

Harmonic superspace from the $AdS_5 \times S^5$ pure spinor formalism

Nathan Berkovits and Thiago Fleury

*ICTP South American Institute for Fundamental Research,
Instituto de Física Teórica, UNESP — Univ. Estadual Paulista,
Rua Dr. Bento T. Ferraz 271, 01140-070, São Paulo, SP, Brasil*

E-mail: nberkovi@ift.unesp.br, tfleury@ift.unesp.br

ABSTRACT: On-shell supergravity vertex operators in an $AdS_5 \times S^5$ background are described in the pure spinor formalism by the zero mode cohomology of a BRST operator. After expanding the pure spinor BRST operator in terms of the AdS_5 radius variable, this cohomology is computed using $\mathcal{N} = 4$ harmonic superspace variables and explicit superfield expressions are obtained for the behavior of supergravity vertex operators near the boundary of AdS_5 .

KEYWORDS: Superspaces, AdS-CFT Correspondence, BRST Quantization

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1 Introduction

Although the superstring worldsheet action in an $AdS_5 \times S^5$ background is well-known both in the Green-Schwarz [1, 2] and pure spinor [3, 4] formalisms, an explicit superfield construction of $AdS_5 \times S^5$ supergravity vertex operators is still an open problem. Superfield expressions for the dual half-BPS super-Yang-Mills gauge-invariant operators have been constructed using d=4 N=4 harmonic superspace [5, 6] [7], however, analogous superfield expressions for supergravity have only been constructed for the field strengths [8, 9] and not for the supergravity gauge fields that appear in superstring vertex operators.

In the Green-Schwarz formalism, supergravity vertex operators must preserve kappa-symmetry and, in the pure spinor formalism, supergravity vertex operators must preserve BRST invariance. Only the pure spinor formalism will be discussed here, however, it should be possible to extend our results for the Green-Schwarz supergravity vertex operators. In any consistent curved background, Type IIB supergravity vertex operators in the pure spinor formalism are defined by [10]

$$V = \lambda^\alpha \hat{\lambda}^\beta A_{\alpha\beta}(x, \theta, \hat{\theta}) \quad (1.1)$$

where λ^α and $\hat{\lambda}^\beta$ are left and right-moving pure spinors and $A_{\alpha\beta}$ is an $\mathcal{N} = 2$ d=10 bispinor superfield. The BRST operator is $Q = \lambda^\alpha \nabla_\alpha + \hat{\lambda}^\beta \hat{\nabla}_\beta$ where ∇_α and $\hat{\nabla}_\beta$ are the covariant $\mathcal{N} = 2$ d=10 superspace derivatives.

In this paper, we will compute the BRST cohomology by first expanding Q near the boundary of AdS_5 as

$$Q = Q_{-\frac{1}{2}} + Q_{\frac{1}{2}} + Q_{\frac{3}{2}} + \dots \quad (1.2)$$

where Q_n is proportional to z^n and z is the distance from the boundary of AdS_5 . In performing this expansion, it will be convenient to use a $\frac{PSU(2,2|4)}{SO(4,1) \times SO(6)} \times \frac{SO(6)}{SO(5)}$ supercoset description of $AdS_5 \times S^5$ instead of the usual $\frac{PSU(2,2|4)}{SO(4,1) \times SO(5)}$ supercoset description. We will then argue, making some assumptions, that the BRST cohomology is completely determined by the cohomology of the first two terms $Q_{-\frac{1}{2}} + Q_{\frac{1}{2}}$ and, as expected from holography, BRST-invariant vertex operators are determined by their behavior near the boundary of AdS_5 . Finally, we will compute the zero mode cohomology of $Q_{-\frac{1}{2}} + Q_{\frac{1}{2}}$ and express the result in $\mathcal{N} = 4$ d=4 harmonic superspace. In this way, we will obtain explicit superfield expressions for the behavior of supergravity vertex operators near the boundary of AdS_5 .

In section 2 of this paper, we describe the $AdS_5 \times S^5$ pure spinor formalism using the $\frac{PSU(2,2|4)}{SO(4,1) \times SO(6)} \times \frac{SO(6)}{SO(5)}$ supercoset instead of the usual $\frac{PSU(2,2|4)}{SO(4,1) \times SO(5)}$ supercoset. In section 3, we expand the BRST operator as $Q = Q_{-\frac{1}{2}} + Q_{\frac{1}{2}} + Q_{\frac{3}{2}} + \dots$ where Q_n is proportional to z^n and z is the distance from the AdS_5 boundary, and argue that the cohomology is determined by the first two terms $Q_{-\frac{1}{2}} + Q_{\frac{1}{2}}$. In section 4, we restrict to the zero mode cohomology corresponding to supergravity states and explicitly compute the cohomology of $Q_{-\frac{1}{2}} + Q_{\frac{1}{2}}$, thereby obtaining explicit superfield expressions in $\mathcal{N} = 4$ harmonic superspace for the behavior of supergravity vertex operators near the boundary of AdS_5 . In section 5, we summarize our results and discuss possible applications such as computation of the massive spectrum and tree-level scattering amplitudes in $AdS_5 \times S^5$.

2 Pure spinor formalism with $\frac{PSU(2,2|4)}{SO(4,1) \times SO(6)}$ supercoset

In this section, the pure spinor formalism in an $AdS_5 \times S^5$ background will be reviewed. However, instead of representing the worldsheet matter variables with the $AdS_5 \times S^5$ superspace coset $\frac{PSU(2,2|4)}{SO(4,1) \times SO(5)}$, the worldsheet matter variables will be represented by the AdS_5 superspace coset $\frac{PSU(2,2|4)}{SO(4,1) \times SO(6)}$ together with $\frac{SO(6)}{SO(5)}$ variables for S^5 . Although the two superspace cosets are related by a field redefinition, the $\frac{PSU(2,2|4)}{SO(4,1) \times SO(6)} \times \frac{SO(6)}{SO(5)}$ is more convenient for comparing with $\mathcal{N} = 4$ d=4 harmonic superspace since the $\frac{SO(6)}{SO(5)}$ variables transform under $\mathcal{N} = 4$ d=4 supersymmetry in the same manner as harmonic variables.

2.1 Worldsheet variables

The AdS_5 superspace contains 5 bosonic variables denoted $[x^m, z]$ for $m = 0$ to 3, and 32 fermionic variables denoted $[\theta^{\alpha j}, \bar{\theta}_j^{\dot{\alpha}}, \psi_j^{\alpha}, \bar{\psi}^{\dot{\alpha} j}]$ for $(\alpha, \dot{\alpha}) = 1$ to 2 and $j = 1$ to 4. These variables appear in the $\frac{PSU(2,2|4)}{SO(4,1) \times SO(6)}$ supercoset as

$$g = \exp(x^m P_m + i \theta^{\alpha j} q_{\alpha j} + i \bar{\theta}_{\dot{\alpha} j} \bar{q}^{\dot{\alpha} j}) \exp(i \psi_j^{\alpha} s_{\alpha}^j + i \bar{\psi}_{\dot{\alpha}}^j \bar{s}_{\dot{\alpha}}^j) z^D, \quad (2.1)$$

where $[P_m, q_{\alpha j}, \bar{q}_{\dot{\alpha}}^j]$ are the $\mathcal{N} = 4$ d=4 supersymmetry and translation generators, D is the dilatation generator, and $[s^{\alpha j}, \bar{s}_{\dot{\alpha}}^j]$ are the $\mathcal{N} = 4$ d=4 superconformal generators. Under

global PSU(2,2|4) transformations generated by Σ , g transforms by left multiplication as $\delta g = \Sigma g$. And under local SO(4,1) $\times$ SO(6) transformations generated by Ω , g transforms by right multiplication as $\delta g = g\Omega$. Note that with the parameterization of (2.1), when $z \rightarrow 0$ at the AdS_5 boundary, the variables $[x, \theta, \bar{\theta}]$ transform in the usual $\mathcal{N} = 4$ d=4 superconformal manner under global PSU(2,2|4) transformations.

The S^5 space will be parameterized using a unit vector y^J for $J = 1$ to 6 satisfying $y^J y^J = 1$. Using SO(6) Pauli matrices σ_{jk}^J , one can define $y_{jk} = -y_{kj} = y_J \sigma_{jk}^J$ which satisfies the normalization condition that $\frac{1}{8} \epsilon^{jklm} y_{jk} y_{lm} = -1$. Note that $y^{jk} = \frac{1}{2} \epsilon^{jklm} y_{lm}$ and that $y_{jk} y^{kl} = \delta_j^l$.

Finally, one needs to include the left and right-moving pure spinor variables $(\lambda^{\alpha j}, \bar{\lambda}_j^{\dot{\alpha}})$ and $(\hat{\lambda}^{\alpha j}, \hat{\bar{\lambda}}_j^{\dot{\alpha}})$, as well as their conjugate momenta $(w_{\alpha j}, \bar{w}_{\dot{\alpha}}^j)$ and $(\hat{w}_{\alpha j}, \hat{\bar{w}}_{\dot{\alpha}}^j)$. These variables satisfy the pure spinor constraints

$$\begin{aligned} \lambda^{\alpha j} \bar{\lambda}_j^{\dot{\alpha}} &= 0, & \lambda^{\alpha j} \lambda_{\alpha}^k - \frac{1}{2} \epsilon^{jklm} \bar{\lambda}_{\dot{\alpha} l} \bar{\lambda}_m^{\dot{\alpha}} &= 0, \\ \hat{\lambda}^{\alpha j} \hat{\bar{\lambda}}_j^{\dot{\alpha}} &= 0, & \hat{\lambda}^{\alpha j} \hat{\lambda}_{\alpha}^k - \frac{1}{2} \epsilon^{jklm} \hat{\bar{\lambda}}_{\dot{\alpha} l} \hat{\bar{\lambda}}_m^{\dot{\alpha}} &= 0, \end{aligned} \quad (2.2)$$

which are the four-dimensional reduction of the d=10 pure spinor constraints

$$\lambda \gamma^M \lambda = 0, \quad \hat{\lambda} \gamma^M \hat{\lambda} = 0, \quad (2.3)$$

for $M = 0$ to 9. As in ten dimensions, gauge invariance under $\delta w = (\gamma^M \lambda) \Lambda_M$ and $\delta \hat{w} = (\gamma^M \hat{\lambda}) \hat{\Lambda}_M$ implies that w and $\hat{w}$ can only appear in the combinations of either SO(9,1) Lorentz currents $N^{MN} = \frac{1}{4} (w \gamma^{MN} \lambda)$ and $\hat{N}^{MN} = \frac{1}{4} (\hat{w} \gamma^{MN} \hat{\lambda})$, or ghost currents $J_g = (w \lambda)$ and $\hat{J}_g = (\hat{w} \hat{\lambda})$. So as in ten dimensions, there are 11 independent λ 's and $\hat{\lambda}$'s and 11 gauge-invariant w 's and $\hat{w}$'s.

Under the local SO(4,1) $\times$ SO(6) gauge transformations which transform g by right multiplication as $\delta g = g\Omega$, one also must transform the y_{jk} and pure spinor variables. Under the SO(3,1) $\times$ SO(6) subgroup of SO(4,1) $\times$ SO(6), these variables transform in the obvious way as

$$\begin{aligned} \delta y_{jk} &= c_j^l y_{lk} + c_k^l y_{jl}, & \delta \bar{\lambda}_j^{\dot{\alpha}} &= c_{\dot{\beta}}^{\dot{\alpha}} \bar{\lambda}_j^{\dot{\beta}} + c_j^k \bar{\lambda}_k^{\dot{\alpha}}, \\ \delta \lambda_{\alpha}^j &= c_{\alpha}^{\beta} \lambda_{\beta}^j - c_k^j \lambda_{\alpha}^k, & \delta \bar{w}_{\dot{\alpha}}^j &= -c_{\dot{\alpha}}^{\dot{\beta}} \bar{w}_{\dot{\beta}}^j - c_k^j \bar{w}_{\dot{\alpha}}^k, \\ \delta w_j^{\alpha} &= -c_{\beta}^{\alpha} w_j^{\beta} + c_j^k w_k^{\alpha}, & \delta \hat{\bar{\lambda}}_j^{\dot{\alpha}} &= c_{\dot{\beta}}^{\dot{\alpha}} \hat{\bar{\lambda}}_j^{\dot{\beta}} + c_j^k \hat{\bar{\lambda}}_k^{\dot{\alpha}}, \\ \delta \hat{\lambda}_{\alpha}^j &= c_{\alpha}^{\beta} \hat{\lambda}_{\beta}^j - c_k^j \hat{\lambda}_{\alpha}^k, & \delta \hat{\bar{w}}_{\dot{\alpha}}^j &= -c_{\dot{\alpha}}^{\dot{\beta}} \hat{\bar{w}}_{\dot{\beta}}^j - c_k^j \hat{\bar{w}}_{\dot{\alpha}}^k, \\ \delta \hat{w}_j^{\alpha} &= -c_{\beta}^{\alpha} \hat{w}_j^{\beta} + c_j^k \hat{w}_k^{\alpha}, & & \end{aligned} \quad (2.4)$$

where $\Omega = c_k^j R_j^k - \frac{1}{4} (c_{\beta}^{\alpha} (\sigma^{mn})_{\alpha}^{\beta} + c_{\dot{\beta}}^{\dot{\alpha}} (\bar{\sigma}^{mn})_{\dot{\alpha}}^{\dot{\beta}}) M_{mn}$, R_j^k are the SU(4) R-symmetry generators and M_{mn} are the SO(3,1) Lorentz generators. And under the local SO(4,1) transformations which are not contained in SO(3,1), these variables transform as

$$\delta y_{jk} = 0, \quad (2.5)$$

$$\begin{aligned}\delta\lambda_\alpha^j &= -c_{\dot{\alpha}\alpha}y^{jk}\bar{\lambda}_k^{\dot{\alpha}}, & \delta\bar{\lambda}_j^{\dot{\alpha}} &= c^{\alpha\dot{\alpha}}y_{jk}\lambda_\alpha^k, & \delta\hat{\lambda}_\alpha^j &= -c_{\dot{\alpha}\alpha}y^{jk}\hat{\lambda}_k^{\dot{\alpha}}, & \delta\hat{\lambda}_j^{\dot{\alpha}} &= c^{\alpha\dot{\alpha}}y_{jk}\hat{\lambda}_\alpha^k, \\ \delta w_j^\alpha &= c^{\alpha\dot{\alpha}}y_{jk}\bar{w}_\alpha^{\dot{\alpha}}, & \delta\bar{w}_\alpha^{\dot{\alpha}} &= -c_{\dot{\alpha}\alpha}y^{jk}w_k^\alpha, & \delta\hat{w}_j^\alpha &= c^{\alpha\dot{\alpha}}y_{jk}\hat{w}_\alpha^{\dot{\alpha}}, & \delta\hat{w}_\alpha^{\dot{\alpha}} &= -c_{\dot{\alpha}\alpha}y^{jk}\hat{w}_k^{\dot{\alpha}},\end{aligned}$$

where $\Omega = c^{\alpha\dot{\alpha}}i\sigma_{\alpha\dot{\alpha}}^m M_{5m}$ and M_{5m} are the four $\text{SO}(4,1)/\text{SO}(3,1)$ generators. Note that

$$\begin{aligned}\lambda^{Aj} &= [\lambda_\alpha^j, y^{jk}\bar{\lambda}_k^{\dot{\alpha}}], & w_{Aj} &= [w_j^\alpha, -y_{jk}\bar{w}_\alpha^{\dot{\alpha}}], \\ \hat{\lambda}^{Aj} &= [\hat{\lambda}_\alpha^j, y^{jk}\hat{\lambda}_k^{\dot{\alpha}}], & \hat{w}_{Aj} &= [\hat{w}_j^\alpha, -y_{jk}\hat{\bar{w}}_\alpha^{\dot{\alpha}}],\end{aligned}\tag{2.6}$$

transform covariantly as $\text{SO}(4,1) \times \text{SO}(6)$ spinors where $A = (\alpha, \dot{\alpha})$ is an $\text{SO}(4,1)$ spinor index.

2.2 Worldsheet action

To construct the BRST-invariant worldsheet action using the $\frac{\text{PSU}(2,2|4)}{\text{SO}(4,1) \times \text{SO}(6)}$ supercoset g of (2.1), the first step is to define the left-invariant currents $J = g^{-1}\partial g$ and $\bar{J} = g^{-1}\bar{\partial}g$ taking values in the $\text{PSU}(2,2|4)$ Lie algebra. The bosonic currents will be denoted as (J^m, J^5) for the AdS_5 translations, $[J^{\alpha\beta}, J^{\dot{\alpha}\dot{\beta}}, J^{\alpha\dot{\beta}}]$ for the $\text{SO}(4,1)$ rotations, and J_k^j for the $\text{SO}(6)$ rotations. And the fermionic currents will be denoted as $[J^{j\alpha}, J_{\dot{\alpha}j}]$ for the supersymmetries and $[J_j^\alpha, J_{\dot{\alpha}}^j]$ for the superconformal transformations. So the left-invariant current is

$$g^{-1}\partial g = J^m \frac{1}{2}(P_m + K_m) + J^5 D + J^{AB} M_{AB} + J_j^k R_k^j + J^{\alpha j} q_{\alpha j} + J_{\dot{\alpha}j} \bar{q}^{\dot{\alpha}j} + J_j^\alpha s_\alpha^j + J_{\dot{\alpha}}^j \bar{s}_j^{\dot{\alpha}}\tag{2.7}$$

where $M_{AB} = M_{BA}$ are the $\text{SO}(4,1)$ generators, $A = (\alpha, \dot{\alpha})$ is an $\text{SO}(4,1)$ spinor index, and K_m is the generator of special conformal transformations.

In terms of these currents, the ghost-independent contribution to the worldsheet action is

$$\begin{aligned}S_{\text{matter}} &= \int d^2z \left[\frac{1}{2} \eta_{mn} J^m \bar{J}^n + \frac{1}{2} J^5 \bar{J}^5 - \frac{1}{8} (\nabla y)_{jk} (\bar{\nabla} y)^{jk} \right. \\ &\quad - 2J_j^\alpha \bar{J}_\alpha^j - 2J^{\alpha j} \bar{J}_{\alpha j} - 2J_{\dot{\alpha}}^j \bar{J}_j^{\dot{\alpha}} - 2J_{\dot{\alpha}j} \bar{J}^{\dot{\alpha}j} \\ &\quad \left. - y_{jk} J^{\alpha j} \bar{J}_\alpha^k - y^{jk} J_j^\alpha \bar{J}_{\alpha k} + y^{jk} J_{\dot{\alpha}j} \bar{J}_k^{\dot{\alpha}} + y_{jk} J_{\dot{\alpha}}^j \bar{J}^{\dot{\alpha}k} \right],\end{aligned}\tag{2.8}$$

where $(\nabla y)_{jk} = \partial y_{jk} - J_j^l y_{lk} - J_k^l y_{jl}$. The easiest way to verify this action is to use the $\text{SO}(6)$ gauge invariance to gauge-fix $y_{jk} = (\sigma^6)_{jk}$ where $(\sigma^J)_{jk}$ for $J = 1$ to 6 are the $\text{SO}(6)$ Pauli matrices. One can then compare (2.8) with the action written in terms of the $\frac{\text{PSU}(2,2|4)}{\text{SO}(4,1) \times \text{SO}(5)}$ supercoset.

When $y_{jk} = (\sigma^6)_{jk}$, the term $-\frac{1}{8}(\nabla y)_{jk}(\bar{\nabla} y)^{jk}$ reduces to $\frac{1}{2} \sum_{J=1}^5 J^6 \bar{J}^J$ where $J^6 J \equiv \frac{1}{2}(\sigma^6 J)_k^j J_j^k$. Furthermore, the third line of (2.8) reduces to

$$-(\sigma^6)^{jk} \varepsilon_{AB} J_j^A \bar{J}_k^B - (\sigma^6)_{jk} \varepsilon_{AB} J^{Aj} \bar{J}^{Bk}\tag{2.9}$$

where ε_{AB} is the $\text{SO}(4,1)$ -invariant antisymmetric metric, i.e. $\varepsilon_{\alpha\dot{\beta}} = 0$, $\varepsilon_{\alpha\beta} = \epsilon_{\alpha\beta}$ and $\varepsilon_{\dot{\alpha}\dot{\beta}} = \epsilon_{\dot{\alpha}\dot{\beta}}$. In this gauge, one can easily show the equivalence of (2.8) with the ghost-independent

contribution to the action written in terms of the $\frac{\text{PSU}(2,2|4)}{\text{SO}(4,1) \times \text{SO}(5)}$ supercoset which is

$$S_{\text{matter}} = \int d^2z \left[\frac{1}{2} \eta_{ab} J_{(2)}^a \bar{J}_{(2)}^b - \frac{1}{2} \varepsilon_{AB} (\sigma^6)_{JK} (J_{(1)}^{AJ} \bar{J}_{(3)}^{BK} + \bar{J}_{(1)}^{AJ} J_{(3)}^{BK}) \right. \\ \left. + \frac{1}{4} \varepsilon_{AB} (\sigma^6)_{JK} (J_{(1)}^{AJ} \bar{J}_{(3)}^{BK} - \bar{J}_{(1)}^{AJ} J_{(3)}^{BK}) \right] \quad (2.10)$$

where we have used the notation

$$\varepsilon_{AB} (\sigma^6)_{JK} J_{(1)}^{AJ} \bar{J}_{(3)}^{BK} = (\sigma^6)_{ij} \epsilon_{\alpha\beta} J_{(1)}^{\alpha i} \bar{J}_{(3)}^{\beta j} - (\sigma^6)^{ij} \epsilon_{\dot{\alpha}\dot{\beta}} J_{(1)i}^{\dot{\alpha}} \bar{J}_{(3)j}^{\dot{\beta}} \quad (2.11)$$

and similar for the other terms. In (2.10) $a = 0$ to 9 is an $\text{SO}(4,1) \times \text{SO}(5)$ vector index and the currents $[J_{(1)}^{\alpha j}, J_{(1)j}^{\dot{\alpha}}, J_{(2)}^a, J_{(3)}^{\alpha j}, J_{(3)j}^{\dot{\alpha}}]$ are related to the currents of (2.7) by

$$J_{(1)}^{\alpha j} = \sqrt{2} J^{\alpha j} + \sqrt{2} (\sigma^6)^{ji} J_i^{\alpha}, \quad J_{(3)}^{\alpha j} = -\sqrt{2} J^{\alpha j} + \sqrt{2} (\sigma^6)^{ji} J_i^{\alpha}, \quad (2.12) \\ J_{(1)j}^{\dot{\alpha}} = -\sqrt{2} J_j^{\dot{\alpha}} + \sqrt{2} (\sigma^6)_{ji} J^{i\dot{\alpha}}, \quad J_{(3)j}^{\dot{\alpha}} = -\sqrt{2} J_j^{\dot{\alpha}} - \sqrt{2} (\sigma^6)_{ji} J^{i\dot{\alpha}},$$

and $J_{(2)}^a = [J^m, J^5, J^{6J}]$.

Finally, the ghost-dependent contribution to the action is given by

$$S_{\text{ghost}} = \int d^2z \left[w_{Aj} (\bar{\nabla} \lambda)^{Aj} - \hat{w}_{Aj} (\nabla \hat{\lambda})^{Aj} + \frac{1}{2} y^{jl} (\bar{\nabla} y)_{lk} w_{Aj} \lambda^{Ak} - \frac{1}{2} y^{jl} (\bar{\nabla} y)_{lk} \hat{w}_{Aj} \hat{\lambda}^{Ak} \right. \\ \left. - 2N_{mn} \hat{N}^{mn} - 4(y^J N_{Jm})(y_K \hat{N}^{Km}) + 2N_{JK} \hat{N}^{JK} - 4(y^L N_{LJ})(y_M \hat{N}^{MJ}) \right], \quad (2.13)$$

where in the first line of (2.13), λ^{Aj} , w_{Aj} , $\hat{\lambda}^{Aj}$ and $\hat{w}_{Aj}$ are the $\text{SO}(4,1) \times \text{SO}(6)$ spinors defined in (2.6) and

$$w_{Aj} (\bar{\nabla} \lambda)^{Aj} = w_j^{\alpha} \bar{\partial} \lambda_{\alpha}^j + \bar{w}_{\dot{\alpha}}^k y_{ki} \bar{\partial} (y^{ij} \lambda_j^{\dot{\alpha}}) - w_j^{\alpha} \bar{J}_{\alpha}^{\beta} \lambda_{\beta}^j - \bar{w}_{\dot{\alpha}}^k \bar{J}_{\beta}^{\dot{\alpha}} \bar{\lambda}_k^{\beta} \\ + 2w_j^{\alpha} \bar{J}_{\alpha\dot{\alpha}} y^{jk} \bar{\lambda}_k^{\dot{\alpha}} - 2\bar{w}_{\dot{\alpha}}^k y_{kl} \bar{J}^{\dot{\alpha}\alpha} \lambda_{\alpha}^l + w_j^{\alpha} \bar{J}_k^j \lambda_{\alpha}^k + \bar{w}_{\dot{\alpha}}^k y_{ki} \bar{J}_m^i y^{mp} \bar{\lambda}_p^{\dot{\alpha}}, \quad (2.14)$$

and similar for $\hat{w}_{Aj} (\nabla \hat{\lambda})^{Aj}$. In the second line of (2.13), the $\text{SO}(9,1)$ Lorentz currents $N^{MN} = \frac{1}{4} (w \gamma^{MN} \lambda)$ for $M = 0$ to 9 are constructed out of the $\text{SO}(3,1) \times \text{SO}(6)$ spinors $(\lambda^{\alpha j}, \bar{\lambda}_j^{\dot{\alpha}})$ and $(w_{\alpha j}, \bar{w}_{\dot{\alpha}}^j)$ and have been decomposed into their $\text{SO}(3,1) \times \text{SO}(6)$ components as $[N^{mn}, N^{mJ}, N^{JK}]$.

This ghost contribution can be verified by choosing the gauge $y_{jk} = (\sigma^6)_{jk}$ and comparing with the ghost contribution using the $\frac{\text{PSU}(2,2|4)}{\text{SO}(4,1) \times \text{SO}(5)}$ supercoset which is

$$S_{\text{ghost}} = \int d^2z [w_{Aj} \tilde{\nabla} \lambda^{Aj} - \hat{w}_{Aj} \tilde{\nabla} \hat{\lambda}^{Aj} + R_{abcd} \tilde{N}^{ab} \tilde{\hat{N}}^{cd}], \quad (2.15)$$

where $\tilde{\nabla}$ only involves the $\text{SO}(4,1) \times \text{SO}(5)$ connection, $\tilde{N}^{ab}$ is constructed out of the $\text{SO}(4,1) \times \text{SO}(5)$ spinors λ^{Aj} and w_{Aj} , and R_{abcd} is the $AdS_5 \times S^5$ curvature, i.e. $R_{abcd} = \eta_{a[d} \eta_{c]b}$ if (a, b, c, d) are on AdS_5 and $R_{abcd} = -\eta_{a[d} \eta_{c]b}$ if (a, b, c, d) are on S^5 . Since $(\bar{\nabla} \lambda)^{Aj} = \tilde{\nabla} \lambda^{Aj} - \frac{1}{2} \bar{J}^{6J} (\sigma_{6J})_k^j \lambda^{Ak}$, the first line of (2.13) reproduces the first two terms of (2.15) when $y_{jk} = \sigma_{jk}^6$. And by writing the $\text{SO}(4,1) \times \text{SO}(5)$ Lorentz spinors in terms of $\text{SO}(3,1) \times \text{SO}(6)$ spinors, one finds that when $y_{jk} = \sigma_{jk}^6$, the second line of (2.13) reduces to the last term of (2.15).

2.3 BRST operator

Physical closed string states in the pure spinor formalism are described by the cohomology at +2 ghost number of the sum of the left and right-moving BRST operator. In terms of the $\frac{\text{PSU}(2,2|4)}{\text{SO}(4,1) \times \text{SO}(5)}$ supercoset, the BRST operator in an $AdS_5 \times S^5$ background is

$$Q = \int dz \lambda^{AJ} \varepsilon_{AB}(\sigma^6)_{JK} J_{(3)}^{BK} - \int d\bar{z} \hat{\lambda}^{AJ} \varepsilon_{AB}(\sigma^6)_{JK} \bar{J}_{(1)}^{BK}. \quad (2.16)$$

Using the relation of (2.12) for the fermionic currents, one therefore finds that the BRST operator in terms of the $\frac{\text{PSU}(2,2|4)}{\text{SO}(4,1) \times \text{SO}(6)}$ supercoset is

$$Q = \int dz [\lambda^{\alpha j}(\sqrt{2}J_{\alpha j} - \sqrt{2}y_{jk}J_{\alpha}^k) - \bar{\lambda}_{\dot{\alpha} j}(\sqrt{2}J^{\dot{\alpha} j} + \sqrt{2}y^{jk}J_k^{\dot{\alpha}})] \\ - \int d\bar{z} [\hat{\lambda}^{\alpha j}(\sqrt{2}\bar{J}_{\alpha j} + \sqrt{2}y_{jk}\bar{J}_{\alpha}^k) + \hat{\bar{\lambda}}_{\dot{\alpha} j}(\sqrt{2}\bar{J}^{\dot{\alpha} j} - \sqrt{2}y^{jk}\bar{J}_k^{\dot{\alpha}})]. \quad (2.17)$$

Under a BRST transformation of (2.16) a representative g' of the supercoset $\frac{\text{PSU}(2,2|4)}{\text{SO}(4,1) \times \text{SO}(5)}$ transforms as

$$\delta g' = g'(\lambda^{\bar{\alpha}} T_{\bar{\alpha}}^1 + \hat{\lambda}^{\hat{\alpha}} T_{\hat{\alpha}}^3) \quad (2.18)$$

where $T_{\bar{\alpha}}^1$ and $T_{\hat{\alpha}}^3$ are the fermionic generators of the $\text{PSU}(2,2|4)$, $\bar{\alpha} = 1, \dots, 16$ and $\hat{\alpha} = 1, \dots, 16$. These generators are related with $q_{\alpha i}$, $q_{\dot{\alpha}}^i$, s_{α}^j , $s_{\dot{\alpha}}^j$ as

$$T_{\alpha i}^1 = \frac{\sqrt{2}}{4} q_{\alpha i} - \frac{\sqrt{2}}{4} (\sigma^6)_{ij} s_{\alpha}^j, \quad T_{\dot{\alpha}}^{1i} = -\frac{\sqrt{2}}{4} q_{\dot{\alpha}}^i - \frac{\sqrt{2}}{4} (\sigma^6)^{ij} s_{\dot{\alpha} j} \\ T_{\alpha i}^3 = -\frac{\sqrt{2}}{4} q_{\alpha i} - \frac{\sqrt{2}}{4} (\sigma^6)_{ij} s_{\alpha}^j, \quad T_{\dot{\alpha}}^{3i} = -\frac{\sqrt{2}}{4} q_{\dot{\alpha}}^i + \frac{\sqrt{2}}{4} (\sigma^6)^{ij} s_{\dot{\alpha} j} \quad (2.19)$$

Note that in the gauge $y_{jk} = (\sigma^6)_{jk}$, (2.12) and (2.19) satisfy

$$J^{\alpha j} q_{\alpha j} + J_{\dot{\alpha} j} q^{\dot{\alpha} j} + J_j^{\alpha} s_{\alpha}^j + J_{\dot{\alpha}}^j s_{\dot{\alpha}}^j = J^{1\bar{\alpha}} T_{\bar{\alpha}} + J^{3\hat{\alpha}} T_{\hat{\alpha}} \quad (2.20)$$

Using the relations above we can see that the BRST transformation of g under (2.17) is

$$\delta g = g \frac{\sqrt{2}}{4} [i\lambda^{+\alpha j} q_{\alpha j} - i\lambda^{-\alpha j} y_{jk} s_{\alpha}^k + i\bar{\lambda}_{\dot{\alpha} j}^+ \bar{q}^{\dot{\alpha} j} + i\bar{\lambda}_{\dot{\alpha} j}^- y^{jk} \bar{s}_k^{\dot{\alpha}}] \quad (2.21)$$

where $(q_{\alpha}^j, \bar{q}_{\dot{\alpha}}^j)$ and $(s_{\alpha}^j, \bar{s}_{\dot{\alpha}}^j)$ are the $\mathcal{N} = 4$ d=4 supersymmetries and superconformal transformations and

$$\lambda^{-\alpha j} \equiv -i(\lambda^{\alpha j} + \hat{\lambda}^{\alpha j}), \quad \lambda^{+\alpha j} \equiv -i(\lambda^{\alpha j} - \hat{\lambda}^{\alpha j}) \\ \bar{\lambda}_{\dot{\alpha} j}^- \equiv i(\bar{\lambda}_{\dot{\alpha} j} - \hat{\bar{\lambda}}_{\dot{\alpha} j}), \quad \bar{\lambda}_{\dot{\alpha} j}^+ \equiv i(\bar{\lambda}_{\dot{\alpha} j} + \hat{\bar{\lambda}}_{\dot{\alpha} j}). \quad (2.22)$$

Since w_{Aj} and $\hat{w}_{Aj}$ are conjugate to λ^{Aj} and $\hat{\lambda}^{Aj}$, one finds that w_{Aj} and $\hat{w}_{Aj}$ transform under BRST as

$$\delta w_{\alpha j} = \sqrt{2}J_{\alpha j} - \sqrt{2}y_{jk}J_{\alpha}^k, \quad \delta \bar{w}^{\dot{\alpha} j} = -\sqrt{2}J^{\dot{\alpha} j} - \sqrt{2}y^{jk}J_k^{\dot{\alpha}}, \quad (2.23)$$

$$\delta\widehat{w}_{\alpha j} = \sqrt{2}\overline{J}_{\alpha j} + \sqrt{2}y_{jk}\overline{J}_{\alpha}^k, \quad \delta\widehat{w}^{\dot{\alpha}j} = \sqrt{2}\overline{J}^{\dot{\alpha}j} - \sqrt{2}y^{jk}\overline{J}_k^{\dot{\alpha}}.$$

Note that these BRST transformations are defined up to the gauge transformation $\delta w = (\gamma^M \lambda) \Lambda_M$ and $\delta \widehat{w} = (\gamma^M \widehat{\lambda}) \widehat{\Lambda}_M$.

Finally, the BRST transformations of λ^{Aj} , $\widehat{\lambda}^{Aj}$ and y^{jk} are zero. However, note that these variables transform under local $\text{SO}(4,1) \times \text{SO}(6)$ transformations. So if the BRST transformation on g needs to be compensated by a local $\text{SO}(4,1) \times \text{SO}(6)$ transformation in order to preserve a gauge-fixing condition, these variables will transform under the compensating $\text{SO}(4,1) \times \text{SO}(6)$ gauge transformation.

3 Cohomology analysis

In this section, the BRST operator in an $AdS_5 \times S^5$ background will be expanded in terms of the AdS_5 radial variable z . The cohomology will then be shown to be described by the boundary value of the state near $z = 0$.

To simplify the analysis of cohomology, it will be convenient to express the BRST operator in terms of the worldsheet variables $[x, \theta, \psi, z, y, \lambda, \widehat{\lambda}]$ and their canonical momenta $[P_x, P_\theta, P_\psi, P_z, P_y, P_\lambda, P_{\widehat{\lambda}}]$ instead of the worldsheet variables and their time derivatives. As usual, the canonical momenta will be defined as $P_x = \frac{\partial L}{\partial(\partial_\tau x)}$, etc. Note that unlike in the Green-Schwarz formalism which has first and second-class constraints, there are no constraints on the canonical momenta in the pure spinor formalism. Using the Lagrangian of (2.8) and (2.13), one finds, for example, that

$$P_z = -\frac{1}{4z} \left(\frac{\dot{z}}{z} + 2\dot{\theta}^{\alpha j} \psi_{\alpha j} + 2\dot{\theta}_{\dot{\alpha} j} \overline{\psi}^{\dot{\alpha} j} \right) \quad (3.1)$$

where dot means derivative with respect to τ .

Expanding the sum of the left and right-moving BRST operator given in (2.17) in terms of these variables and z , one finds that the BRST operator splits as $Q = Q_{-\frac{1}{2}} + Q_{\frac{1}{2}} + Q_{\frac{3}{2}} + \dots$ where

$$\begin{aligned} Q_{-\frac{1}{2}} &= \left(\frac{\sqrt{2}}{2} \right) z^{-\frac{1}{2}} \lambda^{+\gamma m} y_{mi} P_{\psi_i^\gamma} - \left(\frac{\sqrt{2}}{2} \right) z^{-\frac{1}{2}} \overline{\lambda}_j^{+\dot{\alpha}} y^{ji} P_{\overline{\psi}^{i\dot{\alpha}}} + f(\partial_\sigma X) \\ Q_{\frac{1}{2}} &= \left(\frac{\sqrt{2}}{2} \right) z^{\frac{1}{2}} \lambda^{-\alpha i} (-P_{\theta \alpha i} - i(\sigma^a)_{\alpha \dot{\alpha}} \overline{\theta}_i^{\dot{\alpha}} P_{x^a} + 2\psi_{\alpha i} z P_z + 4\psi_{\alpha k} P_{y^{ij}} y^{jk} - \psi_{\alpha i} P_{y^{jk}} y^{kj} \\ &\quad - 4\psi_{\alpha k} \lambda_\beta^k P_{\lambda_\beta^i} + \psi_{\alpha i} \lambda_\beta^k P_{\lambda_\beta^k} - 4\psi_{\alpha k} \widehat{\lambda}_\beta^k P_{\widehat{\lambda}_\beta^i} + \psi_{\alpha i} \widehat{\lambda}_\beta^k P_{\widehat{\lambda}_\beta^k} - 2\psi_i^\beta \lambda_\beta^j P_{\lambda^{\alpha j}} - 2\psi_{\beta i} \lambda_\alpha^j P_{\lambda_\beta^j} \\ &\quad - 2\psi_i^\beta \widehat{\lambda}_\beta^j P_{\widehat{\lambda}^{\alpha j}} - 2\psi_{\beta i} \widehat{\lambda}_\alpha^j P_{\widehat{\lambda}_\beta^j} + 4\psi_{\alpha k} \overline{\lambda}_i^{\dot{\alpha}} P_{\overline{\lambda}_k^{\dot{\alpha}}} - \psi_{\alpha i} \overline{\lambda}_k^{\dot{\alpha}} P_{\overline{\lambda}_k^{\dot{\alpha}}} + 4\psi_{\alpha k} \widehat{\lambda}_i^{\dot{\alpha}} P_{\widehat{\lambda}_k^{\dot{\alpha}}} - \psi_{\alpha i} \widehat{\lambda}_k^{\dot{\alpha}} P_{\widehat{\lambda}_k^{\dot{\alpha}}} \\ &\quad + 4\psi_{\alpha j} \psi_i^\beta P_{\psi_\beta^j} - 2\psi_{\alpha j} \overline{\psi}_\alpha^j P_{\overline{\psi}_\alpha^i}) + \left(\frac{\sqrt{2}}{2} \right) z^{\frac{1}{2}} \overline{\lambda}_i^{-\dot{\alpha}} (-P_{\overline{\theta}_i^{\dot{\alpha}}} - i\theta^{\beta i} (\sigma^a)_{\beta \dot{\alpha}} P_{x^a} - 2\overline{\psi}_\alpha^i z P_z \\ &\quad - 4\overline{\psi}_\alpha^i P_{y^{ij}} y_{jl} + \overline{\psi}_\alpha^i P_{y_{kj}} y_{jk} - 4\overline{\psi}_\alpha^j \lambda_\alpha^i P_{\lambda_\alpha^j} + \overline{\psi}_\alpha^i \lambda_\alpha^k P_{\lambda_\alpha^k} - 4\overline{\psi}_\alpha^j \widehat{\lambda}_\alpha^i P_{\widehat{\lambda}_\alpha^j} + \overline{\psi}_\alpha^i \widehat{\lambda}_\alpha^k P_{\widehat{\lambda}_\alpha^k} \\ &\quad + 4\overline{\psi}_\alpha^i \widehat{\lambda}_l^{\dot{\beta}} P_{\widehat{\lambda}_l^{\dot{\beta}}} - \overline{\psi}_\alpha^i \widehat{\lambda}_k^{\dot{\beta}} P_{\widehat{\lambda}_k^{\dot{\beta}}} + 4\overline{\psi}_\alpha^i \widehat{\lambda}_l^{\dot{\beta}} P_{\widehat{\lambda}_l^{\dot{\beta}}} - \overline{\psi}_\alpha^i \widehat{\lambda}_k^{\dot{\beta}} P_{\widehat{\lambda}_k^{\dot{\beta}}} - 2\overline{\psi}_\beta^i \overline{\lambda}_k^{\dot{\beta}} P_{\overline{\lambda}_k^{\dot{\alpha}}} + 2\overline{\psi}^{\dot{\beta} i} \overline{\lambda}_{\dot{\alpha} k} P_{\overline{\lambda}_k^{\dot{\beta}}} \\ &\quad - 2\overline{\psi}_\beta^i \widehat{\lambda}_k^{\dot{\beta}} P_{\widehat{\lambda}_k^{\dot{\alpha}}} + 2\overline{\psi}^{\dot{\beta} i} \widehat{\lambda}_{\dot{\alpha} k} P_{\widehat{\lambda}_k^{\dot{\beta}}} - 4\overline{\psi}_\alpha^m \overline{\psi}^{i\dot{\beta}} P_{\overline{\psi}^{m\dot{\beta}}} - 2\psi_{\alpha k} \overline{\psi}_\alpha^k P_{\psi_{\alpha i}}) \end{aligned} \quad (3.2)$$

$$\begin{aligned}
 & + \left(\frac{\sqrt{2}}{2} \right) 2z^{\frac{1}{2}} \lambda^{+\gamma m} y_{mi} (\bar{\psi}_{\dot{\alpha}}^i \bar{\lambda}_k^{\dot{\alpha}} y^{kj} P_{\lambda^{\gamma j}} + \bar{\psi}^{i\dot{\alpha}} y_{kl} \lambda_{\gamma}^l P_{\bar{\lambda}_k^{\dot{\alpha}}} + \bar{\psi}_{\dot{\alpha}}^i \bar{\lambda}_k^{\dot{\alpha}} y^{kj} P_{\hat{\lambda}^{\gamma j}} + \bar{\psi}^{i\dot{\alpha}} y_{kl} \hat{\lambda}_{\gamma}^l P_{\hat{\lambda}_k^{\dot{\alpha}}}) \\
 & - \left(\frac{\sqrt{2}}{2} \right) 2z^{\frac{1}{2}} \bar{\lambda}_m^{+\dot{\alpha}} y^{mi} (\psi_{\alpha i} \bar{\lambda}_k^{\dot{\alpha}} y^{jk} P_{\lambda_{\alpha}^j} + \psi_i^{\beta} \lambda_{\beta}^l y_{kl} P_{\bar{\lambda}_k^{\dot{\alpha}}} + \psi_{\alpha i} \bar{\lambda}_k^{\dot{\alpha}} y^{jk} P_{\hat{\lambda}_{\alpha}^j} + \psi_i^{\alpha} \hat{\lambda}_{\alpha}^l y_{kl} P_{\hat{\lambda}_k^{\dot{\alpha}}}) + g(\partial_{\sigma} X) \\
 Q_{\frac{3}{2}} & = i \left(\frac{\sqrt{2}}{2} \right) z^{\frac{3}{2}} \lambda^{+\gamma m} y_{mi} (\sigma^a)_{\gamma\dot{\alpha}} \bar{\psi}^{i\dot{\alpha}} P_{x^a} - i \left(\frac{\sqrt{2}}{2} \right) z^{\frac{3}{2}} \bar{\lambda}_m^{+\dot{\alpha}} y^{mi} \psi_i^{\beta} (\sigma^a)_{\beta\dot{\alpha}} P_{x^a} + \dots
 \end{aligned}$$

and ... are terms which are at least quadratic in ψ . We have suppressed the $\int d\sigma$ in the expression above and $f(\partial_{\sigma} X)$ and $g(\partial_{\sigma} X)$ denote additional terms that contain sigma derivatives of the fields. We will work to lowest order in α' so possible normal-ordering contributions to Q will be ignored.

Near the AdS_5 boundary, physical states V can be expanded as $V = \sum_{d \geq d_0} V_d$ where V_d is proportional to z^d and V_{d_0} is the leading behavior near $z = 0$. Defining the degree to be the power of z in the expansion, V has a minimum degree d_0 , so one can use standard methods to compute the cohomology of Q . One first computes the cohomology of $Q_{-\frac{1}{2}}$, then computes the cohomology of $Q_{\frac{1}{2}}$ restricted to states in the cohomology of $Q_{-\frac{1}{2}}$, then computes the cohomology of $Q_{\frac{3}{2}}$ restricted to states in the cohomology of $Q_{-\frac{1}{2}} + Q_{\frac{1}{2}}$, etc. This procedure is well defined since the complete BRST operator given in (2.17) is nilpotent, and after performing the z expansion, this implies $\{Q_{\frac{1}{2}}, Q_{\frac{1}{2}}\} + 2\{Q_{-\frac{1}{2}}, Q_{\frac{3}{2}}\} = 0$. So $Q_{\frac{1}{2}}$ is a nilpotent operator when acting on states in the cohomology of $Q_{-\frac{1}{2}}$. The same argument of nilpotency applies to $Q_{\frac{3}{2}}, Q_{\frac{5}{2}}, \dots$

In what follows we will focus on the zero mode BRST cohomology which is relevant for the supergravity states. Because of the usual quartet argument, the zero mode cohomology of

$$Q_{-\frac{1}{2}} = \left(\frac{\sqrt{2}}{2} \right) z^{-\frac{1}{2}} \lambda^{+\gamma m} y_{mi} P_{\psi_i^{\gamma}} - \left(\frac{\sqrt{2}}{2} \right) z^{-\frac{1}{2}} \bar{\lambda}_j^{+\dot{\alpha}} y^{ji} P_{\bar{\psi}^{i\dot{\alpha}}} \quad (3.3)$$

will be assumed¹ to be given by states which are independent of λ^+ and which only depend on ψ in the combination $\lambda^{-\gamma M} \hat{\psi}$ where $\hat{\psi} \equiv y_J (\gamma^J \psi)$. This combination $\lambda^{-\gamma M} \hat{\psi}$ has been written in ten-dimensional notation where $\lambda^{-\bar{\alpha}}$ and $\hat{\psi}^{\bar{\alpha}}$ are $d = 10$ Weyl spinors, $\bar{\alpha} = 1$ to 16, and $M = 0$ to 9. Note that the pure spinor condition (2.3) implies that

$$\lambda^{-\gamma M} \lambda^+ = 0 \quad \text{and} \quad \lambda^+ \gamma^M \lambda^+ + \lambda^{-\gamma M} \lambda^- = 0, \quad (3.4)$$

so that $Q_{-\frac{1}{2}}(\lambda^{-\gamma M} \hat{\psi}) = 0$.

Since states in the cohomology of $Q_{-\frac{1}{2}}$ are independent of λ^+ , the condition $\lambda^+ \gamma^M \lambda^+ + \lambda^{-\gamma M} \lambda^- = 0$ implies that $\lambda^{-\gamma M} \lambda^- = 0$, i.e. λ^- is a pure spinor with 11 independent components. This implies that $\lambda^{-\gamma M} \hat{\psi}$ has only 5 independent components.

¹As shown by A. Mikhailov and R. Xu in [11], this treatment using the quartet argument is too naive and there is one state at ghost-number two in the $Q_{-\frac{1}{2}}$ cohomology that depends on λ^+ which is $(\lambda^+ \gamma^M \hat{\psi})(\lambda^{-\gamma M} \hat{\psi})$ [12]. However, if one allows dependence on the non-minimal pure spinor variables $\tilde{\lambda}$ and r described in the following section, this state in the cohomology of $Q_{-\frac{1}{2}}$ can be represented by $(\lambda^- \tilde{\lambda})^{-2} (\lambda^{-\gamma M} \hat{\psi})(\lambda^{-\gamma N} \hat{\psi})(\lambda^{-\gamma P} \hat{\psi})(r \gamma_{MNP} \tilde{\lambda})$ which is independent of λ^+ .

Therefore, states in the cohomology of $Q_{-\frac{1}{2}}$ depend on the 21 bosonic coordinates $[x, z, y, \lambda^-]$ and 21 fermionic coordinates $[\theta, \lambda^- \gamma^M \hat{\psi}]$.

The next step is to compute the cohomology of $Q_{\frac{1}{2}} + Q_{\frac{3}{2}} + \dots$ when restricted to states in the cohomology of $Q_{-\frac{1}{2}}$. Since states in the cohomology of $Q_{-\frac{1}{2}}$ are independent of λ^+ and only depend on ψ in the combination $\lambda^- \gamma^M \hat{\psi}$, any terms proportional to λ^+ in $Q_{\frac{3}{2}} + \dots$ act as zero when restricted to these states. And any terms proportional to λ^- in $Q_{\frac{3}{2}} + \dots$ are at least cubic in ψ and must act as zero when restricted to these states. This is because any operator which is linear in λ^- and cubic in ψ cannot be expressed in terms of the five $\lambda^- \gamma^M \hat{\psi}$ variables. Thus all the operators in $Q_{\frac{3}{2}} + \dots$ act as zero when restricted to these states. So computing the cohomology of Q reduces to computing the cohomology of $Q_{\frac{1}{2}}$ restricted to states depending on the variables $[x, \theta, z, y, \lambda^-, \lambda^- \gamma^M \hat{\psi}]$.

Since $Q_{\frac{1}{2}}$ has a fixed degree, one only needs to consider vertex operators of a fixed degree to compute its cohomology. If V is the original vertex operator in the cohomology of Q , this will be the term V_{d_0} of lowest degree in V after restricting to states in the cohomology of $Q_{-\frac{1}{2}}$. For this reason, the vertex operator inside the region of validity of the z expansion is determined up to a BRST-trivial quantity by its boundary value V_{d_0} . Holography predicts that V_{d_0} should be dual to a gauge-invariant super-Yang-Mills operator, and the precise relation will be discussed in the next section for the case of supergravity vertex operators which are dual to half-BPS super-Yang-Mills operators.

4 Half-BPS states

4.1 BRST cohomology

In this section, the zero mode BRST cohomology at +2 ghost number will be related to the dual of half-BPS gauge-invariant $d = 4$ $\mathcal{N} = 4$ super-Yang-Mills operators. Zero mode cohomology at +2 ghost number corresponds to supergravity states which, like the dual half-BPS super-Yang-Mills operators, will be expressed using $\mathcal{N} = 4$ d=4 harmonic superspace.

As argued in the previous section, the BRST cohomology is described by states in the cohomology of $Q_{\frac{1}{2}}$ depending on $[x, z, y, \theta, \lambda^-, \lambda^- \gamma^M \hat{\psi}]$ where $\lambda^- \gamma^M \lambda^- = 0$. In what follows we are going to suppress the minus superscript in λ^- . The zero mode contribution to $Q_{\frac{1}{2}}$ written in ten dimensional notation is:

$$Q_{\frac{1}{2}} = z^{\frac{1}{2}} \left[\lambda^{\bar{\alpha}} D_{\bar{\alpha}} + 4(\lambda \gamma^{jk} \hat{\psi}) \frac{\partial}{\partial y^{jk}} + y_{ij}(\lambda \gamma^{ij} \hat{\psi}) \left(2z \frac{\partial}{\partial z} + y^{mt} \frac{\partial}{\partial y^{mt}} - \lambda^{\bar{\alpha}} \frac{\partial}{\partial \lambda^{\bar{\alpha}}} \right) \right] + \tilde{w}^{\bar{\alpha}} r_{\bar{\alpha}} \quad (4.1)$$

where $D_{\bar{\alpha}} = -\frac{\partial}{\partial \theta^{\bar{\alpha}}} - (\theta \gamma^m)_{\bar{\alpha}} \frac{\partial}{\partial x^m}$ is the d=4 dimensional reduction of the d=10 supersymmetric derivative.

In (4.1) we have included the usual non-minimal pure spinor term $\tilde{w}^{\bar{\alpha}} r_{\bar{\alpha}}$ [13] that is not present in (3.2). The inclusion of this additional term is necessary² because, as will

²Although some expressions for vertex operators in the next subsection will depend on $\tilde{\lambda}_{\bar{\alpha}}$ and $r_{\bar{\alpha}}$, it should be noted that there always exists a gauge in which the vertex operator depends only on minimal variables. This is clear from the expression of (1.1). However, to express the vertex operator in terms of harmonic superspace variables, dependence on non-minimal variables appears to be necessary when the supergravity state is dual to a half-BPS state involving four or more super-Yang-Mills fields.

be seen below, some of the results can be expressed as a function of $(\lambda\gamma^M\hat{\psi})$ only after introducing the non-minimal bosonic pure spinor variables $\tilde{\lambda}_{\bar{\alpha}}$ satisfying

$$\tilde{\lambda}_{\bar{\alpha}}(\gamma^M)^{\bar{\alpha}\bar{\beta}}\tilde{\lambda}_{\bar{\beta}} = 0. \quad (4.2)$$

The $\tilde{w}^{\bar{\alpha}}$ are the conjugate momenta of $\tilde{\lambda}_{\bar{\alpha}}$ which act on functions of $\tilde{\lambda}_{\bar{\alpha}}$ as $\frac{\partial}{\partial\tilde{\lambda}_{\bar{\alpha}}}$, and $r_{\bar{\alpha}}$ is a fermionic spinor which satisfies

$$\tilde{\lambda}_{\bar{\alpha}}(\gamma^M)^{\bar{\alpha}\bar{\beta}}r_{\bar{\beta}} = 0. \quad (4.3)$$

To rewrite $Q_{\frac{1}{2}}$ given in (3.2) in the concise form of (4.1), we have performed a few manipulations. Firstly, we have redefined λ in order to adsorb the overall factor of $\frac{\sqrt{2}}{2}$. Secondly, the term $z^{\frac{1}{2}}4(\lambda\gamma^{jk}\hat{\psi})\frac{\partial}{\partial y^{jk}}$ in (4.1) is understood to not act on $(\lambda\gamma^M\hat{\psi})$ even though $(\lambda\gamma^M\hat{\psi})$ depends on y_{ij} . This is the case because we have not explicitly included in (4.1) the terms

$$z^{\frac{1}{2}}[-2\lambda^{\alpha i}\psi_{\alpha j}\bar{\psi}_{\dot{\alpha}}^j P_{\bar{\psi}_{\dot{\alpha}}^i} - 2\bar{\lambda}_i^{\dot{\alpha}}\psi_{\alpha k}\bar{\psi}_{\dot{\alpha}}^k P_{\psi_{\alpha i}} + 4\lambda^{\alpha i}\psi_{\alpha j}\psi_i^{\beta} P_{\psi_j^{\beta}} - 4\bar{\lambda}_i^{\dot{\alpha}}\bar{\psi}_{\dot{\alpha}}^m\bar{\psi}^{i\dot{\beta}} P_{\bar{\psi}^{m\dot{\beta}}}]$$

from (3.2), and one can show using pure spinor conditions for λ that

$$\begin{aligned} &\left(4(\lambda\gamma^{jk}\hat{\psi})\frac{\partial}{\partial y^{jk}} - 2\lambda^{\alpha i}\psi_{\alpha j}\bar{\psi}_{\dot{\alpha}}^j P_{\bar{\psi}_{\dot{\alpha}}^i} - 2\bar{\lambda}_i^{\dot{\alpha}}\psi_{\alpha k}\bar{\psi}_{\dot{\alpha}}^k P_{\psi_{\alpha i}} \right. \\ &\quad \left. + 4\lambda^{\alpha i}\psi_{\alpha j}\psi_i^{\beta} P_{\psi_j^{\beta}} - 4\bar{\lambda}_i^{\dot{\alpha}}\bar{\psi}_{\dot{\alpha}}^m\bar{\psi}^{i\dot{\beta}} P_{\bar{\psi}^{m\dot{\beta}}}\right)(\lambda\gamma^M\hat{\psi}) = 0. \end{aligned} \quad (4.4)$$

Finally, to extract the term $-z^{\frac{1}{2}}y_{ij}(\lambda\gamma^{ij}\hat{\psi})\lambda^{\bar{\alpha}}\frac{\partial}{\partial\lambda^{\bar{\alpha}}}$ in (4.1) from the complicated dependence of $Q_{\frac{1}{2}}$ in (3.2) in $\frac{\partial}{\partial\lambda}$, we have used that terms with λ^+ act as zero on the states in the cohomology of $Q_{-\frac{1}{2}}$ and we have omitted terms that are zero by the pure spinor condition such as

$$\lambda^{-\alpha i}(4\psi_{\alpha k}\bar{\lambda}_i^{\dot{\alpha}} P_{\bar{\lambda}_k^{\dot{\alpha}}} + 4\psi_{\alpha k}\bar{\lambda}_i^{\dot{\alpha}} P_{\hat{\lambda}_k^{\dot{\alpha}}})f(\lambda^{-\bar{\alpha}}) = 0.$$

4.2 Supergravity vertex operators

It will now be shown that supergravity vertex operators in the zero mode cohomology which are proportional to $y_{J_1}\dots y_{J_{N-1}}$ are related to half-BPS operators that are constructed from N super-Yang-Mills fields. We start the analysis of the cohomology of the operator $Q_{\frac{1}{2}}$ considering supergravity vertex operators which are independent of y_{jk} . These states must be annihilated by $2z\frac{\partial}{\partial z} - \lambda\frac{\partial}{\partial\lambda}$ and must be in the +2 ghost-number cohomology of $Q = \lambda^{\bar{\alpha}}D_{\bar{\alpha}}$. Since $\lambda^{\bar{\alpha}}D_{\bar{\alpha}}$ is the four-dimensional reduction of the ten-dimensional BRST operator $Q = \lambda^{\bar{\alpha}}D_{\bar{\alpha}}$, these states are the antifields of super-Yang-Mills described by the bispinor superfield $A_{\bar{\alpha}\bar{\beta}}^*(x, \theta)$ [14]. In other words, the y -independent supergravity vertex operators are

$$V = z\lambda^{\bar{\alpha}}\lambda^{\bar{\beta}}A_{\bar{\alpha}\bar{\beta}}^*(x, \theta) \quad (4.5)$$

where $A_{\bar{\alpha}\bar{\beta}}^*(x, \theta)$ is the dimensional reduction of the ten-dimensional super-Yang-Mills anti-field. Note that the factor of z is required to cancel the BRST transformation of λ and implies that V carries zero dimension (since $A_{\bar{\alpha}\bar{\beta}}^*$ carries dimension +1 in units where x^m and

z carry dimension -1). At zero momentum, the vertex operator of (4.5) can be gauged to

$$\begin{aligned} z\lambda^{\bar{\alpha}}\lambda^{\bar{\beta}}A_{\bar{\alpha}\bar{\beta}}^* &= z[(\lambda\gamma^M\theta)(\lambda\gamma^N\theta)(\theta\gamma_{MN})^{\bar{\alpha}}\psi_{\bar{\alpha}}^* + \\ &+ (\lambda\gamma^M\theta)(\lambda\gamma^N\theta)(\theta\gamma_{MNP}\theta)a^{*p} + (\lambda\gamma^M\theta)(\lambda\gamma^N\theta)(\theta\gamma_{MN}^{[jk]}\theta)\phi_{jk}^*] \end{aligned} \quad (4.6)$$

where a_p^* , ϕ_{jk}^* and ψ_{α}^* are the antifields to the gluon a_m , scalars ϕ^{jk} , and gluino ψ^{α} . So these y -independent operators are the duals to super-Yang-Mills “singleton” operators, i.e. the duals to abelian super-Yang-Mills fields.

We next consider supergravity vertex operators in the cohomology which are linear in y . The simplest example is the operator $V = i\lambda^{\alpha j}\lambda_{\alpha}^k y_{jk}$ which is linear in y . Note that V is real since

$$\lambda^{\alpha i}\lambda_{\alpha}^j y_{ij} = -\bar{\lambda}_{\dot{i}}^{\dot{\alpha}}\bar{\lambda}_{\dot{\alpha}j} y^{ij}. \quad (4.7)$$

And V is annihilated by $Q_{\frac{1}{2}}$ since $\epsilon_{jklm}\lambda^{\alpha j}\lambda_{\alpha}^k\lambda^{\beta l} = 0$ and $\bar{\lambda}_{\dot{j}}^{\dot{\alpha}}\lambda^{\alpha j} = 0$ imply that $(\lambda\gamma^{ij}\hat{\psi})\frac{\partial}{\partial y^{ij}}(\lambda\lambda y) = 0$. Since V is a PSU(2,2|4) scalar, it corresponds to the zero-momentum dilaton that is dual to the super-Yang-Mills action.

To construct the general supergravity vertex operator in the BRST cohomology, recall that gauge-invariant half-BPS operators involving N super-Yang-Mills field strengths are elegantly described in harmonic superspace as [6–8]

$$W^{(N)}(u, x, \theta) = (uu)^{i_1 j_1} \dots (uu)^{i_N j_N} \text{Tr}[W_{i_1 j_1}(x, \theta) \dots W_{i_N j_N}(x, \theta)] \quad (4.8)$$

where $W_{jk}(x, \theta)$ is an $\mathcal{N} = 4$ d=4 superfield satisfying the constraints [15]

$$\begin{aligned} \nabla_{\alpha i} W_{jk} &= \nabla_{\alpha[i} W_{jk]}, \\ \bar{\nabla}_{\dot{\alpha}}^i W_{jk} &= -\frac{2}{3}\delta_{[j}^i \bar{\nabla}_{\dot{\alpha}}^l W_{k]l}. \end{aligned} \quad (4.9)$$

In components,

$$W_{jk} = \phi_{jk} + \theta_{\dot{\alpha}[j}\xi_{k]}^{\dot{\alpha}} + \theta^{\alpha l}\xi_{\alpha}^m\epsilon_{jklm} + \theta_{j\dot{\alpha}}\theta_{k\dot{\beta}}F^{\dot{\alpha}\dot{\beta}} + \epsilon_{jklm}\theta^{\alpha l}\theta^{\beta m}F_{\alpha\beta} + \dots \quad (4.10)$$

where ϕ_{jk} are the scalars, ξ_{α}^k and $\xi_{\dot{\alpha}}^{\dot{k}}$ are the chiral and antichiral gluinos, and $F_{\alpha\beta}$ and $F^{\dot{\alpha}\dot{\beta}}$ are the self-dual and anti-self-dual field strengths. The expression $(uu)^{jk}$ denotes $\epsilon^{JK}u_J^j u_K^k$ where $G = (u_J^j, \bar{u}_{J'}^j)$ are harmonic variables parameterizing the coset $\frac{\text{SU}(4)}{\text{S}(\text{U}(2)\times\text{U}(2))}$ with $j = 1$ to 4 and $J, J' = 1$ to 2. The inverse coset will be defined as $G^{-1} = (\bar{u}_j^J, u_j^{J'})$ where the variables u and $\bar{u}$ satisfy the constraints

$$\begin{aligned} u_J^j \bar{u}_j^K &= \delta_J^K, \quad \bar{u}_{J'}^j u_j^{K'} = \delta_{J'}^{K'}, \quad u_J^j u_j^{K'} = 0, \quad \bar{u}_{J'}^j \bar{u}_j^K = 0, \\ (uu)^{jk} &= \epsilon^{JK}u_J^j u_K^k = \frac{1}{2}\epsilon^{jklm}u_l^{J'}u_m^{K'}\epsilon_{J'K'}. \end{aligned} \quad (4.11)$$

Using the superspace constraints, one finds that $W^{(N)}(u, x, \theta)$ satisfies

$$u_J^j D_{\alpha j} W^{(N)} = u_J^{J'} \bar{D}_{\dot{\alpha}}^j W^{(N)} = 0, \quad (4.12)$$

i.e. $W^{(N)}$ is a G-analytic superfield. Furthermore, since $W^{(N)}$ is independent of $\bar{u}$, it satisfies

$$\left(u_J^j \frac{\partial}{\partial \bar{u}_{J'}^j}\right) W^{(N)} = 0, \quad (4.13)$$

i.e. $W^{(N)}$ is an H-analytic superfield. A superfield that is both G-analytic and H-analytic will be called an analytic superfield for short. So if U(1) charge is defined as $\frac{1}{2}(u \frac{\partial}{\partial u} - \bar{u} \frac{\partial}{\partial \bar{u}})$, half-BPS states constructed from N super-Yang-Mills field strengths are described by analytic superfields of $+N$ U(1) charge.

To construct the duals to these analytic superfields, consider the superspace integral [8]

$$\int d^4x \int du \int d^8(u\theta) W^{(N)}(u, x, \theta) T^{(4-N)}(u, \bar{u}, x, \theta) \quad (4.14)$$

where $\int du$ denotes an integral over the compact space $\frac{\text{SU}(4)}{\text{S}(\text{U}(2) \times \text{U}(2))}$ and using the definitions

$$D'^4 = D^{\alpha J'} D_{\alpha}^{K'} D_{J'}^{\beta} D_{\beta K'}, \quad \bar{D}'^4 = \bar{D}_{\dot{\alpha}}^J \bar{D}^{\dot{\alpha} K} \bar{D}_{\dot{\beta} J} \bar{D}_{\dot{K}}^{\dot{\beta}}, \quad (4.15)$$

where $D_{\alpha J'} = \bar{u}_{J'}^j D_{\alpha j}$ and $\bar{D}_{\dot{\alpha}}^J = \bar{u}_j^J \bar{D}_{\dot{\alpha}}^j$, one can write $\int d^8(u\theta) = D'^4 \bar{D}'^4$.

For the integral to be supersymmetric and non-vanishing, $T^{(4-N)}$ must be a G-analytic (but not necessarily H-analytic) superfield of U(1) charge $(4-N)$. Furthermore, the integral of (4.14) is invariant under the gauge transformation

$$\delta T = \left(u \frac{\partial}{\partial \bar{u}}\right)_J^{J'} \Lambda_{J'}^J \quad (4.16)$$

for any G-analytic superfield $\Lambda_{J'}^J$. So the dual to a half-BPS state constructed from N super-Yang-Mills fields is described by a G-analytic superfield T of U(1) charge $(4-N)$ which is defined up to the gauge transformation of (4.16).

In [8], these superfields $T^{(4-N)}(u, \bar{u}, x, \theta)$ were related to chiral $\mathcal{N} = 4$ d=4 superfields coming from the $AdS_5 \times S^5$ Type IIB chiral field strength at the AdS_5 boundary. However, in this paper, the superfields $T^{(4-N)}$ will instead be related to $AdS_5 \times S^5$ Type IIB gauge superfields $A_{\bar{\alpha}\bar{\beta}}$ which appear in the BRST-invariant supergravity vertex operators $V = \lambda^{\bar{\alpha}} \bar{\lambda}^{\bar{\beta}} A_{\bar{\alpha}\bar{\beta}}$ of (1.1). Near the AdS boundary, BRST-invariant supergravity vertex operators dual to half-BPS states constructed from N super-Yang-Mills fields will have the form $V = z^{2-N} \lambda^{\bar{\alpha}} \bar{\lambda}^{\bar{\beta}} A_{\bar{\alpha}\bar{\beta}}^{(N)}(y, x, \theta, \lambda \gamma^M \hat{\psi})$ up to non-minimal variables and the precise relation between V and T is

$$\begin{aligned} V = z^{2-N} \int du [& (yuu)^{N-1} \Omega^{(0)} T + 8(N-1)(yuu)^{N-2} \Omega^{(1)} T + \\ & + 8^2(N-1)(N-2)(yuu)^{N-3} \Omega^{(2)} T + 8^3(N-1)(N-2)(N-3)(yuu)^{N-4} \Omega^{(3)} T + \\ & + 8^4(N-1)(N-2)(N-3)(N-4)(yuu)^{N-5} \Omega^{(4)} T], \end{aligned} \quad (4.17)$$

where

$$\begin{aligned} \Omega^{(0)} = & \frac{1}{16} (\lambda \tilde{\lambda})^{-2} (\lambda \gamma^M \tilde{D}) (\lambda \gamma^N \tilde{D}) (\lambda \gamma^P \tilde{D}) (\lambda \gamma^S \tilde{D}) (\tilde{\lambda} \gamma_{MNPST} \tilde{\lambda}) v^T \\ & + \frac{1}{2} z^{-\frac{1}{2}} (\lambda \tilde{\lambda})^{-2} (\lambda \gamma^M \tilde{D}) (\lambda \gamma^N \tilde{D}) (\lambda \gamma^P \tilde{D}) (r \gamma_{PNM} \tilde{\lambda}), \end{aligned} \quad (4.18)$$

$$\begin{aligned}\Omega^{(1)} &= \frac{1}{4}(\lambda\tilde{\lambda})^{-2}(\lambda\gamma^M\hat{\psi})(\lambda\gamma^N\tilde{D})(\lambda\gamma^S\tilde{D})(\lambda\gamma^P\tilde{D})(\tilde{\lambda}\gamma_{MNSPT}\tilde{\lambda})v^T \\ &\quad + \frac{3}{2}z^{-\frac{1}{2}}(\lambda\tilde{\lambda})^{-2}(\lambda\gamma^M\hat{\psi})(\lambda\gamma^N\tilde{D})(\lambda\gamma^T\tilde{D})(r\gamma_{TNM}\tilde{\lambda}),\end{aligned}\quad (4.19)$$

$$\begin{aligned}\Omega^{(2)} &= \frac{3}{8}(\lambda\tilde{\lambda})^{-2}(\lambda\gamma^M\hat{\psi})(\lambda\gamma^N\hat{\psi})(\lambda\gamma^S\tilde{D})(\lambda\gamma^P\tilde{D})(\tilde{\lambda}\gamma_{MNSPT}\tilde{\lambda})v^T \\ &\quad + \frac{3}{2}z^{-\frac{1}{2}}(\lambda\tilde{\lambda})^{-2}(\lambda\gamma^M\hat{\psi})(\lambda\gamma^N\hat{\psi})(\lambda\gamma^S\tilde{D})(r\gamma_{SNM}\tilde{\lambda}),\end{aligned}\quad (4.20)$$

$$\begin{aligned}\Omega^{(3)} &= \frac{1}{4}(\lambda\tilde{\lambda})^{-2}(\lambda\gamma^M\hat{\psi})(\lambda\gamma^N\hat{\psi})(\lambda\gamma^S\hat{\psi})(\lambda\gamma^P\tilde{D})(\tilde{\lambda}\gamma_{MNSPT}\tilde{\lambda})v^T \\ &\quad + \frac{1}{2}z^{-\frac{1}{2}}(\lambda\tilde{\lambda})^{-2}(\lambda\gamma^M\hat{\psi})(\lambda\gamma^N\hat{\psi})(\lambda\gamma^S\hat{\psi})(r\gamma_{SNM}\tilde{\lambda}),\end{aligned}\quad (4.21)$$

$$\Omega^{(4)} = \frac{1}{16}(\lambda\tilde{\lambda})^{-2}(\lambda\gamma^M\hat{\psi})(\lambda\gamma^N\hat{\psi})(\lambda\gamma^S\hat{\psi})(\lambda\gamma^P\hat{\psi})(\tilde{\lambda}\gamma_{MNSPT}\tilde{\lambda})v^T. \quad (4.22)$$

In the above formulas, v_M and $\bar{v}_M$ are null vectors with nonzero components defined by $v_J = -\frac{1}{4}\sigma_J^{jk}(uu)_{jk}$ and $\bar{v}_J = -\frac{1}{4}\sigma_J^{jk}(\bar{u}\bar{u})_{jk}$ where σ_J^k are SO(6) Pauli matrices, and $\tilde{D} \equiv \bar{v}_M(\gamma^M D)$.

When $N < 4$, the terms in (4.17) do not depend on $\Omega^{(3)}$ and $\Omega^{(4)}$ and one can choose a gauge such that V is independent of the non-minimal variables. In this gauge, V is equal to (4.17) but with $\Omega^{(0)}$, $\Omega^{(1)}$ and $\Omega^{(2)}$ replaced with

$$\begin{aligned}\Omega_{\min}^{(0)} &= -\frac{1}{4}(\lambda\gamma^M\tilde{D})(\lambda\gamma^N\tilde{D})(\tilde{D}\gamma_{MNP}\tilde{D})v^P, \\ \Omega_{\min}^{(1)} &= -(\lambda\gamma^M\hat{\psi})(\lambda\gamma^N\tilde{D})(\tilde{D}\gamma_{MNP}\tilde{D})v^P \\ &\quad + 24(\lambda\gamma^M\hat{\psi})\bar{v}_M(\lambda\gamma^m\tilde{D})\frac{\partial}{\partial x^m}, \\ \Omega_{\min}^{(2)} &= -\frac{3}{2}(\lambda\gamma^M\hat{\psi})(\lambda\gamma^N\hat{\psi})(\tilde{D}\gamma_{MNP}\tilde{D})v^P \\ &\quad + 48(\lambda\gamma^M\hat{\psi})\bar{v}_M(\lambda\gamma^m\hat{\psi})\frac{\partial}{\partial x^m}.\end{aligned}\quad (4.23)$$

However, for $N \geq 4$, such a gauge is not possible since $\Omega^{(3)}$ and $\Omega^{(4)}$ require non-minimal variables.

Before showing that (4.17) is in the BRST cohomology, it will be interesting to discuss some simple examples of T and the associated vertex operator. When $N = 1$, V becomes independent of y and $\hat{\psi}$ and describes the antifield of (4.5). When $N = 2$, the dilaton vertex operator $V = i\lambda^{\alpha j}\lambda_{\alpha}^k y_{jk}$ of (4.7) is obtained from $T = \prod_{J'=1,2}\prod_{\alpha=1,2}(u_j^{J'}\theta_{\alpha}^j)$, and is dual to the super-Yang-Mills action $\int d^4x \int du \bar{D}^4 \text{Tr}(W^2)$ where D'^4 and $\bar{D}^4$ are defined in (4.15). And when $N = 4$, the vertex operator $V = z^{-2-\frac{1}{2}}(\lambda\tilde{\lambda})^{-2}(\lambda\gamma^M\hat{\psi})(\lambda\gamma^N\hat{\psi})(\lambda\gamma^S\hat{\psi})(r\gamma_{SNM}\tilde{\lambda})$ of footnote 1 is obtained by choosing T to be constant, and is dual to $\int d^4x \int du D'^4 \bar{D}^4 \text{Tr}(W^4)$ which is the supersymmetrization of the $\int d^4x \text{Tr}(F^4)$ term.

The procedure to show that V of (4.17) is in the BRST cohomology is as follows: to zeroth order in $(\lambda\gamma^M\hat{\psi})$ the condition for V to be annihilated by $Q_{\frac{1}{2}}$ is

$$(z^{\frac{1}{2}}\lambda^{\bar{\alpha}}D_{\bar{\alpha}} + \tilde{w}^{\bar{\alpha}}r_{\bar{\alpha}})\Omega^{(0)}T = 0. \quad (4.24)$$

To see that $\Omega^{(0)}T$ given in (4.18) satisfies this equation, first note that $\lambda^{\bar{\alpha}}D_{\bar{\alpha}}$ can be decomposed as

$$\begin{aligned}\lambda^{\bar{\alpha}}D_{\bar{\alpha}} &= \bar{v}_M v_N (\lambda \gamma^M \gamma^N D) + v_M \bar{v}_N (\lambda \gamma^M \gamma^N D) \\ &\equiv \lambda D_1 + \lambda D_2\end{aligned}\quad (4.25)$$

and we can rewrite (4.24) as

$$(z^{\frac{1}{2}}\lambda D_2 + \tilde{w}^{\bar{\alpha}}r_{\bar{\alpha}})\Omega^{(0)}T + z^{\frac{1}{2}}[\lambda D_1, \Omega^{(0)}]T = 0 \quad (4.26)$$

where $[\cdot, \cdot]$ means commutator and $(\lambda D_1)T = 0$ since T is G-analytic. It is easy to see that $[\lambda D_1, \Omega^{(0)}] = 0$ since

$$\{(\lambda D_1), (\lambda \gamma^N \tilde{D})\} = -2(\lambda \gamma^N \gamma^m \gamma^S \lambda) \bar{v}_S \frac{\partial}{\partial x^m} = 0, \quad (4.27)$$

where $\{\cdot, \cdot\}$ is an anticommutator and we have used that λ is a pure spinor.

To show that $(z^{\frac{1}{2}}\lambda D_2 + \tilde{w}^{\bar{\alpha}}r_{\bar{\alpha}})\Omega^{(0)}T = 0$ we first note that using Fierz identities it is possible to rewrite the first term in the right hand side of (4.18) as

$$\begin{aligned}\Omega_{ft}^{(0)} &= -\frac{1}{4}(\lambda \gamma^M \tilde{D})(\lambda \gamma^N \tilde{D})(\tilde{D} \gamma_{MNP} \tilde{D})v^P \\ &\quad - (\lambda \tilde{\lambda})^{-1}(\lambda D_2)(\lambda \gamma^S \tilde{D})(\lambda \gamma^P \tilde{D})(\tilde{\lambda} \gamma_{PS} \tilde{D})\end{aligned}\quad (4.28)$$

where ft means first term. It is easy to see that the second term on the right hand side of the equation above is annihilated by (λD_2) . To see that the first term is also annihilated, it is convenient to choose a Lorentz frame where the only non-vanishing component of λ is λ^{++} which carries $\frac{5}{2}$ charge with respect to a U(1) subgroup of SO(10). In $SU(5) \times U(1)$ notation, an SO(10) chiral spinor $S^{\bar{\alpha}}$ splits as (S^{++}, S_{ab}, S^a) where $a = 1$ to 5 carrying U(1) charge $(\frac{5}{2}, \frac{1}{2}, -\frac{3}{2})$ and an SO(10) vector v^M splits as (v^a, v_a) carrying U(1) charge $(+1, -1)$.

In this Lorentz frame, the first term of (4.28) is

$$C_1(\lambda^{++})^2(\epsilon_{abcde}\tilde{D}^a\tilde{D}^b\tilde{D}^c\tilde{D}^dv^e) + C_2(\lambda^{++})^2(\tilde{D}_{ab}\tilde{D}^a\tilde{D}^b\tilde{D}^cv_c), \quad (4.29)$$

where C_1 and C_2 are constants. Using that $\lambda D_2 = \lambda^{++}(\tilde{D}^a v_a)$ and $v_a v^a = 0$, one verifies that λD_2 annihilates (4.29).

Also one can prove that

$$\tilde{w}^{\bar{\alpha}}r_{\bar{\alpha}}\left((\lambda \tilde{\lambda})^{-2}(\lambda \gamma^M X)(\lambda \gamma^N Y)(\lambda \gamma^P Z)(r \gamma_{PNM} \tilde{\lambda})\right) = 0$$

for any fermionic X, Y and Z which implies

$$\tilde{w}^{\bar{\alpha}}r_{\bar{\alpha}}(\Omega^{(0)}) = -\frac{1}{2}(\lambda \tilde{\lambda})^{-2}(\lambda D_2)(\lambda \gamma^M \tilde{D})(\lambda \gamma^N \tilde{D})(\lambda \gamma^P \tilde{D})(r \gamma_{PNM} \tilde{\lambda})$$

and this variation cancels precisely with the action of $z^{\frac{1}{2}}(\lambda D_2)$ in the second term of the right hand side of $\Omega^{(0)}$ given in (4.18). This completes the proof that $(z^{\frac{1}{2}}\lambda D_2 + \tilde{w}^{\bar{\alpha}}r_{\bar{\alpha}})\Omega^{(0)}T = 0$.

To see that V is BRST closed in higher orders in $(\lambda\gamma^M\hat{\psi})$, one also has to show that

$$\begin{aligned}
 (N-1)[z^{\frac{1}{2}}(\lambda\gamma^M\hat{\psi})v_M\Omega^{(0)}T + (z^{\frac{1}{2}}\lambda^{\bar{\alpha}}D_{\bar{\alpha}} + \tilde{w}^{\bar{\alpha}}r_{\bar{\alpha}})\Omega^{(1)}T] &= 0, \quad (4.30) \\
 (N-1)(N-2)[z^{\frac{1}{2}}(\lambda\gamma^M\hat{\psi})v_M\Omega^{(1)}T + (z^{\frac{1}{2}}\lambda^{\bar{\alpha}}D_{\bar{\alpha}} + \tilde{w}^{\bar{\alpha}}r_{\bar{\alpha}})\Omega^{(2)}T] &= 0, \\
 (N-1)(N-2)(N-3)[z^{\frac{1}{2}}(\lambda\gamma^M\hat{\psi})v_M\Omega^{(2)}T + (z^{\frac{1}{2}}\lambda^{\bar{\alpha}}D_{\bar{\alpha}} + \tilde{w}^{\bar{\alpha}}r_{\bar{\alpha}})\Omega^{(3)}T] &= 0, \\
 (N-1)(N-2)(N-3)(N-4)[z^{\frac{1}{2}}(\lambda\gamma^M\hat{\psi})v_M\Omega^{(3)}T + (z^{\frac{1}{2}}\lambda^{\bar{\alpha}}D_{\bar{\alpha}} + \tilde{w}^{\bar{\alpha}}r_{\bar{\alpha}})\Omega^{(4)}T] &= 0, \\
 (N-1)(N-2)(N-3)(N-4)(N-5)(\lambda\gamma^M\hat{\psi})v_M\Omega^{(4)}T &= 0 \quad (4.31)
 \end{aligned}$$

where the factors of $(\lambda\gamma^M\hat{\psi})v_M$ above come from the BRST variation of y_{jk}/z .

When $N < 4$, only the first two equations of (4.30) need to be satisfied, and one can show that they are satisfied by $(\Omega_{\min}^{(0)}, \Omega_{\min}^{(1)}, \Omega_{\min}^{(2)})$ of (4.23). However, in the rest of the paper, we will not put any restrictions on N and will solve all five equations of (4.30) and (4.31).

Consider the first equation in (4.30). To see that $\Omega^{(0)}$ given in (4.18) and $\Omega^{(1)}$ given in (4.19) satisfy this equation we first note that (4.27) implies $[(\lambda D_1), \Omega^{(1)}] = 0$. We follow the same steps of the discussion above and rewrite the first term in the right hand side of (4.19) using Fierz identities as

$$\begin{aligned}
 \Omega_{ft}^{(1)} &= -(\lambda\gamma^M\hat{\psi})(\lambda\gamma^S\tilde{D})(\tilde{D}\gamma_{MSP}\tilde{D})v^P \\
 &\quad - (\lambda\tilde{\lambda})^{-1}(\lambda\gamma_M\hat{\psi})v^M(\lambda\gamma^S\tilde{D})(\lambda\gamma^P\tilde{D})(\tilde{\lambda}\gamma_{PS}\tilde{D}) \\
 &\quad - 3(\lambda\tilde{\lambda})^{-1}(\lambda D_2)(\lambda\gamma^M\hat{\psi})(\lambda\gamma^S\tilde{D})(\tilde{\lambda}\gamma_{SM}\tilde{D}).
 \end{aligned} \quad (4.32)$$

To show that $(\lambda\gamma^M\hat{\psi})v_M(\Omega_{ft}^{(0)}) + (\lambda D_2)(\Omega_{ft}^{(1)}) = 0$, note that many terms cancel and one only needs to show that

$$\begin{aligned}
 -\frac{1}{4}(\lambda\gamma^M\hat{\psi})v_M(\lambda\gamma^M\tilde{D})(\lambda\gamma^N\tilde{D})(\tilde{D}\gamma_{MNP}\tilde{D})v^P + \\
 -(\lambda D_2)(\lambda\gamma^M\hat{\psi})(\lambda\gamma^S\tilde{D})(\tilde{D}\gamma_{MSP}\tilde{D})v^P = 0.
 \end{aligned} \quad (4.33)$$

By choosing as before a Lorentz frame in which the only non-vanishing component of $\lambda^{\bar{\alpha}}$ is λ^{++} , it is not difficult to see that the two terms in (4.33) are proportional. And we have developed a Mathematica program to fix the coefficients so that the two terms in (4.33) cancel.

We also note that

$$\begin{aligned}
 \tilde{w}^{\bar{\alpha}}r_{\bar{\alpha}}(\Omega^{(1)}) &= -\frac{1}{2}(\lambda\tilde{\lambda})^{-2}(\lambda\gamma_M\hat{\psi})v^M(\lambda\gamma^N\tilde{D})(\lambda\gamma^P\tilde{D})(\lambda\gamma^S\tilde{D})(r\gamma_{SPN}\tilde{\lambda}) \\
 &\quad -\frac{3}{2}(\lambda\tilde{\lambda})^{-2}(\lambda D_2)(\lambda\gamma^M\hat{\psi})(\lambda\gamma^N\tilde{D})(\lambda\gamma^P\tilde{D})(r\gamma_{PNM}\tilde{\lambda})
 \end{aligned} \quad (4.34)$$

and this variation cancels with the action of $z^{\frac{1}{2}}(\lambda D_2)$ on the r dependent term of $\Omega^{(1)}$ and with the action of $z^{\frac{1}{2}}(\lambda\gamma^M\hat{\psi})v_M$ on the r dependent term of $\Omega^{(0)}$. Similar arguments can be given to show that all the equations in (4.30) and in (4.31) are satisfied. One last comment is that although it may seem surprising that $\Omega^{(4)}$ does not depend on r , this follows because

$$\tilde{w}^{\bar{\alpha}}r_{\bar{\alpha}}(\Omega^{(4)}) = -\frac{1}{2}(\lambda\tilde{\lambda})^{-2}(\lambda\gamma_S\hat{\psi})v^S(\lambda\gamma^M\hat{\psi})(\lambda\gamma^N\hat{\psi})(\lambda\gamma^P\hat{\psi})(r\gamma_{PNM}\tilde{\lambda}) \quad (4.35)$$

and this variation is precisely cancelled with the action of $z^{\frac{1}{2}}v_M(\lambda\gamma^M\hat{\psi})$ in the r dependent term of $\Omega^{(3)}$, so the last equation of (4.30) is satisfied.

4.3 Gauge invariance

For (4.17) to be consistent, V must change by a BRST-trivial quantity under the gauge transformation of T . In other words $\delta V = Q_{\frac{1}{2}} \Sigma$ for some Σ when $\delta T = (u \frac{\partial}{\partial \bar{u}})_J^{J'} \Lambda_{J'}^J$. Integrating (4.17) by parts with respect to $D_J^{J'} \equiv (u \frac{\partial}{\partial \bar{u}})_J^{J'}$ one finds that

$$\begin{aligned} \Sigma = z^{2-\frac{1}{2}-N} \int du [(yuu)^{N-1} (A^{(0)})_J^{J'} \Lambda_{J'}^J + 8(N-1)(yuu)^{N-2} (A^{(1)})_J^{J'} \Lambda_{J'}^J \\ + 8^2(N-1)(N-2)(yuu)^{N-3} (A^{(2)})_J^{J'} \Lambda_{J'}^J] \end{aligned} \quad (4.36)$$

where

$$(A^{(0)})_J^{J'} = 3(\lambda\tilde{\lambda})^{-1} (\lambda\gamma^M \tilde{D})(\lambda\gamma^S \tilde{D})(\tilde{\lambda}\gamma_{SM} \tilde{D}_J^{J'}) \quad (4.37)$$

$$\begin{aligned} + 3(\lambda\tilde{\lambda})^{-1} \{ (\lambda\gamma^M \tilde{D}_J^{J'}), (\lambda\gamma^N \tilde{D}) \} (\tilde{\lambda}\gamma_{NM} \tilde{D}), \\ (A^{(1)})_J^{J'} = 6(\lambda\tilde{\lambda})^{-1} (\lambda\gamma^S \hat{\psi})(\lambda\gamma^T \tilde{D})(\tilde{\lambda}\gamma_{TS} \tilde{D}_J^{J'}) \end{aligned} \quad (4.38)$$

$$\begin{aligned} + 24(\lambda\tilde{\lambda})^{-1} (\lambda\gamma_N \hat{\psi}) \bar{v}_T (\lambda\gamma_J^{J'} \gamma^m \gamma^T \gamma^N \tilde{\lambda}) \frac{\partial}{\partial x^m}, \\ (A^{(2)})_J^{J'} = 3(\lambda\tilde{\lambda})^{-1} (\lambda\gamma^S \hat{\psi})(\lambda\gamma^T \hat{\psi})(\tilde{\lambda}\gamma_{TS} \tilde{D}_J^{J'}), \end{aligned} \quad (4.39)$$

$(\tilde{D}_J^{J'})_{\bar{\alpha}} = D_J^{J'}(\tilde{D}_{\bar{\alpha}})$, and $\gamma_J^{J'} = D_J^{J'}(\bar{v}_M \gamma^M)$. In order to construct Σ to satisfy $\delta V = Q_{\frac{1}{2}} \Sigma$ we have to solve the following equations

$$\begin{aligned} (\lambda^{\bar{\alpha}} D_{\bar{\alpha}} + z^{-\frac{1}{2}} \tilde{w}^{\bar{\alpha}} r_{\bar{\alpha}}) (A^{(0)})_J^{J'} \Lambda_{J'}^J &= (-D_J^{J'} \Omega^{(0)}) \Lambda_{J'}^J, \\ (\lambda\gamma^M \hat{\psi}) v_M (A^{(0)})_J^{J'} \Lambda_{J'}^J + (\lambda^{\bar{\alpha}} D_{\bar{\alpha}} + z^{-\frac{1}{2}} \tilde{w}^{\bar{\alpha}} r_{\bar{\alpha}}) (A^{(1)})_J^{J'} \Lambda_{J'}^J &= (-D_J^{J'} \Omega^{(1)}) \Lambda_{J'}^J, \\ (\lambda\gamma^M \hat{\psi}) v_M (A^{(1)})_J^{J'} \Lambda_{J'}^J + (\lambda^{\bar{\alpha}} D_{\bar{\alpha}} + z^{-\frac{1}{2}} \tilde{w}^{\bar{\alpha}} r_{\bar{\alpha}}) (A^{(2)})_J^{J'} \Lambda_{J'}^J &= (-D_J^{J'} \Omega^{(2)}) \Lambda_{J'}^J, \\ (\lambda\gamma^M \hat{\psi}) v_M (A^{(2)})_J^{J'} \Lambda_{J'}^J &= (-D_J^{J'} \Omega^{(3)}) \Lambda_{J'}^J. \end{aligned} \quad (4.40)$$

To see that $(A^{(0)})_J^{J'}$ given in (4.37) satisfies the first equation of (4.40), we first note that at zero momentum (i.e. setting all the anticommutators to zero) we have using Fierz identities

$$\begin{aligned} (-D_J^{J'} \Omega^{(0)}) \Lambda_{J'}^J &= 3(\lambda\tilde{\lambda})^{-1} (\lambda D_2) (\lambda\gamma^M \tilde{D})(\lambda\gamma^S \tilde{D})(\tilde{\lambda}\gamma_{SM} \tilde{D}_J^{J'}) \Lambda_{J'}^J \\ &\quad - 3z^{-\frac{1}{2}} (\lambda\tilde{\lambda})^{-2} (\lambda r) (\lambda\gamma^N \tilde{D})(\lambda\gamma^P \tilde{D})(\tilde{\lambda}\gamma_{PN} \tilde{D}_J^{J'}) \Lambda_{J'}^J \\ &\quad + 3z^{-\frac{1}{2}} (\lambda\tilde{\lambda})^{-1} (\lambda\gamma^M \tilde{D})(\lambda\gamma^P \tilde{D})(r\gamma_{PM} \tilde{D}_J^{J'}) \Lambda_{J'}^J, \end{aligned} \quad (4.41)$$

where we have used that $\Lambda_{J'}^J$ is G-analytic.

Using (4.41) we see that

$$\begin{aligned} (\lambda^{\bar{\alpha}} D_{\bar{\alpha}} + z^{-\frac{1}{2}} \tilde{w}^{\bar{\alpha}} r_{\bar{\alpha}}) (A^{(0)})_J^{J'} \Lambda_{J'}^J \\ = (\lambda D_2 + z^{-\frac{1}{2}} \tilde{w}^{\bar{\alpha}} r_{\bar{\alpha}}) 3(\lambda\tilde{\lambda})^{-1} (\lambda\gamma^M \tilde{D})(\lambda\gamma^S \tilde{D})(\tilde{\lambda}\gamma_{SM} \tilde{D}_J^{J'}) \Lambda_{J'}^J \\ = (-D_J^{J'} \Omega^{(0)}) \Lambda_{J'}^J, \end{aligned} \quad (4.42)$$

so at zero momentum the equation is satisfied. The next step is to consider the case where the commutators are not zero and we have

$$(-D_J^{J'} \Omega^{(0)})_{ac} \Lambda_{J'}^J = 3(\lambda\tilde{\lambda})^{-1} (\lambda D_2) \{ (\lambda\gamma^M \tilde{D}_J^{J'}), (\lambda\gamma^N \tilde{D}) \} (\tilde{\lambda}\gamma_{NM} \tilde{D}) \Lambda_{J'}^J \quad (4.43)$$

$$\begin{aligned}
 & + 3 z^{-\frac{1}{2}} (\lambda \tilde{\lambda})^{-1} \{(\lambda \gamma^M \tilde{D}_{J'}^{J'}), (\lambda \gamma^N \tilde{D})\} (r \gamma_{NM} \tilde{D}) \Lambda_{J'}^J \\
 & - 3 z^{-\frac{1}{2}} (\lambda \tilde{\lambda})^{-2} (\lambda r) \{(\lambda \gamma^M \tilde{D}_{J'}^{J'}), (\lambda \gamma^N \tilde{D})\} (\tilde{\lambda} \gamma_{NM} \tilde{D}) \Lambda_{J'}^J \\
 & - \frac{3}{2} (\lambda \tilde{\lambda})^{-1} \{(\lambda \gamma^M \tilde{D}_{J'}^{J'}), (\lambda \gamma^N \tilde{D})\} v_N (\lambda \gamma^S \tilde{D}) (\tilde{\lambda} \gamma_{SM} \tilde{D}) \Lambda_{J'}^J \\
 & + \frac{3}{2} (\lambda \tilde{\lambda})^{-1} v_M \{(\lambda \gamma^M \tilde{D}_{J'}^{J'}), (\lambda \gamma^N \tilde{D})\} (\lambda \gamma^S \tilde{D}) (\tilde{\lambda} \gamma_{SN} \tilde{D}) \Lambda_{J'}^J \\
 & + \frac{3}{2} \{(\lambda \gamma^M \tilde{D}_{J'}^{J'}), (\lambda \gamma^N \tilde{D})\} (\tilde{D} \gamma_{MNT} \tilde{D}) v^T \Lambda_{J'}^J,
 \end{aligned}$$

where the subscript *ac* means the contribution from the anticommutators. We note that it is possible to rewrite the first three terms of the expression above as

$$\begin{aligned}
 & 3 (\lambda \tilde{\lambda})^{-1} (\lambda D_2) \{(\lambda \gamma^M \tilde{D}_{J'}^{J'}), (\lambda \gamma^N \tilde{D})\} (\tilde{\lambda} \gamma_{NM} \tilde{D}) \Lambda_{J'}^J \\
 & + 3 z^{-\frac{1}{2}} (\lambda \tilde{\lambda})^{-1} \{(\lambda \gamma^M \tilde{D}_{J'}^{J'}), (\lambda \gamma^N \tilde{D})\} (r \gamma_{NM} \tilde{D}) \Lambda_{J'}^J \\
 & - 3 z^{-\frac{1}{2}} (\lambda \tilde{\lambda})^{-2} (\lambda r) \{(\lambda \gamma^M \tilde{D}_{J'}^{J'}), (\lambda \gamma^N \tilde{D})\} (\tilde{\lambda} \gamma_{NM} \tilde{D}) \Lambda_{J'}^J \\
 & = (\lambda D_2 + z^{-\frac{1}{2}} \bar{w} \bar{\alpha} r_{\bar{\alpha}}) 3 (\lambda \tilde{\lambda})^{-1} \{(\lambda \gamma^M \tilde{D}_{J'}^{J'}), (\lambda \gamma^N \tilde{D})\} (\tilde{\lambda} \gamma_{NM} \tilde{D}) \Lambda_{J'}^J \\
 & = (\lambda \bar{\alpha} D_{\bar{\alpha}} + z^{-\frac{1}{2}} \bar{w} \bar{\alpha} r_{\bar{\alpha}}) 3 (\lambda \tilde{\lambda})^{-1} \{(\lambda \gamma^M \tilde{D}_{J'}^{J'}), (\lambda \gamma^N \tilde{D})\} (\tilde{\lambda} \gamma_{NM} \tilde{D}) \Lambda_{J'}^J,
 \end{aligned} \tag{4.44}$$

where we have used

$$\{(\lambda D_1), \{(\lambda \gamma^M \tilde{D}_{J'}^{J'}), (\lambda \gamma^N \tilde{D})\} (\tilde{\lambda} \gamma_{NM} \tilde{D})\} = 0$$

and $(\lambda D_1) \Lambda_{J'}^J = 0$. In order to rewrite the last three terms of (4.43) in a convenient form we first note

$$\{(\lambda \gamma^S \tilde{D}_{J'}^{J'}), (\lambda \gamma^P \tilde{D})\} = 2 \bar{v}_T (\lambda \gamma^S \gamma_{J'}^{J'} \gamma^m \gamma^T \gamma^P \lambda) \frac{\partial}{\partial x^m} \tag{4.45}$$

and this implies

$$\{(\lambda \gamma^M \tilde{D}_{J'}^{J'}), (\lambda D_2)\} = -\{(\lambda \gamma^N \tilde{D}_{J'}^{J'}), (\lambda \gamma^M \tilde{D})\} v_N. \tag{4.46}$$

Using the identity of (4.27) we have

$$\{(\lambda \gamma^N \tilde{D}_{J'}^{J'}), (\lambda D)\} = 0$$

and so

$$\{(\lambda \gamma^S \tilde{D}_{J'}^{J'}), (\lambda D_2)\} = -\{(\lambda \gamma^S \tilde{D}_{J'}^{J'}), (\lambda D_1)\}. \tag{4.47}$$

Also noting that

$$\begin{aligned}
 & 3 (\lambda \tilde{\lambda})^{-1} (\lambda \gamma^M \tilde{D}_{J'}^{J'}) (\lambda \gamma^S \tilde{D}) \{(\lambda D_1), (\tilde{\lambda} \gamma_{SM} \tilde{D})\} \\
 & + 6 \{(\lambda D_1), (\lambda \gamma^S \tilde{D})\} (\tilde{D} \gamma_S \tilde{D}_{J'}^{J'}) \\
 & = \frac{3}{2} \{(\lambda \gamma^M \tilde{D}_{J'}^{J'}), (\lambda \gamma^N \tilde{D})\} (\tilde{D} \gamma_{MNT} \tilde{D}) v^T,
 \end{aligned}$$

we see that

$$-\frac{3}{2} (\lambda \tilde{\lambda})^{-1} \{(\lambda \gamma^M \tilde{D}_{J'}^{J'}), (\lambda \gamma^N \tilde{D})\} v_N (\lambda \gamma^S \tilde{D}) (\tilde{\lambda} \gamma_{SM} \tilde{D}) \Lambda_{J'}^J \tag{4.48}$$

$$\begin{aligned}
& + \frac{3}{2} (\lambda \tilde{\lambda})^{-1} v_M \{ (\lambda \gamma^M \tilde{D}_J^{J'}), (\lambda \gamma^N \tilde{D}) \} (\lambda \gamma^S \tilde{D}) (\tilde{\lambda} \gamma_{SN} \tilde{D}) \Lambda_{J'}^J \\
& + \frac{3}{2} \{ (\lambda \gamma^M \tilde{D}_J^{J'}), (\lambda \gamma^N \tilde{D}) \} (\tilde{D} \gamma_{MNT} \tilde{D}) v^T \Lambda_{J'}^J \\
& = \{ (\lambda D_1), (A^{(0)})_J^{J'} \},
\end{aligned}$$

where we used the fact that the first part f_p of $(A^{(0)})_J^{J'}$ (i.e. the part that is independent of the commutators) can be written in the form given below using Fierz identities

$$\begin{aligned}
(A_{f_p}^{(0)})_J^{J'} &= 3 (\lambda \tilde{\lambda})^{-1} (\lambda \gamma^M \tilde{D}_J^{J'}) (\lambda \gamma^S \tilde{D}) (\tilde{\lambda} \gamma_{SM} \tilde{D}) \\
&+ 6 (\lambda \gamma^M \tilde{D}) (\tilde{D} \gamma_M \tilde{D}_J^{J'}).
\end{aligned}$$

Using (4.42), (4.44) and (4.48) we see that

$$(\lambda \bar{\alpha} D_{\bar{\alpha}} + z^{-\frac{1}{2}} \tilde{w} \bar{\alpha} r_{\bar{\alpha}}) (A^{(0)})_J^{J'} \Lambda_{J'}^J = (-D_J^{J'} \Omega^{(0)}) \Lambda_{J'}^J.$$

Similar arguments can be used to show that all the equations given in (4.40) are satisfied.

5 Summary and possible applications

In this paper, we expanded the BRST operator near the boundary of AdS_5 and explicitly computed the zero mode cohomology corresponding to supergravity states. The leading behavior near the AdS_5 boundary of supergravity vertex operators was expressed in $\mathcal{N} = 4$ $d = 4$ harmonic superspace in (4.17) and was shown to be BRST-closed and gauge invariant.

There are several possible applications of our results. One possible application is to generalize our methods to massive $AdS_5 \times S^5$ vertex operators and to compute the spectrum. This generalization would require allowing dependence on nonzero modes of the worldsheet variables in the analysis of the BRST cohomology. It would be interesting to compare the resulting vertex operator for the Konishi state with the pure spinor and RNS vertex operators proposed in [16] and [17].

Another possible application is to use the supergravity vertex operators to compute tree-level superstring scattering amplitudes corresponding to planar super-Yang-Mills correlation functions. For generic tree amplitudes, these computations would probably require working out the behavior of the supergravity vertex operators away from the boundary of AdS_5 . However, knowing the boundary behavior may be enough for computing certain terms in disc amplitudes with one closed string supergravity vertex operator in the bulk and N open string super-Yang-Mills vertex operators located on D-branes near the AdS_5 boundary. For the supergravity vertex operator constructed from $T^{(4-N)}$ in (4.17), one expects the resulting disc amplitude to contain terms proportional to $\int d^4x \int du \int d^8(u\theta) W^{(N)}(u, x, \theta) T^{(4-N)}(u, \bar{u}, x, \theta)$ of (4.14).

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