IFT

Instituto de Física Teórica
Universidade Estadual Paulista

DISSERTAÇÃO DE MESTRADO IFT–D.006/10

MHV Tree Amplitudes in Super-Yang-Mills and in Superstring Theory

Gustavo Machado Monteiro

Orientador

Prof. Dr. Nathan Jacob Berkovits

Abril de 2010

Agradecimentos

Seria uma grande injustiça não começar agradecendo aos meus pais Nilson e Jacqueline. A meu ver, o fato de eles serem meus pais já seria motivo suficiente para esse agradecimento. Entretanto, mais do que isso, muito obrigado por serem aqueles quem sempre posso contar. Agradeço também pelo apoio, pelas vezes que precisei de motivação, ânimo, mas, principalmente, pelos sempre valiosos conselhos e por terem moldado essa personalidade teimosa para ser o que sou hoje.

Ao meu orientador Nathan Berkovits pela paciência comigo durante esse dois anos. Mesmo tendo uma personalidade deveras teimosa, ele soube me ensinar como ser claro, objetivo e como se deve ser um pesquisador, mesmo que muitas vezes eu não consiga por em prática tais ensinamentos. Agradeço ainda o incentivo e toda ajuda prestada para o meu próximo passo a partir daqui.

À minha irmã que ao decorrer dos anos se tornou minha melhor amiga, sem perder aquela típica implicância de irmã.

Aos meus amigos das antigas, Gui, Gu, Glauber e Yuri, por me mostrarem que amizade não se mede por distância ou freqüência.

Aos meus amigos das "modernas", Thiago, Mateus e Neto, por fazerem questão de sempre me lembrar o quão chato e incoveniente eu posso ser. Além de proporcionarem muitos momentos agradáveis durante esses dois anos.

Aos meus amigos de faculdade, Nagata, Aliança e Bebê, pela confiança irrestrita, pelo apoio nos momentos difíceis e por mostrarem que sempre posso contar com vocês.

A Tiago e Daniel Noce, pela parceria e amizade e por acabarem sendo minha "família" em São Paulo, sempre me apoiando e algumas vezes torcendo pela minha desgraça para que pudessem rir (exaustivamente) de mim.

A mis alumnos de español, John y Camilo, por estar en mi lado cuando lo necesitaba y por burlarse de mi acento Guatemalteco. Y a Google mediante la traducción de esta frase.

A Ana e Flávia, pela grande ajuda nesse próximo passo e por me emprestarem seus ouvidos muitas vezes para ouvir minhas reclamações. E à arretada da Fran pelas horas de descontração e sua hospitalidade.

Aos "irmãos" de grupo, Madruga (Rafael) e Bicho Pau (Daniel), pela discussões sempre frutíferas sobre física e por sempre acreditarem em mim, o que facilitava minhas mentiras.

Por fim, porém essencial, ao CNPq pelo apoio financeiro.

"No que diz respeito ao empenho, ao compromisso, ao esforço, à dedicação, não existe meio termo. Ou você faz uma coisa bem feita ou não faz"

Ayrton Senna da Silva

Resumo

A presente dissertação trata-se de uma revisão sobre amplitudes MHV na teoria de Yang-Mills e em sua extensão maximalmente supersimétrica. A demonstração da fórmula geral para tais amplitudes é dada no terceiro capítulo.

Uma vez que o formalismo de twistors se apresenta como um interessante arcabouço teórico para o estudo de tais amplitudes, uma discussão sobre o mesmo é considerada, bem como sua extensão supersimétrica.

Ao final, o cálculo de quatro e cinco pontos para amplitudes MHV em teoria de supercordas é apresentado. Para tal, foi utilizada a prescrição fornecida por Berkovits e Maldacena em [11].

Palavras Chaves: Amplitudes MHV, Teorias de Yang-Mills, Teorias de Yang-Mills Maximalmente Supersimétricas, Twistors e Teoria de Supercordas.

Áreas do conhecimento: Teoria Quântica de Campos; Teoria de Cordas; Supersimetria.

Abstract

In the present work, we have provided a review about MHV amplitudes in Yang-Mills and maximally supersymmetric Yang-Mills theory. A proof of MHV formula was given in the third chapter.

Since twistor formalism provides an interesting framework to study such amplitudes, a discussion about it is also considered as well as its supersymmetric extension.

At the end, we have computed MHV four-point and five-point gluon tree amplitudes in superstring theory, using a prescription given by Berkovits and Maldacena [11].

Contents

Chapter 1

Introduction

The physics scenario in the middle of last century was considerably different from what we have nowadays. In the 60's, after the success of QED, people attempted to use the quantum field theory recipe to study the interaction between mesons and baryons. Actually, problems emerged because perturbation theory fails to describe such interaction. Moreover, at that period, the distinction between weak and strong interaction was not clear at all.

In 1968, Gabriel Veneziano conjectured a formula to four-meson scattering amplitude that reproduces the resonances observed in experimental data. After some time, physicists realize this amplitude comes from a quantum theory of relativistic open strings. In fact, string theory was born as a theory to describe mesons.

In the 70's, the physicists realized that the $SU(3)$ Yang-Mills theory seems to be the correct description of strong interactions, mainly after Gross, Politzer and Wilczek calculate the beta function of this theory. Such computation shows us this theory has asymptotic freedom; what gave the physicists an insight about the quark confinement. Up to 1984, string theory was almost totally abandoned, when it reappeared as the most promising theory of quantum gravity.

Nowadays, much has been talked about the Large Hadron Collider. Since it collides protons, QCD definitely reaches a high position in comparison to other interactions, becoming the most important force in such scattering. This collision may produce gluon jets of great energy. If we extrapolate the energy of the process, we will see that the tree-level n-gluon scattering amplitudes will dominate $(n-2)$-jet product. However, a technical problem emerges here. When n increases, these amplitudes will have more and more terms, even at tree level. Nevertheless, if we consider the gluons having a well-defined helicity, simplifications occurs and the computations becomes rather simple. For example, to four-gluon scattering, we must consider just one amplitude and sum over the permutations among the external legs; to five-gluon scattering, we must consider just two different amplitudes. Furthermore, such amplitudes have an amazing simple form.

These amplitudes regarded in four and five-gluon scattering are examples of maximally helicity violating amplitudes, or MHV amplitudes for short. These MHV amplitudes have been providing an intense object of study for the last six years in maximally supersymmetric Yang-Mills theory and in superstring theory as well.

In the next chapter of this present work, we will present some background information about little group in different signatures, momentum and helicity vectors in spinor form. After this, an introduction about twistor formalism is given. In fact, that is essential to follow the most recent papers about the subject. In some cases, the action on twistor space is the starting point. Twistor theory is used to prove the conformal invariance of the MHV tree amplitudes and the way to recover space-time quantities from twistor space is also provided. At the end of this chapter, the subject of supertwistors will be introduced heuristically.

In chapter three, we will demonstrate the BCFW recursion relation and will use it to proved MHV formula. Some authors begin with BCFW recursion relation to study some features about massless theories. A more detailed discussion on MHV amplitudes in maximally supersymmetric Yang-Mills theory is presented, where we find the supersymmetric Ward identities to MHV amplitudes.

We will not cover subjects as equivalences between MHV amplitudes in gauge theories and amplitudes in superstring theory. However, some remarks on MHV amplitudes in superstring theory are, in some sense, essential. So, in the chapter four, we will present the prescription given by Berkovits and Maldacena [11] to compute MHV amplitudes in superstring theory. We will show that it reproduces the correct result to low energies and computations for four-point and five-point MHV tree amplitude will be also presented.

In the most papers about the subject, the notation $\langle i\ j\rangle$ and $[i\ j]$ is used instead $(\lambda_i\lambda_j)$ and $(\widetilde{\lambda}_i\widetilde{\lambda}_j)$, however, we decided to follow the convention adopted by Wess and Bagger [17].

Chapter 2

MHV Amplitudes in Twistor Space

Following the spirit of S-matrix theory, we are going to focus on the observables arising from n-gluon scattering amplitudes. The quantities that characterize the gluons are their momenta and helicities. Of course these quantities are not directly measurable, since we cannot find gluons in free state, however, at very high energies, four and five-gluon scattering dominate two and three-jet production in hadron colliders and these amplitudes are highly dependent from gluon helicities and momenta.

It would be interesting to study the behavior of the scattering process that most violate the initial state of helicity. In terms of crossing symmetry, we can consider all gluons as incoming and, at this configuration, the maximally violating helicity (MHV) amplitude would be obtained when all incoming gluons have the same helicity. In 1986, Parke and Taylor [1] conjectured that this scattering amplitude always vanishes at tree level as well as the amplitude with just one gluon of different helicity. Then, it was named MHV amplitudes the first nonvanishing configuration at leading order; that is, the amplitudes that have two gluons with opposite helicities from the others. More than this, they proposed (and checked for the first cases) an expression to n-gluon scattering in MHV regimen. This conjecture was proved by Berends and Giele [2] in 1988.

Although, these works greatly simplified the old Feynman graphs procedure, they just really caught the most of physicists' attention when E. Witten [9], in 2003, found an equivalence between these amplitudes and the D-instanton expansion of a certain string theory.

In this chapter, we will assume this conjecture to be true without prove and explore its properties.

2.1 A Survey on Helicity Spinors

Before initiate, we must consider some properties of Lorentz group acting on a flat space-time manifold. We are going to focus on a Minkowski space-time with signature $(+ - --)$, but, at some point in this chapter, it will be convenient to extend the analysis considering other signatures.

Points on space-time are labeled by four real coordinates and they are in one-to-one correspondence with 2×2 Hermitian matrices, in such way that we can write down this map explicitly as follows:

$$\left(x^0, \overrightarrow{x}\right) \mapsto x^0\mathbf{1} - \overrightarrow{x} \cdot \overrightarrow{\sigma}, \tag{2.1}$$

with $\overrightarrow{\sigma}$ being the Pauli matrices. Defining the four-dimension notation, $\sigma^\mu = (\mathbf{1}, \overrightarrow{\sigma})$, the inverse map can be obtained by:

$$x_\mu = \frac{1}{2} tr\left(\sigma^\mu \mathbf{x}\right). \tag{2.2}$$

It is straightforward to observe that taking the determinant of this matrix $(\mathbf{x})$, we get hold of the norm of the position vector. Since we are considering matrices, the action of Lorentz group is translated into:

$$\mathbf{x}' = A\mathbf{x}A^\dagger, \tag{2.3}$$

where $A, A^\dagger$ must belong to $SL\left(2, \mathbb{C}\right)$, to keep $\mathbf{x}'$ Hermitian and preserve the vector norm.

We must note that A and $-A$ correspond to the same transformation, so, it means the Lorentz group $SO\left(3,1\right)$ is isomorphic to quotient group $SL\left(2, \mathbb{C}\right)/\mathbb{Z}_2$. However, $-A$ cannot be continuous expanded from identity matrix, then, we can equivalently say that $SO\left(3,1\right)$ is locally isomorphic to $SL\left(2, \mathbb{C}\right)$. We gain another structure when consider this new representation of Lorentz group. Actually, $SL(2, \mathbb{C})$ vector representation, or spinor, is absent in the $SO\left(3,1\right)$ one.

Equivalent representations are those, such that, we reach one from other by similarity transformation. From trace cyclic property, it implies that the trace of both matrix representations must be the same. Thus, $SL\left(2, \mathbb{C}\right)$ admits two inequivalents representations; one is the complex conjugate from the other. Depending on which representation the spinor belongs, it can be classified into negative $(1/2, 0)$ and positive chirality $(0, 1/2)$. A space-time vector is translated into $(1/2, 1/2)$ representation.

Although our space-time is, at least locally, Minkowskian with signature $(+ - --)$, we often have to perform a Wick rotation in the timelike coordinates to properly define quantities on path integrals. To do so, we must complexify the entries, that is, we must allow the coordinates as well as the vector components to be complex. Minkowski space appears as just a section of this bigger space $(\mathbb{C}^4, \eta)$. There are two more sections of importance, one is the Euclidean section, which the spacial entries are pure imaginary, and the one whose signature is $(+ + - -)$, where one spacial entry is pure imaginary.

After this complexification, the Lorentz group becomes locally isomorphic to $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$. It is easy to see from (2.3), because the Hermitian condition need not be satisfied. The most general transformation that preserves the norm is $\mathbf{x}' = A\mathbf{x}B$. Until the end of this section, all analysis will hold in this complexified Minkowski space. Looking at its sections, we find that the Euclidean one is isomorphic to quaternion space* and we obtain that $SO(4)$ is locally isomorphic to $SU(2) \times SU(2)$. Their spinor representations are independent and pseudoreal†. To signature $(+ + - -)$, we must consider, for convenience, $x^2 \to ix^2$. Doing this, we see that $\mathbf{x}$ becomes real, so, we must restrict $A, B \in SL(2, \mathbb{R})$ and the spinor representations being independent and real. In other words, $SO(2, 2)$ is locally isomorphic to $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$.

Because we are studying gluons, we shall consider the massless particle momentum. Once the momentum is lightlike, we find that $\det(\mathbf{p}) = p_\mu p^\mu = 0$, what means that $\mathbf{p}$ necessarily has a zero eigenvalue and is not rank two, then, it can be written as:

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha \widetilde{\lambda}_{\dot{\alpha}}. \tag{2.4}$$

It is clear that if a lightlike vector p is given, the expression (2.4) does not suffice to determine λ and $\widetilde{\lambda}$ itself. Both spinors can be determined only up to a scaling:

$$\lambda \to u\lambda, \ \widetilde{\lambda} \to u^{-1}\widetilde{\lambda}, \tag{2.5}$$

for $u \in \mathbb{C}^*$. In signature $(+ - - -)$, to $p_{\alpha\dot{\alpha}}$ be Hermitian, it is needed $\widetilde{\lambda} = \pm\overline{\lambda}$ ‡ and u is restricted to satisfy $|u| = 1$. As mentioned before, for signature $(+ + - -)$, $\lambda, \widetilde{\lambda}$ must be independent and $\lambda, \widetilde{\lambda}, u \in \mathbb{R}^*$.

Using the determinant formula, we can determine by intuition the "metric tensor" to spinors, or, equivalently, how to raise and lower indices:

*Note that $\mathbf{x}\mathbf{x}^\dagger = |\mathbf{x}|\,\mathbf{1}$, where $|\mathbf{x}| = \det(\mathbf{x})$.

†Actually, this brief discussion on Euclidean section is for completeness, it will not be used in the rest of this work, since a lightlike vector cannot exist in Euclidean signature.

‡The negative sign holds when $p^0 < 0$.

$$\det\left(p_{\alpha\dot{\alpha}}\right)=\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\lambda_\alpha\widetilde{\lambda}_{\dot{\alpha}}\lambda_\beta\widetilde{\lambda}_{\dot{\beta}},$$

where $\epsilon^{\alpha\beta}$ is the totally antisymmetric tensor. The quantity $\epsilon_{\alpha\beta}\lambda^\alpha\omega^\beta$ is an invariant under Lorentz group, what implies that the Levi-Civita tensor acts as a metric in spinor space. Applying the same mapping to space-time metric tensor as did to position vector, that is, $x_{\alpha\dot{\alpha}}=x_\mu\sigma^\mu_{\alpha\dot{\alpha}}$, we find that, in spinor form, it becomes:

$$\eta_{\alpha\dot{\alpha}\beta\dot{\beta}}=\eta_{\mu\nu}\sigma^\mu_{\alpha\dot{\alpha}}\sigma^\nu_{\beta\dot{\beta}}=\epsilon_{\alpha\dot{\alpha}}\epsilon_{\beta\dot{\beta}}.$$

Thus, the inner product of two vectors is $p\cdot q=\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}p_{\alpha\dot{\alpha}}q_{\beta\dot{\beta}}$. Since we are considering massless particles, both Casimirs of the Lorentz group collapse to zero, so, it is complicated to talk about spin of these massless particles. In fact, the Pauli-Lubanski vector becomes proportional to momentum vector and we are able to associate a spin projection on the momentum direction, what is called helicity. Since helicity vector has the same orientation than momentum, we find:

$$\varepsilon\cdot p=0. \tag{2.6}$$

In terms of spinors, we have two possible solutions: either ε is proportional to λ or proportional to $\widetilde{\lambda}$. Each solution corresponds to a sign of the helicity. Following the convention found on papers about the subject, we define positive and negative helicity as:

$$\varepsilon^+_{\alpha\dot{\alpha}}=\frac{\varepsilon_\alpha\widetilde{\lambda}_{\dot{\alpha}}}{(\varepsilon\lambda)},\ \varepsilon^-_{\alpha\dot{\alpha}}=\frac{\lambda_\alpha\widetilde{\varepsilon}_{\dot{\alpha}}}{(\widetilde{\lambda}\widetilde{\varepsilon})}. \tag{2.7}$$

The factors $(\varepsilon\lambda)^{-1}$ and $(\widetilde{\lambda}\widetilde{\varepsilon})^{-1}$ appear to keep $\varepsilon^+\cdot\varepsilon^-=1$. Here, we used the notation $(\varepsilon\lambda)=\epsilon_{\alpha\beta}\varepsilon^\alpha\lambda^\beta$ and $(\widetilde{\lambda}\widetilde{\varepsilon})=\epsilon_{\dot{\alpha}\dot{\beta}}\widetilde{\lambda}^{\dot{\alpha}}\widetilde{\varepsilon}^{\dot{\beta}}$. The gauge invariance implies that the amplitude have to be invariant under the following transformations:

$$\varepsilon\to\varepsilon+\alpha\lambda,\ \widetilde{\varepsilon}\to\widetilde{\varepsilon}+\beta\widetilde{\lambda}, \tag{2.8}$$

with $\alpha,\beta\in\mathbb{C}$.

Under (2.5), ε^+ has the same scaling as λ^{-2}, what implies that λ carries helicity $-1/2$. Using this in order to determine the helicity of one particle in the scattering amplitude, we intuitively[§] have:

[§]It can be directly observed when the amplitude $A\left(\lambda_i,\widetilde{\lambda}_i,h_i\right)$ is an homogeneous function of λ_i or $\widetilde{\lambda}_i$.

$$\left(\lambda_i^{\alpha}\frac{\partial}{\partial\lambda_i^{\alpha}}-\widetilde{\lambda}_i^{\dot{\alpha}}\frac{\partial}{\partial\widetilde{\lambda}_i^{\dot{\alpha}}}\right)\widehat{\mathcal{A}}\left(\lambda_i,\widetilde{\lambda}_i,h_i\right)=-2h_i\widehat{\mathcal{A}}\left(\lambda_i,\widetilde{\lambda}_i,h_i\right). \tag{2.9}$$

This equation holds for helicity to massless particles of any spin. Writing $\widehat{A}(\lambda_i,\widetilde{\lambda}_i,h_i)$ in a way that energy-momentum conservation is found explicitly, we have:

$$\widehat{\mathcal{A}}\left(\lambda_i,\widetilde{\lambda}_i,h_i\right)=i\,(2\pi)^4\,\delta^4\left(\sum_{i=1}^{n}\lambda_i^{\alpha}\widetilde{\lambda}_i^{\dot{\alpha}}\right)\mathcal{A}\left(\lambda_i,\widetilde{\lambda}_i,h_i\right). \tag{2.10}$$

It is easy to observe that $\mathcal{A}(\lambda_i,\widetilde{\lambda}_i,h_i)$ satisfies the same equation as $\widehat{\mathcal{A}}(\lambda_i,\widetilde{\lambda}_i,h_i)$. The expression (2.9) will be deduce in a more formal and elegant way from twistor formalism.

2.2 Description of MHV Amplitudes

At this point, the reader can argue that, differently than was stated, the gluon cannot be characterized just by its momentum and helicity, because it has one extra quantum number, what we call color. In fact, the reason to the color be neglected from this analysis is because it can be stripped off from the amplitude. We can factorize a term $tr\,(T^{a_1}T^{a_2}...T^{a_n})$ from each scattering configuration and total amplitude becomes the sum over all possibilities:

$$\mathcal{M}=\sum_{perm}\widehat{\mathcal{A}}\left(\lambda_i,\widetilde{\lambda}_i,h_i\right)tr\,(T^{a_1}T^{a_2}...T^{a_n}), \tag{2.11}$$

where the T^{a_i}s are the generators of $SU\,(N)$, correspondent to i-th particle. Of course, this final amplitude must have Bose symmetry. To induce the expression (2.11), we must note that the vertices of the theory are proportional to $tr\left(T^aT^bT^c\right)$ or $tr\left(T^aT^bT^cT^d\right)$. If one of the indices label the propagator, it will be contracted. Then, the amplitude must be proportional to such products:

$$tr\left(T^a...T^d\right)tr\left(T^d...T^b\right).$$

Using the relation obtained for $SU\,(N)$ group, in which:

$$(T^a)^I_J\,(T^a)^K_L=\delta^I_L\delta^K_J-\frac{1}{N}\delta^I_J\delta^K_L, \tag{2.12}$$

we recover (2.11). We can see that the trace contribution cancels inside the summatory. Let the S-matrix initial state be the gluons 1 and 2. By crossing symmetry, the amplitude changes the sign when we permutate them. Since the four-gluon vertices vanish, the term coming from the trace contribution becomes $tr\left(T^{a_1}T^{a_2}\right)$, that is symmetric under such exchange. A heuristic way to observe this is starting from $U(N)$, because we do not have such term. However, $U(N)$ decomposes into $SU(N)\times U(1)$, but $U(1)$ has trivial S-matrix, in other words, it does not admit self-interaction, so, the trace term must not contribute.

The general picture states that, instead of working on the old-fashioned Feynman diagrams, we are dealing with the double line ones, in which, each line corresponds to "color propagation".

We can infer by the chapter description that MHV amplitudes completely determine four and five-gluon scattering. To four-gluon amplitude, we have:

$$\mathcal{A}\left(++++\right)=\mathcal{A}\left(+++-\right)=\mathcal{A}\left(+---\right)=\mathcal{A}\left(----\right)=0,$$

as well as their permutations. The unique contributions come from $\mathcal{A}\left(++--\right)$ and its permutations. For five-gluon scattering amplitude, we must just look at $\mathcal{A}\left(++---\right)$, $\mathcal{A}\left(+++--\right)$ and the permutations of each one. This is of great simplicity, if we compare to the usual procedure based on Feynman diagrams.

Although we have being cited MHV amplitudes many times, we have not shown Parke and Taylor proposition. To do so, suppose that gauge bosons r and s $(1\leq r<s\leq n)$ have negative helicity and the others have positive helicity. Tree level amplitude for this process is:

$$\mathcal{A}=g^{n-2}\frac{\left(\lambda_r\lambda_s\right)^4}{\prod\limits_{i=1}^{n}\left(\lambda_i\lambda_{i+1}\right)}, \tag{2.13}$$

where $n>3$ and g is the gauge coupling constant; in addition, we set $\lambda_{n+1}=\lambda_1$ to condense the notation. Note that this amplitude has the requisite homogeneity in each variable. As expected, to positive helicity gluons, λ_i $(i\neq r,s)$ has -2 homogeneous degree, because it appears twice in the denominator. And, for $i=r,s$, λ_i has homogeneous degree $+2$, since it appear twice in the denominator and four times in the numerator. If gauge bosons r and s have positive helicity and the others have negative helicity, the amplitude becomes:

$$\mathcal{A}=g^{n-2}\frac{(\widetilde{\lambda}_r\widetilde{\lambda}_s)^4}{\prod\limits_{i=1}^{n}(\widetilde{\lambda}_i\widetilde{\lambda}_{i+1})}. \tag{2.14}$$

The case of $n=4$ is really interesting, because the same amplitude seems to be holomorphic and antiholomorphic at the same time. It appears to have a contradiction, since, for example, the amplitude $\mathcal{A}\left(1^+,2^+,3^-,4^-\right)$ is a special case of both constructions:

$$\mathcal{A} = g^2 \frac{(\lambda_1 \lambda_2)^4}{\prod_{i=1}^{4} (\lambda_i \lambda_{i+1})}, \tag{2.15}$$

and,

$$\mathcal{A} = g^2 \frac{(\widetilde{\lambda}_3 \widetilde{\lambda}_4)^4}{\prod_{i=1}^{4} \left(\widetilde{\lambda}_i \widetilde{\lambda}_{i+1}\right)}. \tag{2.16}$$

The resolution of this misleading is instructive if we realize that momentum conservation constrains holomorphic parts to antiholomorphic ones. To be more specific, we shall consider $\sum \lambda_i^\alpha \widetilde{\lambda}_i^{\dot{\alpha}} = 0$, and take inner product with λ_j and $\widetilde{\lambda}_k$. For any j and k we have:

$$\sum_{i=1}^{4} (\lambda_j \lambda_i) \left(\widetilde{\lambda}_i \widetilde{\lambda}_k\right) = 0. \tag{2.17}$$

Setting, for example, $j = 1$, $k = 3$ and $j = 2$, $k = 4$, we get, respectively, $(\lambda_1 \lambda_2) (\widetilde{\lambda}_2 \widetilde{\lambda}_3) = -(\lambda_4 \lambda_1) (\widetilde{\lambda}_3 \widetilde{\lambda}_4)$ and $(\lambda_1 \lambda_2) (\widetilde{\lambda}_4 \widetilde{\lambda}_1) = -(\lambda_2 \lambda_3) (\widetilde{\lambda}_3 \widetilde{\lambda}_4)$. Replacing these identities into (2.15), we obtain:

$$\widehat{\mathcal{A}} = i g^2 (2\pi)^4 \delta^4 \left(\sum \lambda_i^\alpha \widetilde{\lambda}_i^{\dot{\alpha}}\right) \frac{(\widetilde{\lambda}_3 \widetilde{\lambda}_4)^4}{\prod_i \left(\widetilde{\lambda}_i \widetilde{\lambda}_{i+1}\right)} \frac{(p_1 \cdot p_2)}{(p_3 \cdot p_4)}.$$

We have multiplied (2.15) by a delta of momentum conservation, because these identities no longer hold when the total momentum is not conserved. Using these momentum conservation once more, we find that $p_1 + p_2 = -(p_3 + p_4)$. Taking the square of these equality, we show that (2.15) is equal to (2.16), proving that there are no inconsistences.

2.3 Twistor Theory and Conformal Invariance

Since all previous discussion is about massless particles, null lines become geometrically important features. Nevertheless, $p_{\alpha\dot{\alpha}}$ is not enough to determine completely such lines. Actually, if we wish to assign a set of coordinates to a null line, geometrically, we may take into account its moment about the origin, or, in physical terms, the angular momentum of the particle whose trajectory is the null line in question, in addition to $p_{\alpha\dot{\alpha}}$.

$$M^{\alpha\dot{\alpha}\beta\dot{\beta}} = x^{\alpha\dot{\alpha}} \lambda^\beta \widetilde{\lambda}^{\dot{\beta}} - x^{\beta\dot{\beta}} \lambda^\alpha \widetilde{\lambda}^{\dot{\alpha}}. \tag{2.18}$$

However, this ten quantities $(p^\tau, M^{\rho\sigma})$ form a highly redundant representation, because, in addition to them, we have an extra constrain:

$$\epsilon_{\nu\rho\sigma\tau} p^\rho M^{\sigma\tau} = 0, \tag{2.19}$$

which reduce four real degrees of freedom and together with p_τ being a lightlike vector, we find this set of (Plücker-Grassmann) coordinates just as five independent real numbers. It is convenient, from this point, translate the analysis above to spinor form. Making use of the relation $x^{\alpha\dot{\alpha}}\lambda^\beta = x^{\beta\dot{\alpha}}\lambda^\alpha + \epsilon^{\beta\alpha}x^{\gamma\dot{\alpha}}\lambda_\gamma$ and defining:

$$\mu^{\dot{\alpha}} = i x^{\alpha\dot{\alpha}}\lambda_\alpha, \tag{2.20}$$

$$\widetilde{\mu}^{\alpha} = -i x^{\alpha\dot{\alpha}}\widetilde{\lambda}_\alpha, \tag{2.21}$$

we obtain that equation (2.18) becomes:

$$M^{\alpha\dot{\alpha}\beta\dot{\beta}} = i\left[\epsilon^{\alpha\beta}\mu^{(\dot{\alpha}}\widetilde{\lambda}^{\dot{\beta})} - \epsilon^{\dot{\alpha}\dot{\beta}}\lambda^{(\alpha}\widetilde{\mu}^{\beta)}\right]. \tag{2.22}$$

Before continue, we must focus on the differences that arise here in working on $(+---)$ or $(++--)$ signature. In Minkowski space-time, since $\widetilde{\mu}$, $\widetilde{\lambda}$ are complex conjugated from μ, λ, we totally determine a null line by the quantity that is called twistor:

$$Z^A \equiv \begin{pmatrix} \mu_{\dot{\alpha}} \\ \lambda^\alpha \end{pmatrix}. \tag{2.23}$$

Forgetting the previous motivation for a while, we define the twistor space $\mathbb{T}$ as the set of all Z^A whose definition is (2.23), where μ, λ are general (complex) spinors. Although we built twistor by null lines, not all twistors in $\mathbb{T}$ can be associated to light rays in space-time. Introducing the dual of twistor by:

$$\widetilde{Z}_A \equiv \begin{pmatrix} \widetilde{\lambda}^{\dot{\alpha}} \\ \widetilde{\mu}_\alpha \end{pmatrix}, \tag{2.24}$$

and defining the contraction in such way that:

$$Z^A\widetilde{Z}_A = \mu_{\dot{\alpha}}\widetilde{\lambda}^{\dot{\alpha}} + \lambda^\alpha\widetilde{\mu}_\alpha, \tag{2.25}$$

we find from incident relations (2.20) and (2.21) that the condition to twistor represents a null line on space-time is $Z^A\widetilde{Z}_A = 0$. This subset of $\mathbb{T}$ is called the space of null twistors ($\mathbb{N}$). We may observe that twistor space $\mathbb{T}$ is isomorphic to $\mathbb{C}^4$, so $\mathbb{N}$ has seven real dimensions. It implies that a null line is being characterized by seven independent real parameter. In order to fix this misleading, we must note that a scaling in the momentum $p_{\alpha\dot{\alpha}}$ determines the same light ray we had before the scaling. It reflects that a null line cannot be associated to just one point on $\mathbb{N}$. This redundancy can be solved if we regard Z^A as equivalent to αZ^A, with α complex. The complex projective space ($\mathbb{CP}$) is the quotient space whose the equivalence relation, which defines it, is $(z^1, z^2, ..., z^n) \sim (\alpha z^1, \alpha z^2, ..., \alpha z^n)$ to $\alpha \in \mathbb{C}^*$. Thus, we define the projective twistor space $\mathbb{PT}$ as an isomorphic space to $\mathbb{CP}^3$. The same form, $\mathbb{PN}$ is the subspace of $\mathbb{PT}$ that satisfies $Z^A \widetilde{Z}_A = 0$. This space is characterized by five real parameters.

It implies that entire light rays are represented as one point on projective twistor space! It is impressive, but we may ask ourselves: how about points on space-time? What are they on $\mathbb{PN}$? A point P on space-time is determined by the intersection of light rays. These light rays form a future and a past light cone on space-time. Of course, the past cone completely establish the future one, so, we shall consider just the past cone without the point P, in order to determine the referred point. However, each light ray on this past cone without P is mapped onto one point on projective null twistor space and, since this mapping is not one-to-one, it is convenient to define an equivalence relation among points over the same null line, in such way that the map from this quotient space to $\mathbb{PN}$ is a bijection. Now, it is possible to map points on this quotient space to points on $\mathbb{PN}$ and this map must be independent on the quotient space representatives. Thus, we can choose these representatives by points lying on the past cone whose time coordinate of each one is equal. This quotient space is isomorphic to $\mathbb{S}^2$, then, points on space-time is represented on projective null twistor space by Riemann spheres. That is why people often consider twistor theory as an attempt to provide a non-local picture.

To finish these geometric remarks about correspondences between space-time and twistor space, we must realize that all this analysis holds when $\lambda \neq 0$, however, it possible to define a nonvanishing null twistor when $\lambda = 0$. We might ask what can be associated to this null twistor. Actually, we will not go so deep into details because it plays no role in what we have proposed to this present work, but, from equation (2.20), we see that these twistors represent limiting "light rays at infinity". This light rays do not lie in Minkowki space, but they belongs to conformal compactified Minkowski space in which we added a light cone at space-time infinity.

The quantity $Z^A \widetilde{Z}_A$ must be real, because $\widetilde{\lambda}^{\dot{\alpha}} \mu_{\dot{\alpha}} = \overline{(\lambda^\alpha \widetilde{\mu}_\alpha)}$. Yet, we have no guarantees that $Z^A \widetilde{Z}_A \geq 0$, so, $\mathbb{PN}$ splits the projective twistor space into two parts: subspace of right-handed twistors ($Z^A \widetilde{Z}_A > 0$), subspace of left-handed twistor ($Z^A \widetilde{Z}_A < 0$). Up to now, the subspaces of right-handed and left-handed twistors have loose physical meaning. To understand it, we should look at the Pauli-Lubanski vector:

$$S = * \left(\frac{1}{2} p \wedge M\right). \tag{2.26}$$

Taking the square of this expression, we find:

$$S_\rho S^\rho = -\frac{1}{2} M^{\tau\sigma} M_{\tau\sigma} p^\kappa p_\kappa + M^{\tau\sigma} M_{\kappa\sigma} p_\tau p^\kappa.$$

Using the definitions (2.4), (2.22) and keeping the terms which are proportional to $(\lambda\lambda)\,(\widetilde{\lambda}\widetilde{\lambda})$, because, as it was stated, the Pauli-Lubanski vector becomes proportional to massless particle momentum, we obtain the following result:

$$S^{\alpha\dot{\alpha}} S_{\alpha\dot{\alpha}} = \frac{1}{4} \left(Z^A \widetilde{Z}_A\right)^2 \left(\lambda^\alpha \lambda_\alpha\right) \left(\widetilde{\lambda}^{\dot{\alpha}} \widetilde{\lambda}_{\dot{\alpha}}\right). \tag{2.27}$$

This expression provides the particle helicity is given by:

$$h = \frac{1}{2} Z^A \widetilde{Z}_A. \tag{2.28}$$

Obviously, we must not expect this holds without ambiguities at quantum level, because we have not worried about product ordering. However, twistor commutation rules can be determined by Poincaré algebra. From $[p^\rho, p^\sigma] = 0$, we find commutation rules between λ and $\widetilde{\lambda}$. Using:

$$[M^{\rho\sigma}, p^\tau] = i \left(\eta^{\rho\tau} p^\sigma - \eta^{\sigma\tau} p^\rho\right), \tag{2.29}$$

we are able to establish the commutators between λ and μ, λ and $\widetilde{\mu}$, $\widetilde{\lambda}$ and μ, $\widetilde{\lambda}$ and $\widetilde{\mu}$. The last commutator, that is, $[\mu, \widetilde{\mu}]$ is finally obtained by:

$$[M^{\rho\sigma}, M^{\nu\tau}] = i \left(\eta^{\rho\tau} M^{\sigma\nu} - \eta^{\sigma\nu} M^{\rho\tau} + \eta^{\rho\nu} M^{\sigma\tau} - \eta^{\sigma\tau} M^{\rho\nu}\right). \tag{2.30}$$

Thus, it is possible to determine twistor algebra:

$$\left[Z^A, Z^B\right] = [\widetilde{Z}_A, \widetilde{Z}_B] = 0, \tag{2.31}$$

and

$$[Z^A, \widetilde{Z}_B] = \delta^A_B. \tag{2.32}$$

In quantized form, the helicity operator must be symmetrized, then, equation (2.28) have to be modified to:

$$h = \frac{1}{4} \left(Z^A \widetilde{Z}_A + \widetilde{Z}_A Z^A\right). \tag{2.33}$$

Commutator (2.32) provides that, in Z-picture, the operator $\widetilde{Z}_A$ is represented by $-\frac{\partial}{\partial Z^A}$. Making use of this on previous equation:

$$h = -\frac{1}{2}\left(Z^A \frac{\partial}{\partial Z^A} + 2\right). \tag{2.34}$$

In terms of components, we can write this down as:

$$h = -\frac{1}{2}\left(\lambda^\alpha \frac{\partial}{\partial \lambda^\alpha} + \mu^{\dot{\alpha}} \frac{\partial}{\partial \mu^{\dot{\alpha}}} + 2\right). \tag{2.35}$$

It was used in the last expression that $\mu^{\dot{\alpha}} \frac{\partial}{\partial \mu^{\dot{\alpha}}} = \mu_{\dot{\alpha}} \frac{\partial}{\partial \mu_{\dot{\alpha}}}$. From the identifications $\mu_{\dot{\alpha}} \to \frac{\partial}{\partial \widetilde{\lambda}^{\dot{\alpha}}}$ and $\widetilde{\lambda}^{\dot{\alpha}} \to -\frac{\partial}{\partial \mu_{\dot{\alpha}}}$, we recover (2.9) by (2.35).

All analysis about quantization and helicity operator remains valid to signature $(++--)$, because we obtain Poincaré algebra to this signature simply replacing $\eta^{\rho\sigma}$ for the metric of such space into commutations rules (2.29) and (2.30). However, a null line cannot be entirely characterized by a point in this new twistor space; λ is real and μ is pure imaginary, so, $\mathbb{T}$ is isomorphic to $\mathbb{C}^2$ or, equivalently, $\mathbb{R}^4$ and four real parameters are not enough to totally describe the null line. Furthermore, the same scaling redundancy also holds, but now, with a real parameter. Then, we have to make this space a projective space $\left(\mathbb{PT} \cong \mathbb{RP}^3\right)$ and to determine a null line completely, we must have $\mathbb{PT} \times \mathbb{PT}^*$.

If real projective twistor space is unable to totally determine a light ray by itself, it seems to have no gain in consider it. Yet, since scattering amplitudes are functions just of λ and $\widetilde{\lambda}$, they can be Fourier transformed to twistor space, that is, to amplitudes dependent just on Z or $\widetilde{Z}$, and signature $(++--)$ is appropriated to do so, because we can properly define this Fourier transform. On the other hand, defining the same thing to signature $(+---)$ yields a highly non-trivial picture¶ (at least for me). Often, in an unappropriated manner, we shall denote $f(Z)$ as a holomorphic function and $f(\widetilde{Z})$ as an antiholomorphic function, even when $\widetilde{\lambda}$ and $\widetilde{\mu}$ are not complex conjugated to λ and μ, respectively.

Using the fact $(\mu\widetilde{\lambda})$ is pure imaginary in signature $(++--)$, we are able to define the amplitude in twistor space for each gluon by:

$$\widetilde{\mathcal{A}}\left(Z_i\right) = i\left(2\pi\right)^4 \int \prod_j \left[\frac{d^2\widetilde{\lambda}_j}{\left(2\pi\right)^2}\right] \exp\left(\sum \mu_i \widetilde{\lambda}_i\right) \delta^4\left(\sum \lambda_i^\alpha \widetilde{\lambda}_i^{\dot{\alpha}}\right) \mathcal{A}\left(\lambda_i, \widetilde{\lambda}_i\right). \tag{2.36}$$

MHV amplitudes are invariant under action of Poincaré group, and we expect to be conformal invariant as well, because beta function vanishes at three level. In

¶Anticipating things, to signature $(+---)$, an integral in $\widetilde{\lambda}$ must be a contour integral over a pole of the function.

order to confirm these suspicious, we will check them on twistor space. Momentum operator to a system of particles is just the sum of each particle momentum operator, as well as to other conformal generators. In quantum twistor space they become:

$$P_{\alpha\dot{\alpha}} = -\sum_i \lambda_{i\alpha}\frac{\partial}{\partial\mu_i^{\dot{\alpha}}}, \tag{2.37}$$

$$M_{\alpha\dot{\alpha}\beta\dot{\beta}} = i\left(\epsilon_{\dot{\alpha}\dot{\beta}}J_{\alpha\beta} + \epsilon_{\alpha\beta}J_{\dot{\alpha}\dot{\beta}}\right). \tag{2.38}$$

In the last equation, we have split the Lorentz generators in a direct sum of each $SL(2,\mathbb{R})$ generators. The explicit form for the J's are:

$$J_{\alpha\beta} = \frac{1}{2}\sum_i \left(\lambda_{i\alpha}\frac{\partial}{\partial\lambda_i^{\beta}} + \lambda_{i\beta}\frac{\partial}{\partial\lambda_i^{\alpha}}\right), \tag{2.39}$$

$$J_{\dot{\alpha}\dot{\beta}} = -\frac{1}{2}\sum_i \left(\mu_{i\dot{\alpha}}\frac{\partial}{\partial\mu_i^{\dot{\beta}}} + \mu_{i\dot{\beta}}\frac{\partial}{\partial\mu_i^{\dot{\alpha}}}\right). \tag{2.40}$$

In order to determine the dilatation generator, we must consider the conformal algebra. From $[P_{\alpha\dot{\alpha}}, D] = P_{\alpha\dot{\alpha}}$ and $[M_{\alpha\dot{\alpha}\beta\dot{\beta}}, D] = 0$, we find:

$$D = -\frac{1}{2}\sum_i \left(\lambda_i^{\alpha}\frac{\partial}{\partial\lambda_i^{\alpha}} - \mu_i^{\dot{\alpha}}\frac{\partial}{\partial\mu_i^{\dot{\alpha}}}\right). \tag{2.41}$$

The special conformal generator can be obtained at the same way. Using $[D, K_{\alpha\dot{\alpha}}] = K_{\alpha\alpha}$, this operator, in terms of twistor variables becomes:

$$K_{\alpha\dot{\alpha}} = -\sum_i \mu_{i\dot{\alpha}}\frac{\partial}{\partial\lambda_i^{\alpha}}. \tag{2.42}$$

Action of (2.37) on (2.36) is obvious and provides us:

$$\begin{aligned} P^{\alpha\dot{\alpha}}\left[\widetilde{\mathcal{A}}(Z_i)\right] &= \frac{i\,(2\pi)^4}{(2\pi)^{2n}}\int\prod_j\left[d^2\widetilde{\lambda}_j\right]\exp\left(\sum\mu_i\widetilde{\lambda}_i\right)\times \\ &\quad\times\left(\sum\lambda_i^{\alpha}\widetilde{\lambda}_i^{\dot{\alpha}}\right)\delta^4\left(\sum\lambda_i^{\alpha}\widetilde{\lambda}_i^{\dot{\alpha}}\right)\mathcal{A}\left(\lambda_i,\widetilde{\lambda}_i\right). \end{aligned}$$

Nevertheless, $x\delta(x)$ acting in any test function gives zero. To prove translation invariance it is not necessary to know the form of $\mathcal{A}(\lambda_i,\widetilde{\lambda}_i)$, because the amplitude is written in a way that momentum conservation is found explicitly. It does not hold to others generators, so, we must determinate which MHV amplitude we will consider. As (2.36) privileges λ instead of $\widetilde{\lambda}$, we ought to focus on amplitude (2.13).

Replacing the delta function by its integral representation and applying (2.40) to this amplitude, we have:

$$
\begin{aligned}
J_{\dot{\alpha}\dot{\beta}}\left[\widetilde{\mathcal{A}}\left(Z_i\right)\right] &= -\frac{i\left(2\pi\right)^4}{\left(2\pi\right)^{2n}}\sum_i \mu_{i(\dot{\alpha}}\int d^4x\prod_{j=1}^{n}\frac{\partial}{\partial\mu_i^{\dot{\beta})}}\delta^2\left(\mu_j^{\dot{\delta}}-ix^{\delta\dot{\delta}}\lambda_{j\delta}\right)\mathcal{A}\left(\lambda_i\right), \\
J_{\dot{\alpha}\dot{\beta}}\left[\widetilde{\mathcal{A}}\left(Z_i\right)\right] &= -\frac{\left(2\pi\right)^4}{2\left(2\pi\right)^{2n}}\sum_i \mu_{i\dot{\alpha}}\int d^4x\prod_{j=1}^{n}\frac{\partial\left[\delta^2\left(\mu_j^{\dot{\delta}}-ix^{\delta\dot{\delta}}\lambda_{j\delta}\right)\right]}{\partial(x^{\alpha\dot{\beta}}\lambda_{i\alpha})}\mathcal{A}\left(\lambda_i\right)+ \\
&\quad +\dot{\alpha} \quad\leftrightarrow\quad \dot{\beta}.
\end{aligned}
$$

Making use of the chain rule and integrating by parts:

$$
\begin{aligned}
J_{\dot{\alpha}\dot{\beta}}\left[\widetilde{\mathcal{A}}\left(Z_i\right)\right] &= -\frac{\left(2\pi\right)^4}{2\left(2\pi\right)^{2n}}\sum_i \mu_{i\dot{\alpha}}\int d^4x\frac{\partial^2 x^{\gamma\dot{\gamma}}}{\partial x^{\gamma\dot{\gamma}}\partial(x^{\alpha\dot{\beta}}\lambda_{i\alpha})}\prod_{j=1}^{n}\delta^2\left(\mu_j^{\dot{\delta}}-ix^{\delta\dot{\delta}}\lambda_{j\delta}\right)\mathcal{A}\left(\lambda_i\right)+ \\
&\quad +\dot{\alpha} \quad\leftrightarrow\quad \dot{\beta}.
\end{aligned}
$$

It proves the invariance of the amplitude under the action of Lorentz generators $J_{\dot{\alpha}\dot{\beta}}$, because $\frac{\partial^2 x^{\gamma\dot{\gamma}}}{\partial x^{\gamma\dot{\gamma}}\partial(x^{\alpha\dot{\beta}}\lambda_{i\alpha})}=0$. Doing the same for $J_{\alpha\beta}$, we might note that:

$$
\begin{aligned}
J^{\alpha\beta}\left[\widetilde{\mathcal{A}}\left(Z_i\right)\right] &= \frac{i\left(2\pi\right)^4}{\left(2\pi\right)^{2n}}J^{\alpha\beta}\left[\mathcal{A}\left(\lambda_i\right)\right]\int d^4x\prod_{j=1}^{n}\delta^2\left(\mu_j^{\dot{\delta}}-ix^{\delta\dot{\delta}}\lambda_{j\delta}\right)+ \\
&\quad +\frac{i\left(2\pi\right)^4}{\left(2\pi\right)^{2n}}\mathcal{A}\left(\lambda_i\right)\int d^4x\prod_{j=1}^{n}J^{\alpha\beta}\left[\delta^2\left(\mu_j^{\dot{\delta}}-ix^{\delta\dot{\delta}}\lambda_{j\delta}\right)\right]. \qquad (2.43)
\end{aligned}
$$

Considering just the fist term for a while, we have:

$$
J^{\alpha\beta}\left[\mathcal{A}\left(\lambda_i\right)\right]=-\mathcal{A}\left(\lambda_i\right)\left\{\sum_i\left[\frac{\lambda_i^{(\alpha}\lambda_{i+1}^{\beta)}}{(\lambda_i\lambda_{i+1})}+\frac{\lambda_i^{(\alpha}\lambda_{i-1}^{\beta)}}{(\lambda_i\lambda_{i-1})}\right]-4\frac{\lambda_r^{(\alpha}\lambda_s^{\beta)}}{(\lambda_r\lambda_s)}-4\frac{\lambda_r^{(\alpha}\lambda_s^{\beta)}}{(\lambda_s\lambda_r)}\right\},
$$

which clearly vanishes. Now, we must prove the second term on (2.43) is zero.

$$
\begin{aligned}
\int d^4x\prod_{j=1}^{n}J^{\alpha\beta}\left[\delta^2\left(\mu_j^{\dot{\delta}}-ix^{\delta\dot{\delta}}\lambda_{j\delta}\right)\right] &= \sum_i\lambda_i^{(\alpha}\int d^4x\prod_{j=1}^{n}x^{\beta)\dot{\gamma}}\frac{\partial\left[\delta^2\left(\mu_j^{\dot{\delta}}-ix^{\delta\dot{\delta}}\lambda_{j\delta}\right)\right]}{\partial(x^{\gamma\dot{\gamma}}\lambda_{i\gamma})}, \\
&= -\sum_i i\frac{\partial}{\partial\mu_i^{\dot{\gamma}}}\lambda_i^{(\alpha}\int d^4x\prod_{j=1}^{n}x^{\beta)\dot{\gamma}}\delta^2\left(\mu_j^{\dot{\delta}}-ix^{\delta\dot{\delta}}\lambda_{j\delta}\right)
\end{aligned}
$$

It show us the amplitude is Poincaré invariant, as we have already known. In order to check the amplitude dilatation symmetry we must note that:

$$D = -\sum_i \left(h_i + \lambda_i^\alpha \frac{\partial}{\partial \lambda_i^\alpha} \right) - n = -\sum_i \left(\lambda_i^\alpha \frac{\partial}{\partial \lambda_i^\alpha} \right) - 2\left(n-2\right). \tag{2.44}$$

And, using:

$$\sum_i \left(\lambda_i^\alpha \frac{\partial}{\partial \lambda_i^\alpha} \right) \mathcal{A}\left(\lambda\right) = -2\left(n-4\right)\mathcal{A}\left(\lambda\right),$$

we have that the action of the dilatation operator on $\widetilde{\mathcal{A}}\left(Z_i\right)$ provides:

$$\begin{aligned} D\left[\widetilde{\mathcal{A}}\left(Z\right)\right] &= \frac{i\left(2\pi\right)^4}{\left(2\pi\right)^{2n}} \mathcal{A}\left(\lambda\right) \sum_i \int d^4x \prod_{j=1}^n \left(\lambda_i^\alpha \frac{\partial}{\partial \lambda_i^\alpha} \right) \left[\delta^2 \left(\mu_j^{\dot{\delta}} - i x^{\delta\dot{\delta}} \lambda_{j\delta} \right) \right] + 4\widetilde{\mathcal{A}}\left(Z\right), \\ D\left[\widetilde{\mathcal{A}}\left(Z\right)\right] &= \frac{i\left(2\pi\right)^4}{\left(2\pi\right)^{2n}} \mathcal{A}\left(\lambda\right) \sum_i \int d^4x \left[x^{\beta\dot{\alpha}} \lambda_{i\beta} \frac{\partial x^{\gamma\dot{\gamma}}}{\partial\left(x^{\alpha\dot{\alpha}}\lambda_{i\alpha}\right)} \right] \times \\ &\quad \times \prod_{j=1}^n \frac{\partial \left[\delta^2 \left(\mu_j^{\dot{\delta}} - i x^{\delta\dot{\delta}} \lambda_{j\delta} \right) \right]}{\partial x^{\gamma\dot{\gamma}}} + 4\widetilde{\mathcal{A}}\left(Z\right). \end{aligned}$$

Integrating by parts, we see that $D[\widetilde{\mathcal{A}}\left(Z\right)] = 0$. To conclude this analysis, we must look at special conformal generators acting on this amplitude.

$$\begin{aligned} K_{\alpha\dot{\alpha}}\left[\widetilde{\mathcal{A}}\left(Z\right)\right] &= \frac{i\left(2\pi\right)^4}{\left(2\pi\right)^{2n}} \int d^4x K_{\alpha\dot{\alpha}}\left[\mathcal{A}\left(\lambda_i\right)\right] \prod_{j=1}^n \delta^2 \left(\mu_j^{\dot{\delta}} - i x^{\delta\dot{\delta}} \lambda_{j\delta} \right) + \\ &\quad + \frac{i\left(2\pi\right)^4}{\left(2\pi\right)^{2n}} \mathcal{A}\left(\lambda_i\right) \int d^4x \prod_{j=1}^n K_{\alpha\dot{\alpha}} \left[\delta^2 \left(\mu_j^{\dot{\delta}} - i x^{\delta\dot{\delta}} \lambda_{j\delta} \right) \right]. \end{aligned}$$

Focusing on the first term:

$$K^{\alpha\dot{\alpha}}\left[\mathcal{A}\left(\lambda_i\right)\right] = \mathcal{A}\left(\lambda_i\right) \left\{ \sum_i \left[\frac{\mu_i^{\dot{\alpha}} \lambda_{i+1}^\alpha}{\left(\lambda_i \lambda_{i+1}\right)} + \frac{\mu_i^{\dot{\alpha}} \lambda_{i-1}^\alpha}{\left(\lambda_i \lambda_{i-1}\right)} \right] - 4 \frac{\mu_r^{(\dot{\alpha}} \lambda_s^{\alpha)}}{\left(\lambda_r \lambda_s\right)} - 4 \frac{\lambda_r^{(\alpha} \mu_s^{\beta)}}{\left(\lambda_s \lambda_r\right)} \right\}.$$

Since it can be put inside the integral, using the delta function, we can replace $\mu_j^{\dot{\delta}}$ by $i x^{\delta\dot{\delta}} \lambda_{j\delta}$. From identity $\epsilon^{\alpha\beta}\left(\lambda_i \lambda_j\right) = \lambda_i^\beta \lambda_j^\alpha - \lambda_i^\alpha \lambda_j^\beta$, we find:

$$K^{\alpha\dot{\alpha}}\left[\mathcal{A}\left(\lambda_i\right)\right] = -4 i x^{\alpha\dot{\alpha}} \mathcal{A}\left(\lambda_i\right).$$

Making use of the same trick to second term:

$$K_{\alpha\dot{\alpha}}\left[\delta^2\left(\mu_j^{\dot{\delta}} - ix^{\delta\dot{\delta}}\lambda_{j\delta}\right)\right] = -ix_{\beta\dot{\alpha}}\lambda^{\beta}x_{\alpha\dot{\beta}}\frac{\partial x^{\gamma\dot{\gamma}}}{\partial\left(x_{\eta\dot{\beta}}\lambda^{\eta}\right)}\frac{\partial}{\partial x^{\gamma\dot{\gamma}}}\left[\delta^2\left(\mu_j^{\dot{\delta}} - ix^{\delta\dot{\delta}}\lambda_{j\delta}\right)\right].$$

After integration by parts, one can show that $K_{\alpha\dot{\alpha}}[\widetilde{\mathcal{A}}(Z)] = 0$ and the amplitude is conformal invariant as we expected.

Despite of the beauty of its geometrical structure, twistor theory must bring us simplicity to be kept as a useful tool to study Yang-Mills, or, mainly, maximally supersymmetric Yang-Mills theory, since this theory is superconformal. In order to provide this simplicity, we must observe that, in working on twistor space, we break the symmetry between positive and negative helicities, because we are privileging a chiral representation instead of the anti-chiral one. However, if we switch the analysis to dual twistor space, we reverse things in such way that functions on it do not have homogenous degree $(-2h-2)$ anymore, but $(2h-2)$. If we consider holomorphic parts to describe particles of helicity minus one and antiholomorphic parts to helicity plus one, the result function has vanishing homogenous degree.

Regarding the four gluon amplitude (2.15) and Fourier transforming negative helicities to twistor space and positive ones to dual twistor space, we have:

$$\begin{aligned}\widetilde{\mathcal{A}}\left(Z_1, Z_2, \widetilde{Z}_3, \widetilde{Z}_4\right) &= \frac{ig^2}{(2\pi)^4}\int_{\mathbb{R}^{12}} d^4x d^2\widetilde{\lambda}_1 d^2\widetilde{\lambda}_2 d^2\lambda_3 d^2\lambda_4 (\lambda_1\lambda_2)^3 \times \\ &\times \frac{e^{\left[\left(\mu_1\widetilde{\lambda}_1\right)+\left(\mu_2\widetilde{\lambda}_2\right)+(\lambda_3\widetilde{\mu}_3)+(\lambda_4\widetilde{\mu}_4)+ix_{\alpha\dot{\alpha}}\sum\lambda_i^{\alpha}\widetilde{\lambda}_i^{\dot{\alpha}}\right]}}{(\lambda_2\lambda_3)(\lambda_3\lambda_4)(\lambda_4\lambda_1)}.\end{aligned}$$

As we are dealing with two-dimensional vector spaces, we can expand λ_3, λ_4 in terms of λ_1, λ_2 and integrate over these coefficients.

$$\begin{aligned}\lambda_3 &= a_3\lambda_1 + b_3\lambda_2, \\ \lambda_4 &= a_4\lambda_1 + b_4\lambda_2.\end{aligned}$$

Doing the same thing to $\widetilde{\lambda}_1$ and $\widetilde{\lambda}_2$, we have that, in terms of $\widetilde{\lambda}_3$ and $\widetilde{\lambda}_4$, they are given as:

$$\begin{aligned}\widetilde{\lambda}_1 &= \widetilde{a}_1\widetilde{\lambda}_3 + \widetilde{b}_1\widetilde{\lambda}_4, \\ \widetilde{\lambda}_2 &= \widetilde{a}_2\widetilde{\lambda}_3 + \widetilde{b}_2\widetilde{\lambda}_4.\end{aligned}$$

Under these transformations, the measure becomes:

$$d^2\widetilde{\lambda}_1 d^2\widetilde{\lambda}_2 d^2\lambda_3 d^2\lambda_4 = [(\lambda_1\lambda_2)(\widetilde{\lambda}_3\widetilde{\lambda}_4)]^2\ d^2a\ d^2b\ d^2\widetilde{a}\ d^2\widetilde{b}.$$

Substituting these into the amplitude and integrating over $d^2\widetilde{a}\ d^2\widetilde{b}$, we get:

$$\begin{aligned}\widetilde{\mathcal{A}}\left(Z_1, Z_2, \widetilde{Z}_3, \widetilde{Z}_4\right) &= \frac{ig^2}{(2\pi)^4}\int_{\mathbb{R}^4} d^2a\ d^2b \frac{e^{\left[a_3\left(Z_1\widetilde{Z}_3\right)+b_3\left(Z_2\widetilde{Z}_3\right)+a_4\left(Z_1\widetilde{Z}_4\right)+b_4\left(Z_2\widetilde{Z}_4\right)\right]}}{a_3b_4\left(a_3b_4-b_3a_4\right)}\times \\ &\quad \times\int_{\mathbb{R}^4} d^4x\delta[(\mu_1\widetilde{\lambda}_3)+i(\lambda_1x\widetilde{\lambda}_3)]\delta[(\mu_1\widetilde{\lambda}_4)+i(\lambda_1x\widetilde{\lambda}_4)]\times \\ &\quad \times\delta[(\mu_2\widetilde{\lambda}_3)+i(\lambda_2x\widetilde{\lambda}_3)]\delta[(\mu_2\widetilde{\lambda}_4)+i(\lambda_2x\widetilde{\lambda}_4)][(\widetilde{\lambda}_3\widetilde{\lambda}_4)(\lambda_1\lambda_2)]^2.\end{aligned}$$

However, to solve the integral on x, we might note that:

$$d\left(\lambda_1x\widetilde{\lambda}_4\right)d\left(\lambda_1x\widetilde{\lambda}_4\right)d\left(\lambda_2x\widetilde{\lambda}_3\right)d\left(\lambda_2x\widetilde{\lambda}_4\right) = \left[(\widetilde{\lambda}_3\widetilde{\lambda}_4)(\lambda_1\lambda_2)\right]^2 d^4x.$$

Integrating over the rest of the variables and using that $sgn\,(y) = \int_{\mathbb{R}} \frac{da}{a}\exp\,(iay)$, we find the astonishing result from four-gluon scattering amplitude:

$$\widetilde{\mathcal{A}}\left(Z_1, Z_2, \widetilde{Z}_3, \widetilde{Z}_4\right) = \frac{ig^2}{(2\pi)^4} sgn\left[\left(Z_1\widetilde{Z}_3\right)\left(Z_2\widetilde{Z}_3\right)\left(Z_1\widetilde{Z}_4\right)\left(Z_2\widetilde{Z}_4\right)\right], \tag{2.45}$$

which, of course, provides an amazing simple result in comparison to result obtained by the conventional techniques. This intricate space in which we are considering so twistor space as its dual is called ambitwistor space.

2.4 Supersymmetric Extension to Twistor Space

It would be interesting to recover space-time field from a function on twistor space. According to previous considerations, a particle with helicity h must be associated to a homogenous holomorphic function of degree $(-2h-2)$ on twistor space. In order to obtain a space-time scalar field, we must note that the x-dependence comes from incident relations and to have a field with homogenous degree zero we must integrate over λ's. If we consider signature $(+---)$, the integral cannot be an integral over all domain, because the later is $\mathbb{C}^2$ and we cannot cover all domain integrating just on λ. However, we are still able to define a line integral in $\mathbb{C}^2$ and its Lorentz invariant measure must be $\lambda^\alpha d\lambda_\alpha$. By the homogenous degree of the integrand, we might infer that it has at least has two poles, then, this integral must be a contour integral over one of these poles:

$$\phi(x) = \frac{1}{2\pi i}\oint_{\mathcal{C}} \lambda^{\alpha} d\lambda_{\alpha} f(Z). \tag{2.46}$$

Being more careful and specific, we shall define this contour in a more appropriated manner. Since we are avoiding the situation in which secondary spinor part of the twistor vanishes, that is, $\lambda = 0$, we can, for example, restrict ourselves to subspace of $\mathbb{CP}^3$ where $\lambda^1 \neq 0$, and switch to inhomogenous coordinates $\left(\frac{\mu_{\dot{1}}}{\lambda_1}, \frac{\mu_{\dot{2}}}{\lambda_1}, 1, \frac{\lambda_2}{\lambda_1}\right)$. Doing this, we can see that the domain of λ is not more $\mathbb{C}^2$, but $\mathbb{CP}^1$. The contour $\mathcal{C}$ must be diffeomorphic to $\mathbb{S}^1$ and belong to $\mathbb{CP}^1$. Because $\mathbb{CP}^1 \cong \mathbb{S}^2$, we know the domain cannot be entirely covered by just one chart, we need at least two chats in order to do so. Then, we can manage to put each pole in different hemispheres of the two-sphere and set the contour to be the equator, by appropriate choice of $\mathbb{CP}^1$ coordinate set.

In fact, from these results, we are able to prove that a space-time point is associated to a Riemann sphere on projective twistor space in a more formal way. As stated before, a set of intersecting light rays determine a point on space time. Considering just light rays which belong to non-compactified Minkowski space, what means $\lambda \neq 0$, we can define inhomogenous coordinates at $\mathbb{CP}^3$ as above. However, light rays must satisfy incident relations, so, a point in space-time is determined just by λ up to a scaling in twistor space. It proves that x is mapped to $\mathbb{CP}^1$ in $\mathbb{CP}^3$.

We can observe that a scalar field defined by (2.46) satisfies massless Klein-Gordon equation.

$$\partial^{\alpha\dot{\alpha}}\partial_{\alpha\dot{\alpha}}\phi(x) = \frac{1}{2\pi i}\oint_{\mathcal{C}} \lambda^{\alpha} d\lambda_{\alpha} (\lambda\lambda) \frac{\partial^2 f(Z)}{\partial\mu_{\dot{\alpha}}\partial\mu^{\dot{\alpha}}} = 0.$$

The extension to other fields is rather straightforward. For negative helicities, $f(Z)$ have homogenous degree $(-2h-2)$, so, to integrand have homogenous degree -2, we must have:

$$\phi_{\alpha_1\alpha_2\ldots\alpha_{2h}}(x) = \frac{1}{2\pi i}\oint_{\mathcal{C}} \lambda^{\beta} d\lambda_{\beta} \lambda_{\alpha_1}\lambda_{\alpha_2}\ldots\lambda_{\alpha_{2h}} f(Z). \tag{2.47}$$

Obviously, equation of motion to this field is:

$$\partial^{\alpha_1\dot{\alpha}}\phi_{\alpha_1\alpha_2\ldots\alpha_{2h}}(x) = 0.$$

To positive helicity fields, we must increase the degree of homogeneous function, so:

$$\phi_{\dot{\alpha}_1\dot{\alpha}_2\ldots\dot{\alpha}_{2h}}(x) = \frac{1}{2\pi i}\oint_{\mathcal{C}} \lambda^{\alpha} d\lambda_{\alpha} \frac{\partial}{\partial\mu^{\dot{\alpha}_1}}\frac{\partial}{\partial\mu^{\dot{\alpha}_2}}\ldots\frac{\partial}{\partial\mu^{\dot{\alpha}_{2h}}} f(Z). \tag{2.48}$$

The same form, equation of motion to this field is:

$$\partial^{\alpha\dot{\alpha}_1}\phi_{\dot{\alpha}_1\dot{\alpha}_2\ldots\dot{\alpha}_{2h}}(x) = 0.$$

A further remark on Maxwell field strength or Yang-Mills one in perturbative regimen is required. The field strength is written as $F = dA$ and it becomes $f = p \wedge \varepsilon$ in momentum space. Translating to spinor form, we split the field strength into a self-dual and an anti-self-dual part. The self-dual represents the negative helicity gluon and the anti-self-dual the positive helicity gluon. We can see this from $f = p \wedge \varepsilon$ and consider either $\varepsilon = \varepsilon^{+}$ or $\varepsilon = \varepsilon^{-}$. To former, we get $f_{\alpha\dot{\alpha}\beta\dot{\beta}} \propto \epsilon_{\alpha\beta}\widetilde{\lambda}_{\dot{\alpha}}\widetilde{\lambda}_{\dot{\beta}}$ and, equivalently, to later, the result is $f_{\alpha\dot{\alpha}\beta\dot{\beta}} \propto \epsilon_{\dot{\alpha}\dot{\beta}}\lambda_{\alpha}\lambda_{\beta}$. So, the entire field strength must be expressed as a direct sum of them:

$$F_{\alpha\dot{\alpha}\beta\dot{\beta}} = \epsilon_{\alpha\beta}\tilde{\Phi}_{\dot{\alpha}\dot{\beta}} + \epsilon_{\dot{\alpha}\dot{\beta}}\Phi_{\alpha\beta}. \tag{2.49}$$

Twistor formalism segregate different helicities and it is suitable to theories that naturally distinct chiral parts from antichiral ones like supersymmetric theories. As it is well-known, supersymmetry is a symmetry that relates fermions and bosons. The generators of these transformations take bosons in fermions and vice-versa. The algebra associated to them satisfies the following anticommutation rules:

$$\begin{aligned} \left\{Q^{I}_{\alpha}, \overline{Q}_{\dot{\alpha}J}\right\} &= 2\sigma^{\nu}_{\alpha\dot{\alpha}}p_{\nu}\delta^{I}_{J}, && (2.50)\\ \left\{Q^{I}_{\alpha}, Q^{J}_{\beta}\right\} &= \left\{\overline{Q}_{\dot{\alpha}I}, \overline{Q}_{\dot{\beta}J}\right\} = 0. && (2.51) \end{aligned}$$

Indices I, J go from 1 to $\mathcal{N}$ and indicate the number of supersymmetries the theory have. In order to focus on representations of this supersymmetric algebra corresponding to massless one-particle state, we will start by setting, for convenience, the space-time coordinates in such way that momentum becomes $(E, 0, 0, -E)$. Using this on anticommutation rules, we find the unique nonvanishing anticommutator is:

$$\left\{Q^{I}_{1}, \overline{Q}_{\dot{1}J}\right\} = 4E\delta^{I}_{J}. \tag{2.52}$$

However, it is nothing but $\mathcal{N}$ copies of a Clifford algebra we are used to when considering fermion canonical quantization. Defining:

$$\begin{aligned} a^{I} &= \frac{1}{2\sqrt{E}}Q^{I}_{1}, && (2.53)\\ \left(a^{I}\right)^{\dagger} &= \frac{1}{2\sqrt{E}}\overline{Q}_{\dot{1}I}, && (2.54) \end{aligned}$$

we have a "creation and annihilation" operator algebra, so, it must have an state $|\Omega\rangle$ that is annihilated by any of $\left(a^I\right)^\dagger$. The same form, there must be a state $|\Omega^*\rangle$ that is annihilated by every a^I.

All possible one-particle states are obtained from the action of these operators on both sates, that is, $|\Omega\rangle$, $a^I\,|\Omega\rangle$,..., $a^1...a^{\mathcal{N}}\,|\Omega\rangle$ and $|\Omega^*\rangle$, $(a^I)^\dagger\,|\Omega^*\rangle$,..., $(a^1)^\dagger...(a^{\mathcal{N}})^\dagger\,|\Omega^*\rangle$. As a^I and $(a^I)^\dagger$ take bosons in fermions and vice-versa, the action of a^I in a state must raise the helicity by 1/2 and the action of $(a^I)^\dagger$ must lower it by 1/2. It infers that $|\Omega\rangle$ as well as $|\Omega^*\rangle$ are eigenstates of helicity. If $|\Omega\rangle$ has helicity $-h$, we will have $\mathcal{N}$ states of helicity $-h+1/2$, $\mathcal{N}(\mathcal{N}-1)/2$ states of helicity $-h+1$ and so forth. Generally, to helicity $-h+p/2$, we find $\binom{\mathcal{N}}{p}$ states. On the other hand, by CPT invariance, $|\Omega^*\rangle$ must have helicity h, and to helicity $h-p/2$, we find the same $\binom{\mathcal{N}}{p}$ states. It is straightforward to note that when $h=\mathcal{N}/4$ we cannot split the states in a chiral multiplet and in an anti-chiral one. In fact, we can identify $a^1...a^{\mathcal{N}}\,|\Omega\rangle$ to $|\Omega^*\rangle$, because they have the same helicity. This forces us to consider all particles in the same multiplet. So, maximally supersymmetric Yang-Mills theory has $\mathcal{N}=4$ supersymmetries. By the way, this theory has great implications in Superstring theory and we will consider that until the end of this chapter.

From the considerations above, we can say that we are able to obtain all helicities through $h=-1$ in $\mathcal{N}=4$ supersymmetric Yang-Mills. As we put all this fields in a multiplet, we need label the action of a^I by a fermionic parameter. Denoting this Grassmanian parameter as η_I, we are able to define the multiplet as:

$$F = a_- + \eta_J \bar{s}^J + \eta_J\eta_K\phi^{JK} + \frac{\epsilon^{IJKL}}{3!}\eta_J\eta_K\eta_L s_I + \eta_1\eta_2\eta_3\eta_4 a_+. \tag{2.55}$$

Quantities a_- and a_+ are the self-dual and anti-self-dual gluon respectively. The fermions $\bar{s}^J$, s_J are $(-1/2)$-helicity and $(1/2)$-helicity gluinos, and ϕ^{JK} are the six scalars. Then, we can guess the helicity operator form to this theory:

$$h = -1 + \frac{1}{2}\sum_I \eta_I \frac{\partial}{\partial \eta_I}. \tag{2.56}$$

Clearly, it includes a dependence of η_I in $f(Z)$. This heuristic procedure shows us that, in the $\mathcal{N}=4$ supersymmetric Yang-Mills, the formalism of twistor must be extended to supertwistors, whose are defined by such a quantity:

$$\mathcal{Z}^{A'} \equiv \begin{pmatrix} \mu_{\dot{\alpha}} \\ \lambda^{\alpha} \\ \eta_I \end{pmatrix}. \tag{2.57}$$

The set of all these supertwistors introduces the supertwistor space $\widehat{\mathbb{T}} = \mathbb{C}^{4|4}$

with for bosonic coordinates (Z^A) and four fermionic ones (η_I). Yet, for the same arguments we provided in the discussion about twistor space, it becomes convenient to consider, instead of $\widehat{\mathbb{T}}$, the projective supertwistor space $\widehat{\mathbb{PT}}$, defined by the equivalence relation $(Z^A, \eta_I) \sim (tZ^A, t\eta_I)$ for nonzero complex t. Thus, the projective supertwistor space is a copy of the supermanifold $\mathbb{CP}^{3|4}$.

Helicity operator in $\mathbb{CP}^{3|4}$ can be written by (2.35) as well and it constrains the form of functions on supertwistor space:

$$\mathcal{Z}_i^{A'} \frac{\partial}{\partial \mathcal{Z}_i^{A'}} \hat{f}\left(\mathcal{Z}^{A'}\right) = 0. \tag{2.58}$$

In other words, we might say that for each external particle the scattering amplitude is homogenous in supertwistor coordinates.

The conformal generators remain unchanged in this extended space, however, in addition, there are sixteen supersymmetry generators (2.50). The term $\frac{1}{2}\sum \eta_I \frac{\partial}{\partial \eta_I}$ in helicity operator means sort of a number operator, i.e., $\sum (a^I)^\dagger (a^I)$. Thus, we are able to associate $\frac{\partial}{\partial \eta_I}$ to (a^I) and η_I to $(a^I)^\dagger$. From (2.53) and (2.54), we can obtain the form of supersymmetry generator in supertwistor space:

$$Q^I_\alpha = \lambda_\alpha \frac{\partial}{\partial \eta_I}, \tag{2.59}$$

$$\overline{Q}_{\dot{\alpha} I} = -\eta_I \frac{\partial}{\partial \mu^{\dot{\alpha}}}. \tag{2.60}$$

But, the superconformal algebra is not closed if we include just these generators. We must add more sixteen fermionic ones that come from $[K, Q]$ and $[K, \overline{Q}]$:

$$S_{\alpha I} = \eta_I \frac{\partial}{\partial \lambda^\alpha}, \tag{2.61}$$

$$\overline{S}^I_{\dot{\alpha}} = \mu_{\dot{\alpha}} \frac{\partial}{\partial \eta_I}. \tag{2.62}$$

Finally, the last set of generators are the ones which provide the R-symmetry. In fact, this $\mathcal{N} = 4$ super Yang-Mills comes from a $\mathcal{N} = 1$ super Yang-Mills in ten dimensions. Nevertheless, before compactification, the fields had been invariant under $SO(9,1)$ and, after compactification, the $SO(6)$ symmetry arising for six-dimensional compactified manilfold before compactification appears as an internal symmetry and, since $SO(6) \cong SU(4)$, the R-symmetry becomes a global $SU(4)$ symmetry. Generators of $SU(4)$ are traceless, so we complete the set of superconformal generator with:

$$R_I^J = \eta_I \frac{\partial}{\partial \eta_J} - \frac{1}{4}\delta_I^J \eta_K \frac{\partial}{\partial \eta_K}. \tag{2.63}$$

All this analysis shows us that we do not need to focus on the external particles by themselves, but, rather, we can consider the multiplet in a condensed form. So, MHV amplitudes to this Yang-Mills theory takes the following form:

$$\widehat{\mathcal{A}} = ig^{n-2}(2\pi)^4 \delta^4\left(\sum \lambda_i^\alpha \widetilde{\lambda}_i^{\dot{\alpha}}\right)\delta^8\left(\sum \lambda_i^\alpha \widetilde{\eta}_i^I\right)\prod_i \frac{1}{(\lambda_i \lambda_{i+1})}. \tag{2.64}$$

The quantity $\widetilde{\eta}_i^I$ is the eigenvalue of operator $\frac{\partial}{\partial \eta_I}$. We recover (2.13) after integration over $d^4\eta_r d^4\eta_s$, when r and s label the self-dual gluons. Moving to signature $(++--)$ in order to properly define the Fourier transform, we find the amplitude in supertwistor space by:

$$\widetilde{\mathcal{A}}(\mathcal{Z}) = \int \prod_i \left[\frac{d^2\widetilde{\lambda}_i d^4\widetilde{\eta}_i}{(2\pi)^2}\right] \exp\left[\sum_i \left(\mu_i \widetilde{\lambda}_i + \eta_{iI}\widetilde{\eta}_i^I\right)\right]\widehat{\mathcal{A}}\left(\lambda, \widetilde{\lambda}, \widetilde{\eta}\right). \tag{2.65}$$

Using the integral form of deltas functions, we find:

$$\widetilde{\mathcal{A}}(\mathcal{Z}) = ig^{n-2}\int d^4y d^8\theta \prod_{i=1}^n \left[\frac{\delta^2\left(\mu_i^{\dot{\alpha}} - iy^{\alpha\dot{\alpha}}\lambda_{i\alpha}\right)\delta^4\left(\eta_{iI} + \theta_{\alpha I}\lambda_i^\alpha\right)}{(\lambda_i \lambda_{i+1})}\right]. \tag{2.66}$$

It provides the incident relations on supertwistor space. However, since we are in a chiral representation, $y_{\alpha\dot{\alpha}}$ must not be identified with space-time point $x_{\alpha\dot{\alpha}}$ itself, but it translated from $i\theta_{\alpha I}\overline{\theta}^I_{\dot{\alpha}}$. It is easy to see this when we consider Q^I_α in $\left(y, \theta, \overline{\theta}\right)$ representation:

$$Q^I_\alpha = \frac{\partial}{\partial \theta^\alpha_I}. \tag{2.67}$$

By $\eta_{iI} = \theta^\alpha_I \lambda_{i\alpha}$, we recover (2.59).

Equivalently as before, we can get the superfield from a function on a supertwistor space ($\mathbb{RP}^{3|4}$ or $\mathbb{CP}^{3|4}$), using the incident relations and integrating over λ:

$$\Phi\left(y, \theta, \overline{\theta}\right) = \oint_{\mathcal{C}} \frac{\lambda^\alpha d\lambda_\alpha}{(2\pi)^2}\tilde{F}\left(\lambda^\gamma, iy^{\alpha\dot{\alpha}}\lambda_\alpha, \theta_I^\beta \lambda_\beta\right). \tag{2.68}$$

Chapter 3

Proving Parke and Taylor Proposition

In the last chapter, we assumed MHV formula to be true without prove. In this chapter, we will prove this conjecture following Brito, Cachazo, Feng and Witten's proof [7], provided in 2005. This demonstration is based on a recursion relation found by the former three authors [6] in 2004.

After proved Parke and Taylor proposition to Yang-Mills theory, we will focus on properties that arise when a theory satisfies BCFW recursion relations, studying its vertices and checking its consistency conditions. Some remarks on $\mathcal{N} = 4$ super-Yang-Mills theories are considered.

A general proof to MHV formula in maximally supersymmetric Yang-Mills theory is not given, because when consider MHV amplitude with gluons on the external legs, the demonstration is the same to the theory with no supersymmetries. Differences appears when we consider external legs of gluinos or scalar fields, however a comment on them will be given.

3.1 BCFW Recursion Relation and Cauchy Theorem

All analysis of last chapter holds when gluon momenta are lightlike, however, it is not true at all, because we can find virtual gluon propagators that obviously do not satisfies this condition. It seems to be an obstacle in treating gluon scattering at such way. An interesting manner to handle this is to allow gluon momentum to be complex, but still lightlike. When working on complexified Minkoswki space-time, λ and $\widetilde{\lambda}$ are independent, what permits propagator momenta to be on shell. To make this argument clear, let the S-matrix initial state be two gluons with momenta p_1 and p_2, for example. Forgetting the contribution of the four-gluon vertex, we can see that the three-gluon vertex propagator must carry momentum $p_3 = -(p_1 + p_2)$. On shell conditions state $p_1^2 = p_2^2 = p_3^2 = 0$, what, in terms of spinors, can be written as:

$$(\lambda_1\lambda_2)(\widetilde{\lambda}_1\widetilde{\lambda}_2) = (\lambda_2\lambda_3)(\widetilde{\lambda}_2\widetilde{\lambda}_3) = (\lambda_3\lambda_1)(\widetilde{\lambda}_3\widetilde{\lambda}_1) = 0. \tag{3.1}$$

However, if $\widetilde{\lambda}_1$ and $\widetilde{\lambda}_3$ are proportional to $\widetilde{\lambda}_2$, they must be proportional to each other as well. In complexified Minkowski space, we can have two non-trivial solutions to equation (3.1), either $(\lambda_i\lambda_j)$ vanishes or $(\widetilde{\lambda}_i\widetilde{\lambda}_j)$ does.

In order to propagator have complex momentum, we must analytically continue the momenta of, at least, two gluons in scattering amplitude. A way to do so is setting p_r and p_s to $p_r + zq$ and $p_s - zq$, respectively; with q being a lightlike vector, z a complex number and:

$$p_r \cdot q = p_s \cdot q = 0.$$

Clearly, it satisfies momentum conservation. This analytical continuation can be reached by choosing:

$$\lambda_r \rightarrow \lambda_r + z\lambda_s, \tag{3.2}$$
$$\widetilde{\lambda}_s \rightarrow \widetilde{\lambda}_s - z\widetilde{\lambda}_r. \tag{3.3}$$

This new amplitude will depend on z and we recover the amplitude we are looking for when this complex number are set to zero. However, we can obtain $\mathcal{A}(z=0)$ by the residue of the function $z^{-1}\mathcal{A}(z)$. If we suppose that:

$$\lim_{|z|\to\infty} \mathcal{A}(z) = 0,$$

we find:

$$\oint_{\mathcal{C}} \frac{\mathcal{A}(z)}{z} dz = 0, \tag{3.4}$$

when the contour goes to infinity. Using Cauchy theorem, this integral is also equal to the sum of the integrand residues. It is straightforward to observe that poles of this function come from propagators having z-dependence. Because q is lightlike, one can see that these poles are single poles.

In principle, we are able to split this scattering in two parts and, in order to propagator carries z-dependence, each part must have one particle with complex momentum. So, the amplitude acquires the following form:

$$\mathcal{A}(z) = \mathcal{A}_L(z)\frac{1}{P_L^2(z)}\mathcal{A}_R(z).$$

In above expression $P_L(z) = p_L + zq$ is the propagator momentum, $\mathcal{A}_L(z)$ and $\mathcal{A}_R(z)$ are the amplitudes of the left and the right diagram after the cut, respectively. From (3.4), we have:

$$\mathcal{A}(0) = -\sum_{poles}\lim_{z\to z_L}\left[\mathcal{A}_L(z)\frac{1}{zP_L^2(z)}\mathcal{A}_R(z)(z-z_L)\right]. \tag{3.5}$$

Poles are placed in $z_L = -\frac{p_L^2}{2p_L\cdot q}$. Then, we find the BCFW recursion relation:

$$\mathcal{A}(0) = \sum_{poles}\left[\mathcal{A}_L(z_L)\frac{1}{p_L^2}\mathcal{A}_R(z_L)\right]. \tag{3.6}$$

It is interesting to note that we found a recursion relation between tree level amplitudes.

3.2 Helicity Dependent Yang-Mills Vertices

In order to prove MHV formula using the BCFW recursion relation, we must know the form of helicity dependent Yang-Mills vertices. Since in signature $(++--)$ the momentum spinors are independent, it becomes convenient to work in this signature, otherwise, we must complexify the space-time in the same way as we did in the last section. In Feynman gauge, the action to pure Yang-Mills theory is:

$$S = -\frac{1}{4}\int tr\,(F\wedge *F) - \frac{1}{2}\int tr\,[(d*A)\wedge *(d*A)], \tag{3.7}$$

where $F = dA - igA\wedge A$ and $A = A_\mu^a T^a dx^\mu$. In terms of components, the three-gluon vertex becomes:

$$\lim_{x',x''\to x}\frac{\delta^3 S}{\delta A_\mu^a(x)\,\delta A_\nu^b(x')\,\delta A_\sigma^c(x'')} = ig\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}\right)\int d^4x'\partial_\rho\delta^4(x-x')\,tr\left(T^aT^bT^c\right).$$

However, Lorentz indices are not observables, so, we must contract this vertex with the gluon helicity vectors:

$$ig\left[(\varepsilon_1\cdot p_3)(\varepsilon_2\cdot\varepsilon_3) - (\varepsilon_2\cdot p_3)(\varepsilon_1\cdot\varepsilon_3)\right]tr\left(T^aT^bT^c\right).$$

Doing the same to four-gluon vertex, we find:

$$g^2 \left[(\varepsilon_1 \cdot \varepsilon_3) (\varepsilon_2 \cdot \varepsilon_4) - (\varepsilon_2 \cdot \varepsilon_3) (\varepsilon_1 \cdot \varepsilon_4) \right] tr \left(T^a T^b T^c T^d \right) .$$

Let us focus on the last expression for a while. If we consider all gluons with the same helicity, it is easy to see that this vertex vanishes. By (2.7), we are free to choose helicity spinors ε_α, $\widetilde{\varepsilon}_{\dot{\alpha}}$. Choosing all of them equal, helicity vectors will be proportional to each other and the vertex clearly vanishes. The same analysis holds when we have three gluons carrying the same helicity and one carrying opposite helicity.

To vertices with two anti-self-dual gluons and two self-dual ones, we must note that cyclic property of the trace infers:

$$\mathcal{A}(++--) = \mathcal{A}(-++-) = \mathcal{A}(--++) = \mathcal{A}(+--+) .$$

So, we only need to consider $\mathcal{A}(++--)$. Setting $\varepsilon_1^\alpha = \varepsilon_2^\alpha$ and $\widetilde{\varepsilon}_3^{\dot{\alpha}} = \widetilde{\varepsilon}_4^{\dot{\alpha}}$, we can easily see that these vertices vanish. Nevertheless, it is still left to regard $\mathcal{A}(+-+-) = \mathcal{A}(-+-+)$. To see that they are zero, we must choose, for example, $\varepsilon_1^\alpha = \varepsilon_3^\alpha = \lambda_2^\alpha$ and $\widetilde{\varepsilon}_2^{\dot{\alpha}} = \widetilde{\varepsilon}_4^{\dot{\alpha}}$.

Now, we have to ponder about three-gluon vertices. As stated before, to this "amplitude" satisfies momentum conservation and on-shell conditions, either $(\lambda_i \lambda_j)$ or $(\widetilde{\lambda}_i \widetilde{\lambda}_j)$ products vanish. Vertices in which all gluons carries the same helicity, we just have one kind of such products on denominator, so they will not cause problems to us. Assigning $\varepsilon_1^\alpha = \varepsilon_2^\alpha = \varepsilon_3^\alpha$ or $\widetilde{\varepsilon}_1^{\dot{\alpha}} = \widetilde{\varepsilon}_2^{\dot{\alpha}} = \widetilde{\varepsilon}_3^{\dot{\alpha}}$ we find that these kind of vertices are zero.

However, to vertices with two self-dual and one anti-self-dual gluons, or vice-versa, we must be a little bit careful, because we will have so $(\lambda_i \lambda_j)$ as $(\widetilde{\lambda}_i \widetilde{\lambda}_j)$ products in the denominator. We have to impose that these vertices are finite, so, we must choose carefully which of the products can be set to zero in order to satisfies this condition.

Supposing that gluons 2 and 3 have negative helicities and assuming $\widetilde{\varepsilon}_2^{\dot{\alpha}} = \widetilde{\varepsilon}_3^{\dot{\alpha}}$, the amplitude becomes:

$$\begin{aligned}
\mathcal{A}(+--) &= ig\, (\varepsilon_2 \cdot p_1)(\varepsilon_1 \cdot \varepsilon_3) , \\
\mathcal{A}(+--) &= ig \frac{(\varepsilon_1 \lambda_3)(\widetilde{\lambda}_1 \widetilde{\varepsilon}_2)}{(\varepsilon_1 \lambda_1)(\widetilde{\lambda}_3 \widetilde{\varepsilon}_2)} \frac{(\widetilde{\varepsilon}_2 \widetilde{\lambda}_1)(\lambda_2 \lambda_1)}{(\widetilde{\lambda}_2 \widetilde{\varepsilon}_2)} .
\end{aligned}$$

To not have a trivial solution, we must assume $(\widetilde{\lambda}_i \widetilde{\lambda}_j) = 0$, so, we cannot simply assign $\widetilde{\varepsilon}_2 = \widetilde{\lambda}_1$. Setting $\varepsilon_1 = \lambda_2$ and using that:

$$\begin{aligned}
(\lambda_2\lambda_1)\,(\widetilde{\lambda}_1\widetilde{\varepsilon}_2) &= -\,(\lambda_2\lambda_3)\,(\widetilde{\lambda}_3\widetilde{\varepsilon}_2),\\
(\lambda_3\lambda_1)\,(\widetilde{\lambda}_1\widetilde{\varepsilon}_2) &= -\,(\lambda_3\lambda_2)\,(\widetilde{\lambda}_2\widetilde{\varepsilon}_2),
\end{aligned}$$

we get:

$$\mathcal{A}\,(+\,-\,-) = -ig\frac{(\lambda_2\lambda_3)^3}{(\lambda_1\lambda_2)\,(\lambda_3\lambda_1)}. \tag{3.8}$$

The amplitude preserves MHV formula! Then, we can "guess" the amplitude $\mathcal{A}\,(-\,+\,+)$ *, simply replacing λ_i by $\widetilde{\lambda}_i$.

$$\mathcal{A}\,(-\,+\,+) = ig\frac{(\widetilde{\lambda}_2\widetilde{\lambda}_3)^3}{(\widetilde{\lambda}_1\widetilde{\lambda}_2)(\widetilde{\lambda}_3\widetilde{\lambda}_1)}. \tag{3.9}$$

3.3 MHV Formula from BCFW Recursion Relation

We will use BCFW recursion relation in order to prove Parke and Taylor conjecture. First of all, we need to demonstrate that a scattering amplitude with all gluons carrying the same helicity vanishes. In the split diagram, the propagator will have positive helicity in one part and negative in the other by crossing symmetry. Then, one of these graphics will have all gluons carrying the same helicity and the other will have one gluon having opposite helicity from the others. It means that if we prove that an amplitude in which n gluons carry the same helicity vanishes, so the same amplitude with $n+1$ gluons will.

Since the four-gluon vertices are zero, the four-gluon scattering depends just on three-gluon vertices. However, a three-vertex in which all gluons have the same helicity is zero. It proves, by recurrence, that a n-gluon scattering amplitude with all external legs have the same helicity is zero.

The next sort of amplitude we will focus is the amplitude where one gluon have opposite helicity from the others. After the splitting, we have two possibilities, either we have one diagram with all gluon carrying the same helicity and one MHV diagram or two diagrams where we have one gluon with opposite helicity from others. The first possibility vanishes and let us focus on the last possibility.

Before continue, we must remember that to satisfies BCFW recursion relation the analytically continued amplitude have to be zero when the complex parameter goes

*The minus factor drops out because, in the limit of signature $(+\,-\,-\,-)$, $\mathcal{A}\,(+\,-\,-) = [\mathcal{A}\,(-\,+\,+)]^*$.

to infinity. A naïve power counting indicates how we have to deform the momentum of the two gluons. Each propagator carries a z^{-1} dependence and each vertex carries a z dependence. If we have n propagators, we will have $n+1$ vertices. So, naïvely, we have:

$$\lim_{|z|\to\infty} \mathcal{A}(z) \propto z.$$

However, the helicity vectors also depend on z. Then, if we deform the momentum of an anti-self-dual gluon by $\lambda_r \to \lambda_r + z\lambda_s$ and the momentum of a self-dual gluon by $\widetilde{\lambda}_s \to \widetilde{\lambda}_s - z\widetilde{\lambda}_r$, we include a z^{-2} dependence on the amplitude and we can apply BCFW recursion relation.

In order to prove that the amplitudes $\mathcal{A}(--...+...-)$ and $\mathcal{A}(++...-...+)$ vanish, we need the result of $\mathcal{A}(--+-)$ and $\mathcal{A}(++-+)$. Considering $\mathcal{A}(--+-)$ and deforming p_1 and p_3 to satisfy BCFW recursion relation, we can observe that poles appear in $\mathbf{s}$ and $\mathbf{t}$ channel. We will denote the analytically continued quantities by a hat over them. Let $\widehat{\lambda}_L\widehat{\widetilde{\lambda}}_L$ be the complex propagator, the amplitude becomes:

$$\mathcal{A}(--+-) = g^2 \frac{(\lambda_1\lambda_2)^3}{\left(\lambda_2\widehat{\lambda}_L\right)\left(\widehat{\lambda}_L\lambda_1\right)} \frac{1}{(\lambda_1\lambda_2)\left(\widetilde{\lambda}_1\widetilde{\lambda}_2\right)} \frac{\left(\widehat{\lambda}_L\lambda_4\right)^3}{\left(\widehat{\lambda}_L\widehat{\lambda}_3\right)\left(\widehat{\lambda}_3\lambda_4\right)} + (2\leftrightarrow 4)\,.$$

Making use of (2.17) carefully to not regard terms from propagator on-shell conditions:

$$\begin{aligned}
(\widehat{\lambda}_L\lambda_3)(\widetilde{\lambda}_3\widetilde{\lambda}_2) &= -(\widehat{\lambda}_L\lambda_4)(\widetilde{\lambda}_4\widetilde{\lambda}_2)\\
(\lambda_4\widehat{\lambda}_3)(\widetilde{\lambda}_3\widetilde{\lambda}_1) &= -(\lambda_4\widehat{\lambda}_L)(\widetilde{\lambda}_L\widetilde{\lambda}_1)\\
(\lambda_1\lambda_L)(\widetilde{\lambda}_L\widetilde{\lambda}_1) &= -(\lambda_1\lambda_2)(\widetilde{\lambda}_2\widetilde{\lambda}_1)\\
(\lambda_2\lambda_L)(\widetilde{\lambda}_L\widetilde{\lambda}_1) &= -z_L(\lambda_2\lambda_1)(\widetilde{\lambda}_3\widetilde{\lambda}_1)
\end{aligned}$$

Replacing these relations on the amplitude and using that $z_L = \frac{(\widetilde{\lambda}_1\widetilde{\lambda}_2)}{(\widetilde{\lambda}_2\widetilde{\lambda}_3)}$, we find:

$$\mathcal{A}(--+-) = g^2 \frac{(\widetilde{\lambda}_3\widetilde{\lambda}_1)(\widetilde{\lambda}_3\widetilde{\lambda}_2)^2}{(\widetilde{\lambda}_1\widetilde{\lambda}_2)^3(\widetilde{\lambda}_4\widetilde{\lambda}_2)}\left[(\lambda_4\lambda_3) - \frac{(\widetilde{\lambda}_1\widetilde{\lambda}_2)}{(\widetilde{\lambda}_2\widetilde{\lambda}_3)}(\lambda_4\lambda_1)\right] + (2\leftrightarrow 4)\,. \tag{3.10}$$

However, from momentum conservation, the term inside the square brackets is zero and it proves the amplitude is zero. The procedure to demonstrate that $\mathcal{A}(+++-) = 0$ is the same.

Finally, we are going to consider MHV amplitudes. After the splitting, one diagram will be a MHV diagram and the other one will be a diagram where one gluon have opposite helicity from the others. Before stating that this amplitude vanishes, we must remember there is one kind of nonzero diagram which have one gluon with opposite helicity from the others, i.e., the three-gluon amplitude. Thus, in order to have a nonvanishing configuration, we must split the diagram in such a way that one of the graphs will be a MHV amplitude and the other is a three-gluon amplitude. Then, supposing the MHV formula is true, the amplitude $\mathcal{A}(+-++...+-)$, after deforming p_1 and p_n is:

$$\mathcal{A}(+-++...+-) = g^{n-2}\frac{(\widetilde{\lambda}_1\widehat{\widetilde{\lambda}}_L)^3}{(\widetilde{\lambda}_2\widehat{\widetilde{\lambda}}_L)(\widetilde{\lambda}_1\widetilde{\lambda}_2)}\frac{1}{(\lambda_1\lambda_2)\left(\widetilde{\lambda}_1\widetilde{\lambda}_2\right)}\frac{(\widehat{\lambda}_L\lambda_n)^3}{(\widehat{\lambda}_L\lambda_3)(\lambda_3\lambda_4)...(\lambda_{n-1}\lambda_n)}.$$

Other contributions might arise when we left the gluon n in the three-gluon amplitude. Nevertheless, they vanish because we will have a term like:

$$\frac{(\widetilde{\lambda}_i\widehat{\widetilde{\lambda}}_L)^3}{(\widehat{\widetilde{\lambda}}_n\widehat{\widetilde{\lambda}}_L)(\widetilde{\lambda}_i\widehat{\widetilde{\lambda}}_n)},$$

that is zero from (2.17). Focusing on the products involving hat quantities, we have:

$$\begin{aligned}
(\lambda_3\widehat{\lambda}_L)(\widehat{\widetilde{\lambda}}_L\widetilde{\lambda}_1) &= -(\lambda_3\lambda_2)(\widetilde{\lambda}_2\widetilde{\lambda}_1),\\
(\lambda_n\widehat{\lambda}_L)(\widehat{\widetilde{\lambda}}_L\widetilde{\lambda}_2) &= -(\lambda_n\lambda_1)(\widetilde{\lambda}_1\widetilde{\lambda}_2),\\
(\lambda_n\widehat{\lambda}_L)(\widehat{\widetilde{\lambda}}_L\widetilde{\lambda}_1) &= -(\lambda_n\lambda_2)(\widetilde{\lambda}_2\widetilde{\lambda}_1).
\end{aligned}$$

Then, the amplitude becomes:

$$\mathcal{A}(+-++...+-) = g^{n-2}\frac{(\lambda_n\lambda_2)^4}{(\lambda_1\lambda_2)(\lambda_2\lambda_3)...(\lambda_{n-1}\lambda_n)(\lambda_n\lambda_1)}. \tag{3.11}$$

Since the three-gluon MHV amplitude satisfies MHV formula as proved before, it concludes the proof that n-gluon MHV scattering has such that form.

3.4 Heuristic Deviation and Maximally Supersymmetric Extension

Let our starting point be the BCFW recursion relation. We will name by constructible theory a theory of massless particle that satisfies BCFW recursion relation and the three-particle vertices completely determine the four-particle amplitude.

Again, working on signature $(++--)$, either $(\lambda_i\lambda_j)$ or $(\widetilde{\lambda}_i\widetilde{\lambda}_j)$ vanish, but Lorentz invariance restricts the three-particle amplitude to be a generic function of $(\lambda_i\lambda_j)$ and $(\widetilde{\lambda}_i\widetilde{\lambda}_j)$. We can conclude that this amplitude split into a sum of a function dependent just on $(\lambda_i\lambda_j)$ and another one dependent just on $(\widetilde{\lambda}_i\widetilde{\lambda}_j)$. In an abuse of notation, we will call such functions as holomorphic and anti-holomorphic part of the amplitude, even λ and $\widetilde{\lambda}$ not being complex conjugated from each other. So, the three-vertices of a massless-particle theory have the following form:

$$\mathcal{A} = \mathcal{A}_H\left[(\lambda_1\lambda_2),(\lambda_2\lambda_3),(\lambda_3\lambda_1)\right] + \mathcal{A}_A\left[(\widetilde{\lambda}_1\widetilde{\lambda}_2),(\widetilde{\lambda}_2\widetilde{\lambda}_3),(\widetilde{\lambda}_3\widetilde{\lambda}_1)\right]. \tag{3.12}$$

We will initiate from a generic description and then particularize to theories of interest. Applying the helicity operator onto last expression, we obtain:

$$\begin{aligned}
\left(\lambda_i^{\alpha}\frac{\partial}{\partial\lambda_i^{\alpha}} + 2h_i\right)\mathcal{A}_H\left[(\lambda_1\lambda_2),(\lambda_2\lambda_3),(\lambda_3\lambda_1)\right] &= 0,\\
\left(\widetilde{\lambda}_i^{\dot\alpha}\frac{\partial}{\partial\widetilde{\lambda}_i^{\dot\alpha}} - 2h_i\right)\mathcal{A}_A\left[(\widetilde{\lambda}_1\widetilde{\lambda}_2),(\widetilde{\lambda}_2\widetilde{\lambda}_3),(\widetilde{\lambda}_3\widetilde{\lambda}_1)\right] &= 0.
\end{aligned}$$

Supposing the amplitude is an homogenous function, a particular solutions to previous equations are given by:

$$\begin{aligned}
\mathcal{F} &= (\lambda_1\lambda_2)^{h_3-h_1-h_2}(\lambda_2\lambda_3)^{h_1-h_2-h_3}(\lambda_3\lambda_1)^{h_2-h_3-h_1},\\
\mathcal{G} &= (\widetilde{\lambda}_1\widetilde{\lambda}_2)^{-h_3+h_1+h_2}(\widetilde{\lambda}_2\widetilde{\lambda}_3)^{-h_1+h_2+h_3}(\widetilde{\lambda}_3\widetilde{\lambda}_1)^{-h_2+h_3+h_1}.
\end{aligned}$$

If we write the general solutions to these equations as $\mathcal{A}_H = \kappa_H\mathcal{F}$ and $\mathcal{A}_A = \kappa_A\mathcal{G}$, we find that κ_H and κ_A must be annihilated by $\lambda_i^{\alpha}\frac{\partial}{\partial\lambda_i^{\alpha}}$ and $\widetilde{\lambda}_i^{\dot\alpha}\frac{\partial}{\partial\widetilde{\lambda}_i^{\dot\alpha}}$ respectively. Discarding solution like delta function, the only result for κ_H and κ_A are constants. We, then, infer the exact three-particle amplitude:

$$\mathcal{A} = \kappa_H\mathcal{F} + \kappa_A\mathcal{G}. \tag{3.13}$$

To determine the constants, we have to impose the amplitude is finite when the products $(\lambda_i\lambda_j)$ and $(\widetilde{\lambda}_i\widetilde{\lambda}_j)$ are taken to zero. Simple inspection shows that if $h_1+h_2+h_3$, is negative then we must set $\kappa_A = 0$ in order to avoid the such problem, while if $h_1+h_2+h_3$ is positive then κ_H must be zero.

Specifically, for a renormalizable theory of several spin one particles, we find that every possible "three-gluon" vertices are:

$$
\begin{aligned}
\mathcal{A}\left(a^{-}, b^{-}, c^{-}\right) &= \kappa_H'^{abc}(\lambda_1\lambda_2)(\lambda_2\lambda_3)(\lambda_3\lambda_1), \\
\mathcal{A}\left(a^{+}, b^{-}, c^{-}\right) &= \kappa_H^{abc}\frac{(\lambda_2\lambda_3)^3}{(\lambda_1\lambda_2)(\lambda_3\lambda_1)}, \\
\mathcal{A}\left(a^{+}, b^{+}, c^{+}\right) &= \kappa_A'^{abc}(\widetilde{\lambda}_1\widetilde{\lambda}_2)(\widetilde{\lambda}_2\widetilde{\lambda}_3)(\widetilde{\lambda}_3\widetilde{\lambda}_1). \\
\mathcal{A}\left(a^{-}, b^{+}, c^{+}\right) &= \kappa_A^{abc}\frac{(\widetilde{\lambda}_2\widetilde{\lambda}_3)^3}{(\widetilde{\lambda}_1\widetilde{\lambda}_2)(\widetilde{\lambda}_3\widetilde{\lambda}_1)}.
\end{aligned}
$$

From here, we also can see that three-vertices with all "gluons" carrying the same helicity must vanish, because the coupling constant must have inverse of mass dimension and we do not have such a constant in the theory. Moreover, due to crossing symmetry, the amplitude must be invariant under the index permutations, it implies that, for odd spin, the coupling constant must be completely antisymmetric in its indices. Then, a theory of less than three massless particles of odd spin must have a trivial three-particle S-matrix. Other consideration is to the theory be Hermitian in signature $(+---)$ we must have $\kappa_H^{abc} = \left(\kappa_A^{abc}\right)^*$.

For simplicity, let us split κ_H^{abc} and κ_A^{abc} in a dimensionless parameter f_{abc} and a coupling constant $g^\dagger$. The deformation to complex momentum must be independent of which particles are considered, since the amplitude becomes zero at $|z| \to \infty$. So, we shall consider the amplitude $\mathcal{A}\left(a^{+}, b^{-}, c^{+}, d^{-}\right)$. Deforming p_1 and p_4, we have:

$$
\begin{aligned}
\mathcal{A}^{(1,4)}\left(a^{+}, b^{-}, c^{+}, d^{-}\right) &= g^2\sum_e \left[f_{abe}f_{ecd}\frac{(\lambda_2\lambda_4)^4}{(\lambda_1\lambda_2)(\lambda_2\lambda_3)(\lambda_3\lambda_4)(\lambda_4\lambda_1)} + \right. \\
&\left. + f_{ace}f_{ebd}\frac{(\lambda_2\lambda_4)^3}{(\lambda_1\lambda_3)(\lambda_3\lambda_2)(\lambda_4\lambda_1)} \right].
\end{aligned}
$$

Now, doing the same to p_1 and p_2, we find:

$$
\begin{aligned}
\mathcal{A}^{(1,2)}\left(a^{+}, b^{-}, c^{+}, d^{-}\right) &= g^2\sum_e \left[f_{ade}f_{ecb}\frac{(\lambda_2\lambda_4)^4}{(\lambda_1\lambda_2)(\lambda_2\lambda_3)(\lambda_3\lambda_4)(\lambda_4\lambda_1)} + \right. \\
&\left. + f_{ace}f_{edb}\frac{(\lambda_2\lambda_4)^3}{(\lambda_1\lambda_2)(\lambda_3\lambda_1)(\lambda_4\lambda_3)} \right].
\end{aligned}
$$

However, both quantities should be equal and it implies:

$$
\mathcal{A}^{(1,4)}\left(a^{+}, b^{-}, c^{+}, d^{-}\right) - \mathcal{A}^{(1,2)}\left(a^{+}, b^{-}, c^{+}, d^{-}\right) = 0.
$$

†Obviously, the coupling constant g is dimensionless as well, however, it may run its value over the energy scales. We are prohibiting this to f_{abc}.

Using that $(\lambda_1\lambda_2)(\lambda_3\lambda_4) + (\lambda_1\lambda_4)(\lambda_2\lambda_3) = (\lambda_1\lambda_3)(\lambda_2\lambda_4)$, we finally get:

$$\frac{(\lambda_2\lambda_4)^4}{(\lambda_1\lambda_2)(\lambda_2\lambda_3)(\lambda_3\lambda_4)(\lambda_4\lambda_1)} \sum_e (f_{abe}f_{ecd} + f_{ade}f_{ebc} + f_{ace}f_{ebd}) = 0.$$

This condition is nothing but the Jacobi identity! We have found that the four-particle consistency condition implies that a theory of several vector bosons can be non-trivial only if the dimensionless parameters f_{abc} are the structure constants of a Lie algebra.

In principle, we can simply extend this analysis to $\mathcal{N} = 4$ supersymmetric Yang-Mills by adding more fields with different spin and considering all possible vertices of the theory. However, complications might appear such as the complex amplitude to scalar goes as a constant in infinity. The origin of these problems is because the helicity eigenstates are not eigenstates of supersymmetry generators and it makes the supersymmetry not explicit. In fact, if we do not have the supersymmetry explicit we need to be really careful to not make such a deformation that not preserves it. So, instead of considering all amplitudes one-by-one, we should regard all possible scattering at the same time. In other words, the external legs must be the superfields, instead the fields. To clear these arguments, we must state that we can represent all one-particle state in the theory in the of the analog of coherent states. Defining the two types of coherent states in analogy to what we have learned in the first undergraduate course about quantum mechanics:

$$\left|\widetilde{\eta}, \lambda, \widetilde{\lambda}\right\rangle = e^{\overline{Q}_I^{\dot{\alpha}} \widetilde{w}_{\dot{\alpha}} \widetilde{\eta}^I} \left|-1, \lambda, \widetilde{\lambda}\right\rangle, \tag{3.14}$$

$$\left|\eta, \lambda, \widetilde{\lambda}\right\rangle = e^{Q_\alpha^I w^\alpha \eta_I} \left|+1, \lambda, \widetilde{\lambda}\right\rangle. \tag{3.15}$$

The spinors w^α and $\widetilde{w}_{\dot{\alpha}}$ satisfies $(w\lambda) = (\widetilde{w}\widetilde{\lambda}) = 1$. Clearly, we can define $\widetilde{w}_{\dot{\alpha}}$ just up to a shift $\widetilde{w}_{\dot{\alpha}} \sim \widetilde{w}_{\dot{\alpha}} + \widetilde{c}\widetilde{\lambda}_{\dot{\alpha}}$ and the same holds to w^α. As we have already known from section **2.4**:

$$Q_\alpha^I \left|+1\right\rangle = \lambda_\alpha \left|+1/2\right\rangle^I, \tag{3.16}$$

$$\overline{Q}_I^{\dot{\alpha}} \left|-1\right\rangle = \widetilde{\lambda}^{\dot{\alpha}} \left|-1/2\right\rangle_I. \tag{3.17}$$

The shift does not affect the coherent state, since $\widetilde{\lambda}_{\dot{\alpha}}\overline{Q}_I^{\dot{\alpha}} \left|-1\right\rangle = 0$. From now on, we will neglect the momentum labels, unless it would be necessary to keep them. The η and $\widetilde{\eta}$ representations are equivalents and it is easy to see that the former diagonalizes $\overline{Q}$ and Q is diagonalized by the later.

$$
\begin{aligned}
Q^I_\alpha \left|\widetilde{\eta}\right\rangle &= \lambda_\alpha \widetilde{\eta}^I \left|\widetilde{\eta}\right\rangle, && (3.18)\\
\overline{Q}^{\dot\alpha}_I \left|\eta\right\rangle &= \widetilde{\lambda}^{\dot\alpha}\eta_I \left|\eta\right\rangle. && (3.19)
\end{aligned}
$$

Since the representation are equally valid, we need to relate them via a Grassmann Fourier transform:

$$
\begin{aligned}
\left|\widetilde{\eta}\right\rangle &= \int d^4\eta e^{\eta\widetilde{\eta}} \left|\eta\right\rangle, && (3.20)\\
\left|\eta\right\rangle &= \int d^4\widetilde{\eta} e^{\widetilde{\eta}\eta} \left|\widetilde{\eta}\right\rangle. && (3.21)
\end{aligned}
$$

We can consider each external state either in $\widetilde{\eta}$ representation or in η one, so, generally, the amplitude might dependent on both representations, that is, $\widetilde{\eta}_i$ and η_j. Nevertheless, in some cases, it may be convenient to consider all external states in the same representation, let it be $\widetilde{\eta}$. The amplitude will depend just on $\widetilde{\eta}$ and it must be invariant under supersymmetric transformations. So, we need to know how the coherent state behaves under supersymmetry transformations. Making use of (3.18) and (3.20), we find:

$$
\begin{aligned}
e^{Q^I_\alpha \zeta^\alpha_I} \left|\widetilde{\eta}\right\rangle &= e^{\widetilde{\eta}^I(\lambda\zeta_I)} \left|\widetilde{\eta}\right\rangle, && (3.22)\\
e^{\overline{Q}^{\dot\alpha}_I \widetilde{\zeta}^I_{\dot\alpha}} \left|\widetilde{\eta}\right\rangle &= \left|\widetilde{\eta} + (\widetilde{\lambda}\widetilde{\zeta})\right\rangle. && (3.23)
\end{aligned}
$$

Regarding η instead of $\widetilde{\eta}$, the result will be reversed. Using these state properties under supersymmetry, we obtain that the amplitude must satisfies:

$$
\mathcal{A}\left(\widetilde{\eta}_i\right) = \mathcal{A}\left(\widetilde{\eta}_i + (\widetilde{\lambda}_i\widetilde{\zeta})\right), \tag{3.24}
$$

and

$$
\mathcal{A}\left(\widetilde{\eta}_i\right) = e^{-\zeta\sum_j \lambda_j\widetilde{\eta}_j}\mathcal{A}\left(\widetilde{\eta}_i\right). \tag{3.25}
$$

Expression (3.24) tells us that we can set to zero up to two $\widetilde{\eta}$, by translation. To see that, let $\widetilde{\lambda}_1$ and $\widetilde{\lambda}_2$ be non-proportional spinors, then, we can write:

$$
\widetilde{\zeta}^I = a^I\widetilde{\lambda}_1 + b^I\widetilde{\lambda}_2.
$$

After the translation, we obtain $\widetilde{\eta}_1 \to \widetilde{\eta}_1 + (\widetilde{\lambda}_1\widetilde{\lambda}_2)b$ and $\widetilde{\eta}_2 \to \widetilde{\eta}_2 - (\widetilde{\lambda}_1\widetilde{\lambda}_2)a$. It is possible to cancel $\widetilde{\eta}_1$ and $\widetilde{\eta}_2$ by choosing:

$$\widetilde{\zeta}^I = \frac{\widetilde{\eta}_2^I \widetilde{\lambda}_1 - \widetilde{\eta}_1^I \widetilde{\lambda}_2}{(\widetilde{\lambda}_1 \widetilde{\lambda}_2)}. \tag{3.26}$$

By the other hand, expression (3.25) just implies that, necessarily, $\sum_j \lambda_j \widetilde{\eta}_j = 0$. That indicates the amplitude is proportional to a delta function, as in the case of translation invariance of the amplitude. Then:

$$\mathcal{A}\left(\widetilde{\eta}_i\right) = \delta^8 \left(\sum\nolimits_j \lambda_j \widetilde{\eta}_j\right) \widetilde{\mathcal{A}}\left(\widetilde{\eta}_i\right). \tag{3.27}$$

We might ask about how the application of the operator $e^{\overline{Q}_I^{\dot{\alpha}} \widetilde{\zeta}_{\dot{\alpha}}^I}$ affects this amplitude. In fact, a translation of $\widetilde{\eta}_i$ will provide a shift on the argument of the delta function. However, this increment will be equal to $\widetilde{\zeta} \sum_j \lambda_j \widetilde{\lambda}_j$, that is zero by momentum conservation.

From this amplitude formula, we can explicitly see that a deformation on λ_r does not preserve the supersymmetry. In order to keep it, we have to modify BCFW recursion relation to consider the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. To cancel the contribution coming from momentum analytic continuation, we have to deform $\widetilde{\eta}_s \to \widetilde{\eta}_s - z\widetilde{\eta}_r$.

The next check is to see if the complex amplitude vanishes at infinity. However, we can set $\widetilde{\eta}_s$ and $\widetilde{\eta}_r$ to zero and the form of $\widetilde{\zeta}$ to do so is:

$$\widetilde{\zeta}^I = \frac{\widetilde{\eta}_s^I (z) \widetilde{\lambda}_r - \widetilde{\eta}_r^I \widetilde{\lambda}_s (z)}{(\widetilde{\lambda}_r \widetilde{\lambda}_s (z))} = \frac{\widetilde{\eta}_s^I \widetilde{\lambda}_r - \widetilde{\eta}_r^I \widetilde{\lambda}_s}{(\widetilde{\lambda}_r \widetilde{\lambda}_s)}.$$

It means that $\widetilde{\zeta}^I$ carries no z-dependence and we are able to we excluded the $\widetilde{\eta}$-dependence from the external states that have complex momentum. Because of this, the behavior at infinity of the scattering is the same as we have to Yang-Mills theory.

3.5 MHV Amplitudes in $\mathcal{N} = 4$ Supersymmetric Yang-Mills

Supersymmetry constrains the form of scattering amplitude, so, MHV amplitudes in maximally supersymmetric Yang-Mills must satisfies (3.27). It is easy to see that if all external states of a tree-level scattering are gluons, we will just have gluons on the propagators and the amplitude becomes exactly the same as in pure Yang-Mills theory. Thus, we can use the amplitude in $\mathcal{N} = 4$ super-Yang-Mills to obtain the MHV formula.

First of all, we must prove that the amplitudes $\mathcal{A}(++...+)$ and $\mathcal{A}(+...+-)$ vanish. The positive helicity gluon comes from fourth-order terms in η, then, the amplitude $\mathcal{A}(++...+)$ is obtained by:

$$\mathcal{A}(++...+) = \int d^4\eta_1...d^4\eta_n \mathcal{A}(\eta_1, ..., \eta_n) . \tag{3.28}$$

It is always possible to set $\eta_1 = 0$, by supersymmetric transformations, and the integration over η_1 gives a zero result because the integrand does not have η_1-dependence. To the amplitude $\mathcal{A}(+...+-)$, we must have a integral over $\widetilde{\eta}_n$, because the self-dual gluon comes from the fourth-order term in this variable.

$$\mathcal{A}(+...+-) = \int d^4\eta_1 d^4\eta_2...d^4\widetilde{\eta}_n \mathcal{A}(\eta_1, \eta_{2,}..., \widetilde{\eta}_n) . \tag{3.29}$$

Again, we can use the supersymmetry transformations to translate η_1 and η_2 to zero. It modifies the other η's as well, but it will give no contribution to the analysis because they will be integrated. From the last section analysis, we find:

$$\mathcal{A}(+...+-) = \int d^4\eta_1 d^4\eta_2...d^4\widetilde{\eta}_n e^{\widetilde{\eta}_n(A\eta_1+B\eta_2)} \mathcal{A}(0, 0, \eta_3', ..., \widetilde{\eta}_n) .$$

It vanishes, because the only dependence on η_1 and η_2 comes from $A\eta_1 + B\eta_2$ and the Grassmann integral over this orthogonal combination gives zero. Finally, considering the MHV case, we have the amplitude given by:

$$\mathcal{A}(+...+--) = \int d^4\eta_1 d^4\eta_2...d^4\widetilde{\eta}_{n-1} d^4\widetilde{\eta}_n \mathcal{A}(\eta_1, \eta_{2,}..., \widetilde{\eta}_{n-1}, \widetilde{\eta}_n) . \tag{3.30}$$

Doing the same procedure as before, we get:

$$\mathcal{A}(\eta_1, \eta_{2,}..., \widetilde{\eta}_{n-1}, \widetilde{\eta}_n) = e^{[\widetilde{\eta}_{n-1}(A_1\eta_1+B_1\eta_2)+\widetilde{\eta}_n(A_2\eta_1+B_2\eta_2)]} \mathcal{A}(0, 0, \eta_3', ..., \widetilde{\eta}_{n-1}, \widetilde{\eta}_n) .$$

However, it is nothing more than a changing of representation. We are Fourier transforming the $\widetilde{\eta}_{n-1}$ and $\widetilde{\eta}_n$ representations into the $(A_1\eta_1 + B_1\eta_2)$ and $(A_2\eta_1 + B_2\eta_2)$ ones. Then, renaming $(A_1\eta_1 + B_1\eta_2)$ and $(A_2\eta_1 + B_2\eta_2)$ by η_{n-1} and η_n, we find the amplitude in terms of this new variables as:

$$\begin{aligned} \mathcal{A}(+...+--) &= \int d^4\eta_{n-1} d^4\eta_n...d^4\widetilde{\eta}_{n-1} d^4\widetilde{\eta}_n \frac{\partial(\eta_1\eta_2)}{\partial(\eta_{n-1}\eta_n)} \times \\ &\quad \times e^{(\widetilde{\eta}_{n-1}\eta_{n-1}+\widetilde{\eta}_n\eta_n)} \mathcal{A}(0, 0, \eta_3', ..., \widetilde{\eta}_{n-1}, \widetilde{\eta}_n) . \end{aligned}$$

Before determining the Jacobian, we need to know the factors A_1, A_2, B_1 and B_2. Using the analogous of equation (3.26) to ζ, we are able to determine such factors. Then, we can obtain the Jacobian:

$$\frac{\partial\left(\eta_1\eta_2\right)}{\partial\left(\eta_{n-1}\eta_n\right)}=\left[\frac{\left(\lambda_n\lambda_1\right)\left(\lambda_{n-1}\lambda_2\right)-\left(\lambda_n\lambda_2\right)\left(\lambda_{n-1}\lambda_1\right)}{\left(\lambda_1\lambda_2\right)^2}\right]^4.$$

Yet, we have that $\left(\lambda_n\lambda_1\right)\left(\lambda_{n-1}\lambda_2\right)-\left(\lambda_n\lambda_2\right)\left(\lambda_{n-1}\lambda_1\right)=\left(\lambda_1\lambda_2\right)\left(\lambda_{n-1}\lambda_n\right)$. Substituting it on the expression to $\mathcal{A}\left(+...+--\right)$, we find:

$$\begin{aligned}\mathcal{A}\left(+...+--\right) &= \left[\frac{\left(\lambda_{n-1}\lambda_n\right)}{\left(\lambda_1\lambda_2\right)}\right]^4\int d^4\eta_{n-1}d^4\eta_n...d^4\widetilde{\eta}_{n-1}d^4\widetilde{\eta}_n\times\\ &\quad\times e^{\left(\widetilde{\eta}_{n-1}\eta_{n-1}+\widetilde{\eta}_n\eta_n\right)}\mathcal{A}\left(0,0,\eta_3',...,\widetilde{\eta}_{n-1},\widetilde{\eta}_n\right).\end{aligned}$$

We can recognize this as the η representation of $\mathcal{A}\left(--+...+\right)$. So, we find the Ward identities for MHV amplitudes:

$$\mathcal{A}\left(+...+--\right)=\left[\frac{\left(\lambda_{n-1}\lambda_n\right)}{\left(\lambda_1\lambda_2\right)}\right]^4\mathcal{A}\left(--+...+\right). \tag{3.31}$$

We can easy observe that this implies:

$$\mathcal{A}\left(+...-^r...-^s...+\right)=\left(\lambda_r\lambda_s\right)^4\widetilde{\mathcal{A}}. \tag{3.32}$$

The function $\widetilde{\mathcal{A}}$ is independent of the minus helicity state entries. From the result of Yang-Mills theory, we know that:

$$\widetilde{\mathcal{A}}=\prod_{i=1}^n\left(\lambda_i\lambda_{i+1}\right)^{-1}. \tag{3.33}$$

Chapter 4

MHV Tree Amplitudes in Superstring Theory

Twistor-string theory has been successfully used to recover tree-level gluon scattering amplitudes. So, since in the most of the cases it deals with MHV amplitudes, some remarks on them in superstring theory are, in some sense, essential. However, in this chapter, we will follow another direction to consider MHV amplitudes in superstring theory.

In 2008, Berkovits and Maldacena [11] provided an interesting prescription to compute MHV amplitudes in superstring theory. Although it holds just to MHV amplitudes, this formula greatly simplify the computations. That is because when consider a superstring formalism such as pure spinor or RNS, many more fields must be taken into account and no simplifications occur before consider all momenta in four dimensions and all integrated vertex operator helicities to be positive and four-dimensional as well. Calculus of MHV amplitudes using this prescription is given in this chapter. It is shown they provide the correct results to four-point and five-point amplitude as well as to the limit when $\alpha' \to 0$.

4.1 MHV Tree Amplitude Prescription

As we saw, supersymmetry fixes the form of MHV amplitude in $\mathcal{N} = 4$ supersymmetric Yang-Mills as the following:

$$\widehat{\mathcal{A}}\left(\lambda_i, \widetilde{\lambda}_i, \widetilde{\eta}_i\right) = g^{n-2}\delta^4\left(\sum \lambda_i^{\alpha}\widetilde{\lambda}_i^{\dot{\alpha}}\right)\delta^8\left(\sum \lambda_i^{\alpha}\widetilde{\eta}_i^{I}\right)\mathcal{A}(\lambda_i, \widetilde{\lambda}_i).$$

This result arises just by a pure supersymmetry analysis and should be valid at any energy scale. So, generally, $\mathcal{A}(\lambda_i, \widetilde{\lambda}_i)$ ought to depend on α' and must lead us the well-known result:

$$\mathcal{A}(\lambda_i,\widetilde{\lambda}_i)=\prod_{i=1}^{n}\left(\lambda_i\lambda_{i+1}\right)^{-1},$$

when $\alpha'\to 0$. Since the fields on external legs live in four dimension and we are not considering loop corrections, kinematics shows us that all analysis is restricted to four dimensions. Momentum conservation must hold in every step of the scattering. A way to constrain the external fields live in four dimensions is to attach open string endpoints on N parallel **D3**-branes placing at the same point on the compactified six-dimension space.

The conjecture states that MHV superstring amplitude has the form:

$$\mathcal{A}(\lambda_i,\widetilde{\lambda}_i)=\frac{1}{(\lambda_1\lambda_2)(\lambda_2\lambda_3)(\lambda_3\lambda_1)}\left\langle V_1\left(z_1\right)V_2\left(z_2\right)V_3\left(z_3\right)\prod_{r=4}^{n}\int_{z_{r-1}}^{z_1}dz_rU_r\left(z_r\right)\right\rangle, \quad (4.1)$$

where the non-integrated vertex operators is $V_r\left(z_r\right)=\exp\left[i\lambda_r^{\alpha}\widetilde{\lambda}_r^{\dot{\alpha}}x_{\alpha\dot{\alpha}}\left(z_r\right)\right]$ and the integrated one is:

$$U_r\left(z_r\right)=\left[\varepsilon_r^{\alpha}\widetilde{\lambda}_r^{\dot{\alpha}}\partial x_{\alpha\dot{\alpha}}\left(z_r\right)+i\left(\widetilde{\lambda}_r\psi\right)\left(\widetilde{\lambda}_r\widetilde{\psi}\right)\right]V_r\left(z_r\right). \quad (4.2)$$

Different as before, we set $(\varepsilon_r\lambda_r)=1$, that is the reason to not appear the normalization factor on the denominator. By a gauge transformation, $\varepsilon_r\to\varepsilon_r+u\lambda_r$, $U_r\left(z_r\right)$ changes by a total derivative. The angle brackets mean the path integral expectation value of the vertex operator product. After fixing the gauge symmetries for the action, the conformal Killing vectors can be used to fix the three vertex operator to arbitrary points (z_1,z_2,z_3) on the boundary.

In order to continue, we must know the field OPEs. The $x_{\alpha\dot{\alpha}}$ fields satisfy the usual OPE:

$$x_{\alpha\dot{\alpha}}\left(y\right)x_{\beta\dot{\beta}}\left(z\right)\sim-\alpha'\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\left(\ln|y-z|+\ln|y-\bar{z}|\right). \quad (4.3)$$

The Grassmannian fields $\psi_{\dot{\alpha}}$ and $\widetilde{\psi}_{\dot{\beta}}$ have conformal weight $(1/2,0)$ and satisfy the OPE:

$$\psi_{\dot{\alpha}}\left(y\right)\widetilde{\psi}_{\dot{\beta}}\left(z\right)\sim\frac{\alpha'\epsilon_{\dot{\alpha}\dot{\beta}}}{y-z}. \quad (4.4)$$

Note that relabeling $\psi_{\dot{\alpha}}$ and $\widetilde{\psi}_{\dot{\beta}}$ to $\psi_{\alpha\dot{\alpha}}$, the integrated vertex operator becomes the standard RNS vertex operator for a self-dual gluon.

This prescription seems to be in some how relate to the open self-dual string with $\mathcal{N}=2$ worldsheet supersymmetry. There is just a single physical state in this theory spectrum and this state corresponds to a self-dual gluon, in signature $(++--)$. Its action is the same action that generates the OPEs (4.3), (4.4):

$$S = \frac{1}{\alpha'} \int d^2z \left(\frac{1}{2} \partial x^{\alpha\dot{\alpha}} \overline{\partial} x_{\alpha\dot{\alpha}} + \widetilde{\psi}^{\dot{\alpha}} \overline{\partial} \psi_{\dot{\alpha}} \right).$$

The superconformal generators with conformal weight $(2,0)$ are:

$$\begin{aligned} T &= \frac{1}{2} \left(\partial x^{\alpha\dot{\alpha}} \partial x_{\alpha\dot{\alpha}} + \widetilde{\psi}^{\dot{\alpha}} \partial \psi_{\dot{\alpha}} + \psi^{\dot{\alpha}} \partial \widetilde{\psi}_{\dot{\alpha}} \right), \\ G^+ &= \psi_{\dot{\alpha}} \partial x^{+\dot{\alpha}}, \quad G^- = \widetilde{\psi}_{\dot{\alpha}} \partial x^{-\dot{\alpha}}, \quad J = \psi^{\dot{\alpha}} \widetilde{\psi}_{\dot{\alpha}}. \end{aligned} \tag{4.5}$$

The superconformal primary field $V = \exp\left(i p_{\alpha\dot{\alpha}} x^{\alpha\dot{\alpha}}\right)$ can be associated to the physical self-dual Yang-Mills state and the integrated vertex operator is obtained from V by:

$$\int dz G^- G^+ V = \int dz \lambda^- \left[\widetilde{\lambda}_{\dot{\alpha}} \partial x^{+\dot{\alpha}} + i \lambda^+ \left(\widetilde{\lambda} \psi \right) \left(\widetilde{\lambda} \widetilde{\psi} \right) \right] e^{ip\cdot x}. \tag{4.6}$$

We have the freedom to choose the helicity vector conveniently, so, if we set $\varepsilon^+ = 0$ and $\varepsilon^- = (\lambda^+)^{-1}$, we find that the last equation is equal to $\lambda^+ \lambda^- \int dz U(z)$. However, similarities broke down when considering the n-point amplitude prescription. To open self-dual string the expression is

$$\mathcal{A}_{\mathcal{N}=2} = \left\langle \left(G^+ V_1(z_1) \right) \left(G^+ V_2(z_2) \right) V_3(z_3) \prod_{r=4}^{n} \int dz_r U_r(z_r) \right\rangle. \tag{4.7}$$

In the last equation, the fermionic zero-mode measure factor is $\left\langle \psi_{\dot{\alpha}} \psi^{\dot{\alpha}} \right\rangle = 1$ and the superconformal generators have been twisted so that $\psi_{\dot{\alpha}}$ carries zero conformal weight. This amplitude vanishes when $n > 3$ as it is expected for self-dual Yang-Mills tree amplitudes.

The n-point tree amplitude (4.1) is slightly different and provides the right result to MHV amplitude in low-energy limit. The superconformal generators are untwisted in this prescription.

4.2 The $\alpha' \to 0$ Limit of the Amplitude

In this section, we will to prove this prescription reproduces the known result when $\alpha' \to 0$. To do so, it is convenient to express the vertex operator in a "supersymmetric form":

$$U_r(z_r) = i\int d\chi_r \int d\widetilde{\chi}_r \exp\left[i\lambda_r^\alpha \widetilde{\lambda}_r^{\dot{\alpha}} x_{\alpha\dot{\alpha}}(z_r) + \chi_r\left(\widetilde{\lambda}\psi(z_r)\right) + \right.$$
$$\left. +\widetilde{\chi}_r\left(\widetilde{\lambda}\widetilde{\psi}(z_r)\right) - i\chi_r\widetilde{\chi}_r\varepsilon_r^\alpha\widetilde{\lambda}_r^{\dot{\alpha}}\partial x_{\alpha\dot{\alpha}}(z_r)\right], \tag{4.8}$$

where χ_r and $\widetilde{\chi}_r$ are Grassmann parameters. Evaluating the contractions, using that $\langle\prod V_r\rangle \propto \prod_{r<s} |z_r - z_s|^{\alpha' p_r\cdot p_s}$ and choosing ε_r properly to have $(\varepsilon_r\varepsilon_s) = 0$ for all r and s, we obtain:

$$\mathcal{A}(\lambda_p\widetilde{\lambda}_p) = \frac{1}{(\lambda_1\lambda_2)(\lambda_2\lambda_3)(\lambda_3\lambda_1)}\prod_{r=4}^{n}\int dz_r\int d\chi_r\int d\widetilde{\chi}_r\prod_{i,j}|z_i - z_j|^{\alpha' p_i\cdot p_j}\times$$
$$\times\exp\left[\alpha'\frac{\left(\widetilde{\lambda}_i\widetilde{\lambda}_j\right)}{z_i - z_j}\left(\chi_i\widetilde{\chi}_i(\varepsilon_i\lambda_j) + \chi_j\widetilde{\chi}_j(\varepsilon_j\lambda_i) + \chi_i\widetilde{\chi}_j + \chi_j\widetilde{\chi}_i\right)\right]. \tag{4.9}$$

We can see that we do not have double poles when $(z_i - z_j) \to 0$, because:

$$\exp[\rho] = 1 + \rho,$$

with $\rho = \alpha'\frac{(\widetilde{\lambda}_i\widetilde{\lambda}_j)}{z_i - z_j}\left(\chi_i\widetilde{\chi}_i(\varepsilon_i\lambda_j) + \chi_j\widetilde{\chi}_j(\varepsilon_j\lambda_i) + \chi_i\widetilde{\chi}_j + \chi_j\widetilde{\chi}_i\right)$. Since each term in the exponential is proportional to α', these terms can only contribute in the limit $\alpha' \to 0$ if there appear factors of $(\alpha')^{-1}$ coming from the integration over z_r. Such factors can arise from contact terms when $z_{r-1} \to z_r$, since

$$\int_{z_{r-1}}^{z_{r-1}+\varepsilon} dz_r\,|z_r - z_{r-1}|^{\alpha' p_r\cdot p_{r-1} - 1} \to \frac{1}{\alpha' p_r\cdot p_{r-1}}.$$

So, contributions come from $(z_r - z_{r-1})^{-1}$ terms, that is, contractions involving adjacent vertex operators. After integrating over the Grassmannian parameters, we have:

$$\lim_{\alpha'\to 0}\mathcal{A} = \lim_{\alpha'\to 0}\prod_{r=4}^{n}\int dz_r\frac{|z_r - z_{r-1}|^{\alpha' p_r\cdot p_{r-1}}|z_1 - z_n|^{\alpha' p_1\cdot p_n}}{(\lambda_1\lambda_2)(\lambda_2\lambda_3)(\lambda_3\lambda_1)}\times$$
$$\times\sum_{s=0}^{n-3}\left[\prod_{q=4}^{n-s}\frac{\alpha'\left(\widetilde{\lambda}_q\widetilde{\lambda}_{q-1}\right)(\varepsilon_q\lambda_{q-1})}{z_q - z_{q-1}}\prod_{t=n-s+1}^{n}\frac{\alpha'\left(\widetilde{\lambda}_t\widetilde{\lambda}_{t+1}\right)(\varepsilon_t\lambda_{t+1})}{z_t - z_{t+1}}\right],$$

$$\lim_{\alpha'\to 0}\mathcal{A} = \frac{1}{(\lambda_1\lambda_2)(\lambda_2\lambda_3)(\lambda_3\lambda_1)}\sum_{s=0}^{n-3}\left[\prod_{q=4}^{n-s}\frac{\left(\widetilde{\lambda}_q\widetilde{\lambda}_{q-1}\right)(\varepsilon_q\lambda_{q-1})}{p_q\cdot p_{q-1}}\times\right.$$
$$\left.\times\prod_{t=n-s+1}^{n}\frac{\left(\widetilde{\lambda}_t\widetilde{\lambda}_{t+1}\right)(\varepsilon_t\lambda_{t+1})}{-p_t\cdot p_{t+1}}\right],$$

Then, we can see that it lead us the correct result:

$$\lim_{\alpha'\to 0}\mathcal{A}=\prod_{r=1}^{n}\left(\lambda_r\lambda_{r+1}\right)^{-1}.$$

4.3 Computation for Four-Point and Five-Point Gluon Amplitudes

The next step is to consider the proposition to compute n-point MHV tree amplitude in superstring theory.

Evaluating the contractions to four-point amplitude, we find:

$$\mathcal{A}(\lambda_p\widetilde{\lambda}_p)=\frac{-i\alpha'}{(\lambda_1\lambda_2)(\lambda_2\lambda_3)\,(\lambda_3\lambda_1)}\int_{z_3}^{z_1}dz_4\sum_{q=1}^{3}\frac{(\varepsilon_4\lambda_q)\,(\widetilde{\lambda}_4\widetilde{\lambda}_q)}{z_4-z_q}\prod_{r<s}|z_r-z_s|^{\alpha'(\lambda_r\lambda_s)(\widetilde{\lambda}_r\widetilde{\lambda}_s)}. \tag{4.10}$$

Setting, for convenience, (z_1,z_2,z_3) to $(1,\infty,0)$:

$$\lim_{z_2\to\infty}\prod_{r>q}|z_r-z_q|^{\alpha' s_{rq}}=|z_4|^{\alpha' s_{43}}\,|z_4-1|^{\alpha' s_{41}}.$$

In the last expression, we defined $s_{qr}=p_q\cdot p_r$ and used that $\sum\limits_r s_{rq}=0$, by (2.17). Choosing $\varepsilon_4^\alpha=\lambda_3^\alpha\left(\lambda_3\lambda_4\right)^{-1}$ and renaming the Mandelstam variables s_{12}, s_{23} and s_{13} to s, t and u, respectively, we have:

$$\begin{aligned}\mathcal{A}(\lambda_p\widetilde{\lambda}_p)&=\frac{i\alpha'(\widetilde{\lambda}_4\widetilde{\lambda}_1)}{(\lambda_1\lambda_2)(\lambda_2\lambda_3)\,(\lambda_3\lambda_4)}\int_0^1 dz_4\,(1-z_4)^{\alpha' t-1}\,z_4^{\alpha' s},\\ \mathcal{A}(\lambda_p\widetilde{\lambda}_p)&=i\frac{\Gamma\left(\alpha' s+1\right)\Gamma\left(\alpha' t+1\right)}{\Gamma\left(-\alpha' u+1\right)}\prod_{r=1}^{4}\left(\lambda_r\lambda_{r+1}\right)^{-1},\end{aligned} \tag{4.11}$$

which is the correct open superstring four-point amplitude. Clearly, we recover the maximally supersymmetric Yang-Mills result setting $\alpha'=0$, however, it might be interesting to regard the low energy behavior of (4.11). Using that $\Gamma\left(1+\varepsilon\right)\approx 1-\gamma\varepsilon+\frac{1}{2}\left[\gamma^2+\zeta\left(2\right)\right]\varepsilon^2$, with γ being the Euler-Mascheroni constant, we find:

$$\mathcal{A}(\lambda_p\widetilde{\lambda}_p)=\left[1+(\alpha')^2\,\zeta\left(2\right)st\right]\prod_{r=1}^{4}\left(\lambda_r\lambda_{r+1}\right)^{-1}. \tag{4.12}$$

Now, we will consider the five-point amplitude computation. After evaluating the contractions, we obtain:

$$
\begin{aligned}
\mathcal{A}(\lambda_p\widetilde{\lambda}_p) &= \frac{(\alpha')^2}{(\lambda_1\lambda_2)(\lambda_2\lambda_3)\,(\lambda_3\lambda_1)}\int_{z_3}^{z_1} dz_4\int_{z_4}^{z_1} dz_5\prod_{p,r}|z_p-z_r|^{\alpha' s_{pr}}\left[\frac{(\widetilde{\lambda}_4\widetilde{\lambda}_5)^2}{(z_4-z_5)^2}+\right.\\
&\left.-\sum_{q\neq 4}\frac{(\varepsilon_4\lambda_q)\,(\widetilde{\lambda}_4\widetilde{\lambda}_q)}{z_4-z_q}\sum_{t\neq 5}\frac{(\varepsilon_5\lambda_t)\,(\widetilde{\lambda}_5\widetilde{\lambda}_t)}{z_5-z_t}-\frac{(\varepsilon_4\varepsilon_5)\,(\widetilde{\lambda}_4\widetilde{\lambda}_5)}{\alpha'\,(z_4-z_5)^2}\right]. \qquad (4.13)
\end{aligned}
$$

If we set $\varepsilon_4=\upsilon\varepsilon_5$, it is easy to see that we have no double poles. In addition, choosing $\varepsilon_5=(\lambda_3\lambda_5)^{-1}\lambda_3$, there are no more poles at z_3. Then, fixing (z_1,z_2,z_3) by $(\infty,0,1)$:

$$
\lim_{z_1\to\infty}\left(|z_1|^{\alpha' s_{12}}\,|z_1-1|^{\alpha' s_{13}}\,|z_1-z_4|^{\alpha' s_{14}}\,|z_1-z_5|^{\alpha' s_{15}}\right)=1.
$$

Making the substitutions $z_4=x^{-1}$ and $z_5=(xy)^{-1}$, after some manipulation, we have:

$$
\begin{aligned}
\mathcal{A}(\lambda_p\widetilde{\lambda}_p) &= \frac{(\alpha')^2}{\prod\limits_{r=1}^{5}(\lambda_r\lambda_{r+1})}\int_0^1 dx\int_0^1 dy\frac{\mathcal{I}(x,y)}{x}\left[\frac{(\lambda_3\lambda_2)(\lambda_4\lambda_5)(\lambda_5\lambda_1)}{(\lambda_3\lambda_1)(\lambda_3\lambda_5)}\right]\times\\
\times &\quad \left[\frac{(\widetilde{\lambda}_4\widetilde{\lambda}_2)(\lambda_3\lambda_1)(\widetilde{\lambda}_1\widetilde{\lambda}_5)}{y}+\frac{(\widetilde{\lambda}_4\widetilde{\lambda}_5)(\lambda_5\lambda_1)(\widetilde{\lambda}_1\widetilde{\lambda}_2)}{1-y}\right], \qquad (4.14)
\end{aligned}
$$

where,

$$
\mathcal{I}(x,y)=x^{\alpha' s_{23}}y^{\alpha' s_{15}}\,(1-x)^{\alpha' s_{34}}\,(1-y)^{\alpha' s_{45}}\,(1-xy)^{\alpha' s_{35}}\,.
$$

Defining the linear independent integrals

$$
\begin{aligned}
f_1 &\equiv \int_0^1 dx\int_0^1 dy\frac{\mathcal{I}(x,y)}{xy},\\
f_2 &\equiv \int_0^1 dx\int_0^1 dy\frac{\mathcal{I}(x,y)}{1-xy},
\end{aligned}
$$

after integrating by parts, we reach the result to five-point gluon scattering amplitude:

$$
\mathcal{A}(\lambda_p\widetilde{\lambda}_p)=\frac{(\alpha')^2}{\prod\limits_{r=1}^{5}(\lambda_r\lambda_{r+1})}\left[s_{15}s_{23}f_1+(\lambda_3\lambda_2)(\widetilde{\lambda}_3\widetilde{\lambda}_5)(\lambda_5\lambda_1)(\widetilde{\lambda}_1\widetilde{\lambda}_2)f_2\right]. \qquad (4.15)
$$

Nevertheless, using that:

$$
\begin{aligned}
f_1 &= \frac{1}{\alpha'^2 s_{23} s_{51}} - \zeta(2)\left(\frac{s_{45}}{s_{23}} + \frac{s_{34}}{s_{51}}\right) + \mathcal{O}(\alpha'), \\
f_2 &= \zeta(2) + \mathcal{O}(\alpha'),
\end{aligned}
$$

we are able to determine the low energy behavior of the amplitude:

$$
\mathcal{A}(\lambda_p \widetilde{\lambda}_p) = \frac{1 + \alpha'^2 \zeta(2)\left[(\lambda_3\lambda_2)(\widetilde{\lambda}_3\widetilde{\lambda}_5)(\lambda_5\lambda_1)(\widetilde{\lambda}_1\widetilde{\lambda}_2) - s_{45}s_{51} - s_{23}s_{34}\right]}{\prod_{r=1}^{5}(\lambda_r\lambda_{r+1})}. \tag{4.16}
$$

Since supersymmetry prohibes the term F^3 in the effective action obtained from superstring theory, the amplitude must not have a linear dependence on α', as observed.

Chapter 5

Conclusions

In the second chapter, we considered MHV tree amplitudes in twistor space. From (2.45), we could see the simplicity of a scattering amplitude form when properly Fourier transformed to twistor space. So, if we are able to define Yang-Mills and $\mathcal{N} = 4$ supersymmetric Yang-Mills vertices in twistor space, we can compute scattering amplitudes in a simpler way than we use to. Once we have the expressions on twistor space, we can go back to space time by a contour integral. Other remark is that computations on supertwistor space have simpler form, because functions defined on it must have homogenous degree zero.

In the third chapter, after proved MHV formula using the BCFW recursion relation, we obtained the supersymmetric Ward identities to MHV amplitudes. In this analysis, these amplitudes seem to emerge naturally from a maximally supersymmetric context. In fact, all discussion can be applyied to $\mathcal{N} = 8$ supergravity as well.

Both of these maximally supersymmetric theories arise from the correspondent $\mathcal{N} = 1$ ten-dimensional theory massless modes after compactification. So, it might be interesting to consider amplitudes in higher dimensions in order to understand how MHV amplitudes arise from a ten-dimensional scattering. Other possible analysis is to find the differences between scattering that provides MHV amplitudes and the other ones that produce non-MHV amplitudes.

In the last chapter, we used (4.1) to compute four-point and five-point MHV tree amplitudes in superstring theory. However, there are some remarks about this kind of amplitude with more external states. Even this prescription simplifies the computation, the calculus to six-point amplitude increases considerably the complexity. That is because we have many more contractions to evaluate. In [10], we can see that the n-point MHV amplitude in superstring theory is given in terms of $(n-3)!$ linearly independent integrals. Also, an open question is we do not know how to derive such prescription from a superstring formalism.

Another interesting direction to follow comes from [13], where Berkovits considered the ten-dimensional super-Yang-Mills in terms of supertwistors. In this paper, he found a relation between the twistor superfield $\Phi(Z)$ and the ten-dimensional super-Yang-Mills vertex operator $\lambda^\alpha A_\alpha(x, \theta)$. This vertex operator is the same one that appears in the superstring pure spinor formalism, but in super-Yang-Mills case the momentum spinor play the role of pure spinor ghosts. In principle, we can use this framework to study the ten-dimensional amplitudes that originate the MHV amplitudes in four dimensions.

Bibliography

[1] S. Parke and T. Taylor, An Amplitude For N Gluon Scattering, *Phys. Rev. Lett.* **56** (1986) 2459 .

[2] F. A. Berends and W. T. Giele, Recursive Calculations For Processes With N Gluons, *Nucl. Phys. B***306** (1988) 759.

[3] R. Penrose, Twistor Algebra, *J. Math. Phys.* **8** (1967) 345.

[4] R. Penrose, The Central Programme Of Twistor Theory, *Chaos, Solitons, and Fractals* **10** (1999) 581.

[5] A. Ferber, Supertwistors And Conformal Supersymmetry, *Nucl. Phys. B***132** (1978) 55.

[6] R. Britto, F. Cachazo and B. Feng, New Recursion Relations for Tree Amplitudes of Gluons, *Nucl. Phys. B* **715**, 499 (2005), [hep-th/0412308].

[7] R. Britto, F. Cachazo, B. Feng and E. Witten, Direct proof of tree-level recursion relation in Yang-Mills theory, [arXiv:hep-th/0501052].

[8] F. Cachazo, P. Svrcek and E. Witten, MHV vertices and tree amplitudes in gauge theory, *JHEP* **0409**, 006 (2004) [arXiv:hep-th/0403047].

[9] E. Witten, Perturbative gauge theory as a string theory in twistor space, *Commun. Math. Phys.* **252**, 189 (2004) [arXiv:hep-th/0312171].

[10] S. Stieberger and T. R. Taylor, Supersymmetry Relations and MHV Amplitudes in Superstring Theory, *Nucl.Phys.B***793**:83-113,2008, [arXiv:0708.0574] [hep-th]

[11] N. Berkovits and J. Maldacena, "'Fermionic T-Duality, Dual Superconformal Symmetry, and the Amplitude/Wilson Loop Connection"' [arXiv:0807.3196] [hep-th].

[12] N. Berkovits, Self-Dual Super-Yang-Mills as a String Theory, *JHEP* **0405** (2004) 034, [hep-th/0403280]

[13] N. Berkovits, "‘Ten-Dimensional Super-Twistors and Super-Yang-Mills"’ [arXiv:hep-th/0910.1684v2]

[14] N. Arkani-Hamed, F. Cachazo and J. Kaplan, "‘What is the Simplest Quantum Field Theory?"’ [arXiv:0808.1446v2] [hep-th]

[15] N. Arkani-Hamed and J. Kaplan, *JHEP* **0804**, 076 (2008) [arXiv:0801.2385] [hep-th]].

[16] P. Benincasa, F. Cachazo, "‘Consistency Conditions On The S-Matrix Of Massless Particles"’ [arXiv:0705.4305v2] [hep-th].

[17] J. Wess e J. Bagger, "‘Supersymmetry and Supergravity"’ *Princeton University Press*, Princeton, 1992.

[18] J. Polchinski, "‘String Theory - an Introduction to the Bosonic String"’, vol. **1** *Cambridge University Press*, Cambridge, 2001.

[19] G. t Hooft, A Planar Diagram Theory For Strong Interactions, *Nucl. Phys. B***72** (1974) 461.