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Analysis of flow harmonics and particle correlations in relativistic heavy ion collisions

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Analysis of flow harmonics and particle correlations in relativistic heavy ion collisions

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RESUMO

A descrição hidrodinâmica das colisões de íons pesados relativísticos desempenha um papel fundamental na compreensão das propriedades da matéria QCD. A avaliação do fluxo coletivo tem uma importância significativa na área de pesquisa pertinente. Em particular, vários fatores específicos podem causar incertezas na medição. Entre outros, a multiplicidade de eventos apresenta flutuações significativas, enquanto a excentricidade inicial também mostra um comportamento flutuante. A presente tese tem como principal objetivo estudar os assuntos relevantes. Primeiramente, focamos no impacto das flutuações de multiplicidade nas correlações entre pares de hádrons e investigamos métodos eficazes para suprimir essas flutuações durante as medições de correlação de partículas. Primeiramente, focamos no impacto das flutuações de multiplicidade nas correlações entre pares de hádrons e investigamos métodos eficazes para suprimir essas flutuações durante as medições de correlação de partículas.

Quando as flutuações de multiplicidade são suficientemente reduzidas, simulações de Monte Carlo revelam características distintas escondidas dentro das flutuações que seriam esmagadoras de outra forma. Além disso, propomos que este esquema de normalização se alinhe naturalmente com os correladores de múltiplas partículas, que podem ser avaliados usando q-vetores. Notavelmente, mesmo que a função de distribuição de probabilidade das condições iniciais seja isotrópica, flutuações estatísticas no perfil de energia ainda resultam em um ϵ anisotrópico diferente de zero. Por outro lado, harmônicos de fluxo derivados de espectros de hádrons usando correladores de múltiplas partículas exibem inevitavelmente uma variância não negligenciável devido à estatística limitada. Uma vez que seu objetivo é extrair informações sobre flutuações nas condições iniciais através de harmônicos de fluxo e suas flutuações, flutuações devidas à multiplicidade finita podem desempenhar um papel na interferência desses esforços. A este respeito, exploramos a natureza e o impacto de tais flutuações nos estados inicial e final, ambos parecem ser flutuações estatísticas originadas de um número finito de números quânticos no sistema subjacente. Além disso, simulações numéricas foram realizadas. Por fim, exploramos a possibilidade de usar estimadores de máxima verossimilhança (MLE) para medir harmônicos de fluxo. Para uma multiplicidade finita dada, o MLE pode estimar simultaneamente todos os parâmetros desconhecidos da camada subjacente e mitigar eficazmente a variância da medição. Além disso, o método fornece uma maneira de avaliar uma classe específica de harmônicos híbridos, o que não é diretamente viável com métodos baseados principalmente em correlações de partículas. Os resultados foram analisados usando o método de Wald, razões de verossimilhança e testes fracionários de hipóteses. Além disso, foi feita uma comparação entre os harmônicos de fluxo obtidos pelo MLE e aqueles obtidos pelos métodos de correlação de partículas e plano de evento. A dependência dos harmônicos de fluxo extraídos na multiplicidade de eventos individuais e no número total de eventos foi analisada. Os resultados mostram que este método pode resolver eficazmente a limitação da aceitabilidade do detector. Além disso, elaboramos um cenário fictício em que o plano do evento não é uma quantidade bem definida na função de distribuição. Para o último caso, o MLE tem um desempenho melhor do que os estimadores de correlação de duas partículas. Concluímos que o MLE

oferece uma alternativa significativa aos métodos de análise de fluxo existentes.

PALAVRAS-CHAVE: hidrodinâmica relativística. fluxo coletivo. multiplicidade. flutuações estatísticas.

ABSTRACT

The hydrodynamic description of relativistic heavy ion collisions plays a pivotal role in comprehending the properties of QCD matter. The assessment of collective flow holds significant importance in the pertinent research area. In particular, numerous specific factors potentially cause uncertainties in the measurement. Among others, the multiplicity of events exhibits significant fluctuations, while the initial eccentricity also displays fluctuating behavior. The present thesis primarily aims to study the relevant subjects. Firstly, we focus on the impact of multiplicity fluctuations on correlations between hadron pairs and investigate effective methods to suppress these fluctuations during particle correlation measurements. Specifically, by employing appropriate normalization techniques, we can evaluate particle correlations in a manner that is independent of the event multiplicity. When multiplicity fluctuations are sufficiently reduced, Monte Carlo simulations reveal distinct features hidden within the otherwise overwhelming fluctuations. Moreover, we propose that this normalization scheme naturally aligns with multi-particle correlators, which can be assessed using q -vectors. Furthermore, the event-by-event fluctuations have significant implications in relativistic heavy ion collisions. Notably, even though the probability distribution function of the initial conditions is isotropic, statistical fluctuations in the energy profile still result in a non-zero anisotropic ϵ . On the other hand, flow harmonics derived from hadron spectra using multi-particle correlators unavoidably exhibit non-negligible variance due to limited statistics. Since its goal is to extract information about fluctuations in initial conditions through flow harmonics and their fluctuations, fluctuations owing to finite multiplicity may play a role in interfering with such efforts. In this regard, we explore the nature and impact of such fluctuations in the initial and final states, both of which appear to be statistical fluctuations originating from a finite number of quantum numbers in the underlying system. We provide a detailed analysis of initial state eccentricities' characteristics for smooth and event-wise fluctuating initial conditions, elucidating their distinct influence on the resulting flow harmonics. Additionally, numerical simulations were conducted. Lastly, We explored the possibility of using maximum likelihood estimators (MLE) to measure flow harmonics. For a given finite multiplicity, the MLE can simultaneously estimate all unknown parameters of the underlying layer and effectively mitigate measurement variance. Furthermore, the method provides a way to evaluate a specific class of hybrid harmonics, which is not directly feasible with methods mainly based on particle correlations. Results were analyzed using the Wald method, likelihood ratios, and fractional tests of hypotheses. Additionally, a comparison was made between the flow harmonics obtained by MLE and those obtained by the particle correlation and event plane methods. The dependence of the extracted flow harmonics on the multiplicity of individual events and the total number of events was analyzed. The results show that this method can effectively solve the shortcoming of detector acceptability. Furthermore, we elaborate on a fictional scenario where the event plane is not a well-defined quantity in the distribution function. For the latter case, MLE performs better than two-particle correlation estimators. We conclude that MLE offers a meaningful alternative to existing flow analysis methods.

KEYWORDS: relativistic hydrodynamic. collective flow. multiplicity. statistic fluctuations.

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LISTA DE ABREVIATURAS E SIGLAS

UNESP	Universidade Estadual Paulista
MLE	Maximum likelihood Estimator
QGP	Quark-Gluon Plasma
SPH	Smoothed Particle Hydrodynamics
MC	Monte Carlo
EOS	Equation of state
IC	Initial conditions
NeXuS	Initial condition for quark-gluon plasma evolution
SPheRIO	Smoothed Particle hydrodynamical evolution of Relativistic heavy-ION collisions
EPOS	phenomenological approach
RHIC	Relativistic Heavy Ion Collider
LHC	Large Hadron Collider
CLT	Central limit theorem

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1 INTRODUCTION

According to the Standard Model of particle physics, quarks and gluons are fundamental degrees of freedom that exhibit strong interactions. Quarks carry color charge and interact through the exchange of gluons. The Standard Model of particles has unraveled the mysteries surrounding the fundamental building blocks and interactions of matter. However, as we progress further, we venture into extreme conditions and delve deeper into the microscopic world to uncover more profound truths (COTTINGHAM; GREENWOOD, 2007). Relativistic heavy ion collisions have emerged as a valuable avenue for investigating the extreme states of matter in the early universe, with particular emphasis on exploring the properties of Quark-Gluon Plasma (QGP), presenting an intriguing scientific endeavor.

Relativistic hydrodynamics is a viable theoretical framework for describing the temporal evolution of strongly coupled quark-gluon plasmas generated in relativistic heavy ion collisions (ROMATSCHE, 2010; GALE; JEON; SCHENKE, 2013; HEINZ; SNELLINGS, 2013; HIRANO et al., 2013; KODAMA; STOCKER; XU, 2014; SOUZA; KOIDE; KODAMA, 2016; FLORKOWSKI; HELLER; SPALINSKI, 2018). Plasmas are commonly treated as a continuum in macroscopic models, which is pivotal for comprehending the underlying physics that govern empirical observations.

The primary objective of relativistic hydrodynamics is to elucidate the particle spectrum characterized by low and medium transverse momentum, with particular emphasis on their collective attributes such as flow harmonics and correlations (OLLITRAULT, 1992; VOLOSHIN; ZHANG, 1996; OLLITRAULT, 1998; BORGHINI; DINH; OLLITRAULT, 2001a; TAKAHASHI et al., 2009; ANDRADE et al., 2012; LUZUM, 2011). From an experimental perspective, the measured azimuthal distribution plays a pivotal role in establishing a comprehensive understanding of an "ideal" liquid, initially introduced at RHIC (ACKERMANN et al., 2001). Consequently, the analysis of azimuthal anisotropy resulting from nuclear collisions has emerged as a prominent avenue for extracting crucial insights into the fundamental characteristics of the underlying physical system (ARSENE et al., 2005; BACK et al., 2005; ADAMS et al., 2005b; AAD et al., 2012; CHATRCHYAN et al., 2013a; CHATRCHYAN et al., 2012).

The anisotropic distribution of particles in momentum space is characterized by the flow harmonic v_n . Specifically, we have the following single particle distribution function (VOLOSHIN; ZHANG, 1996; VOLOSHIN; POSKANZER; SNELLINGS, 2010)

$$f_1(\phi) = \frac{1}{2\pi} \left[1 + \sum_{n=1} 2v_n \cos n(\phi - \Psi_n) \right], \quad (1.1)$$

where ϕ represents the azimuth angle of the emitted particles, and the reference direction Ψ_n for a given order n is called the event plane. Elliptical flow v_2 and triangular flow v_3 explain many important features of the observation system compared to other flows. In particular, v_2 is mainly due to the geometric almond shape of the initial system (OLLITRAULT, 1992), while v_3 is due to event-by-event fluctuations in the initial conditions (ALVER; ROLAND, 2010). In the literature,

extensive investigations have been conducted to examine the correlation between initial geometric anisotropy and final flow harmonics, particularly focusing on potential deviations from linearization, eccentricity, flow fluctuations, and their interrelationships. In practice, numerous numerical methods have been developed to ascertain the values of flow harmonics v_n from experimental data. The conventional event plane method (VOLOSHIN; ZHANG, 1996) aims to estimate the event plane Ψ_n in Eq. (1.1) and subsequently evaluate the flow harmonics. The primary characteristic of this method is closely associated with the inherent inability to directly experimentally measure the reaction plane. Many other methods are mainly based on the correlation of particles and the concepts of q-vectors and cumulants (DANIELEWICZ; ODYNEC, 1985; BORGHINI; DINH; OLLITRAULT, 2001a; BILANDZIC; SNELLINGS; VOLOSHIN, 2011). The utilization of particle correlations offers the advantage of canceling out event planes within the formalism equation. Furthermore, cumulants can be expressed concisely using generating functions. It is worth noting that the sign of the multi-particle cumulants can exhibit harmonics of varying orders, as extensively discussed in several pertinent studies. Such methods include particle cumulants, Lee-Yang zero (BHALERAO; BORGHINI; OLLITRAULT, 2003; BHALERAO; BORGHINI; OLLITRAULT, 2004; BASTID et al., 2005) and symmetric cumulants (BILANDZIC et al., 2014) (BORGHINI; DINH; OLLITRAULT, 2001a; BORGHINI; DINH; OLLITRAULT, 2001c; BILANDZIC; SNELLINGS; VOLOSHIN, 2011), and more recent generalizations (BHALERAO; OLLITRAULT; PAL, 2013; FRANCESCO et al., 2017; MORDASINI et al., 2020).

However, the multiplicity fluctuations in the experiment bring challenges to the measurement of harmonic coefficients (BOZEK; MEHRABPOUR, 2020; JIA; RADHAKRISHNAN; ZHOU, 2016). Multiplicity fluctuations are due to the probabilistic nature of particle production, which can lead to differences in harmonic coefficients between different events. Therefore, in harmonic measurements, we must carefully consider the effect of multiplicity fluctuations on the results to obtain accurate physical information. In addition, statistical fluctuations can also affect the measured harmonic coefficients due to limited particle samples. Such fluctuations may lead to an increase in the uncertainty of the measurement results, so how to accurately estimate the impact of fluctuations needs to be considered in the analysis.

There are two essential aspects of flow analysis. The first one is non-flow, a generic term used to refer to the collective behavior that cannot be attributed to independent emission based on the single particle distribution function Eq. (1.1). For instance, momentum conservation leads to deviation (CHAJECKI; LISA, 2009) in particle correlation compared to the case where the particle spectrum is entirely governed by the flow. Generally, it is understood that the impact due to non-flow becomes more insignificant as one considers particle correlation involving more hadrons (BORGHINI; DINH; OLLITRAULT, 2001c; BORGHINI; DINH; OLLITRAULT, 2001b) or introduces a rapidity gap among the particles (ADLER et al., 2003; VOLOSHIN, 2006a; VOLOSHIN, 2006b). The second one is related to the finite multiplicity in individual events. As a matter of fact, even though the hadrons are emanated independently according to the one-particle distribution function, the second factor leads to a certain degree of statistical uncertainty. Such uncertainty is distinct from those caused by the fluctuations in the initial conditions, which typically possess some physical origin regarding the

underlying microscopic model. To be precise, from a hydrodynamic perspective, the latter is manifestly by the fluctuations in the geometry of the initial energy distribution on an event-by-event basis. In particular, the statistical error of the particle correlation method might even be larger than that of the event plane method (VOLOSHIN; POSKANZER; SNELLINGS, 2010). Besides, for events with lower multiplicity, the multi-particle correlation is likely constrained by more significant statistical uncertainty.

In practice, even though the accessible number of events is significant, for an individual event, the multiplicity is finite. When averaging over the events, the variance of the extracted harmonic coefficients remains finite even though the particles emanated independently according to a well-defined one-particle distribution function. Such a degree of statistical uncertainty will present itself as flow fluctuations. In practice, it might become indistinguishable from those caused by the event-by-event fluctuations in the initial conditions discussed above. The latter is understood to reflect the physics of the underlying microscopic model, different from its counterpart of statistical origin.

With the deepening of research, we began to consider how to deal with the influence of multiplicity fluctuation and statistical fluctuation on harmonic measurement. In this context, Maximum Likelihood Estimator (MLE) becomes a possible solution. MLE is a statistical method based on the best fit to a probability distribution that can help us measure harmonic coefficients more precisely and thus improve our knowledge of the nature of QGP.

MLE is a commonly used statistical method for estimating unknown parameter values from known observational data to maximize the probability of occurrence of observational data. MLE has a wide range of applications in a variety of fields, including statistics, machine learning, physics, and more. The core idea of MLE is to estimate the unknown parameters in the model by finding the parameter value that makes the observed data appear with the highest probability under the given model. MLE plays an important role in statistical inference, and it can be used to estimate various types of parameters, such as mean, variance, probability distribution parameters, etc. In machine learning, MLE is often used to train models, and adjust model parameters by maximizing the likelihood function of observed data to make the model better fit the actual data. In physics, MLE can be used to fit experimental data, estimate physical quantities, and even measure harmonic coefficients in particle physics experiments. While MLE is a powerful estimation method in many situations, it has some limitations and caveats. For example, when the sample data is small or the model complexity is high, overfitting problems may occur. In addition, the results of MLE may be affected by the initial parameter values, so the initial parameter values need to be carefully selected in practical applications to avoid falling into a local optimal solution.

2 RELATIVISTIC HEAVY ION COLLISIONS

The field of relativistic fluid mechanics investigates the principles governing the motion of fluids in the presence of significant relativistic effects. This typically encompasses scenarios characterized by high energy density, rapid velocities, intense gravitational forces, or powerful magnetic fields. QGP is a fundamental component within the framework of the Standard Model (RISCHKE; PüRSÜN; MARUHN, 1995). In this state, quarks and gluons undergo deconfinement, transitioning from being bound within the nucleus to forming a liberated plasma. The QGP is widely acknowledged as the primordial state of the universe (TAWFIK et al., 2011) immediately following the Big Bang, and its existence has been experimentally observed in high-energy physics investigations.

Relativistic fluid mechanics is very closely related to QGP. In high-energy heavy-ion collision experiments, the produced QGP typically moved at relativistic velocities (NONAKA, 2015). Consequently, its dynamic behavior is best described using relativistic hydrodynamics. For example, the expansion, cooling, and Hadronization (transition from QGP to Hadron matter) processes of QGP can be simulated and understood by equations and models of relativistic fluid dynamics. In recent years, relativistic hydrodynamics has become an important tool for studying the properties and dynamic behavior of QGP. Numerical simulation methods, particularly numerical codes for relativistic hydrodynamics such as MUSIC (SCHENKE; JEON; GALE, 2010) and SPHERIO codes, have been extensively employed to simulate the evolution of QGP generated by heavy ion collisions.

Subsequently, a concise introduction will be provided on the physical variables employed within the framework of relativity theory, which greatly facilitates the investigation of relative fluid nuclear physics. Additionally, an overview will be given on the fluid mechanics model and its associated processes, along with details regarding our utilization of a fluid mechanics code.

2.1 BASIC VARIABLE

Relativistic heavy-ion collisions refer to collisions between high-energy heavy ions. The "relativistic" here means that the particle speed during the collision is close to the speed of light. In this case, Newtonian mechanics is no longer applicable, and special relativity needs to be used. It is a theory of physics that can describe high-speed particles. When the speed of an object is close to the speed of light, its mass, length and time will undergo relativistic changes, and these changes can be described by the Lorentz transformation in special relativity. In heavy ion collisions, since the particles travel very close to the speed of light, their mass increases, while length and time contract and expand. These changes have important effects on both the dynamics of the collision and the properties of the resulting particles, so special relativity must be used to properly describe and understand this process. In addition, special relativity also includes the effects of electromagnetic interactions, which are essential for describing charged Collisions between heavy ions are very important. The special theory of relativity is necessary because it can correctly describe the dynamic process, mass energy change, space time effect and electromagnetic interaction in heavy ion collisions.

2.1.1 Basic Conventions and Notation

In special relativity, the concepts of spacetime and velocity are very different from what we understand in classical physics (EINSTEIN, 1922). In classical physics, time and space are two separate concepts. But in special relativity, time and space are unified into a four-dimensional spacetime. This four-dimensional space-time consists of three spatial dimensions and one time dimension, usually denoted as (ct, x, y, z) , where c is the speed of light. The space-time coordinates of a point x are represented by an inverse vector whose component is x :

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, \mathbf{x}) = (ct, x, y, z), \quad (2.1)$$

Here we will use the space-time metrix tensor $g_{\mu\nu}$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (2.2)$$

The covariant vector x_ν is related to the contravariant vector x^μ , transformed by the metric tensor $g_{\mu\nu}$

$$x_\nu = (x_0, x_1, x_2, x_3) = g_{\mu\nu}x^\mu = (ct, -x, -y, -z), \quad (2.3)$$

We can choose different reference frames to describe an event, so the same event may have different four coordinates in different reference frames. But that doesn't matter, the principle of relativity states that the laws of physics take the same form in all frames of reference. Therefore, we can use the Lorentz transformation to describe in different reference systems. The Lorentz transformation is a mathematical transformation used in the theory of relativity to convert physical quantities between different inertial systems. Consider two inertial frames S and S' , with S' moving relative to S along the x -axis with a constant velocity v . The mathematical expression of the Lorentz transformation is

$$ct' = \gamma(ct - \beta x), \quad (2.4)$$

$$x' = \gamma(x - \beta ct), \quad (2.5)$$

$$y' = y, \quad (2.6)$$

$$z' = z. \quad (2.7)$$

where ct, x, y, z is the coordinate of the event in the S system. And ct', x', y', z' is the coordinate of the event in the S' system. v is the velocity of S' relative to the S system. $\beta = v/c$, γ is the Lorentz factor.

Since the four-speed is the derivative of the four-coordinate to the intrinsic time (or called the intrinsic time of the object) τ , the relationship between the four-speed and the four-coordinate can be expressed by Lorentz transformation

$$U^\mu = \frac{dx^\mu}{d\tau} = \gamma(c, \vec{v}). \quad (2.8)$$

Assume that in the S' frame of reference, the four-velocity of the object is

$$U'^{\mu} = (\gamma'c, \gamma'v'_x, \gamma'v'_y, \gamma'v'_z), \quad (2.9)$$

Then we can use the Lorentz transformation to get v'_x, v'_y, v'_z and γ'

$$v'_x = \frac{v_x - v}{1 - vv_x/c^2}, \quad (2.10)$$

$$v'_y = \frac{v_y}{\gamma(1 - vv_x/c^2)}, \quad (2.11)$$

$$v'_z = \frac{v_z}{\gamma(1 - vv_x/c^2)}. \quad (2.12)$$

Similarly, according to the Lorentz transformation, the four momentums in the S' reference frame can be expressed as:

$$p'_x = \gamma(p_x - \beta E), \quad (2.13)$$

$$p'_y = p_y, \quad (2.14)$$

$$p'_z = p_z. \quad (2.15)$$

And the expression for energy, from $E = \gamma mc^2, E' = \gamma' mc^2$, changes according to Lorentz to

$$E' = \gamma(E - \beta p_x c). \quad (2.16)$$

It is convenient to use these kinematic variables, which have simple properties, under changes in the reference frame. These are kinematic variables with simple properties under the Lorentz transformation. They are commonly used.

2.1.2 Rapidity and pseudorapidity

In relativistic nuclear physics, rapidity, pseudorapidity and transverse momentum are also important parameters used to describe particle dynamics. These parameters are widely used in high-energy physics experiments, especially in collider experiments. The rapidity of a particle is defined by its energy-momentum components p_0 and p_z as

$$y = \frac{1}{2} \ln\left(\frac{p_0 + p_z}{p_0 - p_z}\right). \quad (2.17)$$

An important property of the rapidity y is that it is additive under the Lorentz transformation. That is, for two inertial systems that move relative to each other along the z -axis, the velocity between them is v , then the transformation relationship of the rapidity of a particle in the two inertial systems is

$$y' = y + \tanh^{-1}(v). \quad (2.18)$$

Among them, y' is the value of the particle's rapidity in the moving inertial frame, y is the value of the particle's rapidity in the stationary inertial frame, and v is the relative velocity between the two inertial frames. This property means that the rapidity is invariant under the Lorentz transformation, that is to say, the rapidity is a Lorentz invariant. This makes rapidity a very useful parameter for describing the dynamical properties of energetic particles. However, it is worth noting that the rapidity is invariant only under the Lorentz transformation along the z -axis. The rapidity is not constant for the Lorentz transformation along the x -axis or the y -axis. In practical applications, a convenient coordinate system is usually chosen so that the movement of particles mainly occurs in the direction of the z -axis, so as to take advantage of this property of rapidity. We can define the transverse mass of the particle

$$m_T^2 = m^2 + p_T^2 = p_0^2 - p_z^2, \quad (2.19)$$

where m is the remaining mass of the particle, and

$$p_T^2 = p_x^2 + p_y^2. \quad (2.20)$$

The energy p_0 and the longitudinal momentum p_z of the particle can be expressed by the rapidity:

$$p_0 = m_T \cosh(y), \quad (2.21)$$

$$p_z = m_T \sinh(y). \quad (2.22)$$

Therefore, as long as we measure the energy and the longitudinal momentum, we can get the rapidity of a particle. However, in many experiments, what we can directly measure is the angle θ of the detected particle with respect to the beam axis

$$\theta = \arccos\left(\frac{p_z}{|\mathbf{p}|}\right). \quad (2.23)$$

Another variable that is more widely used is pseudorapidity. In order to describe the velocity of a particle, two quantities of the particle need to be measured, such as its energy and longitudinal momentum. In many experiments, only the angle of the detected particle relative to the beam axis can be measured. In this case, it is convenient to utilize this information to characterize the detected particle using the pseudorapidity variable. The pseudorapidity variable of particle is defined as

$$\eta = -\ln[\tan(\theta/2)], \quad (2.24)$$

Among them, θ is the angle between the particle momentum p and the beam axis. According to the momentum, the pseudorapidity variable can be written as

$$\eta = \frac{1}{2} \ln\left(\frac{|\mathbf{p}| + p_z}{|\mathbf{p}| - p_z}\right). \quad (2.25)$$

Then we can express in terms of pseudorapidity

$$|\mathbf{p}| = p_T \cosh(\eta), \quad (2.26)$$

$$p_z = p_T \sinh(\eta). \quad (2.27)$$

2.1.3 Particle azimuth

Another very common and important quantity is the particle's azimuth angle. The azimuthal angle (ϕ) is an angle that describes the direction of the particle's momentum. It is usually defined as the angle between the projection of the particle momentum on a plane perpendicular to the beam direction and a reference direction on that plane. In high-energy physics experiments, the spherical coordinate system (r, θ, ϕ) is usually used to describe the momentum of particles. In this coordinate system, r is the radial distance, θ is the polar angle, and ϕ is the azimuth angle. The relationship between the particle's momentum components p_x, p_y, p_z and p_T, p, θ, ϕ is:

$$p_x = p_T \cos \phi, \quad (2.28)$$

$$p_y = p_T \sin \phi, \quad (2.29)$$

$$p_z = p \cos \theta, \quad (2.30)$$

$$p_T = p \sin \theta. \quad (2.31)$$

In heavy ion collision experiments, it is usually necessary to reconstruct the event plane, that is, the plane perpendicular to the beam direction, to analyze the geometric structure and flow properties of the event. Azimuthal angle is an important parameter used to reconstruct the event plane. At the same time, in the detector, the trajectory of the particle usually needs to be reconstructed by measuring the azimuth angle of the particle at different positions.

2.2 HYDRODYNAMICS IN HIGH-ENERGY ION COLLISIONS

High-energy nuclear-nuclear collision physics is a subject that studies the interaction between atomic nuclei under high-energy conditions (MCLERRAN, 2008). Its development has become an interdisciplinary subject of nuclear physics, particle physics, statistical physics, relativistic fluid dynamics and astrophysics. The goal is to create an environment of extremely high temperature and density, similar to the early stages of the Big Bang, which would make it possible to create a QGP.

QGP is a special state of matter in which quarks and gluons are no longer bound inside Hadrons (like protons and neutrons), but are able to flow freely. This state is thought to be the state of the universe on the microsecond timescale after the Big Bang. However, due to the limitation of the degrees of freedom of quarks, the existence time of QGP is very short, which makes it impossible to directly observe QGP. Therefore, scientists employed some indirect methods to probe the existence and properties of QGPs. A common approach is to use a hydrodynamic model to simulate the process of high-energy nuclear-nuclear collisions, and extract the information of the QGP from the simulation results. A fluid dynamics model is a mathematical model that describes the motion of a fluid (eg gas,

liquid, plasma). In the scenario of a high-energy nuclear-nuclear collision, a hydrodynamic model is used to simulate the evolution process of the resulting QGP, including its expansion, freeze-out, etc.

Therefore, by using a hydrodynamic model, we can simulate the entire evolution of high-energy nuclear-nuclear collisions and obtain evidence for the nature and existence of QGP from them. So in this section we will briefly introduce the fluid mechanics model and the code of the hydrodynamic model.

2.2.1 Evolution of High Energy Nuclear Collisions

Collisions of heavy ions with ultra-relativistic energies were used to create QGP in the laboratory. The evolution of heavy-ion collisions is often described by a series of stages, which can in principle be broken down. They include:

1) Two atomic nuclei are moving towards each other at close to the speed of light. At this stage, the kinetic energy of atomic nuclei is extremely high, and their energy and momentum can be described by the relativistic energy-momentum relationship: $E^2 = p^2c^2 + m^2c^4$, where E is energy, p is momentum, c is the speed of light, and m is mass. Due to relativistic effects, the two nuclei appear as two thin disks in a view perpendicular to the direction of motion. This is because, as an object approaches the speed of light, its length will decrease due to Lorentz contraction.

2) Once a collision occurs, the colliding nuclei (neutrons and protons) will lose a large amount of energy in an inelastic process. Since the energy of each nucleon in the center-of-mass system is high enough, even after the collision, the baryons (neutrons and protons) still have enough momentum to leave the collision region. The energy lost by colliding baryons is deposited in a small colliding region of space, and this high-energy-density environment leads to the formation of new matter, known as the QGP. QGP is a state of matter in which quarks and gluons are no longer confined inside Hadrons, but move freely within a certain volume.

3) Soon after the QGP is formed, the system will be in local thermal equilibrium, that is, different parts of the system will have different temperatures and densities, but the temperature and density of each part will be uniform. This system in local thermal equilibrium will then evolve according to the laws of relativistic fluid dynamics. Relativistic fluid dynamics is the theory describing the motion of matter at high energy densities, high temperatures and relativistic velocities. The initial conditions have an important influence on the evolution process, because they determine the properties of the matter produced after the collision, such as temperature, density, pressure, etc.

4) As the system expands, the temperature will drop to a critical value, which will cause the quarks to become confined in Hadrons again. This process is commonly known as the quark-hadron phase transition. After the phase transition, the system will continue to expand according to the laws of hydrodynamics.

5) As the temperature decreases further, two important temperature points will be reached. The first is the chemical freezing temperature, at which inelastic collisions cease and the species of particles no longer change. The second is the thermal freezing temperature, at which elastic collisions cease, determining the particle multiplicity and spectrum.

6) When the fluid becomes colder and thinner, the particles stop interacting and hadron emission

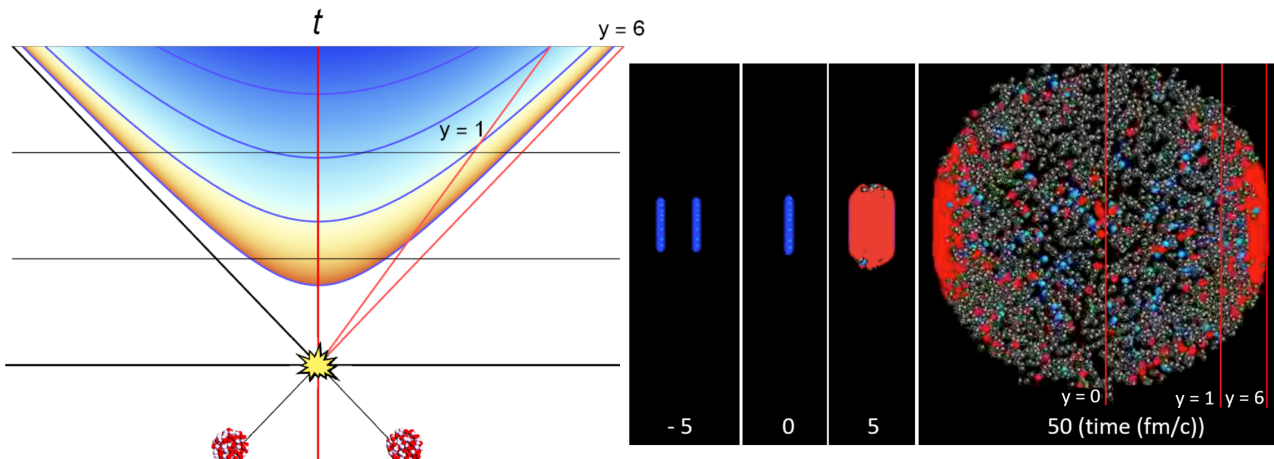


Fig. 1 – (left) Space-time picture of a heavy ion collision, whereby the color gives an indication of the temperature of the plasma formed. Dynamics takes place as a function of proper time (blue curves), which is why plasma forms later at higher rapidities. (right) Snapshots of a central 2.76 TeV PbPb collision at different times (different horizontal slices of the space-time picture on the left) with hadrons (blue and grey spheres) as well as QGP (red) (BUSZA; RAJAGOPAL; SCHEE, 2018).

occurs. What we observe in the final state are free particles. These free particles can be detected by experimental equipment, providing important information about the collision process.

Fig. 1 is a simple process display. Below we will describe in detail from several points we are interested in.

2.2.2 Initial conditions

In hydrodynamic models, the initial conditions are the starting point for the simulation and usually correspond to a very early moment after heavy ion collisions. These initial conditions usually include the following aspects. Modeling with the initial state is a necessary prerequisite for determining QGP properties, and is of great interest in its own right. The Monte Carlo (MC) Glauber method can be used to infer $\langle N_{part} \rangle$, $\langle N_{coll} \rangle$ and $\langle T_{AA} \rangle$ within a specific centrality/multiplicity range for Pb-Pb and p-Pb collisions. This approach assumes that the nucleons in the nucleus are located according to the Woods-Saxon density distribution. Nucleons move in undisturbed orbits whether or not they interact with other nucleons. The criterion for inelastic nucleon-nucleon interactions depends on the cross-section of the individual interactions, which can be inferred from experimental measurements of pp collisions. These assumptions are particularly relevant to heavy ion collisions at the LHC in a transparent state, meaning that the nuclear remnants following collisions travel in straight lines. Another output of this method is the initial state-space eccentricity ϵ_n in the (transverse) xy plane.

It depends on how the initial state of matter is distributed. It is an essential input to the hydrodynamic model chain (described later), since a non-zero value of ϵ_n produces anisotropic flow. Other initial state models go a step further. For example, the IP-Glasma model uses the MC Glauber assumption, but then uses the QCD-based Color Glass Condensate ansatz to determine the spatial gluon distribution in the nucleon. It attempts to model these distributions and how they overlap in heavy-ion collisions

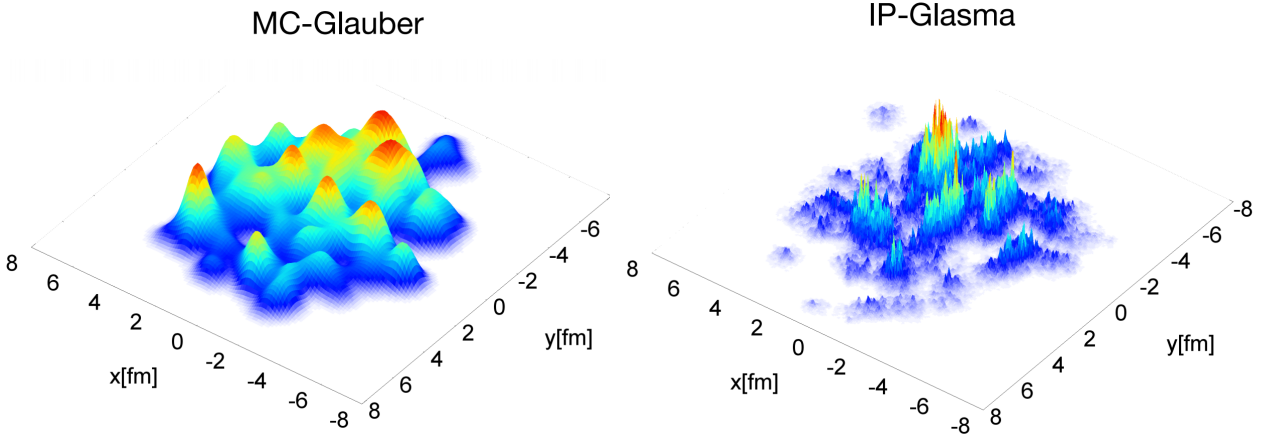


Fig. 2 – The xy distributions of the initial energy density (arbitrary units) from the MC-Glauber and IP-Glasma models for a heavy-ion collision (SCHENKE; TRIBEDY; VENUGOPALAN, 2012).

in the context of the nuclear saturation effect, which governs the lateral extent of the massive low- x gluon-filled nuclei in high-energy collisions. Saturation effects determine the distribution of the vast majority of components in the low- x region and their corresponding spatial distributions. Compared to the MC Glauber method, where this additional feature affects the values, IP-Glasma generally predicts higher values within a specific centrality range. The energy density comparison of MC-Glauber (derived from nucleons) and IP-Glasma (derived from gluons) on the xy plane is shown in Fig. 2. Knowledge of the partial initial momentum distribution is also an essential prerequisite for the study of hard processes in elementary and nuclear collisions. These distributions can be constrained using UPC collision measures and appropriate theoretical comparisons. The parameterization of this parton distribution, expressed as a function of the parton's fractional momentum x and the momentum transfer Q^2 , is usually modeled starting from the results of deep inelastic scattering experiments. In atomic nuclei, typical depletion effects, such as shadowing, are prominent at low x (< 0.1) and have a dramatic effect on the generation of hard processes. Anti-shadowing effects lead to an excess of partial molecules relative to pp collisions.

2.2.3 Equation of state

In high-energy heavy ion collisions, the formation and evolution of QGP is a complicated process. To simulate this process, hydrodynamic models are usually used. The basic assumption of this model is that in the early stages of the collision, the QGP reaches local thermal equilibrium, that is, its four velocities and thermodynamic quantities (such as temperature, density, pressure, etc.) are determined at each point in space. These local equilibrium initial conditions and the equation of state (EOS) of the QGP are the basis for simulating the evolution of the QGP.

Under initial conditions and an equation of state, QGP follows a hydrodynamic expansion governed by the $(4 + i)$ conservation equation

$$\partial_\mu T^{\mu\nu} = 0, \quad (2.32)$$

$$\partial_\mu n_i^\mu = 0. \quad (2.33)$$

where n_i^μ are conserved charges like baryons, singularities etc. energy momentum is

$$T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu - pg^{\mu\nu}. \quad (2.34)$$

where ϵ is the energy density, p is the pressure, and u^μ is the four velocity of the fluid.

The main task of the hydrodynamic model is to solve these partial differential equations. Also, due to their highly non-linear nature, it is really difficult to find an exact analytical resolution. However, these simple analytical solutions obtained under certain ideal conditions (such as symmetry or one-dimensional configuration) have their limitations in real-world situations. We want to observe general flow behavior, which represents the dynamics of the system when subjected to a specific equation of state, initial conditions, and decoupling process. Therefore, we turn to a numerical method that can handle any kind of geometry, even if the solution is not so precise. In order to solve the hydrodynamic equations of motion, it is best to start from the variational formula.

First, we review the derivation from the Lagrangian equations of motion under constraints (HAMA; KODAMA; SOCOLOWSKI JR., 2005). The dynamics of relativistic fluids can be used to describe

$$I = \int d^4x \{-\epsilon\}. \quad (2.35)$$

In the formula, $\epsilon = \epsilon(n, s)$ is the appropriate energy density, and n, s are the appropriate baryon density and entropy density, respectively. Considering the constraints, namely the law of conservation of the four-speed law and the law of normalization

$$\partial_\mu(nu^\mu) = 0, \quad (2.36)$$

$$\partial_\mu(su^\mu) = 0, \quad (2.37)$$

$$u^\mu u_\mu = 1. \quad (2.38)$$

We can write the variational formula in terms of Lagrangian multipliers

$$\delta \int d^4x [-\epsilon + \lambda \partial_\mu(nu^\mu) + \xi \partial_\mu(su^\mu) - \frac{1}{2}w(u^\mu u_\mu - 1)] = 0. \quad (2.39)$$

where λ, ξ and w are Lagrange multipliers and arbitrary functions of x . The efficient Lagrange equation can be obtained from the equation 2.39

$$\mathcal{L}_{eff}(n, s, u^\mu, \lambda, \xi, w) = -\epsilon - nu^\mu \partial_\mu \lambda - snu^\mu \partial_\mu \xi - \frac{1}{2}w(u^\mu u_\mu - 1). \quad (2.40)$$

By changing n, s, u_μ , the equation of motion is obtained

$$-u - u^\mu \partial_\mu \lambda = 0, \quad (2.41)$$

$$-T - u^\mu \partial_\mu \xi = 0, \quad (2.42)$$

$$-n \partial_\mu \lambda - s \partial_\mu \xi - w u^\mu = 0. \quad (2.43)$$

and combine the constraints in and follow the Gibbs-Duhem relation

$$dp = sdT + ndu. \quad (2.44)$$

The equations of motion 2.41 can be written in the standard form of the relativistic hydrodynamic equations 2.32. We can write the effective Lagrangian as the simple Lagrangian

$$\mathcal{L}_{eff}(\vec{a}(t), \frac{d\vec{a}(t)}{dt}) = - \int d\mathbf{r} \epsilon(n, s). \quad (2.45)$$

Now we can get the equations of motion in the form of the Euler-Lagrange equations, which are equations for the variable $a_i(t)$. It is convenient to solve the equations of motion with such a simple form and a small number of parameters. In this case, smoothed particle hydrodynamics is one of the most suitable numerical methods for solving the continuity equation.

2.2.4 Particle production and Measurement

In heavy ion reactions, as energy increases, different types of particles are produced, including muons, baryons, and even heavier hadrons. This is because the collective effect of nucleons (i.e. protons and neutrons) and their Fermi motion allow particles to be created below their production threshold. This means that even at very low energies, some particles that normally require higher energies to create can be produced. For example, in free nucleon-nucleon collisions certain particles may require higher energies to be produced, but in heavy ion collisions these particles may be produced at lower energies due to collective effects and Fermi motion.

In addition, at very high energies, some previously unknown exotic particles, such as exotic particles, may also be produced. Strange particles are particles that contain strange quarks. Strange quarks are heavy quarks that don't normally occur in common particles. However, in high-energy heavy-ion collisions, due to their extremely high energies, these exotic particles may be produced.

These rich possibilities provide physicists with a unique platform for studying high-energy phenomena, exploring new laws of physics, and discovering new particles. Heavy ion beams are the most energetic ion beams ever produced by accelerators, which offer unique possibilities for experimental investigations.

When heavy ions collide at the center of mass, many particles are produced. The type and number of these particles depend on the energy of the collision, the angle, and the type of nucleon. For example, in high-energy collisions, large numbers of mesons and baryons may be produced. And at higher energies, exotic particles and heavy hadrons may be produced. These particles will exist at very high speed and temperature in the initial state after the collision, and then through different interactions and decay processes, more stable particles and states of matter will be formed.

The detection of particle spectrum refers to the measurement of the type, energy, momentum and angular distribution of the generated particles. This is an important part of high-energy physics experiments.

First, a detector is a device used to detect particles. Commonly used particle detectors include: scintillation counters, semiconductor detectors, drift chambers, wire detectors, ring scintillation

counters, electromagnetic calorimeters, time-of-flight detectors, etc. As a particle passes through a detector, it interacts with the detector material, generating some signal, such as an electrical charge, photon, or other secondary particle. These signals are detected by detectors and converted into electrical signals. The electrical signals are amplified and processed before being collected by a data acquisition system. The data acquisition system stores and analyzes the collected data to obtain information such as energy, momentum, angle and type of particles. Here, the particle's energy and momentum are measured by analyzing the signal it generates in a detector. For example, in an electromagnetic calorimeter, particles interact with the detector material to create electron-positron pairs, which generate a charge signal. This signal is proportional to the energy of the particle. In a drift chamber or a silk detector, the trajectory of a particle can be reconstructed, and thus its momentum. The particle's angle, on the other hand, is measured by analyzing its trajectory in the detector.

2.3 HYDRODYNAMICS MODEL AND SPHERIO

Analytical solutions are often difficult to find due to the highly nonlinear nature of the hydrodynamic equations. Therefore, numerical methods are usually employed to solve these equations. Such as smoothed particle hydrodynamics (SPH) method, to solve these equations. The SPH method is a particle method that represents a fluid as a set of particles, each particle having a certain mass, position and velocity. The SPH method uses kernel functions to describe the interaction between particles, thus avoiding the use of grids. This makes the SPH method particularly suitable for handling fluids with complex geometries or large deformations. By solving the hydrodynamic equations, the evolution process of QGP can be obtained, and then the multiplicity, energy spectrum, transverse momentum spectrum and other observations of the final particles can be obtained. These observations can be compared with experimental data, thus providing insight into the nature of QGP.

2.3.1 Smoothed Particle Dynamics algorithm

SPH is a particle method for computational fluid dynamics originally developed to solve problems in astronomy but is now widely used in a variety of hydrodynamic applications, including nuclear physics (LUCY, 1977; GINGOLD; MONAGHAN, 1977). The basic idea of SPH is to represent the fluid as a set of discrete particles, each particle representing a volume of the fluid. These particles can move, deform, and interact to exchange mass, momentum, and energy.

Compared with previous meshing methods, SPH has the following advantages:

1)Adaptability: A significant advantage of the SPH method is its adaptive nature. SPH particles can freely move and change relative positions, which allows the method to automatically adapt to different regions of the flow field, assigning more particles to important regions of the flow, such as regions with large gradient changes, or discontinuous regions of the flow field.

2)Handle free interface:The SPH method is suitable for modeling fluid flow problems with complex boundaries and free surfaces. Because it is a Lagrangian method, the particles can move freely, which allows the method to model the motion of fluid surfaces well without the use of complex meshing techniques.

3)Handling large deformations: The SPH method is suitable for simulating large deformations and large displacements. Because it is a Lagrangian method, the particles move with the fluid, which allows the method to simulate large deformations and large displacements of the fluid well without problems with mesh distortion.

4)Parallel Computing: The SPH method is suitable for parallel computing. Because the interaction between particles is local, each particle only interacts with neighboring particles, which enables the method to be well decomposed into multiple subtasks, which are assigned to different computing nodes for calculation.

5)SPH is a meshless method, meaning it does not require a mesh to be generated beforehand. This feature is very useful for simulating flow fields or structures with complex geometries, such as nonlinear events such as rupture and collision. This also makes the SPH method applicable to the simulation of fluid-structure interaction problems.

6)Adapt to irregular borders: The SPH method can be easily adapted to irregular boundaries. Because the boundary is represented by particles, the particles can adapt to any shape of the boundary, which enables the method to simulate flow fields with irregular boundaries well.

Below we will briefly introduce the basic structure of SPH. We start with the mathematical identity

$$f(\mathbf{r}) = \int f(\mathbf{r}')\delta(\mathbf{r} - \mathbf{r}')d\mathbf{r}', \quad (2.46)$$

where $f(\mathbf{r})$ is any function defined on the three-dimensional coordinate system $\mathbf{r}$ on the volume V . Similarly, $\delta\mathbf{r}$ is the Dirac function, and $\mathbf{r}'$ is a dummy variable that also ranges over V . We can generalize the delta function to a so-called smoothing kernel W with feature width h (called the smoothing length), such that

$$\lim_{h \rightarrow 0} W(\mathbf{r}, h) = \delta\mathbf{r}. \quad (2.47)$$

Affected by standardization

$$\int_V W(\mathbf{r}, h)d\mathbf{r}' = 1. \quad (2.48)$$

By expanding $W(\mathbf{r} - \mathbf{r}', h)$ into a Taylor series, it can be shown that for the symmetric kernel $W(\mathbf{r} - \mathbf{r}', h) = W(\mathbf{r}' - \mathbf{r}, h)$, Equation 2.46 becomes

$$f(\mathbf{r}) = \int_V f(\mathbf{r}')W(\mathbf{r} - \mathbf{r}', h)d\mathbf{r}' + \mathcal{O}(h^2). \quad (2.49)$$

Discretizing this continuous field onto a series of particles of (potentially variable) mass $m = \rho(r')dr'$, the original identity equation 2.46 becomes

$$f(\mathbf{r}) \approx \sum_i \frac{m_i}{\rho_i} f(\mathbf{r}_i) W(\mathbf{r} - \mathbf{r}', h), \quad (2.50)$$

where $f(r_i)$, m_i and $\rho_i = \rho(r_i)$ are the scalar values, mass and density of the i -th particle, and i are all

particles within the smoothing kernel. Equation 2.50 thus expresses a discrete approximation of the continuous scalar field f at position $\mathbf{r}$ in the computational domain V , and is thus the basis for all SPH forms.

In specific nuclear physics, the above form will be changed into a more specific expression. The first is the calculation of the density

$$\rho = \sum_j m_j W(\mathbf{r}_i - \mathbf{r}_j, h), \quad (2.51)$$

The second is the discrete form of the momentum equation

$$\frac{dv_i}{dt} = - \sum_j m_j \left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2} \right) \nabla W(\mathbf{r}_i - \mathbf{r}_j, h), \quad (2.52)$$

and the energy equation

$$\frac{du_i}{dt} = \frac{P_i}{\rho_i^2} \sum_j m_j v_{ij} \cdot \nabla W(\mathbf{r}_i - \mathbf{r}_j, h). \quad (2.53)$$

To solve the numerical discretized equations, we resort to SPheRIO code.

2.3.2 SPheRIO code

In the above, we explored how the SPH method can parameterize the material flow through discrete Lagrangian coordinates (also known as SPH particles). This method exploits the entropy representation of the SPH model and derives the equations of motion through variational principles to determine the degrees of freedom of the SPH. The notable advantage of this approach lies in its capability to handle diverse geometries and dynamic behaviors, such as effectively managing shock wave phenomena through appropriate selection of SPH particle sizes, thereby avoiding numerical complications.

The SPheRIO code was used to perform hydrodynamic simulations of relativistic high-energy collisions when the hot and dense matter produced in high-energy collisions is assumed to be in local equilibrium (HAMA; KODAMA; SOCOLOWSKI JR., 2005; TAKAHASHI et al., 2009; ANDRADE et al., 2010; QIAN et al., 2007b; QIAN et al., 2007a). We employ integrated circuit representations of distributions encompassing fluid velocities, energy-momentum tensors, baryon numbers, singularities, etc., as input parameters for the SPheRIO code. Initial conditions can be generated by using some event generators in the simulation, such as HIJING (GYULASSY; RISCHKE; ZHANG, 1997), VNI (SCHLEI; STROTTMAN, 1999), URASiMA (NONAKA; HONDA; MUROYA, 2000), Ne-XuS (DRESCHER et al., 2002). At the same time, an equation of state describing the thermodynamic relationship is also required, which can reduce the degrees of freedom of the hydrodynamic equation of motion.

As the system expands and cools, it enters the decoupling phase of the hydrodynamic model, when the constituent particles eventually stop interacting and are emitted as free particles. Therefore, at the end of the hydrodynamic evolution, we need to specify the decoupling in the SPheRIO code

to determine where and when the freeze exits. There are three different freezing schemes: thermal freezing, kinetic freezing and chemical freezing, and continuous emission (CE).

Overall, we can study the properties of the QGP through the initial conditions, equation of state and decoupling criterion of SPheRIO. Many studies have shown that SPheRIO is able to reproduce experimental results in several ways.

3 COLLECTIVE FLOW AND ITS MEASUREMENT

In high-energy heavy ion collision experiments, two heavy ions collide at high speeds, producing a large number of particles. The motion characteristics of these particles, such as the distribution of momentum and angles, will be influenced by their initial state. For example, if the spatial distribution of the initial state is inhomogeneous, the distribution of particles in the final state will also exhibit this inhomogeneity. This inhomogeneity can be studied by measuring the collective flow of particles in the final state.

Collective flow refers to the collective motion behavior of particles in the final state caused by the spatial inhomogeneity of the initial state during high-energy heavy ion collisions. This phenomenon has been confirmed by a large number of experimental data, and has also been deeply studied theoretically. Collective flow can be divided into longitudinal flow and transverse flow, and transverse flow can be divided into radial flow and anisotropic flow. Collective flow can be divided into two types, longitudinal flow and anisotropic flow. Longitudinal flow refers to the collective motion of generated particles along the beam direction of the colliding nucleus. Longitudinal flow can be analyzed by particle velocity distribution or velocity distribution. Lateral flow refers to the collective motion of generated particles across the cross-section of the collider nucleus. Lateral flow is usually divided into different modes, such as radial flow, elliptical flow, trilobal flow, quatrefoil flow, etc. It is the most basic lateral flow mode, which means that the generated particle population has an orderly movement along the radial direction from the collision point. Radial flow is characterized by a distinct peak in the transverse momentum distribution of the generated particles.

Anisotropic flow is a special case of lateral flow, which means that in non-central collisions, since the shape of heavy ions is not completely symmetrical, the lateral momentum distribution of the generated particles will show anisotropy. For example, elliptical flow, three-leaf flow, four-leaf flow, etc.

For the measurement of anisotropic flow, many methods have been developed, the two most classic methods are: event plane and particle correlation method. The event plane method is a global method, which is to measure the angular distribution of the generated particles on the cross section, to determine an event plane, and then analyze the anisotropic flow of particles through the event plane. Particle correlation method is a local method to analyze anisotropic flow by measuring the relative angular distribution between two particles.

However, the measurement of anisotropic flow is not as easy as imagined, and it is easily affected by many aspects, such as the multiplicity of particles or the influence of statistical fluctuations, etc. This section will simply pass through the relevant quantities of collective flow And the method of measuring collective flow and the content of measuring impact are introduced.

3.1 ECCENTRICITY AND ITS FLUCTUATION

The development of anisotropic flow is governed by anisotropy in the pressure gradient, which in turn depends on the shape and structure of the initial density distribution. The latter can be characterized

by a set of harmonic eccentricity coefficients ϵ_n . In each collision, all eccentricity coefficients are usually nonzero, driving an anisotropic flow component of any harmonic order whose magnitude and direction vary with the event fluctuation. The statistical distribution of ϵ_n and ϕ_n controls the statistical distribution of the resulting anisotropic flow and its direction Ψ_n . In each collision, all eccentricity coefficients are usually non-zero, driving the anisotropic flow component of any harmonic order, whose magnitude and direction fluctuate with events.

Many studies have been devoted to studying the relationship between the initial geometric anisotropy and the final state flow harmonics. Through the analysis of the initial conditions of fluid dynamics, the anisotropy of the initial energy is represented by the complex eccentricity ϵ_n . Here we assume that the density The center coincides with the origin

$$\epsilon_n = \frac{\langle z^n \rangle}{\langle |z|^n \rangle} \equiv \frac{\int dx dy \rho(z) z^n}{\int dx dy \rho(z) |z|^n}, \quad (3.1)$$

where $z = x + iy$. In practice, even if the number of events is important, the diversity of particles is limited for a single event. When these events are averaged, the variance of the extracted harmonic coefficients remains bounded, even though particles are emitted independently according to a well-defined single-particle distribution function. This level of statistical uncertainty will manifest itself as flow fluctuations. In practice, it may be indistinguishable from fluctuations in the initial conditions discussed above due to event fluctuations. The latter is understood to reflect the physics of the underlying microscopic model, distinct from its statistical counterpart.

We will discuss eccentric fluctuations due to the limited number of participants that end up forming "nuggets" in the initial energy distribution, referring to regions formed by fluctuations in the initial energy distribution. In the case of heavy ion collisions, fluctuations in the initial energy distribution can affect the distribution and kinetic properties of the resulting particles. This feature is widely manifested in many well-known event generators, such as MC Glauber (MILLER et al., 2007), CGC MC-KLN (KHARZEEV; NARDI, 2001; KHARZEEV; LEVIN, 2001; DRESCHER; NARA, 2007b; DRESCHER; NARA, 2007a), NeXUS (DRESCHER et al., 2002; DRESCHER et al., 2001), EPOS (WERNER; LIU; PIEROG, 2006; WERNER; KARPENKO; PIEROG, 2011; WERNER et al., 2014), etc. That is, the initial conditions fluctuating from event to event usually have the characteristics of granular energy distribution.

Density fluctuations even above an isotropic density distribution break the symmetry of the average distribution. For present purposes we may consider that the initial conditions consist of sources caused by partial binary collisions. For mathematical convenience, these sources and their fluctuations are given the same form and are spatially uncorrelated. When the number N of sources becomes significant, we can exploit the multivariate central limit theorem (CLT), where the number of variables is governed by the relative degrees of freedom of the initial distribution (OLLITRAULT, 1992). This is because CLT involves a multivariate normal distribution that eventually converges to a mean among independently distributed random variables sampled. The latter can be used as coordinates on the horizontal plane in practical applications. In order to derive the distribution of eccentricity ϵ_2 , it is possible to transform the variables into new variables including eccentricity. Where feasible, synthetic

eccentric distributions were obtained by integrating the remaining variables.

In what follows, we consider a simplified scenario to illustrate the eccentricity caused by fluctuations. Specifically, we adopt the same point source model (OLLITRAULT, 1992; GRÖNQVIST; BLAIZOT; OLLITRAULT, 2016), where the source is point-like and the probability distribution satisfies the mean Gaussian distribution

$$p(z_i) = \frac{1}{\pi R_0^2} \exp\left(-\frac{|z_i|^2}{R_0^2}\right), \quad (3.2)$$

where the location of source i is denoted by z_i , where $i = 1, \dots, n$. In this case, the composite probability distribution of $|\epsilon_2|$ can be derived analytically, resulting in

$$P(|\epsilon_2|) = (N-2)|\epsilon_2| \left(1 - |\epsilon_2|^2\right)^{\frac{N}{2}-2}, \quad (3.3)$$

where N is the number of point sources. It is easy to prove this by integrating Eq. (3.3)

$$E[|\epsilon_2|] = \frac{\sqrt{\pi}\Gamma(\frac{1}{2})}{2\Gamma(\frac{N+1}{2})}, \quad (3.4)$$

$$E[|\epsilon_2|^2] = \frac{2}{N}, \quad (3.5)$$

$$E[|\epsilon_2|^4] = \frac{8}{2N + N^2}. \quad (3.6)$$

where for a positive interger M , the Γ function simplifies $\Gamma(M) = (M-1)!$. The variance of $|\epsilon_2|$ are readily given by Eq. (3.6) as

$$\text{Var}[|\epsilon_2|] = \frac{4}{N^2} - \frac{\pi\Gamma(\frac{N}{2})^2}{4\Gamma(\frac{N+1}{2})^2}. \quad (3.7)$$

The expected value and variance discussed here refer to the average between different events.

Moreover, the cumulants of the eccentricity are found to be (OLLITRAULT, 1992)

$$\epsilon_2\{2\} = \sqrt{E[|\epsilon_2|^2]} = \sqrt{\frac{2}{N}}, \quad (3.8)$$

$$\epsilon_2\{4\} = \left(2E[|\epsilon_2|^2]^2 - E[|\epsilon_2|^4]\right)^{1/4} = \left(\frac{16}{N^2(N+2)}\right)^{1/4}. \quad (3.9)$$

The above results show that the eccentricity does not disappear due to the limited number of participants. In the continuum limit, however, eccentricity does not persist, since obviously ϵ , its variance and cumulant disappear in the limit $N \rightarrow \infty$. In this regard, it is worth noting that the eccentricity arising from geometric fluctuations in the initial state does not necessarily materialize in a discrete manner. Even in the limit $N \rightarrow \infty$, continuous fluctuations in the initial conditions lead to non-disappearing eccentricities.

3.2 FLOW HARMONICS

Since the initial conditions affect all particles differently, it leads to true multi-particle effects, often referred to as anisotropic collective flow. Studying anisotropic flows in nuclear collisions provides unique and valuable information on the general physics of system evolution and multiparticle production. This means that at high energies, one does not need to rotate to the flow axis to study flow patterns, but one can use a plane transverse to the beam axis. Therefore, we only discuss anisotropic lateral flows with particle azimuth distribution at fixed or pseudorapidity.

The anisotropic flows corresponding to the first two harmonics play such an important role that we use special terms for them respectively: directed flow and elliptical flow. The term "directed" comes from the fact that such a flow has a direction, while the term "elliptical" comes from the fact that in polar coordinates, an azimuthal distribution with a non-zero second harmonic represents an ellipse. It is also recognized that the created QGP fireball fluctuates event-by-event due to nucleons within colliding nuclei constantly changing their positions. The collective expansion of thermal systems transforms initial spatial inhomogeneities and deformations into anisotropic momentum distributions of the resulting particles, which can be quantitatively evaluated by various flow observations.

The anisotropic flow coefficient v_n and its associated flow angle Ψ are defined as (VOLOSHIN; ZHANG, 1996)

$$v_n e^{in\Psi_n} \equiv \langle e^{in\phi} \rangle, \quad (3.10)$$

Note that v_n describes a collision. In the left equation, it defines the "pt integral" or "total" anisotropic flow, averaged over all particles of a given type (identified by species or not) at that event. The equation on the right defines the "pt-differential" flow and only averages over particles with a given transverse momentum. v_1 , v_2 , and v_3 are called directed flow, elliptical flow, and triangular flow, respectively.

Due to the limited number of particles emitted by a single event, statistically accurate measurements of v_n and Ψ_n can only be obtained from particle distributions averaged over many events. We define the mean flow coefficient v_n and the mean flow angle Ψ_n by Fourier decomposition of the experimentally determined event mean particle distribution

$$\frac{d\bar{N}}{d\phi} = \frac{\bar{N}}{2\pi} \left[1 + \sum_{n=1} 2v_n \cos n(\phi - \Psi_n) \right], \quad (3.11)$$

Experimentally, the most direct evidence for hydrodynamic behavior comes from the observation of anisotropic flow. The largest anisotropic flow coefficient is v_2 , the elliptical flow. Because collisions are usually not perfectly central, this results in an inhomogeneous, often elliptical, initial mass distribution. Over time, this inhomogeneity translates into an inhomogeneity in the momentum distribution of the particles, which is elliptical flow.

3.3 EVENT-PLANE METHOD TO MEASURE HARMONICS

The event plane method is a method for analyzing heavy ion collision data, especially for analyzing the collective flow of particles (POSKANZER; VOLOSHIN, 1998). In non-central heavy ion collisions, the overlapping regions of the colliding nuclei are usually elliptical, which leads to anisotropic momentum distribution of the produced species. The event plane method defines an "event plane" based on this anisotropy, and then obtains the flow coefficient by measuring the azimuth distribution of particles relative to the event plane.

The event plane is usually defined as the plane perpendicular to the direction in which the collision occurred. For elliptical flows, the event plane is usually perpendicular to the direction of the major axis of the ellipse. However, the orientation of the event plane cannot be measured directly from experiments and can only be estimated by analyzing the momentum distribution of the particles produced by the collision. The essence of the method is to first estimate the reaction plane. The estimated response plane, we call the event plane.

Since the finite number of particles detected yields finite resolution in measuring the angle of the event plane, these coefficients must be corrected to their level relative to the true reaction plane. This is achieved by dividing the observation coefficient by the event plane resolution estimated from the correlation of the planes of independent sub-events (subgroups of particles used to determine the event plane). The resolution obtained from sub-events can be converted to the resolution of the whole event through the multiplicity dependence of resolution described below.

The main advantage of the Fourier method is that, using the Fourier coefficients estimated from the observed event plane, corrections can be made for the resolution of the event plane caused by the finite multiplicity of events. This method uses the anisotropic flow itself to determine the event plane. This also means that, for each harmonic of the anisotropic flow, the event plane can be determined independently.

The event flow vector Q_n and the event plane angle Ψ_n from the n -th harmonic of the distribution are defined by the equations

$$Q_n \cos(n\Psi_n) = X_n = \sum_i w_i \cos(n\phi_i), \quad (3.12)$$

$$Q_n \sin(n\Psi_n) = Y_n = \sum_i w_i \sin(n\phi_i). \quad (3.13)$$

where the summation is performed over all particles in the event. Note that the event plane angle Ψ_n determined from the n th harmonic is in the range $0 \leq \Psi_n < 2\pi/n$. Once the orientation of the event plane is determined, the flow coefficient v_n can be calculated

$$\Psi_n = (\tan^{-1} \frac{\sum_i w_i \sin(n\phi_i)}{\sum_i w_i \cos(n\phi_i)})/n, \quad (3.14)$$

The summation is over the i particles used in the event plane determination, w_i are the weights. In general, the weights are also optimized to give the best possible response plane resolution. Sometimes it can be achieved by selecting a specific type of particle, or weighting with the particle's lateral

momentum, etc.

However, due to experimental limitations and the influence of non-flow effects, the measured event plane orientation may have certain deviations. This bias is usually measured by calculating the "resolution" of the event plane. The average cosine value is less than 1, so this correction always increases the flow coefficient. The resolution depends on the harmonics used to determine the event plane m and the order of the calculated coefficient n . In general, the determination accuracy of v_n is better by using the event plane (Ψ_n) determined by the same harmonics ($m = n, k = 1$). The resolution of the event plane is usually denoted by R_n , defined as

$$R_n = \langle \cos[n(\Psi_n^a - \Psi_n^b)] \rangle, \quad (3.15)$$

where Ψ_n^a and Ψ_n^b are the directions of the event plane estimated separately from two different sub-events (e.g. different intervals of velocity or different intervals of lateral momentum). The resolution of the event plane can be increased in several ways. For example, more particles can be used to estimate the orientation of the event plane, or the event can be divided into multiple sub-events to estimate the orientation of the event plane independently and then averaged. This reduces the influence of non-flow effects and thus improves the resolution of the event plane. The final flow-harmonic coefficient is

$$v_n = \frac{v_n^{obs}}{R_n}. \quad (3.16)$$

3.4 MULTI-PARTICLE AZIMUTHAL CORRELATIONS

The event plane approach is somewhat problematic because the response plane fluctuates on an event-by-event basis and cannot be directly measured experimentally. One can circumvent the difficulty by forming particle pairs, in which the event plane is canceled. In addition to the traditional event plane method, there is another method that has been used for a long time - the particle correlation method. Both the two-particle correlation method and the event plane method are important methods for analyzing the collective flow of particles. But the advantage of the particle correlation method is that the two-particle correlation method extracts the flow coefficient directly from the experimental data without knowing the orientation of the event plane. The influence of non-flow effects can be estimated and corrected by comparing the correlation functions with mixed events or random events. A multi-particle correlation method called generic framework will be introduced below. This comes up many times in our calculations.

We take the two-particle and multi-particle orientation correlations as our fundamental observations, which we can illustrate with a simple example of the two-particle orientation correlations for the harmonics n and $-n$. For a dataset consisting of M azimuths, $\phi_1, \phi_2, \dots, \phi_M$, we have

$$\langle 2 \rangle_{2,-2} \equiv \langle e^{i2(\phi_1 - \phi_2)} \rangle = \langle \cos n(\phi_1 - \phi_2) \rangle \quad (3.17)$$

$$= \frac{1}{M(M-1)} \sum_{i \neq j} \cos n(\phi_i - \phi_j), \quad (3.18)$$

. The constraint $i = j$ removes the contribution in the autocorrelation of each sum by definition. The

1_{st} and 2_{nd} order moments of $\langle 2 \rangle_n$ are respectively

$$\mu = v_2^2, \quad (3.19)$$

$$\sigma^2 = \frac{1 + v_2^2}{M(M-1)} + 2 \frac{M-2}{M(M-1)} v_2^2 (1 + v_4) + \frac{(M-2)(M-3)}{M(M-1)} v_2^4 - v_2^4. \quad (3.20)$$

These are analytical expressions for the mean and variance of the two-particle azimuth correlation, valid for the general Fourier-like p.d.f. 1.1 parameterized with all harmonics v_n . The meaning of variance will be discussed in the next section. Through the expression of two-particle correlation, we can easily deduce the form of multi-particle correlation

$$\begin{aligned} \langle m \rangle_{n_1, n_2, \dots, n_k} &\equiv \langle e^{i(n_1 \phi_1 + n_2 \phi_2 + \dots + n_k \phi_k)} \rangle \\ &\equiv \frac{\sum_{k\text{-tuples}} w_1 w_2 \dots w_k e^{i(n_1 \phi_1 + n_2 \phi_2 + \dots + n_k \phi_k)}}{\sum_{k\text{-tuples}} w_1 w_2 \dots w_k}, \end{aligned} \quad (3.21)$$

In the above definition, M is the multiplicity of an event, ϕ marks the azimuth angle of the generated particle, and w marks the weight of the particle, its physical meaning and use cases will be elaborated. We enforced the condition $k_1 = k_2 = \dots = k_m$ in the summary to remove trivial and non-negligible contributions from all possible autocorrelations in all sums. We emphasize that we believe that any relevant technique used in the analysis of anisotropic flows is not reliable and if it is not available.

The particle weights presented here can be used to remove systematic biases caused by various types of detector inefficiencies. Well-known examples of particle weights are the so-called ϕ -weights w_ϕ , which deal with the systematic bias due to non-uniform acceptance of azimuth angles, and p_T -weights w_{p_T} , which deal with the non-uniform lateral momentum reconstruction efficiency of the resulting particles. In general, we allow the particle weight w to be the most general function of azimuth, transverse momentum, pseudorapidity, particle type, etc.

$$w = w(\phi, p_T, \eta, \dots). \quad (3.22)$$

The Generic framework is done through Q-vector, as follows. For events with multiplicity M , we construct the following two sets

$$\text{azimuthal angles} : \phi_1, \phi_2, \dots, \phi_M, \quad (3.23)$$

$$\text{weights} : w_1, w_2, \dots, w_M. \quad (3.24)$$

From these two sets, we compute a weighted Q-vector for each event as complex numbers, defined as

$$Q_{n,p} \equiv \sum_k^M w_k^p e^{in\phi_k}. \quad (3.25)$$

We first observe that both the numerator and the denominator of Eq 3.21 are simply related. That is, to give a result dependent on the harmonics $n_1, n_2, \dots, n_m$ using the numerator, setting all harmonics

$n_1, n_2, \dots = 0$ as the denominator. Separate the numerator and denominator

$$N\langle m \rangle \equiv \sum_{k\text{-tuples}} w_{k_1} w_{k_2} \cdots w_{k_m} e^{i(n_1 \phi_{k_1} + n_2 \phi_{k_2} + \cdots + n_m \phi_{k_m})}, \quad (3.26)$$

$$\begin{aligned} D\langle m \rangle &\equiv \sum_{k\text{-tuples}} w_{k_1} w_{k_2} \cdots w_{k_m} \\ &= N\langle m \rangle_{0,0,\dots,0}. \end{aligned} \quad (3.27)$$

Then according to the defined Q-vector, we can simply express the numerator and denominator of the above formula. Here we can list the results of two-particle correlation and three-particle correlation:

$$N_{\langle 2 \rangle_{n_1, n_2}} = Q_{n_1,1} Q_{n_2,1} - Q_{n_1+n_2,2}, \quad (3.28)$$

$$D_{\langle 2 \rangle_{n_1, n_2}} = Q_{0,1}^2 - Q_{0,2}, \quad (3.29)$$

$$\begin{aligned} N_{\langle 3 \rangle_{n_1, n_2, n_3}} &= Q_{n_1,1} Q_{n_2,1} Q_{n_3,1} - Q_{n_1+n_2,2} Q_{n_3,1} - Q_{n_2,1} Q_{n_1+n_3,2}, \\ &\quad - Q_{n_1,1} Q_{n_1+n_3,2} + 2Q_{n_1+n_2+n_3,3} \end{aligned} \quad (3.30)$$

$$\begin{aligned} D_{\langle 3 \rangle_{n_1, n_2, n_3}} &= N_{\langle 3 \rangle_{0,0,0}} \\ &= Q_{0,1}^3 - 3Q_{0,2} Q_{0,1} + 2Q_{0,3} \end{aligned} \quad (3.31)$$

It is worth noting that higher order correlations will contain more items. A simple loop is given in (BILANDZIC et al., 2014) to represent all higher order terms, which is very easy to implement. Afterwards, the harmonic measurement results of the given M multiple events can be obtained through the formula Eq 3.21, as long as we choose $n_1 = -n_2 = n$.

In the appendix A we give the recursive advanced algorithm of the program. it is worth noting, In real calculations, we will not use the method listed in the above formula, but directly adopt the recursive algorithm in the appendix, because it can be calculated from $n = 2$ to any high order.

3.5 TUBE MODEL AND TWO-PARTICLE CUMULANT

The peripheral tube model provides an intuitive picture of the two-particle correlation of the generation of the triangular flow and the origin of the ridge structure. The method quantifies the essential features under event-by-event fluctuating initial conditions through some local variables, rather than some global deformation parameters, such as eccentricity. Based on heuristic arguments and numerical simulations, it is demonstrated that the local expansion of some high-energy points near the transverse profile boundary mainly produces triangular flow. Each hot spot extends longitudinally to form a tubular structure. Within the framework of the peritube model, the origin of ridge and shoulder structures can be consistently understood. On the other hand, it is not straightforward to visualize the physical mechanisms in terms of the essentially linear relationship between eccentricity and flow harmonics at different orders. Specifically, numerical simulations have not convincingly established linear response relationships of various orders for the azimuthal expansion between the initial conditions and the final flow. In this regard, the peripheral tube model can explain the correlated

features observed in dual particles. A more decisive distinction to the different perspectives is the evaluation of particle correlations as a function of emission position, which has hitherto been infeasible.

The initial conditions for the peripheral tube model are similar to those generated by the event generators NEXUS and EPOS. Specifically, the initial energy distribution is the sum of two contributions: the background tube and the peripheral tube. In principle, the former can be obtained by averaging the distribution of many events from the same centrality class. The latter consist of isolated high-energy spots near the surface of the system, called peripheral tubes. As described below, the rapid expansion of the system is deflected by the surrounding tube, causing multiple fluctuations and some flow harmonics associated with that tube. It is worth noting that this tube-related phenomenon is local because it is not perceived by the rest of the system. Subsequently, through the superposition of individual tubes, the properties of the entire system are obtained.

Regarding collective flow, the analytical version of the peripheral tube model assumes that the hydrodynamic evolution that produces two-particle correlations consists of two parts. The first is related to the background energy distribution of the initial conditions, which largely dominates the resulting flow harmonics. The distribution of background particles is mainly controlled by the almond shape, which mainly produces elliptical flows. On an event basis, it is also subject to diversity fluctuations. Specifically, the single-particle distribution is not given directly using the Fourier expansion of equation 1.1. A single fluctuating event is the sum of two terms, the distribution of the background and the distribution of the tube

$$\frac{dN}{d\varphi}(\varphi, \varphi_{\text{tube}}) = \frac{dN_{\text{bgd}}}{d\varphi}(\varphi) + \frac{dN_{\text{tube}}}{d\varphi}(\varphi, \varphi_{\text{tube}}), \quad (3.32)$$

where

$$\frac{dN_{\text{bgd}}}{d\varphi}(\varphi) = \frac{N_{\text{bgd}}}{2\pi}(1 + 2v_2^{\text{bgd}} \cos(2\varphi)), \quad (3.33)$$

$$\frac{dN_{\text{tube}}}{d\varphi}(\varphi, \varphi_{\text{tube}}) = \frac{N_{\text{tube}}}{2\pi} \sum_{n=2,3} 2v_n^{\text{tube}} \cos(n[\varphi - \varphi_{\text{tube}}]). \quad (3.34)$$

In equation 3.33 we consider a simplified scenario where the background flow is essentially elliptical, parameterizing the flow's harmonics v_2^{bgd} and the overall multiplicity N_{bgd} . The contribution from the peripheral tube is measured relative to its angular position ϕ_{tube} . To reproduce the desired structure in two-particle correlations, we introduced a minimal number of Fourier components, encoded in the v_2 and v_3 tubes. Notably, contributions related to surrounding tubes were found to be independent across different events, mainly when event averaging was performed on mixed event correlations. We also note that in this model the triangular flow is generated entirely by the tube, whose axis of symmetry is aligned with a part of the elliptical flow, both of which are related to the tube position ϕ_{tube} . We assume that the flows from the background are much more important than those generated by the test tube. Thus, the event plane Ψ_2 is dominated by the elliptic flow of the background v_2^{bgd} .

According to the method of STAR experiments (FENG, 2008; AGAKISHIEV et al., 2021; AGA-

KISHIEV et al., 2014), the subtracted dihadron correlation is defined as

$$\left\langle \frac{dN_{\text{pair}}}{d\Delta\varphi}(\varphi_s) \right\rangle = \left\langle \frac{dN_{\text{pair}}}{d\Delta\varphi}(\varphi_s) \right\rangle^{\text{prop}} - \left\langle \frac{dN_{\text{pair}}}{d\Delta\varphi}(\varphi_s) \right\rangle^{\text{mix}}, \quad (3.35)$$

where ϕ_s is the angle at which the particle is triggered, which is no different from most calculations. By using equation, the proper two-particle correlation can be computed in the form

$$\left\langle \frac{dN_{\text{pair}}}{d\Delta\varphi} \right\rangle^{\text{prop}} = \int \frac{d\varphi_{\text{tube}}}{2\pi} f(\varphi_{\text{tube}}) \frac{dN^T}{d\varphi}(\varphi_s, \varphi_{\text{tube}}) \frac{dN^A}{d\varphi}(\varphi_s + \Delta\varphi, \varphi_{\text{tube}}), \quad (3.36)$$

where $f(\varphi_{\text{tube}})$ is the distribution function of the tube, and the superscripts T and A denote “trigger” and “associated” particles, respectively. For example, in real-world scenarios, $f(\varphi_{\text{tube}})$ may have a small fraction of elliptic modulation due to almond-shaped initial conditions.

The two-particle correlation $dN_{\text{pair}}/d\Delta\varphi$ mixing of combined mixing events can be calculated by using the cumulant or ZYAM method (AJITANAND et al., 2005; ADLER et al., 2006). While these two approaches yield very similar results in our model, evaluating cumulants is more illustrative. By simply taking $f(\varphi_{\text{tube}}) = 1$, the resulting two-particle correlation is

$$\begin{aligned} \left\langle \frac{dN_{\text{pair}}}{d\Delta\varphi}(\varphi_s) \right\rangle^{(\text{cmlt})} &= \frac{\langle N_{\text{bgd}}^2 \rangle - \langle N_{\text{bgd}} \rangle^2}{(2\pi)^2} (1 + 2v_2^{\text{bgd}} \cos(2\varphi_s))(1 + 2v_2^{\text{bgd}} \cos(2(\Delta\varphi + \varphi_s))) \\ &+ \left(\frac{N_{\text{tube}}}{2\pi} \right)^2 \sum_{n=2,3} 2(v_n^{\text{tube}})^2 \cos(n\Delta\varphi), \end{aligned} \quad (3.37)$$

The two terms on the r.h.s. eq 3.37 are modulated by the background flow harmonic v_2^{bgd} and the harmonic from the v_n pipe. The first term on r.h.s. is dominated by the variance of the background multiplicity, which is $\langle N_{\text{bgd}}^4 \rangle - \langle N_{\text{bgd}}^2 \rangle^2$. In other words, background multiplicity from appropriate events does not exactly cancel out from background multiplicity from mixed events, and residuals are proportional to overall multiplicity fluctuations. The first term competes with the second term, which is proportional to N_{tube} , which measures event-by-event local fluctuations. The physical content of the second term exists only on the deflected flow associated with the peripheral tube. The size of N_{tube} does not affect the event multiplicity when the summation starts from $n = 2$. Although the second term does not change the overall multiplicity of the system, it alters the flow harmonics, causing local flow and multiplicity fluctuations.

The above model can be placed in the context of the observed event plane and centrality dependence of two-particle correlations (QIAN et al., 2013; CASTILHO et al., 2017; CASTILHO et al., 2018). In particular the above expression eq 3.37 explicitly depends on φ_s , which can be used to study the event plane dependence of the correlation. The triangular dependence of the background contribution on ϕ_s suggests that it contributes oppositely to out-of-plane triggers than to in-plane triggers. In the out-of-plane direction, there is an overall suppression of the amplitude of the correlation and a double-peaked structure on the outside. Specifically, when considering the in-plane triggering with

$\varphi_s = 0$ and the out-of-plane triggering $\varphi_s = \pi/2$, the evolution of the correlation structure for the outer side $\Delta\phi \sim \pi$ can be easily deduced. In fact, experimental data (FENG, 2008; AGAKISHIEV et al., 2021; AGAKISHIEV et al., 2014) show that as φ_s increases, the overall correlation decreases, while the side correlation evolves from unimodal to bimodal. Since these observed features are consistent with the results obtained from the analysis, the peripheral model is simple but still meaningful. We note that such parameters remain largely unchanged if the ZYAM scheme is used to subtract the contribution of mixed events.

4 SUPPRESSION OF THE MULTIPLICITY FLUCTUATIONS IN PARTICLE CORRELATION

A relevant feature of collective flows is their fluctuations. On the one hand, these fluctuations carry key information about the initial eccentricity associated with the partial-degree-of-freedom dynamics governed by the underlying microscopic model. On the other hand, even if an event has an infinite number of events, for a single event, the finite multiplicity introduces additional fluctuations (QIAN et al., 2023; YE et al., 2023) due to statistical uncertainty. Specifically, the variance of flow harmonics remains bounded when we average over a large number of events. The latter does not disappear even if the event-by-event fluctuations in the initial conditions are completely eliminated. Although the two types of fluctuations are distinct in physical origin, in practice they can be indistinguishable from an empirical point of view.

Particle correlations at low and medium transverse momenta are another relevant observation that provides valuable information about the properties of the resulting medium. In practice, the associated di-hadron yields are represented by the pseudorapidity difference $\Delta\eta$ and the azimuthal spacing $\Delta\phi$. Stronger enhancements are observed compared to high lateral momentum (ADAMS et al., 2005a; ABELEV et al., 2009; ALVER et al., 2010a; ALVER et al., 2010b; AAMODT et al., 2011; AAD et al., 2012). Proximal correlations are often referred to as "ridges", a manifestation of pseudorapidity long-term correlations (ADLER et al., 2003; ADAMS et al., 2006). The structure has a narrow $\Delta\phi$ peak and a considerable extension at $\Delta\eta$ of about a few units. Furthermore, on the other side, the observed correlations exhibit interesting features, often referred to as "shoulders". Specifically, a doublet is observed in certain ranges of central magnitude and lateral momentum. These structures, in most cases, can be explained as a medium effect due to the hydrodynamic evolution of the system and the triangular flow.

This phenomenon was explored in greater depth by studying its trigger angle dependence and centrality dependence. The STAR collaboration reported the measurement of the two-particle azimuth correlation (FENG, 2008) as a function of the trigger particle's azimuth about the event plane, denoted by $\phi_s = |\phi_{tr} - \Psi_{EP}|$, evaluating different triggers and the associated transverse momentum p_T . Later, the analysis (AGAKISHIEV et al., 2014; AGAKISHIEV et al., 2021) was further developed in an attempt to separate the "jet" from the "ridge". In particular, the ridge yield is extracted by introducing a fast gap $|\Delta\eta|$ when counting hadron pairs. In carrying out this procedure, it is also assumed that the ridges have uniformity in the $|\Delta\eta|$ direction, but the jets do not. Nonetheless, even with such refinement applied, residual correlations exhibit some interesting characteristics. Recent studies show that the lateral structure evolves from an in-plane direction to an out-of-plane direction. More specifically, the correlations near and far from the trigger particle varied with ϕ_s . When the trigger particle moves from in-plane to out-of-plane, the proximal peak decreases, and the correlation of the outer peak may evolve from a single peak to a double peak as the value of ϕ_s increases.

In addition to the A+A collisions discussed above, recently CMS (KHACHATRYAN et al., 2015; KHACHATRYAN et al., 2010; CHATRCHYAN et al., 2013b; CHATRCHYAN et al., 2013c), ATLAS (AAD et al., 2013; AAD et al., 2016) and Alice (ABELEV et al., 2013; ABELEV et al., 2015;

ACHARYA et al., 2021) have also recently observed long-range correlations in small systems associated with p+A and p+p collisions and collective flow. Notably, the observed ridges are more pronounced in high-multiplicity events. Interestingly, hydrodynamic calculations using the SPheRIO code (QIAN et al., 2007a; ANDRADE et al., 2006; ANDRADE et al., 2008; GARDIM et al., 2011; GARDIM et al., 2012; HAMA et al., 2008; JR. et al., 2004) provided a good description of the ridges observed in A+A collisions. In a peripheral tube model was proposed to Describe the observations of the hydrodynamic method (ANDRADE et al., 2010; ANDRADE et al., 2012) for ridges and shoulders. The model is further elaborated, discussing event-plane dependence (QIAN et al., 2013) and centrality dependence (CASTILHO et al., 2017; CASTILHO et al., 2018), and considering fast gases. For the peripheral tube model, the evolution of the lateral correlation structure is attributed to the interaction between the triangular flow and multiple fluctuations. In particular, we find that, as a cumulant, the resulting two-particle correlation consists of two contributions. Multiple fluctuations in the background dominate the first fluctuation, while the second can be attributed to the peripheral tube. However, the size of the second term did not affect the overall diversity. The observed evolution of the laterally correlated structure can be explained as an interaction between the two contributions. This idea has been confirmed and further explored in realistic simulations.

With regard to the two-particle correlations in the peripheral tube model, the results discussed above show that the contribution related to the overall multiplicity fluctuations is separate from the remaining contributions related to eccentricity under the initial conditions. The information carried by the latter can be extracted if the multiplicity fluctuations are eliminated. The purpose of this study is to propose a scheme to suppress multiple functions from two-particle correlations. Thus, one can directly obtain the particle correlations associated with the initial geometric fluctuations.

4.1 RELATED SETTINGS OF SPHERIO FLUID DYNAMICS

In this section, we elaborate on the scheme for computing multiple fluctuations associated with two particles. Numerical simulations were performed using the hydrodynamic code SPheRIO (HAMA; KODAMA; SOCOLOWSKI JR., 2005; HAMA; KODAMA; QIAN, 2021). These calculations were performed for initial conditions produced by placing a high energy tube on top of the background energy density distribution, reminiscent of the main features provided by microscopic event generators such as NeXuS (DRESCHER et al., 2002; DRESCHER et al., 2001) and EPOS (WERNER; LIU; PIEROG, 2006; WERNER; KARPENKO; PIEROG, 2011) . Specifically, fitting an almond shape to initial conditions for a 20%-40% centrality window yields following energy distribution

$$\epsilon = \epsilon_{\text{bgd}} + \epsilon_{\text{tube}}, \quad (4.1)$$

where

$$\epsilon_{\text{bgd}} = \epsilon_0(9.33 + 7r^2 + 2r^4)e^{-r^{1.8}}, \quad (4.2)$$

with

$$r = \sqrt{0.41x^2 + 0.186y^2}.$$

and the profile of the high energy tube is calibrated to that of a typical peripheral tube in NeXuS IC

$$\epsilon_{\text{tube}} = 12e^{-(x-x_{\text{tube}})^2-(y-y_{\text{tube}})^2}, \quad (4.3)$$

where the tube is located at a given value of energy density close to the surface, whose coordinates on the transverse plane read

$$\begin{aligned} x_{\text{tube}} &= \frac{r_{\text{tube}} \cos \theta}{\sqrt{0.41 \cos^2 \theta + 0.186 \sin^2 \theta}} \\ y_{\text{tube}} &= \frac{r_{\text{tube}} \sin \theta}{\sqrt{0.41 \cos^2 \theta + 0.186 \sin^2 \theta}}. \end{aligned} \quad (4.4)$$

Among them, the azimuth angle of tube θ is randomized in different events, and the parameter r_{tube} is selected as 4.3 fm in the calculation. Using the above parameterized initial conditions, the hydrodynamic equations were then solved with the SPHeRIO code and the resulting two-particle correlations were calculated. Event-by-event fluctuations for initial conditions by introducing a normal distribution in the background.

$$\epsilon_0 \sim N(\bar{\epsilon}_0, \sigma^2), \quad (4.5)$$

where the mean $\bar{\epsilon}_0 = 1.0$, and the variance $\sigma^2 = 0.01$, at the initial instant $\tau = 1.0$ fm/c. Here, the position variables, r, x, y , are expressed in the unit of fm, and the energy densities ϵ and ϵ_0 are in the unit of $\text{GeV} \cdot \text{fm}^{-3}$. Initial conditions are assigned to SPHeRIO, where the hydrodynamic equations are numerically solved event-by-event, accounting for fluctuations in the initial energy distribution. We generate 1000 events to simulate the 200 A GeV Au-Au collisions. At the end of the hydrodynamic evolution of each event, hadron emission was performed in a Cooper-Frey recipe using a Monte Carlo generator. In our calculations, the Monte Carlo process is performed 50 times for a given event. Subsequently, hadron decay was also considered.

In Fig. 3, we present the initial conditions of the peripheral tube model and the time evolution at different time instants. As the system expands, the flow is observed to be deflected by the high-energy tube, which creates a local triangular component. The velocity vectors indicate that the flow near the tube points mainly in two directions, forming an angle of about 60 degrees, observed in the profiles at $\tau = 5.0$ and 7.0 fm/c. The latter is also demonstrated in synthetic two-particle correlations, as described below.

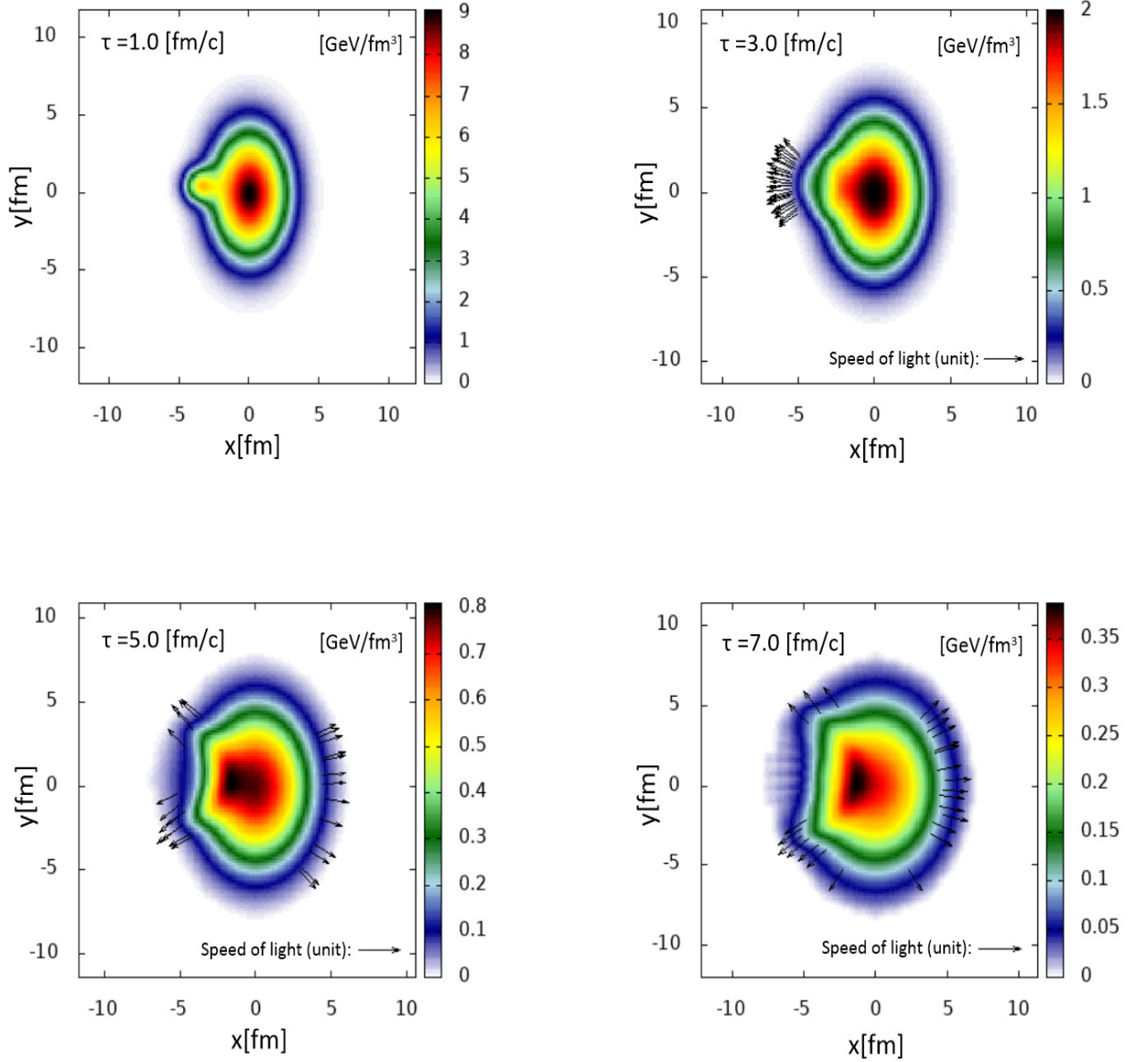


Fig. 3 – The peripheral tube model's initial conditions and the temporal evolutions evaluated at different instants $\tau = 1.0, 3.0, 5.0$, and 7.0 fm/c. The color scale describes the energy density expressed in $\text{GeV} \cdot \text{fm}^{-3}$. The vectors represent the fluid velocity, whose magnitude is shown relative to that of the speed of light, indicated by the vector at the bottom-right corner.

4.2 STRUCTURE THAT SUPPRESSES MULTIPLICITY FLUCTUATIONS

To assess two-hadron correlations, two-particle cumulants are used. The background modulation is then subtracted from the appropriate two-particle correlation according to Eq. 3.32. In Fig. 4 we present the subtracted two-particle correlation in the solid black curve. The resulting correlation structures were evaluated and displayed for different trigger angles from in-plane to out-of-plane orientation. Calculated correlations as a function of $\Delta\phi$ vary with different trigger directions, especially at a

distance. In the in-plane direction, the correlation has a single peak that is wider on the outer side than on the proximal side. Correlation structures on the outside are continuously involved in the formation of doublets in out-of-plane directions. This property is consistent with measurements of stars.

However, one can also perform numerical simulations by manually removing event-by-event fluctuations in the background. This can be achieved by assigning $\sigma = 0$ in Eq. (4.5). The rest of the calculation results remain essentially the same. The dashed red curve in Fig. 4 represents the subtracted two-particle correlation in the results.

Compared with the solid black curve, it can be observed that the evolution of the related structure from the in-plane to out-of-plane direction is clearly suppressed. In fact, the two-particle correlations mostly remain constant. This can be intuitively understood by looking at Eq. (3.37). Eq. (3.37) for the first term of the r.h.s. is proportional to background multiplicity fluctuations, mainly governed by the fluctuation Eq. (4.5) under initial conditions. Eq. (3.37) for the second term of the r.h.s. is dominated by the local flow deflection associated with the tube, which remains constant from event to event. Because we can remove the population fluctuations Eq. (4.5) by using $\sigma = 0$ in Eq. (3.37) regarding the first term of r.h.s. is significantly suppressed. Hence, the two-particle correlation is less pronounced in magnitude and is essentially dominated by the tube-related elliptical and triangular flow components. In other words, numerical simulations show that the analysis leads to Eq. (3.37) being relevant.

Based on the above arguments, we propose the following scheme to calculate the correlation of two particles using cumulants, as shown in Fig. 5. Instead of using all emitted hadrons, the multiplicity of different events is normalized to their average. Specifically, if an event has more hadrons than the average, the excess hadrons are discarded. These hadrons will be randomly selected. On the other hand, if an event has fewer hadrons than the average, this deficiency will be compensated for by a second random selection of some hadrons. With the above strategy, one always has the same number of hadron pairs when calculating the cumulant. Multiplicity fluctuations will be effectively eliminated.

In Fig. 4, empty blue circles represent the two-particle correlations that arise when implementing the proposed suppression scheme. The result is satisfactory because the empty blue circle is basically on top of the dashed red curve. The subtracted two-particle correlation does not change significantly in the in-plane and out-of-plane directions. Compared to the analytic result, it is understood to be essentially governed by the r.h.s. of Eq. (3.37).

Before concluding this section, we note that there is another perspective to understand the normalization described above, in general, multi-particle correlators and cumulants can be analyzed in terms of moments.

$$\mu_{n_1, n_2, \dots, n_N} = \langle e^{i(n_1\varphi_1 + n_2\varphi_2 + \dots + n_N\varphi_N)} \rangle, \quad (4.6)$$

Similar to normalization for two-particle correlations, the moments defined in the q -vector must be computed in a multiplicity-independent manner. For example, the calculation involves one of the

following

$$\begin{aligned}
N\langle 2 \rangle_{n_1, n_2} &= Q_{n_1, 1} Q_{n_2, 1} - Q_{n_1 + n_2, 2} \\
&= \sum_{k_1=1}^M e^{in_1 \varphi_{k_1}} \sum_{k_2=1}^M e^{in_2 \varphi_{k_2}} - \sum_{k_3=1}^M e^{i(n_1 + n_2) \varphi_{k_3}},
\end{aligned} \tag{4.7}$$

must be normalized by the denominator

$$D\langle 2 \rangle_{n_1, n_2} = N\langle 2 \rangle_{0, 0}, \tag{4.8}$$

so that multiplicity does not enter the ratio $N\langle 2 \rangle_{n_1, n_2} / D\langle 2 \rangle_{n_1, n_2}$. Also, the event-average part of the cumulant does not explicitly involve multiplicity and its fluctuations. As an example, the two-particle cumulant is defined as follows (BORGHINI; DINH; OLLITRAULT, 2001c)

$$\langle \langle e^{in(\varphi_1 - \varphi_2)} \rangle \rangle = \langle e^{in(\varphi_1 - \varphi_2)} \rangle - \langle e^{in\varphi_1} \rangle \langle e^{-in\varphi_2} \rangle, \tag{4.9}$$

where the average $\langle \dots \rangle$ includes the average involving all the hadrons for a given event and the event average. For the latter, the difference in multiplicity between different events should not play a role.

So, we elaborate a scheme to suppress multiple fluctuations in two-particle correlations. In the computation of particle correlations, suppression is achieved by sufficient normalization polymorphism when pairing proper and mixed events. Thus, the cumulant is obtained in a multiplicity-independent manner. As the multiple fluctuations are fully extracted, the resulting two-particle correlations show major features governed by the peripheral tube model. The latter were numerically calculated by means of Monte Carlo simulations. This normalization scheme is naturally consistent with multi-particle correlators that are usually evaluated in terms of ratios of q-vectors.

The correlation of particles in the environment dominated by collective flow is mainly analyzed. Although there is essentially no difference from an empirical point of view, particle correlations arise due to different physical origins. In relativistic heavy-ion collisions, collective flow, conservation laws, relics of microjets, and HBT interferometry all contribute to the eventual two-particle correlation, especially the "ridge". To efficiently obtain relevant physical content, observables must be developed to suppress or isolate specific fluctuation-independent contributions. In this regard, we can exploit empirical observations to extract information about initial states that also have intrinsic fluctuations due to the underlying microscopic physical system. Attempts to distinguish the contributions of different physical mechanisms have been ongoing efforts from both theoretical and experimental perspectives. The role of collective flow on particle correlation and the associated accumulation method were first elaborated and generalized by Boggini, Olitero et al (BORGHINI; DINH; OLLITRAULT, 2001a; BORGHINI; DINH; OLLITRAULT, 2001c). proposed the use of particle correlations to extract flow harmonic information, and in practice further developed (BILANDZIC; SNELLINGS; VOLOSHIN, 2011) vectors in addition to generating functions.

These studies are mainly based on the assumption that collective flow completely dominates the observed particle correlations. As described in the Ref. (BORGHINI; DINH; OLLITRAULT, 2001a),

with increasing event multiplicity and considering multi-particle correlations, the effects of non-flow are generally suppressed by flow. Experimentalists use these accumulation methods extensively to obtain flow harmonics involving averages between events of different multiplicity, where multiplicity fluctuations inevitably play a role. In the Ref. (BILANDZIC; SNELLINGS; VOLOSHIN, 2011), it was suggested that a weight could be assigned to the event average, and that the best approach might be to compute the cumulant at a fixed multiplicity and then average over the entire sample of events. For a given centrality window, this is because the multiple fluctuations mainly come from statistical sources. The latter can be attributed to either the finite size of the centrality window or the finite multiplicity (MA et al., 2020b; MA et al., 2020a; QIAN et al., 2023; YE et al., 2023), which are largely independent of the physics of interest. In this regard, the proposed method is inspired by the above observations and the analysis encountered in (QIAN et al., 2013).

In our calculations, we only elaborate on two-particle cumulant. But following this train of thought, one may generalize the two-particle cumulant

$$\langle e^{i2(\varphi_1 - \varphi_2)} \rangle,$$

to the four-particle cumulant

$$\langle e^{i2(\varphi_1 + \varphi_2 - \varphi_3 - \varphi_4)} \rangle = \langle e^{i2(\varphi_1 + \varphi_2 - \varphi_3 - \varphi_4)} \rangle - \langle e^{i2(\varphi_1 - \varphi_3)} \rangle \langle e^{i2(\varphi_2 - \varphi_4)} \rangle - \langle e^{i2(\varphi_1 - \varphi_4)} \rangle \langle e^{i2(\varphi_2 - \varphi_3)} \rangle,$$

using the same reasoning that was employed to derive Eq. (3.37). Again, mixed event correlation will not include any contribution between the background and the tube, as it cancels out in the event-by-event average. Intuitively, the resulting expression will involve higher moments of the multiplicity fluctuations such as $\langle N_{\text{bgd}}^4 \rangle - \langle N_{\text{bgd}}^2 \rangle^2$, which is expressed in terms of angle differences $\varphi_1 - \varphi_4$, $\varphi_2 - \varphi_3$, and $\varphi_1 - \varphi_3$. Unfortunately, to our knowledge, it seems that such measurements are not straightforward and have not been performed experimentally.

Regarding the application of the proposed method, the present study involves a scheme primarily devoted to suppressing the multiplicity fluctuations in A+A collisions. For such a scenario, events are typically characterized by high multiplicity, ensuring that collective flow emerges and dominates the resultant particle correlations. Moreover, the numerical simulations presented in this work show that the proposed scheme can indeed effectively suppress the multiplicity and extract relevant information on particle correlations. The good agreement with the theoretical results is not difficult to understand because the latter's dynamics are primarily hydrodynamics. It is also noted that one can always normalize residual correlations by the number of particle pairs to compare to the simulations or experimental data. On the other hand, one must point out that one encounters a somewhat different physical picture for small systems. The magnitude of the multiplicity of the underlying event may determine whether collectivity plays a role in the produced matter. Events with different multiplicities are likely to be governed by entirely distinct physical mechanisms. Even in the framework of the tube model, if one randomizes the number of tubes to the extent that it affects the overall initial geometric shape of the underlying event, the multiplicity fluctuations will be entangled with those of flow harmonics and substantially affecting the validity of the approach. Therefore, special care

should be taken regarding the apparent suppression of multiplicity fluctuations between such events by employing the proposed algorithm, which might indiscriminately eliminate crucial physical content. Nonetheless, we understand that the proposed approach can be applied to the same extent as cumulant method. Even with a reasonable degree of flow fluctuations, the procedure can be performed to extract flow harmonics, as the experimentalist does. In this context, the method can be viewed as an alternative to the existing approaches. Specifically, when comparing its deviation from the others, a measure of uncertainty between available means to extract flow harmonics from the data. Regarding numerical simulations, it is relevant to perform more detailed analyses regarding different profiles of initial fluctuations, the dependence of the resulting correlations on transverse momentum cuts, and possible comparisons between models of hydrodynamic and transport origins. We plan to carry out such studies in future works.

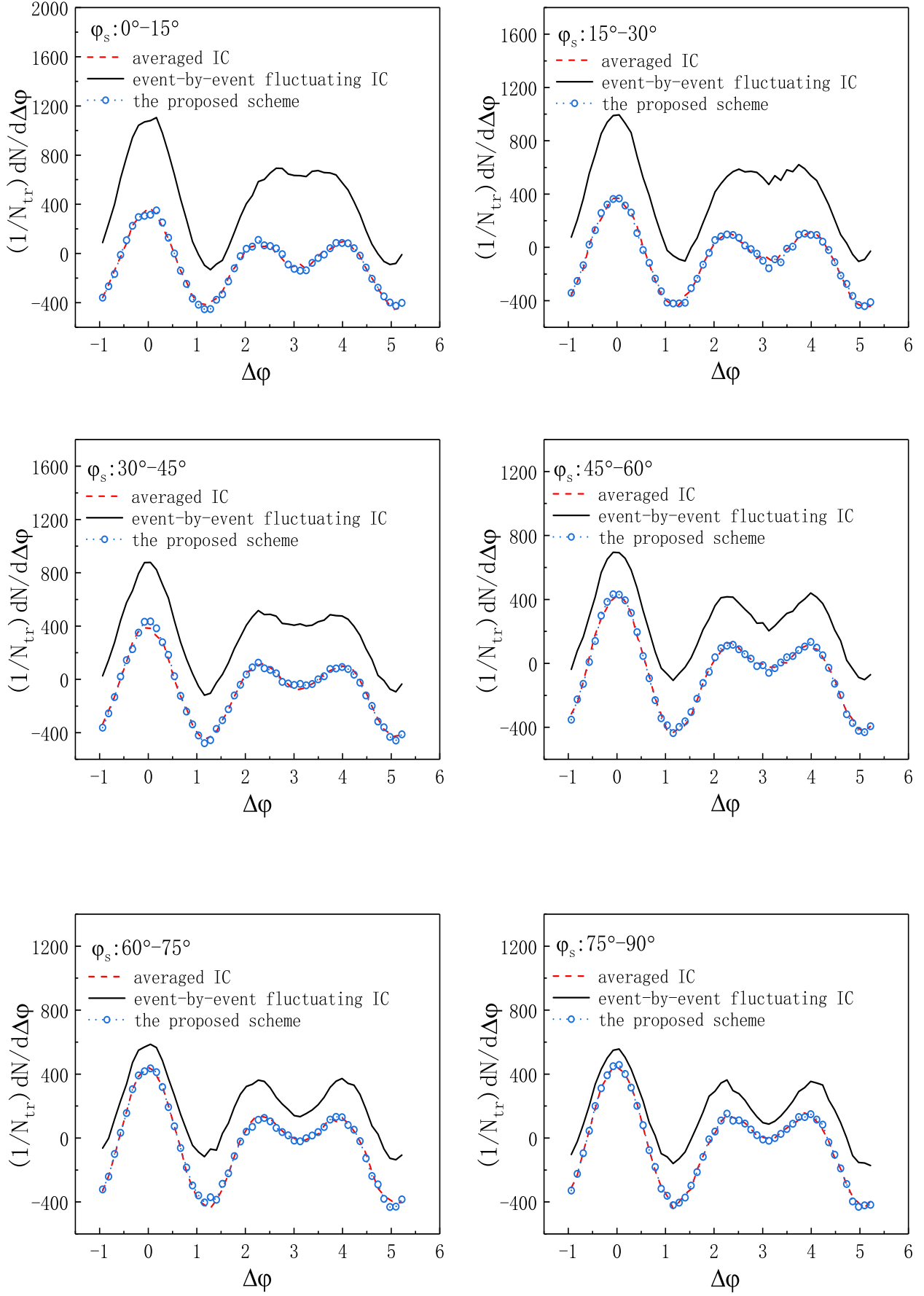


Fig. 4 – The subtracted two-particle correlations for three different scenarios.

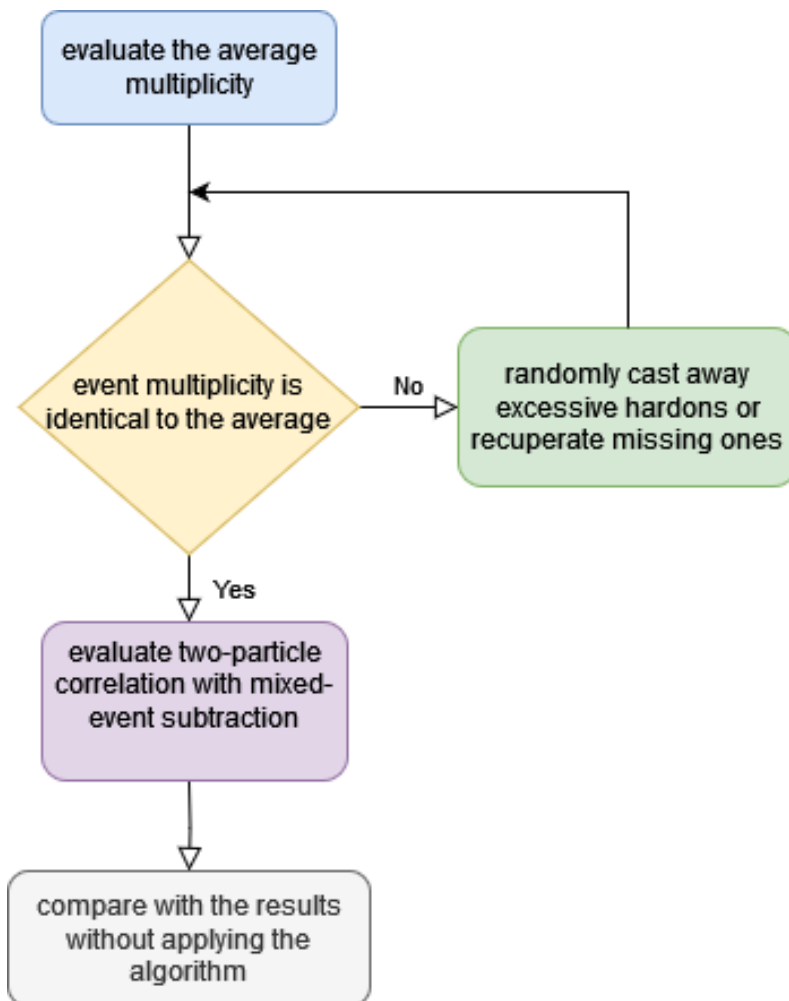


Fig. 5 – The flowchart of the proposed suppression algorithm.

5 ON STATISTIC FLUCTUATIONS IN HEAVY ION COLLISIONS

In this section, we will discuss the flow fluctuations due to the statistical and initial geometric fluctuations mentioned in the previous section. In a real event, the flow fluctuations, shown in terms of the variance of the flow harmonics, carry information about initial state geometry and final state statistical fluctuations.

In Fig. 6, we present the elliptic and triangular flow coefficients for events generated under smooth and fluctuating initial conditions. For fluctuating initial conditions, they are prepared according to Eq. (3.2), where $N = 10$ is assumed for the number of gold nuggets

$$z_0 = \frac{\int dx dy z \rho(z)}{\int dx dy \rho(z)}. \quad (5.1)$$

This leads to $\epsilon_1 = 0$ for all the events. One can readily verify the numerical implementation by considering the case $N = 2$, which, by the definition Eq. (3.1), one has $\epsilon_2 = 1$, free of the event-by-event fluctuations. For simplicity, we adopt the usual assumption that the genuine flow harmonics are proportional to the eccentricities,

$$v_n = \mathcal{C}_v \epsilon_n. \quad (5.2)$$

For illustrative purposes, it is further assumed that the proportional constant $\mathcal{C}_v = 1$. These flow coefficients, in turn, determine the one-particle distribution given by Eq. (1.1). In the present calculation, the flow harmonics in Eq. (1.1) are truncated at $n = 5$. We employ a Monte Carlo procedure to generate the hardons to simulate the realistic events with a finite multiplicity M . Subsequently, the values of v_n are estimated by the square root of $\widehat{v}_n^2$, which is a straightforward generalization of Eq. (6.3). To be specific,

$$\widehat{2}_{\{n,-n\}} \equiv \widehat{v}_n^2 = \frac{1}{M(M-1)} \sum_{i \neq j} \cos n(\varphi_i - \varphi_j), \quad (5.3)$$

where in our calculations, $n = 2, 3$. As a result, the estimated flow harmonics bear the statistical uncertainty due to finite multiplicity in the particle emission according to Eq. (1.1). The results are presented as a scatter plot of estimated flow harmonics as a function of the event multiplicity, v_n vs. M .

From Fig. 6, it can be seen that flow harmonics are affected by fluctuations, which decrease with increasing event multiplicity. As expected, geometric fluctuations in the initial state create additional uncertainties in flow harmonics on top of these statistical states. This turns out to be that the scatter of smooth initial condition events is mostly distributed in a narrower region above the scatter produced by fluctuating initial conditions. Furthermore, this difference becomes more significant as the event diversity increases.

For events under smooth initial conditions, the variance of flow harmonics decreases with increasing

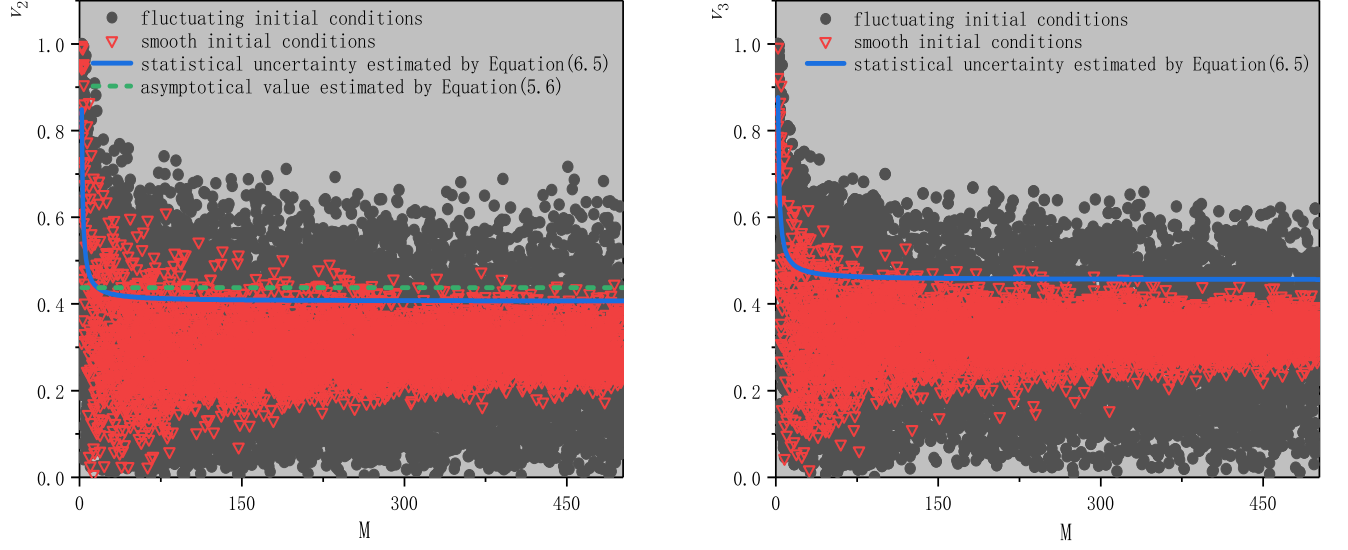


Fig. 6 – The extracted elliptic and triangular flow coefficients for the events of smooth and fluctuating initial conditions, as a function of multiplicity per event M for $N = 10$. The values of v_2 are estimated using $\sqrt{\widehat{v_n^2}}$ given by Eq. (5.3). The hollow red triangles represent the results obtained by the smooth initial condition, and the filled black circles are those by the identical point-source model. The solid blue curve estimates the statistical uncertainties using Eq. (6.5). The dashed green curve shows the asymptotical value of the square root of the variance given by Eq. (5.6) due to event-by-event fluctuations.

event multiplicity and disappears at $M \rightarrow \infty$. On the other hand, for events generated by fluctuating initial conditions, although the variances of flow harmonics decrease with event multiplicity, they approach a constant value, as discussed and evaluated further below.

The quantitative values in the table confirm the above results in Tab. 1, where one estimates elliptic and triangular flows and their variances. To show the robustness of our conclusions, we also performed calculations using a more significant number $N = 50$. The latter results are shown in Tab. 2. The shimmer can be observed, and the main feature remains: limited multiplicity that creates additional flux variance based on event-by-event initial conditions.

In Fig. 7, we present the distribution of elliptical flow and triangular flow. Likewise, distributions associated with event-by-event fluctuating initial conditions have more significant deviations than distributions associated with smooth initial conditions. As the number of event multiplicity increases, the variance decreases due to the suppression of statistical uncertainty, which is consistent with the results shown in Fig. 6 and Tab. 2.

Also, the difference between fluctuating and smooth initial conditions is more pronounced. Since the probability distribution of flow is a relevant observable that can be measured experimentally, this distinction may lead to interesting physical implications. We now proceed to analyze the above-mentioned flow fluctuations from an analytical perspective. In particular, we reevaluate Eqs. (6.4)

Tabela 1 – The calculated flow harmonics and their fluctuations using the identical point-source model with $N = 10$. The calculations are carried out for smooth and fluctuating initial conditions with event multiplicity $M = 10, 20, 50, 100$, and 500.

M	IC	$E[v_2]$	$\text{Var}[v_2]$	$E[v_3]$	$\text{Var}[v_3]$
10	smooth	0.375	3.467×10^{-2}	0.393	3.039×10^{-2}
	fluctuating	0.397	3.647×10^{-2}	0.409	3.359×10^{-2}
20	smooth	0.316	2.119×10^{-2}	0.338	1.870×10^{-2}
	fluctuating	0.341	2.633×10^{-2}	0.359	2.480×10^{-2}
50	smooth	0.283	1.116×10^{-2}	0.321	9.532×10^{-3}
	fluctuating	0.307	2.044×10^{-2}	0.335	2.125×10^{-2}
100	smooth	0.287	6.533×10^{-3}	0.328	4.778×10^{-3}
	fluctuating	0.292	1.812×10^{-2}	0.324	1.984×10^{-2}
500	smooth	0.295	2.410×10^{-3}	0.331	1.288×10^{-3}
	fluctuating	0.282	1.696×10^{-2}	0.319	1.833×10^{-2}
1000	smooth	0.297	2.041×10^{-3}	0.333	8.467×10^{-4}
	fluctuating	0.289	1.769×10^{-2}	0.317	1.832×10^{-2}

Tabela 2 – The same as Tab. 1, but the calculations are carried out for $N = 50$.

M	IC	$E[v_2]$	$\text{Var}[v_2]$	$E[v_3]$	$\text{Var}[v_3]$
10	smooth	0.312	2.332×10^{-2}	0.342	2.712×10^{-2}
	fluctuating	0.348	2.879×10^{-2}	0.347	2.744×10^{-2}
20	smooth	0.249	1.723×10^{-2}	0.275	1.461×10^{-2}
	fluctuating	0.251	1.769×10^{-2}	0.287	1.944×10^{-2}
50	smooth	0.193	7.953×10^{-3}	0.216	8.454×10^{-3}
	fluctuating	0.215	1.114×10^{-2}	0.242	1.341×10^{-2}
100	smooth	0.173	4.774×10^{-3}	0.207	5.051×10^{-3}
	fluctuating	0.186	8.352×10^{-3}	0.221	1.136×10^{-2}
500	smooth	0.167	9.794×10^{-4}	0.217	1.069×10^{-3}
	fluctuating	0.169	6.995×10^{-3}	0.215	1.070×10^{-2}
1000	smooth	0.174	5.576×10^{-4}	0.213	4.839×10^{-4}
	fluctuating	0.170	7.396×10^{-3}	0.209	9.398×10^{-3}

and (6.5) in the presence of initial eccentricity fluctuations.

$$E \left[\widehat{2}_{\{2,-2\}} \right] = \mathcal{C}_v^2 E [|\epsilon_2|^2] = \frac{2\mathcal{C}_v^2}{N}, \quad (5.4)$$

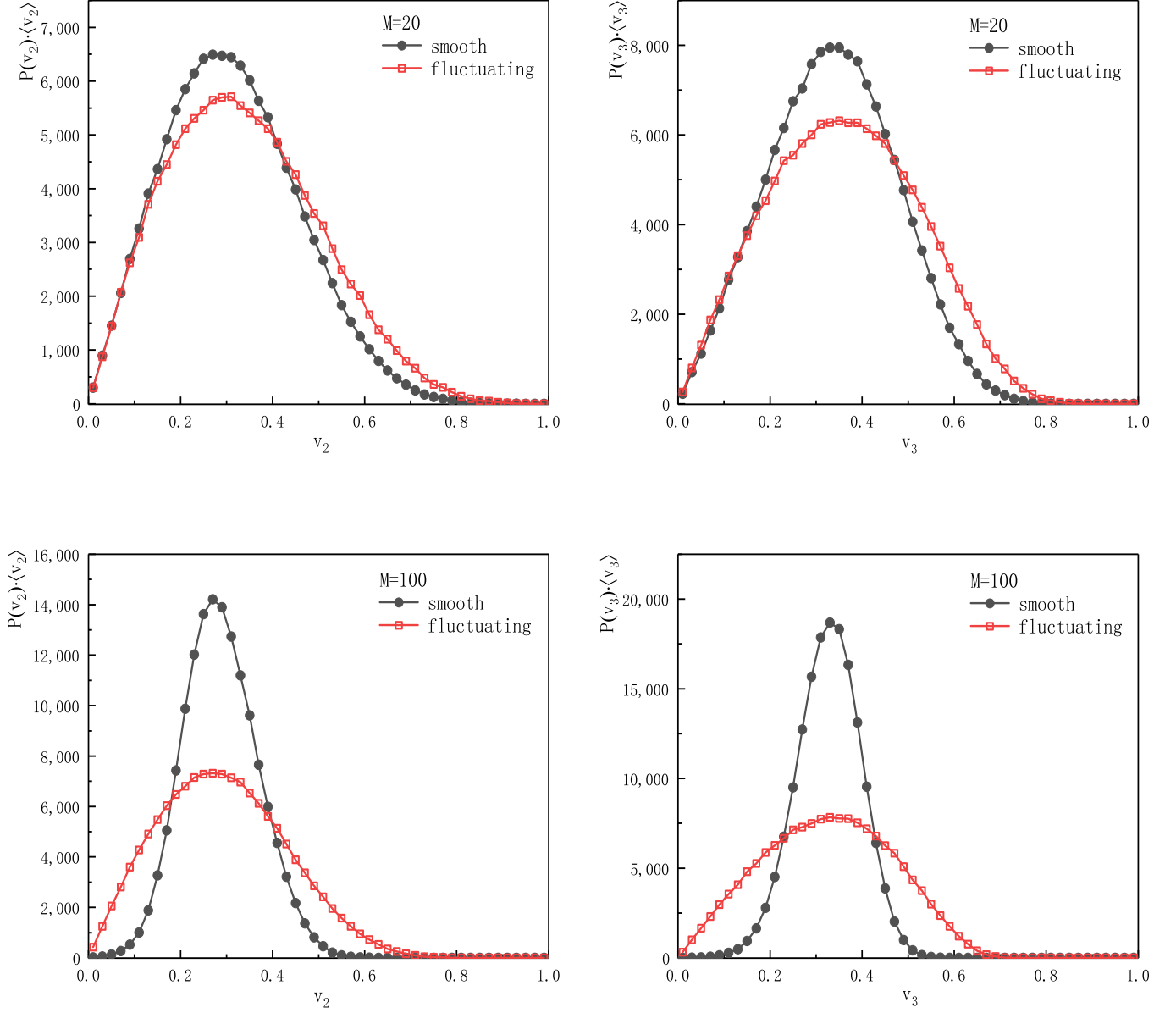


Fig. 7 – The probability distributions of triangular flow extracted for events of smoothed (black filled dotted) and event-by-event fluctuating (red empty squares) initial conditions generated with $N = 10$. The calculations are carried out by considering different event multiplicities $M = 20$ (top row) and $M = 100$ (bottom row).

Similarly, the variance Eq. (6.5) gives

$$\begin{aligned}
 \text{Var} \left[\hat{2}_{\{2,-2\}} \right] &= \frac{1 + \mathcal{C}_v^2 \text{E} [|\epsilon_2|^2]}{M(M-1)} + 2 \frac{M-2}{M(M-1)} (\mathcal{C}_v^2 \text{E} [|\epsilon_2|^2] + \mathcal{C}_v^3 \text{E} [|\epsilon_2|^2 |\epsilon_4|]) \\
 &+ \frac{(M-2)(M-3)}{M(M-1)} \mathcal{C}_v^4 \text{E} [|\epsilon_2|^4] - \mathcal{C}_v^4 \text{E} [|\epsilon_2|^2]^2 \\
 &= \frac{1 + \frac{2\mathcal{C}_v^2}{N}}{M(M-1)} + 2 \frac{M-2}{M(M-1)} \left(\frac{2\mathcal{C}_v^2}{N} + \mathcal{C}_v^3 \text{E} [|\epsilon_2|^2 |\epsilon_4|] \right) \\
 &+ \frac{(M-2)(M-3)}{M(M-1)} \frac{8\mathcal{C}_v^4}{2N + N^2} - \frac{4\mathcal{C}_v^4}{N^2}.
 \end{aligned} \tag{5.5}$$

At the limit of infinite multiplicity, the above result gives

$$\lim_{M \rightarrow \infty} \text{Var} \left[\widehat{2}_{\{2,-2\}} \right] = \frac{8\mathcal{C}_v^4}{2N + N^2} - \frac{4\mathcal{C}_v^4}{N^2}, \quad (5.6)$$

As long as N remains finite, it will not disappear. For numerical simulations, its asymptotic value of Eq. (5.6) corresponds to flow variance due entirely to inter-event fluctuations. When the event multiplicity M becomes quite significant, it becomes dominant and thus the statistical uncertainty in the estimate of Eq. (6.5) disappears. On the other hand, flow fluctuations are completely governed by the statistical uncertainty for the events generated by smooth initial conditions. Compared with the results presented above in Figs. Fig. 6 and Fig. 7, the deviations of flow harmonics due to initial geometric fluctuations and statistical uncertainties are observed to be in accordance with the analytic results.

From a statistical point of view, we have carefully studied fluctuations in flow harmonics in this work. In particular, we explore two different types of averages. For the first event, we assume that the diversity of each event is finite, but consider an infinite number of events. For the second case, an average is taken for a given event, considering particles at the hydrodynamic limit. Even assuming that the elliptic flow v_2 is well defined in Eq. (1.1) without any fluctuations, it inevitably produces a finite variance in the case of events with a limited diversity. In this case, we point out that one has to employ a specific scheme to estimate flow harmonics, which is, by definition, a statistical estimator. It is inevitably subject to finite variance due to finite statistics. From a physical point of view, we argue that, for realistic events, flow fluctuations, shown as flow harmonic variances, carry key information about initial state geometry and final state statistical fluctuations. In practice, these two types of variance will be mixed and may be indistinguishable due to fluctuations in the initial state from which the latter is understood to carry essential information about the underlying microphysical system. Event-by-event flow fluctuations are of a purely statistical nature. For example, if the multiplicity approaches infinity, this statistical uncertainty will disappear.

One of the motivations for extensive numerical simulations of heavy-ion collisions in terms of experimental observations, especially collective flow and particle correlation, is to distinguish the underlying microscopic mechanisms of the initial conditions. From a hydrodynamic point of view, these differences can manifest themselves in terms of fluctuations and long-term correlations, which in turn may be encoded by eccentricity and its fluctuations in initial conditions. In this regard, it is generally accepted that fluid dynamics translates eccentricity primarily linearly into final-state flow fluctuations. As mentioned above, we believe that fluctuations in the final flow consist of two components. Therefore, if one tries to extract information about fluctuations in initial conditions and their fluctuations due to finite statistics through flow harmonics, the limited multiplicity may interfere with such efforts. Event fluctuations have long been an important topic in the study of relativistic heavy ion collisions. Current research focuses on statistical fluctuations, especially their interaction with those arising from finite quanta of correlated systems. Furthermore, as pointed out in the main text, under the initial conditions taken in the continuum limit, geometric fluctuations may introduce eccentricity. Much effort has been devoted to the latter case. It is a potentially interesting topic to systematically assess or differentiate flow fluctuations associated with different causes, which could

lead to further implications. As shown in Fig. 7, the difference in flow variance between the smooth and event-wise fluctuating initial conditions is more significant as the event multiplicity increases, which is due to the suppression of the statistical uncertainty caused by the multiplicity. This result may be interesting physically because of its observable implications. Furthermore, based on experimental data, the possibility of extracting flow harmonics using a statistical estimator with less variance is also a topic worthy of further exploration.

6 MLE AS A FLOW ESTIMATOR

As mentioned in the previous section, due to the finiteness of the multiplicity of events, the statistical fluctuations it brings cannot be ignored. This uncertainty is different from the uncertainty caused by fluctuations in the initial conditions. In particular, the statistical error of the particle correlation method may even be larger than that of the event plane method. In addition, for events with low multiplicity, the multi-particle correlation may be more affected.

To elucidate the aforementioned statement, let us consider the subsequent illustrative example. By assuming that the flow is constituted by independent particle emission according to

$$f(\phi_1, \phi_2) = f_1(\phi_1)f_1(\phi_2), \quad (6.1)$$

where f_1 is defined by Eq. (1.1), one has

$$\langle 2 \rangle_{2,-2} \equiv \langle e^{i2(\phi_1 - \phi_2)} \rangle = \langle \cos 2(\phi_1 - \phi_2) \rangle + i \langle \sin 2(\phi_1 - \phi_2) \rangle = v_2^2, \quad (6.2)$$

where $\langle \dots \rangle \equiv \int d\phi_1 d\phi_2 f(\phi_1, \phi_2) \dots$ is evaluated by averaging out all the particle pairs of a given event, in accordance with the probability given by the one-particle distribution function, at the limit of infinity multiplicity. One notes that the imaginary part of Eq. (6.2) vanishes by employing the orthogonal relationship of sinusoidal functions. On the other hand, for an event of a finite number of multiplicity M , it is natural to estimate v_2^2 by a discrete version of Eq. (6.2), namely,

$$\hat{v}_2^2 \equiv \frac{1}{M(M-1)} \sum_{i \neq j} \cos 2(\phi_i - \phi_j), \quad (6.3)$$

where the summation enumerates all distinct $M(M-1)$ ordered pairs. In this regard, one can view Eq. (6.3) as a *statistical estimator* (RICE, 2006) $\hat{\theta}$ of a physical quantity $\theta = v_2^2$ for an individual event. Given the underlying distribution, as a statistical estimator, its *quality* is usually measured in terms of its expected value μ and variance σ^2 . In the case of Eq. (6.3), one has

$$\mu \equiv \mathbb{E} [\hat{v}_2^2] = v_2^2, \quad (6.4)$$

$$\sigma^2 \equiv \text{Var} [\hat{v}_2^2] = \frac{1 + v_2^2}{M(M-1)} + 2 \frac{M-2}{M(M-1)} v_2^2 (1 + v_4) + \frac{(M-2)(M-3)}{M(M-1)} v_2^4 - v_2^4. \quad (6.5)$$

Here, Eq. (6.4) indicates that the estimator Eq. (6.3) is unbiased. On the other hand, Eq. (6.5) clearly shows that the statistical uncertainty does not vanish as long as the multiplicity M is finite. Different from Eq. (6.3), Eq. (6.2) evaluates the expectation at the limit of infinite multiplicity. On the other hand, any given flow evaluation scheme must be applied to realistic events with finite multiplicity and, therefore, can always be viewed as a specific choice of statistical estimator. As a result, it is inevitably subject to some statistical uncertainty.

The present study is mainly motivated to explore the above aspect in flow measurements. We

examine the possibility of utilizing the MLE as a flow estimator. The MLE is a well-known approach to estimating the parameters of a hypothetical probability distribution when observed data is given. Its goal is achieved by maximizing the associated likelihood function in the parameter space. In other words, the estimated parameters, for our case, the flow harmonics, ensure that the observed data is most probable. The method is widely used in statistical inference due to its intuitive and flexible nature. In particular, as an asymptotically normal estimator, its efficiency is guaranteed in the sense that it is more accurate than any other estimator at the limit of a significant sample size. In terms of convergence, the method is either unbiased or asymptotically unbiased. Apparently, the context of relativistic heavy ion collisions meets most of the characteristics of MLE. In particular, the measurements performed at RHIC and LHC have accumulated significant events for different collision systems at different centralities.

6.1 MEASUREMENT OF FLOW HARMONICS USING PARTICLE CORRELATIONS

As mentioned above, the most prominent approaches to extract the flow harmonics are based on particle correlations (POSKANZER; VOLOSHIN, 1998). The cornerstone of such an approach is based on the following relation regarding k -particle correlation (BHALERAO; LUZUM; OLLITRAULT, 2011)

$$\langle k \rangle_{n_1, \dots, n_k} \equiv \langle e^{i(n_1\phi_1 + \dots + n_k\phi_k)} \rangle = v_{n_1} \dots v_{n_k} e^{i(n_1\Psi_{n_1} + \dots + n_k\Psi_{n_k})}, \quad (6.6)$$

where $\langle \dots \rangle$ is an average over distinct tuples of particles, again assuming independent particle emission according to Eq. (1.1) particles of a given event, at the limit of infinity multiplicity.

To concentrate on v_n , one usually chooses a specific set of $(n_1, \dots, n_k)$ so that $\sum_{j=1}^k n_j = 0$ (BHALERAO; LUZUM; OLLITRAULT, 2011), and therefore, all the coefficients involving the event planes cancel out in the exponential. Otherwise, the event plane correlator will become a part of the formalism (BHALERAO; OLLITRAULT; PAL, 2013). For instance, in the case of two-particle correlation $k = 2$, one may consider $n_1 = -n_2 = n$. Therefore, we have

$$\langle 2 \rangle_{n, -n} \equiv \langle e^{in(\phi_1 - \phi_2)} \rangle = \langle \cos n(\phi_1 - \phi_2) \rangle = v_n^2 \quad (6.7)$$

For an event with finite multiplicity M , where the azimuth angles of the measured particles read $\phi_1, \phi_2, \dots, \phi_M$, it is intuitive to adopt the following estimator (BILANDZIC et al., 2014)

$$\hat{v}_n^2 = \frac{1}{M(M-1)} \sum_{i \neq j} \cos n(\phi_i - \phi_j). \quad (6.8)$$

Subsequently, the first two moments of the estimator are found to be (BILANDZIC et al., 2014)

$$\mathbb{E} [\hat{v}_n^2] = v_n^2, \quad (6.9)$$

$$\text{Var} [\hat{v}_n^2] = \frac{1 + v_{2n}^2}{M(M-1)} + 2 \frac{M-2}{M(M-1)} v_n^2 (1 + v_{2n}) + \frac{(M-2)(M-3)}{M(M-1)} v_n^4 - v_n^4, \quad (6.10)$$

which is an unbiased estimator that has a finite variance that decreases with increasing multiplicity. This result readily falls back to Eqs. (6.4) and (6.5) given above.

To proceed, one may either proceed to evaluate cumulant using the formalism of generating functions, as first proposed by Borghini *et al.* (BORGHINI; DINH; OLLITRAULT, 2001a; BORGHINI, 2003; BORGHINI, 2007), or evaluate the multi-particle correlation straightforwardly (BILANDZIC; SNELLINGS; VOLOSHIN, 2011).

The above formalism can also be generalized to include weighted average (BILANDZIC *et al.*, 2014), where a specific weight w_k is associated with k th particle in the correlation. In particular, we have

$$\langle k \rangle_{n_1, n_2 \dots n_k} \equiv \langle e^{i(n_1 \phi_1 + n_2 \phi_2 + \dots + n_k \phi_k)} \rangle, \quad (6.11)$$

which is intuitively given by

$$\frac{\sum_{k\text{-tuples}} w_1 w_2 \dots w_k e^{i(n_1 \phi_1 + n_2 \phi_2 + \dots + n_k \phi_k)}}{\sum_{k\text{-tuples}} w_1 w_2 \dots w_k} \equiv \frac{N_{\langle k \rangle_{n_1, n_2 \dots n_k}}}{D_{\langle k \rangle_{n_1, n_2 \dots n_k}}}, \quad (6.12)$$

at finite multiplicity, where the summation is carried out for all distinct tuples, from which any auto-correlation should be removed.

In practice, the numerator and denominator of Eq. (6.12) can be expressed by employing the Q-vectors (DANIELEWICZ; ODYNIEC, 1985), defined as

$$Q_{n,p} \equiv \sum_{j=1}^M w_j^p e^{in\phi_j}, \quad (6.13)$$

where p is an exponent that can be chosen conveniently to simplify the resultant expressions. As an example, for $k = 2$, one has

$$N_{\langle 2 \rangle_{n_1, n_2}} = Q_{n_1, 1} Q_{n_2, 1} - Q_{n_1 + n_2, 2}, \quad (6.14)$$

$$D_{\langle 2 \rangle_{n_1, n_2}} = Q_{0, 1}^2 - Q_{0, 2}, \quad (6.15)$$

which falls back to (the real part of) the r.h.s. of Eq. (6.8) for $w_j = 1$ and $n_1 = -n_2 = n$.

Again, we note that Eqs. (6.12) and (6.14) are essentially estimators for a given event with finite multiplicity obtained experimentally, which, in turn, are subject to statistical uncertainties. To make the above generic statements concrete, we consider two simple examples. For an event with multiplicity M , let $k = 2$, $w_1 = w_2 = 1$ and $n_1 = -n_2 = n$, we have

$$E [N_{\langle 2 \rangle_{n_1, n_2}}] = M(M-1)v_n^2, \quad (6.16)$$

$$\text{Var} [N_{\langle 2 \rangle_{n_1, n_2}}] = M(M-1) \{ (1 + v_{2n}^2) + 2(M-2)v_n^2(1 + v_{2n}) - 2(2M-3)v_n^4 \}. \quad (6.17)$$

As a second example, consider $w_1 = w_2 = w_3 = 1$, $n_1 = 2$, $n_2 = 3$, $n_3 = -5$, and let us assume

$\Psi_2 = \Psi_3 = \Psi_5 = \Psi$ for simplicity, one finds

$$\mathbb{E} [N_{\langle 3 \rangle_{2,3,-5}}] = M(M-1)(M-2)v_2v_3v_5, \quad (6.18)$$

$$\begin{aligned} \text{Var} [N_{\langle 3 \rangle_{2,3,-5}}] &= M(M-1)(M-2) \{ [v_4v_6v_{10} + v_2^2v_4 + v_3^2v_6 + v_1^2v_{10} + 2v_2v_3v_5] \\ &+ 2(M-3) [(v_2^2v_5^2 + v_2v_3v_5v_6) + (v_2v_3v_4v_5 + v_3^2v_5^2) + (v_2^2v_3^2 + v_2v_3v_5v_{10})] \\ &+ (M-3) [(v_3^2v_4v_{10} + v_3^4) + (v_2^2v_6v_{10} + v_2^4) + (v_4v_5^2v_6 + v_5^4)] \\ &+ (M-3)(M-4) [2v_2v_3v_5^3 + 2v_2v_3^3v_5 + 2v_2^3v_3v_5 + v_3^2v_4v_5^2 + v_2^2v_5^2v_6 + v_2^2v_3^2v_{10}] \\ &+ (M-3)(M-4)(M-5)v_2^2v_3^2v_5^2 - M(M-1)(M-2)v_2^2v_3^2v_5^2 \}. \end{aligned} \quad (6.19)$$

Although a bit tedious, evaluating the last expression is straightforward by considering different combinations when one, two, or three particles from the two ordered triple (consisting of distinct particles, respectively) coincide. Eqs. (6.17) and (6.19) illustrate the fact that these estimators are subject to finite uncertainties and do not vanish, owing to finite statistics. The event planes do not coincide, and therefore, the r.h.s. of the three-particle correlation Eq. (6.18) actually furnishes the event plane correlation (BHALERAO; LUZUM; OLLITRAULT, 2011).

In this regard, one might speculate that MLE could serve as an alternative flow estimator, and such a possibility will be explored in the remainder of the thesis. The definition and elementary properties of the MLE are elaborated on in the following subsection. Subsequently, the approach is implemented numerically. The Monte Carlo simulations are performed, and the results will be compared to the conventional techniques, such as the particle correlation and event-plane methods.

6.2 MLE AND ITS APPLICATION AS A FLOW ESTIMATOR

In this subsection, we turn our attention to the MLE. For a given set of observations $y \equiv (y_1, y_2, \dots, y_M)$, we assume that they are sampled from a joint probability distribution governed by several unknown parameters $\theta \equiv (\theta_1, \theta_2, \dots, \theta_m)$. As mentioned in the introduction, one considers the following likelihood function $\mathcal{L}$ at the observed data

$$\mathcal{L}(\theta) \equiv \mathcal{L}(\theta; y) = f(y; \theta). \quad (6.20)$$

which is the joint probability density for the given observation evaluated at the parameters θ . The goal of MLE is to determine the parameters for which the observed data attains the highest joint probability, namely

$$\hat{\theta}_{\text{MLE}} = \arg \max_{\theta \in \Theta} \mathcal{L}(\theta), \quad (6.21)$$

where Θ is the domain of the parameters. In particular, for independent and identically distributed (i.i.d.) random variables, $f(y; \theta)$ is given by

$$f(y; \theta) = \prod_{j=1}^M f^{\text{uni}}(y_j; \theta). \quad (6.22)$$

Indeed, the above scheme can be readily applied in the context of collective flow in heavy-ion collisions. Considering an event consisting of M particles, the likelihood function reads

$$\mathcal{L}(\theta; \phi_1, \dots, \phi_M) = f(\phi_1, \dots, \phi_M; \theta) = \prod_{j=1}^M f_1(\phi_j; \theta), \quad (6.23)$$

where the likelihood function $\mathcal{L}$ is governed by the one-particle distribution function Eq. (6.1). The last equality is based on the assumption that the particles' azimuthal angles are i.i.d. variables, which gives Eq. (6.1) in the case of two particles. The parameters of the distribution, $\theta = (v_1, \Psi_1, v_2, \Psi_2, \dots)$, are the flow harmonics and event planes.

In practice, one often chooses the objective function to be the log-likelihood function ℓ

$$\ell(\theta; \phi_1, \dots, \phi_M) = \log \mathcal{L}(\theta; \phi_1, \dots, \phi_M) = \sum_{j=1}^M \log f_1(\phi_j; \theta). \quad (6.24)$$

Numerical calculations indicate that Eq. (6.24) is more favorable than Eq. (6.23), although as the multiplicity M becomes more significant, the appropriate implementation should be adopted to avoid increasing truncation error.

The maximum of ℓ occurs at the same value of θ as does the maximum of $\mathcal{L}$. For ℓ that is differentiable in its domain Θ , the necessary conditions for the occurrence of a maximum are

$$\frac{\partial \ell}{\partial \theta_1} = \dots = \frac{\partial \ell}{\partial \theta_m} = 0. \quad (6.25)$$

In this work, we will primarily focus on the dominant harmonic coefficients such as v_2 and v_3 even though the associated event planes are obtained simultaneously from Eq. (6.25).

As discussed in the introduction, MLE has asymptotical normality, which attains the Cramér-Rao lower bound when the sample size increases. In other words, no consistent estimator has a lower asymptotic mean squared error than the MLE. In the context of relativistic heavy-ion collisions, all the events of a given multiplicity asymptotically form a (multivariate) normal distribution

$$\hat{\theta}_{\text{MLE}} \sim N(\theta_0, (I_M(\theta_0))^{-1}), \quad (6.26)$$

where θ_0 represents the “true” value, and $I_M(\theta)$ is the Fisher information matrix, defined as

$$I_M(\theta) \equiv E_\theta \left[-\frac{d^2}{d\theta^2} \ell(\theta; \phi_1, \dots, \phi_M) \right], \quad (6.27)$$

where the expectation E_θ is taken with respect to the distribution of $f(\phi_1, \dots, \phi_M; \theta)$. For i.i.d. data, the Fisher information possesses the form

$$I_M(\theta) = M I_1(\theta), \quad (6.28)$$

where I_1 is the Fisher information matrix for a single observation. As a result, the standard deviation

Tabela 3 – The estimated flow harmonics v_n and event planes Ψ_n using MLE from a single event.

θ	v_2	v_3	v_4	Ψ_2	Ψ_3	Ψ_4
true values	0.2	0.08	0.25	2.25	0.0323	0.925
MLE	0.228	0.0782	0.253	2.36	0.0461	0.966

Tabela 4 – The results of the Wald, likelihood ratio, and score tests of hypotheses.

Wald	likelihood ratio	score
4.981	2.061	6.394

of MLE is expected to be roughly proportional to $\frac{1}{\sqrt{M}}$. These properties can be further quantified using the Wald, likelihood ratio, and score tests, which will be performed in the numerical simulations.

6.3 NUMERICAL RESULTS AND COMPARISON WITH OTHER METHODS

Based on the discussions in the preceding sections, we employ the Monte Carlo generator to simulate events and estimate the flow coefficients using the MLE method. The quality of the estimation is evaluated by the standard tests of hypotheses. Also, the results are compared with those obtained using the other approaches, namely, the particle correlation and event plane methods.

One primarily considers events consisting of independent particle emissions according to the one-particle distribution function Eq. (1.1), which is further truncated at the fourth order, and the directed flow is ignored

$$f_1(\phi) = \frac{1}{2\pi} [1 + 2v_2 \cos 2(\phi - \Psi_2) + 2v_3 \cos 3(\phi - \Psi_3) + 2v_4 \cos 4(\phi - \Psi_4)], \quad (6.29)$$

where the event planes Ψ_n are randomized between distinct events, and the harmonics are taken as

$$\begin{aligned} v_2 &= 0.2, \\ v_3 &= 0.08, \\ v_4 &= 0.25. \end{aligned} \quad (6.30)$$

We use the above parameters to generate a specific number of events with a given multiplicity. The MLE method is then employed to extract the flow harmonics. To validate the MLE approach, we aim to see whether the estimation of v_2 will be undermined by some other parameter that plays a more significant role in the underlying probability distribution. In this regard, we have particularly chosen a large and unphysical value for v_4 . We have verified that the results below are not sensitive to this specific choice. Besides, for simplicity, we do not consider the flow's momentum or rapidity dependence; therefore, only the particle's azimuthal angles are involved in the calculations.

As a first example, we consider one single event with a multiplicity $M = 1000$. In Tab. 3, we show the estimated parameters. The corresponding results using the Wald, likelihood ratio, and score tests are shown in Tab. 4.

The quality of the estimation is further evaluated by the statistical tests of hypotheses (NEYMAN,

1952), which are presented in Tab. 4. In particular, we perform three types of hypothesis testing: Wald, likelihood ratio, and score tests. Roughly speaking, all these tests assess constraints on statistical parameters by quantifying the deviation between the unrestricted estimate and its hypothesized value. Analysis such as confidence intervals for the estimator can be facilitated by the fact that all three tests asymptotically approach the χ^2 distribution, a sum of the squares of normal distributions under the null hypothesis. Intuitively, in terms of the log-likelihood function, one measures the precision of the estimation in terms of the horizontal and vertical distances between the estimator $\hat{\theta}$ and the true value θ_0 . Specifically, the Wald test assess how far away the estimator locates in the horizontal direction, and the likelihood ratio test gives that in the vertical direction, namely,

$$t_W = (\hat{\theta} - \theta_0)^T \hat{V}_M^{-1} (\hat{\theta} - \theta_0), \quad (6.31)$$

and

$$t_{LR} = 2 [\ell(\hat{\theta}) - \ell(\theta_0)]. \quad (6.32)$$

where the covariant matrix $\hat{V}_M$ is

$$\hat{V}_M = [I_M(\theta_0)]^{-1}, \quad (6.33)$$

where I_M is given by Eq. (6.27), the Fisher information under the null hypothesis and ℓ is the log-likelihood function, and the factor “2” ensures that it converges asymptotically to a χ^2 -distribution.

The score test measures the slope at the true value

$$t_S = S(\theta_0)^T \hat{V}_M S(\theta_0), \quad (6.34)$$

where

$$S(\theta) = \nabla_{\theta} \ell(\theta; \phi_1, \dots, \phi_M), \quad (6.35)$$

is the gradient of the log-likelihood function,

For the specific event, the Fisher information $I_1(\theta_0)$ is numerically found to be

$$\begin{pmatrix} 2.148 & 0.479 & 0.106 & -0.247 & -0.015 & 0.579 \\ 0.479 & 3.140 & 0.479 & -0.107 & -0.038 & 0.239 \\ 0.106 & 0.479 & 2.819 & -0.208 & -0.065 & 0.084 \\ -0.247 & -0.107 & -0.208 & 0.655 & 0.024 & -0.348 \\ -0.015 & -0.038 & -0.065 & 0.024 & 0.155 & 0.004 \\ 0.579 & 0.239 & 0.084 & -0.348 & 0.004 & 2.915 \end{pmatrix}, \quad (6.36)$$

where the dimension of the matrix is determined by that of θ . The MLE results manifestly sit within the 95% confident interval.

We have observed that, even for a single event, the flow harmonics estimated by the MLE method

are reasonable. In what follows, we compared the obtained flow harmonics with those extracted using particle correlations. Also, since the method gives a more favorable asymptotic mean squared error, it might be used to achieve a minimized statistical uncertainty when compared to other approaches. In this regard, we consider a scenario where the elliptic flow fluctuates on an event-by-event basis. To be specific, one replaces the first line of Eq. (6.30) with the following Gaussian distribution

$$v_2 \sim N(\mu = 0.2, \sigma^2 = 0.01). \quad (6.37)$$

The numerical results on the comparison between the two-particle correlation and MLE methods are presented in Tabs. 5 and 6. First, for each event of a given multiplicity, particles are drawn independently according to Eq. (6.29) using the parameters given by Eq. (6.30) and estimate the v_2 for individual events by the MLE and then calculate the average. In the calculations, for simplicity, we adopt the values estimated by the event-plane method for the remaining flow harmonics and the event planes. This applies to the results presented in Tabs. 5, 6, 9, and Fig. Fig. 8 in the current subsection, and also to Figs. Fig. 9-Fig. 12 below in Sec. 6.4, as well as to Tab. 10 in Sec. 6.5. For the remaining applications of MLE, the dimension of the parameter space θ is six-dimensional.

The resulting elliptic flow is obtained by the event average of MLE estimations and shown in Tab. 5. The corresponding variance of v_2 among different events, a measure of flow fluctuations, is also evaluated. The results obtained by the MLE method are compared with those using the two-particle correlation formula Eq. (6.8). Specifically, for the latter, the elliptic flow is estimated by taking the square root of the event average of the r.h.s. of Eq. (6.8), in accordance with the practice (NADDERD; MILOSEVIC; WANG, 2021). Also, the variance of the flow is evaluated by that of $\sqrt{\widehat{v_2^2}}$ for individual events, following Refs. (AGAKISHIEV et al., 2021). It is found that both methods give a reasonable estimation of the flow harmonics. As the multiplicity M increases, the variance extracted from the MLE method becomes smaller. It is noted that the values of the mean and variance numerically converge as long as one uses a significant number of events. The results presented in Tab. 5 are obtained using $N = 1000$ events. We also note that in this case, the origin of the variance of the elliptic flow is entirely statistical because v_2 in the one-particle distribution is well-defined. In this sense, the column of $\text{Var}[v_2^2]$ denoted by “true value” is filled with zeros.

For a realistic scenario, on the other hand, it is understood that the measured flow fluctuations also contain the part which is governed by the event-by-event fluctuations of initial geometry. The latter reflects the underlying microscopic physical model. This motivated us to consider the event-by-event fluctuation given in Eq. (6.37). The corresponding results are shown in Tab. 6. Compared with the results presented in the foregoing table, the average v_2 obtained using MLE is closer to the true value. This is understood as the particle correlation method estimates v_n^2 and contains a fraction of flow fluctuations (OLLITRAULT; POSKANZER; VOLOSHIN, 2009; CHATRCHYAN et al., 2013b). Regarding the variance, we note that the value shown in the last column becomes less than $\sigma^2 = 0.01$ since, in the Monte Carlos process, all negative harmonic coefficients are discarded. By definition, the MLE cannot entirely remove the flow fluctuations due to statistical uncertainty. Nonetheless, in principle, it attains a more accurate value than other estimators, particularly when the multiplicity becomes more significant.

Tabela 5 – The average and variance of the estimated elliptic flow v_2 using the two-particle correlation and MLE methods. The results are obtained using $N = 1000$ events and for different multiplicities per event. The averages $\mu[\dots]$ and variance $\text{Var}[\dots]$ of the estimators are evaluated on an event-by-event basis.

M	$\sqrt{\mu[\widehat{v}_2^2]}$	$\mu[\widehat{v}_2]$		$\text{Var}[\sqrt{\widehat{v}_2^2}]$	$\text{Var}[\widehat{v}_2]$	
	particle correlation	MLE	true value	particle correlation	MLE	true value
500	0.1978	0.2015	0.2	9.58×10^{-4}	7.77×10^{-4}	0
1000	0.1994	0.2021	0.2	4.78×10^{-4}	3.05×10^{-4}	0
2000	0.1997	0.2014	0.2	2.11×10^{-4}	1.78×10^{-4}	0

Tabela 6 – The same as Tab. 5, but the elliptic flow is generated by including additional Gaussian fluctuations governed by Eq. (6.37).

M	$\sqrt{\mu[\widehat{v}_2^2]}$	$\mu[\widehat{v}_2]$		$\text{Var}[\sqrt{\widehat{v}_2^2}]$	$\text{Var}[\widehat{v}_2]$	
	particle correlation	MLE	true value	particle correlation	MLE	true value
500	0.2221	0.2113	0.2	9.69×10^{-3}	1.11×10^{-2}	8.90×10^{-3}
1000	0.2252	0.2132	0.2	8.86×10^{-3}	1.03×10^{-2}	8.90×10^{-3}
2000	0.2252	0.2135	0.2	8.79×10^{-3}	1.03×10^{-2}	8.90×10^{-3}

In order to make the above discussions about flow fluctuations more concrete, in Fig. Fig. 8, we show the scatter plots for extracted v_2^2 and v_3^2 for individual events using both methods. In the calculations, the events are generated whose multiplicities are drawn randomly according to a uniform distribution up to $M = 5000$. The estimated flow harmonics are shown as a function of event multiplicity. From Fig. Fig. 8, it is observed that the estimations of MLE primarily possess a smaller statistical uncertainty when compared to other methods, as the hollow red triangles are confined mainly within the region determined by the filled black circles. These results are consistent with those shown above in Tabs. 5. It is also noted that v_n^2 derived from the particle correlation might be negative, but the values of v_n estimated by MLE are always found positive in our numerical calculations.

Now, we present the results on the event planes extracted by the MLE method. Since the particle correlation method does not provide an estimation of the event planes, the obtained results are compared with a conventional event-plane approach (POSKANZER; VOLOSHIN, 1998), adopting the following formula

$$\widehat{\Psi}_n = \frac{1}{n} \text{atan} \left(\sum_{j=1}^M \sin n\phi_j, \sum_{j=1}^M \cos n\phi_j \right). \quad (6.38)$$

The results are presented in Tab. 7.

To verify the statistical robustness of the method, we explore the multiplicity and event number dependence of the MLE method. The results are presented in Tabs. 8 and 9. To study the asymptotical normality of the MLE approach, one shows in Tab. 8 the average and variance of the estimator for the flow harmonics evaluated at different values of multiplicity. It is shown that as the multiplicity increases, all the estimations of flow harmonics become more precise while the variance decreases, as

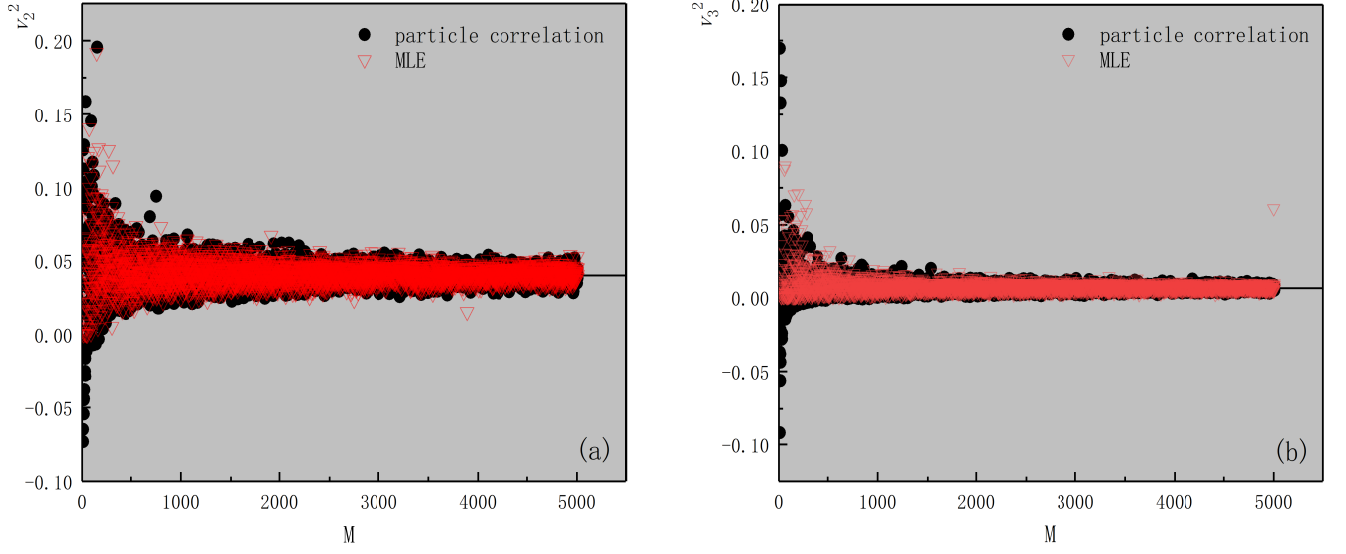


Fig. 8 – (Color Online) The estimated flow harmonics v_2^2 and v_3^2 using the MLE and particle correlation methods, as functions of the event multiplicity M . The hollow red triangles represent the results obtained by the MLE, while the filled black circles are those by the particle correlation method.

Tabela 7 – The event planes Ψ_n obtained by MLE and event-plane methods. The results are obtained for a single event but with different multiplicities.

	Ψ_2			Ψ_3			Ψ_4		
M	500	1000	2000	500	1000	2000	500	1000	2000
true value	1.185	2.943	2.435	1.519	1.695	1.389	0.682	1.272	0.802
event plane method	1.130	2.986	2.462	1.774	1.844	1.389	0.685	1.242	0.797
MLE	1.134	2.983	2.458	1.849	1.864	1.408	0.659	1.245	0.790

Tabela 8 – The dependence of the MLE method on the event multiplicity M . The calculations are carried out with $N = 1000$ events.

M	500	1000	2000
$\mu[\hat{v}_2]$	0.205	0.202	0.202
$\text{Var}[\hat{v}_2]$	8.95×10^{-4}	4.16×10^{-4}	2.27×10^{-4}
$\mu[\hat{v}_3]$	0.0847	0.0843	0.0830
$\text{Var}[\hat{v}_3]$	1.11×10^{-3}	4.32×10^{-4}	1.95×10^{-4}
$\mu[\hat{v}_4]$	0.258	0.252	0.252
$\text{Var}[\hat{v}_4]$	8.99×10^{-4}	4.31×10^{-4}	2.15×10^{-4}

expected. On the other hand, Tab. 8 shows the dependence of the event average and variance of MLE on the event number. It is indicated that both the average and variance of the estimator become stable as long as the event number is big enough. This is readily understood as, by definition, the events are independently sampled from the likelihood distribution, whose behavior is therefore governed by the central limit theorem.

Tabela 9 – The dependence of the MLE method on the number of events N . The calculations are carried out with multiplicity $M = 1000$.

N	500	1000	2000
$\mu [\hat{v}_2]$	0.2023	0.2022	0.2026
$\text{Var} [\hat{v}_2]$	4.08×10^{-4}	4.27×10^{-4}	4.41×10^{-4}

Before closing this subsection, we note that since the MLE evaluates the flow harmonics in a way that is independent of specific particle combinations, it naturally furnishes certain harmonic products that are not obtained straightforwardly from the particle correlation analysis. Moreover, owing to the invariance property of MLE, a product of the MLEs is the MLE of the product of harmonic coefficients. We will further address the last point in the following subsection in the context of detector inefficiency.

6.4 MLE AND DETECTOR INEFFICIENCY

In practice, a detector's acceptance might not be uniform, potentially leading to non-negligible systematic bias in anisotropic flow analysis. In terms of particle correlation and Q-vectors, this issue has been addressed by various authors (BILANDZIC et al., 2014). This subsection explores how the MLE method can be applied to scenarios involving detector inefficiency. In particular, we elaborate on a scheme to compensate for the detector's nonuniform acceptance by incorporating *weight* into the formalism of the likelihood function. The proposed scheme is then implemented and illustrated by considering two specific detector inefficiencies. Moreover, we show that MLE can be employed to evaluate certain harmonic products that are not straightforwardly embraced by the particle correlation approach.

One may quantify the nonuniformity by the detector's acceptance rate as a function of the azimuthal angle, $g(\phi) \leq 1$, where $g(\phi) = 1$ corresponds to a perfect detector. For simplicity, we also ignore the dependence of the acceptance on transverse momentum and rapidity. For the MLE scheme, the likelihood function can be modified to adapt to this inefficiency by including a weight factor defined by $w(\phi) = 1/g(\phi)$. Subsequently, one generalizes Eqs. (6.23) and (6.24) to the form

$$\ell'(\theta; \phi_1, \phi_2, \phi_3 \cdots \phi_M) = \log \mathcal{L}'(\theta; \phi_1, \phi_2, \phi_3 \cdots \phi_M) = \sum_{j=1}^M w(\phi_j) \log f_1(\phi_j; \theta). \quad (6.39)$$

It is noted that the weight factor gives a correction in the power of the one-particle distribution function, compensating for the suppressed multiplicity. Using Eq. (6.39), the MLE calculations remain essentially unchanged.

As mentioned above, the particle correlation method is mostly applicable to a specific class of set of correlator Eq. (6.6) where $\sum_{j=1}^k n_j = 0$. In other words, the quantity is isotropic because the event planes cancel out entirely in the remaining expression.

In what follows, we illustrate the scheme through two specific forms of detector acceptance. By

employing the MLE method, we consider the following products of flow harmonics

$$\begin{aligned}
\langle 2 \rangle &\equiv \langle 2 \rangle_{-2,2} = v_2^2 = 0.04, \\
\langle 3 \rangle &\equiv \langle 3 \rangle_{-2,-2,4} = v_2^2 v_4 = 0.01, \\
\langle 4 \rangle &\equiv \langle 4 \rangle_{-3,-2,2,3} = v_2^2 v_3^2 = 0.000256, \\
\langle 2' \rangle &\equiv v_2 v_3 = 0.016, \\
\langle 3' \rangle &\equiv v_2^2 v_3 = 0.0032, \\
\langle 4' \rangle &\equiv v_2^2 v_3 v_4 = 0.0008,
\end{aligned} \tag{6.40}$$

where the last equality gives the true values. It is noted that the first three quantities are essentially multi-particle correlators of the form Eq. (6.6), which can also be evaluated¹ using the particle correlation scheme as in (BILANDZIC et al., 2014). For instance, the event average $\langle 2 \rangle_{-2,2}$ vanishes due to the event average carried out for the terms involving the event planes. However, the last three quantities in Eq. (6.40) do not fit into this category as the corresponding correlator vanishes. Specifically, using the MLE approach, one estimates v_n and their products for individual events, and the event averages $\langle v_2^2 v_4 \rangle$ and $\langle v_2^2 v_3 \rangle$ can be evaluated in a similar fashion irrelevant to any specific particle tuple.

Our first example is a simplified scenario according to (BILANDZIC; SNELLINGS; VOLOSHIN, 2011), where the detector's acceptance is given by a piecewise function

$$g_1(\phi) = \begin{cases} 0.5 & \pi/3 < \phi \leq 2\pi/3 \\ 1.0 & \text{otherwise} \end{cases}. \tag{6.41}$$

The results are shown in Figs. Fig. 9 and Fig. 10. In the left plot of Fig. Fig. 9, we present the distributions of the detected particles according to the Monte Carlo procedure. These particles are then utilized by the MLE method to estimate the quantities given in Eq. (6.40). As shown on the right plot of Fig. Fig. 9, for all six quantities, the MLE with weight is shown to give appropriate compensation for the detector inefficiency. The results of the first three quantities defined in Eq. (6.40) are mainly consistent with the results obtained by using Q-vectors (BILANDZIC et al., 2014). For the last three quantities, where reasonable corrections are also achieved, the evaluation using MLE is somewhat unique in its own right. We also show in Fig. Fig. 10 the distributions of the estimated v_2 and v_3 on an event-by-event basis. For both cases, it is demonstrated clearly that the expected value and probability distributions of the relevant quantities are properly recuperated.

We proceed to elaborate on a more realistic detector's acceptance function given by the following form

$$g_2(\phi) = \begin{cases} 1 + \exp(-\phi/7) \sin 2(\phi + 0.5) & 1.07 < \phi \leq 2.64 \text{ or } 4.21 < \phi \leq 5.78 \\ 1.0 & \text{otherwise} \end{cases}. \tag{6.42}$$

The above acceptance's function mimics a realistic detector where the acceptance is suppressed at the

¹ As in (BILANDZIC et al., 2014), the second row in Eqs. (6.40) is obtained by considering $\Psi_2 = \Psi_4$.

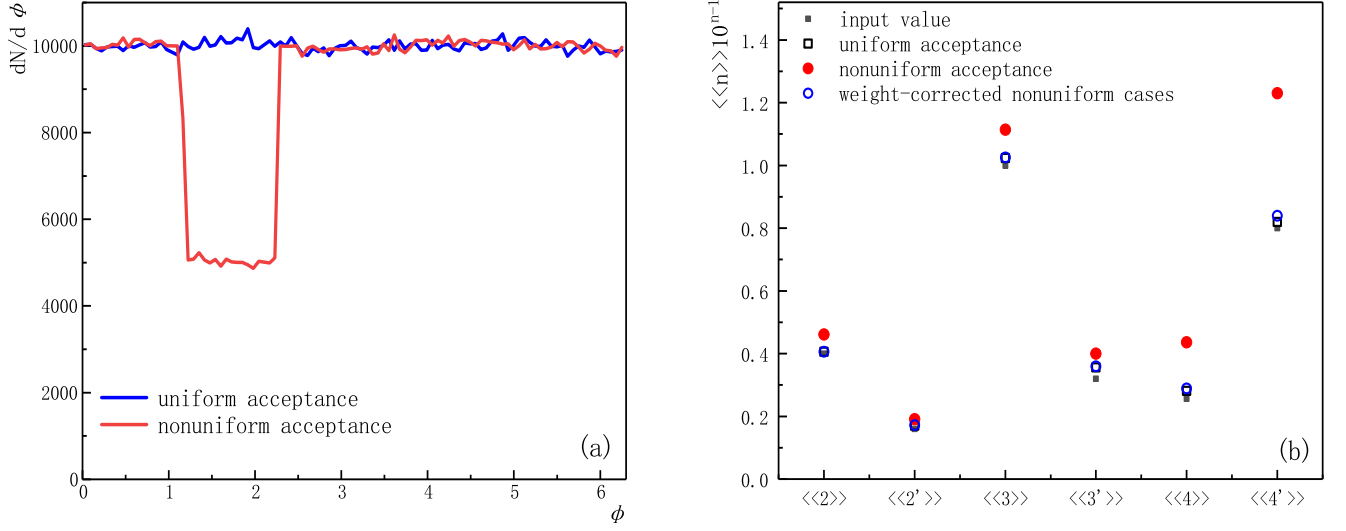


Fig. 9 – (Color Online) The results for a simplified detector's acceptance given by a step function Eq. (6.41). (a): The particle generated and observed by the detector with a uniform (blue line) and nonuniform (red line) azimuthal acceptance. (b): The resultant products of flow harmonics for uniform, nonuniform, and weight-corrected nonuniform cases.

forward and backward directions (ALT et al., 2003). Fig. Fig. 11 shows the distribution of the detected particles and the resulting estimation of harmonic products. Also, Fig. Fig. 12 gives the distribution of the estimations for v_2 and v_3 regarding individual events. Again, it is clearly seen that the MLE method does a reasonable job of estimating these quantities and their probability distributions while coping with detector inefficiency, particularly for the ones which cannot be furnished straightforwardly by the particle correlation method.

6.5 A FICTITIOUS SCENARIO WITH FLUCTUATING EVENT PLANE

In this section, we elaborate on a fictitious scenario where the event plane in Eq. (1.1) is not a well-defined quantity in the distribution function. In particular, we consider that the event plane is subjected to a normal distribution

$$\Psi_2 \sim N(\mu, \sigma^2). \quad (6.43)$$

where μ is sampled uniformly from the interval $\mu \in (0, \pi)$ and $\sigma^2 = \pi/12$.

Using the two-particle correlation estimator Eq. (6.3) and MLE, the results are presented in Tab. 10. Since a parameter of the distribution function, i.e., the event plane, is governed by a probability distribution, the estimation scheme should be taken with a grain of salt. As expected, the estimated v_2 becomes less accurate as the deviation σ^2 becomes more significant. Nonetheless, the estimations of v_2 are numerically convergent for both approaches and are mostly reasonable when the fluctuations

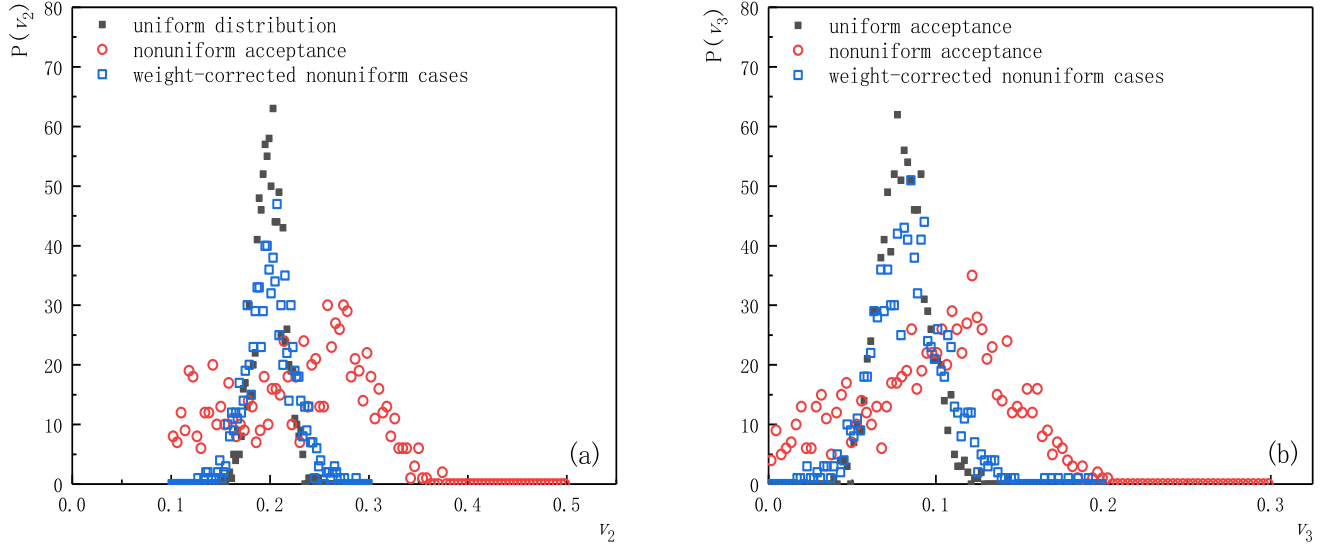


Fig. 10 – (Color Online) The distributions of the estimated v_2 and v_3 on an event-by-event basis using the MLE method, evaluated for the detector’s acceptance given by Eq. (6.41).

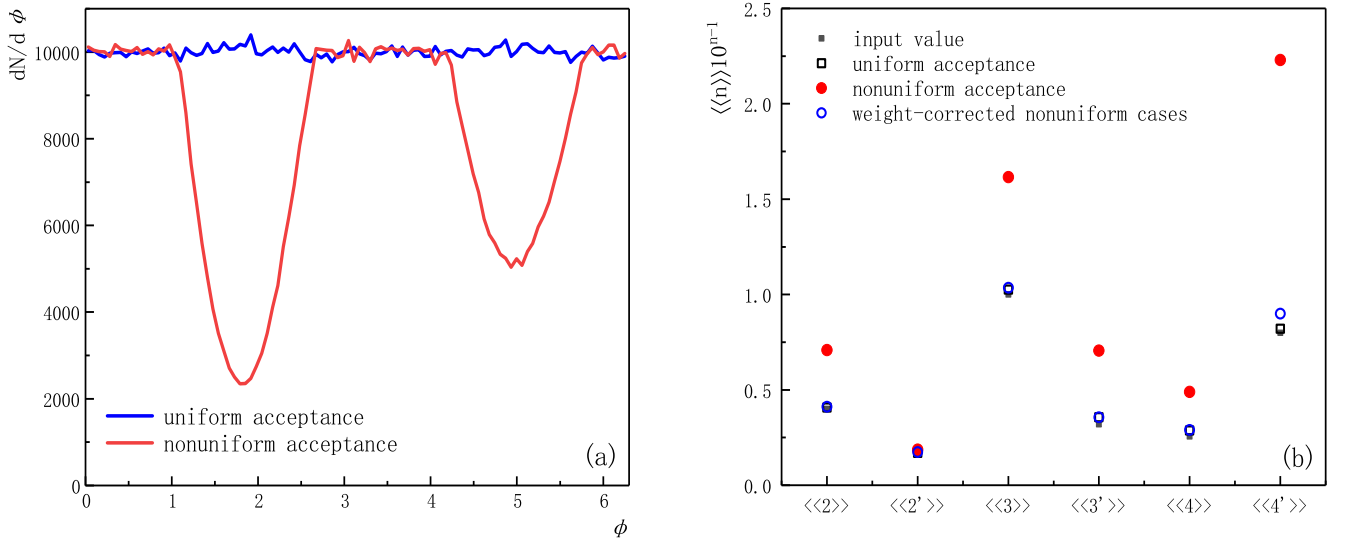


Fig. 11 – (Color Online) The same as Fig. Fig. 9, but for a more realistic nonuniform acceptance given by Eq. (6.42). (a): The particle generated and observed by the detector with a uniform (blue line) and nonuniform (red line) azimuthal acceptance. (b): The resultant products of flow harmonics for uniform, nonuniform, and weight-corrected nonuniform cases.

are not significant. When compared with two-particle correlation, MLE yields essentially better results. Specifically, if one assumes that Ψ_2 is a fixed parameter in the one-particle distribution, MLE gives slightly better results than those from particle correlation. However, if one considers that Ψ_2 is generated by a distribution where the variance σ^2 is taken as an unknown parameter as the remaining ones, MLE (dubbed as “MLE-mod” in Tab. 10) yields better results. We note that such an extension does not seem straightforward in the case of the particle-correlation estimator.

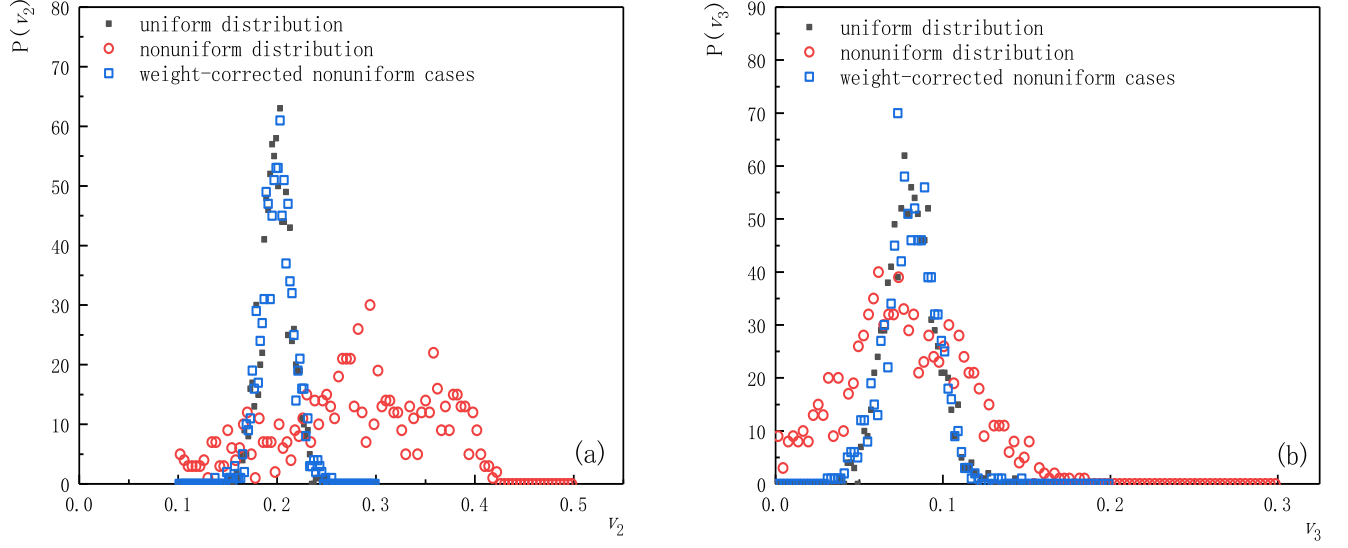


Fig. 12 – (Color Online) The same as Fig. Fig. 10, but for the detector's acceptance given by Eq. (6.42).

Tabela 10 – The estimated elliptic flow using the particle correlation and MLE estimators where the event-plane Ψ_2 is generated by considering additional Gaussian fluctuations given by Eq. (A.6).

σ^2		$\pi/12$		
M	particle correlation	MLE	MLE-mod	true value
100	0.1174	0.1227	0.1805	0.2
500	0.1501	0.1554	0.2010	0.2
σ^2		$\pi/24$		
100	0.1171	0.1198	0.1794	0.2
500	0.1556	0.1573	0.2156	0.2

7 CONCLUSION

In summary, in this thesis, we explored the influence of hydrodynamics on the measurement of anisotropic flow. And try to use MLE as another estimator to measure anisotropic flow.

To this end, we investigate the possibility of suppressing the effects of event-by-event multiplicity fluctuations. This study was carried out with high energy tube models and hadron correlators. Suppression is achieved by adequate normalization polymorphism when forming pairs of appropriate and mixed events in the computation of particle correlations.

As the multiple fluctuations are fully extracted, the resulting two-particle correlations show major features governed by the peripheral tube model. The latter were numerically calculated by means of Monte Carlo simulations. Furthermore, this normalization scheme is naturally consistent with multi-particle correlators that are usually evaluated in terms of ratios of q -vectors.

Afterwards, we turned our focus to a statistical perspective. From a statistical point of view, we carefully study the fluctuations of the flow harmonics. In particular, we explore two different types of averages. For the first event, we assume that the diversity of each event is finite, but consider an infinite number of events. For the second case, an average is taken for a given event, considering particles at the hydrodynamic limit. Even assuming elliptic flow v_2 without any fluctuations, it inevitably produces a finite variance in the case of events with a finite diversity. In this case, we point out that one has to employ a specific scheme to estimate flow harmonics, which is, by definition, a statistical estimator. It is inevitably subject to finite variance due to finite statistics. From a physical point of view, for realistic events, flow fluctuations, shown as flow harmonic variances, carry key information about initial state geometry and final state statistical fluctuations. In practice, these two types of variance will be mixed and, due to fluctuations in the initial state, probably indistinguishable from the latter, which is understood to carry essential information about the underlying microscopic physical system. Part of the event-by-event flow fluctuations is of a purely statistical nature.

In the last part, we think of the possibility of using different estimators from the variance brought by statistics. we study the possibility of employing the MLE as a flow estimator. The proposed estimator possesses three interesting features. First, due to its asymptotical normality, MLE provides a smaller variance when comparing other estimators. Since a part of flow fluctuations comes from the finite multiplicity of realistic collision events, this feature can be utilized to suppress undesirable statistical uncertainties. Second, the proposed approach provides a means to access specific mixed harmonics, which cannot be straightforwardly obtained using methods primarily based on particle correlations. Third, the MLE flow estimator is robust for the fictitious scenario where some parameters of the probability distribution are not well-defined. The resultant flow harmonics obtained using MLE were compared with those derived using other existing methods. The obtained results were analyzed using the Wald, likelihood ratio, and score tests of hypotheses. We also explore the dependence of extracted flow harmonics on the event multiplicity and number of events. Moreover, it is shown that the proposed approach works efficiently to deal with the deficiency in detector acceptability. Therefore, it is argued that the MLE furnishes a meaningful alternative to the existing approaches.

We also acknowledge that the proposed scheme encounters certain limitations or drawbacks. First, it is computationally expensive in its present form. The algorithm complexity of the MLE approach increases essentially linearly with the multiplicity, similar to that of the particle-correlation method. Besides, the MLE's computational time increases geometrically with the dimension of the parameter space. Second, although MLE is asymptotically normal, unbiased, and consistent, which generally attains more accurate values than other estimators, numerical results also seem to reveal a certain degree of bias at finite multiplicity. To our knowledge, mathematically confirming whether the flow MLE is a biased estimator is not straightforward and has not been addressed in this study. Lastly, the present study has been primarily focused on the integrated flow. To generalize the scheme to differential flow may also face further challenges such as, among others, the computational feasibility. Besides, it is interesting to examine the performance of MLE when non-flow plays a significant role. We plan to explore these topics in future studies.

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APÊNDICE A – HIGH-ORDER PARTICLE CORRELATION ALGORITHM

As mentioned earlier, due to the promotion of the order, the promotion of related projects will also increase exponentially. Direct evaluation of higher order correlators quickly becomes impractical. Therefore, we need algorithms that represent all higher-order correlators in a recursive form. By looking for the existence of the numerator

$$N\langle 1 \rangle_{n_1} = Q_{n_1,1}. \quad (\text{A.1})$$

$$N\langle 2 \rangle_{n_1,n_2} = N\langle 1 \rangle_{n_1} Q_{n_2,1} - Q_{n_1+n_2,2} \quad (\text{A.2})$$

$$N\langle 3 \rangle_{n_1,n_2,n_3} = N\langle 2 \rangle_{n_1,n_2} Q_{n_3,1} - N\langle 1 \rangle_{n_1} Q_{n_2+n_3,2} - N\langle 1 \rangle_{n_2} Q_{n_1+n_3,2} + 2Q_{n_1+n_2+n_3,3} \quad (\text{A.3})$$

$$N\langle 4 \rangle_{n_1,n_2,n_3,n_4} = N\langle 3 \rangle_{n_1,n_2,n_3} Q_{n_4,1} - N\langle 2 \rangle_{n_1,n_2} Q_{n_3+n_4,2} - N\langle 2 \rangle_{n_1,n_3} Q_{n_2+n_4,2} \quad (\text{A.4})$$

$$- N\langle 2 \rangle_{n_2,n_3} Q_{n_1+n_4,2} + 2N\langle 1 \rangle_{n_1} Q_{n_2,n_3,n_4,3} + 2N\langle 1 \rangle_{n_2} Q_{n_1,n_3,n_4,3} \quad (\text{A.5})$$

$$+ 2N\langle 1 \rangle_{n_3} Q_{n_1,n_2,n_4,3} - 6Q_{n_1+n_2+n_3+n_4,4}. \quad (\text{A.6})$$

It's clear that $N\langle m \rangle_{n_1,n_2,\dots,n_m}$ is determined through ordered partitions of the numbers $n_1, n_2, \dots, n_m$.

We can use this property to calculate $N\langle m \rangle_{n_1,n_2,\dots,n_m}$ for any m

$N\langle 1 \rangle'_{n_1} : \text{return } Q_{n_1,1}$

$N\langle m \rangle'_{n_1,n_2,\dots,n_m} :$

$C \leftarrow 0$

for $k \leftarrow (m-1), 1$ do

for each combination $c = c_1, \dots, c_k$ of $n_1, \dots, n_{m-1}$ do

$q \leftarrow \sum_{j \text{ not in } c} n_j$

$C \leftarrow C + (-1)^{m-k} (m-k-1)! \times N\langle k \rangle'_{c_1,\dots,c_k} \times Q_{q,m-k}$

end of each c

end for k

return C .

It can be seen that the calculation of the innermost layer is not affected, so through this recursive method. We can quickly obtain high-order calculation results with the least amount of calculation.

A different recursive relation can be developed by examining 3.26 itself. The innermost sum can be reformulated without the constraint of being distinct from any other index, as demonstrated below.

$$\begin{aligned} N\langle m \rangle &\equiv \sum_{k\text{-tuples}} w_{k_1} w_{k_2} \cdots w_{k_m} e^{i(n_1 \phi_{k_1} + n_2 \phi_{k_2} + \cdots + n_m \phi_{k_m})} \\ &\times \left(\sum_{k_m=1}^M w_{k_m} e^{in_m \phi_{k_m}} - \sum_{j=1}^{m-1} w_{k_j} e^{in_m \phi_{k_j}} \right). \end{aligned} \quad (\text{A.7})$$

This can be expanded into the following recursive formula, where, however, one must be careful to set

the power of the weights equal to the number of summands

$$\begin{aligned}
 N\langle m \rangle_{n_1, n_2, \dots, n_m} &= Q_{n_m, 1} N\langle m-1 \rangle_{n_1, n_2, \dots, n_{m-1}} - N\langle m-1 \rangle_{n_1+n_m, n_2, \dots, n_{m-1}} \\
 &- N\langle m-1 \rangle_{n_1, n_2+n_m, \dots, n_{m-1}} - \dots - N\langle m-1 \rangle_{n_1, n_2, \dots, n_{m-1}+n_m}. \quad (\text{A.8})
 \end{aligned}$$

An optimized version of this recursive formula, which ensures that unique terms are evaluated only once, is shown here

```

 $N\langle 1 \rangle_{c_1}''$  : return  $Q_{n_1, c_1}$ 
 $N\langle m \rangle_{n_1, n_2, \dots, n_m}''(c_1, \dots, c_m)$  :
     $C \leftarrow Q_{n_m, c_m} \times N\langle m-1 \rangle_{n_1, \dots, n_{m-1}}''(c_1, \dots, c_{m-1})$ 
    if  $c_m \leq 1$  then
        for  $i \leftarrow 1, m-1$  do
             $C \leftarrow C - c_i \times N\langle m-1 \rangle_{n_1, \dots, n_i+n_m, \dots, n_{m-1}}''(c_1, \dots, c_i+1, \dots, c_{m-1})$ 
        end for i
    end if
    return C.

```

Through such an algorithm, we can easily calculate the high-order particle correlation cumulant, and it will not bring too much pressure to the program.