

Low-frequency absorption cross section of the electromagnetic waves for extreme Reissner-Nordström black holes in higher dimensions

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We investigate the low-frequency absorption cross section of the electromagnetic waves for extreme Reissner-Nordström black holes in higher dimensions. We first construct the exact solutions to the relevant wave equations in the zero-frequency limit. In most cases it is possible to use these solutions to find the transmission coefficients of partial waves in the low-frequency limit. We use these transmission coefficients to calculate the low-frequency absorption cross section in five and six spacetime dimensions. We find that this cross section is dominated by the modes with $\ell = 2$ in the spherical-harmonic expansion rather than those with $\ell = 1$, as might have been expected, because of the mixing between the electromagnetic and gravitational waves. We also find an upper limit for the low-frequency absorption cross section in dimensions higher than six.

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I. INTRODUCTION

Motivated by proposals of gravity theories in the context of extra dimensions, the absorption cross section of spherically symmetric black holes in higher dimensions has been investigated for different fields. The scalar case has been analyzed in Refs. [1,2], the fermionic case in Refs. [1–3], and the gravitational case in Ref. [4]. The massless scalar case has also been investigated by one of the present authors for stationary black holes [5], whereas the massive scalar case has been studied for Reissner-Nordström black holes in Ref. [6]. Investigations for the scalar field in charged black hole spacetimes can also be found in Ref. [7].

Sometime ago, the present authors studied the behavior of the absorption cross section of the massless vector field for extreme Reissner-Nordström black holes in arbitrary dimensions [8], generalizing a result of Gubser in four dimensions [9]. However, the massless vector field equation in the Reissner-Nordström background does not describe the electromagnetic field because of the mixing between the electromagnetic and gravitational waves [10–14]. In Ref. [15], which followed up the work in Ref. [16], the absorption cross section of electromagnetic waves for Reissner-Nordström black holes in four dimensions was studied, and for the extreme case it was found that some modes with $\ell = 2$ in the spherical-harmonic

expansion contributed at the same order of the frequency ω as the modes with $\ell = 1$ because of the mixing between the electromagnetic and gravitational waves.

In this paper, we study the low-frequency behavior of the absorption cross section of the electromagnetic waves for the higher-dimensional extreme Reissner-Nordström black holes. In particular, we determine the low-frequency behavior of the cross section in five and six spacetime dimensions and find that it is dominated by some modes with $\ell = 2$ rather than those with $\ell = 1$. The rest of this paper is organized as follows. In Sec. II, we present the exact solutions to the equations describing the system of electromagnetic and gravitational waves found by Kodama and Ishibashi [17] in the zero-frequency limit. In Sec. III, we discuss the low-frequency behavior of the transmission coefficients of partial waves. In Sec. IV, we find the low-frequency absorption cross section in dimensions five and six and its upper bound in dimensions higher than six. In Sec. V, we summarize our results and make some concluding remarks.

II. EXACT SOLUTIONS TO THE ZERO-FREQUENCY EQUATIONS

As in four dimensions, the modes for the electromagnetic waves and the part of the gravitational waves that mix with the electromagnetic waves have angular dependence given either by vector or scalar spherical harmonics. Following Chandrasekhar [18], we call the modes described by vector spherical harmonics the axial modes and those described by the scalar spherical harmonics the

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polar modes. (In Ref. [17] they are called the vector-type and scalar-type perturbations, respectively. They are also called the even- and odd-parity perturbations, respectively.) The line element of the $p + 2$ dimensional extreme Reissner-Nordström black hole (with $p \geq 2$) is given by [19]

$$ds^2 = -f(r)dt^2 + [f(r)]^{-1}dr^2 + r^2d\Omega^2, \quad (1)$$

where $d\Omega^2$ is the line element of the unit p -dimensional sphere, and where

$$f(r) = \left[1 - \left(\frac{r_H}{r}\right)^{p-1}\right]^2. \quad (2)$$

Let us first present the radial wave equations for the modes describing the electromagnetic and gravitational perturbations in this spacetime, which are special cases of the general equations found in Ref. [17]. The radial functions $\Phi_{a\pm}(r)$ describing the axial perturbations satisfy

$$\left[f(r)\frac{d}{dr}f(r)\frac{d}{dr} + \omega^2 - \frac{f(r)}{r^2}V_{a\pm}(r)\right]\Phi_{a\pm}(r) = 0, \quad (3)$$

where

$$V_{a\pm} = \left(\ell + \frac{p}{2} - 1\right)\left(\ell + \frac{p}{2}\right) + \frac{p(5p-2)}{4}\left(\frac{r_H}{r}\right)^{2p-2} + \left(-\frac{p^2+2}{2} \pm \Delta\right)\left(\frac{r_H}{r}\right)^{p-1}, \quad (4)$$

$$\Delta \equiv \sqrt{(p^2-1)^2 + 2p(p-1)(\ell-1)(\ell+p)}. \quad (5)$$

We have $\ell \geq 1$ for Φ_{a+} whereas $\ell \geq 2$ for Φ_{a-} . These functions are given in terms of the radial functions Φ_{ae} and Φ_{ag} which describe the electromagnetic and gravitational axial perturbations, respectively, by

$$\Phi_{a+} = \cos\psi_\ell^{(a)}\Phi_{ae} + \sin\psi_\ell^{(a)}\Phi_{ag}, \quad (6)$$

$$\Phi_{a-} = -\sin\psi_\ell^{(a)}\Phi_{ae} + \cos\psi_\ell^{(a)}\Phi_{ag}, \quad (7)$$

where (with $|\psi_\ell^{(a)}| < \pi/4$)

$$\sin 2\psi_\ell^{(a)} = \sqrt{\frac{2p(\ell-1)(\ell+p)}{2p(\ell-1)(\ell+p) + (p-1)(p+1)^2}}. \quad (8)$$

The radial functions $\Phi_{p\pm}(r)$ describing the polar perturbations satisfy

$$\left[f(r)\frac{d}{dr}f(r)\frac{d}{dr} + \omega^2 - \frac{f(r)}{r^2}V_{p\pm}\right]\Phi_{p\pm}(r) = 0, \quad (9)$$

where, with the definition

$$\xi \equiv (r_H/r)^{p-1}, \quad (10)$$

we have

$$V_{p+} = \frac{1}{4(p\xi - \ell - p)^2} \{(\ell + p)^2(2\ell + p - 2)(2\ell + p) - 2(\ell + p)(4\ell^2 + 3p^2\ell - 4p\ell - 2p^2)\xi + [(p^2 - 6p + 8)\ell^2 - 6p^3\ell - 6p^4]\xi^2 + [(-4p^2 + 6p^3)\ell + 8p^4 - 4p^3]\xi^3 + (2p^3 - 3p^4)\xi^4\}. \quad (11)$$

Interestingly, the potential V_{p-} is obtained from V_{p+} by letting $\ell \rightarrow -\ell - p + 1$. We have $\ell \geq 1$ for Φ_{p+} and $\ell \geq 2$ for Φ_{p-} . These functions are given in terms of the radial functions Φ_{pe} and Φ_{pg} which describe the electromagnetic and gravitational polar perturbations, respectively, by

$$\Phi_{p+} = \cos\psi_\ell^{(p)}\Phi_{pe} + \sin\psi_\ell^{(p)}\Phi_{pg}, \quad (12)$$

$$\Phi_{p-} = -\sin\psi_\ell^{(p)}\Phi_{pe} + \cos\psi_\ell^{(p)}\Phi_{pg}, \quad (13)$$

where (with $|\psi_\ell^{(p)}| < \pi/4$)

$$\sin 2\psi_\ell^{(p)} = \sqrt{\frac{4(\ell-1)(\ell+p)}{4(\ell-1)(\ell+p) + (p+1)^2}}. \quad (14)$$

A more useful form is

$$\sin^2\psi_\ell^{(p)} = \frac{\ell-1}{2\ell+p-1}. \quad (15)$$

In all cases the zero-frequency radial equation with $\omega = 0$ can be written in terms of the variable ξ defined by Eq. (10) as

$$\left[\xi^2(1-\xi)^2\frac{d^2}{d\xi^2} + \xi(1-\xi)\left(1-3\xi + \frac{1-\xi}{p-1}\right)\frac{d}{d\xi} - \frac{V(\xi)}{(p-1)^2}\right]\Phi = 0, \quad (16)$$

where $V(\xi)$ is $V_{a\pm}$ or $V_{p\pm}$, and Φ is $\Phi_{a\pm}$ or $\Phi_{p\pm}$.

For the axial case Eq. (16) can be transformed to the standard hypergeometric equation by multiplying Φ by an appropriate factor. We find in this manner the following solutions:

$$\Phi_{a\pm}^{(1)} = \xi^{-(2\ell+p)/[2(p-1)]}(1-\xi)^{\lambda_\pm^{(1)}}F\left(\lambda_\pm^{(1)} - \frac{\ell+p}{p-1}, \lambda_\pm^{(1)} - \frac{\ell-2p+1}{p-1}; 2(\lambda_\pm^{(1)}+1); 1-\xi\right), \quad (17)$$

where

$$\lambda_{\pm}^{(1)} = -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{(p-1)^2}(p^2 - 1 + (\ell-1)(\ell+p) \pm \Delta)}. \quad (18)$$

Since $p^2 - 1 + (\ell-1)(\ell+p) \pm \Delta$ are increasing functions of ℓ for $\ell \geq 1$ and are non-negative for $\ell = 1$, the constants $\lambda_{\pm}^{(1)}$ are both real. The other independent solutions can be chosen as

$$\Phi_{a\pm}^{(2)} = \xi^{-(2\ell+p)/[2(p-1)]}(1-\xi)^{\lambda_{\pm}^{(2)}} F\left(\lambda_{\pm}^{(2)} - \frac{\ell+p}{p-1}, \lambda_{\pm}^{(2)} - \frac{\ell-2p+1}{p-1}; 2(\lambda_{\pm}^{(2)} + 1); 1-\xi\right), \quad (19)$$

where

$$\lambda_{\pm}^{(2)} = -\frac{1}{2} - \sqrt{\frac{1}{4} + \frac{1}{(p-1)^2}(p^2 - 1 + (\ell-1)(\ell+p) \pm \Delta)}, \quad (20)$$

if $2\lambda_{\pm}^{(2)}$ is not an integer. If it is an integer, the solutions $\Phi_{a\pm}^{(2)}$ are not valid, but valid solutions can be generated by the standard method, and they behave like $(1-\xi)^{\lambda_{\pm}^{(2)}}$ for $|1-\xi| \ll 1$.

Now, let us examine the radial equations for the polar perturbations. Remarkably, the following functions are solutions to Eq. (16) with $V(\xi) = V_{p+}(\xi)$:

$$\Phi_{p+}^{(1)} = \xi^{-(2\ell+p)/[2(p-1)]}(1-\xi)^{(\ell+p-1)/(p-1)} \frac{1}{\ell+p-p\xi}, \quad (21)$$

$$\Phi_{p+}^{(2)} = \xi^{(2\ell+p-2)/[2(p-1)]}(1-\xi)^{-(\ell+2p-2)/(p-1)} \times \left(\frac{1}{\ell+p-p\xi} + a_{\ell+} + b_{\ell+\xi} \right), \quad (22)$$

where

$$a_{\ell+} = \frac{3p}{\ell^2} - \frac{(p-2)(p+1)}{(\ell+p)(p-1)^2} + \frac{2p+3}{\ell(p-1)}, \quad (23)$$

$$b_{\ell+} = -\frac{3p}{\ell^2} - \frac{p(p+1)}{(\ell+p)^2(p-1)^2} + \frac{2p+3}{(\ell+p)(p-1)} - \frac{2p+3}{\ell(p-1)}. \quad (24)$$

These solutions are linearly independent for all $p \geq 2$ and $\ell \geq 1$. The solutions with $V(\xi) = V_{p-}(\xi)$ can be

obtained from these by replacing ℓ by $-\ell - p + 1$ unless $2\ell = p - 1$:

$$\Phi_{p-}^{(1)} = \xi^{(2\ell+p-2)/[2(p-1)]}(1-\xi)^{-\ell/(p-1)} \frac{1}{p\xi + \ell - 1}, \quad (25)$$

$$\Phi_{p-}^{(2)} = \xi^{-(2\ell+p)/[2(p-1)]}(1-\xi)^{(\ell-p+1)/(p-1)} \times \left(\frac{1}{p\xi + \ell - 1} + a_{\ell-} + b_{\ell-\xi} \right), \quad (26)$$

where

$$a_{\ell-} = -\frac{3p}{(\ell+p-1)^2} - \frac{(p-2)(p+1)}{(\ell-1)(p-1)^2} + \frac{2p+3}{(\ell+p-1)(p-1)}, \quad (27)$$

$$b_{\ell-} = \frac{3p}{(\ell+p-1)^2} + \frac{p(p+1)}{(\ell-1)^2(p-1)^2} + \frac{2p+3}{(\ell-1)(p-1)} - \frac{2p+3}{(\ell+p-1)(p-1)}. \quad (28)$$

If $2\ell = p - 1$, $\Phi_{p-}^{(2)}$ is proportional to $\Phi_{p-}^{(1)}$. In this case, an independent second solution is given by

$$\Phi_{p-}^{(2')} = \xi^{-(2p-1)/[2(p-1)]}(1-\xi)^{-1/2} \frac{1}{2p\xi + p - 3} \times \left[-\frac{1}{2}(p-3)^2 - (5p-3)(p-3)\xi + 9(p-1)^2\xi^2 \log \frac{\xi}{1-\xi} \right]. \quad (29)$$

III. TRANSMISSION COEFFICIENTS FOR LOW FREQUENCIES

The wave equations (3) and (9) take the form

$$\left[\frac{d^2}{dr_*^2} + \omega^2 - \frac{f(r)}{r^2} V(r) \right] \Phi = 0, \quad (30)$$

where the Regge-Wheeler tortoise coordinate r_* is defined by

$$\frac{dr_*}{dr} = \left[1 - \left(\frac{r_H}{r} \right)^{p-1} \right]^{-2}. \quad (31)$$

We note that

$$r_* \sim r \quad \text{if } r \gg r_H, \quad (32)$$

$$r_* \sim -\frac{r_H^2}{(p-1)^2(r-r_H)} \quad \text{if } r - r_H \ll r_H. \quad (33)$$

In all cases, we have

$$f(r)V(r) \approx \left(\ell + \frac{p}{2} - 1\right)\left(\ell + \frac{p}{2}\right), \quad r \gg r_H. \quad (34)$$

Hence the large r behavior of the solutions relevant to the absorption process is given by

$$\Phi \approx \Phi_{r \gg r_H} = \sqrt{\frac{\pi \omega r}{2}} \left[H_{\ell+(p-1)/2}^{(2)}(\omega r) + (1 - b_\ell(\omega)) H_{\ell+(p-1)/2}^{(1)}(\omega r) \right], \quad (35)$$

where $b_\ell(\omega)$ is some smooth complex function. Near the horizon, i.e. for r_* large and negative, Eq. (30) can be approximated by

$$\left[\frac{d^2}{dr_*^2} + \omega^2 - \frac{\nu_\ell^2 - \frac{1}{4}}{r_*^2} \right] \Phi \approx 0, \quad (36)$$

where

$$\nu_\ell = \sqrt{\frac{V(r_H)}{(p-1)^2} + \frac{1}{4}}. \quad (37)$$

Hence, near the horizon we have

$$\Phi \approx \Phi_{r \approx r_H} = D_\ell(\omega) \sqrt{\frac{-\pi \omega r_*}{2}} H_{\nu_\ell}^{(1)}(-\omega r_*). \quad (38)$$

Note that $\Phi_{r \gg r_H} \approx e^{-i\omega r + i\gamma_\ell} + (1 - b_\ell(\omega))e^{i\omega r - i\gamma_\ell}$, where $\gamma_\ell = \frac{\pi}{2}(\ell + \frac{p-2}{2})$ and $\Phi_{r \approx r_H} \approx D_\ell(\omega)e^{-i\omega r_*}$ up to a phase factor for $|\omega r_*| \gg 1$. Thus, the transmission coefficient is given by

$$\mathcal{P}_\ell \equiv |D_\ell(\omega)|^2. \quad (39)$$

We also have by energy conservation

$$|D_\ell(\omega)|^2 = 2 \operatorname{Re}[b_\ell(\omega)] - |b_\ell(\omega)|^2. \quad (40)$$

The low-frequency behavior of the transmission coefficients can be found as follows. Noting that $1 - \xi \approx (p-1)(r - r_H)/r_H$ for $r \approx r_H$, we find from Eq. (38), that for small ω (with $|\omega r_*| \ll 1$), the solutions behave near the horizon (with ξ fixed) as follows:

$$\Phi_{r \approx r_H} \approx -iD_\ell(\omega) \frac{\Gamma(\nu_\ell)}{\sqrt{\pi}} \left[\frac{2(p-1)}{\omega r_H} \right]^{\nu_\ell - 1/2} (1 - \xi)^{\nu_\ell - 1/2} \quad \text{if } \nu_\ell > 0, \quad (41)$$

and

$$\Phi_{r \approx r_H} \approx iD_\ell(\omega) \left[\sqrt{\frac{2\omega r_H}{\pi(p-1)}} \log \omega r_H \right] (1 - \xi)^{-1/2} \quad \text{if } \nu_\ell = 0, \quad (42)$$

where we have used the fact that if $-\omega r_* \ll 1$ and $r_* \approx -r_H/[(p-1)(1-\xi)]$, then $\omega r_H \ll 1 - \xi$. We note that a second independent solution of Eq. (36) behaves like $(1 - \xi)^{-\nu_\ell - 1/2}$ if $\nu_\ell > 0$, and $(1 - \xi)^{-1/2} \log(1 - \xi)$ if $\nu_\ell = 0$ near the horizon. That is, a second independent solution diverges faster as $\xi \rightarrow 1$. Thus, the $\omega = 0$ solution that diverges more slowly as $\xi \rightarrow 1$ corresponds to the $\omega \rightarrow 0$ limit of the solution relevant to the absorption process, and it behaves like $(1 - \xi)^{\nu_\ell - 1/2}$, where ν_ℓ is given by Eq. (37). Therefore, the relevant solutions are $\Phi_{a\pm}^{(1)}$ in Eq. (17), $\Phi_{p+}^{(1)}$ in Eq. (21), $\Phi_{p-}^{(2)}$ in Eq. (26) if $2\ell > p-1$, and $\Phi_{p-}^{(1)}$ in Eq. (25) if $2\ell \leq p-1$.

To find the behavior of the solutions for $r \gg r_H$ (i.e. for $\xi \ll 1$) in the limit $\omega \rightarrow 0$, we first note that Eq. (35) can be written as

$$\Phi_{r \gg r_H} = \sqrt{2\pi\omega r} \left[\left(1 - \frac{b_\ell(\omega)}{2}\right) J_{\ell+(p-1)/2}(\omega r) - \frac{ib_\ell(\omega)}{2} N_{\ell+(p-1)/2}(\omega r) \right]. \quad (43)$$

We assume that the limit of $\omega^{-2\ell-p+1}|b_\ell(\omega)|$ as $\omega \rightarrow 0$ either exists or is infinite. For small ω and for small ξ , we have

$$\Phi_{r \gg r_H} \approx \frac{\sqrt{4\pi}}{\Gamma(\ell + \frac{p+1}{2})} \left(\frac{\omega r_H}{2} \right)^{\ell+p/2} \xi^{-(2\ell+p)/[2(p-1)]} \quad \text{if } \lim_{\omega \rightarrow 0} \omega^{-2\ell-p+1}|b_\ell(\omega)| < \infty. \quad (44)$$

It can be seen from Eqs. (17), (21), (25), and (26), that all partial waves except for the “−” polar modes with $2\ell \leq p-1$ fall into this category. In each of these cases, there is a solution $\Phi_\ell(\xi)$ to the $\omega = 0$ equation such that

$$\Phi_\ell(\xi) \approx \begin{cases} (1 - \xi)^{\nu_\ell - 1/2} & \text{for } 1 - \xi \ll 1 \\ \alpha_\ell \xi^{-(2\ell+p)/[2(p-1)]} & \text{for } \xi \ll 1, \end{cases} \quad (45)$$

where α_ℓ is a nonzero constant. By comparing these equations with Eqs. (41), (42), and (44) we find the low-frequency transmission coefficients as

$$\mathcal{P}_\ell = |D_\ell(\omega)|^2 = \frac{4\pi^2}{|\alpha_\ell|^2 (p-1)^{2\nu_\ell - 1} |\Gamma(\nu_\ell) \Gamma(\ell + \frac{p+1}{2})|^2} \times \left(\frac{\omega r_H}{2} \right)^{2(\nu_\ell + \ell) + p - 1} \quad \text{if } \nu_\ell > 0. \quad (46)$$

We do not need the case $\nu_\ell = 0$ here.

The values of ν_ℓ and α_ℓ for the axial modes can be found from Eq. (17) as

$$\nu_\ell^{(a\pm)} = \sqrt{\frac{1}{4} + \frac{1}{(p-1)^2} (p^2 - 1 + (\ell-1)(\ell+p) \pm \Delta)}, \quad (47)$$

$$\alpha_\ell^{(a\pm)} = \frac{\Gamma(2(\lambda_\pm^{(1)} + 1))\Gamma(1 + \frac{2\ell}{p-1})}{\Gamma(\lambda_\pm^{(1)} + \frac{\ell+3p-2}{p-1})\Gamma(\lambda_\pm^{(1)} + \frac{\ell-1}{p-1})}. \quad (48)$$

Equation (47) can also be obtained from Eq. (37). The $\nu_\ell^{(a\pm)}$ are increasing as functions of ℓ because $(\ell - 1)(\ell + p) \pm \Delta$ are increasing. For the “+” polar modes, we find the values of ν_ℓ and α_ℓ from Eq. (21) as

$$\nu_\ell^{(p+)} = \frac{\ell}{p-1} + \frac{3}{2}, \quad (49)$$

$$\alpha_\ell^{(p+)} = \frac{\ell}{\ell + p}. \quad (50)$$

For the “−” polar modes with $2\ell > p - 1$, we find from Eq. (26)

$$\nu_\ell^{(p-)} = \frac{\ell}{p-1} - \frac{1}{2}, \quad (51)$$

$$\alpha_\ell^{(p-)} = \frac{(\ell - 1)(2\ell - p + 1)}{(\ell + p - 1)(2\ell + p - 1)}. \quad (52)$$

For the “−” polar modes with $2\ell \leq p - 1$ the $\omega = 0$ solutions relevant to the absorption process is given by Eq. (25). We find that they behave like $\xi^{(2\ell+p-2)/[2(p-1)]}$ rather than $\xi^{-(2\ell+p)/[2(p-1)]}$ for small ξ . This apparent difficulty is resolved by noting that

$$\begin{aligned} \Phi_{r \gg r_H} &\approx i \frac{b_\ell(\omega)}{\sqrt{\pi}} \left(\frac{2}{\omega r_H} \right)^{\ell+p/2-1} \\ &\times \Gamma\left(\ell + \frac{p-1}{2}\right) \xi^{(2\ell+p-2)/[2(p-1)]} \\ &\text{if } \lim_{\omega \rightarrow 0} \omega^{-2\ell-p+1} |b_\ell(\omega)| = \infty. \end{aligned} \quad (53)$$

Thus, the partial waves for the “−” polar modes with $2\ell \leq p - 1$ fall into this category. In each of these cases, one can rescale the solutions in Eq. (25) so that they satisfy

$$\Phi_\ell(\xi) \approx \begin{cases} (1 - \xi)^{\nu_\ell-1/2} & \text{for } 1 - \xi \ll 1 \\ \alpha_\ell \xi^{(2\ell+p-2)/[2(p-1)]} & \text{for } \xi \ll 1 \end{cases} \quad (54)$$

for some α_ℓ . The transmission coefficients $\mathcal{P}_\ell = |D_\ell(\omega)|^2$ in these cases satisfy

$$\begin{aligned} |D_\ell(\omega)|^2 &= \frac{|\Gamma(\ell + \frac{p-1}{2})|^2 |b_\ell(\omega)|^2}{|\alpha_\ell|^2 |\Gamma(\nu_\ell)|^2 (p-1)^{2\nu_\ell-1}} \left(\frac{\omega r_H}{2} \right)^{-2(\ell-\nu_\ell)-p+1} \\ &\text{if } \nu_\ell > 0, \end{aligned} \quad (55)$$

and

$$\begin{aligned} |D_\ell(\omega)|^2 &= \frac{(p-1) |\Gamma(\ell + \frac{p-1}{2})|^2 |b_\ell(\omega)|^2}{4 |\alpha_\ell|^2 \log^2 \omega r_H} \left(\frac{\omega r_H}{2} \right)^{-2\ell-p+1} \\ &\text{if } \nu_\ell = 0. \end{aligned} \quad (56)$$

We find from Eq. (25) that for the “−” polar modes with $2\ell \leq p - 1$

$$\nu_\ell^{(p-)} = \frac{1}{2} - \frac{\ell}{p-1}, \quad (57)$$

$$\alpha_\ell^{(p-)} = \frac{p + \ell - 1}{\ell - 1}. \quad (58)$$

It is not possible to determine the low-frequency behavior of the transmission coefficients, $\mathcal{P}_\ell^{(p-)}$, in these cases. However, it is possible to find their upper limits using Eqs. (40), (55), and (56). Thus, for $2\ell \leq p - 1$,

$$\mathcal{P}_\ell^{(p-)} \leq 4C_\ell^{-1} \left(\frac{\omega r_H}{2} \right)^{2(\ell-\nu_\ell)+p-1}, \quad (59)$$

where C_ℓ is the coefficient of $|b_\ell(\omega)|^2 (\frac{\omega r_H}{2})^{-2(\ell-\nu_\ell)-p+1}$ in Eqs. (55) and (56). Here, terms that tend to zero faster than $\omega^{2(\ell-\nu_\ell)+p-1}$ as $\omega \rightarrow 0$ have been neglected.

IV. LOW-FREQUENCY ABSORPTION CROSS SECTION

Let $\mathcal{P}_\ell^{(a\pm)}$ and $\mathcal{P}_\ell^{(p\pm)}$ be the transmission coefficients of the partial waves in the four different modes. Then the total absorption cross section of an electromagnetic wave with frequency ω can be expressed as

$$\begin{aligned} \sigma &= \frac{(2\pi)^p}{p\Omega_p \omega^p} \\ &\times \left\{ \sum_{\ell=1}^{\infty} [M_\ell^{(p)} \mathcal{P}_\ell^{(p+)} \cos^2 \psi_\ell^{(p)} + M_\ell^{(a)} \mathcal{P}_\ell^{(a+)} \cos^2 \psi_\ell^{(a)}] \right. \\ &\times \left. \sum_{\ell=2}^{\infty} [M_\ell^{(p)} \mathcal{P}_\ell^{(p-)} \sin^2 \psi_\ell^{(p)} + M_\ell^{(a)} \mathcal{P}_\ell^{(a-)} \sin^2 \psi_\ell^{(a)}] \right\}, \end{aligned} \quad (60)$$

where

$$\Omega_p = \frac{2\pi^{(p+1)/2}}{\Gamma(\frac{p+1}{2})}, \quad (61)$$

is the surface area of the p -dimensional unit sphere, and where

$$M_\ell^{(p)} = \frac{(2\ell + p - 1)(\ell + p - 2)!}{(p - 1)! \ell!}, \quad (62)$$

$$M_\ell^{(a)} = \frac{(2\ell + p - 1)(\ell + p - 1)!}{(\ell + 1)(\ell + p - 2)(p - 2)!(\ell - 1)!} \quad (63)$$

are the multiplicities of the scalar and vector spherical harmonics, respectively, on the p dimensional sphere (see, e.g. Refs. [20,21]). The mixing angles $\psi_\ell^{(a)}$ and $\psi_\ell^{(p)}$ are given by Eqs. (8) and (15), respectively. For $p = 2, 3$ and 4, the low-frequency of all partial waves have been given in Sec. III, and hence we can find the low-frequency behavior of the total absorption cross section in these

dimensions. For $p \geq 5$ we can only give the upper limit of the low-frequency absorption cross section.

We first treat the $p = 2$ case, i.e. the four-dimensional case, which has been studied in Ref. [15]. Since the transmission coefficients $\mathcal{P}_\ell^{(a\pm)}$ and $\mathcal{P}_\ell^{(p\pm)}$ behave like $\omega^{2(\ell+\nu_\ell)+1}$ at low frequencies, we only need the modes with the lowest value of $\ell + \nu_\ell$. As is well known [18], we have $\mathcal{P}_\ell^{(a\pm)} = \mathcal{P}_\ell^{(p\pm)}$ for $p = 2$. We indeed find

$$\nu_\ell^{(a+)} = \nu_\ell^{(p+)} = \ell + \frac{3}{2}, \quad (64)$$

$$\nu_\ell^{(a-)} = \nu_\ell^{(p-)} = \ell - \frac{1}{2}, \quad (65)$$

$$\alpha_\ell^{(a+)} = \alpha_\ell^{(p+)} = \frac{\ell}{\ell + 2}, \quad (66)$$

$$\alpha_\ell^{(a-)} = \alpha_\ell^{(p-)} = \frac{(\ell - 1)(2\ell - 1)}{(\ell + 1)(2\ell + 1)}. \quad (67)$$

The modes with the lowest value of $\ell + \nu_\ell$ are the “+” modes with $\ell = 1$ and the “−” modes with $\ell = 2$. The transmission coefficients for these modes are all equal and take the value $\frac{4}{9}(\omega r_H)^8$ [22]. (Interestingly, the transmission coefficients for the “−” modes can be obtained from those for the “+” modes by reducing the value of ℓ by 1.) Then, from Eq. (60) we find

$$\sigma|_{\text{low } \omega} = \frac{16\pi r_H^2}{9}(\omega r_H)^6, \quad p = 2. \quad (68)$$

For $3 \leq p \leq 4$, we only need to determine the smallest among $\nu_\ell^{(a\pm)} + \ell$ and $\nu_\ell^{(p\pm)} + \ell$ as stated above. Since $\nu_\ell^{(a\pm)}$ and $\nu_\ell^{(p\pm)}$ are all increasing functions of ℓ , all we need to find is the smallest of the following four numbers:

$$\nu_1^{(a+)} - 1 = \sqrt{\frac{1}{4} + \frac{2(p+1)}{p-1}} - 1, \quad (69)$$

$$\nu_2^{(a-)} = \left[\frac{1}{4} - \frac{\sqrt{(p-1)(p^3 + 3p^2 + 3p - 1)}}{(p-1)^2} + \frac{p^2 + p + 1}{(p-1)^2} \right]^{1/2}, \quad (70)$$

$$\nu_1^{(p+)} - 1 = \frac{1}{p-1} + \frac{1}{2}, \quad (71)$$

$$\nu_2^{(p-)} = \frac{2}{p-1} - \frac{1}{2}. \quad (72)$$

In the appendix we prove the following inequalities:

$$\nu_2^{(p-)} < \nu_2^{(a-)} < \nu_1^{(p+)} - 1 < \nu_1^{(a+)} - 1, \quad p \geq 3. \quad (73)$$

Hence, the low-frequency absorption cross section is dominated by the “−” polar modes with $\ell = 2$. Thus, we find the low-frequency absorption cross section as

$$\sigma|_{\text{low } \omega} = \frac{2\pi(p-1)^2 A_H}{[\Gamma(\frac{p+3}{2(p-1)})]^2} \left(\frac{\omega r_H}{2(p-1)} \right)^{4/(p-1)+2}, \quad 3 \leq p \leq 4, \quad (74)$$

where $A_H = \Omega_p r_H^p$ is the horizon area.

For $p \geq 5$ we can only find the upper limit for the low-frequency absorption cross section. We can use exactly the same argument as above to conclude that the “−” polar modes give the upper bound which tends to zero most slowly as $\omega \rightarrow 0$. We find

$$\sigma|_{\text{low } \omega} \leq \frac{A_H}{4\pi} (\omega r_H)^3 \log^2 \omega r_H, \quad p = 5, \quad (75)$$

and

$$\sigma|_{\text{low } \omega} \leq \frac{2(p-1)^2 A_H}{\pi} \left[\Gamma\left(\frac{p-5}{2(p-1)}\right) \right]^2 \times \left(\frac{\omega r_H}{2(p-1)} \right)^{4/(p-1)+2}, \quad p \geq 6. \quad (76)$$

The upper limit (75) is clearly consistent with the assumption that the low-frequency absorption cross section is given by Eq. (74) also for $p = 5$. It is also possible for this equation to be valid for $p \geq 6$ because the right-hand side is $\cos^2 \frac{2\pi}{p-1}$ times the upper limit given by Eq. (76). In Fig. 1, we summarize our results for the low-frequency absorption cross section as a function of p .

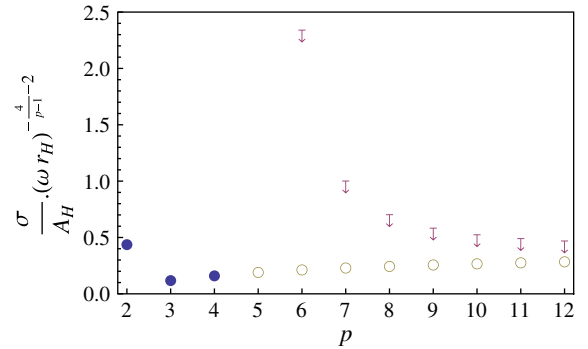


FIG. 1 (color online). The low-frequency absorption cross section of electromagnetic waves for extreme Reissner-Nordström black holes in $p + 2$ dimensions is exhibited. The full circles represent the exact values obtained for $2 \leq p \leq 4$ as given by Eqs. (68) and (74). The downward arrows indicate the upper limits obtained for $p \geq 6$ as given in Eq. (76). The upper limit for $p = 5$ is not displayed because of the extra $\log^2 \omega r_H$ factor, which diverges as $\omega r_H \rightarrow 0$. The conjectured values for the cross sections with $p \geq 5$, as discussed below Eq. (76), are plotted using empty circles.

V. SUMMARY AND DISCUSSIONS

In this paper, we investigated the low-frequency absorption cross section of the electromagnetic waves for extreme Reissner-Nordström black holes in higher dimensions using the work of Kodama and Ishibashi [17]. It was found that the low-frequency behavior of the absorption cross section is dominated by modes with $\ell = 2$ rather than those with $\ell = 1$ due to the mixing between the electromagnetic and gravitational waves in five and six spacetime dimensions, i.e. for $p = 3$ and 4. For dimensions higher than six we only found upper limits for the low-frequency behavior of the absorption cross sections, which are again dominated by modes with $\ell = 2$. These upper limits are consistent with the assumption that the expression for the low-frequency absorption cross section for dimensions five and six are valid in dimensions higher than six as well.

In addition, our investigation revealed some remarkable features of the equations governing the gravitational and electromagnetic perturbations in the extreme Reissner-Nordström background. Considering the complexity of these equations, particularly of those governing the polar perturbations, it is already remarkable that they allow simple solutions. We also noted that the potentials for the two decoupled equations for the polar perturbations are related by the transformation $\ell \leftrightarrow -\ell - p + 1$. It is likely that these features are only part of a more profound structure in perturbations of higher-dimensional Reissner-Nordström black holes. It will be interesting to find whether this is indeed the case.

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APPENDIX A: PROOF OF INEQUALITY (73)

First we note that

$$\begin{aligned} (p-1)^2[(\nu_2^{(a-)})^2 - (\nu_2^{(p-)})^2] \\ = p^2 + 3p - 5 - \sqrt{(p-1)(p^3 + 3p^2 + 3p - 1)}. \end{aligned} \quad (\text{A1})$$

Since $\nu_2^{(a-)} > 0$ for all $p \geq 2$, the inequality

$$p^2 + 3p - 5 > \sqrt{(p-1)(p^3 + 3p^2 + 3p - 1)} \quad (\text{A2})$$

will imply that $\nu_2^{(p-)} < \nu_2^{(a-)}$. By squaring both sides and subtracting one from the other, we find that this inequality is equivalent to

$$(p-2)(4p^2 + 7p - 12) > 0, \quad (\text{A3})$$

which holds for $p \geq 3$.

Next we note

$$\begin{aligned} (p-1)^2[-(\nu_2^{(a-)})^2 + (\nu_1^{(p+)})^2] \\ = \sqrt{(p-1)(p^3 + 3p^2 + 3p - 1)} - (p^2 + 1). \end{aligned} \quad (\text{A4})$$

This is positive since

$$\begin{aligned} (p-1)(p^3 + 3p^2 + 3p - 1) - (p^2 + 1)^2 \\ = 2p(p-2)(p+1) > 0. \end{aligned} \quad (\text{A5})$$

Since $\nu_1^{(p+)} - 1 > 0$ for $p \geq 2$, this implies that $\nu_2^{(a-)} < \nu_1^{(p+)} - 1$.

Finally, we find

$$(\nu_1^{(a+)})^2 - (\nu_1^{(p+)})^2 = \frac{p-2}{(p-1)^2} > 0, \quad p \geq 3. \quad (\text{A6})$$

Since $\nu_1^{(a+)} > 0$, this inequality implies $\nu_1^{(p+)} < \nu_1^{(a+)}$.

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