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Lotka–Volterra models with fractional diffusion

Michele O. Alves
Departamento de Matemática, Universidade Estadual de Londrina, 86057-970 Londrina – PR, Brazil (michelealves@uel.br)

Marcos T. O. Pimenta
Departamento de Matemática e Computação, Faculdade de Ciências e Tecnologia, Universidade Estadual Paulista, 19060-900 Presidente Prudente – SP, Brazil (pimenta@fct.unesp.br)

Antonio Suárez
Departamento de Ecuaciones Diferenciales y Análisis Numérico, Faculdad de Matemáticas, Universidad de Sevilla, Calle Tarfia s/n, 41012 Sevilla, Spain (suarez@us.es)

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We study Lotka–Volterra models with fractional Laplacian. To do this we study in detail the logistic problem and show that the sub–supersolution method works for both the scalar problem and for systems. We apply this method to show the existence and non-existence of positive solutions in terms of the system parameters.

Keywords: fractional Laplacian; Lotka–Volterra models; sub–supersolution method

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1. Introduction

In this paper we study the following systems:

$$\left.\begin{aligned} (-\Delta)^{\alpha} u &= u(\lambda - u - bv) && \text{in } \Omega, \\ (-\Delta)^{\beta} v &= v(\mu - v - cu) && \text{in } \Omega, \\ u = v &= 0 && \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geqslant 1$, is a bounded and regular domain, $\lambda, \mu, b, c \in \mathbb{R}$ and $\alpha, \beta \in (0, 1)$. Here, u and v denote the densities of two species in Ω (the habitat), which is surrounded by inhospitable areas due to the homogeneous Dirichlet boundary conditions. In (1.1) we assume that the species diffuse following the fractional Laplacian (see § 2, where we define this non-local operator).

When $\alpha = \beta = 1$, (1.1) is the classical Lotka–Volterra system with random walk, which has been widely studied in recent years in the following cases: competition ($b, c > 0$), predator–prey ($b > 0$ and $c < 0$) and symbiosis ($b, c < 0$) (see [8] and the references therein).

Fractional operators are used in different contexts: physics, finance and ecology (see [15,21] for the ecological meaning of fractional diffusion). For many years, non-oriented animal movement was modelled by classical Brownian motion. However, it seems that when a species is searching for resources the strategy based on Lévy flights (supported in long jumps) could be more appropriate in some situations. This kind of strategy is optimal for the location of targets that are randomly and sparsely distributed, but Brownian motion is optimal when resources are abundant. The Lévy diffusion processes are generated by fractional powers of the Laplacian $(-\Delta)^\gamma$ for $\gamma \in (0,1)$.

We are interested in the existence of non-negative solutions of (1.1). It is clear that (1.1) possesses the trivial solution $(u,v)=(0,0)$ for all $\lambda,\mu \in \mathbb{R}$, since when $u \equiv 0$ (respectively, $v \equiv 0$) v (respectively, u) verifies an equation of the following type:

$$\left.\begin{aligned} (-\Delta)^\gamma w + c(x)w = w(\sigma - w) \quad & \text{in } \Omega, \\ w = 0 \qquad\qquad & \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.2}$$

where $\gamma \in (0,1)$, $\sigma \in \mathbb{R}$ and $c \in L^\infty(\Omega)$. This is the classical logistic equation, studied in [19,20] with homogeneous Dirichlet and Neumann boundary conditions, respectively, with $\gamma = \frac{1}{2}$ in both papers. To study this equation, we analyse the eigenvalue problem

$$\left.\begin{aligned} (-\Delta)^\gamma w + c(x)w = \lambda w \quad & \text{in } \Omega, \\ w = 0 \qquad & \text{on } \partial\Omega. \end{aligned}\right\} \tag{1.3}$$

We study the existence of a principal eigenvalue, the unique eigenvalue of (1.3) having a positive eigenfunction, denoted by $\lambda_1[\gamma;c]$. This problem has been analysed in [1,19] (for $\gamma = \frac{1}{2}$) and in [20] for the Neumann case. We study some properties of this eigenvalue and of the associated eigenfunction.

Then, we prove that (1.2) has a positive solution if and only if $\sigma > \lambda_1[\gamma;c]$; moreover, it is unique and we denote it by $\theta_{[\gamma,\sigma-c]}$. Furthermore, we attempt to give an ecological interpretation of the result, comparing our results with that obtained in the local operator case, in which the fractional Laplacian is substituted by the classical Laplacian operator.

To show the existence, we employ the sub–supersolution method. Note that this method was used for the nonlinear fractional diffusion problem (see, for instance, [3,9]; in both these papers, the method is the consequence of a maximum principle and a classical iterative argument). However, we present a different proof, which is also valid, with minor technical changes, for systems.

Once we have studied (1.2) in detail, we shall analyse the existence of solutions with both positive components of (1.1). To do this, we apply the sub–supersolution method, first showing that it works for systems, and then applying it to (1.1). Thus, we must find appropriate sub–supersolutions of (1.1) using the results obtained for the logistic equations. We prove the following results. There exists at least one positive solution of (1.1) if

- $b,c>0$ or $b,c<0$ and $bc < C(\alpha,\beta)$ for some positive constant (detailed in § 6) and λ and μ verify

$$\lambda > \lambda_1[\alpha; b\theta_{[\beta,\mu]}] \quad \text{and} \quad \mu > \lambda_1[\beta; c\theta_{[\alpha,\lambda]}], \tag{1.4}$$

or

- $b > 0$ and $c < 0$ and λ and μ verify

$$\lambda > \lambda_1[\alpha; b\theta_{[\beta,\mu - c\theta_{[\alpha,\lambda]}]}] \quad \text{and} \quad \mu > \lambda_1[\beta; c\theta_{[\alpha,\lambda]}]. \tag{1.5}$$

We show that conditions (1.4) and (1.5) define regions on the (λ, μ)-plane.

The paper is organized as follows. In § 2 we present the functional setting necessary for the remainder of the work. Section 3 is devoted to the eigenvalue problem: we study the existence and main properties of the principal eigenvalue. In § 4 we study (1.2). The sub–supersolution method for systems is shown in § 5. Finally, in § 6 we study the existence of a positive solution of (1.1).

2. Preliminaries

We begin by introducing the functional framework necessary to develop the theory, and recover some known results about the different forms to define the fractional power of the Laplacian with Dirichlet boundary condition.

2.1. Functional setting

Consider a smooth bounded domain $\Omega \subset \mathbb{R}^N$. Since in bounded domains there are some non-equivalent definitions of the fractional Laplacian operator, let us explain what we mean by the symbol $(-\Delta)^\alpha$. For $u \in C_0^\infty(\Omega)$ such that $u = \sum_{k=1}^\infty b_k\varphi_k$, where λ_k, φ_k are the eigenpairs of $(-\Delta, H_0^1(\Omega))$ (with λ_k repeated as many times as its multiplicity and $\{\varphi_k\}$ forming an orthonormal basis of $L^2(\Omega)$), we define

$$(-\Delta)^\alpha u := \sum_{k=1}^\infty \lambda_k^\alpha b_k \varphi_k.$$

Then the operator $(-\Delta)^\alpha$ is defined on

$$D((-\Delta)^\alpha) = \left\{ u \in L^2(\Omega); \sum_{k=1}^\infty \lambda_k^\alpha b_k^2 < +\infty \right\}$$

by density.

Now, let us consider the half cylinder

$$\mathcal{C} := \Omega \times (0, +\infty)$$

with base Ω, and denote its lateral boundary by

$$\partial_L \mathcal{C} := \partial\Omega \times [0, +\infty).$$

We set $(x, y) \in \mathcal{C}$, $x \in \Omega$ and $y > 0$ and define

$$\mathcal{H}^\alpha(\mathcal{C}) := \{v \in H^1(\mathcal{C});\ \|v\|_\alpha < +\infty\},$$
$$\mathcal{H}_0^\alpha(\mathcal{C}) := \{v \in \mathcal{H}^\alpha(\mathcal{C});\ v = 0 \text{ on } \partial_L\mathcal{C}\},$$

where

$$\|v\|_\alpha := \left(k_\alpha^{-1}\int_{\mathcal{C}} y^{1-2\alpha}|\nabla v|^2\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2}, \quad k_\alpha = \frac{2^{1-2\alpha}\Gamma(1-\alpha)}{\Gamma(\alpha)}, \quad \alpha\in(0,1),$$

and Γ is the Gamma function. It is not difficult to see that $\mathcal{H}_0^\alpha(\mathcal{C})$ is a Hilbert space when endowed with the norm $\|\cdot\|_\alpha$, which comes from the following inner product:

$$\langle v,w\rangle_\alpha = k_\alpha^{-1}\int_{\mathcal{C}} y^{1-2\alpha}\nabla v\cdot\nabla w\,\mathrm{d}x\,\mathrm{d}y.$$

Consider the following subspace of the fractional Sobolev space $H^\alpha(\Omega)$:

$$\mathcal{V}_0^\alpha(\Omega) := \{\mathrm{tr}_\Omega\, v;\ v\in\mathcal{H}_0^\alpha(\mathcal{C})\}.$$

This is a Banach space when endowed with the norm

$$\|u\|_{\mathcal{V}_0^\alpha(\Omega)} := \left(\|u\|^2_{L^2(\Omega)} + \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2},$$

where tr_Ω is the trace operator defined by

$$\mathrm{tr}_\Omega\, v = v(\cdot,0) \quad \text{for } v\in\mathcal{H}_0^\alpha(\mathcal{C}).$$

Moreover, by the trace theorem (see [9, proposition 2.1]) and embeddings for fractional Sobolev spaces (see [12, theorem 6.7]) it follows that

$$\|\mathrm{tr}_\Omega\, v\|_{L^p(\Omega)} \leqslant C\|v\|_\alpha \quad \forall v\in\mathcal{H}_0^\alpha(\mathcal{C}),\ p\in(1,2_\alpha), \tag{2.1}$$

where $2_\alpha = 2N/(N-2\alpha)$.

By [9, proposition 2.1] it holds that

$$\mathcal{V}_0^\alpha(\Omega) = \left\{u\in L^2(\Omega);\ u=\sum_{k=1}^\infty b_k\varphi_k \text{ satisfying } \sum_{k=1}^\infty b_k^2\lambda_k^\alpha < +\infty\right\}.$$

As far as the scalar non-local problem

$$\left.\begin{aligned} (-\Delta)^\alpha u &= f(x,u) &&\text{in } \Omega,\\ u &= 0 &&\text{on } \partial\Omega,\end{aligned}\right\} \tag{2.2}$$

is concerned, the approach we shall follow is by associating with (2.2) an $(N+1)$-dimensional local problem in $\mathcal{C}$. This can be done by considering the procedure to obtain a local realization of $(-\Delta)^\alpha$, described below.

As proved in [9, § 2.1], for each $u\in\mathcal{V}_0^\alpha(\Omega)$ there exists a unique $v\in\mathcal{H}_0^\alpha(\mathcal{C})$, called its *$\alpha$-harmonic extension* such that

$$\begin{aligned} -\operatorname{div}(y^{1-2\alpha}\nabla v) &= 0 &&\text{in } \mathcal{C},\\ v &= 0 &&\text{on } \partial_L\mathcal{C},\\ v(\cdot,0) &= u &&\text{on } \Omega.\end{aligned}$$

Moreover, if $u=\sum_{k=1}^\infty b_k\varphi_k$ is its spectral decomposition, then

$$v(x,y) = \sum_{k=1}^\infty b_k\varphi_k(x)\psi(\lambda_k^{1/2}y) \quad \forall(x,y)\in\mathcal{C}, \tag{2.3}$$

where ψ solves the Bessel equation

$$\left.\begin{aligned} \psi'' + \frac{1-2\alpha}{s}\psi' = \psi, \quad s > 0, \\ -\lim_{s\to 0^+} s^{1-2\alpha}\psi'(s) = k_\alpha, \\ \psi(0) = 1. \end{aligned}\right\} \tag{2.4}$$

Let $u \in \mathcal{V}_0^\alpha(\Omega)$ and let $v \in \mathcal{H}_0^\alpha(\mathcal{C})$ be its α-harmonic extension. Define the functional

$$\frac{1}{k_\alpha}\frac{\partial v}{\partial y^\alpha}\bigg|_{\Omega\times\{0\}} \in \mathcal{V}_0(\Omega)^*$$

by

$$\left\langle \frac{1}{k_\alpha}\frac{\partial v}{\partial y^\alpha}(\cdot,0), g \right\rangle := \frac{1}{k_\alpha}\int_{\mathcal{C}} y^{1-2\alpha}\nabla v\cdot\nabla\tilde{g}\,\mathrm{d}x\,\mathrm{d}y,$$

where $\tilde{g}$ is the α-harmonic extension of $g \in \mathcal{V}_0^\alpha(\Omega)$ and

$$\frac{\partial v}{\partial y^\alpha}(x,0) = -\lim_{y\to 0^+} y^{1-2\alpha}\frac{\partial v}{\partial y}(x,y) \quad \forall x \in \Omega.$$

Then we can define an operator $A_\alpha\colon \mathcal{V}_0^\alpha(\Omega) \to \mathcal{V}_0^\alpha(\Omega)^*$ such that

$$A_\alpha u := \frac{1}{k_\alpha}\frac{\partial v}{\partial y^\alpha}\bigg|_{\Omega\times\{0\}},$$

where v is the α-harmonic extension of u to $\mathcal{C}$. Let us prove that the operators A_α and $(-\Delta)^\alpha$ are in fact the same, i.e. that, for all $u \in \mathcal{V}_0^\alpha(\Omega)$,

$$A_\alpha u = \sum_{k=1}^\infty b_k\lambda_k^\alpha\varphi_k, \quad \text{where } u = \sum_{k=1}^\infty b_k\varphi_k.$$

By linearity, it is enough to prove that, for all φ_k,

$$\left\langle \frac{1}{k_\alpha}\frac{\partial v}{\partial y^\alpha}(\cdot,0), \varphi_k \right\rangle = \langle(-\Delta)^\alpha u, \varphi_k\rangle_{L^2(\Omega)} \quad \forall k \in \mathbb{N},$$

where v is the α-harmonic extension of u.

For $u \in \mathcal{V}_0^\alpha(\Omega)$ and $k \in \mathbb{N}$, let v and $\tilde{\varphi}_k$ be the α-harmonic extensions of u and φ_k, respectively. By (2.3),

$$v(x,y) = \sum_{k=1}^\infty b_k\varphi_k(x)\psi(\lambda_k^{1/2}y) \quad \text{and} \quad \tilde{\varphi}_k(x,y) = \varphi_k(x)\psi(\lambda_k^{1/2}y).$$

Now, integration by parts and properties of φ_k imply that, for each $y > 0$,

$$\int_\Omega y^{1-2\alpha}\nabla_x v(x,y)\cdot\nabla_x\tilde{\varphi}_k(x,y)\,\mathrm{d}x = y^{1-2\alpha}b_k(\lambda_k\psi(\lambda_k^{1/2}y)^2 + \psi'(\lambda_k^{1/2}y)^2)$$

holds. Then, by (2.4),

$$\begin{aligned}\left\langle \frac{1}{k_\alpha}\frac{\partial v}{\partial y^\alpha}(\cdot,0),\varphi_k\right\rangle &= \frac{1}{k_\alpha}\int_{\mathcal{C}} y^{1-2\alpha}\nabla v\cdot\nabla\tilde{\varphi}_k\,\mathrm{d}x\,\mathrm{d}y\\ &= \frac{1}{k_\alpha}\int_0^{+\infty} y^{1-2\alpha}b_k(\lambda_k\psi(\lambda_k^{1/2}y)^2+\psi'(\lambda_k^{1/2}y)^2)\,\mathrm{d}y\\ &= \frac{1}{k_\alpha}\lim_{\eta\to 0^+} y^{1-2\alpha}\lambda_k^{1/2}b_k\psi'(\lambda_k^{1/2}y)\psi(\lambda_k^{1/2}y)|_{y=\eta}\\ &= b_k\lambda_k^\alpha\\ &= \langle(-\Delta)^\alpha u,\varphi_k\rangle_{L^2(\Omega)}.\end{aligned}$$

Hence, in (2.2) we shall understand $(-\Delta)^\alpha$ to mean A_α.

For simplicity, without loss of generality, we can assume throughout the paper that $k_\alpha = 1$. Then we have the following.

Definition 2.1. $u \in \mathcal{V}_0(\Omega)$ is a weak solution of (2.2) if $u = \operatorname{tr}_\Omega v$, where $v \in \mathcal{H}_0^\alpha(\mathcal{C})$ is a weak solution of

$$\begin{aligned}-\operatorname{div}(y^{1-2\alpha}\nabla v) = 0 \quad &\text{in } \mathcal{C},\\ \frac{\partial v}{\partial y^\alpha}(x,0) = f(x,v(x,0)) \quad &\text{on } \Omega.\end{aligned}$$

In this case, v is such that

$$\int_{\mathcal{C}} y^{1-2\alpha}\nabla v\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega f(x,v(x,0))\psi(x,0)\,\mathrm{d}x \quad \forall\psi\in\mathcal{H}_0^\alpha(\mathcal{C}). \tag{2.5}$$

2.2. Maximum principle

Throughout the paper, the following maximum principle will be very useful (see [9, lemma 2.5] for a related result).

Proposition 2.2. *Let $d \in L^\infty(\Omega)$ and $v \in \mathcal{H}^\alpha(\mathcal{C})$ such that $v \geqslant 0$ in $\partial_L\mathcal{C}$ and*

$$\begin{aligned}-\operatorname{div}(y^{1-2\alpha}\nabla v) \geqslant 0 \quad &\text{in } \mathcal{C},\\ \frac{\partial v}{\partial y^\alpha}(x,0) + d(x)v(x,0) \geqslant 0 \quad &\text{on } \Omega.\end{aligned}$$

(a) *Assume that $d \geqslant 0$ in Ω; then $v \geqslant 0$ in $\mathcal{C}$.*

(b) *Assume that $v \geqslant 0$ in $\mathcal{C}$; then either $v \equiv 0$ or $v > 0$ in $\mathcal{C}$.*

Proof.

(a) The proof follows just by using $-v^-$ as a test function, where $v = v^+ + v^-$.

(b) In this case we follow the proof of [6, lemma 4.9]. Define

$$w(x,y) := \exp\{Ay^{2\alpha}\}v(x,y).$$

Then, w satisfies

$$\begin{aligned}-\operatorname{div}(y^{1-2\alpha}\nabla(\exp\{-Ay^{2\alpha}\}w)) \geqslant 0 \quad &\text{in } \mathcal{C},\\ \frac{\partial w}{\partial y^\alpha}(x,0) + (d(x)+2A\alpha)w(x,0) \geqslant 0 \quad &\text{on } \Omega.\end{aligned}$$

We can choose A such that $d(x)+2A\alpha \leqslant 0$ in Ω, and so

$$\frac{\partial w}{\partial y^{\alpha}}(x,0) \geqslant 0 \quad \text{in } \Omega.$$

Taking $R>0$, we now consider the even extension of w in $\Omega \times (-R,R)$, defined by

$$\tilde{w}(x,y) = \begin{cases} w(x,y) & \text{if } y>0, \\ w(x,-y) & \text{if } y \leqslant 0. \end{cases}$$

We can show that

$$-\operatorname{div}(|y|^{1-2\alpha}\nabla(\exp\{-A|y|^{2\alpha}\}\tilde{w})) \geqslant 0 \quad \text{in } \Omega \times (-R,R).$$

Now, define the problem

$$\begin{gathered} -\operatorname{div}(|y|^{2\alpha}\nabla(\exp\{-A|y|^{2\alpha}\}h)) = 0 \quad \text{in } \Omega \times (-R,R), \\ h = \tilde{w} \quad \text{on } (\Omega \times \{-R\}) \cup (\Omega \times \{R\}). \end{gathered}$$

The above problem possesses a solution by [13] (see also [6, theorem 3.2]) and by the maximum principle we get that

$$h \leqslant \tilde{w} \quad \text{in } \Omega \times (-R,R).$$

On the other hand, by the strong maximum principle (see [13, lemma 2.3.5]), we conclude that

$$h>0.$$

This completes the proof. □

Remark 2.3. Observe that proposition 2.2 can equivalently be stated as follows: assume $d \in L^{\infty}(\Omega)$ and $(-\Delta)^{\alpha}u + d(x)u \geqslant 0$ in Ω and $u \geqslant 0$ on $\partial\Omega$. Then,

(a) if $d \geqslant 0$ in Ω, then $u \geqslant 0$ in Ω,

(b) assuming that $u \geqslant 0$ in Ω, either $u \equiv 0$ or $u>0$ in Ω.

2.3. Regularity results

The following result follows by [10, lemma 3.3] (see also [2, proposition 5.1]).

Lemma 2.4. *Assume that $f \in C(\bar{\Omega} \times \mathbb{R})$ and that there exists a constant C and $p \in (2, 2N/(N-2\alpha))$ such that*

$$|f(x,t)| \leqslant C(1+|t|^{p-1}), \quad x \in \Omega, \ t \in \mathbb{R}.$$

If $v \in \mathcal{H}_0^{\alpha}(\mathcal{C})$ is a solution of (2.5) and $u = \operatorname{tr}_{\Omega} v$, then $v \in L^{\infty}(\mathcal{C}) \cap C^{\sigma}(\bar{\mathcal{C}})$ and $u \in C^{\sigma}(\bar{\Omega})$ for some $\sigma \in (0,1)$.

Consider now the linear problem

$$\left.\begin{aligned} (-\Delta)^{\alpha}u &= g(x) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}\right\} \tag{2.6}$$

The following result is taken from [10, lemma 3.2] (see also [7]).

Lemma 2.5. *Assume that $g \in H^{-\alpha}(\Omega)$ and $v \in \mathcal{H}_0^\alpha(\mathcal{C})$ is a solution of (2.6) and $u = \mathrm{tr}_\Omega v$. Then we have the following.*

(a) *If $g \in L^r(\Omega)$ for $r > N/(2\alpha)$, then $v \in L^\infty(\mathcal{C})$ and $u \in L^\infty(\Omega)$.*

(b) *If $g \in L^\infty(\Omega)$, then $v \in C^\sigma(\bar{\mathcal{C}})$ and $u \in C^\sigma(\bar{\Omega})$ for some $\sigma \in (0,1)$.*

(c) *If $g \in C^\sigma(\bar{\Omega})$ and $g\rfloor_{\partial\Omega} \equiv 0$, then $v \in C^{1,\sigma}(\bar{\mathcal{C}})$ and $u \in C^{1,\sigma}(\bar{\Omega})$ for some $\sigma \in (0,1)$.*

(d) *If $g \in C^{1,\sigma}(\bar{\Omega})$ and $g\rfloor_{\partial\Omega} \equiv 0$, then $v \in C^{2,\sigma}(\bar{\mathcal{C}})$ and $u \in C^{2,\sigma}(\bar{\Omega})$ for some $\sigma \in (0,1)$.*

3. The eigenvalue problem

Given $c \in L^\infty(\Omega)$, we study the eigenvalue problem

$$\left.\begin{aligned} (-\Delta)^\alpha u + c(x)u = \lambda u \quad & \text{in } \Omega, \\ u = 0 \quad & \text{on } \partial\Omega, \end{aligned}\right\} \tag{3.1}$$

where $\alpha \in (0,1)$ and $\lambda \in \mathbb{R}$. Recall that $u \in \mathcal{V}_0^\alpha(\Omega)$ is an eigenfunction associated with an eigenvalue λ of (3.1) if and only if $u = \mathrm{tr}_\Omega v$, where $v \in \mathcal{H}_0^\alpha(\mathcal{C})$ is a solution of

$$\left.\begin{aligned} -\operatorname{div}(y^{1-2\alpha}\nabla v) = 0 \quad & \text{in } \mathcal{C}, \\ v = 0 \quad & \text{on } \partial_L\mathcal{C}, \\ \frac{\partial v}{\partial y^\alpha}(x,0) + c(x)v(x,0) = \lambda v(x,0) \quad & \text{on } \Omega. \end{aligned}\right\} \tag{3.2}$$

In the following result, we show the existence of principal eigenvalue and positive eigenfunction of (3.1) and their main properties.

Theorem 3.1. *There exists a principal eigenvalue of (3.1), denoted by $\lambda_1[\alpha; c]$. This eigenvalue is simple and possesses a unique eigenfunction Φ_1 of (3.2), up to multiplicative constants, which can be taken to be positive. Moreover, the principal eigenfunction Φ_1 is strongly positive, and $\lambda_1[\alpha; c]$ is the only eigenvalue of (3.1) possessing a positive eigenfunction. If we define $\varphi_1 := \mathrm{tr}_\Omega \Phi_1$, we have that*

$$\varphi_1 \in C^\sigma(\bar{\Omega}) \quad \text{and} \quad \Phi_1 \in L^\infty(\mathcal{C}) \cap C^\sigma(\bar{\mathcal{C}}) \quad \text{for some } \sigma \in (0,1).$$

Furthermore, the map from $c \in L^\infty(\Omega) \mapsto \lambda_1[\alpha; c]$ is increasing.

Proof. For each $v \in \mathcal{H}_0^\alpha(\mathcal{C})$ such that $\mathrm{tr}_\Omega v \neq 0$ in $L^2(\Omega)$, let us consider

$$J(v) := \left(\int_{\mathcal{C}} y^{1-2\alpha}|\nabla v|^2 \,\mathrm{d}x\,\mathrm{d}y + \int_\Omega c(x)v(x,0)^2 \,\mathrm{d}x\right)\left(\int_\Omega v(x,0)^2 \,\mathrm{d}x\right)^{-1} \tag{3.3}$$

and note that J is bounded from below. In fact, the trace theorem and the boundedness of c imply that

$$\begin{aligned}\int_{\mathcal{C}} y^{1-2\alpha}|\nabla v|^2\,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega} c(x)v(x,0)^2\,\mathrm{d}x \\ \geqslant C\int_{\Omega} v(x,0)^2\,\mathrm{d}x + \int_{\Omega} c(x)v(x,0)^2\,\mathrm{d}x \\ \geqslant K\int_{\Omega} v(x,0)^2\,\mathrm{d}x,\end{aligned}$$

where $K \in \mathbb{R}$, for every such v.

Let us define

$$\lambda_1[\alpha;c] := \inf\{J(v);\ v \in \mathcal{H}_0^{\alpha}(\mathcal{C}) \text{ and } \operatorname{tr}_{\Omega} v \neq 0 \text{ in } L^2(\Omega)\}. \tag{3.4}$$

Let $(v_n)_{n\in\mathbb{N}} \subset \mathcal{H}_0^{\alpha}(\mathcal{C})$ be such that $\int_{\Omega} v_n(x,0)^2\,\mathrm{d}x = 1$ and $J(v_n) \to \lambda_1[\alpha;c]$. It is straightforward to see that $(v_n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}_0^{\alpha}(\mathcal{C})$, and hence there exists $w \in \mathcal{H}_0^{\alpha}(\mathcal{C})$ such that $w_n \rightharpoonup w$ in $\mathcal{H}_0^{\alpha}(\mathcal{C})$. Since $\mathcal{H}_0^{\alpha}(\mathcal{C}) \hookrightarrow \mathcal{V}_0^{\alpha}(\Omega)$ continuously and $\mathcal{V}_0^{\alpha}(\Omega) \hookrightarrow L^2(\Omega)$ compactly, $\int_{\Omega} w(x,0)^2\,\mathrm{d}x = 1$. Just by imitating the arguments of [14, § 8.12], one can show that $(v_n)_{n\in\mathbb{N}}$ is a Cauchy sequence which strongly converges to v in $\mathcal{H}_0^{\alpha}(\mathcal{C})$. Hence, $J(v) = \lambda_1[\alpha;c]$.

If $\psi \in \mathcal{H}_0^{\alpha}(\mathcal{C})$, setting $\varphi(t) = J(v + t\psi)$, it follows that

$$\begin{aligned}0 &= \varphi'(0) \\ &= \int_{\mathcal{C}} y^{1-2\alpha}\nabla v\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega} c(x)v(x,0)\psi(x,0)\,\mathrm{d}x \\ &\quad - \lambda_1[\alpha;c]\int_{\Omega} v(x,0)\psi(x,0)\,\mathrm{d}x.\end{aligned}$$

Hence, v is a solution of (3.2) with $\lambda = \lambda_1[\alpha;c]$ and it is therefore an eigenfunction associated with $\lambda_1[\alpha;c]$.

Of course, the definition implies that $\lambda_1[\alpha;c]$ is the smallest eigenvalue of (3.2).

Now let us prove that the eigenfunctions have at least $C^{\gamma}(\bar{\Omega})$ regularity, where $\gamma = \min\{1, 2\alpha\}$. This follows easily from lemmas 2.4 and 2.5 once we prove that $\|\operatorname{tr}_{\Omega}\phi\|_{L^{\infty}(\Omega)} < +\infty$ for every eigenfunction ϕ. On the other hand, this L^{∞} estimate can be obtained by a standard application of the Moser iteration technique, which we describe below.

Let $v \in \mathcal{H}_0^{\alpha}(\mathcal{C})$, satisfying (3.2) for some λ, and let $M > 0$. Defining $v_M = \min\{v, M\}$, note that it is an $\mathcal{H}_0^{\alpha}(\mathcal{C})$ function. Let $b > 0$ be a constant to be chosen conveniently, and let us take v_M^b as a test function in (3.2). Defining $e(x) := (\lambda - c(x))$ it follows that

$$b\int_{\mathcal{C}} y^{1-2\alpha}v_M^{b-1}|\nabla v_M|^2\,\mathrm{d}x\,\mathrm{d}y = \int_{\Omega} e(x)v(x,0)v_M(x,0)^b\,\mathrm{d}x,$$

which implies that

$$\frac{4b}{(b+1)^2}\int_{\mathcal{C}} y^{1-2\alpha}|\nabla(v_M^{(b+1)/2})|^2\,\mathrm{d}x\,\mathrm{d}y \leqslant \int_{\Omega} e(x)v(x,0)^{b+1}\,\mathrm{d}x.$$

By the trace theorem and the embedding of fractional Sobolev spaces, we have that

$$\frac{4b}{(b+1)^2}\|\mathrm{tr}_\Omega\, v_M^{(b+1)/2}\|^2_{L^{2_\alpha}(\Omega)} \leqslant C\|\mathrm{tr}_\Omega\, v\|^{b+1}_{L^{b+1}(\Omega)}.$$

Considering $M \to +\infty$ and using Fatou's lemma, we have that

$$\frac{4b}{(b+1)^2}\|\mathrm{tr}_\Omega\, v^{(b+1)/2}\|^2_{L^{2_\alpha}(\Omega)} \leqslant C\|\mathrm{tr}_\Omega\, v\|^{b+1}_{L^{b+1}(\Omega)}.$$

Then it follows that

$$\|\mathrm{tr}_\Omega\, v\|_{L^{2_\alpha(b+1)/2}(\Omega)} \leqslant \left(C\frac{(b+1)^2}{4b}\right)^{1/(b+1)}\|\mathrm{tr}_\Omega\, v\|_{L^{b+1}(\Omega)}. \tag{3.5}$$

Let us consider a sequence $(\eta_k)_k$ defined by $\eta_0 = 2$ and $\eta_k = (2_\alpha/2)\eta_{k-1}$ for $k \geqslant 1$. Taking b in (3.5) such that $b + 1 = \eta_{k-1}$, we have that

$$\|\mathrm{tr}_\Omega\, v\|_{L^{\eta_k}(\Omega)} \leqslant \left(C\frac{\eta_{k-1}^2}{4(\eta_{k-1}-1)}\right)^{1/\eta_{k-1}}\|\mathrm{tr}_\Omega\, v\|_{L^{\eta_{k-1}}(\Omega)}.$$

Iterating this expression in k, we get that

$$\|\mathrm{tr}_\Omega\, v\|_{L^{\eta_k}(\Omega)} \leqslant \prod_{j=0}^{k-1}\left(C\frac{\eta_j^2}{4(\eta_j-1)}\right)^{1/\eta_j}\|\mathrm{tr}_\Omega\, v\|_{L^2(\Omega)}.$$

Note that there exists a constant $C > 0$ such that $z^2/4(z-1) \leqslant Cz$ for all $z \geqslant 1$. Taking into account the fact that $\eta_j = 2_\alpha^j/2^{j-1}$, it follows that

$$\begin{aligned}\|\mathrm{tr}_\Omega\, v\|_{L^{\eta_k}(\Omega)} &\leqslant \prod_{j=0}^{k-1}\left(C\frac{2_\alpha^j}{2^{j-1}}\right)^{2^{j-1}/2_\alpha^j}\|\mathrm{tr}_\Omega\, v\|_{L^2(\Omega)}\\ &\leqslant (2_\alpha C)^{A_k}\prod_{j=0}^{k-1}(\delta^{1-j})^{\delta^{j-1}/2\alpha}\|\mathrm{tr}_\Omega\, v\|_{L^2(\Omega)},\end{aligned}$$

where $\delta = 2/2_\alpha \in (0,1)$ and

$$A_k = \frac{1}{2_a}\sum_{j=1}^{k-1}\delta^{j-1}.$$

Now, since $0 < \delta < 1$, the series in A_k converges and

$$\prod_{j=0}^{k-1}(\delta^{1-j})^{\delta^{j-1}/2\alpha} < +\infty.$$

Now, observing that $\eta_k \to +\infty$, it follows that $\|\mathrm{tr}_\Omega\, v\|_{L^\infty(\Omega)} < +\infty$.

If v is a minimizer for J, then it is straightforward to see that $|v|$ is too. Taking a constant $M > 0$ such that $M + c(x) > 0$ in Ω, proposition 2.2 implies that $|v| > 0$ in $\mathcal{C}$. Since v is regular, it follows that v cannot change sign. Consequently, two of them cannot be orthogonal, and $\lambda_1[\alpha; c]$ is simple.

Applying the same procedure to $\lambda_1[\alpha; c]$ proves that (denoting by V_j the eigenspace associated with the jth eigenvalue) the higher eigenvalues can be characterized as

$$\lambda_j = \inf\{J(u); u \neq 0, \langle u, v\rangle_{L^2(\Omega)} = 0 \ \forall v \in \operatorname{span}[V_1, \dots, V_{j-1}]\}.$$

This characterization with the positiveness of the first eigenfunction implies that the first eigenvalue is the only one that has a single-signed eigenfunction.

In order to complete the proof, note that the variational characterization of the eigenvalues still implies that if $c_1, c_2 \in L^\infty(\Omega)$ and $c_1 < c_2$ in Ω. Then $\lambda_1[\alpha; c_1] < \lambda_1[\alpha; c_2]$. In fact, let $w \in \mathcal{H}_0^\alpha(\mathcal{C})$ such that $\operatorname{tr}_\Omega w \neq 0$ in $L^2(\Omega)$ and $J(w) = \lambda_1[\alpha; c_2]$. Note that

$$\left(\int_{\mathcal{C}} y^{1-2\alpha}|\nabla w|^2 \, \mathrm{d}x \, \mathrm{d}y + \int_\Omega c_1(x) w(x,0)^2 \, \mathrm{d}x\right)\left(\int_\Omega w(x,0)^2 \, \mathrm{d}x\right)^{-1} < \left(\int_{\mathcal{C}} y^{1-2\alpha}|\nabla w|^2 \, \mathrm{d}x \, \mathrm{d}y + \int_\Omega c_2(x) w(x,0)^2 \, \mathrm{d}x\right)\left(\int_\Omega w(x,0)^2 \, \mathrm{d}x\right)^{-1},$$

which completes the proof. □

Analysing the behaviour of $\lambda_1[\alpha; c]$ with respect to the weights is a challenging problem (see, for example, [20, § 3]). We would, however, like to study $\lambda_1[\alpha; c]$ in a particular case. When $c \equiv 0$ we set $\lambda_1[\alpha] := \lambda_1[\alpha; 0]$. Finally, for $\alpha = 1$ we denote by $\lambda_1[1; c]$ the principal eigenvalue of the local operator $-\Delta + c(x)$ under homogeneous Dirichlet boundary conditions and set $\lambda_1 := \lambda_1[1; 0]$. Recall that $\lambda_1[\alpha] = \lambda_1^\alpha$.

Remark 3.2. Given $c \in L^\infty(\Omega)$, we define

$$c_L := \operatorname*{ess\,inf}_\Omega c(x) \quad \text{and} \quad c_M := \operatorname*{ess\,sup}_\Omega c(x).$$

Note that, by the definition of J and the fact that $\lambda_1[\alpha; c]$ minimizes J, it follows that

$$\lambda_1[\alpha] + c_L \leqslant \lambda_1[\alpha; c] \leqslant \lambda_1[\alpha] + c_M.$$

It is not difficult to show that when $c \in \mathbb{R}$ we get

$$\lambda_1[\alpha; c] = \lambda_1[\alpha] + c = \lambda_1^\alpha + c.$$

In the following we show the dependence of $\lambda_1[\alpha; c]$ for $N = 1$ with respect to the domain $\Omega = B_r = (-r, r)$. Denote by $\lambda_1[\alpha; c; B_r]$ the principal eigenvalue of (3.1) in B_r and by $\lambda_1[1; c; B_r]$ the principal eigenvalue of the $-\Delta + c$ in B_r, i.e. the principal eigenvalue of

$$-\Delta v + c(x)v = \lambda_1[1; c; B_r]v \quad \text{in } B_r, \qquad v = 0 \quad \text{on } \partial B_r. \tag{3.6}$$

With this notation, we can prove the following.

Proposition 3.3. *It holds that*

$$\lambda_1[\alpha; c; B_r] r^{2\alpha} = \lambda_1[\alpha; r^{2\alpha} c(r\cdot); B_1] \tag{3.7}$$

and

$$\lambda_1[1; c; B_r]r^2 = \lambda_1[1; r^2 c(r\cdot); B_1]. \tag{3.8}$$

Consequently,

$$\lim_{r\to 0} \lambda_1[\alpha; c; B_r]r^{2\alpha} = \lambda_1[\alpha; 0; B_1] = (\lambda_1[1; 0; B_1])^\alpha. \tag{3.9}$$

Proof. By the definition of $\lambda_1[\alpha; c; B_r]$, there exists v such that

$$\left.\begin{aligned} -\operatorname{div}(y^{1-2\alpha}\nabla v) &= 0 \quad \text{in } B_r \times (0,\infty),\\ v &= 0 \quad \text{on } \partial B_r \times (0,\infty),\\ \frac{\partial v}{\partial y^\alpha}(x,0) + c(x)v(x,0) &= \lambda_1[\alpha; c; B_r]v(x,0) \quad \text{on } B_r. \end{aligned}\right\} \tag{3.10}$$

The change of variables

$$z = \frac{x}{r}, \quad t = \frac{y}{r} \quad \text{and} \quad w(z,t) = v(zr, tr),$$

transforms (3.10) into

$$\left.\begin{aligned} -\operatorname{div}(t^{1-2\alpha}\nabla w) &= 0 \quad \text{in } B_1 \times (0,\infty),\\ w &= 0 \quad \text{on } \partial B_1 \times (0,\infty),\\ \frac{\partial w}{\partial t^\alpha}(z,0) + r^{2\alpha}c(rz)w(z,0) &= r^{2\alpha}\lambda_1[\alpha; c; B_r]w(z,0) \quad \text{on } B_1. \end{aligned}\right\} \tag{3.11}$$

This concludes the proof of (3.7).

In a similar way, under the change of variable

$$z = \frac{x}{r}, \quad w(z) = v(zr)$$

in (3.6), we get (3.8). Obtaining (3.9) from (3.7) is trivial. □

Let us compare the eigenvalues of the Laplacian and fractional Laplacian for the case when $N = 1$, $c \in \mathbb{R}$ and $\Omega = B_r$.

LEMMA 3.4. *Assume $c \in \mathbb{R}$. Then,*

$$\lambda_1[\alpha; c; B_r] > (\textit{resp.} \ <, =)\ \lambda_1[1; c; B_r] \iff r > (\textit{resp.} \ <, =)\ \sqrt{\lambda_1[1; B_1]}.$$

On the other hand, $\alpha \mapsto \lambda_1[\alpha; c; B_r]$ is decreasing when $r > \sqrt{\lambda_1[1; 0; B_1]}$ and increasing when $r < \sqrt{\lambda_1[1; 0; B_1]}$.

Proof. Observe that

$$\lambda_1[1; c; B_r]r^2 = \lambda_1[1; r^2 c(r\cdot); B_1],$$

and so, if c is a constant,

$$\lambda_1[1; c; B_r] = \frac{\lambda_1[1; 0; B_1]}{r^2} + c,$$

and by proposition 3.3 we get

$$\lambda_1[\alpha; c; B_r] = \frac{\lambda_1[\alpha; 0; B_1]}{r^{2\alpha}} + c = \left(\frac{\lambda_1[1; 0; B_1]}{r^2}\right)^\alpha + c.$$

This concludes the result. □

REMARK 3.5. Recall that $\lambda_1[1; 0; B_1] = \frac{1}{4}\pi^2$.

4. The logistic equation

In this section, we want to study the logistic equation

$$\left.\begin{aligned} (-\Delta)^\alpha u + c(x)u &= \lambda u - u^2 \quad \text{in } \Omega, \\ u &= 0 \quad\quad\quad\quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{4.1}$$

where $\alpha \in (0, 1)$ and $c \in L^\infty(\Omega)$, or, equivalently,

$$\left.\begin{aligned} -\operatorname{div}(y^{1-2\alpha}\nabla v) &= 0 \quad \text{in } \mathcal{C}, \\ v &= 0 \quad \text{on } \partial_L\mathcal{C}, \\ \frac{\partial v}{\partial y^\alpha}(x, 0) + c(x)v(x, 0) &= \lambda v(x, 0) - v(x, 0)^2 \quad \text{on } \Omega. \end{aligned}\right\} \tag{4.2}$$

THEOREM 4.1. *Equation (4.1) possesses a positive solution if and only if* $\lambda > \lambda_1[\alpha; c]$. *Moveover, if it exists, this is the unique positive solution and we denote it by* $\theta_{[\alpha,\lambda-c]}$. *Furthermore,* $\theta_{[\alpha,\lambda-c]} \in C^{2,\sigma}(\bar{\Omega})$ *for some* $\sigma \in (0, 1)$*, and the following property holds: if we denote by* φ_1 *the principal eigenfunction associated with* $\lambda_1[\alpha; c]$ *such that* $\|\varphi_1\|_\infty = 1$*, then*

$$(\lambda - \lambda_1[\alpha; c])\varphi_1(x) \leqslant \theta_{[\alpha,\lambda-c]}(x) \leqslant \lambda - c_L \quad \forall x \in \Omega. \tag{4.3}$$

REMARK 4.2. A similar result holds for (4.2). In this case, we denote by $\Theta_{[\alpha,\lambda-c]}$ the unique positive solution of (4.2), i.e. $\theta_{[\alpha,\lambda-c]} = \operatorname{tr}_\Omega \Theta_{[\alpha,\lambda-c]}$. Moreover, $\Theta_{[\alpha,\lambda-c]} \in C^{2,\sigma}(\bar{\mathcal{C}}) \cap L^\infty(\mathcal{C})$.

In the proof of theorem 4.1 we shall apply the well known sub–supersolution method. Despite the definitions and results for this subject in the fractional setting being a rather standard adaptation of the sub–supersolution method to second-order operators, we present them here for completeness.

Let us consider problem (2.2), which is associated with the extension problem

$$\left.\begin{aligned} \operatorname{div}(y^{1-2\alpha}\nabla v) &= 0 \quad\quad\quad\quad \text{in } \mathcal{C}, \\ v &= 0 \quad\quad\quad\quad \text{on } \partial_L\mathcal{C}, \\ \frac{\partial v}{\partial y^\alpha}(x, 0) &= f(x, v(x, 0)) \quad \text{on } \Omega, \end{aligned}\right\} \tag{4.4}$$

where $f \in C(\bar{\Omega} \times \mathbb{R})$. Recall the definition of the solution of (4.4) (definition 2.1).

Definition 4.3. We say that $(\underline{v}, \bar{v})$ is a sub–supersolution of (4.4) if $\underline{v}, \bar{v} \in \mathcal{H}^\alpha(\mathcal{C})$, $\underline{u} := \operatorname{tr}_\Omega \underline{v}$, $\bar{u} := \operatorname{tr}_\Omega \bar{v} \in L^\infty(\Omega)$ and

(a) $\underline{v} \leqslant \bar{v}$ in $\mathcal{C}$ and $\underline{v} \leqslant 0 \leqslant \bar{v}$ on $\partial_L \mathcal{C}$.

(b) for all $\psi \in \mathcal{H}_0^\alpha(\mathcal{C})$, $\psi \geqslant 0$, it holds that

$$\int_{\mathcal{C}} y^{1-2\alpha} \nabla \underline{v} \cdot \nabla \psi \, \mathrm{d}x \, \mathrm{d}y \leqslant \int_\Omega f(x, \underline{v}(x,0)) \psi(x,0) \, \mathrm{d}x \tag{4.5}$$

and

$$\int_{\mathcal{C}} y^{1-2\alpha} \nabla \bar{v} \cdot \nabla \psi \, \mathrm{d}x \, \mathrm{d}y \geqslant \int_\Omega f(x, \bar{v}(x,0)) \psi(x,0) \, \mathrm{d}x. \tag{4.6}$$

Theorem 4.4. *Assume that* $(\underline{v}, \bar{v})$ *is a sub–supersolution of (4.4). Then, there exists a solution* v *of (4.4) such that*

$$\underline{v} \leqslant v \leqslant \bar{v} \quad \text{in } \mathcal{C}.$$

Consequently, there exists a solution $u \in \mathcal{V}_0^\alpha(\Omega)$ *of (2.2) such that*

$$\underline{u} = \operatorname{tr}_\Omega \underline{v} \leqslant u \leqslant \bar{u} = \operatorname{tr}_\Omega \bar{v} \quad \text{in } \Omega.$$

Proof. Let $\underline{v}$, $\bar{v}$ be such that (4.5) and (4.6) hold, respectively. For $x \in \Omega$ and $t \in \mathbb{R}$, let us define

$$\tilde{f}(x,t) := \begin{cases} f(x, \underline{u}(x)) & \text{if } t \leqslant \underline{u}(x), \\ f(x,t) & \text{if } \underline{u}(x) \leqslant t \leqslant \bar{u}(x), \\ f(x, \bar{u}(x)) & \text{if } t \geqslant \bar{u}(x), \end{cases}$$

and consider the problem

$$\left.\begin{aligned} \operatorname{div}(y^{1-2\alpha} \nabla v) &= 0 \quad \text{in } \mathcal{C}, \\ v &= 0 \quad \text{on } \partial_L \mathcal{C}, \\ \frac{\partial v}{\partial y^\alpha}(x,0) &= \tilde{f}(x, v(x,0)) \quad \text{on } \Omega. \end{aligned}\right\} \tag{4.7}$$

Observe that by the definition of $\tilde{f}$ we have that

$$\left| \int_\Omega \tilde{f}(x, u(x,0)) \psi(x,0) \, \mathrm{d}x \right| \leqslant C \|\psi(x,0)\|_{L^2(\Omega)} \tag{4.8}$$

for some positive constant C and for all $u \in \mathcal{H}^\alpha(\mathcal{C})$ and $\psi \in \mathcal{H}_0^\alpha(\mathcal{C})$. Here, we have used that $\underline{u}, \bar{u} \in L^\infty(\Omega)$ and $f \in C(\bar{\Omega} \times \mathbb{R})$.

First, we show that (4.7) possesses at least one solution. Define the operator

$$T \colon \mathcal{H}_0^\alpha(\mathcal{C}) \mapsto (\mathcal{H}_0^\alpha(\mathcal{C}))'$$

given by

$$(Tu, v) = \int_{\mathcal{C}} y^{1-2\alpha} \nabla u \cdot \nabla v \, \mathrm{d}x \, \mathrm{d}y - \int_\Omega \tilde{f}(x, u(x,0)) v(x,0) \, \mathrm{d}x \quad \forall u, v \in \mathcal{H}_0^\alpha(\mathcal{C}).$$

We study some properties of the map T.

- T is a bounded map. It is clear, using (4.8), that if u belongs to a bounded set of $\mathcal{H}_0^\alpha(\mathcal{C})$, then $T(u)$ is also bounded in $(\mathcal{H}_0^\alpha(\mathcal{C}))'$.
- T is pseudo-monotone: given a sequence $u_n \rightharpoonup u$ in $\mathcal{H}_0^\alpha(\mathcal{C})$ such that

$$\limsup (Tu_n, u_n - u) \leqslant 0,$$

we must show that

$$\liminf (Tu_n, u_n - v) \geqslant (Tu, u - v) \quad \forall v \in \mathcal{H}_0^\alpha(\mathcal{C}). \tag{4.9}$$

Observe that from (4.8) we have that

$$\left| \int_\Omega \tilde{f}(x, u_n(x,0))(u_n(x,0) - u(x,0))\,\mathrm{d}x \right| \leqslant C\|u_n - u\|_{L^2(\Omega)} \to 0.$$

Hence, using the fact that $u_n \rightharpoonup u$ in $\mathcal{H}_0^\alpha(\mathcal{C})$,

$$\begin{aligned} 0 &\geqslant \limsup (Tu_n, u_n - u) \\ &= \limsup \int_{\mathcal{C}} y^{1-2\alpha} \nabla u_n \cdot \nabla(u_n - u) \\ &= \limsup \|u_n\|_\alpha^2 - \|u\|_\alpha^2. \end{aligned}$$

We can conclude that

$$\|u\|_\alpha^2 \geqslant \limsup \|u_n\|_\alpha^2 \geqslant \liminf \|u_n\|_\alpha^2 \geqslant \|u\|_\alpha^2,$$

and then

$$\lim \|u_n\|_\alpha^2 = \|u\|_\alpha^2.$$

Consequently, $u_n \to u$ in $\mathcal{H}_0^\alpha(\mathcal{C})$ and we get that

$$\liminf (Tu_n, u_n - v) = \liminf\{(Tu_n, u_n - u) + (Tu_n, u - v)\} \geqslant (Tu, u - v).$$

- T is coercive, i.e.

$$\lim_{\|v\|_\alpha \to \infty} \frac{(T(v), v)}{\|v\|_\alpha} = \infty.$$

It is clear that

$$(T(v), v) \geqslant \|v\|_\alpha^2 - C\|v\|_{L^2(\Omega)}^2,$$

whence it follows that T is coercive.

Then, we can conclude from [16, ch. 2, theorem 2.7] that there exists a solution of (4.7), i.e. $T(v) = 0$. Now, we show that

$$v \in [\underline{v}, \bar{v}],$$

and hence v is solution of (4.4). Indeed, define $\tilde{v} := \underline{v} - v$. Note that, for all $\psi \in \mathcal{H}_0^\alpha(\mathcal{C})$, $\psi \geqslant 0$,

$$\int_{\mathcal{C}} y^{1-2\alpha} \nabla \tilde{v} \cdot \nabla \psi \,\mathrm{d}x\,\mathrm{d}y \leqslant \int_\Omega (f(x, \underline{v}(x,0)) - \tilde{f}(x, v(x,0)))\psi(x,0)\,\mathrm{d}x.$$

Taking $\psi = (\underline{v} - v)^+$, we have that

$$\int_{\mathcal{C}} y^{1-2\alpha} |\nabla \tilde{v}^+|^2 \,\mathrm{d}x\,\mathrm{d}y \leqslant 0.$$

Then $\underline{v} \leqslant v$ in $\mathcal{C}$, and in a similar way one can prove that $v \leqslant \bar{v}$. □

Now let us present the proof of theorem 4.1.

Proof of theorem 4.1. First consider a positive solution $u \in \mathcal{V}_0^\alpha(\Omega)$ of (4.1), and consider $v \in \mathcal{H}_0^\alpha$ solution of (4.2). If $\lambda - c_L \leqslant 0$, then by the maximum principle it follows that $v \leqslant 0$. So, assume that $\lambda - c_L > 0$. Taking $\psi = (v - (\lambda - c_L))^+$ in (4.2), we can show that

$$v \leqslant \lambda - c_L \quad \text{in } \mathcal{C}.$$

By lemma 2.4, we have that $u \in L^\infty(\Omega)$; and then, using lemma 2.5 we obtain that u and v are regular functions.

Now, suppose that there exists a positive solution $u \in \mathcal{V}_0^\alpha(\Omega)$ of (4.1) for some $\lambda \in \mathbb{R}$. Then note that u is a positive solution of (3.1) with $c(x)$ substituted by $(c(x) + u(x))$. Then, by theorem 3.1,

$$\lambda = \lambda_1[\alpha; c + u] > \lambda_1[\alpha; c].$$

Now let us prove that $\lambda > \lambda_1[\alpha; c]$ is sufficient for the existence of a positive solution. Let $\Omega \subset\subset \Omega'$, Ω' be an open bounded set, $\mathcal{C}' = \Omega' \times (0, +\infty)$ and $E \in \mathcal{H}_0^\alpha(\mathcal{C}')$ be the unique positive solution of

$$\left.\begin{aligned} \operatorname{div}(y^{1-2\alpha}\nabla v) &= 0 \quad \text{in } \mathcal{C}', \\ v &= 0 \quad \text{on } \partial_L \mathcal{C}', \\ \frac{\partial v}{\partial y^\alpha}(x, 0) &= 1 \quad \text{in } \Omega'. \end{aligned}\right\} \tag{4.10}$$

Define

$$e(x) := \operatorname{tr}_{\Omega'} E.$$

Observe that, from the regularity results, $e \in L^\infty(\Omega')$ and by proposition 2.2 we get that $E > 0$.

Note, in particular, that we can extend $\psi \in \mathcal{H}_0^\alpha(\mathcal{C})$, in such a way that $\psi \in \mathcal{H}_0^\alpha(\mathcal{C}')$, and then the following holds:

$$\int_{\mathcal{C}} y^{1-2\alpha} \nabla E \cdot \nabla \psi \,\mathrm{d}x\,\mathrm{d}y = \int_\Omega \psi(x, 0)\,\mathrm{d}x.$$

Let us take $\bar{v} = KE$, where K is a positive constant to be chosen. Note that $\bar{v}$ is a supersolution of (4.2) if and only if, for all $\psi \in \mathcal{H}_0^\alpha(\mathcal{C})$, $\psi \geqslant 0$,

$$\int_{\mathcal{C}} y^{1-2\alpha} \nabla E \cdot \nabla \psi \,\mathrm{d}x\,\mathrm{d}y + K \int_\Omega c(x) E(x, 0) \psi(x, 0)\,\mathrm{d}x \geqslant \int_\Omega (\lambda E(x, 0) - K E(x, 0)^2) \psi(x, 0)\,\mathrm{d}x.$$

This is equivalent to

$$\int_{\Omega} \psi(x,0)(Ke(x)^2 + e(x)(c(x) - \lambda) + 1)\,\mathrm{d}x \geqslant 0 \quad \forall \psi \in \mathcal{H}_0^\alpha(\mathcal{C}),\ \psi \geqslant 0.$$

It suffices that $Ke(x)^2 + e(x)(c_L - \lambda) + 1 \geqslant 0$ almost everywhere in Ω, which is possible by choosing K large enough.

For the subsolution, let us take $\underline{v} = \varepsilon \Psi_1$, where $\varepsilon > 0$ is a constant to be chosen and $\Psi_1 \in \mathcal{H}_0^\alpha(\mathcal{C})$ is a positive eigenfunction associated with $\lambda_1[\alpha; c]$. Then, for all $\psi \in \mathcal{H}_0^\alpha$, $\psi \geqslant 0$, writing $\lambda_1 = \lambda_1[\alpha; c]$, we have

$$\begin{aligned}\int_{\mathcal{C}} y^{1-2\alpha} \nabla \underline{v} \cdot \nabla \psi \,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega} c(x)\underline{v}(x,0)\psi(x,0)\,\mathrm{d}x &= \varepsilon \int_{\Omega} \lambda_1 \varphi_1 \psi(x,0)\,\mathrm{d}x \\ &\leqslant \int_{\Omega} \varepsilon \varphi_1 \psi(x,0)(\lambda - \varepsilon\varphi_1)\,\mathrm{d}x\end{aligned}$$

if and only if

$$\varepsilon \varphi_1 \leqslant (\lambda - \lambda_1) \quad \text{in } \Omega, \tag{4.11}$$

where we have defined $\varphi_1 = \mathrm{tr}_\Omega \Psi_1$. Since $\varphi_1 \in \mathcal{V}_0^\alpha(\Omega)$, $\varphi_1 \in L^\infty(\Omega)$ and $\varphi_1 > 0$ in Ω, (4.11) is possible and it follows that we have a sub–supersolution pair if $\varepsilon > 0$ is small enough. Now theorem 4.4 implies the existence of a solution if $\lambda > \lambda_1[\alpha; c]$.

To prove the uniqueness of positive solution, all the arguments of [4] (see also [5]) can be adapted to the fractional setting (see [3, lemma 5.2] or [19, proposition 4.2]).

Then, there exists a solution $\theta_{[\alpha,\lambda-c]} \in \mathcal{V}_0^\alpha(\Omega)$ of (4.1) if and only if $\lambda > \lambda_1[\alpha; c]$.

We now prove (4.3). The first inequality follows since $\varepsilon\varphi_1$ is a subsolution for all $\varepsilon \in (0, \lambda - \lambda_1[\alpha; c]]$. For the second, note that $\theta_{[\alpha,\lambda-c]} \leqslant \lambda - c_L$. □

To compare different solutions of the logistic equation we need the following result.

Proposition 4.5. *Assume that $\underline{v}$ is a bounded subsolution of (4.2). Then*

$$\mathrm{tr}_\Omega \underline{v} \leqslant \theta_{[\alpha,\lambda-c]}.$$

Proof. Since $\underline{v}$ is bounded, it is clear that we can choose $K > 0$ such that KE is a supersolution of (4.2) and $\underline{v} \leqslant KE$. By uniqueness, we conclude that $\underline{v}(x,0) \leqslant \theta_{[\alpha,\lambda-c]}$. □

As a direct consequence of proposition 4.5, we deduce the following.

Corollary 4.6. *If $\lambda_1 \leqslant \lambda_2$ and $c_2 \leqslant c_1$ in Ω, then $\theta_{[\alpha,\lambda_1-c_1]} \leqslant \theta_{[\alpha,\lambda_2-c_2]}$.*

Let us give an interesting biological interpretation of this result, comparing it with the linear diffusion case. Recall that the classical logistic equation

$$\left.\begin{aligned}-\Delta u + c(x)u = \lambda u - u^2 \quad &\text{in } \Omega, \\ u = 0 \quad &\text{on } \partial\Omega,\end{aligned}\right\} \tag{4.12}$$

possesses a unique positive solution if and only if

$$\lambda > \lambda_1[1; -c].$$

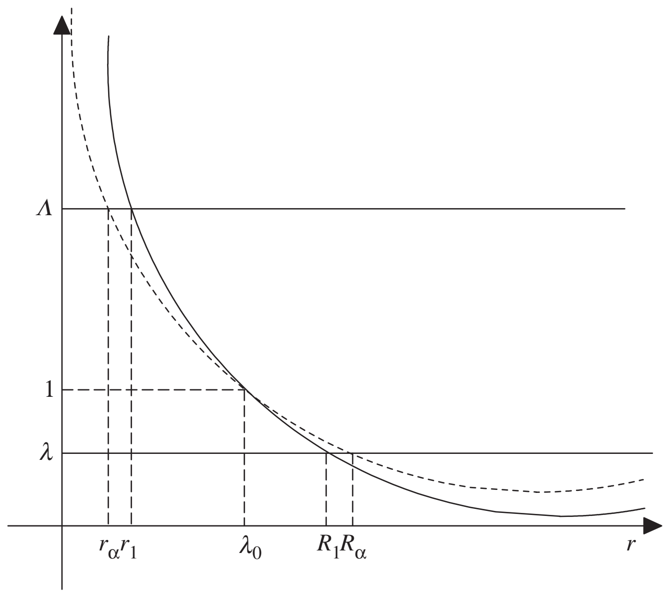

Figure 1. The solid line represents the map $G_1(r) = \lambda_1[1; c; B_r]$ and the dotted line represents $G_\alpha(r) = \lambda_1[\alpha; c; B_r]$. $\lambda_0 = \sqrt{\lambda_1}$.

Let us compare this result with that obtained for (4.1) in the particular case when $N = 1$, $c \in \mathbb{R}$ and $\Omega = B_r$. In figure 1 we represent $G_1(r) := \lambda_1[1; c; B_r]$ by the solid line and by $G_\alpha(r) := \lambda_1[\alpha; c; B_r]$ the dotted line with $c = 0$ (a similar representation can be made with $c \neq 0$). Take Λ large ($\Lambda > 1$). Then, there exist $r_\alpha < r_1$ such that

$$\Lambda = G_1(r_1) = G_\alpha(r_\alpha).$$

Then, we have the following.

(a) If $r < r_\alpha$, for (4.1) and (4.12) the species dies.

(b) If $r > r_1$, the species persists in both cases.

(c) Assume that $r \in (r_\alpha, r_1)$. Then, the species disappears in the local diffusion and it persists in the fractional diffusion case.

Now, assume λ is small ($\lambda < 1$). Then, there exist $R_1 < R_\alpha$ such that

$$\lambda = G_1(R_1) = G_\alpha(R_\alpha).$$

Moreover, we have the following.

(a) If $r < R_1$ for (4.1) and (4.12), the species dies.

(b) If $r > R_\alpha$, the species persists in both cases.

(c) Assume that $r \in (R_1, R_\alpha)$. Then, the species disappears in the fractional diffusion and it persists in the local diffusion case.

Hence, in the case of favourable habitats (abundant resources) there exist domains such that the species with fractional diffusion persists, while the species with linear diffusion dies. On the other hand, for unfavourable habitats, there exist domains when the opposite occurs.

5. The sub–supersolution method for systems

In this section we extend the sub–supersolution method employed in the last section to the system setting. Let us consider

$$\left.\begin{aligned} (-\Delta)^{\alpha}u &= f(x,u,v) \quad \text{in } \Omega,\\ (-\Delta)^{\beta}v &= g(x,u,v) \quad \text{in } \Omega,\\ u = v &= 0 \qquad\qquad \text{on } \partial\Omega, \end{aligned}\right\} \tag{5.1}$$

where $f, g \in C^0(\bar{\Omega} \times \mathbb{R}^2)$ and $\alpha, \beta \in (0, 1)$.

Definition 5.1. We say that $(u, v) \in \mathcal{V}_0^{\alpha}(\Omega) \times \mathcal{V}_0^{\beta}(\Omega)$ is a solution of (5.1) if there exists $(U, V) \in \mathcal{H}_0^{\alpha}(\mathcal{C}) \times \mathcal{H}_0^{\beta}(\mathcal{C})$ such that $\operatorname{tr}_{\Omega} U := u$, $\operatorname{tr}_{\Omega} V := v$ and (U, V) is a solution of

$$\left.\begin{aligned} \operatorname{div}(y^{1-2\alpha}\nabla U) = \operatorname{div}(y^{1-2\beta}\nabla V) &= 0 \quad \text{in } \mathcal{C},\\ U = V &= 0 \quad \text{on } \partial_L\mathcal{C},\\ \frac{\partial U}{\partial y^{\alpha}}(x,0) &= f(x, U(x,0), V(x,0)) \quad \text{on } \Omega,\\ \frac{\partial V}{\partial y^{\beta}}(x,0) &= g(x, U(x,0), V(x,0)) \quad \text{on } \Omega. \end{aligned}\right\} \tag{5.2}$$

Definition 5.2. We say that $\underline{U}, \bar{U} \in \mathcal{H}^{\alpha}(\mathcal{C})$, $\underline{V}, \bar{V} \in \mathcal{H}^{\beta}(\mathcal{C})$ is a pair of sub–supersolutions of (5.1) if

$$\underline{u} := \operatorname{tr}_{\Omega} \underline{U}, \qquad \bar{u} := \operatorname{tr}_{\Omega} \bar{U}, \qquad \underline{v} := \operatorname{tr}_{\Omega} \underline{V}, \qquad \bar{v} := \operatorname{tr}_{\Omega} \bar{V} \in L^{\infty}(\Omega),$$

and

(a) $\underline{U} \leqslant \bar{U}$ and $\underline{V} \leqslant \bar{V}$ in $\mathcal{C}$ and $\underline{U} \leqslant 0 \leqslant \bar{U}$ and $\underline{V} \leqslant 0 \leqslant \bar{V}$ on $\partial_L\mathcal{C}$,

(b) for all $(\psi, \phi) \in \mathcal{H}_0^{\alpha}(\mathcal{C}) \times \mathcal{H}_0^{\beta}(\mathcal{C})$, $\psi, \phi \geqslant 0$ and $(u, v) \in [\underline{U}, \bar{U}] \times [\underline{V}, \bar{V}]$, the following hold:

$$\begin{aligned} \int_{\mathcal{C}} y^{1-2\alpha}\nabla\underline{U} \cdot \nabla\psi \,\mathrm{d}x\,\mathrm{d}y &\leqslant \int_{\Omega} f(x, \underline{U}(x,0), v(x,0))\psi(x,0)\,\mathrm{d}x,\\ \int_{\mathcal{C}} y^{1-2\alpha}\nabla\bar{U} \cdot \nabla\psi \,\mathrm{d}x\,\mathrm{d}y &\geqslant \int_{\Omega} f(x, \bar{U}(x,0), v(x,0))\psi(x,0)\,\mathrm{d}x,\\ \int_{\mathcal{C}} y^{1-2\beta}\nabla\underline{V} \cdot \nabla\phi \,\mathrm{d}x\,\mathrm{d}y &\leqslant \int_{\Omega} f(x, u(x,0), \underline{V}(x,0))\phi(x,0)\,\mathrm{d}x,\\ \int_{\mathcal{C}} y^{1-2\beta}\nabla\bar{V} \cdot \nabla\phi \,\mathrm{d}x\,\mathrm{d}y &\geqslant \int_{\Omega} f(x, u(x,0), \bar{V}(x,0))\phi(x,0)\,\mathrm{d}x, \end{aligned}$$

where $[\underline{U}, \bar{U}] = \{w \in \mathcal{H}^{\alpha}(\mathcal{C});\ \underline{U} \leqslant w \leqslant \bar{U} \text{ in } \mathcal{C}\}$ and $[\underline{V}, \bar{V}]$ is defined analogously.

THEOREM 5.3. *Assume that there exists a pair* $(\underline{U}, \bar{U})$, $(\underline{V}, \bar{V})$ *of sub–supersolutions of (5.2). Then, there exists a solution* $(U, V) \in \mathcal{H}_0^\alpha(\mathcal{C}) \times \mathcal{H}_0^\beta(\mathcal{C})$ *of (5.1) such that*

$$\underline{U} \leqslant U \leqslant \bar{U}, \quad \underline{V} \leqslant V \leqslant \bar{V} \quad \text{in } \mathcal{C}.$$

Moreover, there exists a solution $(u, v) \in \mathcal{V}_0^\alpha(\Omega) \times \mathcal{V}_0^\beta(\Omega)$ *of (5.1) such that* $\underline{u} \leqslant u \leqslant \bar{u}$ *in* Ω *and* $\underline{v} \leqslant v \leqslant \bar{v}$ *in* Ω.

Proof. The proof is similar to theorem 4.4. Define the operators T_1 and T_2 by

$$T_1(w) = \begin{cases} \underline{u} & \text{if } w \leqslant \underline{u}, \\ w & \text{if } \underline{u} \leqslant w \leqslant \bar{u}, \\ \bar{u} & \text{if } w \geqslant \bar{u}, \end{cases} \qquad T_2(z) = \begin{cases} \underline{v} & \text{if } z \leqslant \underline{u}, \\ z & \text{if } \underline{v} \leqslant z \leqslant \bar{v}, \\ \bar{v} & \text{if } z \geqslant \bar{v}, \end{cases}$$

and the functions by

$$\tilde{f}(x, u, v) = f(x, T_1(u), T_2(v)), \qquad \tilde{g}(x, u, v) = g(x, T_1(u), T_2(v)).$$

Consider the problem

$$\left.\begin{aligned} \operatorname{div}(y^{1-2\alpha}\nabla U) = \operatorname{div}(y^{1-2\beta}\nabla V) = 0 \quad & \text{in } \mathcal{C}, \\ U = V = 0 \quad & \text{on } \partial_L \mathcal{C}, \\ \frac{\partial U}{\partial y^\alpha}(x, 0) = \tilde{f}(x, U(x, 0), V(x, 0)) \quad & \text{on } \Omega, \\ \frac{\partial V}{\partial y^\beta}(x, 0) = \tilde{g}(x, U(x, 0), V(x, 0)) \quad & \text{on } \Omega. \end{aligned}\right\} \tag{5.3}$$

First, we prove that (5.3) has at least one solution. To do this, consider the space

$$\mathcal{H} := \mathcal{H}_0^\alpha(\mathcal{C}) \times \mathcal{H}_0^\beta(\mathcal{C})$$

with the norm $\|(u, v)\| = \|u\|_\alpha + \|v\|_\beta$ and the map $T\colon \mathcal{H} \mapsto (\mathcal{H})'$ defined by

$$\begin{aligned} (T(u, v), (w, z)) = \bigg(& \int_{\mathcal{C}} y^{1-2\alpha} \nabla u \cdot \nabla w \, \mathrm{d}x \, \mathrm{d}y - \int_\Omega \tilde{f}(x, u(x, 0)) w(x, 0) \, \mathrm{d}x, \\ & \int_{\mathcal{C}} y^{1-2\beta} \nabla v \cdot \nabla z \, \mathrm{d}x \, \mathrm{d}y - \int_\Omega \tilde{g}(x, v(x, 0)) z(x, 0) \, \mathrm{d}x \bigg). \end{aligned}$$

Now, we can just follow the arguments of theorem 4.4 and show that there exists a solution (U, V) of (5.3), i.e. $T(U, V) = (0, 0)$. Again, we can prove that (U, V) is a solution of (5.1), as it suffices to show that $(U, V) \in [\underline{U}, \bar{U}] \times [\underline{V}, \bar{V}]$. Define $\tilde{U} = \underline{U} - U$. Then, taking $T_2(V)$ in the definition of the subsolution, we get that, for all $\psi \in \mathcal{H}_0^\alpha$, $\psi \geqslant 0$,

$$\int_{\mathcal{C}} y^{1-2\alpha} \nabla \tilde{U} \cdot \nabla \psi \, \mathrm{d}x \, \mathrm{d}y \leqslant \int_\Omega [f(x, \underline{U}, T_2(V)) - \tilde{f}(x, U, V)] \psi(x, 0) \, \mathrm{d}x \leqslant 0.$$

Taking $\psi = (\underline{U} - U)^+$ we get that $\underline{U} \leqslant U$. The same argument can be applied to the other inequalities. □

6. Application to the Lotka–Volterra systems

In this section we apply the above results to system (1.1), or equivalently, to the system

$$\left.\begin{aligned} \operatorname{div}(y^{1-2\alpha}\nabla U) = \operatorname{div}(y^{1-2\beta}\nabla V) = 0 \quad &\text{in } \mathcal{C}, \\ U = V = 0 \quad &\text{on } \partial_L\mathcal{C}, \\ \frac{\partial U}{\partial y^\alpha}(x,0) = U(x,0)(\lambda - U(x,0) - bV(x,0)) \quad &\text{in } \Omega, \\ \frac{\partial V}{\partial y^\beta}(x,0) = V(x,0)(\mu - V(x,0) - cU(x,0)) \quad &\text{in } \Omega. \end{aligned}\right\} \tag{6.1}$$

First, we deduce some bounds of the solutions of (1.1).

PROPOSITION 6.1.

(a) *Assume that $b, c > 0$ and let (u, v) be a positive solution of (1.1). Then,*

$$u \leqslant \theta_{[\alpha,\lambda]}, \qquad v \leqslant \theta_{[\beta,\mu]}.$$

(b) *Assume that $b > 0$ and $c < 0$ and let (u, v) be a positive solution of (1.1). Then,*

$$u \leqslant \theta_{[\alpha,\lambda-b\theta_{[\beta,\mu]}]} \leqslant \theta_{[\alpha,\lambda]}, \qquad \theta_{[\beta,\mu]} \leqslant v \leqslant \theta_{[\beta,\mu-c\theta_{[\alpha,\lambda]}]}.$$

(c) *Assume that $b, c < 0$ and let (u, v) be a positive solution of (1.1). Then,*

$$\theta_{[\alpha,\lambda]} \leqslant u, \qquad \theta_{[\beta,\mu]} \leqslant v.$$

Proof.
(a) Assume that $b, c > 0$ and let (u, v) be a positive solution of (1.1), i.e. $(u, v) = (\operatorname{tr}_\Omega U, \operatorname{tr}_\Omega V)$, (U, V) being a solution of (6.1). With similar reasoning to that used in theorem 4.1 we can show that $U, V \in L^\infty(\mathcal{C})$. Moreover, $u \in L^\infty(\Omega)$. It is thus clear that U is a bounded subsolution of (4.1) with $c \equiv 0$. Then, $U \leqslant \Theta_{[\alpha,\lambda]}$, and so

$$u \leqslant \theta_{[\alpha,\lambda]} \quad \text{in } \Omega.$$

In a similar way, we can show that $v \leqslant \theta_{[\beta,\mu]}$.

(b) It is easy to show that $u \leqslant \theta_{[\alpha,\lambda]}$ and $\theta_{[\beta,\mu]} \leqslant v$. The latter inequality shows that $\Theta_{[\beta,\mu]}$ is a subsolution of $(-\Delta)^\beta v = v(\mu - v - cu)$. Moreover, using that $V \geqslant \Theta_{[\beta,\mu]}$, we can show that U is a subsolution of (4.2) with $c(x) = -b\theta_{[\beta,\mu]}$, and so $u \leqslant \theta_{[\alpha,\lambda-b\theta_{[\beta,\mu]}]}$.

(c) This is shown in a similar way to the statements above. □

COROLLARY 6.2.

(a) *Assume that $b, c > 0$. If there exists a positive solution of (1.1), then $\lambda > \lambda_1[\alpha]$ and $\mu > \lambda_1[\beta]$.*

(b) *Assume that $b > 0$ and $c < 0$. If there exists a positive solution of (1.1), then $\lambda > \lambda_1[\alpha; b\theta_{[\beta,\mu]}]$ and $\mu > \lambda_1[\beta; c\theta_{[\alpha,\lambda]}]$.*

We now introduce some notation. Denote by E_α the unique positive solution of (4.10) in $\mathcal{C}$ and $e_\alpha = \operatorname{tr}_\Omega E$. We define

$$C(\alpha,\beta) := \left(\frac{e_\alpha}{e_\beta}\right)_M \left(\frac{e_\beta}{e_\alpha}\right)_M.$$

Our main result is as follows.

THEOREM 6.3.

(a) *Assume* $b, c > 0$ *(the* competitive *case). Assume also that* $\lambda > \lambda_1[\alpha]$ *and* $\mu > \lambda_1[\beta]$. *If* (λ,μ) *verifies*

$$\lambda > \lambda_1[\alpha; b\theta_{[\beta,\mu]}] \quad \text{and} \quad \mu > \lambda_1[\beta; c\theta_{[\alpha,\lambda]}], \tag{6.2}$$

then there exists at least one coexistence state of (1.1).

(b) *Assume that* $b > 0$ *and* $c < 0$ *(the* predator–prey *case). If* (λ,μ) *verifies*

$$\lambda > \lambda_1[\alpha; b\theta_{[\beta,\mu-c\theta_{[\alpha,\lambda]}]}] \quad \text{and} \quad \mu > \lambda_1[\beta; c\theta_{[\alpha,\lambda]}], \tag{6.3}$$

then there exists at least one coexistence state of (1.1).

(c) *Assume that* $b < 0$, $c < 0$ *and* $bc < C(\alpha,\beta)$ *(the* symbiosis *case). If* (λ,μ) *verifies (6.2), then there exists at least one coexistence state of (1.1).*

Proof.
(a) Assume that $b, c > 0$. We take the following sub–supersolutions:

$$(\underline{U}, \bar{U}) = (\Theta_{[\alpha,\lambda-b\theta_{[\beta,\mu]}]}, \Theta_{[\alpha,\lambda]}), \qquad (\underline{V}, \bar{V}) = (\Theta_{[\beta,\mu-c\theta_{[\alpha,\lambda]}]}, \Theta_{[\beta,\mu]}).$$

Indeed, observe that, for $\psi \in \mathcal{H}_0^\alpha(\mathcal{C})$, $\psi \geqslant 0$,

$$\begin{aligned}\int_{\mathcal{C}} y^{1-2\alpha} \nabla \bar{U} \cdot \nabla \psi \, \mathrm{d}x \, \mathrm{d}y &= \int_\Omega \bar{U}(x,0)(\lambda - \bar{U}(x,0))\psi(x,0)\,\mathrm{d}x \\ &\geqslant \int_\Omega \bar{U}(x,0)(\lambda - \bar{U}(x,0) - bV(x,0))\psi(x,0)\,\mathrm{d}x\end{aligned}$$

for all $V \in [\underline{V}, \bar{V}]$.

On the other hand, observe that if $V \in [\underline{V}, \bar{V}]$, then $V \leqslant \Theta_{[\beta,\mu]}$, and so

$$V(x,0) \leqslant \theta_{[\beta,\mu]}.$$

Hence, for $\psi \in \mathcal{H}_0^\alpha(\mathcal{C})$, $\psi \geqslant 0$,

$$\begin{aligned}\int_{\mathcal{C}} y^{1-2\alpha} \nabla \underline{U} \cdot \nabla \psi \, \mathrm{d}x \, \mathrm{d}y &= \int_\Omega \underline{U}(x,0)(\lambda - \underline{U}(x,0) - b\theta_{[\beta,\mu]})\psi(x,0)\,\mathrm{d}x \\ &\leqslant \int_\Omega \bar{U}(x,0)(\lambda - \bar{U}(x,0) - bV(x,0))\psi(x,0)\,\mathrm{d}x\end{aligned}$$

for all $V \in [\underline{V}, \bar{V}]$.

We can proceed with $\underline{V}$ and $\bar{V}$ in a similar way.

Finally, observe that, due to (6.2), $\underline{U} > 0$ and $\underline{V} > 0$. Moreover, since $b, c > 0$, we have $\underline{U} \leqslant \bar{U}$ and $\underline{V} \leqslant \bar{V}$ in $\mathcal{C}$.

(b) Assume that $b > 0$, $c < 0$ and (6.3) holds. Now, we take a pair of sub–supersolutions

$$(\underline{U}, \bar{U}) = (\Theta_{[\alpha,\lambda-b\bar{V}(x,0)]}, \Theta_{[\alpha,\lambda]}), \qquad (\underline{V}, \bar{V}) = (\Theta_{[\beta,\mu]}, \Theta_{[\beta,\mu-c\theta_{[\alpha,\lambda]}]}).$$

First, since $b > 0$ and $c < 0$ it is clear that $\underline{U} \leqslant \bar{U}$ and $\underline{V} \leqslant \bar{V}$, and by (6.3) we get that $\underline{U} > 0$ and $\underline{V} > 0$.

It is not difficult to show that $\underline{V}$ and $\bar{U}$ are sub–supersolutions. Consider $\bar{V}$. We have that, for $\phi \in \mathcal{H}_0^\alpha(\mathcal{C})$, $\phi \geqslant 0$,

$$\begin{aligned}\int_{\mathcal{C}} y^{1-2\alpha}\nabla\bar{V}\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}y &= \int_\Omega \bar{V}(x,0)(\mu-\bar{V}(x,0)-c\theta_{[\alpha,\lambda]})\phi(x,0)\,\mathrm{d}x \\ &\geqslant \int_\Omega \bar{V}(x,0)(\mu-\bar{V}(x,0)-cU(x,0))\phi(x,0)\,\mathrm{d}x\end{aligned}$$

for all $U \in [\underline{U}, \bar{U}]$ because $c < 0$.

Finally, we consider $\underline{U}$. In this case, we have

$$\begin{aligned}\int_{\mathcal{C}} y^{1-2\alpha}\nabla\underline{U}\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}y &= \int_\Omega \underline{U}(x,0)(\lambda-\underline{U}(x,0)-b\bar{V}(x,0))\phi(x,0)\,\mathrm{d}x \\ &\leqslant \int_\Omega \bar{U}(x,0)(\lambda-\bar{U}(x,0)-bV(x,0))\phi(x,0)\,\mathrm{d}x\end{aligned}$$

for all $V \in [\underline{V}, \bar{V}]$.

(c) Assume $b, c < 0$, $bc < C(\alpha, \beta)$ and (6.2) holds. Take

$$(\underline{U}, \bar{U}) = (\Theta_{[\alpha,\lambda-b\theta_{[\beta,\mu]}]}, M_1E_\alpha), \qquad (\underline{V}, \bar{V}) = (\Theta_{[\beta,\mu-c\theta_{[\alpha,\lambda]}]}, M_2E_\beta),$$

where M_1, M_2 are positive constants to be chosen and E_α is the unique solution of (4.10). It is easy to show that $\underline{U}$ and $\underline{V}$ are subsolutions. On the other hand, $\bar{U}$ and $\bar{V}$ are supersolutions provided that

$$M_1e_\alpha^2 \geqslant e_\alpha\lambda + bM_2e_\alpha e_\beta - 1 \quad \text{and} \quad M_2e_\beta^2 \geqslant e_\beta\mu + cM_1e_\alpha e_\beta - 1 \quad \forall x \in \Omega.$$

Since $bc < C(\alpha, \beta)$, we can take M_1 and M_2 to be large. □

Remark 6.4. Conditions (6.2) and (6.3) define a region in the (λ, μ)-plane that could eventually be empty. There are detailed studies of these regions in the $\alpha = \beta = 1$ case (see, for example, [8, 11, 17, 18]). The latter are beyond the scope of this paper; we point out only that if $b > 0$, then

$$\mu \in [\lambda_1[\beta], \infty) \mapsto \lambda_1[\alpha; b\theta_{[\beta,\mu]}] \in \mathbb{R}$$

is an increasing map, because $\mu \mapsto \theta_{[\beta,\mu]}$ is increasing and $c \mapsto \lambda_1[\alpha; c]$ is also increasing.

Similarly, it is decreasing when $b < 0$.

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