \left(\dot{\vec{x}}_l \cdot \dot{\vec{x}}_l\right) \,. \tag{6.4}$$

The second term on the right side of the above equation gives twice the kinetic energy of the system and the first can be written in terms of the moment of inertia $I = \sum_{l=0}^{n} m_l \left(\vec{x}_l \cdot \vec{x}_l\right)$ of system. The moment of inertia should not vary over time as the system is in dynamic equilibrium nullifying the term

$$\frac{d^2}{dt^2}\sum_{l=0}^{n} m_l \left(\vec{x}_l \cdot \vec{x}_l\right) = 0 \,. \tag{6.5}$$

In turn the left side of the equation reduces the potential energy of the system due to the shape of (6.2). Thus, the equation (6.4) results in the virial theorem

$$2T + V = 0 \,. \tag{6.6}$$

6.2 Virial mass of cluster of galaxies in Minkowski

Galaxy clusters are structures formed by gravitationally linked galaxies, such galaxies have non-relativistic velocities. The virial theorem can be applied to estimate the total mass of a galaxy cluster as long as the cluster is in a configuration where the condition (6.5) is valid. For this we will consider the time average values of the kinetic energy and the potential energy of the cluster. The time average kinetic energy of the cluster is given by

$$T = \frac{1}{2}M\left\langle v^2\right\rangle \ , \tag{6.7}$$

where $M = \sum_{l=0}^{n} m_l$ is the total mass of the cluster and $\left\langle v^2\right\rangle$ is the average square velocity of galaxies relative to the center of mass of the cluster. Now the average potential energy in the cluster time is equal to

$$V = -\frac{1}{2}GM^2\left\langle 1/r\right\rangle \ , \tag{6.8}$$

where $\langle 1/r\rangle$ is the average inverse separation of time between galaxies. Substituting the equations (6.7) and (6.8) in the virial theorem (6.6), and isolating M we find the total mass of the cluster

$$M = \frac{2\left\langle v^2\right\rangle}{G\left\langle 1/r\right\rangle}\,, \tag{6.9}$$

whose average values $\left\langle v^2\right\rangle$ and $\langle 1/r\rangle$ are known.

6.3 The de Sitter-modified virial mass of cluster of galaxies

If the background spacetime is assumed to be Minkowski, the mass of a cluster of galaxies is given by [3]

$$M_m = \frac{2\left\langle v_m^2\right\rangle}{G\left\langle 1/r\right\rangle}\,, \tag{6.10}$$

with the Doppler velocities of the galaxies v_m given by equation (5.19). On the other hand, according to the de Sitter-invariant special relativity, each individual galaxy of a cluster moves in a local de Sitter spacetime. This means that the mass of the cluster in this case is written as

$$M_{ds} = \frac{2\left\langle v_{ds}^2\right\rangle}{G\left\langle 1/r\right\rangle}\,, \tag{6.11}$$

with the Doppler velocities of the galaxies v_{dS} given by equation (5.25). Using the identity (5.26), the relation between the two masses is found to be

$$M_{ds} = \frac{M_m}{n^2(\Lambda)}\,, \tag{6.12}$$

where $n(\Lambda)$ is given by equation (5.8). Since $n(\Lambda) \geqslant 1$, the de Sitter-modified mass M_{ds} of the cluster will be smaller than the usual mass M_m obtained in the case of a Minkowski background.

7 Conclusions

As discussed in Section 1, the virial mass-to-light ratio M/L of clusters of galaxies, and in particular of the Coma cluster, is much larger than the mass-to-light ratio of the visible regions of individual galaxies. Estimates of M/L have generally given results of the order $10^2\, h\, M_\odot/L_\odot$. This discrepancy between the gravitational and the luminous masses is usually attributed to the presence of dark matter in the clusters, which is not taken into account in the mass obtained from luminosity. However, as we have just seen, this is not necessarily true. To see that, let us first recall that the bulge of a typical spiral galaxy has a slow decaying mass density ρ . In the Keplerian region, on the other hand, it decays faster, vanishing outside the border of the galaxy. Owing to this property, the whole mass of the galaxy can be assumed to be in the bulge [2]. Considering a bulge average mass density of the order [27]

$$\rho_0 \simeq 2 \times 10^{-13}\ \mathrm{Kg\, m^{-3}}\,, \tag{7.1}$$

the corresponding cosmological term, according to equation (4.13), is

$$\Lambda_0 = \frac{4\pi G}{c^2}\rho_0 \simeq 1.8 \times 10^{-39}\ \mathrm{m}^{-2}\,. \tag{7.2}$$

Assuming furthermore that τ is of the order of the diameter of the galactic bulge,

$$\tau \simeq 6\ \mathrm{Kpc} \simeq 10^{20}\ \mathrm{m}\,, \tag{7.3}$$

the expansion factor (5.8) becomes

$$n^2(\Lambda) \simeq 1.4 \times 10^2\,. \tag{7.4}$$

As a consequence, the de Sitter modified square velocity of galaxies is two orders of magnitude smaller than the usual Minkowskian result:

$$v_{ds}^2 \simeq 10^{-2}\, v_m^2\,. \tag{7.5}$$

Accordingly, the de Sitter modified virial mass of clusters of galaxies will also be two orders of magnitude smaller than the usual Minkowskian result:

$$M_{ds} \simeq 10^{-2}\, M_m\,. \tag{7.6}$$

In this case, the de Sitter modified virial mass will agree with the mass obtained from luminosity, and no dark matter would be necessary to explain the dynamics of the galaxy cluster. Considering that the existence of dark matter, as well as its nature, are as yet an unsolved problems of cosmology, the alternative solution proposed here, which is based on a modified gravity theory fully obtained from first principles, can be considered a viable approach for the missing mass problem.

Finally, it should be remarked that, due to the presence of an exponential function in the definition (5.8) of the expansion factor $n(\Lambda)$, a small change in the mass density (7.1), even keeping the order of magnitude, would change the numerical coefficient of relation (7.6) substantially. This means that fine-tuning is necessary to achieve the desired result. Conversely, we can resort to the notion of naturalness and argue that there are physically acceptable values for ρ_0 and τ that yields the physically relevant results.

Bibliography

[1] F. Zwicky, *Die Rotverschieb ung von extragalaktischen Nebeln*, Helv. Phys. Acta **6**, 110 (1933). For an English version, see F. Zwicky, *The redshift of extragalactic nebulae*, Gen. Rel. Grav. **41**, 207 (2009).

[2] J. Binney and S. Tremaine, *Galactic Dynamics* (Princeton University Press, Princeton, 1987).

[3] S. Weinberg, *Cosmology* (Oxford University Press, Oxford, 2008), Section 1.9.

[4] R. G. Carlberg, H. K. C. Yee, and E. Ellingson, *The average mass and light profiles of galaxy clusters*, Astrophys. J. **478**, 462 (1997); arXiv:astro-ph/9512087.

[5] H. Andernach, M. Plionis, O. López-Cruz, E. Tago, and S. Basilakos, *The Cluster M/L Ratio and the Value of* $\Omega_{\rm m}$, Astron. Soc. Pacific Conf. Ser. **329**, 289 (2005); arXiv:astro-ph/0407098.

[6] V. C. Rubin, N. Thonnard and W. K. Ford Jr, *Rotational Properties of 21 SC Galaxies with a Large Range of Luminosities and Radii, From NGC 4605 (R=4kpc) to UGC 2885 (R=122kpc)*, Astrophys. J. **238**, 471 (1980).

[7] V. C. Rubin, D. Burstein, W. K. Ford Jr, and N. Thonnard, *Rotation Velocities of 16 SA Galaxies and a Comparison of Sa, Sb, and SC Rotation Properties*, The Astrophysical Journal **289**, 81 (1985).

[8] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, 1973.

[9] F. Gürsey, in F. Gürsey (editor), *Group Theoretical Concepts and Methods in Elementary Particle Physics*, Istanbul Summer School of Theoretical Physics (Gordon and Breach, New York, 1964).

[10] S. Coleman, *Aspects of Symmetry* (Cambridge University Press, Cambridge, 1985).

[11] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).

[12] A. Araujo, H. Jennen, J. G. Pereira, A. C. Sampson and L. L. Savi, *On the spacetime connecting two aeons in conformal cyclic cosmology*, Gen. Rel. Grav. **47**, 151 (2015); arXiv:1503.05005.

[13] R. Aldrovandi, J. P. Beltrán Almeida and J. G. Pereira, *de Sitter special relativity*, Class. Quantum Grav. **24**, 1385 (2007); arXiv:gr-qc/0606122.

[14] S. Cacciatori, V. Gorini and A. Kamenshchik, *Special relativity in the 21*st *century*, Ann. Phys. (Berlin) **17**, 728 (2008); arXiv:gr-qc/0807.3009.

[15] J. Dixmier, *Représentations intégrables du groupe de de Sitter* Bul. Soc. Math. Franc. **89**, no. 9 (1961).

[16] J. G. Pereira and A. C. Sampson, *de Sitter geodesics: reappraising the notion of motion*, Gen. Rel. Grav. **44**, 1299 (2012); arXiv:1110.0965.

[17] R. Aldrovandi and J. G. Pereira, *An Introduction to Geometrical Physics*, second edition (World Scientific, Singapore, 2016), Chapter 37.

[18] J. G. Pereira and D. F. López, *An Improved Framework for Quantum Gravity*, Universe **6**, 243 (2020); arXiv:2012.09075.

[19] R. Sharpe, *Differential Geometry: Cartan's Generalization of Klein's Erlangen Program* (Springer, Berlin, 1997).

[20] C. G. Callan, S. Coleman and R. Jackiw, *A new improved energy-momentum tensor*, Ann. Phys. (NY) **59**, 42 (1970).

[21] R. Aldrovandi and J. G. Pereira, *Is Physics Asking for a New Kinematics?* Int. J. Mod. Phys. D **17**, 2485 (2008); arXiv:0805.2584.

[22] A. G. Riess *et al*, *Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant*, Ap. J. **116**, 1009 (1998); arXiv:astro-ph/9805201.

[23] S. Perlmutter *et al*, *Measurements of* Ω *and* Λ *from 42 High-Redshift Supernovae*, Ap. J. **517**, 565 (1999); arXiv:astro-ph/9812133.

[24] P. de Bernardis *et al*, *A Flat Universe from High-Resolution Maps of the Cosmic Microwave Background Radiation*, Nature **404**, 955 (2000); arXiv:astro-ph/0004404.

[25] H. Jennen, *Cartan geometry of spacetimes with non-constant cosmological function* Λ, Phys. Rev. D **90**, 084046 (2014); arXiv:1406.2621.

[26] A. Araujo, D. F. López and J. G. Pereira, *de Sitter invariant special relativity and the dark energy problem*, Class. Quantum Grav. **34**, 115014 (2017); arXiv:1704.02120.

[27] Paul J. McMillan, *The mass distribution and gravitational potential of the Milky Way*, MNRAS **465**, 76 (2017); arXiv:1608.00971.