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On linear combinations of L-orthogonal polynomials associated with distributions belonging to symmetric classes☆

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Abstract

This paper deals with the classes $S^3(\omega, \beta, b)$ of strong distribution functions defined on the interval $[\beta^2/b, b]$, $0 < \beta < b \leqslant \infty$, where $2\omega \in \mathbb{Z}$. The classification is such that the distribution function $\psi \in S^3(\omega, \beta, b)$ has a (reciprocal) symmetry, depending on ω, about the point β. We consider properties of the L-orthogonal polynomials associated with $\psi \in S^3(\omega, \beta, b)$. Through linear combination of these polynomials we relate them to the L-orthogonal polynomials associated with some $\phi \in S^3(1/2, \beta, b)$.

Keywords: L-orthogonal polynomials; Symmetric distribution functions; Three term recurrence relations

1. Introduction

Let $0 \leqslant a < b \leqslant \infty$ and let ψ be a bounded nondecreasing function defined on $[a, b]$, with infinitely many points of increase on $[a, b]$, such that the moments $\mu_k = \int_a^b t^k \, d\psi(t)$, $k = 0, \pm 1, \pm 2, \ldots$, all exist. We refer to ψ as a strong distribution function on $[a, b]$ or we simply refer to $d\psi$ as a strong distribution on $[a, b]$.

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It is known (see for example [15]) that the monic polynomials $B_n^{(l)}(t),\ n \geqslant 0$, defined for each $l \in \mathbb{Z}$, by the L-orthogonality condition

$$\int_a^b t^{-n+s+l} B_n^{(l)}(t)\,\mathrm{d}\psi(t) = 0, \quad 0 \leqslant s \leqslant n-1, \quad \text{for } n \geqslant 1, \tag{1}$$

exist and if we write $\rho_n^{(l,s)} = \int_a^b t^{-n+s+l} B_n^{(l)}(t)\,\mathrm{d}\psi(t)$ then $(-1)^n \rho_n^{(l,-1)} > 0$ and $\rho_n^{(l,n)} > 0$.

For any given integer l, the sequence of Laurent polynomials $\{t^{-\lfloor (n+1)/2 \rfloor} B_n^{(l)}(t)\}$, form a sequence of orthogonal Laurent polynomials in relation to the distribution $\mathrm{d}\psi$. Polynomials such as $B_n^{(l)}(t)$ were introduced in [11] in order to solve the strong Stieltjes moment problem and later these polynomials and the orthogonal Laurent polynomials were explored in many articles. Applications of these polynomials include differential equations and electrostatics [7], quadrature rules [10,15], two-point Padé approximation [5,6,9,12] and moment preserving approximation [2]. For a survey on orthogonal Laurent polynomials and strong moment theory see [8]. For simplicity, we refer to the polynomials $B_n^{(l)}(t)$ as Laurent-orthogonal polynomials or L-orthogonal polynomials.

The polynomials $B_n^{(l)}(t)$ satisfy the recurrence relation

$$B_{n+1}^{(l)}(t) = (t - \beta_{n+1}^{(l)}) B_n^{(l)}(t) - \alpha_{n+1}^{(l)}\, t\, B_{n-1}^{(l)}(t), \quad n \geqslant 1 \tag{2}$$

with $B_0^{(l)}(t) = 1,\ B_1^{(l)}(t) = t - \beta_1^{(l)}$, where $\beta_1^{(l)} = \mu_l/\mu_{l-1}$, and

$$\alpha_{n+1}^{(l)} = \frac{\rho_n^{(l,n)}}{\rho_{n-1}^{(l,n-1)}} > 0, \quad \beta_{n+1}^{(l)} = -\alpha_{n+1}^{(l)} \frac{\rho_{n-1}^{(l,-1)}}{\rho_n^{(l,-1)}} > 0, \quad n \geqslant 1.$$

Note that $B_n^{(l)}(0) = (-1)^n \beta_n^{(l)} \beta_{n-1}^{(l)} \cdots \beta_1^{(l)} \neq 0$.

The coefficients $\alpha_n^{(l)}$ and $\beta_n^{(l)}$ can be generated using the q–d algorithm (see [12]), given by

$$\beta_n^{(l)} + \alpha_{n+1}^{(l)} = \beta_n^{(l+1)} + \alpha_n^{(l+1)}, \quad \beta_{n+1}^{(l+1)} \alpha_{n+1}^{(l)} = \beta_n^{(l)} \alpha_{n+1}^{(l+1)}, \quad n \geqslant 1$$

with $\alpha_1^{(l)} = 0$ and $\beta_1^{(l)} = \mu_l/\mu_{l-1}$, for all values of l. These two relations in the algorithm also provide

$$\frac{\beta_n^{(l)}}{\beta_{n+1}^{(l+1)}} = \frac{\alpha_{n+1}^{(l)}}{\alpha_{n+1}^{(l+1)}} = \frac{\gamma_n^{(l)}}{\gamma_{n+1}^{(l)}}, \quad n \geqslant 1, \tag{3}$$

where $\gamma_n^{(l)} = \beta_n^{(l)} + \alpha_{n+1}^{(l)}$. The polynomials $B_n^{(l)}(t)$ also satisfy other relations (see [13]) such as

$$B_n^{(l+1)}(t) = B_n^{(l)}(t) - \alpha_{n+1}^{(l)} B_{n-1}^{(l)}(t), \quad n \geqslant 1, \tag{4}$$

$$B_n^{(l+1)}(t) = (B_{n+1}^{(l)}(t) + \beta_{n+1}^{(l)} B_n^{(l)}(t))/t, \quad n \geqslant 0, \tag{5}$$

$$B_n^{(l-1)}(t) = (t B_n^{(l)}(t) - B_{n+1}^{(l)}(t))/\gamma_{n+1}^{(l-1)}, \quad n \geqslant 0 \tag{6}$$

and

$$B_n^{(l-1)}(t) = (\beta_n^{(l-1)} B_n^{(l)}(t) + \alpha_{n+1}^{(l-1)}\, t\, B_{n-1}^{(l)}(t))/\gamma_n^{(l-1)}, \quad n \geqslant 1. \tag{7}$$

As shown in [11] for $B_n^{(0)}(t)$, the zeros of the polynomials $B_n^{(l)}(t)$ are real, distinct and they lie inside (a, b).

Let $a = \beta^2/b$, $0 < \beta < b \leqslant \infty$ and ω be such that $2\omega \in \mathbb{Z}$. We say that the distribution function ψ belongs to the class $S^3(\omega, \beta, b)$ (or $\psi \in S^3(\omega, \beta, b)$) if ψ is a strong distribution function on $[a, b]$ with the additional property

$$\frac{\mathrm{d}\psi(t)}{t^{\omega}} = -\frac{\mathrm{d}\psi(\beta^2/t)}{(\beta^2/t)^{\omega}}, \quad t \in [a, b]. \tag{8}$$

Clearly Eq. (8) implies that $\mu_m = \beta^{2(m+\omega)}\mu_{-m-2\omega}$. The notation $S^3(\omega, \beta, b)$ was introduced in [4], where some properties of the polynomials $B_n^{(r)}(t)$ associated with $\psi \in S^3(\omega, \beta, b)$ were considered. For example, it was shown that for any $l \in \mathbb{Z}$,

$$\frac{t^n B_n^{(l)}(\beta^2/t)}{B_n^{(l)}(0)} = B_n^{(1-2\omega-l)}(t), \quad n \geqslant 0, \tag{9}$$

$$\beta_n^{(l)}\beta_n^{(1-2\omega-l)} = \beta^2, \quad \frac{\beta_n^{(1-2\omega-l)}}{\beta_{n+1}^{(l)}} = \frac{\alpha_{n+1}^{(1-2\omega-l)}}{\alpha_{n+1}^{(l)}} = \frac{\gamma_n^{(1-2\omega-l)}}{\gamma_{n+1}^{(l-1)}}, \quad n \geqslant 1. \tag{10}$$

In this article we give information on polynomials obtained as linear combinations of the L-orthogonal polynomials $B_n^{(0)}(t)$ when the associated distribution function $\psi \in S^3(\omega, \beta, b)$. The results given in this article provide technics for obtaining examples of strong distributions with explicitly known information on their L-orthogonal polynomials.

2. Linear combinations

Let r be a positive integer. We consider the sequence of real monic polynomials $\{B_n(\lambda_{n,1}, \ldots, \lambda_{n,r}; t)\}$, $n \geqslant r$, defined by

$$B_n(\lambda_{n,1}, \ldots, \lambda_{n,r}; t) = B_n^{(0)}(t) + \sum_{k=1}^{r} \lambda_{n,k} B_{n-k}^{(0)}(t),$$

where $\lambda_{n,1}, \ldots, \lambda_{n,r} \in \mathbb{R}$. Then it follows from the L-orthogonality relation (1) that

$$\int_a^b t^{-n+s} B_n(\lambda_{n,1}, \ldots, \lambda_{n,r}; t)\, \mathrm{d}\psi(t) = 0, \quad r \leqslant s \leqslant n-1. \tag{11}$$

It is known (see [4]) that the polynomials $B_n(\lambda_{n,1}, \ldots, \lambda_{n,r}; t)$ has at least $n-r$ zeros of odd multiplicity inside (a, b).

Lemma 1. *Let r be a positive integer, let $Q_n(t)$ be a real monic polynomial of degree $n > r$ such that*

$$\int_a^b t^{-n+s} Q_n(t)\, \mathrm{d}\psi(t) = 0, \quad r \leqslant s \leqslant n-1.$$

Then there exist r real numbers $\lambda_{n,1}, \ldots, \lambda_{n,r}$ such that $Q_n(t) = B_n(\lambda_{n,1}, \ldots, \lambda_{n,r}; t)$.

Proof. We can write $Q_n(t) = \sum_{j=0}^{n} c_j B_j^{(0)}(t)$ with $c_n = 1$. Hence,

$$\int_a^b t^{-n+s} Q_n(t)\,\mathrm{d}\psi(t) = \sum_{j=0}^{n} c_j \int_a^b t^{-n+s} B_j^{(0)}(t)\,\mathrm{d}\psi(t) = 0, \quad r \leqslant s \leqslant n-1.$$

Setting $s=n-1, n-2, \ldots, r+1, r$, and using the L-orthogonality relation (1), we obtain the homogeneous triangular system of $n-r$ equations in the $n-r$ unknowns $c_0, c_1, \ldots, c_{n-r-1}$

$$\begin{pmatrix} \rho_0^{(0,-1)} & & & \\ \rho_0^{(0,-2)} & \rho_1^{(0,-1)} & & \\ \vdots & \vdots & \ddots & \\ \rho_0^{(0,-n+r)} & \rho_1^{(0,-n+r+1)} & \cdots & \rho_{n-r-1}^{(0,-1)} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-r-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since the diagonal elements do not vanish, the only solution is $c_0 = c_1 = \cdots = c_{n-r-1} = 0$. By setting $c_{n-j} = \lambda_{n,j}$ for $j = 1, 2, \ldots, r$, we then obtain the required result. □

By using the above lemma we can prove the following result.

Theorem 1. *Given a positive integer r, let $\psi \in S^3(\omega, \beta, b)$, where $\omega = (1-r)/2$. If $n \geqslant r$ then for any $\lambda_{n,1}, \ldots, \lambda_{n,r} \in \mathbb{R}$ such that $B_n(\lambda_{n,1}, \ldots, \lambda_{n,r}; 0) \neq 0$, there exist $\eta_{n,1}, \ldots, \eta_{n,r} \in \mathbb{R}$ that satisfy*

$$\frac{t^n B_n(\lambda_{n,1}, \ldots, \lambda_{n,r}; \beta^2/t)}{B_n(\lambda_{n,1}, \ldots, \lambda_{n,r}; 0)} = B_n(\eta_{n,1} \ldots, \eta_{n,r}; t).$$

Proof. For $n = r$ the result is obvious. For $n > r$ setting $t = \beta^2/t$ in (11) and using (8) it follows that

$$\int_a^b t^{n-s-2\omega} B_n(\lambda_{n,1}, \ldots, \lambda_{n,r}; \beta^2/t)\,\mathrm{d}\psi(t) = 0, \quad r \leqslant s \leqslant n-1.$$

Since $r = 1-2\omega$, then $t^{-s-2\omega}$ for $s = r, r+1, \ldots, n-1$ is equivalent to t^{-n+s} for $s = n-1, n-2, \ldots, r$, and we obtain that

$$\int_a^b t^{-n+s} \left(\frac{t^n B_n(\lambda_{n,1}, \ldots, \lambda_{n,r}; \beta^2/t)}{B_n(\lambda_{n,1}, \ldots, \lambda_{n,r}; 0)} \right) \mathrm{d}\psi(t) = 0, \quad r \leqslant s \leqslant n-1.$$

Hence from Lemma 1 the result follows. □

These results can be found in [17] for the special case $r = 1 - 2\omega = 1$.

In this work, based on the result of Theorem 1, we seek for the values of the parameters $\lambda_{n,1}, \ldots, \lambda_{n,r} \in \mathbb{R}$ that satisfy

$$\frac{t^n B_n(\lambda_{n,1}, \ldots, \lambda_{n,r}; \beta^2/t)}{B_n(\lambda_{n,1}, \ldots, \lambda_{n,r}; 0)} = B_n(\lambda_{n,1}, \ldots, \lambda_{n,r}; t).$$

Now we give some definitions which we use throughout this work.

If a monic polynomial $R_n(t)$ of degree n satisfy

$$\frac{t^n R_n(\beta^2/t)}{R_n(0)} = R_n(t), \tag{12}$$

we call it a β-inversive polynomial.

Since $|R_n(0)| = \beta^n$, the zeros of $R_n(t)$ are symmetrically (inversely) positioned about β and/or $-\beta$. Hence, if n is odd then $R_n(t)$ has a zero of odd multiplicity at $t = \beta$ or at $t = -\beta$.

We classify the β-inversive polynomials into two types:

- *β-inversive polynomial of type A*, when $R_n(0) = (-\beta)^n$. This means that $R_n(t)$ cannot have a zero of odd multiplicity at $t = -\beta$,
- *β-inversive polynomial of type B*, when $R_n(0) = -(-\beta)^n$. This means that $R_n(t)$ must have a zero of odd multiplicity at $t = -\beta$.

We note that when 2ω is an odd integer, since all the zeros of $B_n^{(l)}(t)$ are inside (a, b), then from (9) the polynomials $B_n^{((1-2\omega)/2)}(t)$ are β-inversive of type A. In particular, when $\omega = 1/2$ the polynomials $B_n^{(0)}(t)$ are β-inversive of type A.

In [1], Andrade et al. studied the polynomials satisfying (12) for which $R_n(0) = (-\beta)^n$. Based on results found in [1] we can state the following.

Lemma 2. *Let $\{\tilde{R}_m(t)\}$ be a sequence of monic polynomials where $\tilde{R}_m(t)$ is of degree m and β-inversive of type A. Then given any monic polynomial $R_n(t)$, $n \geqslant 1$ with the β-inversive property* (12), *the following holds*:

(1) *If $R_n(t)$ is β-inversive of type A then there exist unique constants $\xi_0, \xi_1, \ldots, \xi_s \in \mathbb{R}$, $0 \leqslant s \leqslant \lfloor n/2 \rfloor$, with $\xi_0 = 1$, such that*

$$R_n(t) = \sum_{k=0}^{s} \xi_k t^k \tilde{R}_{n-2k}(t).$$

(2) *If $R_n(t)$ is β-inversive of type B then there exist unique constants $\xi_0, \xi_1, \ldots, \xi_s \in \mathbb{R}$, $0 \leqslant s \leqslant \lfloor (n-1)/2 \rfloor$, with $\xi_0 = 1$, such that*

$$R_n(t) = (t + \beta) \sum_{k=0}^{s} \xi_k t^k \tilde{R}_{n-1-2k}(t).$$

The proof of part 1 of this lemma can be found in [1]. The proof of part 2 follows from part 1, since $R_n(t)$ is β-inversive of type B implies $(t + \beta)^{-1} R_n(t)$ is β-inversive of type A.

3. Polynomials $B_n(\lambda_{n,1}; t)$ when $\psi \in S^3(0, \beta, b)$

We know that the zeros of the polynomials $B_n(\lambda_{n,1}; t)$ for any strong distribution function ψ, defined on $[a, b]$, are real, distinct and at least $n - 1$ of these zeros are inside (a, b).

According to Theorem 1, it is appropriate to consider the polynomials $B_n(\lambda_{n,1}; t)$ when the associated distribution function ψ is in $S^3(0, \beta, b)$. This was done in Sri Ranga et al. [17] where they have shown that for any $n \geqslant 1$,

$$\text{if} \quad \lambda_{n,1} = \beta_n^{(0)} \pm \sqrt{\beta_n^{(0)} \gamma_n^{(0)}} \quad \text{then} \quad \frac{t^n B_n(\lambda_{n,1}; \beta^2/t)}{B_n(\lambda_{n,1}; 0)} = B_n(\lambda_{n,1}; t).$$

In particular,

- if $\lambda_{n,1}^A = \beta_n^{(0)} - \sqrt{\beta_n^{(0)} \gamma_n^{(0)}}$ then $\frac{t^n B_n(\lambda_{n,1}^A; \beta^2/t)}{(-\beta)^n} = B_n(\lambda_{n,1}^A; t)$, i.e., $B_n(\lambda_{n,1}^A; t)$ is a β-inversive polynomial of type A.
- if $\lambda_{n,1}^B = \beta_n^{(0)} + \sqrt{\beta_n^{(0)} \gamma_n^{(0)}}$ then $\frac{t^n B_n(\lambda_{n,1}^B; \beta^2/t)}{(-\beta)^n} = -B_n(\lambda_{n,1}^B; t)$, i.e., $B_n(\lambda_{n,1}^B; t)$ is a β-inversive polynomial of type B, and its only zero which lies outside $(\beta^2/b, b)$ is equal to $-\beta$.

In [16] Sri Ranga has shown that

$$B_n(\lambda_{n,1}^A; t) = \tilde{B}_n^{(0)}(t) \quad \text{and} \quad B_n(\lambda_{n,1}^B; t) = (t + \beta)\hat{B}_{n-1}^{(0)}(t),$$

where $\tilde{B}_n^{(0)}(t)$ and $\hat{B}_n^{(0)}(t)$ are polynomials defined by (1), respectively, with the distributions

$$\mathrm{d}\tilde{\psi}(t) = \frac{t}{t+\beta}\,\mathrm{d}\psi(t) \quad \text{and} \quad \mathrm{d}\hat{\psi}(t) = (t+\beta)\,\mathrm{d}\psi(t),$$

both of class $S^3(1/2, \beta, b)$. Note that both $\tilde{B}_n^{(0)}(t)$ and $\hat{B}_n^{(0)}(t)$ are β-inversive polynomials of type A.

4. Polynomials $B_n(\lambda_{n,1}, \lambda_{n,2}; t)$ when $\psi \in S^3(-1/2, \beta, b)$

Following Theorem 1 we consider the polynomials $B_n(\lambda_{n,1}, \lambda_{n,2}; t) = B_n^{(0)}(t) + \lambda_{n,1} B_{n-1}^{(0)}(t) + \lambda_{n,2} B_{n-2}^{(0)}(t)$ when the associated distribution function $\psi \in S^3(-1/2, \beta, b)$.

Since $\omega = -1/2$ in (9), we have that

$$\frac{t^n B_n^{(0)}(\beta^2/t)}{B_n^{(0)}(0)} = B_n^{(2)}(t), \qquad \frac{t^n B_n^{(1)}(\beta^2/t)}{B_n^{(1)}(0)} = B_n^{(1)}(t), \qquad n \geqslant 1, \tag{13}$$

and from (10) we obtain $\beta_n^{(1)} = \beta$, $n \geqslant 1$. Hence, $\{B_n^{(1)}(t)\}$ form a sequence of β-inversive polynomials of type A.

Theorem 2. *Let $n \geqslant 2$. Let the parameters $\lambda_{n,1}, \lambda_{n,2} \in \mathbb{R}$ be such that the β-inversive property*

$$\frac{t^n B_n(\lambda_{n,1}, \lambda_{n,2}; \beta^2/t)}{B_n(\lambda_{n,1}, \lambda_{n,2}; 0)} = B_n(\lambda_{n,1}, \lambda_{n,2}; t)$$

holds. Then

(1) *the polynomial* $B_n(\lambda_{n,1}, \lambda_{n,2}; t)$ *is* β*-inversive of type A if* $\lambda_{n,1}$ *and* $\lambda_{n,2}$ *satisfy*

$$(\alpha_{n+1}^{(0)} + \lambda_{n,1})\beta_{n-1}^{(0)} - \lambda_{n,2} = 0,$$

(2) *the polynomial* $B_n(\lambda_{n,1}, \lambda_{n,2}; t)$ *is* β*-inversive of type B if*

$$\lambda_{n,1} = \beta_n^{(0)} + \beta \quad \textit{and} \quad \lambda_{n,2} = -\beta\alpha_n^{(0)}.$$

Proof. The idea for proving this theorem comes from the results of Lemma 2. That is, we expand $B_n(\lambda_{n,1}, \lambda_{n,2}; t)$ in terms of a sequence of polynomials which are β-inversive of type A.

Since $B_n(\lambda_{n,1}, \lambda_{n,2}; t) = B_n^{(0)}(t) + \lambda_{n,1}B_{n-1}^{(0)}(t) + \lambda_{n,2}B_{n-2}^{(0)}(t)$, using (4), (6) and (7) we obtain

$$B_n(\lambda_{n,1}, \lambda_{n,2}; t) = B_n^{(1)}(t) + \frac{\alpha_{n+1}^{(0)} + \lambda_{n,1}}{\gamma_{n-1}^{(0)}}(\beta_{n-1}^{(0)}B_{n-1}^{(1)}(t) + \alpha_n^{(0)}\, tB_{n-2}^{(1)}(t)) + \frac{\lambda_{n,2}}{\gamma_{n-1}^{(0)}}(tB_{n-2}^{(1)}(t) - B_{n-1}^{(1)}(t)).$$

Let $C_n^{(1)}=C_n^{(1)}(\lambda_{n,1}, \lambda_{n,2})=(\alpha_{n+1}^{(0)}+\lambda_{n,1})\beta_{n-1}^{(0)}-\lambda_{n,2}$, and $C_n^{(2)}=C_n^{(2)}(\lambda_{n,1}, \lambda_{n,2})=(\alpha_{n+1}^{(0)}+\lambda_{n,1})\alpha_n^{(0)}+\lambda_{n,2}$. Hence the previous equation can be written as

$$B_n(\lambda_{n,1}, \lambda_{n,2}; t) = B_n^{(1)}(t) + \frac{1}{\gamma_{n-1}^{(0)}}C_n^{(1)}B_{n-1}^{(1)}(t) + \frac{1}{\gamma_{n-1}^{(0)}}C_n^{(2)}tB_{n-2}^{(1)}(t). \tag{14}$$

Hence, if $t^nB_n(\lambda_{n,1}, \lambda_{n,2}; \beta^2/t)/(-\beta)^n = B_n(\lambda_{n,1}, \lambda_{n,2}; t)$, then from (13) and (14) we have

$$B_n(\lambda_{n,1}, \lambda_{n,2}; t) = B_n^{(1)}(t) - \frac{t}{\beta\gamma_{n-1}^{(0)}}C_n^{(1)}B_{n-1}^{(1)}(t) + \frac{1}{\gamma_{n-1}^{(0)}}C_n^{(2)}tB_{n-2}^{(1)}(t).$$

Comparing the two above expressions for $B_n(\lambda_{n,1}, \lambda_{n,2}; t)$ we conclude that $C_n^{(1)} = 0$. This concludes part 1 of the theorem.

To prove part 2 of the theorem let $D_n^{(1)}=D_n^{(1)}(\lambda_{n,1})=\lambda_{n,1}-(\beta_n^{(0)}+\beta)$ and $D_n^{(2)}=D_n^{(2)}(\lambda_{n,2})=\lambda_{n,2}+\beta\alpha_n^{(0)}$. Then we first obtain from (4) and (5) that

$$B_n(\lambda_{n,1}, \lambda_{n,2}; t) = (t+\beta)B_{n-1}^{(1)}(t) + D_n^{(1)}B_{n-1}^{(0)}(t) + D_n^{(2)}B_{n-2}^{(0)}(t).$$

Now applying (6) and (7), this becomes

$$B_n(\lambda_{n,1}, \lambda_{n,2}; t) = (t+\beta)B_{n-1}^{(1)}(t) + \frac{1}{\gamma_{n-1}^{(0)}}(\beta_{n-1}^{(0)}D_n^{(1)} - D_n^{(2)})B_{n-1}^{(1)}(t) + \frac{1}{\gamma_{n-1}^{(0)}}(\alpha_n^{(0)}D_n^{(1)} + D_n^{(2)})tB_{n-2}^{(1)}(t). \tag{15}$$

If $B_n(\lambda_{n,1}, \lambda_{n,2}; t)$ is β-inversive polynomial of type B, then from (13) and (15) we obtain

$$B_n(\lambda_{n,1}, \lambda_{n,2}; t) = (t+\beta)B^{(1)}_{n-1}(t) + \frac{t}{\beta\gamma^{(0)}_{n-1}}(\beta^{(0)}_{n-1}D^{(1)}_n - D^{(2)}_n)B^{(1)}_{n-1}(t) - \frac{1}{\gamma^{(0)}_{n-1}}(\alpha^{(0)}_n D^{(1)}_n + D^{(2)}_n)tB^{(1)}_{n-2}(t).$$

Comparing the two above expressions for $B_n(\lambda_{n,1}, \lambda_{n,2}; t)$ we get

$$(\beta^{(0)}_{n-1}D^{(1)}_n - D^{(2)}_n)(t-\beta)B^{(1)}_{n-1}(t) = 2\beta(\alpha^{(0)}_n D^{(1)}_n + D^{(2)}_n)tB^{(1)}_{n-2}(t),$$

which is possible only if $D^{(1)}_n = D^{(2)}_n = 0$, concluding part 2 of the theorem. □

Remark 1. Theorem 2 gives us that, if $\psi \in S^3(-1/2, \beta, b)$ then the polynomial

$$\begin{aligned} B_n(\lambda^B_{n,1}, \lambda^B_{n,2}; t) &= B^{(0)}_n(t) + \lambda^B_{n,1}B^{(0)}_{n-1}(t) + \lambda^B_{n,2}B^{(0)}_{n-2}(t) \\ &= (t+\beta)B^{(1)}_{n-1}(t) \end{aligned}$$

is β-inversive of type B. Here $\lambda^B_{n,1} = \beta^{(0)}_n + \beta$ and $\lambda^B_{n,2} = -\beta\alpha^{(0)}_n$.

Remark 2. Theorem 2 also gives us that, if $\psi \in S^3(-1/2, \beta, b)$ then the polynomial

$$\begin{aligned} B_n(\lambda^A_{n,1}, \lambda^A_{n,2}; t) &= B^{(0)}_n(t) + \lambda^A_{n,1}B^{(0)}_{n-1}(t) + (\alpha^{(0)}_{n+1} + \lambda^A_{n,1})\beta^{(0)}_{n-1}B^{(0)}_{n-2}(t) \\ &= B^{(1)}_n(t) + (\alpha^{(0)}_{n+1} + \lambda^A_{n,1})tB^{(1)}_{n-2}(t) \end{aligned} \tag{16}$$

is β-inversive of type A, for any $\lambda^A_{n,1} \in \mathbb{R}$ and $\lambda^A_{n,2} = (\alpha^{(0)}_{n+1} + \lambda^A_{n,1})\beta^{(0)}_{n-1}$. The last equality of the above equation comes from (14) by taking $C^{(1)}_n(\lambda_{n,1}, \lambda_{n,2}) = 0$. Hence, for example, if $\bar{\lambda}^A_{n,1} = -\alpha^{(0)}_{n+1}$ and $\bar{\lambda}^A_{n,2} = 0$, then

$$\begin{aligned} B_n(\bar{\lambda}^A_{n,1}, \bar{\lambda}^A_{n,2}; t) &= B^{(0)}_n(t) - \alpha^{(0)}_{n+1}B^{(0)}_{n-1}(t) \\ &= B^{(1)}_n(t) \end{aligned}$$

is β-inversive polynomial of type A. Similarly, for example, if $\hat{\lambda}^A_{n,1} = 0$ and $\hat{\lambda}^A_{n,2} = \alpha^{(0)}_{n+1}\beta^{(0)}_{n-1}$, then

$$\begin{aligned} B_n(\hat{\lambda}^A_{n,1}, \hat{\lambda}^A_{n,2}; t) &= B^{(0)}_n(t) + \alpha^{(0)}_{n+1}\beta^{(0)}_{n-1}B^{(0)}_{n-2}(t) \\ &= B^{(1)}_n(t) + \alpha^{(0)}_{n+1}tB^{(1)}_{n-2}(t) \end{aligned}$$

is β-inversive polynomial of type A.

Another interesting choice for $\lambda^A_{n,1}$ is the one that makes the polynomial in (16) to have a zero at the point $t = -\beta$. Because of the symmetry of the polynomial in (16) and since only two zeros of $B_n(\lambda^A_{n,1}, \lambda^A_{n,2}; t)$ can be out side of the interval (a, b), the zero at $t = -\beta$ must be of multiplicity 2. Letting $t = -\beta$ in (16) we then obtain

$$\lambda^A_{n,1} + \alpha^{(0)}_{n+1} = \frac{B^{(1)}_n(-\beta)}{\beta B^{(1)}_{n-2}(-\beta)}.$$

Corollary 1. *Let* $\tilde{\lambda}^A_{n,1}$, *for* $n \geqslant 2$, *be such that*

$$\tilde{\lambda}^A_{2,1} + \alpha^{(0)}_3 = 4\beta + \alpha^{(1)}_2, \quad \tilde{\lambda}^A_{3,1} + \alpha^{(0)}_4 = 4\beta + \alpha^{(1)}_2 + \alpha^{(1)}_3$$

$$\tilde{\lambda}^A_{n,1} + \alpha^{(0)}_{n+1} = 4\beta + \alpha^{(1)}_{n-1} + \alpha^{(1)}_n - \frac{\alpha^{(1)}_{n-2}\alpha^{(1)}_{n-1}}{\tilde{\lambda}^A_{n-2,1} + \alpha^{(0)}_{n-1}}, \quad n \geqslant 4.$$

Then the monic polynomial $\tilde{B}^{(0)}_{n-2}(t) = (t+\beta)^{-2} B_n(\tilde{\lambda}^A_{n,1}, (\tilde{\lambda}^A_{n,1} + \alpha^{(0)}_{n+1})\beta^{(0)}_{n-1}; t)$, $n \geqslant 2$ *is* β*-inversive of type A and*

$$\int_a^b t^{-n+s} \tilde{B}^{(0)}_n(t)\, \mathrm{d}\phi(t) = 0, \quad 0 \leqslant s \leqslant n-1, \tag{17}$$

where $\phi \in S^3(1/2, \beta, b)$ *is such that* $\mathrm{d}\phi(t) = (t+\beta)^2\, \mathrm{d}\psi(t)$. *Furthermore*

$$\tilde{B}^{(0)}_n(t) = (t-\beta)\tilde{B}^{(0)}_{n-1}(t) - \tilde{\alpha}^{(0)}_n t \tilde{B}^{(0)}_{n-2}(t), \quad n \geqslant 2,$$

with $\tilde{B}^{(0)}_0(t) = 1$ *and* $\tilde{B}^{(0)}_1(t) = t - \beta$, *where*

$$\tilde{\alpha}^{(0)}_n = \beta^{(0)}_{n+2} + \alpha^{(0)}_{n+2} - \beta - (\tilde{\lambda}^A_{n+2,1} - \tilde{\lambda}^A_{n+1,1}), \quad \text{for } n \geqslant 2. \tag{18}$$

Proof. From the three term recurrence relation (2) for $B^{(1)}_n(t)$ we obtain the first result of the corollary. From (11) we know that

$$\int_a^b t^{-n+s} B_n(\tilde{\lambda}^A_{n,1}, (\tilde{\lambda}^A_{n,1} + \alpha^{(0)}_{n+1})\beta^{(0)}_{n-1}; t)\, \mathrm{d}\psi(t) = 0, \quad 2 \leqslant s \leqslant n-1.$$

Since

$$(t+\beta)^2 \tilde{B}^{(0)}_{n-2}(t) = B_n(\tilde{\lambda}^A_{n,1}, (\tilde{\lambda}^A_{n,1} + \alpha^{(0)}_{n+1})\beta^{(0)}_{n-1}; t), \quad n \geqslant 2, \tag{19}$$

we then obtain the L-orthogonality relation (17).

To obtain the last part of the corollary, we first compare the coefficients of t^{n-1} on both sides in (19) and obtain

$$\tilde{b}_{n-2,n-3} - \tilde{b}_{n-3,n-4} = b_{n,n-1} - b_{n-1,n-2} + \tilde{\lambda}^A_{n,1} - \tilde{\lambda}^A_{n-1,1},$$

where $B^{(0)}_n(t) = t^n + b_{n,n-1}t^{n-1} + \cdots + b_{n,0}$ and $\tilde{B}^{(0)}_n(t) = t^n + \tilde{b}_{n,n-1}t^{n-1} + \cdots + \tilde{b}_{n,0}$. On the other hand, from the three term recurrence relations for $B^{(0)}_n(t)$ and $\tilde{B}^{(0)}_n(t)$ we obtain, respectively,

$$b_{n,n-1} - b_{n-1,n-2} = -(\beta^{(0)}_n + \alpha^{(0)}_n) \quad \text{and} \quad \tilde{b}_{n-2,n-3} - \tilde{b}_{n-3,n-4} = -(\beta + \tilde{\alpha}^{(0)}_{n-2}).$$

These results immediately lead to the required result of the corollary. □

Example 1. We consider the strong distribution function ψ given by

$$\mathrm{d}\psi(t) = \frac{1}{t\sqrt{b-t}\sqrt{t-a}}\, \mathrm{d}t,$$

where $0 < \beta < b < \infty$, $\beta = \sqrt{ab}$, which belongs to the class $S^3(-1/2, \beta, b)$.

For this distribution function, from results given in [3] one can determine that

$$\beta_1^{(0)}=\frac{\beta^2}{\beta+2\alpha},\quad \alpha_2^{(0)}=\frac{2\beta\alpha}{\beta+2\alpha},\quad \beta_2^{(0)}=\beta\frac{\beta+2\alpha}{\beta+\alpha},\quad \alpha_3^{(0)}=\alpha\frac{\beta+2\alpha}{\beta+\alpha},$$

$$\beta_n^{(0)}=\beta,\quad \alpha_{n+1}^{(0)}=\alpha,\quad n\geqslant 3,$$

where $\alpha=(\sqrt{b}-\sqrt{a})^2/4$. Also

$$\alpha_2^{(1)}=2\alpha,\quad \beta_n^{(1)}=\beta,\quad \alpha_{n+2}^{(1)}=\alpha,\quad n\geqslant 1.$$

We now use Corollary 1 to obtain information on the polynomials $\tilde{B}_n^{(0)}(t)$ which satisfy the L-orthogonality property

$$\int_a^b t^{-n+s}\tilde{B}_n^{(0)}(t)\,\mathrm{d}\phi(t)=0,\quad 0\leqslant s\leqslant n-1,$$

where $\mathrm{d}\phi(t)=[(t+\beta)^2/(t\sqrt{b-t}\sqrt{t-a})]\,\mathrm{d}t$. We have

$$\tilde{B}_{n-2}^{(0)}(t)=(t+\beta)^{-2}B_n(\tilde{\lambda}_{n,1}^A,(\tilde{\lambda}_{n,1}^A+\alpha_{n+1}^{(0)})\beta_{n-1}^{(0)};t),\quad n\geqslant 2,$$

where $\tilde{\lambda}_{2,1}^A=4\beta+\alpha\beta/(\beta+\alpha)$, $\tilde{\lambda}_{3,1}^A=4\beta+2\alpha$, $\tilde{\lambda}_{4,1}^A+\alpha=4\beta+2\alpha-\alpha^2/(2\beta+\alpha)$,

$$\tilde{\lambda}_{n,1}^A+\alpha=4\beta+2\alpha-\frac{\alpha^2}{\tilde{\lambda}_{n-2,1}^A+\alpha},\quad n\geqslant 5.$$

We can write $\tilde{\lambda}_{n,1}^A$, $n\geqslant 4$, explicitly. Note that for $j=0,1$ and $n\geqslant 2$,

$$\frac{\alpha}{\tilde{\lambda}_{2n+j,1}^A+\alpha}=\frac{1}{2(1+2\beta/\alpha)}\,\frac{1}{-2(1+2\beta/\alpha)}-\cdots-\frac{1}{2(1+2\beta/\alpha)}\,\frac{1}{-2(1+2\beta/\alpha)+\kappa_j},$$

where $\kappa_0=-(1+2\beta/\alpha)$ and $\kappa_1=1$. The expression on the right-hand side is a continued fraction of order n (n terms).

We now use the following known result:

$$\frac{U_{n-1}(x)}{U_n(x)}=\frac{1}{2x}\,\frac{1}{-2x}-\cdots-\frac{1}{2x}\,\frac{1}{-2x},$$

where $U_n(x)=\sin((n+1)\theta)/\sin(\theta)$, with $x=\cos\theta$, is the nth degree Chebyshev polynomial of the second kind and the expression on the right hand side is a continued fraction of order n. From this result and from the properties of the numerators and denominators of continued fractions, we then obtain

$$\tilde{\lambda}_{2n-1,1}^A+\alpha=\alpha\frac{U_{n-1}(1+2\beta/\alpha)+U_{n-2}(1+2\beta/\alpha)}{U_{n-2}(1+2\beta/\alpha)+U_{n-3}(1+2\beta/\alpha)}=\alpha\frac{U_{2n-2}\left(\sqrt{1+\beta/\alpha}\right)}{U_{2n-4}\left(\sqrt{1+\beta/\alpha}\right)},$$

$$\tilde{\lambda}_{2n,1}^A+\alpha=\alpha\frac{U_n(1+2\beta/\alpha)-(1+2\beta/\alpha)U_{n-1}(1+2\beta/\alpha)}{U_{n-1}(1+2\beta/\alpha)-(1+2\beta/\alpha)U_{n-2}(1+2\beta/\alpha)}=\alpha\frac{T_n(1+2\beta/\alpha)}{T_{n-1}(1+2\beta/\alpha)},$$

for $n\geqslant 2$. Here $T_n(x)=\cos(n\theta)$ is the nth degree Chebyshev polynomial of the first kind.

From (18) the coefficients of the recurrence relation for $\tilde{B}_n^{(0)}(t)$ are found to be

$$\tilde{\alpha}_n^{(0)} = \alpha - (\tilde{\lambda}_{n+2,1}^A - \tilde{\lambda}_{n+1,1}^A), \quad n \geqslant 2.$$

Then we can write the coefficients $\tilde{\alpha}_{2n}^{(0)}$ and $\tilde{\alpha}_{2n+1}^{(0)}$, $n \geqslant 1$, explicitly as

$$\begin{aligned}\tilde{\alpha}_{2n}^{(0)} &= \alpha + \frac{2(\alpha+\beta)}{T_n(1+2\beta/\alpha)(U_{n-1}(1+2\beta/\alpha)+U_{n-2}(1+2\beta/\alpha))} \\ &= \alpha + \frac{2(\alpha+\beta)}{T_n(1+2\beta/\alpha)U_{2n-2}\left(\sqrt{1+\beta/\alpha}\right)},\end{aligned}$$

$$\begin{aligned}\tilde{\alpha}_{2n+1}^{(0)} &= \alpha - \frac{2(\alpha+\beta)}{T_n(1+2\beta/\alpha)\,(U_n(1+2\beta/\alpha)+U_{n-1}(1+2\beta/\alpha))} \\ &= \alpha - \frac{2(\alpha+\beta)}{T_n(1+2\beta/\alpha)U_{2n}\left(\sqrt{1+\beta/\alpha}\right)}.\end{aligned}$$

Example 2. The log-normal distribution (see [14]) is given by

$$\mathrm{d}\psi(t) = \frac{\sqrt{q}}{2\kappa\sqrt{\pi}}\,\mathrm{e}^{-(\ln(t)/2\kappa)^2}\,\mathrm{d}t, \quad t \in (0,\infty)$$

with $0<q<1$, $q=\mathrm{e}^{-2\kappa^2}$. This is a very interesting distribution function in the sense that ψ can be classified as any one of the class $S^3(\omega,\beta,\infty)$ by taking $\beta = q^{(\omega-1)}$. The choice $\omega = -\frac{1}{2}$, then means that ψ belongs to the class $S^3(-1/2, q^{-3/2}, \infty)$.

The moments of this distribution function are explicitly given in [14]. Hence, assuming $\psi \in S^3(-1/2, q^{-3/2}, \infty)$, we obtain using the q–d algorithm

$$\beta_n^{(l)} = q^{-(1/2+l)} \quad \alpha_{n+1}^{(l)} = q^{-(1/2+l)}(q^{-n}-1), \quad n \geqslant 1.$$

Application of Corollary 1 gives

$$\tilde{B}_{n-2}^{(0)}(t) = (t+q^{-\frac{3}{2}})^{-2} B_n(\tilde{\lambda}_{n,1}^A, (\tilde{\lambda}_{n,1}^A + \alpha_{n+1}^{(0)})\beta_{n-1}^{(0)}; t), \quad n \geqslant 2,$$

satisfying the L-orthogonality

$$\int_0^\infty t^{-n+s}\,\tilde{B}_n^{(0)}(t)(t+q^{-\frac{3}{2}})^2 \frac{\sqrt{q}}{2\kappa\sqrt{\pi}}\,\mathrm{e}^{-(\ln(t)/2\kappa)^2}\,\mathrm{d}t = 0, \quad 0 \leqslant s \leqslant n-1.$$

The numbers $\tilde{\lambda}_{n,1}^A$ can be generated by

$$\tilde{\lambda}_{2,1}^A + q^{-1/2}(q^{-2}-1) = q^{-3/2}(q^{-1}+3),$$

$$\tilde{\lambda}_{3,1}^A + q^{-1/2}(q^{-3}-1) = q^{-3/2}(q^{-1}+q^{-2}+2)$$

and, for $n \geqslant 4$,

$$\tilde{\lambda}_{n,1}^A + q^{-1/2}(q^{-n}-1) = q^{-3/2}(q^{-(n-3)}+q^{-(n-2)}+2) - \frac{q^{-3}(q^{-(n-3)}-1)(q^{-(n-2)}-1)}{\tilde{\lambda}_{n-2,1}^A + q^{-\frac{1}{2}}(q^{-(n-2)}-1)}.$$

5. Polynomials $B_n(\lambda_{n,1}, \lambda_{n,2}, \lambda_{n,3}; t)$ when $\psi \in S^3(-1, \beta, b)$

The idea presented in the previous sections can be extended for the linear combination $B_n(\lambda_{n,1}, \lambda_{n,2}, \lambda_{n,3}; t) = B_n^{(0)}(t) + \lambda_{n,1} B_{n-1}^{(0)}(t) + \lambda_{n,2} B_{n-2}^{(0)}(t) + \lambda_{n,3} B_{n-3}^{(0)}(t)$, where ψ must be assumed to be a $S^3(-1, \beta, b)$ distribution function. We give in Theorem 3 the main results for this case.

In (9) letting $\omega = -1$, we have

$$\frac{t^n B_n^{(1)}(\beta^2/t)}{B_n^{(1)}(0)} = B_n^{(2)}(t), \quad n \geqslant 1, \tag{20}$$

and from (10), for example, that $\beta_n^{(1)} \beta_n^{(2)} = \beta^2$, $n \geqslant 1$. It is also easy to prove that

$$B_n^{(1)}(0) = (-1)^n \beta_1^{(1)} \beta_2^{(1)} \cdots \beta_n^{(1)} = (-\beta)^n \frac{\sqrt{\beta_n^{(1)}}}{\sqrt{\gamma_n^{(1)}}}. \tag{21}$$

Theorem 3. *Let $n \geqslant 3$. Let the parameters $\lambda_{n,1}, \lambda_{n,2}, \lambda_{n,3} \in \mathbb{R}$ be such that the β-inversive property*

$$\frac{t^n B_n(\lambda_{n,1}, \lambda_{n,2}, \lambda_{n,3}; \beta^2/t)}{B_n(\lambda_{n,1}, \lambda_{n,2}, \lambda_{n,3}; 0)} = B_n(\lambda_{n,1}, \lambda_{n,2}, \lambda_{n,3}; t)$$

holds. Let $\sigma_n^{(1)-} = \beta_n^{(1)} - \sqrt{\beta_n^{(1)} \gamma_n^{(1)}}$ and $\sigma_n^{(1)+} = \beta_n^{(1)} + \sqrt{\beta_n^{(1)} \gamma_n^{(1)}}$. Then for any $\lambda_{n,1} \in \mathbb{R}$,

(1) *the polynomial $B_n(\lambda_{n,1}, \lambda_{n,2}, \lambda_{n,3}; t)$ is β-inversive of type A if*

$$\begin{aligned} \lambda_{n,2} &= \lambda_{n,1}(\beta_{n-1}^{(0)} + \sigma_{n-2}^{(1)-}) + \alpha_{n+1}^{(0)}(\beta_{n-1}^{(0)} + \sigma_{n-2}^{(1)-}) - (\gamma_{n-1}^{(0)} + \sigma_{n-2}^{(1)-})\sigma_n^{(1)-}, \\ \lambda_{n,3} &= \lambda_{n,1}\beta_{n-2}^{(0)}\sigma_{n-2}^{(1)-} + \beta_{n-2}^{(0)}\sigma_{n-2}^{(1)-}(\alpha_{n+1}^{(0)} - \sigma_n^{(1)-}), \end{aligned} \tag{22}$$

(2) *the polynomial $B_n(\lambda_{n,1}, \lambda_{n,2}, \lambda_{n,3}; t)$ is β-inversive of type B if*

$$\begin{aligned} \lambda_{n,2} &= \lambda_{n,1}(\beta_{n-1}^{(0)} + \sigma_{n-2}^{(1)+}) + \alpha_{n+1}^{(0)}(\beta_{n-1}^{(0)} + \sigma_{n-2}^{(1)+}) - (\gamma_{n-1}^{(0)} + \sigma_{n-2}^{(1)+})\sigma_n^{(1)+}, \\ \lambda_{n,3} &= \lambda_{n,1}\beta_{n-2}^{(0)}\sigma_{n-2}^{(1)+} + \beta_{n-2}^{(0)}\sigma_{n-2}^{(1)+}(\alpha_{n+1}^{(0)} - \sigma_n^{(1)+}). \end{aligned} \tag{23}$$

Proof. By using relations (4), (6) and (7) we obtain

$$\begin{aligned} B_n(\lambda_{n,1}, \lambda_{n,2}, \lambda_{n,3}; t) = B_n^{(1)}(t) &+ \frac{\alpha_{n+1}^{(0)} + \lambda_{n,1}}{\gamma_{n-1}^{(0)}}(\beta_{n-1}^{(0)} B_{n-1}^{(1)}(t) + \alpha_n^{(0)}\, t B_{n-2}^{(1)}(t)) \\ &+ \frac{\lambda_{n,2}}{\gamma_{n-1}^{(0)}}(t B_{n-2}^{(1)}(t) - B_{n-1}^{(1)}(t)) + \frac{\lambda_{n,3}}{\gamma_{n-2}^{(0)}}(t B_{n-3}^{(1)}(t) - B_{n-2}^{(1)}(t)). \end{aligned}$$

This can be written as

$$B_n(\lambda_{n,1},\lambda_{n,2},\lambda_{n,3};t)=B_n^{(1)}(t)+\frac{1}{\gamma_{n-1}^{(0)}}((\alpha_{n+1}^{(0)}+\lambda_{n,1})\beta_{n-1}^{(0)}-\lambda_{n,2})B_{n-1}^{(1)}(t)$$
$$+\frac{1}{\gamma_{n-1}^{(0)}}((\alpha_{n+1}^{(0)}+\lambda_{n,1})\alpha_n^{(0)}+\lambda_{n,2})tB_{n-2}^{(1)}(t)$$
$$+\frac{\lambda_{n,3}}{\gamma_{n-2}^{(0)}}(tB_{n-3}^{(1)}(t)+B_{n-2}^{(1)}(t)).$$

Now, by using (2) and (3) we can write

$$B_n(\lambda_{n,1},\lambda_{n,2},\lambda_{n,3};t)=B_n^{(1)}(t)+F_nB_{n-1}^{(1)}(t)+G_ntB_{n-2}^{(1)}(t)+H_ntB_{n-3}^{(1)}(t), \tag{24}$$

where

$$\begin{aligned}F_n&=F_n(\lambda_{n,1},\lambda_{n,2},\lambda_{n,3})=\frac{1}{\gamma_{n-1}^{(0)}}\left((\alpha_{n+1}^{(0)}+\lambda_{n,1})\beta_{n-1}^{(0)}-\lambda_{n,2}+\frac{\lambda_{n,3}}{\beta_{n-2}^{(0)}}\right),\\ G_n&=G_n(\lambda_{n,1},\lambda_{n,2},\lambda_{n,3})=\frac{1}{\gamma_{n-1}^{(0)}}\left((\alpha_{n+1}^{(0)}+\lambda_{n,1})\alpha_n^{(0)}+\lambda_{n,2}-\frac{\lambda_{n,3}}{\beta_{n-2}^{(0)}}\right),\\ H_n&=H_n(\lambda_{n,3})=\frac{\lambda_{n,3}}{\beta_{n-2}^{(0)}}.\end{aligned} \tag{25}$$

To prove part 1 of the theorem we use (20) and (21) and we get

$$\frac{t^nB_n(\lambda_{n,1},\lambda_{n,2},\lambda_{n,3};\beta^2/t)}{(-\beta)^n}=\frac{\beta_n^{(1)}}{\sqrt{\beta_n^{(1)}\gamma_n^{(1)}}}B_n^{(2)}(t)-\frac{F_n}{\sqrt{\beta_n^{(1)}\gamma_n^{(1)}}}tB_{n-1}^{(2)}(t)$$
$$+\frac{\beta_{n-2}^{(1)}G_n}{\sqrt{\beta_{n-2}^{(1)}\gamma_{n-2}^{(1)}}}tB_{n-2}^{(2)}(t)-\frac{H_n}{\sqrt{\beta_{n-2}^{(1)}\gamma_{n-2}^{(1)}}}t^2B_{n-3}^{(2)}(t).$$

Since $t^nB_n(\lambda_{n,1},\lambda_{n,2},\lambda_{n,3};\beta^2/t)/(-\beta)^n=B_n(\lambda_{n,1},\lambda_{n,2},\lambda_{n,3};t)$, by using (4) and (5) we have

$$B_n(\lambda_{n,1},\lambda_{n,2},\lambda_{n,3};t)=\frac{\beta_n^{(1)}}{\sqrt{\beta_n^{(1)}\gamma_n^{(1)}}}(B_n^{(1)}(t)-\alpha_{n+1}^{(1)}B_{n-1}^{(1)}(t))$$
$$-\frac{F_n}{\sqrt{\beta_n^{(1)}\gamma_n^{(1)}}}(B_n^{(1)}(t)+\beta_n^{(1)}B_{n-1}^{(1)}(t))$$
$$+\frac{\beta_{n-2}^{(1)}G_n}{\sqrt{\beta_{n-2}^{(1)}\gamma_{n-2}^{(1)}}}t(B_{n-2}^{(1)}(t)-\alpha_{n-1}^{(1)}B_{n-3}^{(1)}(t))$$
$$-\frac{H_n}{\sqrt{\beta_{n-2}^{(1)}\gamma_{n-2}^{(1)}}}t(B_{n-2}^{(1)}(t)+\beta_{n-2}^{(1)}B_{n-3}^{(1)}(t)).$$

Equivalently,

$$B_n(\lambda_{n,1},\lambda_{n,2},\lambda_{n,3};t)=\frac{\beta_n^{(1)}-F_n}{\sqrt{\beta_n^{(1)}\gamma_n^{(1)}}}B_n^{(1)}(t)-\frac{\beta_n^{(1)}(\alpha_{n+1}^{(1)}+F_n)}{\sqrt{\beta_n^{(1)}\gamma_n^{(1)}}}B_{n-1}^{(1)}(t)+\frac{\beta_{n-2}^{(1)}G_n-H_n}{\sqrt{\beta_{n-2}^{(1)}\gamma_{n-2}^{(1)}}}tB_{n-2}^{(1)}(t)-\frac{\beta_{n-2}^{(1)}(H_n+\alpha_{n-1}^{(1)}G_n)}{\sqrt{\beta_{n-2}^{(1)}\gamma_{n-2}^{(1)}}}tB_{n-3}^{(1)}(t). \quad (26)$$

We now compare the two expressions (24) and (26) for $B_n(\lambda_{n,1},\lambda_{n,2},\lambda_{n,3};t)$ to get information about F_n, G_n and H_n. Since $B_n(\lambda_{n,1},\lambda_{n,2},\lambda_{n,3};t)$ is monic we must have

$$F_n=\beta_n^{(1)}-\sqrt{\beta_n^{(1)}\gamma_n^{(1)}}. \quad (27)$$

Since $\gamma_n^{(1)}=\alpha_{n+1}^{(1)}+\beta_n^{(1)}$ Eq. (27) implies $F_n=-\beta_n^{(1)}(\alpha_{n+1}^{(1)}+F_n)/\sqrt{\beta_n^{(1)}\gamma_n^{(1)}}$. Hence, subtracting (26) from (24) we get

$$\left(G_n-\frac{\beta_{n-2}^{(1)}G_n-H_n}{\sqrt{\beta_{n-2}^{(1)}\gamma_{n-2}^{(1)}}}\right)B_{n-2}^{(1)}(t)+\left(H_n+\frac{\beta_{n-2}^{(1)}(H_n+\alpha_{n-1}^{(1)}G_n)}{\sqrt{\beta_{n-2}^{(1)}\gamma_{n-2}^{(1)}}}\right)B_{n-3}^{(1)}(t)=0.$$

Considering the coefficient of t^{n-2}, we conclude that

$$H_n=G_n\left(\beta_{n-2}^{(1)}-\sqrt{\beta_{n-2}^{(1)}\gamma_{n-2}^{(1)}}\right). \quad (28)$$

Hence results (22) of the theorem follows from (25) and conditions (27) and (28).

Results (23) of the theorem can be obtained in a similar way using the relation $t^nB_n(\lambda_{n,1},\lambda_{n,2},\lambda_{n,3};\beta^2/t)/(-\beta)^n=-B_n(\lambda_{n,1},\lambda_{n,2},\lambda_{n,3};t)$. □

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