DISSERTAÇÃO DE MESTRADO IFT–D.003/22

Worldsheet complex structure in the bosonic string sigma model

Vinícius Bernardes da Silva

Advisor

Andrey Yuryevich Mikhaylov

January 2022

S586w

Silva, Vinícius Bernardes da
Worldsheet complex structure in the bosonic string sigma model / Vinícius Bernardes da Silva. – São Paulo, 2022
84 f.

Dissertação (mestrado) – Universidade Estadual Paulista (Unesp), Instituto de Física Teórica (IFT), São Paulo
Orientador: Andrei Yuryevich Mikhaylov

1. Modelos de corda. 2. Física matemática. 3. Teoria das supercordas. I. Título

Sistema de geração automática de fichas catalográficas da Unesp. Biblioteca do Instituto de Física Teórica (IFT), São Paulo. Dados fornecidos pelo autor(a).

Dedico esta tese ao meu pai,
Paulo *(in memoriam)*.

Agradecimentos

Quero agradecer meu orientador Andrei Mikhailov, que ao longo do curso de mestrado me ensinou muito sobre pesquisa. Seu jeito de pensar me influenciou em várias áreas da vida. Também agradeço meu colega durante esses dois anos, Eggon Viana, que me estimulou a querer melhorar sempre. Nossas conversas foram muito importantes, sobre física e sobre a vida.

Agradeço a todos os colegas e professores que me acolheram na chegada ao IFT. Foi um desafio fazer a maior parte do mestrado de forma remota, então também os agradeço por proporcionaram boas conversas e convivência, mesmo à distância. As discussões com Lucas, Dennis, Lefundes, Vidal e João foram muito frutíferas.

Muito obrigado à CAPES pelo financiamento confiado ao meu desenvolvimento ao longo do curso, que se conclui com esta dissertação. Finalmente, sou muito grato à minha família que me deu suporte constante. Especialmente, obrigado a Nana, Wali, Laura, Maira e Dani.

Resumo

O sigma-model da corda bosônica é uma teoria de calibre – é invariante sob reparametrizações da folha de mundo da corda –, portanto deve-se fixar o calibre para quantizá-lo. Uma das formas de se fazer isso é usando o formalismo BV, que permite a quantização de teorias com uma simetria fermiônica, como a simetria BRST. Define-se o espaço de fase BV, contendo os campos e seus anticampos. Para fixar um calibre, escolhemos uma subvariedade do espaço de fase BV.

O calibre de Polyakov (ou calibre conforme) é definido ao fixar a estrutura complexa (ou métrica) na folha de mundo. No formalismo BV, esse procedimento define um número infinito de operadores bracket, que têm um papel nas deformações da teoria. Além disso, o operador BRST nesse calibre não é nilpotente, quando os campos estão off-shell, i.e. antes de usar equações de movimento.

O anticampo da estrutura complexa surge neste calibre como o fantasma b, que é necessário para computar amplitudes, através de uma integração pelo espaço de moduli da folha de mundo. A transformação BRST do fantasma b é a expressão off-shell do tensor de energia-momento da teoria. Após ressomar os termos que dão origem ao operadores bracket, vemos que as deformações podem ser linearizadas, o que implica na fatorização holomórfica das amplitudes.

O formalismo da supercorda de espinores puros é a generalização da corda bosôncia no calibre de Polyakov. Portanto, não há necessidade de fixar-se um calibre nesse formalismo. Diferentement da corda bosônica, o operador BRST é nilpotent mesmo off-shell. O fantasma b é definido de forma que a sua transformação BRST gera a expressão on-shell do tensor de energia-momento.

Após a introdução no Capítulo 1, introduzimos fundamentos matemáticos no Capítulo 2. O formalismo BV é apresentado no Capítulo 3, e no Capítulo 4 o utilizamos para fixar fixar o calibre e quantizar o sigma-model da corda bosônica. Fazemos também comentários sobre os operadores bracket. No Capítulo 5 introduzimos a supercorda de espinores puros, e comparamos o formalismo com a corda bosônica.

Palavras Chaves: Formalismo BV; Formalismo BRST; Corda bosôncia; Estrutura complexa; Supercorda de espinores puros.

Áreas do conhecimento: Física; Teoria de cordas; Física Matemática.

Abstract

The bosonic string sigma model is a gauge theory – it is invariant under worldsheet diffeomorphisms, so it must be gauge fixed to be quantized. One way to do it is using BV formalism, which allows one to quantize theories with a fermionic symmetry, such as the BRST symmetry. One defines a BV phase space, with fields and their antifields. The gauge fixing procedure consists in a choice of submanifold in the BV phase space, and it is well defined off-shell.

The Polyakov gauge (or conformal gauge) is defined by fixing the worldsheet complex structure (or metric) of the bosonic string. In BV formalism, this procedure defines an infinite number of derived brackets that play a role in deformations of the theory. Moreover, the BRST operator in this gauge in not nilpotent off-shell. This condition is substituted by constraints among the brackets.

The antifield of the complex structure arises as the b ghost, which is needed to compute amplitudes, by integrating over the moduli space of the worldsheet. The BRST transformation of the b ghost is the energy-momentum of the theory defined off-shell. After resumming the terms that give rise to the higher brackets, we see that the deformations are linearized, and give rise to the holomorphic factorization.

The pure spinor superstring is a generalization of the bosonic string in Polyakov gauge, so there is no need to gauge fix. Different from the bosonic string, the BRST operator is nilpotent off-shell. The b ghost is defined in such a way that its BRST transformation yields the energy-momentum tensor only on-shell.

After the introduction in Chapter 1, we introduce the mathematical foundations in Chapter 2. The BV formalism is presented in Chapter 3, and in Chapter 4 we use it to gauge fix and quantize the bosonic string sigma-model. We make comments about the derived brackets. In Chapter 5 we introduce the pure spinor superstrings, and compare it with the bosonic string.

Keywords: BV formalism; BRST formalism; Bosonic string; Complex structure; Pure spinor superstring.

Fields of knowledge: Physics; String theory; Mathematical physics.

Contents

Chapter 1

Introduction

To quantize a gauge theory, we need to appropriately put a constraint on the fields. The BRST formalism is a correct approach that is well-known. But it is more insightful to use a formalism where deformations of the theory have geometrical meaning, such as the Batalin-Vilkovisky formalism. The BV formalism can be applied to any theory described by an action and a fermionic symmetry, such as the BRST symmetry. The space of fields is extended to the BV phase space, containing the fields and its antifields, and the action and BRST operator are encoded in the BV action. The gauge fixing procedure consists in a choice of submanifold in the BV phase space [1]. Once the choice is made, the action and the fermionic symmetry can be transformed into a gauge fixed action, plus a possibly infinite number of bracket operations in the space of fields, one of these being a BRST operator. The additional bracket operations are non-linear terms of the BV action, and their presence is not clear from the BRST point of view, though they are important when one considers deformations of the theory.

The bosonic string sigma model is a gauge theory – invariant under diffeomorphisms – that describes the dynamics of a freely propagating string. A BV action can be constructed with the action and the BRST operator that generates diffeomorphisms [2]. To describe interaction of strings, one must introduce vertex operators that correspond to physical states. Their introduction is equivalent to small deformations of the free theory, and the BV formalism constrains them to preserve the BRST structure of the theory. The introduction of vertex operators is described as a deformation of the BV action, and their introduction changes not only the action, but possibly the BRST operator as well.

In order to compute scattering amplitudes of strings, one must sum probabilities over the moduli space of the theory, which is the space of all topologies and metrics modulo rescaling and diffeomorphism on the worldsheet. In bosonic string theory, the metric is a field that is gauge fixed, so to perform the moduli space integration it is necessary to consider deformations that change the metric. These deformations are trivial, in the sense that the theory doesn't change under them – they are equivalent to a change in the gauge fixing condition. In BV formalism, the trivial deformations can be parameterized, and the fields that generate these deformations in bosonic strings are the b ghosts.

The presence of non-linear terms with higher brackets affects the theory only up to equations of motion, so the BV formalism allows one to understand the off-shell aspects of the BRST structure of the bosonic string. Moreover, there are OPEs which are contact terms – which are delta functions on the worldsheet – that can be removed by resumming the deformations of the BV action. This procedure reduces the BRST structure to an action and a BRST operator only, and implies in the holomorphic factorization of the amplitudes [3].

In the pure spinor formalism of superstring theory, the action describes the worldsheet embedding in the supersymmetric spacetime, and a BRST-like operator constrains the physical states [4]. Though the model has a BRST-like structure, it doesn't come from gauge fixing. Because there is no need to gauge fix, there are no higher brackets in the BV action of the theory, and the BRST-like operator is nilpotent off-shell by construction. On the other hand, the ghost field that generates deformations along the moduli space is a composite field that generates the energy-momentum only on-shell [5, 6].

Chapter 2

Mathematical foundations

In this section we'll present the mathematical definitions that will come to use in the subsequent chapters. Some of the content of this chapter is built upon basic differential geometry, which is reviewed in Appendix A.

2.1 Supermanifolds, Lie groups and Lie algebras

2.1.1 Exterior algebra and gradings

The space of functions on a real manifold M is given by Fun$M = T^{(0,0)}M$. The elements satisfy an algebra defined by the algebra of multiplication of real numbers. For two functions $F_1, F_2 \in \text{Fun}M$, there is another function $F_3 \in \text{Fun}M$ given by $F_3 = F_1 F_2$. The product of real numbers is commutative, hence the algebra on Fun$M$ is associative as well.

The wedge product defines an algebra of 1-forms, called the exterior algebra, which takes forms $u_1, u_2 \in \Omega^1(M)$ to another form $u_1 \wedge u_2 \in \Omega^2(M)$. By the definition of wedge product, the algebra is anticommutative. Let's define the space of all forms as

$$\Omega^{\circ}(M) = \Omega^0(M) \oplus \Omega^1(M) \oplus \Omega^2(M) \oplus \cdots . \tag{2.1.1}$$

Then we can say that $\Omega^{\circ}(M) = \text{Fun}(T^*M)$. The elements of $\Omega^0(M)$ are functions M, and the elements of $\Omega^1(M)$ are functions on M with values on the cotangent spaces. With the exterior algebra defined by the wedge product, elements of $\Omega^{\circ}(M)$ are considered functions on the cotangent bundle.

An exterior algebra can also be defined without the notion of a wedge product. Let's generalize the real linear space $\mathbb{R}^m$ of dimension m, whose elements satisfy the real commutative algebra. First we write the real space as $\mathbb{R}^m = \mathbb{R}^{m|0}$, and we say that it is a space with m dimensions of even parity. Then we define the real space $\mathbb{R}^{0|n}$ of n dimensions of odd parity, which exactly like $\mathbb{R}^n$, but the components of its elements satisfy an anticommutative algebra, the exterior algebra. Thus for $\theta_1, \theta_2 \in \mathbb{R}^{0|n}$ we have $\theta_1\theta_2 = -\theta_2\theta_1$. Finally we can define the real linear space $\mathbb{R}^{m|n}$ of m even dimensions and n odd dimensions, given by

$$\mathbb{R}^{m|n} = \mathbb{R}^{m|0} \oplus \mathbb{R}^{0|n}. \tag{2.1.2}$$

Elements of $\mathbb{R}^{m|n}$ are written as

$$(x_1, \cdots, x_m, \theta_1, \cdots, \theta_n) \in \mathbb{R}^{m|n}. \tag{2.1.3}$$

The exterior algebra can then be written as the algebra between each of the components, as

$$[x_i, x_j]_- = 0, \quad [\theta_a, \theta_b]_+ = 0, \quad [x_i, \theta_a]_- = 0, \tag{2.1.4}$$

where $[,]_-$ is the commutator and $[,]_+$ the anticommutator.

More generally, we can consider a linear space V which is decomposed in k others as

$$V = V_0 \oplus \cdots \oplus V_{k-1}, \tag{2.1.5}$$

Along with a **grading**, which is a map $\mathrm{gr} : V \to \mathbb{Z}$, that associates to each element $v \in V_k$ to the integer label $\mathrm{gr}(v) = k$. The grading of an element can also be called its degree. We define a product operation which sums the gradings:

$$\mathrm{gr}(uv) = \mathrm{gr}(u) + \mathrm{gr}(v). \tag{2.1.6}$$

An exterior algebra is a linear space with two gradings $V = V_0 + V_1$, with $V_0 = \mathbb{R}^{m|0}$ and $V_1 = \mathbb{R}^{0|n}$. If we have two gradings, we may call it the parity of an element.

2.1.2 Supermanifolds

A **supermanifold** is a manifold that is locally equal to $\mathbb{R}^{m|n}$. It is equipped with charts that take it to linear spaces with odd dimensions, $\phi : U \subset M \to \mathbb{R}^{m|n}$, such that the transition functions $\tau = \phi^{-1} \circ \tilde{\phi}$ are smooth functions on an open subset of $\mathbb{R}^{m|n}$. Given a chart, a supermanifold has coordinates (x^i, θ^α), with gradings

$$\bar{x} = 0, \qquad \bar{\theta} = 1. \tag{2.1.7}$$

We say that a supermanifold is (m, n)-dimensional if it is locally equal to $\mathbb{R}^{m|n}$.

One example of a supermanifold are the odd tangent bundle and the odd cotangent bundle. Consider the tangent bundle TM of an even manifold M, i.e. $(m, 0)$-dimensional. The **odd tangent bundle** ΠTM is defined by flipping the parity of the tangent fiber. In coordinates, we write the elements as $(x, \mathrm{d}x)$, where $x \in M$ have grading $\mathrm{gr}x = 0$, and the tangent vectors $\mathrm{d}x$ are odd: $\mathrm{grd}x = 1$.

The **odd cotangent bundle** ΠT^*M of a manifold M is the cotangent bundle with flipped parity in the cotangent fiber. It can be seen as an incorporation of the wedge product algebra to the manifold. In fact, if we write the elements as $(x, \mathrm{d}x)$, and consider a function $F \in \mathrm{Fun}\Pi T^*M$, it can be expanded in powers of the cotangent vectors as

$$F(x, \mathrm{d}x) = F_0(x) + F_1^A(x)\mathrm{d}x_A + \cdots + F_n^{A_1 \cdots A_n}\mathrm{d}x_{A_1} \cdots \mathrm{d}x_{A_n}, \tag{2.1.8}$$

so it is a formal sum of forms $F_i \in \Omega^i(M)$ on the base of ΠT^*M.

The de Rham operator on M is defined as a vector field on ΠT^*M, given by

$$\mathrm{d} = \mathrm{d}x^A \frac{\partial}{\partial x^A}. \tag{2.1.9}$$

For a vector field $v \in TM$, we define the iota operator ι_v as a vector field on ΠT^*M, by

$$\iota_v = v^A \frac{\partial}{\partial \mathrm{d}x^A}. \tag{2.1.10}$$

With these operators we are able to define the Cartan magic formula

$$\mathcal{L}_v = [\iota_v, \mathrm{d}], \tag{2.1.11}$$

which holds for action of Lie derivative on forms on M. The commutator takes into account the odd parity of d and ι_v (if we take v to be an odd vector field, then ι_v has even parity).

2.1.3 Lie groups and Lie algebras

A group G is a set with a map $G \times G \to G, (g_1, g_2) \mapsto g_1 \cdot g_2$, called the product, that satisfies associativity, i.e.

$$(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3), \tag{2.1.12}$$

an element $1 \in G$ called the identity, that satisfies

$$1 \cdot g = g \cdot 1 = g, \tag{2.1.13}$$

and a map $G \to G, g \mapsto g^{-1}$ called the inverse, which has the property

$$g \cdot g^{-1} = g^{-1} \cdot g = 1. \tag{2.1.14}$$

A Lie group is a manifold equipped with a product, an identity and an inverse that satisfy the above mentioned properties, and such that the product and the inverse are smooth maps.

The general linear group $GL(n)$ is the set of linear transformations on $\mathbb{R}^n$. There is also the general Lie group over the complex numbers, $GL(n, \mathbb{C})$, which is the set of linear transformations on $\mathbb{C}^n$.

The coordinates of $\mathbb{R}^n$ induce the parameterization of $GL(n)$ as $n \times n$ matrices. This allows the straighforward definitions of the determinant, trace and transpose of an element of the group, which are independent of the choice of coordinates.

The orthogonal group $O(n) \subset GL(n)$ is defined by imposing the condition $\det g = \pm 1$ for every $g \in O(n)$. The special orthogonal group $SO(n) \subset O(n)$ is defined by containing only elements with positive determinant.

The Lie algebra of a Lie group G is the tangent space around the identity:

$$\mathfrak{g} = T_1 G. \tag{2.1.15}$$

The elements $v \in \mathfrak{g}$ can be written, for some $(g, \mathrm{d}g) \in \Pi TG$, as

$$v = g^{-1} \mathrm{d}g. \tag{2.1.16}$$

The elements of the Lie algebra can also be seen as small deformations of the group identity. Take a chart of G in which $g = g^A t_A$, where t_A form the basis of the tangent space, and are called generators. Let there be a curve $t \mapsto g_v(t)$ on G such that $g(0) = 1$. It defines the Lie algebra vector v as

$$v = \left.\frac{\partial}{\partial t}\right|_{t=0} g_v(t). \tag{2.1.17}$$

The Lie algebra vectors act as linear maps on G, as

$$vg = \left.\frac{\partial}{\partial t}\right|_{t=0} (g_v(t) \cdot g). \tag{2.1.18}$$

The commutator is a map $[\,\cdot\,,\,\cdot\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ defined from the group product by the action on G, as $[u, v]g$. In terms curves $g_1(t)$ and $g_2(t)$, the definition is

$$\begin{aligned}[v_1, v_2] &= \frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2}\Big(g_1(t_1) \cdot g_2(t_2) - g_2(t_2) \cdot g_1(t_1)\Big)_{t_1=t_2=0} \\ &= v_1 v_2 - v_2 v_1. \end{aligned} \tag{2.1.19}$$

It is proportional to the term that goes with $t_1 t_2$ in the expansion of group elements around the identity in the group product, because the linear terms don't couple the tangent vectors, and thus don't define operations in the algebra. In coordinates, the commutator can be written in terms of structure constants $f^A_{\ BC}$, as in

$$[u, v] = f^A_{\ BC} u^B v^C t_A. \tag{2.1.20}$$

The commutator satisfies antisymmetry

$$[u, v] = -[v, u] \tag{2.1.21}$$

and the Jacobi identity

$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0. \tag{2.1.22}$$

An example of Lie algebra is $\mathfrak{gl}(n)$, the algebra of $GL(n)$. The elements $v \in \mathfrak{gl}(n)$ are represented as n-dimensional matrices, and the commutator is the commutator of the matrices.

2.1.4 Lie superalgebras

A Lie superalgebra is a Lie algebra $\mathfrak{g}$ which is the sum of subalgebras $\mathfrak{g}_i$ with different gradings [7]

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_n, \tag{2.1.23}$$

where an element $v \in \mathfrak{g}_k$ has grading k. The commutator is then a map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that sums the gradings of the elements of $\mathfrak{g}$, as

$$\mathrm{gr}([u, v]) = \mathrm{gr}(u) + \mathrm{gr}(v). \tag{2.1.24}$$

The symmetry relation is generalized to

$$[u, v] = (-1)^{\bar{u}\bar{v}}[v, u] \tag{2.1.25}$$

where $\bar{u}$ and $\bar{v}$ are the signs of the gradings of the elements of the algebra, and the Jacobi identity is

$$[u, [v, w]] + (-1)^{\bar{u}(\bar{v}+\bar{w})}[v, [w, u]] + (-1)^{\bar{w}(\bar{u}+\bar{v})}[w, [u, v]] = 0. \tag{2.1.26}$$

One example of Lie superalgebra is $\mathfrak{gl}(m|n)$, which is a generalization of $\mathfrak{gl}(m)$ to linear transformations on $\mathbb{R}^{m|n}$. We define this algebra as the sum of two subalgebras with different gradings:

$$\mathfrak{gl}(m|n) = \mathfrak{gl}_0(m|n) \oplus \mathfrak{gl}_1(m|n). \tag{2.1.27}$$

The even elements $u = \mathfrak{gl}_0(m|n) = \mathfrak{gl}(m) \times \mathfrak{gl}(n)$ are block diagonal matrices

$$u = \begin{pmatrix} A & 0 \\ 0 & B, \end{pmatrix} \tag{2.1.28}$$

where $A \in \mathfrak{gl}(m)$ and $B \in \mathfrak{gl}(n)$. The odd elements $v \in \mathfrak{gl}_1(m|n)$ are off diagonal matrices

$$v = \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix}. \tag{2.1.29}$$

The commutator $[u, v]$ is the commutator of the matrices, and the result is

$$[u, v] = \begin{pmatrix} 0 & (A - B)C \\ (B - A)D & 0 \end{pmatrix}, \tag{2.1.30}$$

which is an element of grading 1.

2.2 Spinors and supersymmetry algebra

2.2.1 Poincaré group and algebra

The group G_{T} of translations on $\mathbb{R}^m$ is a m-dimensional group, parameterized by translation vectors $a \in \mathbb{R}^m$ as $g(a) \in G_{\mathrm{T}}$, with the group product given by

$$g(a_1) \cdot g(a_2) = g(a_1 + a_2). \tag{2.2.1}$$

The generators P_m of translation are called momenta. For a small parameter a, we define

$$g(a) = 1 - ia^m P_m. \tag{2.2.2}$$

From the group product, we can derived the translation algebra

$$[P_m, P_n] = 0. \tag{2.2.3}$$

The generalization of $SO(n)$ for a space with a metric of signature (m, n) (instead of the Euclidean signature $(n, 0)$) is the Lorentz group $SO(m, n)$. Its fundamental representation consists of $(m + n)$-dimensional matrices that satisfy

$$\Lambda^{\mathrm{T}} \Lambda = g, \tag{2.2.4}$$

where g is the metric. Parameterizing group elements by these matrices, as $g(\Lambda)$, the group product is written as

$$g(\Lambda_1) \cdot g(\Lambda_2) = g(\Lambda_1 \Lambda_2). \tag{2.2.5}$$

To obtain the Lorentz algebra, we parameterize small transformations around the identity as $\Lambda^m_{\ n} = \delta^m_n + \omega^m_{\ n}$, where ω_{mn} is antisymmetric due to the orthogonality condition. The Lorentz generators M^{mn} are then given by

$$g(1 + \omega) = 1 - \frac{i}{2} \omega_{mn} M^{mn}. \tag{2.2.6}$$

The Lorentz algebra is then given by

$$[M^{mn}, M^{pq}] = i(g^{np} M^{mq} - g^{mp} M^{nq} - g^{nq} M^{mp} + g^{mq} M^{np}). \tag{2.2.7}$$

The Poincaré group in n dimensions the product of translation group and Lorentz group: $G^n_{\mathrm{P}} = G^n_{\mathrm{T}} \times SO(p, n-p)$. Its elements are written as $g(a, \Lambda)$, and the group product is

$$g(a_1, \Lambda_1) \cdot g(a_2, \Lambda_2) = g(a_1 + \Lambda_1 a_2, \Lambda_1 \Lambda_2). \tag{2.2.8}$$

We have translation group and Lorentz group as subgroups. The Poincaré algebra is parameterized by the P_m and M_{mn}, and the translation and Lorentz algebras are subalgebras. The commutator left to determine the Poincaré algebra is given by

$$[P^m, M^{pq}] = i(g^{mp} P^q - g^{mq} P^p). \tag{2.2.9}$$

2.2.2 Clifford algebra and spinors

The Clifford algebra in the space $\mathbb{R}^n$, with metric g^{mn}, is a superalgebra $\mathfrak{g}_{\mathrm{cl}}(n)$. An element of grading k is an antisymmetric tensor $\Gamma^{m_1 \cdots m_k}$. We represent elements of degree 1 as p-dimensional matrices Γ^m, where $p = 2^{n/2-1}$ or $p = 2^{(n+1)/2-1}$ depending whether the dimension n is even or odd. In the matrix representation, the elements of degree k are defined by the antisymmetrized product of degree 1 matrices. In particular, elements of degree 2 is given by the matrix commutator $\Gamma^{mn} = \frac{1}{2}[\Gamma^m, \Gamma^n]$.

The algebra commutator $\{\,\cdot\,,\,\cdot\,\}$ is defined so that it subtracts the grading by 2: $\mathrm{gr}(\{u, v\}) = \mathrm{gr}(u) + \mathrm{gr}(v) - 2$. In the matrix representation, the algebra commutator of elements of odd gradings is represented by the matrix anticommutator, whereas for elements of opposite grading it is the matrix commutator.

The Clifford algebra is defined by the relation on degree 1 elements

$$\{\Gamma^m, \Gamma^n\} = 2g^{mn}. \tag{2.2.10}$$

The algebra commutator on degree 2 elements then yields the Lorentz algebra

$$[\Gamma^{mn}, \Gamma^{pq}] = g^{np}\Gamma^{mq} - g^{nq}\Gamma^{mp} - g^{mp}\Gamma^{nq} + g^{mq}\Gamma^{np}, \tag{2.2.11}$$

where the i factor can be made present by properly rescaling the matrices.

The p-dimensional space where the Γ^m matrices act is called spinor space $\mathrm{Spin}(\mathbb{R}^n)$, and its elements θ^α ($\alpha = 1, \cdots, p$) are called spinors. Because Γ^{mn} satisfy the Lorentz algebra, they are generators of the Lorentz group in the spinor representation. On a spinor θ, a rotation $\Lambda = 1 + \omega$ acts as

$$\delta\theta = \frac{1}{2}\omega_{mn}\Gamma^{mn}\theta. \tag{2.2.12}$$

We take translation to act on the spinor space as $\delta\theta = 0$.

We can take the $\Gamma^m_{\alpha\beta}$ to be symmetric. For two spinors ξ and η, we have

$$\xi\Gamma^m\eta = (-1)^{\bar{\xi}\bar{\eta}}\eta\Gamma^m\xi. \tag{2.2.13}$$

In 10 dimensions ($m = 1, \cdots, 10$) the matrices satisfy the Fierz identity,

$$(\Gamma^m)_{\alpha(\beta}(\Gamma_m)_{\gamma\delta)} = 0, \tag{2.2.14}$$

where () indicates symmetrization of the indices. The identity can also be written, for spinors ξ, η, ψ, as

$$(\xi\Gamma^m\eta)\psi\Gamma_m + (-1)^{\bar{\xi}\bar{\eta}+\bar{\xi}\bar{\psi}}(\eta\Gamma^m\psi)\xi\Gamma_m + (-1)^{\bar{\psi}\bar{\xi}+\bar{\psi}\bar{\eta}}(\psi\Gamma^m\xi)\eta\Gamma_m = 0. \tag{2.2.15}$$

Another useful identity is the decomposition of the product

$$\Gamma^m\Gamma^n = \frac{1}{2}\{\Gamma^m, \Gamma^n\} + \frac{1}{2}[\Gamma^m, \Gamma^n] = g^{mn} + \Gamma^{mn}. \tag{2.2.16}$$

The spinor bundle of the manifold M is a manifold $\mathrm{Spin}(M)$ with elements (m, θ), where $m \in M$ and $\theta \in \mathrm{Spin}(\mathbb{R}^n)$, and θ transforms as a spinor under change of coordinates. One can also define a spinor bundle with k sectors: $\mathrm{Spin}^k(M)$, with k spinor linear spaces at each point $m \in M$.

One example of spinor bundle with two sectors is the one over $\mathbb{R}^n$, with the sector of spinors $\mathrm{Spin}(\mathbb{R}^n)$ and the space of antispinors $\overline{\mathrm{Spin}}(\mathbb{R}^n)$, or spinors with opposite chirality. We can also define the two sectors as $\mathrm{Spin}_L(\mathbb{R}^n)$ and $\mathrm{Spin}_R(\mathbb{R}^n)$, for left and right spinors. We write θ^α for left spinors, and $\theta^{\dot{\alpha}}$ for right spinors.

We define the space of Dirac spinors as the sum of left and right spinor spaces:

$$\mathrm{Spin}_{\mathrm{D}}(\mathbb{R}^n) = \mathrm{Spin}_L(\mathbb{R}^n) + \mathrm{Spin}_R(\mathbb{R}^n). \tag{2.2.17}$$

The Dirac matrices are defined as matrices that act on $\mathrm{Spin}_{\mathrm{D}}(\mathbb{R}^n)$ through Pauli matrices Γ^m_L on $\mathrm{Spin}_L(\mathbb{R}^n)$ and Γ^m_R on $\mathrm{Spin}_R(\mathbb{R}^n)$

$$\Gamma^m = \begin{pmatrix} \Gamma^m_L & 0 \\ 0 & \Gamma^m_R \end{pmatrix}. \tag{2.2.18}$$

2.2.3 Supersymmetry algebra

The supersymmetry algebra on an n-dimensional space is a superalgebra which is the direct sum of two linear spaces

$$\mathfrak{g}_{\mathrm{SUSY}} = \mathfrak{g}_{\mathrm{P}} + \mathfrak{g}_Q, \tag{2.2.19}$$

where $\mathfrak{g}_{\mathrm{P}}$ is the Poincaré algebra on an n-dimensional space, and $\mathfrak{g}_Q$ is a p-dimensional space with odd grading, where $p = 2^{n/2-1}$ or $p = 2^{(n+1)/2-1}$ depending whether n is even or odd. The odd parity spinor generator Q_α is an element of $\mathfrak{g}_Q$. The algebra between Q_α and Poincaré generators is given by

$$\begin{aligned} [Q, P^m] &= 0, \\ [Q, M^{mn}] &= \Gamma^{mn} Q. \end{aligned} \tag{2.2.20}$$

For a spinor bundle on a 2-dimensional manifold, we define

$$[Q, Q] = 2\Gamma^m P_m. \tag{2.2.21}$$

If we have left and right sectors we also impose independence between the sectors as

$$[Q_L, Q_R] = 0. \tag{2.2.22}$$

The representation of supersymmetry transformation on a spinor bundle parameterized by (x, θ) is given by

$$\delta x^m = \frac{1}{2}\xi\Gamma^m\theta, \qquad \delta\theta = \xi. \tag{2.2.23}$$

We can also have two supersymmetry generators Q^+ and Q^- over the same bundle sector, with

$$\begin{aligned}
[Q^+, Q^-] &= 0, \\
[Q^+, Q^+] &= 2\Gamma^m P_m, \\
[Q^-, Q^-] &= -2\Gamma^m P_m.
\end{aligned} \tag{2.2.24}$$

We call Q^- a supersymmetry transformation with flipped parity. The representation of this algebra on a spinor bundle parameterized by (x, θ) is

$$\begin{aligned}
\delta_+ x^m &= \frac{1}{2}\xi\Gamma^m\theta, \qquad && \delta_+\theta = \xi, \\
\delta_- x^m &= -\frac{1}{2}\xi\Gamma^m\theta, \qquad && \delta_-\theta = \xi.
\end{aligned} \tag{2.2.25}$$

2.3 Metric and complex structure

2.3.1 Metric

A metric is a symmetric tensor field of rank $(0, 2)$. More precisely, the space of metrics on a manifold M is given by

$$S^2(M) = T^{(0,2)}M/\sim, \tag{2.3.1}$$

with the equivalence relation

$$u_1 \otimes u_2 \sim \frac{1}{2}\left(u_1 \otimes u_2 + u_2 \otimes u_1\right), \tag{2.3.2}$$

which makes the antisymmetric products equivalent to zero.

A metric $g \in S^2(M)$ defines an inner product between vector fields $U, V \in \Gamma(TM)$. The metric is written in coordinates as

$$g = g_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j, \tag{2.3.3}$$

and the vector fields are written in coordinates as

$$U = U_i\frac{\partial}{\partial x^i}, \qquad V = V_i\frac{\partial}{\partial x^i}. \tag{2.3.4}$$

The inner product can be defined in terms of the action on the basis as

$$g(U,V) = g_{ij}U^kV^l(\mathrm{d}x^i \otimes \mathrm{d}x^j)\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right), \tag{2.3.5}$$

where we used linearity of inner products to move U^k and V^l out, and also wrote g as a linear combination of the $S^2(M)$ basis. Then we take the action of the basis of metrics on the basis of vector fields to be

$$(\mathrm{d}x^i \otimes \mathrm{d}x^j)\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right) = \left\langle dx^i, \frac{\partial}{\partial x^k}\right\rangle \left\langle dx^j, \frac{\partial}{\partial x^l}\right\rangle = \delta^i_k \delta^j_l. \tag{2.3.6}$$

The inner products thus gets written in components as

$$g(U,V) = g_{ij}U^iV^j. \tag{2.3.7}$$

A metric $g \in S^2(M)$ also defines a linear map $g : TM \to T^*M$, which takes a tangent vector field to a cotangent vector field. It is given in components by

$$g(V) = g_{ij}V^k\mathrm{d}x^i \otimes \mathrm{d}x^j\left(\frac{\partial}{\partial x^k}\right) = g_{ij}V^i\mathrm{d}x^j. \tag{2.3.8}$$

More generally, the metric can act on a tensor of rank (p,q) and take it to a tensor of rank $(p-1,q+1)$. It does so by acting with a component of its basis $\mathrm{d}x^i$ on a component of the vector field basis ∂_j, yielding the inner product $\langle \mathrm{i}, \partial_j\rangle = \delta^i_j$.

The inverse of the metric $g \in S^2(M)$ is a tensor field $g^{-1} \in T^{(2,0)}M$ such that $gg^{-1} = 1$, where $1 \in T^{(1,1)}M$ is the identity operator given by

$$1 = \sum_{i=1}^{n} \mathrm{d}x^i \otimes \frac{\partial}{\partial x^i}. \tag{2.3.9}$$

The inverse is written in components as

$$g^{-1} = g^{ij}\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}, \tag{2.3.10}$$

so its definition in components is given by $g_{ij}g^{jk} = \delta^k_i$.

2.3.2 Complex structure

A complex structure is a $(1,1)$ tensor field written as

$$I = I^i{}_j \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^j, \tag{2.3.11}$$

that satisfies the constraint

$$I^2 = -1. \tag{2.3.12}$$

It has no real eigenvalues, because its eigenvalues must satisfy $\lambda^2 = -1$. If it is defined on a complex manifold, or if the real manifold where it's defined on is complexified, than we can say its eigenvalues are $\pm i$.

We can see that it can be understood as a rotation by $\pi/2$ at each point, since applying it twice is equivalent to a reflection of the coordinates.

2.3.3 Complex structure in two dimensions

On a 2-dimensional manifold (which we will also call a "worldsheet"), a complex structure can be used to define the Hodge dual operator $\star : \Omega^1(M) \to \Omega^1(M)$, which maps 1-forms to 1-forms by the expression

$$\star u = I \cdot u, \tag{2.3.13}$$

or, in coordinates,

$$\star(u_a \mathrm{d}\sigma^a) = I^a{}_b u_a \mathrm{d}\sigma^b. \tag{2.3.14}$$

The Hodge dual must be connected with a definition of inner product by the property

$$u \wedge \star v = \langle u, v \rangle \mathrm{vol}, \tag{2.3.15}$$

where vol is the volume form. Thus complex structure must be related to a metric. We have

$$\begin{aligned} u \wedge I \cdot u &= I^b{}_c u_a u_b \mathrm{d}\sigma^a \mathrm{d}\sigma^c \\ &= I^b{}_c \epsilon^{ac} u_a u_b \mathrm{d}\sigma^1 \mathrm{d}\sigma^2, \end{aligned} \tag{2.3.16}$$

so the connection with the metric is

$$I^{\alpha}{}_{\beta}\epsilon^{\beta\gamma} = \sqrt{\det g}\, g^{\alpha\gamma}. \tag{2.3.17}$$

Chapter 3

Gauge theories and BV formalism

3.1 Field theory and gauge symmetries

3.1.1 Path integral and OPEs

We consider a field theory as a manifold Φ and a measure μ defined on it. The manifold Φ is the space of fields, and it contains fields $\phi \in \Phi$ (notice that we write it lowercase for the fields, and uppercase for the space of fields). The fields $\phi : \Sigma \to N$ are maps from the worldsheet Σ, which is a two dimensional manifold, to a target space N.

The path integral is defined by a measure of integration μ on Φ, e.g. a density on Φ, which we write as

$$\mu = [d\phi] \exp(-S(\phi)/\hbar), \tag{3.1.1}$$

where $S : \Phi \to \mathbb{R}$ is the action. The path integral is defined as integration over μ and it yields a number defined as

$$\mathcal{P} = \int [d\phi] \exp(-S(\phi)/\hbar). \tag{3.1.2}$$

We call such a theory a sigma model.

Correlation functions are computed by inserting operators in the path integral, and we take operators to be functions of the fields and its derivatives. The correlation function between operators $\mathcal{O}_1, \cdots, \mathcal{O}_n$ is defined as

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \int [d\phi] \mathcal{O}_1 \cdots \mathcal{O}_n \exp(-S(\phi)/\hbar). \tag{3.1.3}$$

Another way to define correlation functions is to think of operators as deformations of the measure. Consider the following trasformation of the action

$$S \mapsto S + \epsilon \mathcal{O}_1 \cdots \mathcal{O}_n, \tag{3.1.4}$$

for some parameter ϵ. For small values of ϵ, the measures is transformed by

$$\mu \mapsto \mu(\epsilon) = \mu \left(1 - \epsilon \mathcal{O}_1 \cdots \mathcal{O}_n\right), \tag{3.1.5}$$

so that the correlation function can be written as function of the deformation, like

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int (\mu - \mu(\epsilon)). \tag{3.1.6}$$

3.1.2 OPEs and Ward identities

A free theory is a theory with a quadratic action on the fields. Let's write the worldsheet coordinates as $(z, \bar{z})$, and write fields as $\phi^a(z)$ (we write only z for short, but it can depend on both z and $\bar{z}$). The action of a free theory depends on a local linear operator $A_{ij}(z)$ as

$$S = \int \mathrm{d}^2 z \frac{1}{2} \phi^i A_{ij} \phi^j. \tag{3.1.7}$$

In a free theory, the correlation function between two fields is the inverse of the quadratic factor, as

$$\langle \phi^i(z) \phi^j(z') \rangle = (A^{-1})^{ij}(z) \delta^2(z - z'). \tag{3.1.8}$$

Some of what follows is presented in [8].

We can consider a correlation function between two local operators (local functions of fields) very close to each other, and expand the result in powers of their distance. This is called the **operator product expansion** between two fields, or OPE. Consider the operator $P(\phi)(z, \bar{z})$ located at $(z, \bar{z})$ in the worldsheet, and an operator $Q(\phi)(0)$ located at $(z, \bar{z}) = 0$. Their operator product expansion is given by

$$P(z, \bar{z}) Q(0) = V_0(z, \bar{z}) + \frac{V_{(1,0)}}{z} + \frac{V_{(0,1)}}{\bar{z}} + \frac{V_{(1,1)}}{z \bar{z}} + \frac{V_{(2,0)}}{z^2} + \cdots, \tag{3.1.9}$$

where the operator $V_0(z, \bar{z})$ is a polynomial in z and $\bar{z}$, and $V_{(m,n)}$ are independent of $(z, \bar{z})$. To determine the operators on the right hand side, one has to compute the correlation function on the left hand side, using Wick theorem to express it in terms of correlation functions of the fields in coordinates ϕ^i.

Consider an action S which is symmetric under the vector field $v \in \text{Vect}\Phi$ in the space of fields. For every operator $U(z)$ that we insert in the path integral, we derive the following identity

$$0 = \int v(Ue^{-S}) = \int ((vU) - (vS)U)\, e^{-S}, \tag{3.1.10}$$

which is the statement that the expected value of the symmetry transformation on U has expected value zero: $\langle vU \rangle = 0$, because of the symmetry $vS = 0$.

Now let the action be $S = \int L$, for a 2-form L on the worldsheet, and let $v(\xi)$ be the generator of a global symmetry, with parameter ξ. The gauged generator $\tilde{v}_\xi$ is defined by promoting the parameter ξ to a function $\xi(z)$ on the worldsheet. The variation of the action is then

$$vS = \int \xi \mathrm{d}j, \tag{3.1.11}$$

for some 1-form j, which is the current of symmetry generated by v. When the fields are on-shell, we have $vS = 0$, and thus the current is conserved: $\mathrm{d}j = 0$.

An important expression is the relation between OPE of an operator U with the current j, which yields the action of the symmetry transformation on U, as

$$j(z)U(0) = \frac{\delta_v U}{z} + \text{other orders}. \tag{3.1.12}$$

A countour integral of the OPE selects only the factor of the simple pole, so we can write

$$\oint jU = \delta U. \tag{3.1.13}$$

We see that we can compute how fields transform under a symmetry transformation by computing their OPEs with the current.

3.1.3 Gauge theories

A **gauge group** is a group that is parameterized by functions on the worldsheet, so its dimension is infinite. The elements of such a group are called gauge transformations.

If a quantum field theory has an action which is invariant under a gauge group, it is said to have a gauge symmetry, and it is called a gauge theory. It is often the case that the correlation functions of such theories are ill-defined, because the path integral sums over redundant field configurations, and is thus infinite, in a sense. To deal with gauge theories we must do a process called gauge fixing.

Let H be the gauge group. Each element $h \in H$ is a gauge transformation that defines an automorphism on the space of fields Φ:

$$\phi \in \Phi \mapsto h(\phi) \in \Phi. \tag{3.1.14}$$

The statement that the action is invariant under H is written as

$$S(h(\phi)) = S(\phi), \quad \text{for all } h \in H, \phi \in \Phi. \tag{3.1.15}$$

The field configurations ϕ and $h(\phi)$ are said to be physically equivalent.

The aim of gauge fixing is to switch the integral over the whole space of fields to an integral which sums only fields which are not physically equivalent among each other. We will call by gauge slice such a space of fields that are not physically equivalent. More rigorously, we define a gauge slice $\Gamma \in \Phi$ as a manifold such that, for all $\phi \in \Gamma$, we have $h(\phi) \notin \Gamma$, for any transformation $h \in H$. This means that any gauge transformation takes a configuration out of the gauge slice.

One way to define a gauge slice is with a constraint. The constraint is defined by a function $K : \Phi \to V$, for some linear space V. Then the gauge slice $\Gamma \subset \Phi$ is defined by

$$K(\phi) = 0 \ \text{ iff } \ \phi \in \Gamma. \tag{3.1.16}$$

The constraint K really defines a gauge slice if it is perpendicular to any gauge transformation, i.e.

$$v^a \frac{\partial}{\partial \phi^a} K = 0, \tag{3.1.17}$$

for element v of the Lie algebra of H, which acts as a small gauge transformation.

The correct way to write the path integral as an integration only over a gauge slice is to change coordinates of the space of fields, so the integration along one gauge slice can be factored out. To change the coordinates correctly we need to know the Jacobian of the transformation, which is non trivial for spaces with infinite dimensionality. To do this properly we need to use BRST formalism to introduce ghost fields, and then we can generalize the procedure using BV formalism.

3.1.4 BRST formalism

To change coordinates in the space of fields Φ, we consider an extension of this space to the BRST field space Φ_{BRST}.

The BRST field space is defined by

$$\Phi_{\mathrm{BRST}} = \frac{\Phi \times \Pi TH}{H}. \tag{3.1.18}$$

Here ΠTH is a tangent space with flipped parity, we denote its elements as $(h, \mathrm{d}h)$. Its action on H is naturally defined.[1] The equivalence relation on eq. (3.1.18) is given by

$$(\phi, h, \mathrm{d}h) \sim (h_0 \cdot \phi, h_0 \cdot h, h_0 \cdot \mathrm{d}h), \tag{3.1.19}$$

for any $h_0 \in H$. If we choose $h_0 = h^{-1}$ to parameterize the space, we identify the isomorphism

$$\Phi_{\mathrm{BRST}} = \Phi \times \Pi\mathfrak{h}, \tag{3.1.20}$$

with $\Phi_{\mathrm{BRST}} \ni (\phi, h^{-1}\mathrm{d}h) = (\phi, c)$. The c coordinates are generators of the gauge transformation (i.e. elements of the gauge algebra), but with flipped parity, and are called ghosts.

There are vector fields on $\Phi \times \Pi TH$ which are equivalent to zero in Φ_{BRST}. They are the generators of small perturbations around $h = 1$, and are parameterized by ghosts c_0 as $h_0 = 1 + c_0 = 1 + (h_0)^{-1}\mathrm{d}h_0$. These null vector fields are given by

$$v_{c_0} = c_0 \cdot \phi \frac{\partial}{\partial \phi} + c_0 \cdot h \frac{\partial}{\partial h} + c_0 \cdot c \frac{\partial}{\partial c}. \tag{3.1.21}$$

The de Rham operator on H is a vector field on ΠTH given by

$$\mathrm{d} = \mathrm{d}h^\alpha \frac{\partial}{\partial h^\alpha}, \tag{3.1.22}$$

and it can be trivially defined on $\Phi \times \Pi TH$ with the same expression, transversal to

[1] The action on elements of ΠTH can be defined as either by the right or by the left. We take action by the left as the equivalence relation. Then the action by the right remains as an operation over Φ_{BRST}.

the direction along Φ. To descend its definition to Φ_{BRST}, we add it to a suitable null vector field v_{c_0} that makes the sum transversal to the direction along H. The resulting vector is proportional to what we define as the BRST operator

$$Q_{\mathrm{BRST}} = c \cdot \phi \frac{\partial}{\partial \phi} + \frac{1}{2}[c,c]\frac{\partial}{\partial c}, \tag{3.1.23}$$

where we used $c \cdot c = \frac{1}{2}[c,c]$.

The BRST quantization of a gauge theory with action $S \in \mathrm{Fun}\Phi$ and gauge group H consists in choosing a gauge fixing constraint $K \in \mathrm{Fun}\Phi$, along with the definition of a BRST action $S_{\mathrm{BRST}} \in \mathrm{Fun}\Phi_{\mathrm{BRST}}|_{K=0}$ which correctly contains the factor coming from the Jacobian in the path integral. Schematically, one needs to find the action such that $\int [\delta\phi] e^{-S} = \left(\int [\delta\phi']\right) \int [\delta\phi_0] e^{-S_{\mathrm{BRST}}}$, where ϕ_0 parameterizes the gauge fixed fields, and ϕ' parameterizes the degrees of freedom that decouple from the action.

The introduction of ghosts to gauge fix makes the path integral well defined. However, to recover the correct degrees of freedom, the physical states must be elements of the cohomology of the Q_{BRST} operator. A clear way to understand the BRST construction is using the BV formalism, which will be discussed in the next section.

3.2 Elements of BV formalism

Now we will introduce the definitions and results of the Batalin-Vilkovisky formalism.

3.2.1 BV phase space

To do the quantization using BV formalism, we start with a space of fields Φ and an action $S \in \mathrm{Fun}\Phi$. If we are using the BRST formalism we consider the action S with the gauge symmetry, and the BRST space of fields Φ_{BRST}, with ghosts.

The BV phase space is a cotangent space with flipped parity $\Phi_{\mathrm{BV}} = \Pi T^*\Phi$. Its elements are $(\phi, \phi^\star)$, where ϕ are the fields (including ghosts) and $\phi^\star$ are covectors in the space of fields, with flipped parity, and are called antifields.

In Φ_{BV} there is a naturally defined symplectic form ω with odd parity, which can be written, using Darboux coordinates, as

$$\omega = \delta\phi^\star_a \wedge \delta\phi^a, \tag{3.2.1}$$

where δ is the differential in the space of fields.

The inversion of the odd symplectic form defines the odd Poisson bracket, or **BV bracket**, as

$$\{\mathcal{F}, \mathcal{G}\} = \mathcal{F}\left(\frac{\overleftarrow{\delta}}{\delta\phi^a}\frac{\overrightarrow{\delta}}{\delta\phi^\star_a} - \frac{\overleftarrow{\delta}}{\delta\phi^\star_a}\frac{\overrightarrow{\delta}}{\delta\phi^a}\right)\mathcal{G}, \tag{3.2.2}$$

for any functions $\mathcal{F}, \mathcal{G} \in \Phi_{\mathrm{BV}}$. The relation between left and right derivatives is

$$\frac{\overrightarrow{\delta}}{\delta\phi}\mathcal{F} = (-1)^{(\bar{\mathcal{F}}+1)\bar{\phi}}\mathcal{F}\frac{\overleftarrow{\delta}}{\delta\phi}. \tag{3.2.3}$$

The BV bracket has the (anti)symmetry property

$$\{\mathcal{F}, \mathcal{G}\} = (-1)^{\bar{\mathcal{F}}\bar{\mathcal{G}}+\bar{\mathcal{F}}+\bar{\mathcal{G}}}\{\mathcal{G}, \mathcal{F}\}, \tag{3.2.4}$$

which means that it is symmetric, but has odd parity. It satisfies the Jacobi identity as

$$\{\mathcal{F}, \{\mathcal{G}, \mathcal{H}\}\} + (-1)^{(\bar{\mathcal{F}}+1)(\bar{\mathcal{G}}+\bar{\mathcal{H}})}\{\mathcal{G}, \{\mathcal{H}, \mathcal{F}\}\} + (-1)^{(\bar{\mathcal{H}}+1)(\bar{\mathcal{F}}+\bar{\mathcal{G}})}\{\mathcal{H}, \{\mathcal{F}, \mathcal{G}\}\} = 0, \tag{3.2.5}$$

which is the usual Jacobi identity, but taking anticommutations into account.

Another meaningful operator on an odd cotangent space is the odd Laplacian operator, or **BV Laplacian**, defined as

$$\Delta = (-1)^{\bar{A}}\frac{\partial}{\partial\phi^A}\frac{\partial}{\partial\phi^\star_A}, \tag{3.2.6}$$

where $\bar{A}$ is the parity of the coordinate ϕ^A in the BV phase space. It is related to the BV bracket by the following property

$$\Delta(\mathcal{F}\mathcal{G}) = (\Delta\mathcal{F})\mathcal{G} + (-1)^{\bar{\mathcal{F}}}\mathcal{F}(\Delta\mathcal{G}) + (-1)^{\bar{\mathcal{F}}}\{\mathcal{F}, \mathcal{G}\}. \tag{3.2.7}$$

Given a function $\mathcal{H} \in \mathrm{Fun}\Phi_{\mathrm{BV}}$ in BV phase space, one defines the Hamiltonian vector field $\{\mathcal{H}, -\}$ as

$$\{\mathcal{H}, -\} = (-1)^{(\bar{\mathcal{H}}+1)\bar{A}} \frac{\delta \mathcal{H}}{\delta \phi^A} \frac{\delta}{\delta \phi^\star_A} + (-1)^{(\bar{\mathcal{H}}+1)(\bar{A}+1)} \frac{\delta \mathcal{H}}{\delta \phi^\star_A} \frac{\delta}{\delta \phi^A}. \tag{3.2.8}$$

An important property is that, for any $\mathcal{H} \in \mathrm{Fun}\Phi_{\mathrm{BV}}$, we have

$$\mathcal{L}_{\{\mathcal{H},-\}}\omega = 0, \tag{3.2.9}$$

where $\omega = \delta\phi^A \delta\phi^\star_A$ is the symplectic form.

3.2.2 Integration on Lagrangian submanifolds

A Lagrangian submanifold $\Phi_{\mathrm{LAG}} \subset \Phi_{\mathrm{BV}}$ is a submanifold with half the dimensionality of Φ_{BV}, and where the odd symplectic form vanishes. The trivial Lagrangian submanifold is the one obtained putting all antifields to zero: $\phi^\star = 0$.

A half-density $\rho_{1/2}$ is a density with weight $1/2$. In particular, the volume form of the space of fields Φ is a half-density on the BV phase space $\Phi_{\mathrm{BV}} = \Pi T^*\Phi$.

The restriction of any half-density of the BV phase space to a Lagrangian submanifold becomes a volume form. We can thus consider, for a half-density $\rho_{1/2}$ and a Lagrangian submanifold Φ_{LAG}, the integral

$$\int_{\Phi_{\mathrm{LAG}}} \rho_{1/2}. \tag{3.2.10}$$

Consider another Lagrangian submanifold Φ'_{LAG} that is an infinitesimal deformation of the original one. Let $\{\Psi, -\}$ be the Hamiltonian vector field on Φ_{BV} that generates the variation, for an odd function $\Psi \in \mathrm{Fun}\Phi_{\mathrm{BV}}$. This deformation preserves the condition $\omega = 0$, and we call such a function a **gauge fermion**. By Stokes theorem, the integral changes by

$$\int_{\mathcal{L}_{\{\Psi,-\}}\Phi_{\mathrm{LAG}}} \rho_{1/2} = \int_{\Phi_{\mathrm{LAG}}} \mathcal{L}_{\{\Psi,-\}}\rho_{1/2}. \tag{3.2.11}$$

The action of the Lie derivative on the half density is

$$\begin{aligned} \mathcal{L}_{\{\Psi,-\}}\rho_{1/2} &= \frac{1}{2}(\mathrm{div}\{\Psi, -\})\rho_{1/2} + \{\Psi, \rho_{1/2}\} \\ &= -(\Delta\Psi)\rho_{1/2} + \{\Psi, \rho_{1/2}\}. \end{aligned} \tag{3.2.12}$$

By using the relation between the BV bracket and BV Laplacian, we get the identity

$$\mathcal{L}_{\{\Psi,-\}}\rho_{1/2} = \; - \Delta(\Psi\rho_{1/2}) + \Psi\Delta\rho_{1/2}. \tag{3.2.13}$$

The first term is a total derivative, so the variation of the integral is

$$\mathcal{L}_{\{\Psi,-\}} \int_{\Phi_{\mathrm{BV}}} \rho_{1/2} = \int_{\Phi_{\mathrm{BV}}} \Psi\Delta\rho_{1/2}. \tag{3.2.14}$$

3.2.3 Classical and quantum master equations

The trivial Lagrangian submanifold, i.e. putting antifields to zero $\phi^\star = 0$, is the original space of fields. The path integral of a quantum field theory with action $S_0 \in \mathrm{Fun}\Phi$ is given by the integration of the measure

$$\mu = [\delta\phi] \exp(-S_0/\hbar). \tag{3.2.15}$$

Consider a half-density $\rho_{1/2}$ such that its restriction to $\phi^\star = 0$ is the measure μ. Then the path integral is given by eq. (3.2.10). Under a deformation of the trivial Lagrangian submanifold, the integral remains constant if the measure satisfies

$$\Delta\rho_{1/2} = 0. \tag{3.2.16}$$

This is the **quantum master equation**.

Let's define the half-density in terms of a BV action $S \in \mathrm{Fun}\Phi_{\mathrm{BV}}$, which equal to the original action when one restricts $\phi^\star = 0$, as

$$\rho_{1/2} = [\delta\phi\delta\phi^\star]^{1/2} \exp(S/\hbar). \tag{3.2.17}$$

In terms of the action, the quantum master equation is written as

$$\frac{1}{2}\{S, S\} + \hbar\Delta S = 0. \tag{3.2.18}$$

It is a condition on the BV action, so that we can perform the path integral in any Lagrangian submanifold, and obtain the same correlation functions. A change in Lagrangian submanifold can be seen as an exchange of degrees of freedom from the fields to the antifields, so it can be used to do gauge fixing of the fields.

The **classical master equation** is the one obtained by putting $\hbar \to 0$ in the quantum master equation, and is given by

$$\{S, S\} = 0. \tag{3.2.19}$$

3.2.4 BRST-BV construction

A theory in BRST formalism consists of a BRST space of fields Φ, containing fields and ghosts, a classical action $S_{\rm cl} \in \Phi$ (the nomenclature helps to remind that this action usually doesn't depend on ghosts), and an odd vector field $Q \in \mathrm{Vect}\Phi$, which is nilpotent:

$$Q^2 = \frac{1}{2}[Q, Q] = 0 \tag{3.2.20}$$

(the brackets are for anticommutator, because Q is odd). The BRST operator Q is a symmetry of the action. Writing the fields in Φ in coordinates as $\phi^a(\sigma)$, where σ are worldspace M coordinates, and a are indices in the target space N, we write the symmetry condition as

$$QS = \int_M Q^a(\phi) \frac{\delta}{\delta\phi^a} S(\phi) = 0. \tag{3.2.21}$$

To apply BV formalism in such a theory, we consider a BV action given by

$$S(\phi, \phi^\star) = S_{\rm cl}(\phi) + \langle Q(\phi), \phi^\star \rangle. \tag{3.2.22}$$

The inner product $\langle\rangle$ contracts tangent fields with cotangent fields (antifields). We will often write it as $S = S_{\rm cl} + \hat{Q}$, where

$$\hat{Q} = \langle Q, \phi^\star \rangle = \int_M Q^a \phi^\star_a \tag{3.2.23}$$

is the generating function in BV phase space of the BRST vector field.

This BV action is linear in antifields, so it is useful to split the classical master equation in orders of antifields and see which conditions on $S_{\rm cl}$ and Q it implies. The classical master equation is

$$\{S, S\} = \{S_{\rm cl} + \hat{Q}, S_{\rm cl} + \hat{Q}\} = 2\{S_{\rm cl}, \hat{Q}\} + \{\hat{Q}, \hat{Q}\} = 0, \tag{3.2.24}$$

where we already used $\{S_{\rm cl}, S_{\rm cl}\} = 0$, because $S_{\rm cl}$ doesn't depend on antifields. At zeroth order in antifields we have

$$\{S_{\mathrm{cl}}, \hat{Q}\} = -QS_{\mathrm{cl}}, \tag{3.2.25}$$

so the symmetry of the classical action under the BRST transformation is implied. At first order in antifields we have

$$\{\hat{Q}, \hat{Q}\} = -\langle [Q, Q], \phi^\star \rangle, \tag{3.2.26}$$

so the classical master equation also constrains Q to be nilpotent. We are thus allowed to apply BV formalism with this action for a BRST system.

3.2.5 Gauge fixing and conormal bundle

The last step in applying BV formalism to a BRST theory is gauge fixing. If the classical action has gauge symmetries, it makes the path integral ill-defined, so we have to constrain some degrees of freedom of the fields. BV formalism allows us to do that by deforming the Lagrangian submanifold to another where the fields are constrained.

Consider a space of fields Φ. A gauge fixing condition is a function $K : \Phi \to V$, where V is some linear space, that defines a gauge fixed space of fields Φ_{gf} in the following way: is $K(\phi) = 0$, $\phi \in \Phi_{\mathrm{gf}}$. From the gauge condition $K = 0$ we get a constraint on tangent vectors of Φ_{gf}:

$$\delta\phi^\mu \frac{\partial K^A}{\partial \phi^\mu} = 0, \tag{3.2.27}$$

where A are components of coordinates in V.

We need to find a Lagrangian submanifold Φ_{LAG} in BV phase space $\Phi_{\mathrm{BV}} = \Pi T^*\Phi$ such that Φ_{gf} is in Φ_{LAG}. We now construct such a space: the **conormal bundle** of Φ_{gf} is a space $\Pi N^*\Phi_{\mathrm{gf}} \subset \Pi T^*\Phi$ defined by

$$\langle \delta\phi, \phi^\star \rangle = 0, \tag{3.2.28}$$

where $\delta\phi \in \Pi T\Phi_{\mathrm{gf}}$ and $\phi^* \in \Pi N^*\Phi$. It is a submanifold of BV phase space such that its base coincides with Φ_{gf}, and the cotangent fiber is constrained to be transversal to variations along Φ_{gf}.

If we define the symplectic potential as $\alpha = \langle \delta\phi, \phi^* \rangle$, we can obtain the symplectic form by $\omega = \delta\alpha$. In the conormal bundle $\Pi N^*\Phi_{\mathrm{gf}}$, we have $\alpha = 0$, and thus also

$\omega = 0$. In order to be sure that the conormal bundle $N^*\Phi_{\text{gf}}$ is be a Lagrangian submanifold, we still have to check that it has the same dimensionality as Φ. We write the dimension of the constrained space as

$$\dim \Phi_{\text{gf}} = \dim \Phi - \dim K(\Phi), \tag{3.2.29}$$

so the dimension of the fiber of the conormal bundle has to be the dimension of the gauge fixing constraint K. We thus parameterize the antifields by $\Psi \in \Pi V$, as

$$\phi^\star_\mu = \Psi_A \frac{\partial K^A}{\partial \phi^\mu}. \tag{3.2.30}$$

This parameterization is consistent with eq. (3.2.28), because of the constraint in eq. (3.2.27).

3.3 Non-linearity and derived brackets

Until now we have only discussed the construction of a BV action which is linear in antifields, and have a BRST-like structure. However, we saw that under a change of coordinates equivalent to a deformation of Lagrangian submanifold, the BV action might acquire non-linear terms on antifields.

Let's consider a general BV action, and expand it in powers of antifields, as

$$S = S_0 + S_1 + S_2 + \cdots , \tag{3.3.1}$$

where S_k is of order k in antifields, as

$$S_k = \frac{1}{k}\sigma_k^{A_1\cdots A_k}\phi^\star_{A_1}\cdots\phi^\star_{A_k}. \tag{3.3.2}$$

The classical master equation imposes constraints between the S_k terms. We have

$$0 = \{S, S\} = \{S_0 + S_1 + \cdots , S_0 + S_1 + \cdots\}. \tag{3.3.3}$$

When we split the equation order by order in power of antifields, we get the list of the conditions

$$\begin{aligned} \{S_0, S_1\} &= 0, \\ \{S_0, S_2\} + \frac{1}{2}\{S_1, S_1\} &= 0, \\ \{S_0, S_3\} + \{S_1, S_2\} &= 0, \\ \{S_0, S_4\} + \{S_1, S_3\} + \frac{1}{2}\{S_2, S_2\} &= 0, \\ &\cdots . \end{aligned} \tag{3.3.4}$$

The generalization can be made at order n, for even n, as

$$\sum_{i=0}^{n/2} \{S_i, S_{n-i}\} = 0, \tag{3.3.5}$$

and for odd n as

$$\frac{1}{2}\{S_{(n+1)/2}, S_{(n+1)/2}\} + \sum_{i=0}^{(n-1)/2} \{S_i, S_{n-i}\} = 0. \tag{3.3.6}$$

At first order, the condition states that the BRST operator $Q = \{S_1, -\}$ is a symmetry of S_0, as

$$\{S_0, S_1\} = \, - Q^a \frac{\delta S_0}{\delta \varphi^a} = 0. \tag{3.3.7}$$

To look at the other equations, we can use the condition that the action S_0 is on-shell. We can easily do it by taking

$$\{S_0, S_k\} = 0. \tag{3.3.8}$$

We are then putting to zero the equations of motion that are perpendicular to the directions of the σ_k indices. That can be seen by writing the condition in coordinates:

$$\sigma_k^{AB_1 \cdots B_{k-1}} \frac{\delta S_0}{\delta \phi^A} = 0. \tag{3.3.9}$$

3.3.1 Derived brackets

We can see that the S_k term defines a k-bracket on the field, i.e. a map from k functions $F_1, \cdots, F_k \in \Phi$ on the space of fields to numbers. The S_k-bracket is given by

$$\{F_1, \cdots, F_k\}_{S_k} := \{\{\cdots\{S_1, F_1\}, \cdots\}, F_k\}. \tag{3.3.10}$$

and is called a derived bracket.

In particular, $\{F\}_{S_1} = \{S_1, F\}$ is simply the action of the vector field generated by S_1 on the function F (in BRST, the vector field is Q). Aĺso, the second order term defines a Poisson-like bracket

$$\{F, G\}_{S_2} = \{\{S_2, F\}, G\}, \tag{3.3.11}$$

which may or may not satisfy Jacobi identity.

Let's check the symmetry of the S_2-bracket. Using Jacobi identity of BV bracket, we compute

$$\begin{aligned} \{F, G\}_{S_2} =& -(-1)^{\bar{F}\bar{G}+\bar{G}}\{\{G, S_2\}F\} - (-1)^{\bar{F}+\bar{G}}\{\{F, G\}, S_2\} \\ =& (-1)^{(\bar{F}+1)(\bar{G}+1)}\{G, F\}_{S_2} \end{aligned} \tag{3.3.12}$$

where we used that $\{F, G\} = 0$, because none of the functions depends on antifields.

We can now check what are the conditions from the classical master equation on the derived brackets. The constraint at first order in antifields is

$$\{S_0, S_2\} + \frac{1}{2}\{S_1, S_1\} = 0, \tag{3.3.13}$$

and it implies

$$\begin{aligned} Q^2 F =& \{\{F\}_{S_1}\}_{S_1} = \{S_1, \{S_1, F\}\} \\ =& -\frac{1}{2}\{F, \{S_1, S_1\}\} = \{F, \{S_0, S_2\}\}. \end{aligned} \tag{3.3.14}$$

We see that this condition makes Q nilpotent only if the on-shell condition $\{S_0, S_2\}$ is applied. It doesn't mean necessarily putting all fields on-shell, but only the fields in the direction of σ_2.

Another important condition is the one at third order, that is

$$\{S_0, S_4\} + \{S_1, S_3\} + \frac{1}{2}\{S_2, S_2\} = 0. \tag{3.3.15}$$

If $S_3 = 0$, and if impose the $\{S_0, S_4\} = 0$ on-shell condition, then $\{S_2, S_2\} = 0$

implies that the S_2 bracket satisfies Jacobi identity.

3.3.2 Deformations of the action

If we want to compute correlation functions between operators, for a theory with an action $S_0 \in \Phi_{\mathrm{BV}}$, we have to consider a suitable deformation, like

$$S_0 \mapsto S_0 + \epsilon V_0, \tag{3.3.16}$$

where $V_0 \in \Phi_{\mathrm{BV}}$ is an operator, and ϵ is a small real parameter.

In BV formalism, a deformation of the theory consists in a deformation of the BV action, which contains more information than just the classical information. Let $S \in \Phi_{\mathrm{BV}}$ be the BV action, and $V \in \Phi_{\mathrm{BV}}$ the deformation operator, so that the deformation is

$$S \mapsto S + \epsilon V. \tag{3.3.17}$$

Every term of the BV action gets deformed. We split the action like $S = \Sigma_k S_k$, so we also split the deformation operator in orders of antifields as

$$V = V_0 + V_1 + V_2 + \cdots . \tag{3.3.18}$$

Then each term gets deformed by

$$S_k \mapsto S_k + \epsilon V_k. \tag{3.3.19}$$

This amounts to a deformation of the classical action S_0 by V_0, but also a deformation of the BRST operator $S_2 = \hat{Q}$ by V_2, and so on.

The classical master equation imposes a restriction on what deformation operators V are allowed. Let's take ϵ to be a small parameter, so we can compute the linear deviation from the classical master equation. It is

$$\frac{1}{2}\{S + \epsilon V, S + \epsilon V\} \approx \frac{1}{2}\{S, S\} + \epsilon\{S, V\} = 0, \tag{3.3.20}$$

so that the deformation operator must satisfy $\{S, V\} = 0$. At each order, the condition becomes

$$\begin{aligned}
\{S_0, V_1\} + \{S_1, V_0\} &= 0, \\
\frac{1}{2}\{S_0, V_2\} + \{S_1, V_1\} + \frac{1}{2}\{S_2, V_1\} &= 0, \\
\{S_0, V_3\} + \{S_1, V_2\} + \{S_2, V_1\} + \{S_3, V_0\} &= 0, \\
&\cdots .
\end{aligned} \tag{3.3.21}$$

We can generalize it: for even n,

$$\{S_{n/2}, V_{n/2}\} + \frac{1}{2}\sum_{i=0}^{n}\{S_i, V_{n-i}\} = 0, \tag{3.3.22}$$

and, for odd n,

$$\sum_{i=0}^{n}\{S_i, V_{n-i}\} = 0. \tag{3.3.23}$$

As an example, take the deformation of the BRST operator as $V_2 = K^A \phi^\star_A$, for a vector field $K \in \mathrm{Vect}\Phi_{\mathrm{BV}}$. The following relation must hold:

$$K^A \frac{\delta S_0}{\delta \phi^A} + Q^A \frac{\delta V_0}{\delta \phi^A} = 0, \tag{3.3.24}$$

i.e. the BRST variation of the deformation of the classical action must cancel the K variation of the classical action.

Another example: consider a linear BV action $S = S_0 + \hat{Q}$. Then, after putting fields on-shell, we must have

$$\{S_1, V_1\} = \; - \langle [Q, K], \phi^\star \rangle = 0, \tag{3.3.25}$$

so the BRST operator and its deformation must commute.

There is a special type of transformation. If the deformation operator is the generating function of a Hamiltonian vector field, i.e.

$$V = \langle \{\Psi, -\}, \phi^\star \rangle = \{\Psi, \phi^\star\}, \tag{3.3.26}$$

then the deformed action is BRST trivial, because it is equivalent to a deformation of Lagrangian submanifold. In other words, it only changes the fixed gauge.

Chapter 4

Bosonic string sigma model

4.1 Formulation with metric

4.1.1 Action and gauge symmetries

The bosonic string sigma model describes the propagation of quantum strings on spacetime. A configuration is described by a worldsheet Σ, which is a smooth 2-dimensional surface embedded in a D-dimensional spacetime. The worldsheet embedding can be described by a parameterization $x : V \subset \mathbb{R}^2 \to M$, where M is spacetime, which takes worldsheet coordinates σ^a to spacetime coordinates $x^i(\sigma^a)$.

The amplitude of a worldsheet configuration is $\exp(-S_{\mathrm{NG}})$, where S_{NG} is the Nambu-Goto action. This action is proportional to the area of the worldsheet surface. With a given parameterization, it is written as

$$S_{\mathrm{NG}}(x) = T \int_\Sigma d^2\sigma \sqrt{\det\left(\frac{\partial x}{\partial\sigma^\alpha} \cdot \frac{\partial x}{\partial\sigma^\beta}\right)}, \tag{4.1.1}$$

where the inner product is given by the spacetime metric. The proportionality constant T is the string tension, which is sometimes written as

$$T = \frac{1}{2\pi\alpha'}, \tag{4.1.2}$$

where α' is then taking as the weighing factor of the action.

There is an another action that describes the same dynamics. Consider a metric g on the worldsheet. The Polyakov action depends on the spacetime coordinates x and on the worldsheet metric γ, and is defined by

$$S_{\mathrm{P}}(x,\gamma) = \frac{T}{2} \int_\Sigma d^2\sigma \sqrt{\det\gamma}\,\gamma^{\alpha\beta}\partial_\alpha x \cdot \partial_\beta x. \tag{4.1.3}$$

The classical equations of motion for the worldsheet metric implies that it becomes the induced spacetime metric on-shell. Putting only the metric on-shell, the Polyakov action reduces to the Nambu-Goto action. This means that the correlation functions involving only the x fields are equivalent for both actions. In other words, we recover the path integral with the Nambu-Goto action if we integrate the Polyakov path integral over worldsheet metric configurations.

Because of the introduction of the metric as a field, the Polyakov action is invariant under diffeomorphisms, i.e. change of coordinates of the worldsheet. The generators of diffeomorphisms are vector fields $v \in \mathrm{Vect}\Sigma$ on the worldsheet, and the action of this symmetry on the fields is

$$\begin{aligned} \mathcal{L}_v x &= v^\alpha \partial_\alpha x, \\ \mathcal{L}_v \gamma_{\alpha\beta} &= v^\gamma \partial_\gamma \gamma_{\alpha\beta} + \partial_\alpha v^\gamma \gamma_{\gamma\beta} + \partial_\beta v^\gamma \gamma_{\alpha\gamma}. \end{aligned} \tag{4.1.4}$$

There is an additional gauge symmetry on the Polyakov action. For any scalar field $\phi \in \mathrm{Fun}\Sigma$, the Weyl transformation is defined as

$$\gamma_{\alpha\beta}(\sigma) \mapsto e^{\phi(\sigma)} \gamma_{\alpha\beta}(\sigma), \tag{4.1.5}$$

i.e. it is a rescaling of the metric independently at each worldsheet point. It is a symmetry precisely because the worldsheet has two dimensions. The generators of the trasformation are also functions on the worldsheet — we can think of them as infinitesimal versions of the transformation. The action of the generator can be written as

$$\delta_\phi \gamma_{\alpha\beta} = \phi \gamma_{\alpha\beta}, \tag{4.1.6}$$

and it defines the trivial Lie algebra of Weyl transformations: $[\delta_{\phi_1}, \delta_{\phi_2}] = 0$.

4.1.2 BV construction

To construct the BRST field space, we introduce the diffeomorphism ghosts $c \in \Pi\mathrm{Vect}\Sigma$, which are vector fields on the worldsheet with odd parity, and the Weyl ghosts $\theta \in \mathrm{Fun}\Sigma$, which are functions on the worldsheet.

The BRST operator is then a vector field $Q \in \mathrm{Vect}\Phi$ in the BRST space of fields, given by

$$Q = \int d^2\sigma \left(\mathcal{L}_c x^m \frac{\delta}{\delta x^m} + (\mathcal{L}_c \gamma_{\alpha\beta} + \theta \gamma_{\alpha\beta}) \frac{\delta}{\delta \gamma_{\alpha\beta}} + \mathcal{L}_c c^\alpha \frac{\delta}{\delta c^\alpha} + \mathcal{L}_c \theta \frac{\delta}{\delta \theta} \right), \tag{4.1.7}$$

where we used that

$$\frac{1}{2}[c,c]^\alpha = c^\beta \partial_\beta c^\alpha = \frac{1}{2}\mathcal{L}_c c^\alpha \tag{4.1.8}$$

and

$$[c,\theta]^\alpha = c^\alpha \partial_\alpha \theta = \mathcal{L}_c \theta. \tag{4.1.9}$$

The BV phase space of the theory of bosonic strings is given by D fields $x \in \mathrm{Fun}\Sigma$, the metric g on the worldsheet, the diffeomorphism ghost $c \in \Pi\mathrm{Vect}\Sigma$, the Weyl ghost $\theta \in \mathrm{Fun}\Sigma$, and their antifields. The antifield of x is a D-plet scalar density of odd parity $x^\star$. The antifield of the metric is a symmetric $(2,0)$-tensor density $(\gamma^\star)^{\alpha\beta}$ on the worldsheet. The antifield of the ghost is a vector field density $c^\star$ on the worldsheet, with even parity (the parity flipping for being BRST ghost and BV antifield add up). Finally, the antifield of Weyl ghost is simply a density $\theta^\star$ on the worldsheet, with flipped parity.

The BV action for a theory in flat space is thus

$$\begin{aligned} S = \int d^2\sigma \bigg(& \frac{T}{2}\sqrt{\det\gamma}\,\gamma^{ab}\partial_a x^m \partial_b x^m(x) \\ & + \mathcal{L}_c x^m x^\star_m + (\mathcal{L}_c \gamma_{\alpha\beta} + \theta\gamma_{\alpha\beta})(\gamma^\star)_{\alpha\beta} + \mathcal{L}_c c^\alpha c^\star_\alpha + \mathcal{L}_c \theta\theta^\star \bigg). \end{aligned} \tag{4.1.10}$$

4.1.3 Gauge fixing: light-cone gauge

As an example of gauge fixing in BV formalism, we will consider the light-cone gauge used in Chapter 1 of [9].

Let's fix the light-cone gauge for an infinite string. First we define the light-cone coordinates

$$x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1). \tag{4.1.11}$$

Then we fix diffeomorphism in σ^0 direction by imposing

$$x^+ = \sigma^0, \tag{4.1.12}$$

we fix Weyl symmetry putting

$$\det\gamma = -1. \tag{4.1.13}$$

To fix diffeomorphism in σ^1 direction we define

$$\mathrm{d}l = \frac{\gamma_{11}\mathrm{d}\sigma^1}{\sqrt{-\det\gamma}} = \gamma_{11}\mathrm{d}\sigma^1 \tag{4.1.14}$$

which is invariant under σ^1 diffeormorphism, so we can fix

$$\sigma^1 = \beta(\sigma^0)\int_0^{\sigma^1}\mathrm{d}l = \beta(\sigma^0)\int_0^{\sigma^1}\gamma_{11}\mathrm{d}\sigma^1, \tag{4.1.15}$$

where β is some function that fixes the proportionality for each σ^0. Differentiating with respect to σ^1 we get $1 = \beta(\sigma^0)\gamma_{11}(\sigma^0, \sigma^1)$, so we get the constraint

$$\partial_1\gamma_{11} = 0. \tag{4.1.16}$$

The line defined by $\sigma^1 = 0$ along the string is arbitrarily fixed when we choose coordinates for the worldsheet. Thus we still have a residual symmetry due to the transformation $\delta\sigma^1 = f(\sigma^0)$, for a function f. To fix it, we impose

$$\gamma_{01}(\sigma^0, 0) = 0, \tag{4.1.17}$$

which fixes perpendicularity between σ^0 and σ^1 at the origin.

Now we are able to write the gauge fixing constraint $K = 0$ as

$$\begin{pmatrix} K^1 \\ K^2 \\ K^3 \\ K^4 \end{pmatrix} = \begin{pmatrix} -\det\gamma - 1 \\ x^+ - \sigma^0 \\ \partial_1\gamma_{11} \\ \gamma_{01}\delta(\sigma^1) \end{pmatrix} \tag{4.1.18}$$

The variations of the constraints are

$$\begin{aligned} \frac{\delta K^1}{\delta\gamma_{\alpha\beta}} &= \gamma^{\alpha\beta}\delta(\sigma - \sigma'), & \frac{\delta K^2}{\delta x^+} &= \delta(\sigma - \sigma'), \\ \frac{\delta K^3}{\delta\gamma_{11}} &= -\partial_1\delta(\sigma - \sigma'), & \frac{\delta K^4}{\delta\gamma_{01}} &= \delta(\sigma^1)\delta(\sigma - \sigma'). \end{aligned} \tag{4.1.19}$$

Then the conormal bundle gets the antifields parameterized by Ψ_A, as

$$\begin{aligned} x_+^\star &= \Psi_2, \\ (\gamma^\star)^{11} &= \partial_1\Psi_3 - \Psi_1\gamma^{11}, \\ (\gamma^\star)^{01} &= \Psi_4\delta(\sigma^1) - \Psi_1\gamma^{01}, \\ (\gamma^\star)^{00} &= \Psi_1\gamma^{00}, \end{aligned} \tag{4.1.20}$$

and $c^* = 0$ and $\theta^* = 0$.

From $\det\gamma = -1$ we get the following form for the metric:

$$\gamma_{\alpha\beta} = \begin{pmatrix} -(1-\gamma_{01}^2)\gamma_{11}^{-1} & \gamma_{01} \\ \gamma_{01} & \gamma_{11} \end{pmatrix}, \tag{4.1.21}$$

and the inverse can be written as

$$\gamma^{\alpha\beta} = \begin{pmatrix} -\gamma_{11} & \gamma_{01} \\ \gamma_{01} & (1-\gamma_{01}^2)\gamma_{11}^{-1} \end{pmatrix}. \tag{4.1.22}$$

We recall that $\gamma_{01}(\sigma^0, 0) = 0$, and γ_{11} is a function of σ^0 only.

The light-cone gauge action is then the BV action restricted to the conormal bundle. Let's write it as

$$S_{\text{LC}} = S_{\text{P}} + S_{\text{gh}}. \tag{4.1.23}$$

The first term comes from the Polyakov action restriction on the submanifold, and it is

$$\begin{aligned} S_{\text{P}} = \frac{T}{2}\int \mathrm{d}^2\sigma\Big(&- 2\gamma_{11}\partial_0 x^- + 2\gamma_{01}\partial_1 x^- + \gamma_{11}\partial_0 x^i\partial_0 x^i \\ &- 2\gamma_{01}\partial_0 x^i\partial_1 x^i - (1-\gamma_{01}^2)\gamma_{11}^{-1}\partial_1 x^i\partial_1 x^i\Big). \end{aligned} \tag{4.1.24}$$

The second term is the one that comes from $\hat{Q}$, and it becomes

$$\begin{aligned} S_{\text{gh}} = \int \mathrm{d}^2\sigma\Big(&- c^0\Psi_2 + \theta(\gamma_{11}\partial_1\Psi_3 + 2\Psi_1 + \gamma_{01}\Psi_4\delta(\sigma^1)) \\ &- (c^\alpha\partial_\alpha\gamma_{11} + 2\partial_1 c^\alpha\gamma_{1\alpha})\partial_1\Psi_3 \\ &- (c^\gamma\partial_\gamma\gamma_{\alpha\beta}\gamma^{\alpha\beta} + 2\partial_\alpha c^\alpha)\Psi_1 \\ &- (c^\alpha\partial_\alpha\gamma_{01} + 2\partial_0 c^\alpha\gamma_{\alpha 1})\Psi_4\delta(\sigma^1)\Big), \end{aligned} \tag{4.1.25}$$

where we used $\gamma_{\alpha\beta}\gamma^{\alpha\beta} = 2$.

We see that the action acquires a term involving ghosts and antifields, which took the degrees of freedom from the gauge fixed fields. In BV formalism they appear from the deformation of the Lagrangian submanifold. The higher order terms of the BV action also change when deforming the path integral, and it modifies the way that the BRST symmetry acts on the fields. We'll explore it in the next section.

4.2 Complex structure as a field

There is another formulation of the bosonic string, which is formulated in a smaller space of fields: the worldsheet metric field is replaced by a complex structure on the worldsheet. The action does not depend on the scale of the metric at each point, so we can write it in terms of the complex structure. This removes Weyl symmetry, so we don't have this gauge symmetry anymore. This construction follows [2].

We recall the relation between Hodge operator and complex structure by acting on the form $\mathrm{d}x$, where d is the de Rham operator in the worldsheet. We have

$$\star \mathrm{d}x = I^{\alpha}{}_{\beta} \partial_\alpha x \; \mathrm{d}\sigma^b. \tag{4.2.1}$$

We are thus able to write the Polyakov action as

$$S_{\mathrm{P}}(x, I) = \frac{T}{2} \int_\Sigma \mathrm{d}x^m \star \mathrm{d}x^n G_{mn}, \tag{4.2.2}$$

where we have explicitly written the spacetime metric as G_{mn}.

Now the BRST operator can be reduced to contain only action of diffeomorphisms, and become

$$Q = \int d^2\sigma \left(\mathcal{L}_c x^m \frac{\delta}{\delta x^m} + \mathcal{L}_c I^{\alpha}{}_{\beta} \frac{\delta}{\delta I^{\alpha}{}_{\beta}} + \mathcal{L}_c c^\alpha \frac{\delta}{\delta c^\alpha} \right). \tag{4.2.3}$$

The BV phase space has a very similar structure to the formulation with the metric: removing Weyl ghost and its antifield, and switching the metric with the complex structure. However, the antifield of the complex structure is the subtlest one, because the complex structure satisfies a constraint.

Let's parameterize the space of all complex structures on a two dimensional manifold (or worldsheet). We'll work here with $z\bar{z}$ coordinates, which are given by

$$z = \sigma^1 + i\sigma^2, \qquad \bar{z} = \sigma^1 - i\sigma^2, \tag{4.2.4}$$

i.e. a change of coordinates to a complexified chart. We will also use the notation $z^\alpha = (z^1, z^2) = (z, \bar{z})$.

Consider the space of mixed $(1,1)$ tensors of the worldsheet, whose elements are

$$I = I^\alpha_{\ \beta}\, dz^\beta \frac{\partial}{\partial z^\alpha}. \tag{4.2.5}$$

The constraint $I^2 + 1 = 0$ defines a submanifold of which the elements are complex structures.

To find a parametrization for I, we parameterize a general $(1,1)$ tensor with four independent parameters in $z\bar{z}$ coordinates as

$$I = \begin{pmatrix} i\rho & im \\ -i\bar{m} & -i\bar{\rho}. \end{pmatrix}. \tag{4.2.6}$$

The constraint is then given by

$$\begin{pmatrix} -\rho^2 + m\bar{m} & im(\rho - \bar{\rho}) \\ i\bar{m}(\rho - \bar{\rho})\bar{m} & -\bar{\rho}^2 + m\bar{m} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.2.7}$$

which is solved by

$$I = \begin{pmatrix} i\sqrt{1 + m\bar{m}} & im \\ -i\bar{m} & -i\sqrt{1 + m\bar{m}} \end{pmatrix}. \tag{4.2.8}$$

The complex structure which corresponds to a flat metric is

$$I_{(0)} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tag{4.2.9}$$

from the relation $I^\alpha_{\ \beta}\epsilon^{\beta\gamma} = \sqrt{\det g}\, g^{\alpha\gamma}$. We get $I = I_{(0)}$ by taking $m\bar{m} = 0$.

We can expand the non-polynomial component in power series on the fields m and $\bar{m}$. Using the expansion

$$\sqrt{1 + m\bar{m}} = 1 + \frac{1}{2} m\bar{m} - \frac{1}{4}(m\bar{m})^2 + \cdots, \tag{4.2.10}$$

we can split I in different orders on the m and $\bar{m}$ parameters, as

$$I = I_{(0)} + I_{(1)} + I_{(2)} + I_{(4)} + \cdots \tag{4.2.11}$$

with

$$\begin{aligned} I_{(0)} &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad I_{(1)} = \begin{pmatrix} 0 & im \\ -i\bar{m} & 0 \end{pmatrix}, \\ I_{(2)} &= \begin{pmatrix} \frac{1}{2}m\bar{m} & 0 \\ 0 & \frac{1}{2}m\bar{m} \end{pmatrix}, \end{aligned} \tag{4.2.12}$$

and so on. Notice that there are no terms of odd order, except for the first order term.

Because of the constraint of the complex structure, its antifield has a gauge symmetry. In Appendix B.1 we describe it and gauge fix it. We are then able to write the complex structure as

$$I = i\sqrt{1+m\bar{m}}\left(\frac{\partial}{\partial z}dz - \frac{\partial}{\partial \bar{z}}d\bar{z}\right) + im\frac{\partial}{\partial \bar{z}}dz - i\bar{m}\frac{\partial}{\partial z}d\bar{z}, \tag{4.2.13}$$

and the antifield of the complex structure as

$$I^{\star} = dzd\bar{z}\,\frac{1}{4}\left(-b\;dz\frac{\partial}{\partial \bar{z}} + \bar{b}\;d\bar{z}\frac{\partial}{\partial z}\right). \tag{4.2.14}$$

The components of $I^{\star}$ are called b and $\bar{b}$ to agree with the standard notation, as they turn out to be the ghosts that generate the energy-momentum tensor.

The BV bracket between $m\bar{m}$ and $b\bar{b}$ show that they are Darboux coordinates:

$$\begin{aligned} &\{m(z), b(z')\} = 4i\delta^2(z-z'), \qquad \{m(z), \bar{b}(z')\} = 0, \\ &\{\bar{m}(z), b(z')\} = 0, \qquad \{\bar{m}(z), \bar{b}(z')\} = 4i\delta^2(z-z'). \end{aligned} \tag{4.2.15}$$

Thus b is the antifield of m and $\bar{b}$ is the antifield of $\bar{m}$, apart from a factor of $4i$. We are able to write the odd symplectic form in Darboux coordinates as

$$\omega = \int \left(\delta x \delta x^{\star} + \delta c \delta c^{\star} + \delta \bar{c} \delta \bar{c}^{\star} - \frac{i}{4}\delta m \delta b - \frac{i}{4}\delta \bar{m} \delta \bar{b}\right). \tag{4.2.16}$$

4.2.1 Expansion around Polyakov gauge

To write the Polyakov action in terms of $m\bar{m}$ let's use the relation $dx = dz\partial x + d\bar{z}\bar{\partial}x$, where $\partial = \partial/\partial z$ and $\bar{\partial} = \partial/\partial\bar{z}$, and the complex structure parameterized as in eq. (4.2.13). Then $I \cdot dx$ can be computed as $\iota_I dx$. We also note that the relation between the $dzd\bar{z}$ and the measure on the worldsheet $d^2z = d\tau d\sigma$ is

$$dzd\bar{z} = (d\tau + id\sigma)(d\tau - id\sigma) = -2id^2z. \tag{4.2.17}$$

Then the classical action is written as

$$S_{cl} = \int d^2z \left(\sqrt{1 + m\bar{m}}\partial x\bar{\partial}x + \frac{1}{2}m(\partial x)^2 + \frac{1}{2}\bar{m}(\bar{\partial}x)^2 \right). \tag{4.2.18}$$

To write the BRST operator in terms of $m\bar{m}$ and $b\bar{b}$, we must write the Lie derivative of the complex structure I in terms of the $m\bar{m}$ parameters, and substitute the gauge fixed $I^\star$ in terms of $b\bar{b}$. We get

$$\begin{aligned}\hat{Q} = \int d^2z \Big(&\mathcal{L}_c x x^\star + \frac{1}{2}\mathcal{L}_c c c^\star + \sqrt{1 + m\bar{m}}(b\bar{\partial}c + \bar{b}\partial\bar{c}) \\ &- \frac{1}{2}\left(c(\partial m)b + \bar{c}(\bar{\partial}m)b - (\partial c)mb + (\bar{\partial}\bar{c})mb\right) \\ &- \frac{1}{2}\left(c(\partial\bar{m})\bar{b} + \bar{c}(\bar{\partial}\bar{m})\bar{b} + (\partial c)\bar{m}\bar{b} - (\bar{\partial}\bar{c})\bar{m}\bar{b}\right) \Big).\end{aligned} \tag{4.2.19}$$

We can rewrite the BRST operator with the derivatives acting on $b\bar{b}$ instead of on $m\bar{m}$. It is

$$\begin{aligned}\hat{Q} = \int d^2z \Big(&\mathcal{L}_c x x^\star + \frac{1}{2}\mathcal{L}_c c c^\star + \sqrt{1 + m\bar{m}}(b\bar{\partial}c + \bar{b}\partial\bar{c}) \\ &+ \frac{1}{2}\left(c(\partial b)m + \bar{c}(\bar{\partial}b)m + 2(\partial c)bm\right) \\ &+ \frac{1}{2}\left(c(\partial\bar{b})\bar{m} + \bar{c}(\bar{\partial}\bar{b})\bar{m} + 2(\bar{\partial}\bar{c})\bar{b}\bar{m}\right) \Big).\end{aligned} \tag{4.2.20}$$

We can now change coordinates of the BV phase space, and let $b\bar{b}$ be fields and $m\bar{m}$ be the antifields. This change of coordinates parameterizes the BV phase space around a different conormal bundle. By putting the antifields to zero, we obtain the **Polyakov gauge**, or conformal gauge. When $m = \bar{m} = 0$, the complex structure is the one related to a flat metric.

We rewrite the BV action as a power series in the antifields

$$S = S_0 + S_1 + S_2 + \cdots, \tag{4.2.21}$$

where the term S_k is of kth power on $x^\star$, $c^\star$ and $m\bar{m}$. We see right away that $x^\star$ and $c^\star$ appear only linearly, so the non-linear terms come from the expansion of S_{cl} and $\hat{Q}$ in terms of $m\bar{m}$.

The term S_0 is

$$S_0 = \int d^2z \left(\partial x \bar{\partial} x + b\bar{\partial} c + \bar{b}\partial\bar{c}\right) \tag{4.2.22}$$

and it is the action we get upon fixing Polyakov gauge.

The first order term is given by

$$\begin{aligned} S_1 = \int d^2z \bigg(& \mathcal{L}_c x x^\star + \frac{1}{2}\mathcal{L}_c c c^\star + \frac{1}{2} m(\partial x)^2 + \frac{1}{2}\bar{m}(\bar{\partial} x)^2 \\ & + \frac{1}{2}\left(c(\partial b)m + \bar{c}(\bar{\partial} b)m + 2(\partial c) b m\right) \\ & + \frac{1}{2}\left(c(\partial \bar{b})\bar{m} + \bar{c}(\bar{\partial}\bar{b})\bar{m} + 2(\bar{\partial}\bar{c})\bar{b}\bar{m}\right)\bigg). \end{aligned} \tag{4.2.23}$$

It defines the BRST operator in Polyakov gauge, which generates BRST symmetry on S_0.

The last term that we want to write is S_2, given by

$$S_2 = \int d^2z \frac{1}{2}\left(\partial x \bar{\partial} x + b\bar{\partial} c + \bar{b}\partial\bar{c}\right) m\bar{m}. \tag{4.2.24}$$

This coefficient of $m\bar{m}$ is a bivector, which defines Poisson brackets in the space of fields.

4.2.2 BRST transformation in Polyakov gauge

The Polyakov gauge was defined as the choice of Lagrangian submanifold given by fixing the complex structure. The fields in this gauge are $x, c, \bar{c}, b$ and $\bar{b}$, and the action is S_0, given in eq. (4.2.22). The equations of motion that come from this action are

$$\begin{aligned} \partial\bar{\partial}x &= 0, \\ \partial\bar{c} &= \bar{\partial}c = 0, \\ \partial\bar{b} &= \bar{\partial}b = 0. \end{aligned} \tag{4.2.25}$$

The term S_1 of the BV action, given by eq. (4.2.23), defines the BRST transformation of the fields. The BRST vector field Q on the space of fields is defined as $S_1 = Q^a(\phi)\phi^\star_a$. Then the x, c and $\bar{c}$ fields transform as

$$\begin{aligned} Qx &= c\partial x + \bar{c}\bar{\partial}x, \\ Qc &= c\partial c + \bar{c}\bar{\partial}c, \\ Q\bar{c} &= c\partial\bar{c} + \bar{c}\bar{\partial}\bar{c}. \end{aligned} \tag{4.2.26}$$

To find the transformations of $b, \bar{b}$ we first carefully write the relevant term in S_1 as

$$\begin{aligned} (QI)I^\star = & -2i\int d^2z\,((QI)^z{}_{\bar{z}}(I^\star)_{z\bar{z}z}{}^{\bar{z}} + (QI)^{\bar{z}}{}_z(I^\star)_{z\bar{z}\bar{z}}{}^z) \\ = & -\frac{1}{2}\int d^2z\,\left((Qm)b + (Q\bar{m})\bar{b}\right) \\ = & \frac{1}{2}\int d^2z\,\left((Qb)m + (Q\bar{b})\bar{m}\right). \end{aligned} \tag{4.2.27}$$

We used the normalization (4.2.15) of the Darboux coordinates of $b\bar{b}$ in the BV phase space, and also the relation $dzd\bar{z} = -2id^2z$. Then, paying attention to the 1/2 term in front of the transformation, we get

$$\begin{aligned} Qb &= c\partial b + \bar{c}\bar{\partial}b + 2(\partial c)b + (\partial x)^2, \\ Q\bar{b} &= c\partial\bar{b} + \bar{c}\bar{\partial}\bar{b} + 2(\bar{\partial}\bar{c})\bar{b} + (\bar{\partial}x)^2. \end{aligned} \tag{4.2.28}$$

The BRST operator defined by these transformations is only nilpotent on-shell, as consequence from the relations between S_0, S_1 and S_2 that come from the classical master equation.

4.3 Energy-momentum tensor and OPEs

The energy-momentum tensor of the theory is defined to be the result of the variation of S_0 with respect to the worldsheet metric. To compute the variation we must consider the minimal coupling of a metric, which recovers the Polyakov action. Then we compute the variation with respect to the metric – or, in our case, to the complex structure, since the action does not depend on the scale of the metric. And then we fix again the complex structure to the one we fixed in Polyakov gauge. So we are able to define the two independent components of the energy-momentum tensor as

$$T = \left(\frac{\delta S}{\delta m}\right)_{\phi^\star=0}, \qquad \bar{T} = \left(\frac{\delta S}{\delta \bar{m}}\right)_{\phi^\star=0}, \tag{4.3.1}$$

where $S = S_0 + S_1 + \cdots$ is the BV action. This shows us that the energy-momentum tensor is defined in terms of S_1, and is related with the BRST operator acting on the b fields, as

$$T = Qb, \qquad \bar{T} = Q\bar{b}. \tag{4.3.2}$$

However the definition of energy-momentum that is often considered is the one defined on-shell. So the standard definition is obtained by using eqs. (4.2.25) in the expressions of eq. (4.2.28), so we get

$$\begin{aligned} T &= (\partial x)^2 + c\partial b + 2(\partial c)b, \\ \bar{T} &= (\bar{\partial} x)^2 + \bar{c}\bar{\partial}\bar{b} + 2(\bar{\partial}\bar{c})\bar{b}. \end{aligned} \tag{4.3.3}$$

Now we can compute OPEs involving the (normal ordered) energy momentum tensor. As we're working in BV formalism, we are able to compute the OPEs off-shell, and put the result on-shell afterwards. We'll see that this procedure yields different results: we can get contact terms in the OPEs.

From the action S_0 in eq. (4.2.22) we compute the basic OPEs

$$\begin{aligned} x^m(z)x^n(0) &= -\frac{1}{2}\delta^{mn}\ln|z|^2 + \text{regular}, \\ b(z)c(0) &= \frac{1}{z} + \text{regular}, \\ \bar{b}(z)\bar{c}(0) &= \frac{1}{\bar{z}} + \text{regular}, \end{aligned} \tag{4.3.4}$$

from which we can compute OPEs of composite fields.

4.3.1 OPE with ghost current

The ghost number current is defined as

$$j = \; :bc:, \quad \bar{j} =: \bar{b}\bar{c}:, \tag{4.3.5}$$

and comes from the $U(1)$ symmetry $b \to e^{ia}b$, $c \to e^{-ia}c$, and analogously for $\bar{b}$ and $\bar{c}$. Then the on-shell OPE of T with j is given by

$$T(z)j(0) = \frac{3}{z^3} + \frac{j}{z^2} + \frac{\partial j}{z} + \text{regular} \tag{4.3.6}$$

We see the conformal weight of j, i.e. the factor multiplying j on the second order pole, is 1. Also, the action of translation on j is ∂j, which we see from the first order pole. This kind of transformation determines that j is a primary operator, in the language of conformal theories [8].

Using the off-shell expression for the energy-momentum tensor, we have the following contributions to the OPE:

$$T(z)j(0) = c(z)\partial b(z)b(0)c(0) + \bar{c}(z)\bar{\partial} b(z)b(0)c(0) + 2\partial c(z)b(z)b(0)c(0). \tag{4.3.7}$$

The first term is obtained by performing contractions, and expanding around $z = 0$. As we are off-shell, the fields are not holomorphic, and we must expand in both z and $\bar{z}$ coordinates. We get

$$\begin{aligned} c(z)\partial b(z)b(0)c(0) &= \frac{1}{z^3} - \frac{:b(0)c(z):}{z^2} - \frac{:\partial b(z)c(0):}{z} \\ &= \frac{1}{z^3} - \frac{:bc:}{z^2} - \frac{\partial :bc:}{z} - \frac{\bar{z}}{z^2} :b\bar{\partial}c: . \end{aligned} \tag{4.3.8}$$

The third term of eq. (4.3.7) is

$$\begin{aligned} 2\partial c(z)b(z)b(0)c(0) &= \frac{2}{z^3} + \frac{2:b(z)c(0):}{z^2} + \frac{2:b(0)\partial c(z):}{z} \\ &= \frac{2}{z^3} + \frac{2:bc:}{z^2} + \frac{2\partial :bc:}{z} + 2\frac{\bar{z}}{z^2} :\bar{\partial} bc: . \end{aligned} \tag{4.3.9}$$

Finally, the second term, which comes strictly from the off-shell contribution, is given by

$$\bar{c}(z)\bar{\partial} b(z)b(0)c(0) = \; :b\bar{c}: \delta^2(z). \tag{4.3.10}$$

Summing all of the terms, the off-shell Tj OPE yields

$$T(z)j(0) = \frac{3}{z^3} + \frac{j}{z^2} + \frac{\partial j}{z} + \frac{\bar{z}}{z^2}(2 : \bar{\partial} bc : - : b\bar{\partial} c :) + \delta^2(z) : b\bar{c} : . \tag{4.3.11}$$

When we put the off-shell contraction on-shell, we almost recover the on-shell contraction:

$$T(z)j(0) = \frac{3}{z^3} + \frac{j}{z^2} + \frac{\partial j}{z} + \delta^2(z) : b\bar{c} : . \tag{4.3.12}$$

The additional term, which is proportional to a delta function, is called a contact term.

4.4 Integration over moduli space

The BV quantization of the Polyakov action is built upon the gauge symmetry of diffeomorphisms (and Weyl transformations, if we work with the metric instead of complex structure). Thus when choosing a Lagrangian submanifold we fix the parameterization of the worldsheet, specifying the complex structure $I^{(0)}$. However we cannot map all complex structures by diffeomorphisms.

The space of complex structures modulo diffeomorphisms is called moduli space. For smooth manifolds we can parameterize the moduli space by the length of the closed curves that cannot be dragged one into the other. For the sphere, every closed curve can be dragged into a point, thus the moduli space is trivial – it has only one element. As for a torus, there are two independent curves – the moduli space has two real dimensions. For higher genus, the real dimension of the moduli space is $6g - 6$.

So to compute correlation function of some operator $\mathcal{O}$ (e.g. the product of local operators) on the worldsheet, we must construct the correlation function on a Lagrangian submanifold, corresponding to a point in moduli space, and then integrate over all moduli. For that, we must define an integration measure

We define a mapping Ω from operators to functions of μ and $\bar{\mu}$, given by

$$\Omega(\mathcal{O})_{\mu,\bar{\mu}} = \left\langle \exp\left(\int (\mu T + \bar{\mu}\bar{T})\right) \mathcal{O} \right\rangle , \tag{4.4.1}$$

where on the right hand side the $\langle\rangle$ brackets mean integration on the Lagrangian submanifold, and $T = Qb$, $\bar{T} = Q\bar{b}$ are components of the energy-momentum tensor.

The parameters μ and $\bar{\mu}$ are the ones that parameterize variations of the fixed complex structure, $I^{(0)}$. To get the integration measure of the moduli space, we expand the exponential in power series, and pick the term with the correct dimensionality, and the same number of b and $\bar{b}$. For instance, for the torus the integration measure is

$$\left\langle \int \mu Q b \int \bar{\mu} Q \bar{b}\ \mathcal{O} \right\rangle, \tag{4.4.2}$$

whereas for $g > 1$ the measure is

$$\left\langle \left(\prod_{i=1}^{3g-3} \int \mu T \int \bar{\mu} \bar{T} \right) \mathcal{O} \right\rangle. \tag{4.4.3}$$

The expression defines a measure on the moduli space, when we take $\mu, \bar{\mu}$ to be infinitesimal variations around the fixed Lagrangian submanifold.

The deformations around the Lagrangian submanifold are generated by the gauge fermion

$$\psi = -\int (\mu b + \bar{\mu} \bar{b}), \tag{4.4.4}$$

which a function on the BV phase that depends only on b and $\bar{b}$. It defines the Hamiltonian vector field

$$\{\psi, -\} = \mu \frac{\delta}{\delta m} + \bar{\mu} \frac{\delta}{\delta \bar{m}}, \tag{4.4.5}$$

which changes $m \mapsto m + \mu$ and $\bar{m} \mapsto \bar{m} + \bar{\mu}$. The Polyakov gauge is slightly changed, and it induces a reparameterization of the BV phase space.

We will later introduce a new parameterization of the space of complex structures, and thus a new parameterization of the BV phase space. The new fields will be also written as μ and $\bar{\mu}$, and they will be useful to understand the deformations along the moduli space.

4.4.1 Dolbeault operator and the $\mu\bar{\mu}$ fields

We have parameterized the complex structure with the fields m and $\bar{m}$, which are Darboux coordinates to the b and $\bar{b}$ fields. To introduce the $\mu\bar{\mu}$ parameterization, we will introduce the Dolbeault operators, which are eigenvectors of the complex structure. We will describe the space of complex structures as the space of deformations of these operators.

In $z\bar{z}$ coordinates we can write the complex structure locally as

$$I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tag{4.4.6}$$

so it has eigenvalues $\pm i$. We define the Dolbeault operators as

$$\partial := \mathrm{d}z\frac{\partial}{\partial z}, \qquad \bar{\partial} := \mathrm{d}\bar{z}\frac{\partial}{\partial \bar{z}}, \tag{4.4.7}$$

which satisfy $\mathrm{d} = \partial + \bar{\partial}$, where d is the de Rham operator. Their action on the X fields are eigenfunctions of the complex structures, as

$$I\partial X = i\partial X, \qquad I\bar{\partial} X = -i\bar{\partial} X. \tag{4.4.8}$$

Then we can write Polyakov action in terms of ∂ and $\bar{\partial}$, which gives

$$S = \frac{-i}{2}\int \partial X \wedge \bar{\partial} X, \tag{4.4.9}$$

which has the form of the Polyakov action with a flat metric.

To parameterize locally the possible gauge fixing choices we can fix the complex structures to different values. It should be equivalent to fixing the complex structure to be flat in different coordinates.

Let's parameterize the family of complex structures by functions μ and $\bar{\mu}$, and call the complex structure $I^{[\mu]}$. For $\mu = \bar{\mu} = 0$ we get the flat one, $I^{[0]}$. The Dolbeault operators ∂ and $\bar{\partial}$ are eigenvectors of $I^{[0]}$. We can then define deformed Dolbeault operators by

$$\begin{aligned} I^{[\mu]}\partial^{[\mu]} &= i\partial^{[\mu]}, \\ I^{[\mu]}\bar{\partial}^{[\mu]} &= -i\bar{\partial}^{[\mu]}. \end{aligned} \tag{4.4.10}$$

We fix their normalization by asking that

$$\mathrm{d} = \partial^{[\mu]} + \bar{\partial}^{[\mu]} \tag{4.4.11}$$

continues to hold.

The way we want to parameterize $\partial^{[\mu]}$ and $\bar{\partial}^{[\mu]}$ is the following. Define the vector fields

$$v^{[\mu]} = \frac{\partial}{\partial z} + \bar{\mu}\frac{\partial}{\partial \bar{z}}, \tag{4.4.12}$$

$$\bar{v}^{[\mu]} = \frac{\partial}{\partial \bar{z}} + \mu\frac{\partial}{\partial z}. \tag{4.4.13}$$

The Dolbeault operators are mixed 2-tensors, so we define them as

$$\partial^{[\mu]} = u^{[\mu]}v^{[\mu]}, \qquad \bar{\partial}^{[\mu]} = \bar{u}^{[\mu]}\bar{v}^{[\mu]}, \tag{4.4.14}$$

for a family of forms $u^{[\mu]}$ and $\bar{u}^{[\mu]}$. Upon the definition of eq. (4.4.13), the forms are fixed by eq. (4.4.11). They are given by

$$u^{[\mu]} = \frac{dz - \mu d\bar{z}}{1-\mu\bar{\mu}}, \qquad \bar{u}^{[\mu]} = \frac{d\bar{z} - \bar{\mu}dz}{1-\mu\bar{\mu}}. \tag{4.4.15}$$

So the Dolbeault operators are explicitly given by

$$\begin{aligned}\partial^{[\mu]} &= \frac{1}{1-\mu\bar{\mu}}(dz - \bar{\mu}d\bar{z})\left(\frac{\partial}{\partial z} + \bar{\mu}\frac{\partial}{\partial \bar{z}}\right),\\ \bar{\partial}^{[\mu]} &= \frac{1}{1-\mu\bar{\mu}}(d\bar{z} - \mu dz)\left(\frac{\partial}{\partial \bar{z}} + \mu\frac{\partial}{\partial z}\right).\end{aligned} \tag{4.4.16}$$

4.4.2 Action in terms of $\mu\bar{\mu}$ fields

The Polyakov action is defined in terms of the complex structure $I^{[\mu]}$ as

$$S_{cl} = -\frac{1}{4}\int dx \wedge I^{[\mu]}dx, \tag{4.4.17}$$

where we make the dependence on μ and $\bar{\mu}$ explicit. Then it is written in terms of the Dolbeault operators as

$$\begin{aligned}S_{cl} &= -\frac{1}{4}\int(\partial^{[\mu]}x + \bar{\partial}^{[\mu]}x) \wedge I^{[\mu]}(\partial^{[\mu]}x + \bar{\partial}^{[\mu]}x)\\ &= \frac{i}{2}\int \partial^{[\mu]}x \wedge \bar{\partial}^{[\mu]}x.\end{aligned} \tag{4.4.18}$$

Using eq. (4.4.16) to write the dependence on $\mu, \bar{\mu}$ explicitly, we get

$$S_{cl} = \int d^2z \left(\frac{1+\mu\bar{\mu}}{1-\mu\bar{\mu}}\frac{\partial x}{\partial z}\frac{\partial x}{\partial \bar{z}} + \frac{\mu}{1-\mu\bar{\mu}}\left(\frac{\partial x}{\partial z}\right)^2 + \frac{\bar{\mu}}{1-\mu\bar{\mu}}\left(\frac{\partial x}{\partial \bar{z}}\right)^2\right). \tag{4.4.19}$$

Then we compare it with the Polyakov action in terms of the $m, \bar{m}$ coordinates (eq. (4.2.18)), which can be written in terms of the Dolbeault operators as

$$S_{cl} = \int d^2z \left(\sqrt{1+m\bar{m}} \frac{\partial x}{\partial z}\frac{\partial x}{\partial \bar{z}} + \frac{m}{2}\left(\frac{\partial x}{\partial z}\right)^2 + \frac{\bar{m}}{2}\left(\frac{\partial x}{\partial \bar{z}}\right)^2 \right). \tag{4.4.20}$$

We can then relate the coordinates $m, \bar{m}$ with the coordinates $\mu, \bar{\mu}$. The relation is

$$m = \frac{2\mu}{1-\mu\bar{\mu}}, \qquad \bar{m} = \frac{2\bar{\mu}}{1-\mu\bar{\mu}}. \tag{4.4.21}$$

This makes a clear connection between the Darboux coordinates $m, \bar{m}$ and the Dolbeault parameters $\mu, \bar{\mu}$.

Other useful relations are

$$\sqrt{1+m\bar{m}} = \frac{1+\mu\bar{\mu}}{1-\mu\bar{\mu}} \tag{4.4.22}$$

and

$$\mu = \frac{m}{1+\sqrt{1+m\bar{m}}}, \qquad \bar{\mu} = \frac{\bar{m}}{1+\sqrt{1+m\bar{m}}}. \tag{4.4.23}$$

The $\mu\bar{\mu}$ fields parameterize the complex structure as

$$I = \frac{i}{1-\mu\bar{\mu}} \begin{pmatrix} 1+\mu\bar{\mu} & 2\mu \\ -2\bar{\mu} & -1-\mu\bar{\mu} \end{pmatrix}. \tag{4.4.24}$$

There must exist fields B and $\bar{B}$ which are Darboux conjugates to μ and $\bar{\mu}$, as

$$\begin{aligned} &\{\mu(z), B(w)\} = \delta(z-w), \qquad &\{\mu(z), \bar{B}(w)\} = 0, \\ &\{\bar{\mu}(z), B(w)\} = 0 \qquad &\{\bar{\mu}(z), \bar{B}(w)\} = \delta(z-w). \end{aligned} \tag{4.4.25}$$

In Appendix (B.3), we show that B and $\bar{B}$ parameterize $I^\star$ by

$$I^\star = \frac{1-\mu\bar{\mu}}{1+\mu\bar{\mu}} \begin{pmatrix} 0 & \bar{B}-\mu^2 B \\ -B+\bar{\mu}^2\bar{B} & 0 \end{pmatrix}. \tag{4.4.26}$$

Now let's write the BV action in terms of $\mu\bar{\mu}$ and $B\bar{B}$ coordinates. The classical action is

$$S_{cl} = \int d^2z \left(\frac{1+\mu\bar{\mu}}{1-\mu\bar{\mu}} \partial x \bar{\partial} x + \frac{\mu}{1-\mu\bar{\mu}} (\partial x)^2 + \frac{\bar{\mu}}{1-\mu\bar{\mu}} (\bar{\partial} x)^2 \right), \tag{4.4.27}$$

and the BRST generator becomes

$$\begin{aligned} \hat{Q} = \int d^2z \Big(&\mathcal{L}_c x x^\star + \frac{1}{2} \mathcal{L}_c c c^\star + (B - \bar{\mu}^2 \bar{B}) \bar{\partial} c + (\bar{B} - \mu^2 B) \partial \bar{c} \\ &+ \mu (c \cdot \partial + 2(\partial c)) B + \bar{\mu} (c \cdot \partial + 2(\bar{\partial} \bar{c})) \bar{B} \Big). \end{aligned} \tag{4.4.28}$$

The BV expansion around Polyakov gauge is

$$\begin{aligned} S_0 &= \int d^2z (\partial x \bar{\partial} x + B \bar{\partial} c + \bar{B} \partial \bar{c}), \\ S_1 &= \int d^2z (\mathcal{L}_c x x^\star + \frac{1}{2} \mathcal{L}_c c c^\star + \mu\big((\partial x)^2 + (c \cdot \partial + 2(\partial c)) B\big) \\ &\qquad + \bar{\mu}\big((\bar{\partial} x)^2 + (c \cdot \partial + 2(\bar{\partial} \bar{c})) \bar{B}\big), \\ S_2 &= \int d^2z \left(2\mu\bar{\mu} \partial x \bar{\partial} x - \mu^2 B \partial \bar{c} - \bar{\mu}^2 \bar{B} \bar{\partial} c \right), \end{aligned} \tag{4.4.29}$$

which is generalized as

$$\begin{aligned} S_k &= \int d^2z (-1)^{(k+1)/2} (\mu\bar{\mu})^{(k-1)/2} \big(\mu (\partial x)^2 + \bar{\mu} (\bar{\partial} x)^2 \big), \quad \text{odd } k, \\ S_k &= \int d^2z (2\mu\bar{\mu})^{k/2} \partial x \bar{\partial} x, \quad \text{even } k. \end{aligned} \tag{4.4.30}$$

We can see that the structure of S_0 and S_1 in $\mu\bar{\mu}$ coordinates is the same as for $m\bar{m}$ coordinates. Thus the two parameterizations of the field space are equal to the view of on-shell BRST formalism, when the higher terms vanish.

4.4.3 Holomorphic factorization

Now we'll make some comments about the results of [3]. They shine a light in the importance of the $\mu\bar{\mu}$ coordinates.

Consider the classical action, written in terms of a linear operator on the worldsheet,

$$S_{cl} = \frac{1}{2} \int d^2z \frac{1}{1-\mu\bar{\mu}} \begin{pmatrix} \partial x & \bar{\partial} x \end{pmatrix} \begin{pmatrix} 2\mu & 1+\mu\bar{\mu} \\ 1+\mu\bar{\mu} & 2\bar{\mu} \end{pmatrix} \begin{pmatrix} \partial x \\ \bar{\partial} x \end{pmatrix}. \tag{4.4.31}$$

The μ and $\bar{\mu}$ fields parameterize the deformations of the action in Polyakov gauge along the moduli space. We write the deformation as the operator insertion $\exp(\delta S(\mu, \bar{\mu}))$, where

$$\delta S = S_{cl}(\mu, \bar{\mu}) - S_{cl}(0, 0). \tag{4.4.32}$$

The action in the Polyakov gauge gives rise to the OPE

$$\partial x^m(z)\bar{\partial} x^n(w) = \frac{1}{2} g^{mn} \delta(z - w), \tag{4.4.33}$$

which is a contact term, i.e. it is a delta function on the wordlsheet. We can resum these terms by acting with an operator on the path integral deformation, using the identity

$$\begin{aligned} \exp\left(\frac{1}{2} G_{\alpha\beta} \frac{\partial}{\partial \partial_\alpha x} \frac{\partial}{\partial \partial_\beta x}\right) \exp\left(-\frac{1}{2} A^{\alpha\beta} \partial_\alpha x \partial_\beta x\right) \\ = \exp\left(-\frac{1}{2}\left(A(1 + GA)^{-1}\right)^{\alpha\beta} \partial_\alpha x \partial_\beta x\right), \end{aligned} \tag{4.4.34}$$

where A is the quadratic operator in δS. We take

$$A = -\frac{2}{1 - \mu\bar{\mu}} \begin{pmatrix} \mu & \mu\bar{\mu} \\ \mu\bar{\mu} & \bar{\mu} \end{pmatrix}, \qquad G = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.4.35}$$

To write the resummed deformation, we define a normal ordering $\times\!\!\times\;\times\!\!\times$ that means that all contact terms are removed. The new way to write the deformation is

$$e^{\delta S} = \times\!\!\times \exp\left(\int d^2z (\mu(\partial x)^2 + \bar{\mu}(\bar{\partial} x)^2)\right) \times\!\!\times . \tag{4.4.36}$$

Under this normal ordering, the classical action then depends linearly on deformations of the complex structure. Moreover, in the expansion of the path integral in powers of μ and $\bar{\mu}$, which parameterize a deformation over the moduli space around the Polyakov gauge, there are only terms μ^k and $\bar{\mu}^k$. Every time there should be $\mu\bar{\mu}$ in the expansion, there is a contact term, then the normal ordering removes it.

This result is called holomorphic factorization: the amplitudes are sums of holomorphic functions – which depend on μ, plus antiholomorphic ones, depending on $\bar{\mu}$. Thus, even though the μ and $\bar{\mu}$ coordinates are not Darboux conjugates to the b and $\bar{b}$ fields, they are the coordinates that decouple in the deformations.

Chapter 5

Pure spinor superstring

The pure spinor superstring is a formalism that describes the propagation of a string in a supersymmetric spacetime. The formalism consists of an action, which is invariant under supersymmetry, and a BRST-like fermionic operator that defines physical states. In this chapter we describe the formalism.

5.1 The target space

In the flat space pure spinor superstring description, the target space locally the product of the supersymmetric spacetime M, which is a fermionic spinor bundle with two sectors (left and right) over an even 10 dimensional manifold (spacetime), with the space of the so called pure spinors. We now describe each of this objects with more detail.

Consider the supermanifold $M^{10|32}$ as the spacetime with 10 even dimensions and 32 odd dimensions. We write its elements as (x, θ_L, θ_R), where x^m are the even coordinates and θ_L^α and $\theta_R^{\dot{\alpha}}$ are the odd coordinates, split in two parts: one called left moving, with 16 coordinates, and the other called right moving, with the remaining 16 degrees of freedom.

Consider the group

$$G = SO(10, \mathbb{C}) \times \mathbb{C} \times SO(10, \mathbb{C}) \times \mathbb{C}. \tag{5.1.1}$$

The spacetime $M^{10|32}$ can be written as a coset space

$$M^{10|32} = \frac{E}{G}, \tag{5.1.2}$$

where E is a principal bundle generated by the free action of G on $M^{10|32}$.

The pure spinor bundle over $M^{10|32}$ is defined by the equivalence relation

$$PS(M) = \frac{E \times V}{G} \tag{5.1.3}$$

where V is given by

$$V = \frac{V_0}{\sim} \times \frac{V_0}{\sim}, \tag{5.1.4}$$

where $V_0 = \text{Spin}(\mathbb{C})$ is a spinor bundle on 10d spacetime, with 16d complex spinors λ^α as elements, and the equivalence relation is the pure spinor constraint, given by

$$\lambda\Gamma^m\lambda = 0, \tag{5.1.5}$$

where $\Gamma^m_{\alpha\beta}$ are the Pauli matrices. The action of G rotates and rescales either the left and right moving spinors. The elements of $PS(M)$ are $(x, \theta_L, \theta_R, \lambda_L, \lambda_R)$.

5.2 The action

The pure spinor string is a sigma model on a 2-dimensional bosonic worldsheet Σ, on the target space $PS(M)$. The fields are functions on the worldsheet with values on $PS(M)$. We call the space of fields $\Phi \ni \Sigma \to PS(M)$.

To define the action, we consider the extension of the space of fields to contain momenta of the fields. The Hamiltonian phase space of fields Φ_H consists of the fields $(x, \theta_L, \theta_R, \lambda_L, \lambda_R)$, and the momenta (P, p_L, p_R, w_L, w_R), which are 1-forms on the worldsheet.

The inner product in the phase space involves an integral over a spatial slice, which we take to be S^1 for closed strings (topologically a circle). The symplectic potential is given by

$$\alpha = P_A\delta\phi^A = \int_{S_1} (P_m\delta x^m + p_L\delta\theta_L + p_R\delta\theta_R + w_L\delta\lambda_L + w_R\delta\lambda_R), \tag{5.2.1}$$

and the symplectic form is $\omega = \delta\alpha$.

The action is given by

$$S_0 = \int \mathrm{d}\tau^+\mathrm{d}\tau^- \left(\frac{1}{2}\partial_+ x\partial_- x + p_{L+}\partial_-\theta_L + p_{R-}\partial_+\theta_R - w_{L+}\partial_-\lambda_L - w_{R-}\lambda_R\right). \tag{5.2.2}$$

The action depends on the momenta $p_{L,R}$ and $w_{L,R}$ only through one component each, because of $\mathrm{d}\tau^+\mathrm{d}\tau^+ = 0$, etc. We write, for short, $p_{L+} = p_+$, $p_{R-} = p_-$, $w_{L+} = w_+$ and $w_{R-} = w_-$.

Varying the action with respect to $\partial_\tau \theta_L$, we get the momentum constraint

$$p_{\theta_L} = -2\frac{\partial S_0}{\partial \partial_\tau \theta_L} = p_+. \tag{5.2.3}$$

Similarly, the momenta of the other spinor fields are p_-, w_+ and w_-. For the momentum of x, the constraint is

$$P_m = -2\frac{\partial S_0}{\partial \partial_\tau x^m} = -\frac{1}{2}\partial_\tau x_m. \tag{5.2.4}$$

The action is quadratic, so the equations of motion are simple:

$$\begin{aligned} & \partial_+\partial_- x^m = 0, \\ \partial_-\theta_L = 0, \qquad & \partial_+\theta_R = 0, \\ \partial_- p_+ = 0, \qquad & \partial_+ p_- = 0, \\ \partial_-\lambda_L = 0, \qquad & \partial_+\lambda_R = 0, \\ \partial_- w_+ = 0, \qquad & \partial_+ w_- = 0. \end{aligned} \tag{5.2.5}$$

Under the on-shell condition, the left fields depend only on τ^+ and the right fields depend only on τ^-.

Because of the pure spinor constraint $\lambda\Gamma\lambda = 0$, the momenta $w_\pm$ have a gauge symmetry under the transformation

$$\begin{aligned} \delta w_+ &= \Lambda_+^m \Gamma_m \lambda_L, \\ \delta w_- &= \Lambda_-^m \Gamma_m \lambda_R. \end{aligned} \tag{5.2.6}$$

for worldsheet 1-form, spacetime vectors, $\Lambda^m = \Lambda_+^m \mathrm{d}\tau^+$ and $\Lambda^m = \Lambda_-^m \mathrm{d}\tau^-$.

5.2.1 Symmetries and charges

For a vector field $v \in \mathrm{Vect}\Phi$ given by

$$v = \int v^A(\phi)\frac{\delta}{\delta\phi^A}, \tag{5.2.7}$$

we define its generating function on the Hamiltonian phase space Φ_H as

$$\hat{v} = \int v^A(\phi) P_A + u(\phi) \tag{5.2.8}$$

where $u \in \text{Fun}\Phi$ is an arbitrary function of the fields, and doesn't change the action of v on Φ, but changes the action in the momenta P_A. The generating functions are also called charges of the symmetries.

The supersymmetry transformation, in a negative momentum representation, on the left sector and right sector, with fermionic spinor parameters ξ_L and ξ_R respectively, is given by the vector fields

$$\begin{aligned} S_L(\xi_L) &= \int \left(-\frac{1}{2}\xi_L \Gamma^m \theta_L \frac{\delta}{\delta x^m} + \xi_L \frac{\delta}{\delta \theta_L} \right), \\ S_R(\xi_R) &= \int \left(-\frac{1}{2}\xi_R \Gamma^m \theta_R \frac{\delta}{\delta x^m} + \xi_R \frac{\delta}{\delta \theta_R} \right), \end{aligned} \tag{5.2.9}$$

so their charges are of the form

$$\begin{aligned} \hat{S}_L(\xi_L) &= \int \left(-\frac{1}{2}\xi_L \Gamma^m \theta_L P_m + \xi_L p_+ \right) + u_L(\xi_L), \\ \hat{S}_R(\xi_R) &= \int \left(-\frac{1}{2}\xi_R \Gamma^m \theta_R P_m + \xi_R p_- \right) + u_R(\xi_R). \end{aligned} \tag{5.2.10}$$

The action on the fields and momenta is computed by $S = \{\hat{S}, -\}$, and the transformations are, for the left supersymmetry only,

$$\begin{aligned} S_L x^m &= -\frac{1}{2}\xi_L \Gamma^m \theta_L, \\ S_L \theta_L &= \xi_L, \\ S_L p_+ &= -\frac{1}{2}\xi_L \Gamma^m P_m - \frac{\delta u_L}{\delta \theta_L}, \end{aligned} \tag{5.2.11}$$

and $S_L\theta_R = S\lambda_L = S\lambda_R = 0$. Also, we assume u_L is a function of x and θ_L only, so that we have $S_L p_- = S_L w_+ = S_L w_- = 0$. We have used the antisymmetry of the Poisson brackets in the Hamiltonian field space: $\{x, P\} = -\{P, x\}$ and $\{\theta, p_+\} = \{p_+, \theta\}$ (the sign from anticommuting θ and p_+ cancels the antisymmetry of the brackets). Another point is that $(\overrightarrow{\delta}/\delta\theta_L)u_L = -u_L(\overleftarrow{\delta}/\delta\theta_L)$.

The action of left sector supersymmetry transformation on the action is

$$\begin{aligned} S_L S_0 &= \int (-S_L x_m \partial_+ \partial_- x^m + S_L p_+ \partial_- \theta_L + p_+ \partial_- S_L \theta_L) \\ &= \int \left(\frac{1}{2}(\xi_L \Gamma_m \theta_L)\partial_+\partial_- x^m + S_L p_+ \partial_- \theta_L \right), \end{aligned} \tag{5.2.12}$$

For the action to be invariant under supersymmetry, we choose the action on p_+ to be

$$S_L p_+ = \frac{1}{2}\xi_L \Gamma^m \partial_+ x_m, \tag{5.2.13}$$

which fixes

$$\begin{aligned}\frac{\delta u_L}{\delta \theta_L} &= -\frac{1}{2}\xi_L \Gamma^m (P_m + \partial_+ x_m)\\ &= -\frac{1}{4}\xi_L \Gamma^m \partial_\sigma x_m.\end{aligned} \tag{5.2.14}$$

We integrate it to obtain

$$u_L = \int \frac{1}{4}(\xi_L \Gamma^m \theta_L)\partial_\sigma x_m, \tag{5.2.15}$$

So the left supersymmetry generator becomes

$$\hat{S}_L = \int \left(\frac{1}{2}(\xi_L \Gamma^m \theta_L)\partial_+ x_m + \xi_L p_+\right). \tag{5.2.16}$$

Analogously, the right supersymmetry charge becomes

$$\hat{S}_R = \int \left(\frac{1}{2}(\xi_R \Gamma^m \theta_R)\partial_- x_m + \xi_R p_-\right). \tag{5.2.17}$$

Now we'll consider the a BRST-like operator Q that is the generator of supersymmetry in the positive momentum representation, and take the pure spinor fields as generators. It is the sum of Q_L, which acts on the left sector, and Q_R, which acts on the right sector. They are given by

$$\begin{aligned}Q_L &= \int \left(\frac{1}{2}\lambda_L \Gamma^m \theta_L \frac{\delta}{\delta x^m} + \lambda_L \frac{\delta}{\delta \theta_L}\right),\\ Q_R &= \int \left(\frac{1}{2}\lambda_R \Gamma^m \theta_R \frac{\delta}{\delta x^m} + \lambda_R \frac{\delta}{\delta \theta_R}\right).\end{aligned} \tag{5.2.18}$$

The charge of the left supersymmetry on the Hamiltonian function is

$$\hat{Q}_L = \int \left(\frac{1}{2}(\lambda_L \Gamma^m \theta_L)P_m + \lambda_L p_+\right) + v_L, \tag{5.2.19}$$

which defines the following transformations on the fields and momenta

$$Q_L x^m = \frac{1}{2}\lambda_L \Gamma^m \theta_L, \qquad Q_L P_m = -\frac{\delta v_L}{\delta x^m},$$
$$Q_L \theta_L = \lambda_L, \qquad Q_L p_+ = \frac{1}{2}\lambda_L \Gamma^m P_m + \frac{\delta v_L}{\delta \theta_L},$$
$$Q_L \lambda_L = 0, \qquad Q_L w_+ = -\frac{1}{2}\theta_L \Gamma^m P_m - p_+ - \frac{\delta v_L}{\delta \lambda_L}. \tag{5.2.20}$$

The left BRST transformation on the action gives us

$$Q_L S_0 = \int \left(\frac{1}{2}(\lambda_L \Gamma^m \partial_- \theta_L)(P_m + \partial_+ x_m) + \frac{\delta v_L}{\delta \theta_L}\partial_- \theta_L \right.$$
$$\left. + \frac{1}{2}(\partial_- \lambda_L \Gamma^m \theta_L)(P_m + \partial_+ x_m) + \frac{\delta v_L}{\delta \lambda_L}\partial_L \lambda_L \right). \tag{5.2.21}$$

To make the action invariant, we take the conditions

$$\frac{\delta v_L}{\delta \theta_L} = -\frac{1}{2}\lambda_L \Gamma^m (P_m + \partial_+ x_m) = \frac{1}{4}\lambda_L \Gamma^m \partial_\sigma x_m,$$
$$\frac{\delta v_L}{\delta \lambda_L} = -\frac{1}{2}\theta_L \Gamma^m (P_m + \partial_+ x_m) = \frac{1}{4}\theta_L \Gamma^m \partial_\sigma x_m, \tag{5.2.22}$$

which fixes $v_L = \frac{1}{4}(\lambda_L \Gamma^m \theta_L)\partial_\sigma x_m$. But, more generally, we can have

$$v_L = \frac{1}{4}(\lambda_L \Gamma^m \theta_L)\partial_\sigma x_m + v_L^0, \tag{5.2.23}$$

with

$$\frac{\delta v_L^0}{\delta \theta_L}\partial_- \theta_L + \frac{\delta v_L^0}{\delta \lambda_L}\partial_- \lambda_L = 0, \tag{5.2.24}$$

so we can take v_L^0 to be any function of θ_L and λ_L. The charge of Q_L is then

$$\hat{Q}_L = \int \left(-\frac{1}{2}(\lambda_L \Gamma^m \theta_L)\partial_+ x_m + \lambda_L p_+ \right) + v_L^0. \tag{5.2.25}$$

Similarly, the charge of Q_R is

$$\hat{Q}_R = \int \left(-\frac{1}{2}(\lambda_R \Gamma^m \theta_R)\partial_- x_m + \lambda_R p_- \right) + v_R^0. \tag{5.2.26}$$

We can see that left supersymmetry and left BRST transformations commute by computing the Poisson brackets of their generating functions:

$$\{\hat{S}_L, \hat{Q}_L\} = -\frac{1}{2}\{\xi_L p_+, (\lambda_L \Gamma^m \theta_L)\partial_+ x_m\} + \frac{1}{2}\{(\xi_L \Gamma^m \theta_L)\partial_+ x_M, \lambda_L p_+\} = 0. \quad (5.2.27)$$

The nilpotency of the action of Q in the Hamiltonian phase space can then be checked by computing the Poisson brackets of the generating function with itself:

$$\{\hat{Q}_L, \hat{Q}_L\} = (\lambda_L \Gamma^m \lambda_L)\partial_+ x_m = 0, \quad (5.2.28)$$

which is zero because the λ_L ghost is a pure spinor.

5.3 BRST operator

Now we are going to define the BRST operator Q, that is used to define the physical states of the theory. We start defining the supersymmetric momentum as a 1-form $\Pi^m = \Pi^m_+ \mathrm{d}\tau^+ + \Pi^m_- \mathrm{d}\tau^-$ given by $\Pi = \mathrm{d}x + u$, where u is a function of $\theta_{L,R}$ fixed so $\Pi^m_\pm$ is invariant under the S_L supersymmetry. We have

$$S_L \Pi^m_+ = -\partial_+ \left(\frac{1}{2}\xi_L \Gamma^m \theta_L\right) + \xi_L \frac{\delta u^m_+}{\delta \theta_L} \quad (5.3.1)$$

and analogously for Π^m_-. To have $\delta\Pi = 0$ we fix it to be, on-shell, [4]

$$\begin{aligned} \Pi^m_+ &= \partial_+ x^m + \frac{1}{2}\theta_L \Gamma^m \partial_+ \theta_L, \\ \Pi^m_- &= \partial_- x^m + \frac{1}{2}\theta_R \Gamma^m \partial_- \theta_R. \end{aligned} \quad (5.3.2)$$

We also define the supersymmetric covariant derivative constraint, or Green-Schwarz constraint, as

$$\begin{aligned} d_+ &= p_+ - \frac{1}{2}\partial_+ x_m \Gamma^m \theta_L - \frac{1}{6}(\theta_L \Gamma_m \partial_+ \theta_L)\Gamma^m \theta_L, \\ d_- &= p_- - \frac{1}{2}\partial_- x_m \Gamma^m \theta_R - \frac{1}{6}(\theta_R \Gamma_m \partial_- \theta_R)\Gamma^m \theta_R, \end{aligned} \quad (5.3.3)$$

which is related to the supersymmetric momentum by

$$\begin{aligned}\{d_+, d_+\} &= -\Gamma_m \Pi_+^m,\\ \{d_-, d_-\} &= -\Gamma_m \Pi_-^m,\\ \{d_+, d_-\} &= 0.\end{aligned} \tag{5.3.4}$$

We can parameterize the fiber of the phase space with the fields $(\Pi_+, d_\pm, w_\pm)$ or $(\Pi_-, d_\pm, w_\pm)$. In terms of these fields, the $p_\pm$ momenta are expressed as

$$\begin{aligned}p_+ &= d_+ + \frac{1}{2}\Pi_{m+}\Gamma^m\theta_L - \frac{1}{12}(\theta_L\Gamma^m\partial_+\theta_L)\Gamma_m\theta_L,\\ p_- &= d_- + \frac{1}{2}\Pi_{m-}\Gamma^m\theta_R - \frac{1}{12}(\theta_R\Gamma^m\partial_-\theta_R)\Gamma_m\theta_R.\end{aligned} \tag{5.3.5}$$

The BRST operator is then defined by the BRST charge, which is the generating function on the phase space, given by $\hat{Q} = \hat{Q}_L + \hat{Q}_R$, with

$$\hat{Q}_L = \int \lambda_L d_+, \qquad \hat{Q}_R = \int \lambda_R d_-. \tag{5.3.6}$$

These charges are the same as the ones of eqs. (5.2.25) and (5.2.26), with $v_L^0 = -(1/6)\Gamma^m\theta_L(\theta_L\Gamma_m\partial_+\theta_L)$, etc.

The action of Q_L on the fields is

$$\begin{aligned}Q_L x^m &= \frac{1}{2}\lambda_L\Gamma^m\theta_L,\\ Q_L\theta_L &= \lambda_L,\\ Q_L\lambda_L &= 0,\end{aligned} \tag{5.3.7}$$

and $Q_L\theta_R = Q_L\lambda_R = 0$. Now we write the transformation on the momenta. On the Green-Schwarz constraint instead, we use $Q_L d_+ = \{\lambda_L d_+, d_+\}$, and get

$$Q_L d_+ = -\lambda_L\Gamma^m\Pi_{m+}, \tag{5.3.8}$$

and $Q_L d_- = 0$. On the momenta of the pure spinor ghost, the transformation is given by

$$Q_L w_+ = -d_+, \tag{5.3.9}$$

and $Q_L w_- = 0$. On the supersymmetric momenta of x we get

$$Q_L \Pi_{m+} = \lambda_L \Gamma^m \partial_+ \theta_L \tag{5.3.10}$$

and $Q_L \Pi_{m-} = 0$.

To verify the nilpotency of Q_L, we act twice on each field. On x, we need to use the pure spinor constraint

$$Q_L^2 x^m = \frac{1}{2} Q_L (\lambda_L \Gamma^m \theta_L) = \frac{1}{2} \lambda_L \Gamma^m \lambda_L = 0. \tag{5.3.11}$$

On θ_L and λ_L, the $Q\lambda_L = 0$ implies the nilpotency. For the pure spinor momentum, we have

$$Q_L^2 w_+ = \, - Q_L d_+ = \lambda_L \Gamma^m \Pi_{m+}, \tag{5.3.12}$$

which is a gauge transformation on w_+, so it can be compensated to zero. On the Green-Schwarz constraint, we get

$$Q_L^2 d_+ = \, - Q_L (\lambda_L \Gamma^m \Pi_{m+}) = -\lambda_L \Gamma^m (\lambda_L \Gamma_m \partial_+ \theta_L). \tag{5.3.13}$$

We use Fierz identity to rewrite it as

$$Q_L^2 d_+ = \frac{1}{2} \partial_+ \theta_L \Gamma^m (\lambda_L \Gamma^m \lambda_L) = 0, \tag{5.3.14}$$

which is zero due to the pure spinor constraint. Finally, on the supersymmetric momentum of x we get

$$\begin{aligned} Q_L^2 \Pi_{m+} &= Q_L (\lambda_L \Gamma^m \partial_+ \theta_L) \\ &= \lambda_L \Gamma^m \partial_+ \lambda_L = \frac{1}{2} \partial_+ (\lambda_L \Gamma^m \lambda_L) = 0, \end{aligned} \tag{5.3.15}$$

where again the pure spinor constraint guarantees nilpotency.

5.4 Energy-momentum tensor and the b ghost

The energy-momentum can be computed by acting with worldsheet diffeomorphism on the action, as $\mathcal{L}_v S_0 = \int v^\alpha \partial_\beta T_\alpha{}^\beta$. The expression is given in terms of Lie derivatives acting on the fields. Under a diffeomorphism generated by a worldsheet vector field v, the action changes by

$$\mathcal{L}_v S_0 = \int (-\mathcal{L}_v x^m \partial_+ \partial_- x_m + \mathcal{L}_v p_+^L \partial_- \theta_L + p_+^L \partial_- \mathcal{L}_v \theta_L + \mathcal{L}_v p_-^R \partial_+ \theta_R + p_-^R \partial_+ \mathcal{L}_v \theta_R - (w, \lambda)), \tag{5.4.1}$$

where (w, λ) represent the ghost terms, which have a similar structure to the fermionic spinor terms. We are now expressing left and right momenta explicitly. Applying the Lie derivatives, we see that the energy-momentum tensor depends on components that are not present in the action:

$$\begin{aligned} T_{++} &= -\frac{1}{2}\partial_+ x^m \partial_+ x_m - p_+^L \partial_+ \theta_L - p_+^R \partial_+ \theta_R + w_+^L \partial_+ \lambda_L + w_+^R \partial_+ \lambda_R, \\ T_{--} &= -\frac{1}{2}\partial_- x^m \partial_- x_m - p_-^R \partial_- \theta_R - p_-^L \partial_- \theta_L + w_-^R \partial_- \lambda_R + w_-^L \partial_- \lambda_L, \end{aligned} \tag{5.4.2}$$

and $T_{+-} = T_{-+} = 0$. The components p_-^L, p_+^R, w_-^L and w_+^R vanish when we put all spinor fields on-shell. Hiding L, R superscripts of the momenta again, we get

$$\begin{aligned} T_{++} &= -\frac{1}{2}\partial_+ x^m \partial_+ x_m - p_+ \partial_+ \theta_L + w_+ \partial_+ \lambda_L, \\ T_{--} &= -\frac{1}{2}\partial_- x^m \partial_- x_m - p_- \partial_- \theta_R + w_- \partial_- \lambda_R, \end{aligned} \tag{5.4.3}$$

In terms of $\Pi_\pm$ and $d_\pm$ we can write it as

$$\begin{aligned} T_{++} &= -\frac{1}{2}\Pi_+^m \Pi_{m+} - d_+ \partial_+ \theta_L + w_+ \partial_+ \lambda_L, \\ T_{--} &= -\frac{1}{2}\Pi_-^m \Pi_{m-} - d_+ \partial_+ \theta_L + w_+ \partial_+ \lambda_L. \end{aligned} \tag{5.4.4}$$

The momenta $w_\pm$ of the pure spinor ghosts have a gauge symmetry. We can express the gauge invariant degrees of freedom of these fields in term of some composite field expressions [5]. The Lorentz current for the ghosts is

$$\begin{aligned} (N_L)_{mn} &= \frac{1}{2} w_+ \Gamma_{mn} \lambda_L, \\ (N_R)_{mn} &= \frac{1}{2} w_- \Gamma_{mn} \lambda_R, \end{aligned} \tag{5.4.5}$$

and the ghost current is

$$J_L = w_+ \lambda_L, \qquad J_R = w_- \lambda_R. \tag{5.4.6}$$

The energy-momentum tensor can be written in terms of these gauge invariant quantities as

$$\begin{aligned} T_{++} &= -\frac{1}{2}\Pi_{+}^{m}\Pi_{+}^{m} - d_{+}\partial_{+}\theta_{L} + \frac{1}{10} : N_{L}^{mn}N_{Lmn} : -\frac{1}{8} : J_{L}J_{L} : +\partial_{+}J_{L}, \\ T_{--} &= -\frac{1}{2}\Pi_{-}^{m}\Pi_{-}^{m} - d_{-}\partial_{-}\theta_{R} + \frac{1}{10} : N_{R}^{mn}N_{Rmn} : -\frac{1}{8} : J_{R}J_{R} : +\partial_{-}J_{R}, \end{aligned} \tag{5.4.7}$$

which is the Sugawara form of the energy-momentum tensor.

5.4.1 Definition of the b ghost

In the bosonic string sigma model, the b ghost arised from the antifield of the complex structure, after gauge fixing to Polyakov gauge. In the case of the pure spinor superstring, we did not fix a gauge – the model is a generalization of the already gauge fixed bosonic string. But, to compute amplitudes, we need to integrate over the moduli space as well, and thus we need to construct the proper Beltrami differential. To do that we must have an expression for the b ghost.

We will define it using the relations

$$Qb_{++} = T_{++}, \qquad Qb_{--} = T_{--}. \tag{5.4.8}$$

The energy-momentum is a worldsheet 2-form with ghost number 0. The BRST operator carries ghost number 1, and is fermionic, so that the b field must be a fermionic worldsheet 2-form with ghost number -1. The only fields with negative ghost number are $w_{\pm}$, with ghost number -1, but they are not gauge invariant.

Let's define fermionic spinor fields G_{++}^{α} and $G_{--}^{\dot{\alpha}}$, 2-forms on the worldsheet, by the expressions

$$QG_{++} = \lambda_L T_{++}, \qquad QG_{--} = \lambda_R T_{--}. \tag{5.4.9}$$

Then we can define the b field as a fractions

$$b_{++} = \frac{G_{++}}{\lambda_L}, \qquad b_{--} = \frac{G_{--}}{\lambda_R}. \tag{5.4.10}$$

To make the fraction well-defined, we pick an arbitrary bosonic spinor C^{α}, and define the b ghost as

$$b_{++} = \frac{C_\alpha G^\alpha_{++}}{C_\beta \lambda^\beta_L}, \qquad b_{--} = \frac{C_\alpha G^\alpha_{--}}{C_\beta \lambda^\beta_R} \tag{5.4.11}$$

From now on we'll hide the L subscripts of the spinor fields.

To define the b ghost that generates the energy-momentum tensor in Sugawara form, in eq. (5.4.7), the expression for G_{++} is given by [5]

$$G^\alpha_{++} = \frac{1}{2}\Pi_{m+}(\Gamma^m d_+)^\alpha - \frac{1}{4}(N_{mn}\Gamma^{mn}\partial_+\theta)^\alpha - \frac{1}{4}J\partial_+\theta^\alpha - \frac{1}{4}\partial^2_+\theta^\alpha. \tag{5.4.12}$$

The BRST transformation on this expression yields the energy-momentum tensor in terms of N_{mn} and J, instead of w. Then the b ghost is expressed as

$$b = \frac{C_\alpha(2\Pi^m_+(\Gamma^m d_+)^\alpha - N_{mn}(\Gamma^{mn}\partial_+\theta)^\alpha - J\partial_+\theta^\alpha - \partial^2_+\theta^\alpha}{4C_\beta\lambda^\beta}. \tag{5.4.13}$$

5.4.2 Derivation of the b ghost and Y formalism

Instead of defining the b ghost that generates the energy-momentum tensor in Sugawara form, we can define the expression that generates the energy-momentum tensor in eq. (5.4.4). This expression for T is given in terms of the momentum w_+ of the pure spinor ghost, which is not gauge-invariant because of the pure spinor constraint.

We start with the ansatz that G_{++} is a polynomial on the fields. As it is a 2-form, we consider a sum of products of the 1-forms w_+, d_+, Π_+ and $\partial_+\theta$. To keep the ghost number zero, for each w_+ we also multiply the term by λ. We also impose the condition that each product of fields must be fermionic. From the possible expressions, let's choose

$$G_{++} = \frac{1}{2}\Pi^m_+\Gamma_m d_+ + \lambda(w_+\partial_+\theta) - \frac{1}{2}w_+\Gamma^m(\lambda\Gamma_m\partial_+\theta). \tag{5.4.14}$$

Now we act with the BRST transformation on G_{++}, which gives

$$\begin{aligned} Q_L G_{++} &= \frac{1}{2}(\lambda\Gamma^m\partial_+\theta)\Gamma_m d_+ - \frac{1}{2}\lambda\Pi^m_+\Pi_{m+} - \lambda(d_+\partial_+\theta) \\ &\quad + \lambda(w_+\partial_+\lambda) + \frac{1}{2}d_+\Gamma^m(\lambda\Gamma_m\partial_+\theta) - \frac{1}{2}w_+\Gamma^m(\lambda\Gamma_m\partial_+\lambda) \\ &= \lambda T_{++}. \end{aligned} \tag{5.4.15}$$

The second, third and fourth terms make up the energy-momentum tensor expression. The last one is zero due to the pure spinor constraint. The two remaining ones cancel each other.

The b ghost is then defined as

$$b_{++} = \frac{1}{2}\Pi_+^m \frac{(C\Gamma_m d_+)}{(C\lambda)} + w_+ \partial_+ \theta - \frac{1}{2}\frac{(C\Gamma^m w_+)(\lambda\Gamma_m \partial_+ \theta)}{(C\lambda)}. \tag{5.4.16}$$

We now introduce a formalism that guarantees that the b ghost depends on the correct number of degrees of freedom of w_+. This construction follows [10]. The Y formalism consists in the choice of an arbitrary constant pure spinor v, and the definition of

$$Y_\alpha = \frac{v_\alpha}{(v\lambda)}, \tag{5.4.17}$$

so that it satisfy the properties

$$Y\lambda = 1, \qquad Y\Gamma^m Y = 0, \tag{5.4.18}$$

so Y is a pure spinor, and is the inverse of λ. Then we define the projector

$$K_\alpha{}^\beta = \frac{1}{2}(\Gamma^m\lambda)_\alpha (Y\Gamma_m)^\beta, \tag{5.4.19}$$

which satisfies $\mathrm{Tr}K = D/2 = 5$, with $D = 10$. Thus K projects from the 16-dimensional spinor space to a 5-dimensional constrained space. In particular, projection of the pure spinor ghost is $K\lambda = 0$, so we can see that the space of pure spinors is 11-dimensional. The space of the momenta w_+ thus has 11 independent components.

If we take the constant spinor C to be the constant pure spinor v of Y formalism, we rewrite the expression for b_{++} in terms of Y and K as

$$b_{++} = \frac{1}{2}\Pi_+^m (Y\Gamma_m d_+) + w_+(1-K)\partial_+\theta. \tag{5.4.20}$$

Writing in this form, it's clear that the operator $(1-K)$ projects the 16 components of w_+ to 11 independent components.

Under the gauge trasformation that acts on w_+, the b ghost transforms as

$$\begin{aligned}\delta b_{++} &= \Lambda_+^n(\lambda\Gamma_n(1-K)\partial_+\theta) \\ &= \Lambda_+^n(\lambda\Gamma_n\partial_+\theta) - \frac{1}{2}\Lambda_+^n(\lambda\Gamma_n\Gamma_m Y)(\lambda\Gamma^m\partial_+\theta). \end{aligned} \tag{5.4.21}$$

Using the identity in eq. (2.2.16), we rewrite the transformation as

$$\delta b_{++} = \frac{1}{2}\Lambda_+^n(\lambda\Gamma_m\Gamma_n Y)(\lambda\Gamma^m\partial_+\theta) = 0, \tag{5.4.22}$$

where we used the Fierz identity $\lambda\Gamma^m(\lambda\Gamma_m\partial_+\theta) = 0$, which vanishes due to the pure spinor constraint. The expression for the b ghost is thus gauge invariant.

We do the same procedure to find b_{--}. Recovering the L and R subscripts, we write the two components as

$$\begin{aligned} b_{++} &= \frac{1}{2}\Pi_+^m(Y_L\Gamma_m d_+) + w_+(1-K_L)\partial_+\theta_L, \\ b_{--} &= \frac{1}{2}\Pi_-^m(Y_R\Gamma_m d_-) + w_-(1-K_R)\partial_-\theta_R. \end{aligned} \tag{5.4.23}$$

Comparison with bosonic string

To compare the BRST structure of the pure spinor superstring with the one of the bosonic string sigma model, we recall that for the bosonic string, the BRST operator was nilpotent only on-shell. The needed equations of motion were the ones for the BRST ghosts: $\bar{\partial}c = \partial\bar{c} = 0$. On the other hand, the relation $Qb = T$ yields the off-shell expression of the energy-momentum tensor.

In the pure spinor superstring, the BRST operator is nilpotent off-shell. However, the b ghost generates the energy-momentum tensor on-shell, in eq. (5.4.3). The equations of motion used to get the on-shell expression are the ones for the spinor fields: the fermionic spacetime spinors, $\partial_-\theta_L = \partial_+\theta_R = 0$, and the pure spinor ghosts, $\partial_-\lambda_L = \partial_+\lambda_R = 0$.

Chapter 6

Conclusions

To quantize gauge theories, it is necessary a formalism that enables gauge fixing, and the BRST formalism is one way to do so. However, the BV formalism is a more general framework that gives geometrical insights to gauge fixing, which happens by a choice of Lagrangian submanifold. Moreover, the space of deformations of BRST-like theories around a submanifold can be easily described. The fundamental elements of a BRST-like theory, which are the action and the BRST operator, have its information spread over an infinite number of brackets that encode the information about the BRST structure and the gauge fixing condition.

The bosonic string sigma-model is a gauge theory, invariant under diffeomorphisms, and it is quantized using BRST formalism. Using BV formalism to describe its BRST structure, we are able to gauge fix to Polyakov gauge by fixing the complex structure on the worldsheet. We can then parameterize the deformations along the moduli space of the worldsheet, around the submanifold corresponding to the gauge. The deformations affect the BRST structure non-linearly, giving rise to bracket operators that dictate how the theory can be deformed, even non-trivially, by the insertion of vertex operators. The nonlinearity is however very subtle, and, under a resummation of contact terms, the trivial deformations can be described linearly. This points to the holomorphic factorization of the amplitudes.

On the Polyakov gauge, the BRST operator is not nilpotent – only upon the use of the equations of motion of the diffeomorphism ghosts: $\bar{\partial} c = \partial \bar{c} = 0$. The energy-momentum tensor can be defined off-shell, and it is generated by the antifield of the complex structure – the b ghost.

Finally, the pure spinor superstring is not a gauge theory, but has a BRST-like structure. It is a generalization of the bosonic string in Polyakov gauge, but the BRST operator has a different nature. It is nilpotent even off-shell, which is imposed by restricting the ghosts to be pure spinors, satisfying the constraint $\lambda\Gamma\lambda = 0$. The b cannot arise as the antifield of a gauge fixed field, and it is defined as a composite field, which generates the on-shell expression for the energy-momentum tensor. To find the expression, we must use the equations of motion for all spinor fields: $\partial_- \theta_L = \partial_+ \theta_R = \partial_- \lambda_L = \partial_+ \lambda_R = 0$.

The analysis of the bosonic string sigma model using BV formalism gives a light to the BRST structure in a geometrical sense, and allows us to comprehend the off-shell aspects of the theory. These tools are important in order to compare the BRST structure of the bosonic string with the one of the pure spinor superstring.

Appendix A

Differential geometry

Differential geometry describes manifolds and geometrical objects defined on it. We start with the definition of a **smooth manifold**, which are a generalization of the linear spaces $\mathbb{R}^d$. They can have different topologies, but still be similar to $\mathbb{R}$ in a local way. Here we present a definition.

Consider a topological set M, with a family of open subsets $\mathcal{U}$. Define the set $\mathcal{A}$ called **atlas** as

$$\mathcal{A} = \{\phi : U \subset M \to \mathbb{R}^d,\ U \in \mathcal{U}\}, \tag{A.0.1}$$

whose elements ϕ are called charts.

Let the n of the charts be labelled by $i = 1, \cdots, n$ as $\phi_i : U_i \subset M \to \mathbb{R}^d$. Define their images to be $V_i = \mathrm{im}\phi_i \in \mathbb{R}^d$, so we can write

$$\phi_i : U_i \subset M \to V_i \subset \mathbb{R}^d. \tag{A.0.2}$$

The intersections of open subsets are defined as $U_{ij} = U_i \cap U_j$ and $V_{ij} = V_i \cap V_j$.

If U_{ij} is not empty, we can define the restriction of ϕ_i on U_{ij} as

$$\phi_{ij} : U_{ij} \subset M \to V_{ij} \subset \mathbb{R}^d. \tag{A.0.3}$$

If the charts of $\mathcal{A}$ are invertible, one can define the inverse

$$\phi_{ij}^{-1} : V_{ij} \subset \mathbb{R}^d \to U_{ij} \subset M. \tag{A.0.4}$$

The **transition function** between charts ϕ_i and ϕ_j is defined as

$$\tau_{ij} : V_{ij} \to V_{ij}, \quad \tau_{ij} = \phi_{ij}^{-1} \circ \phi_{ji}. \tag{A.0.5}$$

The topological set M is called a **smooth manifold** if it has an atlas $\mathcal{A}$ with invertible charts that cover all of the set, i.e.

$$\bigcup_{\phi\in\mathcal{A}} \mathrm{im}\phi^{-1} = M, \tag{A.0.6}$$

such that the transition functions between all charts are smooth functions.

The smooth manifold M has dimension d if the image of the charts have dimension d.

A.1 Tangent and cotangent space

Smooth manifolds are equipped with smoothly changing charts of its open subsets, and thus are locally equivalent to linear spaces. Thus there is the notion of vector at each point of the manifold. A vector a point $x \in M$ in the manifold must be an element of some linear space, which we call the **tangent space** to the manifold at that point. The definition goes as follows.

Consider a curve $f : [0,1] \subset \mathbb{R} \to U \subset M$, which is a function that maps a segment of the real line to the manifold M. Let $x = f(t_0) \in U \subset M$ be one of the points of the manifold where the curve passes. Let $\phi : U \subset M \to \mathbb{R}^d$ be a chart that maps the point x to coordinates $x^i = \phi^i(x)$. The curve in coordinates is given by $f_\phi = \phi \circ f : [0,1] \to U \subset M$.

The curve f defines a tangent vector to x, which is the first order term in the expansion of f around x. In the chart ϕ we have

$$\begin{aligned} f^i_\phi(t_0 + t) &= f^i_\phi(t_0) + t\frac{\partial f^i_\phi}{\partial t}(t_0) + \mathcal{O}(t^2) \\ &= x^i + tv^i + \mathcal{O}(t^2). \end{aligned} \tag{A.1.1}$$

The vector can thus be written in the ϕ chart as

$$v^i = \frac{\partial f^i_\phi}{\partial t}(t_0), \tag{A.1.2}$$

though it should be noted that the definition of a tangent vector only makes sense upon the introduction of a chart.

Now we are able to give a general definition of T_xM, the space of all tangent vectors on a point $x \in M$. We call it the tangent space on $x \in M$, and we define it in terms of an equivalence class of curves that pass through the point x, as

$$T_xM = \{f : U \subset \mathbb{R}, x \in \mathrm{im} f\}/\sim, \tag{A.1.3}$$

where the equivalence relation is

$$f \sim g \quad \text{iff} \quad f_\phi(t) - g_\phi(t) = \mathcal{O}(t^2), \quad \text{for all } \phi. \tag{A.1.4}$$

This means that $\partial f^i_\phi(t_0)/\partial t = \partial g^i_\phi(t_0)/\partial t$, and hence the two curves define the same tangent vector.

To see that the elements of T_xM really are vectors, we can compute their transformation under a coordinate transformation. A change of coordinates is simply a change between charts, say from ϕ to $\tilde{\phi}$, so it is given by a transition function $\tau = \tilde{\phi} \circ \phi^{-1}$, as defined in equation (A.0.5). Then the coordinates of point x in both charts are related by $\tilde{x}^i = \tau^i(x^j)$. We can relate the curve f written in coordinates with ϕ and $\tilde{\phi}$ charts as $f_{\tilde{\phi}} = \tau \circ f_\phi$.

The tangent vector to x defined by f is defined by $\tilde{v}^i = \partial f_{\tilde{\phi}}(t_0)/\partial t$ in the new chart, and so it is related to the one in the original coordinates as

$$\tilde{v}^i = \frac{\partial}{\partial t}\left((\tau \circ f_\phi)(t)\right)_{t=t_0} = \frac{\partial \tau^i(x)}{\partial x^j}\frac{\partial f^j_\phi(t_0)}{\partial t}, \tag{A.1.5}$$

where we used the chain rule in the last. We recognize that $v = \partial f_\phi(t_0)/\partial t$. Then, simplifying the expression by writing $\partial \tau/\partial x = \partial \tilde{x}/\partial x$, we have

$$\tilde{v}^i = \frac{\partial \tilde{x}^i}{\partial x^j} v^j, \tag{A.1.6}$$

which is precisely the way a vector transforms under coordinate transformations.

At each point of a manifold we can also define a **cotangent space**, with elements that are dual to the tangent space. Its elements are called cotangent vectors, or covectors. Under coordinate transformations they should transform in the opposite way that tangent vectors do, so their inner product is coordinate independent.

To define a covector at a point $x \in U \subset M$, consider a function $F : U \subset M \to \mathbb{R}$. With a chart ϕ, it is given in coordinates by the function $F_\phi = F \circ \phi^{-1}$. The cotangent vector defined by F on x is then given by

$$u_i = \frac{\partial F_\phi}{\partial x^i}(x), \tag{A.1.7}$$

in analogy to the definition of tangent vector from a curve.

To check that this is a sensible definition, consider a curve $f : [0,1] \subset \mathbb{R} \to U \subset M$, which passes through x, with $x = f(t_0)$. One can define the composition $F \circ f : [0,1] \to \mathbb{R}$, which has some fixed value $(F \circ f)(t_0)$ at t_0. Then we can expand $F \circ f$ around this fixed point. At first order in t we get

$$\begin{aligned}(F \circ f)(t_0 + t) &= (F \circ f)(t_0) + t\frac{\partial F(x)}{\partial x^i}\frac{\partial f^i(t_0)}{\partial t} + \mathcal{O}(t^2) \\ &= (F \circ f)(t_0) + tv^i u_i + \mathcal{O}(t^2),\end{aligned} \tag{A.1.8}$$

which enables us to define the inner product between u and v as

$$\langle u, v\rangle = \frac{\partial}{\partial t}(F \circ f)(t)_{t=t_0}. \tag{A.1.9}$$

The cotangent vectors at the point $x \in M$ are elements of the cotangent space at x, defined by an equivalence class of functions as

$$T_x^*M = \{F : U \subset M \to \mathbb{R}, x \in U\}/\sim, \tag{A.1.10}$$

with the equivalence relation given by

$$F \sim G \quad \text{iff} \quad F_\phi(x + y) - G_\phi(x + y) = \mathcal{O}(y^2), \quad \text{for all } \phi. \tag{A.1.11}$$

Finally we must check that the cotangent vector has the correct transformation law. For switching from the chart ϕ to the chart $\tilde{\phi}$, which amounts to the coordinate transformation $x^i = \tau^i(x^j)$, a function F undergoes the following transfomation $F_{\tilde{\phi}} = F_\phi \circ \tau^{-1}$ where $\tau^{-1} = \phi \circ \tilde{\phi}^{-1}$ is the inverse transition function. Then we get that the cotangent vector obtained from F transforms as

$$\tilde{u}_i = \frac{\partial}{\partial x^i}((F_\phi \circ \tau^{-1})(y))_{y=x} = \frac{\partial F_\phi(x)}{\partial x^j}\frac{\partial(\tau^{-1})^j(x)}{\partial x^i}. \tag{A.1.12}$$

We identify $u_i = \partial F_\phi(x)/\partial x^i$. We can write $\partial(\tau^{-1})/\partial x{=}\partial x/\partial\tilde{x}$, so we are able to express the transformation as

$$\tilde{u}_i = \frac{\partial x^j}{\partial \tilde{x}^i}u^j, \tag{A.1.13}$$

which is the transformation law of a covector.

A.2 Bundles, tensors and fields

Bundles

The tangent bundle of a manifold M is the set defined by the union of the tangent spaces to all points of M. We write it as

$$TM = \bigcup_{x \in M} T_x M. \tag{A.2.1}$$

Each element of TM is a tangent vector $v \in T_x M$ to a certain point $x \in M$, so we write

$$(x, v) \in TM, \tag{A.2.2}$$

representing each element as a pair.

In the same way, the cotangent bundle of M is the union of the cotangent spaces to all points of M, written as

$$T^*M = \bigcup_{x \in M} T_x^* M. \tag{A.2.3}$$

The elements are cotangent vectors $u \in T_x^* M$ to given points $x \in M$, so we write them as pairs

$$(x, u) \in T^*M. \tag{A.2.4}$$

Tensors

Given two real vector spaces V_1 and V_2, the tensor product between them is the product of the sets, modulo an equivalence relation. We write the product as

$$V_1 \otimes V_2 = \frac{V_1 \times V_2}{\sim}, \tag{A.2.5}$$

where $\otimes$ denotes tensor product and $\times$ denote the cartesian product of sets. For $\lambda_1, \lambda_2 \in \mathbb{R}$, $a_1, a_2 \in V_1$ and $b \in V_2$, the equivalence relation is given by

$$(\lambda_1 a_1 + \lambda_2 a_2, b) \sim \lambda_1 (a_1, b) + \lambda_2 (a_2, b), \tag{A.2.6}$$

which ensures that $V_1 \otimes V_2$ keeps the linearity of V_1. There is also the equivalence relation that expresses the linearity coming from V_2.

We can consider tensor products of bundles of a manifold. A bundle of rank (p, q) is the product

$$T^{(p,q)}M = \underbrace{TM \otimes \cdots \otimes TM}_{\text{p times}} \otimes \underbrace{T^*M \otimes \cdots \otimes T^*M}_{\text{q times}}. \tag{A.2.7}$$

An element of a (p, q) bundle is written as

$$(x; v_1 \otimes \cdots \otimes v_p \otimes u_1 \otimes \cdots \otimes u_p) \in T^{(p,q)}M, \tag{A.2.8}$$

where $x \in M$ and

$$v_1 \otimes \cdots \otimes v_p \otimes u_1 \otimes \cdots \otimes u_p \in T_x^{(p,q)}M \tag{A.2.9}$$

is a tensor of rank (p, q) on the point x. Tangent and cotangent vectors are special types of tensors, with $TM = T^{(1,0)}M$ and $T^*M = T^{(0,1)}M$.

Let a change of coordinates be given by the transformation $x^i \to \tilde{x}^i$. Then the components of a tensor T transform as

$$T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q} \to \frac{\partial \tilde{x}^{i_1}}{\partial x^{k_1}} \cdots \frac{\partial \tilde{x}^{i_p}}{\partial x^{k_p}} \frac{\partial x^{l_1}}{\partial \tilde{x}^{j_1}} \cdots \frac{\partial x^{l_p}}{\partial \tilde{x}^{j_p}} T^{k_1 \cdots k_p}{}_{l_1 \cdots l_q}. \tag{A.2.10}$$

The upper indices are called covariant, and the lower indices are called contravariant. A tensor field of rank $(p, 0)$, for some p, is called a covariant tensor, whereas a tensor field of rank $(0, q)$, for some q, is called a contravariant tensor.

In a given coordinate system, the determinant of a tensor with ranks $(2, 0)$, $(1, 1)$ or $(0, 2)$ is defined as the determinant of the matrix formed by its components. The transformation of the determinant under a coordinate change is then

$$\det T \to \left(\det \frac{\partial \tilde{x}}{\partial x} \right)^{p-q} \det T. \tag{A.2.11}$$

We see that the determinant of covariant and contravariant tensors transform with the inverse factor one of the other. Also, determinants of $(1, 1)$ tensors do not transform.

Fields

We will define a section of the bundle $T^{(p,q)}M$ as a function $V : U \subset M \to T^{(p,q)}M$, for some open subset U, such that $V(x) = (x, v)$, where $v \in T_x^{(p,q)}M$. The space of sections of $T^{(p,q)}M$ is $\Gamma(T^{(p,q)}M)$.

A tensor field of rank (p, q) on $U \subset M$ is defined as a section of $T^{(p,q)}M$. In particular, a vector field on $U \subset M$ is a map $V : U \subset M \to TM$, which maps each point of the manifold to a tangent vector at that point. This means that we have $V(x) \in T_xM$ for all $x \in U \subset M$.

Consider the cotangent vector $u \in T_xM$ at the point $x \in U \subset M$, generated by a function $F : U \subset M \to \mathbb{R}$ as $u = \partial F/\partial x$. Its inner product with the tangent vector $V(x)$, which is the value of the vector field V in the point x, is given by definition by

$$\langle u, V(x)\rangle = \frac{\partial}{\partial t}(F \circ f(x))(t)_{t=t_0} = V^i(x)\frac{\partial F(x)}{\partial x^i}, \tag{A.2.12}$$

where $f(x)$ is the curve that generates the tangent vector, by $V(x) = \partial f/\partial t$. Thus we can represent vector fields as differential operators, like

$$V(x) = V^i(x)\frac{\partial}{\partial x^i}. \tag{A.2.13}$$

Then the operators $\partial/\partial x^i$ form a basis of the vector fields, on a given chart. They transform in the same way as covectors coordinates, so that vector fields are coordinate independent.

The covector fields are functions $W : U \subset M \to T^*M$, which associate to each point $x \in U \subset M$ a cotangent vector $W(x)$. We define with elements $\mathrm{d}x^i$ a basis for the cotangent bundle, so we write

$$W(x) = W_i(x)\mathrm{d}x^i, \tag{A.2.14}$$

which we define to be dual to the tangent bundle basis $\{\partial/\partial x^i\}$, i.e.

$$\left\langle \mathrm{d}x^i, \frac{\partial}{\partial x^j}\right\rangle = \delta^i_j. \tag{A.2.15}$$

With these basic elements we can define a basis for the $T^{(p,q)}M$ bundle, given by the elements

$$\mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_q} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_p}}. \tag{A.2.16}$$

The elements $\partial/\partial x^i$ may also be abbreviated to ∂_i.

Tensor fields of rank $(0,0)$, in $T^{(0,0)}M$, are called scalar fields on M, or simply functions on M. We also write $T^{(0,0)}M = \mathrm{Fun}M$.

A.3 Forms and densities

A form of rank n, or n-form, is a completely antissymetric tensor field of rank $(0,n)$. It can be defined with the wedge product. For two covectors $u_1, u_2 \in T_x^*M$, their wedge product is given by

$$u_1 \wedge u_2 = u_1 \otimes u_2 - u_2 \otimes u_1, \tag{A.3.1}$$

which is the antisymmetrized form of the tensor product. Then n-forms can be written as wedge products of n covector fields. We will define them more rigorously by defining the space of n-forms.

The space of n-forms on M is a subspace of the tensor bundle of rank n, given by

$$\Omega^n(M) = \frac{T^{(0,n)}M}{\sim}, \tag{A.3.2}$$

with the equivalence relation

$$u_1 \otimes \cdots \otimes u_n \sim \frac{1}{n!} u_1 \wedge \cdots \wedge u_n. \tag{A.3.3}$$

This relation makes equivalent to zero any element which is not completely antisymmetric.

Because of the antisymmetry, if n is bigger than the dimension of M, then $\Omega^n(M) = \{0\}$. And if $n = \mathrm{dim}M$, then, besides 0, there are elements which are multiples of

$$\mathrm{vol} = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n, \tag{A.3.4}$$

which is called the volume form (on a given chart). We can then write any element of $\Omega^{\dim M}(M)$ as $\omega = f\mathrm{vol}$, where f is a function which is the only component of ω. The integral over ω, defined on $U \subset M$, is given by

$$\int_U \omega = \int_U d^n x f(x), \tag{A.3.5}$$

in the given chart.

Forms of degree $n < \dim M$ can also be integrated over. Let ω be an n-form with $n < \dim M$, defined over $U \subset M$. Let $N \subset U$ be a submanifold of dimension n. Then the integration of ω over N is defined in terms of the volume form of N.

An density ρ_s of weight $s \in \mathbb{R}$ is an object that, given a coordinate system x^i, is written as a function $\rho_s(x^1, \cdots, x^n)$, but which transforms as

$$\rho'_s(x'^1, \cdots, x'^n) = \left(\det \frac{\partial x^i}{\partial x'^j}\right)^s \rho_s(x^1, \cdots, x^n) \tag{A.3.6}$$

under change of coordinates. In particular, a volume form is a density of weight 1, and a function is a density of weight zero.

A.4 Lie derivative

A vector $v \in \mathrm{Vect}M$ on the manifold M can be understood as the generator of a change of coordinates, of **diffeomorphism**. The infinitesimal change on coordinates x^i is given, in terms of a small real parameter t, by

$$x'^i = x^i + tv^i(x). \tag{A.4.1}$$

This small change is actually a map from the manifold M to the tangent bundle TM, as $x \mapsto v(x)$.

Under a small change of coordinates, a tensor field $A \in T^{(p,q)}M$ has a transformation that can be understood a map from $T^{(p,q)}M$ to the tangent bundle of the tensor fields space, $T(T^{(p,q)}M)$. The map is called the **Lie derivative** of $A \in T^{(p,q)}M$ along $v \in \mathrm{Vect}M$.

To compute a Lie derivative of an object, we consider its transformation law under an infinitesimal change of coordinates. Then we expand the transformation as a linear transformation on the object.

For example, consider a vector field $W \in TM$. Its transformation law under the diffeomorphism $x \to x + tv$, for a small parameter t and a vector field v, is

$$W'^i(x - tv) = \frac{\partial (x - tv)^i}{\partial x^j} W^j(x), \tag{A.4.2}$$

so, at first order in t, we have

$$\mathcal{L}_v W^i = v^j \partial_j W^i - \partial_j v^i W^j. \tag{A.4.3}$$

For a general tensor field $A \in T^{(p,q)}$, the Lie derivative acts as

$$\begin{aligned} \mathcal{L}_v A^{i_1 \cdots i_p}_{j_1 \cdots j_q} = v^k \partial_k A^{i_1 \cdots i_p}_{j_1 \cdots j_q} + \partial_k v^{i_1} A^{k i_2 \cdots i_p}_{j_1 \cdots j_q} + \cdots + \partial_k v^{i_p} A^{i_1 \cdots i_{p-1} k}_{j_1 \cdots j_q} \\ - \partial_{j_1} v^k A^{i_1 \cdots i_p}_{k j_2 \cdots j_q} - \cdots - \partial_{j_q} v^k A^{i_1 \cdots i_p}_{j_1 \cdots j_{q-1} k}. \end{aligned} \tag{A.4.4}$$

To compute the Lie derivative of a density of weight s, we write the transformation law

$$\rho'_s(x - tv) = \left(\det \frac{\partial (x + tv)^i}{\partial x^j} \right)^s \rho_s(x), \tag{A.4.5}$$

which gives us

$$\begin{aligned} \mathcal{L}_v \rho_s &= s \partial_i v^i \rho_s + v^i \partial_i \rho_s \\ &= s (\mathrm{div} v) \rho_s + v^i \partial_i \rho_s. \end{aligned} \tag{A.4.6}$$

Appendix B

Antifield of the complex structure

B.1 Antifields of $m\bar{m}$

Because the complex structure I satisfies a constraint, there is a gauge symmetry on its antifield $I^\star$. Since the constraint has two degrees of freedom, the gauge symmetry will also have two degrees of freedom.

Consider the manifold M of tensor fields $I = I^\alpha_\beta d\sigma^\alpha \partial_\beta$ on the worldsheet. The space of complex structures is a submanifold defined by the constraint $F = 0$, where $F : M \to M$ is given by

$$F = I^2 + 1 = 0. \tag{B.1.1}$$

The antifield $I^\star$ of I is an element of $\Pi T^\star M$, the odd cotangent bundle to the space of $(1,1)$ tensors. For every tangent vector δI of ΠTM, there is a natural inner product $\langle \delta I, I^\star \rangle$. This shows that $I^\star$ must be a $(1,1)$ tensor density, because the inner product involves not only the inner product of $(1,1)$ matrices, but also an integration over the worldsheet.

We can represent a $(1,1)$ tensor density as a $(1,3)$ tensor with antisymmetry in the first two indices, so that we can write, in $z\bar{z}$ coordinates,

$$I^\star = (I^\star)_{z\bar{z}\alpha}{}^\beta \; dz \wedge d\bar{z} \otimes dz^\alpha \frac{\partial}{\partial z^\beta}, \tag{B.1.2}$$

where $dzd\bar{z}$ is a wedge product. The tensor product $\otimes$ can be omitted. The density part then is encoded in the first two indices, which are antisymmetrical. The only four independent components are $(I^\star)_{z\bar{z}\alpha}{}^\beta$. The inner product is expressed as

$$\langle \delta I, I^\star \rangle = \int \delta I I^\star = \int dzd\bar{z} \; (\delta I)^\alpha_\beta (I^\star)_{z\bar{z}\alpha}{}^\beta, \tag{B.1.3}$$

and can be seen as the trace of the matrix product of δI with $I^\star$.

If we take the Polyakov action to be a function of general tensor fields instead of only complex structures. Then we can write the BV action as

$$S = \int \left(-\frac{1}{4} dxI \cdot dx + \mathcal{L}_c x x^\star + \mathcal{L}_c I I^\star + \frac{1}{2}\mathcal{L}_c c c^\star \right). \tag{B.1.4}$$

This action is the Polyakov action string when we restrict the integration over complex structures only, and not over general tensor fields. In this way we implement the constraint $I^2 + 1 = 0$.

B.2 Gauge symmetry of $I^\star$

The antifield $I^\star$ of I acquires a gauge symmetry, from the constraint on I to be a complex structure. By applying Lie derivative on the $F = I^2 + 1$ constraint, we get a constraint on the tangent space

$$\mathcal{L}_c F = \mathcal{L}_c I I + I \mathcal{L}_c I = 0. \tag{B.2.1}$$

Using this constraint we define the transformation

$$\delta_\eta I^\star = \eta I + I \eta, \tag{B.2.2}$$

for a $(1,1)$ tensor density η. This is a gauge symmetry of S, that appears as

$$\begin{aligned} \int \mathcal{L}_c I I^\star &= \int \mathcal{L}_c I \left(I^\star + \eta I + I \eta \right) \\ &= \int \left(\mathcal{L}_c I I^\star + (\mathcal{L}_c I I + I \mathcal{L}_c I)\eta \right), \end{aligned} \tag{B.2.3}$$

where we used the cyclicity of the trace. We can write the gauge parameter η as a $(1,3)$ tensor just like we did for $I^\star$,

$$\eta = \eta_{z\bar{z}\alpha}{}^{\beta}\ dz d\bar{z}\ dz^\alpha \frac{\partial}{\partial z^\beta}. \tag{B.2.4}$$

Let's now write $I^\star$ in components as

$$I^\star = \frac{1}{4} dz d\bar{z} \left(p\ dz \frac{\partial}{\partial z} + \bar{p}\ d\bar{z} \frac{\partial}{\partial \bar{z}} - b\ dz \frac{\partial}{\partial \bar{z}} + \bar{b}\ d\bar{z} \frac{\partial}{\partial z} \right). \tag{B.2.5}$$

We will see that we can choose a gauge slice where $p = \bar{p} = 0$, and we are left with the $b\bar{b}$ fields.

The constraint $F = 0$ has only two independent components. We can see that by noting that a general tensor field has four independent components, but a complex structure I has two free parameters. Thus there are only two independent directions for the gauge transformation. Let's take the gauge parameter η to be diagonal in $z\bar{z}$ coordinates. We write

$$\eta = dzd\bar{z}\left(\eta\ dz\frac{\partial}{\partial z} + \bar{\eta}\ d\bar{z}\frac{\partial}{\partial \bar{z}}\right) \tag{B.2.6}$$

Then a transformation of $I^\star$ when I is a complex structure parameterized by $m\bar{m}$, is given by

$$\begin{aligned}\delta_\eta I^\star = dzd\bar{z}\Big(& 2i\eta\sqrt{1+m\bar{m}}\ dz\frac{\partial}{\partial z} - 2i\bar{\eta}\sqrt{1+m\bar{m}}\ d\bar{z}\frac{\partial}{\partial \bar{z}} \\ & + i(\eta+\bar{\eta})m\ dz\frac{\partial}{\partial \bar{z}} - i(\eta+\bar{\eta})\bar{m}\ d\bar{z}\frac{\partial}{\partial z}\Big).\end{aligned} \tag{B.2.7}$$

We see clearly that we can gauge away the diagonal components of $I^\star$. If we choose η in such a way that

$$\begin{aligned}2i\eta\sqrt{1+m\bar{m}} - p = 0, \\ 2i\bar{\eta}\sqrt{1+m\bar{m}} + \bar{p} = 0,\end{aligned} \tag{B.2.8}$$

then the diagonal components of $I^\star$ become zero.

To see that there are no gauge freedom left, consider now the transformation given by the off-diagonal parameter

$$\eta = dzd\bar{z}\left(H\ d\bar{z}\frac{\partial}{\partial z} + \bar{H}\ dz\frac{\partial}{\partial \bar{z}}\right). \tag{B.2.9}$$

Then the transformation on $I^\star$ is given by

$$\delta_\eta I^\star = i(\bar{H}m - H\bar{m})dzd\bar{z}\left(dz\frac{\partial}{\partial z} + d\bar{z}\frac{\partial}{\partial \bar{z}}\right). \tag{B.2.10}$$

Thus the remaining degrees of freedom can only lead out of the $p = \bar{p} = 0$ gauge slice. We cannot restrict the $b\bar{b}$ fields more.

B.3 Antifields of $\mu\bar{\mu}$

Using the gauge symmetry of $I^\star$ we write it in the form

$$I^\star = \begin{pmatrix} 0 & \beta \\ \bar{\beta} & 0 \end{pmatrix}. \tag{B.3.1}$$

The Darboux relation between a $(1,1)$ tensor field I and its antifield is

$$\{I^\alpha_{\ \beta}(z), (I^\star)_\gamma^{\ \delta}(w)\} = \delta^\alpha_\gamma \delta^\delta_\beta \delta(z-w). \tag{B.3.2}$$

The components of the complex structure are however constrained, so the relation is only valid for the independent components. In this parameterization, the independent parameters are μ and $\bar{\mu}$, but the off-diagonal components of I are also independent. So we want to impose the four Darboux conjugate relations

$$\begin{aligned} \left\{\frac{2i\mu}{1-\mu\bar{\mu}}, \beta\right\} &= 1, & \left\{\frac{2i\mu}{1-\mu\bar{\mu}}, \bar{\beta}\right\} &= 0, \\ \left\{\frac{-2i\bar{\mu}}{1-\mu\bar{\mu}}, \beta\right\} &= 0, & \left\{\frac{-2i\bar{\mu}}{1-\mu\bar{\mu}}, \bar{\beta}\right\} &= 1. \end{aligned} \tag{B.3.3}$$

Using product rule to split the left argument of each bracket, and rearranging the terms, we get the set of linear equations

$$\begin{aligned} (1-\mu\bar{\mu})^2 &= \{\mu, \beta\} + \mu^2\{\bar{\mu}, \beta\}, \\ (1-\mu\bar{\mu})^2 &= \{\bar{\mu}, \bar{\beta}\} + \bar{\mu}^2\{\mu, \bar{\beta}\}, \\ 0 &= \{\bar{\mu}, \beta\} + \bar{\mu}^2\{\mu, \beta\}, \\ 0 &= \{\mu, \bar{\beta}\} + \mu^2\{\bar{\mu}, \bar{\beta}\}. \end{aligned} \tag{B.3.4}$$

The solution is

$$\begin{aligned} \{\mu, \beta\} &= \frac{1-\mu\bar{\mu}}{1+\mu\bar{\mu}}, & \{\bar{\mu}, \bar{\beta}\} &= \frac{1-\mu\bar{\mu}}{1+\mu\bar{\mu}}, \\ \{\mu, \bar{\beta}\} &= -\mu^2\frac{1-\mu\bar{\mu}}{1+\mu\bar{\mu}}, & \{\bar{\mu}, \beta\} &= -\bar{\mu}^2\frac{1-\mu\bar{\mu}}{1+\mu\bar{\mu}}. \end{aligned} \tag{B.3.5}$$

Now we define the Darboux conjugates to μ and $\bar{\mu}$ as B and $\bar{B}$ respectively. To find how they depend on β and $\bar{\beta}$, we use the expressions in (B.3.5) to express the

Darboux conjugate relation as the linear equations

$$
\begin{aligned}
&\frac{\delta B}{\delta \beta} - \mu^2 \frac{\delta B}{\delta \bar{\beta}} = \frac{1+\mu\bar{\mu}}{1-\mu\bar{\mu}}, \qquad \frac{\delta \bar{B}}{\delta \beta} - \mu^2 \frac{\delta \bar{B}}{\delta \bar{\beta}} = 0, \\
&\frac{\delta B}{\delta \bar{\beta}} - \bar{\mu}^2 \frac{\delta B}{\delta \beta} = 0, \qquad \frac{\delta \bar{B}}{\delta \bar{\beta}} - \bar{\mu}^2 \frac{\delta \bar{B}}{\delta \beta} = \frac{1+\mu\bar{\mu}}{1-\mu\bar{\mu}}.
\end{aligned} \tag{B.3.6}
$$

Solving the equations, we are able to express the Darboux coordinates as

$$
B = \frac{\beta + \bar{\mu}^2 \bar{\beta}}{(1-\mu\bar{\mu})^2}, \qquad \bar{B} = \frac{\bar{\beta} + \mu^2 \beta}{(1-\mu\bar{\mu})^2}. \tag{B.3.7}
$$

The relations can be inverted

$$
\beta = \frac{1-\mu\bar{\mu}}{1+\mu\bar{\mu}}(B - \bar{\mu}^2 \bar{B}), \qquad \bar{\beta} = \frac{1-\mu\bar{\mu}}{1+\mu\bar{\mu}}(\bar{B} - \mu^2 B), \tag{B.3.8}
$$

and enables us to parameterize $I^\star$ in terms of the Darboux conjugate as

$$
I^\star = \frac{1-\mu\bar{\mu}}{1+\mu\bar{\mu}} \begin{pmatrix} 0 & \bar{B} - \mu^2 B \\ -B + \bar{\mu}^2 \bar{B} & 0 \end{pmatrix}. \tag{B.3.9}
$$

Bibliography

[1] A. Mikhailov, "Integration over families of Lagrangian submanifolds in BV formalism," 2018, 1611.04978.

[2] A. Mikhailov and A. Schwarz, "Families of gauge conditions in BV formalism," *Journal of High Energy Physics*, vol. 2017, Jul 2017.

[3] V. Bernardes, A. Mikhailov, and E. Viana, "Derived brackets in bosonic string worldsheet sigma-model," 2021, 2111.15104.

[4] N. Berkovits, "Covariant quantization of the superstring," *International Journal of Modern Physics A*, vol. 16, p. 801–811, Feb 2001.

[5] N. Berkovits, "Multiloop amplitudes and vanishing theorems using the pure spinor formalism for the superstring," *Journal of High Energy Physics*, vol. 2004, p. 047–047, Sep 2004.

[6] N. Berkovits and N. Nekrasov, "Multiloop superstring amplitudes from non-minimal pure spinor formalism," *Journal of High Energy Physics*, vol. 2006, p. 029–029, Dec 2006.

[7] V. G. Kac, "A Sketch of Lie Superalgebra Theory," *Commun. Math. Phys.*, vol. 53, pp. 31–64, 1977.

[8] D. Tong, "Lectures on string theory," 2012, 0908.0333.

[9] J. Polchinski, *String theory. Vol. 1: An introduction to the bosonic string*. Cambridge Monographs on Mathematical Physics, Cambridge University Press, 12 2007.

[10] I. Oda and M. Tonin, "Y-formalism in pure spinor quantization of superstrings," *Nuclear Physics B*, vol. 727, p. 176–195, Oct 2005.