# Degenerate and critical Bloch branes

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(Received 27 February 2008; published 15 August 2008)

In the last few years a number of works reported the appearance of thick branes with internal structure, induced by the parameter which controls the interaction between two scalar fields coupled to gravity in (4,1) dimensions in warped space-time with one extra dimension. Here we show that one can implement the control over the brane thickness without needing to change the potential parameter. On the contrary, this is going to be done by means of the variation of a parameter associated with the domain wall degeneracy. We also report the existence of novel and qualitatively different solutions for a critical value of the degeneracy parameter, which could be called critical Bloch branes.

DOI: 10.1103/PhysRevD.78.043526 PACS numbers: 98.80.Cq, 04.65.+e

## I. INTRODUCTION

The problem of the so called thick branes have received a considerable amount of attention during the last years [1–23]. As observed by Campos [1] a few years ago, some kinds of two interacting scalar fields potentials can be used in order to describe the splitting of thick branes due to a first-order phase transition in a warped geometry. In that work, Campos discussed the effect by studying a model without supersymmetry. In a recent work, Bazeia and Gomes [2] discussed the appearance of thick branes by using a two interacting scalar fields model that can be naturally incorporated in supergravity, what was also done by Eto and Sakai [3]. In our work, by using the very same model discussed by Bazeia and Gomes, we show that the reported Bloch branes [2] have, in fact, more general soliton solutions and, as a consequence, the resulting brane can have a much richer structure as, for instance, a degeneracy controlling parameter [3,24]. It is shown that for a convenient choice of this parameter a double wall structure and a corresponding thicker Bloch brane, which we call *degenerate Bloch branes*, shows up. It is important to remark that both in the case of Campos as well as in the case of Bazeia and Gomes, the splitting of the thick branes is controlled by the potential parameters. Here, instead, the splitting is controlled by means of a parameter which is not present in the Lagrangian density; on the contrary it appears in the solutions as a shape controlling parameter. In fact, as asserted in the above, there is a degeneracy in the solution because, in spite of the value of this parameter the energy of the field configuration is precisely the same. In view of this we call it a degeneracy parameter [24]. In fact, Bazeia and collaborators [25] have introduced a one scalar field model with similar properties like the existence of double-walls solutions and thick branes [6], but once again, the brane thickness control is done through a fine-tuning of the potential self-interaction parameters themselves. Here the idea is to get a more robust way of controlling the thickness, and the correspondent distance between the walls while preserving a supersymmetric structure. Finally, we introduce a special solution, at the critical value of the degeneracy parameter, which presents a quite different and interesting behavior for the brane.

On the other hand, finding exact classical solutions, particularly for solitons, is one of the problems on nonlinear models with interacting fields [26–41]. As pointed out by Rajaraman and Weinberg [31], in such nonlinear models more than one time-independent classical solution can exist and each one of them corresponds to a different family of quantum states, which come into play when one performs a perturbation around those classical solutions.

In order to deal with the systems we are going to work here, it is common to use the so-called *trial orbits method* [32], which is a very powerful one presented for finding exact soliton solutions for nonlinear second-order differential equations of models with two interacting relativistic scalar fields in $1 + 1$ dimensions, and it is model independent. A couple of years ago one of us presented a method for finding additional soliton solutions for those special cases whose soliton solutions are the BPS ones [35] and in the last year that approach was extended, allowing more general models [36,37,41]. This last approach is the one we will use in this paper. As a consequence, we present more general soliton solutions and how they are intrinsically related to degenerate and critical Bloch branes. Furthermore, once we show more general solutions which engender thicker branes, we discuss the influence of those solutions in the warp factor and in the fluctuation of the metric around those classical solutions. We also compare our results with those obtained in [2].

This work is organized as follows. In the second section we present the model we are going to work with and review the approach introduced in Ref. [35] to find classical

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soliton solutions. In that section we also obtain the warp factor in a general form. In the third section, a variety of soliton solutions that have been found up to now are constructed by using the method of the second section; we also present the warp factor for each set of solutions. The following section is devoted to a discussion on the stability and the zero modes for each set of soliton solutions in the context of the brane worlds scenario. Finally, we address final comments on the soliton solutions and their consequences and applicability in the brane world scenario.

## II. GRAVITY COUPLED TO TWO INTERACTING SCALAR FIELDS: ANALYTICAL SOLUTIONS

The action we are going to work with is the five-dimensional gravity coupled to two interacting real scalar fields, which can be represented by [13,14]

$$S = \int d^4x dr \sqrt{|g|}\left[-\frac{1}{4}R + \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi + \partial_\mu \chi \partial^\mu \chi) - V(\phi, \chi)\right], \tag{1}$$

where $g \equiv \det(g_{ab})$ and for granting that the four-dimensional space has the Poincaré invariance, it is usually used that

$$ds^2 = g_{ab}dx^a dx^b = e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu - dr^2; \quad a, b = 0, \ldots, 4, \tag{2}$$

where $r$ is the extra dimension, $\eta_{\mu\nu}$ the usual Minkowski metric and $e^{2A(r)}$ is the so-called warp factor. A usual hypothesis is that the warp factor depends only on the extra dimension $r$. Besides, one can also assume that the scalar fields depend only on the extra dimension $r$. Under these assumptions, one can determine the resulting equations of motion for the above system as [2,13,14]

$$\frac{d^2\phi}{dr^2} + 4\frac{dA}{dr}\frac{d\phi}{dr} = \frac{\partial V(\phi, \chi)}{\partial \phi}, \quad \frac{d^2\chi}{dr^2} + 4\frac{dA}{dr}\frac{d\chi}{dr} = \frac{\partial V(\phi, \chi)}{\partial \chi}, \tag{3}$$

and

$$\frac{d^2A}{dr^2} = -\frac{2}{3}\left[\left(\frac{d\phi}{dr}\right)^2 + \left(\frac{d\chi}{dr}\right)^2\right], \quad \left(\frac{dA}{dr}\right)^2 = \frac{1}{6}\left[\left(\frac{d\phi}{dr}\right)^2 + \left(\frac{d\chi}{dr}\right)^2\right] - \frac{1}{3}V(\phi, \chi). \tag{4}$$

As can be demonstrated [2,13], the above set of second-order nonlinear coupled equations has another set of first-order differential equations which shares solutions with it, and that is given by

$$\frac{d\phi}{dr} = \frac{\partial W(\phi, \chi)}{\partial \phi}, \quad \frac{d\chi}{dr} = \frac{\partial W(\phi, \chi)}{\partial \chi}, \quad \frac{dA}{dr} = -\frac{2}{3}W(\phi, \chi), \tag{5}$$

provided that the potential $V(\phi, \chi)$ is restricted to be of a given class of potentials which can be written in terms of a kind of superpotential as

$$V(\phi, \chi) = \frac{1}{2}\left[\left(\frac{\partial W(\phi, \chi)}{\partial \phi}\right)^2 + \left(\frac{\partial W(\phi, \chi)}{\partial \chi}\right)^2\right] - \frac{4}{3}W(\phi, \chi)^2. \tag{6}$$

Note that, once one has solutions of the first two first-order equations for the interacting scalar fields, it becomes a simple task of integration to get $A(r)$ and, as a consequence, determine the warp factor. Now, in order to go further on the analysis it is important to work with a concrete example which, when available, should be one with exact analytical solutions [16]. So, we will work with a superpotential equivalent to that used by Bazeia and collaborators [2]. The idea is to show that there are other solutions which were not analyzed in Ref. [2], and which present quite interesting features. Particularly, we will show that some of them allow one to control the behavior of the warp factor without performing the kind of restriction over the potential parameters as the one did in [2]. The superpotential we are going to work with here is

$$W(\phi, \chi) = \phi\left[\lambda\left(\frac{\phi^2}{3} - a^2\right) + \mu\chi^2\right], \tag{7}$$

which becomes equal to that considered in [2] by choosing $a = 1$, $\lambda = -1$, and $\mu = -r$. From now on, in order to solve the first-order differential equations, we follow the method of Ref. [35] instead of applying the usual trial orbits method [32,38]. For this we note that it is possible to write the relation $d\phi/W_\phi = dr = d\chi/W_\chi$, where the differential element $dr$ is a kind of invariant. Thus, one is lead to

$$\frac{d\phi}{d\chi} = \frac{W_\phi}{W_\chi}. \tag{8}$$

This is in general a nonlinear differential equation relating the scalar fields of the model. If one is able to solve it completely for a given model, the function $\phi(\chi)$ (in fact, it will be the equation for a generic orbit) can be used to eliminate one of the fields, rendering the first-order differential equations uncoupled and equivalent to a single one. Finally, the resulting uncoupled first-order nonlinear equation can be solved in general, even if numerically. By substituting the derivatives of the superpotential (7) with respect to the fields in (8) we have

$$\frac{d\phi}{d\chi} = \frac{\lambda(\phi^2 - a^2) + \mu\chi^2}{2\mu\phi\chi}, \tag{9}$$

which can be rewritten as a linear differential equation,

$$\frac{d\rho}{d\chi} - \frac{\lambda}{\mu\chi} = \chi, \tag{10}$$

by the redefinition of the fields, $\rho = \phi^2 - a^2$. Now, the general solutions are easily obtained as

$$\rho(\chi) = \phi^2 - a^2 = c_0\chi^{\lambda/\mu} - \frac{\mu}{\lambda - 2\mu}\chi^2, \quad \text{for } \lambda \neq 2\mu, \tag{11}$$

and

$$\rho(\chi) = \phi^2 - a^2 = \chi^2[\ln(\chi) + c_1], \quad \text{for } \lambda = 2\mu, \tag{12}$$

where $c_0$ and $c_1$ are arbitrary integration constants. We substitute the above solutions, for instance, in the differential equation for the $\chi$ field, obtaining the following first-order differential equations for the field $\chi(r)$:

$$\frac{d\chi}{dr} = \pm 2\mu\chi\sqrt{a^2 + c_0\chi^{\lambda/\mu} - \frac{\mu}{\lambda - 2\mu}\chi^2}, \qquad \lambda \neq 2\mu, \tag{13}$$

and

$$\frac{d\chi}{dr} = \pm 2\mu\chi\sqrt{a^2 + \chi^2[\ln(\chi) + c_1]}, \qquad \lambda = 2\mu. \tag{14}$$

As a matter of fact, in general, an explicit solution for each one of the above equations cannot be obtained, but one can verify numerically that the solutions belong to the same classes, and some of those classes of solutions can be written in terms of analytical elementary functions. In those last cases one is able to obtain the several types of soliton solutions we discuss in the next section.

Now, let us discuss a bit on the form of the warp factor $e^{2A(r)}$, for which it is necessary to compute the function $A(r)$ by integrating the second equation in (5). Once we will consider many situations, some of them with expressions much more involved than those studied in [2], it should be very convenient to express it in a general form in terms of the field itself, instead of a function of the spatial variable. Furthermore, that would allow one to make qualitative considerations about the behavior of the warp factor. In order to put this idea in a concrete form, we will use the orbit equations (11) and (12), and manipulate the equation for $A(r)$ in order to write it in terms of the field $\chi(r)$. For this we start by noting that after eliminating the dependence of $A(\phi, \chi)$ by using the orbit equation ($\phi \equiv \phi(\chi)$), one obtains

$$\frac{dA(r)}{dr} = \frac{dA(\chi)}{d\chi}\frac{d\chi}{dr} = \frac{dA(\chi)}{d\chi}\frac{\partial W(\phi(\chi), \chi)}{\partial\chi} = -\frac{2}{3}W(\phi(\chi), \chi), \tag{15}$$

which leads to

$$\frac{dA(\chi)}{d\chi} = -\frac{2}{3}\frac{W(\phi(\chi), \chi)}{W_\chi(\phi(\chi), \chi)}, \tag{16}$$

where $W_\chi(\phi(\chi), \chi) \equiv \frac{\partial W(\phi(\chi),\chi)}{\partial\chi}$. Now, substituting the superpotential of the model under analysis we get

$$\frac{dA(\chi)}{d\chi} = -\frac{1}{3\mu\chi}\left[\lambda\left(\frac{\phi(\chi)^2}{3} - a^2\right) + \mu\chi^2\right], \tag{17}$$

which after some simple manipulations using the orbit equation conduces to

$$\frac{dA(\chi)}{d\chi} = \left(\frac{2\lambda a^2}{9\mu}\right)\chi^{-1} - \frac{2}{9}\left(\frac{\lambda - 3\mu}{\lambda - 2\mu}\right)\chi - \left(\frac{\lambda c_0}{9\mu}\right)\chi^{((\lambda/\mu)-1)}, \quad \text{for } \lambda \neq 2\mu, \tag{18}$$

and

$$\frac{dA(\chi)}{d\chi} = \left(\frac{2\lambda a^2}{9\mu}\right)\chi^{-1} - \frac{(3\mu + \lambda c_1)}{9\mu}\chi - \frac{1}{3\mu}\chi\ln(\chi), \quad \text{for } \lambda = 2\mu. \tag{19}$$

Finally we can perform the integration over the field $\chi$, obtaining

$$A(\chi) = \alpha_0 + \left(\frac{2a^2}{9}\right)\ln(\chi) - \frac{1}{9}\left(\frac{\lambda - 3\mu}{\lambda - 2\mu}\right)\chi^2 - \left(\frac{c_0}{9}\right)\chi^{(\lambda/\mu)}, \quad \text{for } \lambda \neq 2\mu, \tag{20}$$

and

$$A(\chi) = \alpha_1 + \left(\frac{2a^2}{9}\right)\ln(\chi) - \frac{(3\mu + \lambda c_1)}{18\mu}\chi^2 - \frac{1}{6\mu}\chi^2\left(\ln(\chi) - \frac{1}{2}\right), \quad \text{for } \lambda = 2\mu, \tag{21}$$

where $\alpha_0$ and $\alpha_1$ are arbitrary integration constants, which are going to be chosen to ensure that $A(r = 0) = 0$. It is important to remark that the above solutions are completely general for this model and, as a consequence, can be used to get the warp factor for an arbitrary choice of the potential parameters, even for the cases where its solution cannot be obtained through analytical elementary functions. The above general approach can be checked, for instance with the case studied by Bazeia and Gomes [2], for which

$$\chi(y) = \pm\sqrt{\frac{1}{s} - 2}\,\text{sech}(2sy), \tag{22}$$

and, from above, one obtains

$$A(y) = \frac{1}{9s}[(1-3s)]\tanh(2sy)^2 - 2\ln(\cosh(2sy)), \quad (23)$$

where we have used the original variables and parameters defined in [2]. This is precisely the result obtained in that work through direct integration in the spatial variable.

From the expressions obtained above, one can clearly see that the behavior of the warp factor is very sensitive to the one of the field $\chi$. For instance, when $\chi(r)$ changes very slowly in a given region, so it will happen with $A(r)$. In a certain manner, one can guess the behavior of the warp factor, simply by observing that for $\chi$.

## III. SOLITON SOLUTIONS AND THEIR WARP FACTORS

In this section we explore in some extent the solutions for Eq. (13). This is done by presenting those resulting soliton solutions for the model under consideration and also by obtaining explicitly the warp factor for each set of soliton solutions in the brane scenario under analysis. Finally, we compare our results with those offered in [2].

Before proceeding, we would like to stress that the model we are working with admits a particular set of solutions which cannot be obtained from the method described in the previous section. That set of classical solutions could be called isolated solutions because they are characterized by $\bar{\chi}_I(r) = 0$, such that there is no sense in writing the differential equation (9) for this case and, consequently, do not furnish any internal structure for the brane [2]. Even though, the system admits a soliton solution given by $\bar{\phi}_I(r) = \pm a\tanh(\lambda a r)$, where the (lower) upper sign refers to a (anti-)kink solution. We will not consider this case in detail, since it is effectively a one field model, and we are primarily interested in the two fields nontrivial solutions.

### A. Bloch walls

The usual set of solutions, baptized as Bloch wall in [2], can be obtained by means of the method described in the previous section. It is obtained when we take $c_0 = 0$ in the expression (13). In this case that equation can be solved analytically for any value of $\lambda$ and $\mu$, provided that $\lambda > 2\mu$ in order to keep the solution real. In this case we get the following solution for $\chi(r)$:

$$\chi_{\rm BW}(r) = a\sqrt{\frac{\lambda - 2\mu}{\mu}}\,{\rm sech}(2\mu a r). \quad (24)$$

One can observe that this solution vanishes when $x \to \pm\infty$. The corresponding kinklike solution for the field $\phi$ is given by

$$\phi_{\rm BW}(r) = \pm a\tanh(2\mu a r), \quad (25)$$

which connects the vacua of the potential. In this case the warp factor of the configuration is the one presented in the previous section in Eq. (23). We call this type of domain wall a BW domain wall to distinguish it from other types of domain wall solutions we are going to present for this model.

At this point it is interesting to note that some of the solutions we are going to explore in the next sections are, in fact, in a different range of the potential parameters as compared with the ones considered in [2]. For instance, when we consider the $\lambda = \mu$, in terms of the parameters used by Bazeia and Gomes, this case would correspond to considering $r = 1$. However, the range of validity of the solutions used by them is $0 < r \leq \frac{1}{2}$.

### B. Degenerate Bloch walls

Others soliton solutions can be found when one considers the integration constant $c_0 \neq 0$. It was found in Ref. [35] that at least in three particular cases the Eq. (13) can be solved analytically. For $c_0 < -2$ and $\lambda = \mu$ it was found that the solutions for the $\chi(r)$ field are lumplike solutions, which vanish when $r \to \pm\infty$. On its turn, the field $\phi(r)$ exhibits a kinklike profile.

These classical solutions can be written as

$$\tilde{\chi}^{(1)}_{\rm DBW}(r) = \frac{2a}{\sqrt{c_0^2 - 4}\cosh(2\mu a r) - c_0}, \quad \text{for } \lambda = \mu,\ c_0 < -2, \quad (26)$$

and

$$\tilde{\phi}^{(1)}_{\rm DBW}(r) = a\frac{\sqrt{c_0^2 - 4}\sinh(2\mu a r)}{\sqrt{c_0^2 - 4}\cosh(2\mu a r) - c_0}, \quad \text{for } \lambda = \mu,\ c_0 < -2. \quad (27)$$

and its warp factor is given by

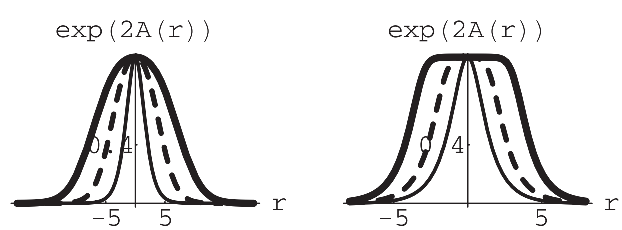


FIG. 1. Warp factor appearing in Ref. [2], with the parameters used there: $s = 0.05$ (thick solid line), 0.1 (dashed line), and 0.3 (thin solid line) (left). Warp factor for the case where $a = 1$, $\lambda = \mu$, $c_0 \neq -2$, and $c_0 = -2.00001$ (thick solid line); $-2.005$ (dashed line); $-3.0$ (thin solid line) (right).

$$e^{2A(r)} = N_\alpha \left[ \frac{2a}{\sqrt{c_0^2 - 4}\cosh(2\mu a r) - c_0} \right]^{(4a^2/9)} \times \exp\left[ \frac{2a(c_0^2 - c_0\sqrt{c_0^2 - 4}\cosh(2\mu a r) - 4a)}{9(\sqrt{c_0^2 - 4}\cosh(2\mu a r) - c_0)^2} \right], \tag{28}$$

where, as we anticipated above, $N_\alpha$ will be chosen in order to get $e^{2A(0)} = 1$, for plotting convenience.

An interesting aspect of these solutions is that, for some values of $c_0 < -2$, $\tilde{\phi}^{(1)}_{\mathrm{DBW}}(r)$ exhibits a double kink profile. We can speak of a formation of a double wall structure, extended along the space dimension. In Fig. 1 we compare the case studied in [2] to some typical profiles of the warp factors in the case where $\lambda = \mu$, both when $c_0$ is close to its critical value ($c_0 = -2$ in this case) and far from it. One can observe the appearance of a more pronounced flat region, where one could speak of a Minkowski-type metric. In fact this "Minkowski sector" becomes larger as $c_0$ approaches its critical value.

Similar behavior is also noted in the classical solutions for $\lambda = 4\mu$ and $c_0 < 1/16$. In this case the field $\chi(r)$ has a lumplike profile given by

$$\tilde{\chi}^{(2)}_{\mathrm{DBW}}(r) = -\frac{2a}{\sqrt{\sqrt{1 - 16c_0}\cosh(4\mu a r) + 1}}, \tag{29}$$

and the solution for the field $\phi(x)$ is

$$\tilde{\phi}^{(2)}_{\mathrm{DBW}}(r) = \sqrt{1 - 16c_0}\, a \frac{\sinh(4\mu a r)}{\sqrt{1 - 16c_0}\cosh(4\mu a r) + 1}, \tag{30}$$

with the corresponding warp factor being

$$e^{2A(r)} = N_\alpha \left[ \frac{-2a}{\sqrt{\sqrt{1 - 16c_0}\cosh(4\mu a r) + 1}} \right]^{(16a^2/9)} \exp\left\{ -\frac{4a^2}{9} \left[ \frac{1 + 32a^2 c_0 + \sqrt{1 - 16c_0}\cosh(4\mu a r)}{(\sqrt{1 - 16c_0}\cosh(4\mu a r) + 1)^2} \right] \right\}, \tag{31}$$

and, once more we choose $N_\alpha$ in order to get $e^{2A(0)} = 1$.

In this last case $\tilde{\phi}^{(2)}_{\mathrm{DBW}}(r)$ also presents a double kink profile for some values of $c_0$. In Fig. 2 the rising can be seen of two peaks at the extremum of the flatten region and an increasing of the distance between the two peaks in the warp factor as $c_0$ approaches its critical value ($c_0 = 1/16$ in this case).

### C. Critical Bloch walls

Finally, a very interesting class of analytical soliton solutions were shown to exist when one takes $\lambda = \mu$ with the critical parameter $c_0 = -2$ and for $\lambda = 4\mu$ with the critical parameter $c_0 = 1/16$, in Eq. (13). The novelty in these cases is the fact that both the $\chi(r)$ and the $\phi(r)$ fields present a kinklike profile and the warp factor presents a remarkable behavior, which can be noted from Fig. 3, where the existence of two "Minkowski-type" regions, separated by a transition one is evident. This argument is reinforced by the behavior of the energy densities of the soliton configurations, as can be noted in Fig. 4, as well as from the stability potential (see Figs. 5 and 6).

We call this set of solutions CBW domain walls. For $\lambda = \mu$ and $c_0 = -2$ the classical solution for the $\chi(r)$ field can be shown to be

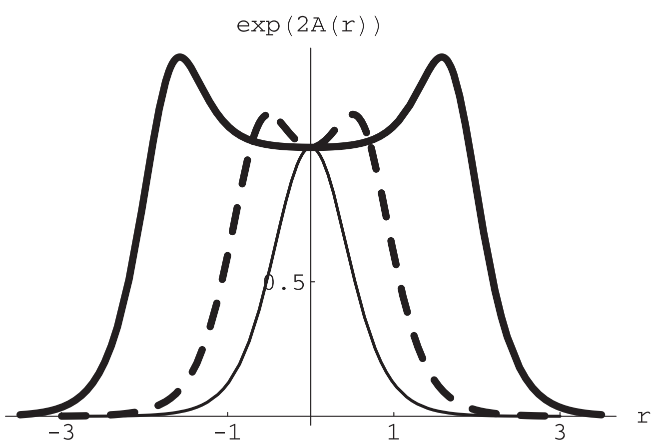


FIG. 2. Warp factor for the case where $a = 1$, $\lambda = 4\mu$, $c_0 \neq 1/16$, and $c_0 = 1/16.0001$ (thick solid line); $1/17$ (dashed line); $1/200$ (thin solid line). Note the appearance of the two peaks, signalizing a richer structure for the zero mode.

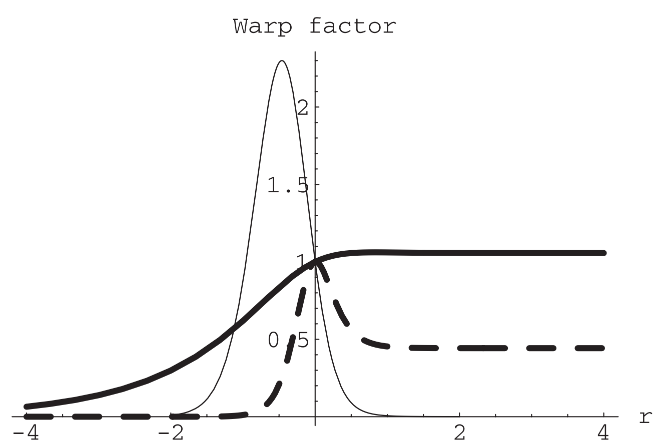


FIG. 3. Warp factor for the case where $\lambda = 4\mu$, $c_0 = 1/16$, $\mu = 1$, $a = 0.6$ (thick solid line); $a = 1.2$ (dashed line); and the case where $a = 2$ and $\mu = 0.2$.

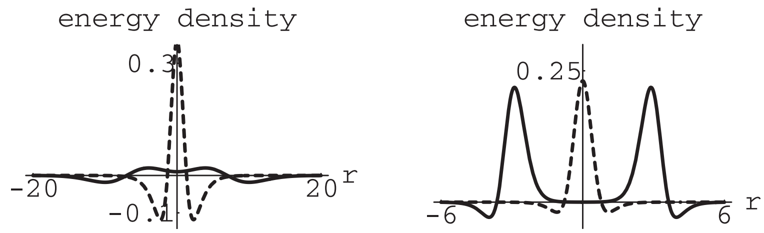


FIG. 4. Energy density of Ref. [2] $s = 0.05$ and 0.30 (left). Energy density for the case where $a = 1$, $\lambda = \mu = 1$, $c_0 \neq -2$, and $c_0 = -2.00001$ (solid line); $-4.0$ (dashed line) (right).

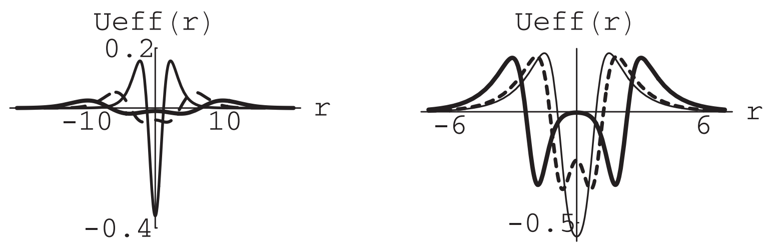


FIG. 5. Comparison of the stability potential of Ref. [2] (left) with $s = 0.05, 0.1, 0.3$ and ours with $\lambda = \mu$, $a = 1$, and $c_0 = -2.001$ (thick solid line); $-2.1$ (dashed line); $-2.5$ (thin solid line).

$$\chi^{(1)}_{\mathrm{CBW}}(r) = \frac{a}{2}[1 \pm \tanh(\mu a r)], \tag{32}$$

and the solution for the $\phi(r)$ field is given by

$$\phi^{(1)}_{\mathrm{CBW}}(r) = \frac{a}{2}[\tanh(\mu a r) \mp 1]. \tag{33}$$

The corresponding warp factor is

$$e^{2A(r)} = N_\alpha\left[\frac{a}{2}[1 \pm \tanh(\mu a r)]\right]^{(2a^2/9)} \times \exp\left\{-\frac{a^2}{9}\left[(1 \pm \tanh(\mu a r))^2 + \frac{c_0}{a}(1 \pm \tanh(\mu a r))\right]\right\}. \tag{34}$$

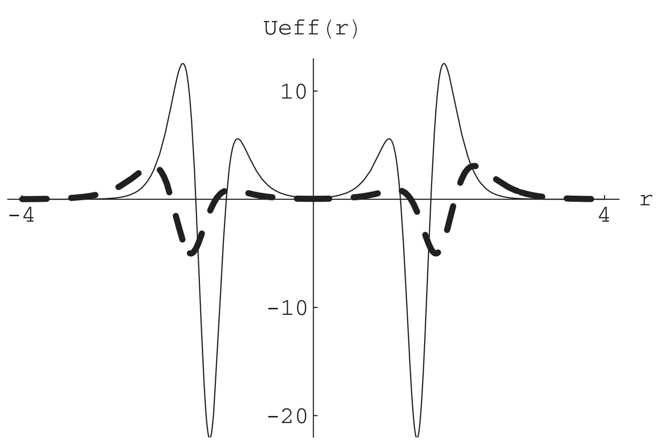


FIG. 6. Stability potential for the case where $c_0 = 1/16.0001$ with $a = 1$ (thin dashed line) and $a = 1.2$ (thick solid line).

For $c_0 = 1/16$ and $\lambda = 4\mu$, the following set of domain walls is obtained:

$$\tilde{\chi}^{(2)}_{\mathrm{CBW}}(x) = -\sqrt{2}a\frac{\cosh(\mu a r) \pm \sinh(\mu a r)}{\sqrt{\cosh(2\mu a r)}}, \tag{35}$$

and

$$\tilde{\phi}^{(2)}_{\mathrm{CBW}}(x) = \frac{a}{2}(1 \mp \tanh(2\mu a r)). \tag{36}$$

The warp factor, on its turn, is found to be given by

$$e^{2A(r)} = N_\alpha\left[\frac{-ae^{\pm\mu a r}}{\sqrt{\cosh(2\mu a r)}}\right]^{(16a^2/9)} \times \exp\left\{-\frac{2a^2}{9}\tanh(2\mu a r)[16a^2 c_0 \tanh(2\mu a r) \mp (1 + 32a^2 c_0)]\right\}. \tag{37}$$

## IV. STABILITY AND ZERO MODES

In general it is quite hard to take into account a full set of fluctuations of the metric around the background in a model where gravity is coupled to scalars. This happens as a consequence of a very intricate system of coupled nonlinear second-order differential equations [2,13,14,16]. Fortunately, however, there is a sector where the metric fluctuations decouple from the scalars, and it comes to be the one associated with the transverse and traceless part of the metric fluctuation [13,14]. This can be shown if one introduces a metric perturbation like

$$ds^2 = e^{2A(r)}(\eta_{\mu\nu} + \varepsilon h_{\mu\nu})dx^\mu dx^\nu - dr^2, \tag{38}$$

and performs small fluctuations on the scalar fields $\phi \to \phi(r) + \varepsilon\tilde{\phi}(r, x_\mu)$ and $\chi \to \chi(r) + \varepsilon\tilde{\chi}(r, x_\mu)$, with $h_{\mu\nu} = h_{\mu\nu}(r, x_\mu)$, and $\varepsilon$ is a small perturbation parameter. Now, keeping the terms in the action up to the second order in $\varepsilon$, as done originally by DeWolfe and collaborators [13,14] for the case with an arbitrary number of scalar fields, as well as by Bazeia and Gomes in the case of two scalar fields [2], one gets the following set of coupled equations for the scalars fluctuations:

$$\begin{aligned} &e^{-2A}\Box\tilde{\phi} - 4\frac{dA}{dr}\frac{d\tilde{\phi}}{dr} - \frac{d^2\tilde{\phi}}{dr^2} + \frac{\partial^2 V}{\partial\phi^2}\tilde{\phi} + \frac{\partial^2 V}{\partial\phi\partial\chi}\tilde{\chi} \\ &\quad = \frac{1}{2}\frac{d\phi}{dr}\eta^{\mu\nu}\frac{dh_{\mu\nu}}{dr}, \\ &e^{-2A}\Box\tilde{\chi} - 4\frac{dA}{dr}\frac{d\tilde{\chi}}{dr} - \frac{d^2\tilde{\chi}}{dr^2} + \frac{\partial^2 V}{\partial\chi^2}\tilde{\chi} + \frac{\partial^2 V}{\partial\phi\partial\chi}\tilde{\phi} \\ &\quad = \frac{1}{2}\frac{d\chi}{dr}\eta^{\mu\nu}\frac{dh_{\mu\nu}}{dr}, \end{aligned} \tag{39}$$

and, for the metric fluctuations, one obtains

$$\begin{aligned} &-\frac{1}{2}\Box h_{\mu\nu} + e^{2A}\left(\frac{1}{2}\frac{d}{dr} + 2\frac{dA}{dr}\right)\frac{dh_{\mu\nu}}{dr} \\ &- \frac{1}{2}\eta^{\alpha\beta}(\partial_\mu\partial_\nu h_{\alpha\beta} - \partial_\mu\partial_\alpha h_{\beta\nu} - \partial_\nu\partial_\alpha h_{\beta\mu}) \\ &+ \frac{1}{2}\eta_{\mu\nu}e^{2A}\frac{dA}{dr}\partial_r(\eta^{\alpha\beta}h_{\alpha\beta}) \\ &\qquad + \frac{4}{3}e^{2A}\eta_{\mu\nu}\left(\frac{\partial V}{\partial\phi}\tilde{\phi} + \frac{\partial V}{\partial\chi}\tilde{\chi}\right) = 0. \end{aligned} \tag{40}$$

One can simplify this last equation by choosing a transverse and traceless $h_{\mu\nu}$, which is done through the use of the projector

$$P_{\mu\nu\alpha\beta} \equiv \tfrac{1}{2}(\pi_{\mu\alpha}\pi_{\nu\beta} + \pi_{\mu\beta}\pi_{\nu\alpha}) - \tfrac{1}{3}\pi_{\mu\nu}\pi_{\alpha\beta}, \tag{41}$$

where $\pi_{\mu\nu} \equiv \eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box}$. In other words, by using that $\bar{h}_{\mu\nu} = P_{\mu\nu\alpha\beta}h^{\alpha\beta}$, one arrives at

$$\frac{d^2\bar{h}_{\mu\nu}}{dr^2} + 4\frac{dA}{dr}\frac{d\bar{h}_{\mu\nu}}{dr} - e^{-2A}\partial_\rho\partial^\rho\bar{h}_{\mu\nu} = 0. \tag{42}$$

Now, performing a sequence of function redefinition

$$\bar{h}_{\mu\nu} \equiv e^{i\vec{k}.\vec{x}}e^{-(3/2)A}\psi_{\mu\nu}, \tag{43}$$

and variable transformation $z = \int e^{-A(r)}dr$ [2], one can recast the above equation into a kind of Schrödinger equation

$$-\frac{d^2\psi_{\mu\nu}}{dz^2} + U_{\rm eff}(z)\psi_{\mu\nu} = k^2\psi_{\mu\nu}, \tag{44}$$

where the effective potential is

$$U_{\rm eff}(z) = \frac{9}{4}\left(\frac{dA}{dz}\right)^2 + \frac{3}{2}\frac{d^2A}{dz^2}. \tag{45}$$

The above differential equation can be factorized as

$$\begin{aligned} a^+a\psi_{\mu\nu}(z) &\equiv \left(\frac{d}{dz} + \frac{3}{2}\frac{dA}{dz}\right)\left(-\frac{d}{dz} + \frac{3}{2}\frac{dA}{dz}\right)\psi_{\mu\nu}(z) \\ &= k^2\psi_{\mu\nu}. \end{aligned} \tag{46}$$

It can be shown that $k^2$ is positive or zero since the resulting Hamiltonian can be factorized as the product of two operators which are adjoint to each other [21]. So the system is stable against linear classical metric fluctuations.

Regarding the stability of the system against classical linear fluctuations of the scalar fields, we notice that Eqs. (39) are quite hard to be analyzed due to the coupling of the fields among themselves and with the metric fluctuations [the term in the right-hand side of Eqs. (39)]. Thus, one can try to simplify the problem by considering only the fluctuations of one of the scalar fields. If we consider, for instance, only the fluctuation of $\chi(r)$, we are left just with the second of the equations in (39). Performing the field transformation $\tilde{\chi}(r) = e^{i\vec{p}.\vec{x}}e^{-(3/2)A}\zeta(z)$, we obtain

$$-\frac{d^2\zeta}{dz^2} + \left(U_{\rm eff}(z) + e^{2A(z)}\frac{\partial^2 V}{\partial\chi^2}\right)\zeta = p^2\zeta, \tag{47}$$

where $U_{\rm eff}(z)$ is given in Eq. (45) and we have taken into account only the zero mode of the metric fluctuation. This can be done as far as the potential $U_{\rm eff}(z)$, the one responsible to localize the gravity, supports only the zero mode as a localized state. Unfortunately, we have not been able to factorize Eq. (47) as a product of two operators which are adjoint to each other, as in Eq. (46). If that is possible, the system is also stable against the fluctuations of at least one of the scalar fields. Furthermore, in general the factor $e^{2A(z)}(\partial^2 V/\partial\chi^2)$ is not positive for all values of the variable $z$, thus we cannot guarantee that the spectrum is positive semidefinite. As far as we know, the question regarding the stability of the scalar fields is an open problem. In Ref. [14] this question was thoroughly examined for the situation where only one active scalar field is present and all the metric fluctuations modes were fully taken into account. The authors show that in some cases the stability of domain walls can be proven, that is $p^2 \geq 0$, although the effective potential cannot be factorized. Unfortunately, the case we are studying here does not belongs to any of those cases.

Returning to the analysis of the metric fluctuations equation, we remark that the zero mode coming from Eq. (44) grants the existence of massless four-dimensional gravitons [2,4,6,16]. In general, the shape of the zero mode is quite similar to the warp factor so that one can think that there is some relation between them. Next we will show that these two quantities really present the same generic shape. With this in mind, we start from Eq. (42), redefine the function $\bar{h}_{\mu\nu}$ as $\bar{h}_{\mu\nu} = e^{-2A(r)}\xi_{\mu\nu}(r)$, and obtain the

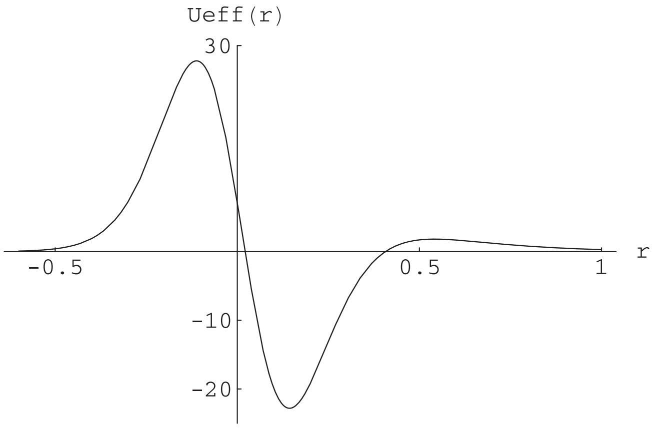


FIG. 7. A typical stability potential for the critical case, both when $\lambda = \mu$ as $\lambda = 4\mu$.

following equation:

$$-\frac{d^2\xi_{\mu\nu}}{dr^2} + 2\left(\frac{d^2A}{dr^2} + 2\left(\frac{dA}{dr}\right)^2\right)\xi_{\mu\nu} - k^2 e^{-2A}\xi_{\mu\nu} = 0, \tag{48}$$

which, for the case of the zero mode ($k^2 = 0$) can be rewritten as

$$\left(\frac{d}{dr} + 2\frac{dA}{dr}\right)\left(-\frac{d}{dr} + 2\frac{dA}{dr}\right)\xi_{\mu\nu} = 0, \tag{49}$$

and finally one can see that the zero-mode solution, apart from a normalization factor, is precisely the warp factor

$$\xi^{(0)}_{\mu\nu} = N_0 e^{2A(r)}\eta_{\mu\nu}. \tag{50}$$

In terms of the coordinate $r$ the effective potential which localizes the gravitation in the brane is written as

$$U_{\mathrm{eff}}(r) = \frac{3}{4}e^{2A}\left(2\frac{d^2A}{dr^2} + 5\left(\frac{dA}{dr}\right)^2\right). \tag{51}$$

Obviously the above potential is equal to the one in the $z$ variable; it will be a kind of rescaled one. However, the general shape and characteristics in both variables are the same. The stability potential is represented in Figs. 5–7. In Fig. 5 we compare the behavior of one of our degenerate cases with those of Bazeia and Gomes [2]. Figure 6 shows clearly the structure of the potential in a situation where two interactive regions are separated by an approximately zero force one. Finally, in Fig. 7, we see that those separated potentials recombine into a single one.

The essential idea in our work is to show that the situation is much richer than that analyzed in [2], and that from a complete set of solutions such as the one we present here, important consequences for the warp factor structure and, consequently, for the brane world scenario, show up. One can cite for instance the fact that one can control, by means of a parameter which is not present in the potential $V(\phi, \chi)$, the region where the metric is approximately flat. Furthermore, in a given critical case, there are two of these regions, separated by a transition one (see Fig. 3).

## V. FINAL REMARKS

In this work we analyze the impact of a general set of solitonic solutions over the characteristics of some models presenting interaction between two scalar fields coupled to gravity in (4,1) dimensions in warped space-time with one extra dimension. Essentially, we explore a larger class of solutions of a model recently studied [2]. In doing so, we have discovered a number of interesting features, for instance, a kind of type-I extreme domain wall as classified in [28], when we dealt with what we called critical domain walls (see Fig. 3).

One very important consequence of our study is that the thickness of the domain walls can be controlled by means of an external parameter (regarding the scalar fields potential), and this can be done without changing the potential parameters, in contrast with what is done in other models [2,6].

Furthermore, one can observe the appearance of a controllable flat region in the warp factor, where one could speak of a Minkowski-type metric region (see Figs. 2 and 4). In fact, in Fig. 4, where the energy density is plotted, we see clearly in the case with $c_0 = -2.000\,01$ that the region where negative energy densities show up is outside of the ''Minkowskian'' one. Thus one could speculate about a possible confining mechanism for the bulk particles in that internal region.

## ACKNOWLEDGMENTS

The authors A. S. D. and M. H. give thanks to CNPq and ACAF to CAPES for partial financial support. We also thank Professor D. Bazeia for introducing us to the matter of solitons and BPS solutions.

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